Annals of Mathematics, **166** (2007), 215–244

Dynamical delocalization in random Landau Hamiltonians

By François Germinet, Abel Klein, and Jeffrey H. Schenker

Abstract

We prove the existence of dynamical delocalization for random Landau Hamiltonians near each Landau level. Since typically there is dynamical localization at the edges of each disordered-broadened Landau band, this implies the existence of at least one dynamical mobility edge at each Landau band, namely a boundary point between the localization and delocalization regimes, which we prove to converge to the corresponding Landau level as either the magnetic field goes to infinity or the disorder goes to zero.

1. Introduction

In this article we prove the existence of dynamical delocalization for random Landau Hamiltonians near each Landau level. More precisely, we prove that for these two-dimensional Hamiltonians there exists at least one energy E near each Landau level such that $\beta(E) \geq \frac{1}{4}$, where $\beta(E)$, the local transport exponent introduced in [GK5], is a measure of the rate of transport in wave packets with spectral support near E. Since typically there is dynamical localization at the edges of each disordered-broadened Landau band, this implies the existence of at least one dynamical mobility edge at each Landau band, namely a boundary point between the localization and delocalization regimes, which we prove to converge to the corresponding Landau level as either the magnetic field goes to infinity or the disorder goes to zero.

Random Landau Hamiltonians are the subject of intensive study due to their links with the integer quantum Hall effect [Kli], for which von Klitzing received the 1985 Nobel Prize in Physics. They describe an electron moving in a very thin flat conductor with impurities under the influence of a constant magnetic field perpendicular to the plane of the conductor, and play an important role in the understanding of the quantum Hall effect [L], [AoA], [T], [H], [NT], [Ku], [Be], [AvSS], [BeES]. Laughlin's argument [L], as pointed out by Halperin [H], uses the assumption that under weak disorder and strong magnetic field the energy spectrum consists of bands of extended states separated

*A.K. was supported in part by NSF Grants DMS-0200710 and DMS-0457474.

by energy regions of localized states and/or energy gaps. (The experimental existence of a nonzero quantized Hall conductance was construed as evidence for the existence of extended states, e.g., [AoA], [T].) Halperin's analysis provided a theoretical justification for the existence of extended states near the Landau levels, or at least of some form of delocalization, and of nonzero Hall conductance. Kunz [Ku] stated assumptions under which he derived the divergence of a "localization length" near each Landau level at weak disorder, in agreement with Halperin's argument. Bellissard, van Elst and Schulz-Baldes [BeES] proved that, for a random Landau Hamiltonian in a tight-binding approximation, if the Hall conductance jumps from one integer value to another between two Fermi energies, then there is an energy between these Fermi energies at which a certain localization length diverges. Aizenman and Graf [AG] gave a more elementary derivation of this result, incorporating ideas of Avron, Seiler and Simon [AvSS]. (We refer to [BeES] for an excellent overview of the quantum Hall effect.) But before the present paper there were no results about nontrivial transport and existence of a *dynamical* mobility edge near the Landau levels.

The main open problem in random Schrödinger operators is delocalization, the existence of "extended states", a forty-year-old problem that goes back to Anderson's seminal article [An]. In three or more dimensions it is believed that there exists a transition from an insulator regime, characterized by "localized states", to a very different metallic regime characterized by "extended states". The energy at which this metal-insulator transition occurs is called the "mobility edge". For two-dimensional random Landau Hamiltonians such a transition is expected to occur near each Landau level [L], [H], [T].

The occurrence of localization is by now well established, e.g., [GoMP], [FrS], [FrMSS], [CKM], [S], [DrK], [KlLS], [AM], [FK1], [A], [Klo1], [CoH1], [CoH2], [FK2], [FK3], [W1], [GD],[KSS], [CoHT], [FLM], [ASFH], [DS], [GK1], [St], [W2], [Klo2], [DSS], [KlK2], [GK3], [U], [AENSS], [BouK] and many more. But delocalization is another story. At present, the only mathematical result for a typical random Schrödinger operator (that is, ergodic and with a locally Hölder-continuous integrated density of states at all energies) is for the Anderson model on the Bethe lattice, where Klein has proved that for small disorder the random operator has purely absolutely continuous spectrum in a nontrivial interval [Kl1] and exhibits ballistic behavior [Kl2]. For lattice Schrödinger operators with slowly decaying random potential, Bourgain proved existence of absolutely continuous spectrum in $d = 2$ and constructed proper extended states for dimensions $d \geq 5$ [Bou1], [Bou2]. For lattice Schrödinger operators, Jaksic and Last [JL] gave conditions under which the existence of singular spectrum can be ruled out, yielding the existence of absolutely continuous spectrum. Two other promising approaches to the phenomena of delocalization do not work directly with spectral analysis of random Schrödinger operators.

The most successful to date has been the analysis of a scaling limit of the time dependent Schrödinger equation up to a disorder dependent finite time scale [ErY], [Che], [ErSY]. It has also been suggested that delocalization could be understood in the context of random matrices [BMR]. However at present only a result on the density of states [DiPS] and a result compatible with delocalization in a modified random matrix model [SZ] have been established.

But what do we mean by delocalization? In the physics literature one finds the expression "extended states," which is often interpreted in the mathematics literature as absolutely continuous spectrum. But the latter may not be the correct interpretation in the case of random Landau Hamiltonians; Thouless [T] discussed the possibility of singular continuous spectrum or even of the delocalization occurring at a single energy. In this paper we rely on the approach to the metal-insulator transition developed by Germinet and Klein [GK5], based on transport instead of spectral properties. It provides a structural result on the dynamics of Anderson-type random operators: At a given energy E there is either dynamical localization ($\beta(E) = 0$) or dynamical delocalization with a nonzero minimal rate of tranport ($\beta(E) \geq \frac{1}{2d}$, with d the dimension). An energy at which such a transition occurs is called a dynamical mobility edge. (The terminology used in this paper differs from the one in [GK4], [GK5], which use strong insulator region for the intersection of the region of dynamical localization with the spectrum, weak metallic region for the region of dynamical delocalization, and transport mobility edge for dynamical mobility edge. Note also that the region of dynamical localization is called the region of complete localization in [GK6].)

We prove that for disorder and magnetic field for which the energy spectrum consists of disjoint bands around the Landau levels (as in the case of either weak disorder or strong magnetic field), the random Landau Hamiltonian exhibits dynamical delocalization in each band (Theorem 2.1). Since the existence of dynamical localization at the edges of these Landau bands is known [CoH2], [W1], [GK3], this proves the existence of dynamical mobility edges. We thus provide a mathematically rigorous derivation of the previously mentioned underlying assumption in Laughlin's argument.

It is worth noting that the results proved here have no implications regarding the spectral type of random Landau Hamiltonians. In fact, there might be only finitely many points, even exactly one point, in each Landau band with $\beta(E) > 0$. Indeed, $\beta(E)$ need not be continuous in E, and *a priori* there is no contradiction between having $\beta(E) \geq \frac{1}{2d}$ and the random Landau Hamiltonian having pure point spectrum almost surely in a neighborhood of E. Thus it may happen that $\beta(E) > 0$ only at a discrete set of points, for example at a single energy in each Landau band, in which case the spectrum of the Hamiltonian would be pure point almost surely. In fact, percolation arguments and numerical results indicate that for a large magnetic field there should be only one delocalized energy, located at the Landau level [ChC]. We prove that these predictions hold asymptotically. That is, for the random Landau Hamiltonian studied in [CoH2], [GK3], we prove that delocalized energies converge to the corresponding Landau level as the magnetic field goes to infinity (Corollary 2.3). We also prove this result as the disorder goes to zero for an appropriately defined random Landau Hamiltonian (Corollary 2.4).

Our proof of dynamical delocalization for random Landau Hamiltonians is based on the use of some decidedly nontrivial consequences of the multiscale analysis for random Schrödinger operators combined with the generalized eigenfunction expansion to establish properties of the Hall conductance. It relies on three main ingredients:

(1) The analysis in [GK5] showing that for an Anderson-type random Schrödinger operator the region of dynamical localization is exactly the region of applicability of the multiscale analysis, that is, the conclusions of the multiscale analysis are valid at every energy in the region of dynamical localization, and that outside this region some nontrivial transport must occur with nonzero minimal rate of transport.

(2) The random Landau Hamiltonian satisfies all the requirements for the multiscale analysis (i.e., the hypotheses in [GK1], [GK5]) at all energies. The only difficulty here is a Wegner estimate at all energies, including the Landau levels, a required hypothesis for applying (1). If the single bump in the Anderson-style potential covers the unit square this estimate was known [CoH2], [HuLMW]. But if the single bump has small support (which is the most interesting case for this paper in view of Corollary 2.3), a Wegner estimate at all energies was only known for the case of rational flux in the unit square [CoHK]. We prove a new Wegner estimate which has no restrictions on the magnetic flux in the unit square (Theorem 5.1). This Wegner estimate holds in appropriate squares with integral flux, hence the length scales of the squares may not be commensurate with the distances between bumps in the Anderson-style potential. This problem is overcome by performing the multiscale analysis with finite volume operators defined with boundary conditions depending on the location of the square (see the discussion in Section 4).

(3) Some information on the Hall conductance, namely: (i) The precise values of the Hall conductance for the (free) Landau Hamiltonian: it is constant between Landau levels and jumps by one at each Landau level, a well known fact (e.g., [AvSS], [BeES]). (ii) The Hall conductance is constant as a function of the disorder parameter in the gaps between the Landau bands, a result derived by Elgart and Schlein [ES] for smooth potentials and extended here to more general potentials (Lemma 3.3). Combining (i) and (ii) we conclude that the Hall conductivity cannot be constant across Landau bands. (iii) The Hall conductance is well defined and constant in intervals of dynamical localization. This is proved here in a very transparent way using a deep consequence of the

multiscale analysis, called SUDEC [GK6, Cor. 3(iii)], derived from a new characterization of the region of dynamical localization [GK6, Theorem 1]. SUDEC is used to show that in intervals of dynamical localization the change in the Hall conductance is given by the (infinite) sum of the contributions to the Hall conductance due to the individual localized states, which is trivially seen to be equal to zero. (See Lemma 3.2. This constancy in intervals of localization was known for discrete operators as a consequence of the quantization of the Hall conductance [BeES], [AG]. An independent but somewhat similar proof for discrete operators with finitely degenerate eigenvalues is found in the recent paper [EGS]. The proof of Lemma 3.2 does not require "*a priori*" knowledge of the nonexistence of eigenvalues with infinite multiplicity; they are controlled using SUDEC. But note that it follows from [GK6, Corollary 1] that the random Landau Hamiltonian has eigenvalues with finite multiplicity in the region of dynamical localization.) Combining (i), (ii) and (iii), we will conclude that there must be dynamical delocalization as we cross a Landau band.

It is worth noting that each of the three ingredients (1), (2) and (3) is based on intensive research conducted over the past 20 years. (1) relies on the ideas of the multiscale analysis, originally introduced by Fröhlich and Spencer [FrS] and further developed in [FrMSS], [Dr], [DrK], [S], [CoH1], [FK2], [GK1]. (2), namely the Wegner estimate, originally proved for lattice operators by Wegner [We], is a key tool for the multiscale analysis, and it has been studied in the continuum in [CoH1], [Klo1], [HuLMW], [CoHN], [HiK], [CoHK]. (3) has a long story in the study of the quantum Hall effect [L], [H], [TKNN], [Ku], [Be], [AvSS], [BeES], [AG], [ES], [EGS].

In this paper we give a simple and self-contained analysis of the Hall conductance based on consequences of localization for random Schrödinger operators. In particular, we do not use the fact that the quantization of the Hall conductance is a consequence of the geometric interpretation of this quantity as a Chern character or a Fredholm index [TKNN, Be, AvSS, BES, AG]. Our analysis applies when the disorder-broadened Landau bands do not overlap (true at either large magnetic field or small disorder); the existence of spectral gaps between the Landau bands allows the calculation of the Hall conductivity in these gaps from its values for the (free) Landau Hamiltonian as outlined in ingredient (3)(ii). In a sequel, we will discuss quantization of the Hall conductance for ergodic Landau Hamiltonians in the region where we have sufficient decay of operator kernels of the Fermi projections, extending to continuous operators an argument given in [AG] for discrete operators. This fact is well known for lattice Hamiltonians [Be, BES, AG], but the details of the proof have been spelled out for continuum operators only in spectral gaps [AvSS]. Combining results from the present paper and its sequel we expect to prove dynamical delocalization for random Landau Hamiltonians in cases when the Landau bands overlap.

This paper is organized as follows: In Section 2 we introduce the random Landau Hamiltonians and state our results. Our main result is Theorem 2.1, the existence of dynamical delocalization for random Landau Hamiltonians near each Landau level. This theorem is restated in a more general form as Theorem 2.2, which is proved in Section 3. In Corollary 2.3 we give a rather complete picture for random Landau Hamiltonians at large magnetic field as in [CoH1], [GK3]: there are dynamical mobility edges in each Landau band, which converge to the corresponding Landau level as the magnetic field goes to infinity. Corollary 2.4 gives a similar picture as the disorder goes to zero; it is proven in Section 6. In Sections 4 and 5 we show that random Landau Hamiltonians satisfy all the requirements for a multiscale analysis; Theorem 5.1 is the Wegner estimate.

Notation. We write $\langle x\rangle := \sqrt{1+|x|^2}$. The characteristic function of a set A will be denoted by χ_A. Given $x \in \mathbb{R}^2$ and $L > 0$ we set

$$\Lambda_L(x) := \left\{y \in \mathbb{R}^2;\ |y-x|_\infty < \tfrac{L}{2}\right\}, \qquad \chi_{x,L} := \chi_{\Lambda_L(x)}, \qquad \chi_x := \chi_{x,1}.$$

$C_c^\infty(I)$ denotes the class of real valued infinitely differentiable functions on $\mathbb{R}$ with compact support contained in the open interval I, with $C_{c,+}^\infty(I)$ being the subclass of nonnegative functions. The Hilbert-Schmidt norm of an operator A is written as $\|A\|_2 = \sqrt{\operatorname{tr} A^*A}$.

Acknowledgements. The authors are grateful to Jean Bellissard, Jean-Michel Combes, Peter Hislop and Frédéric Klopp for many helpful discussions.

2. Model and results

We consider the random Landau Hamiltonian

$$H_{B,\lambda,\omega} = H_B + \lambda V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2), \tag{2.1}$$

where H_B is the (free) Landau Hamiltonian,

$$H_B = (-i\nabla - \mathbf{A})^2 \quad \text{with} \quad \mathbf{A} = \tfrac{B}{2}(x_2, -x_1). \tag{2.2}$$

Here $\mathbf{A}$ is the vector potential and $B > 0$ is the strength of the magnetic field. (We use the symmetric gauge and incorporated the charge of the electron in the vector potential). The parameter $\lambda > 0$ measures the disorder strength, and V_ω is a random potential of the form

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^2} \omega_i\, u(x-i), \tag{2.3}$$

with u a measurable function satisfying $u^-\chi_{0,\varepsilon_u} \le u \le u^+\chi_{0,\delta_u}$ for some $0 < \varepsilon_u \le \delta_u < \infty$ and $0 < u^- \le u^+ < \infty$, and $\omega = \{\omega_i;\ i \in \mathbb{Z}^2\}$ a family of independent, identically distributed random variables taking values in a bounded interval $[-M_1, M_2]$ $(0 \leq M_1, M_2 < \infty,\ M_1 + M_2 > 0)$ whose common probability distribution ν has a bounded density ρ. (We write $(\Omega, \mathbb{P})$ for the underlying probability space, and $\mathbb{E}$ for the corresponding expectation.) Without loss of generality we set $\left\|\sum_{i\in\mathbb{Z}^2} u(x-i)\right\|_\infty = 1$, and hence $-M_1 \leq V_\omega(x) \leq M_2$.

$H_{B,\lambda,\omega}$ is a random operator, i.e., the mappings $\omega \to f(H_{B,\lambda,\omega})$ are strongly measurable for all bounded measurable functions on $\mathbb{R}$. We define the magnetic translations $U_a = U_a(B)$, $a \in \mathbb{R}^2$, by

$$(U_a\psi)(x) = \mathrm{e}^{-i\frac{B}{2}(x_2a_1 - x_1a_2)}\psi(x-a), \tag{2.4}$$

obtaining a projective unitary representation of $\mathbb{R}^2$ on $\mathrm{L}^2(\mathbb{R}^2)$:

$$U_aU_b = \mathrm{e}^{i\frac{B}{2}(a_2b_1 - a_1b_2)}U_{a+b} = \mathrm{e}^{iB(a_2b_1-a_1b_2)}U_bU_a, \quad a, b \in \mathbb{Z}^2. \tag{2.5}$$

We have $U_aH_BU_a^* = H_B$ for all $a \in \mathbb{R}^2$, and for magnetic translation by elements of $\mathbb{Z}^2$ we have the covariance relation:

$$U_aH_{B,\lambda,\omega}U_a^* = H_{B,\lambda,\tau_a\omega} \quad \text{for } a \in \mathbb{Z}^2, \tag{2.6}$$

where $(\tau_a\omega)_i = \omega_{i-a}$, $i \in \mathbb{Z}^2$. It follows that $H_{B,\lambda,\omega}$ is a $\mathbb{Z}^2$-ergodic random self-adjoint operator on $\mathrm{L}^2(\mathbb{R}^2)$; hence there exists a nonrandom set $\Sigma_{B,\lambda}$ such that $\sigma(H_{B,\lambda,\omega}) = \Sigma_{B,\lambda}$ with probability one, and the decomposition of $\sigma(H_{B,\lambda,\omega})$ into pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum is also independent of the choice of ω with probability one [KM1], [PF].

The spectrum $\sigma(H_B)$ of the Landau Hamiltonian H_B consists of a sequence of infinitely degenerate eigenvalues, the Landau levels:

$$B_n = (2n-1)B, \quad n = 1, 2, \ldots. \tag{2.7}$$

It will be convenient to set $B_0 = -\infty$. A simple argument shows that

$$\Sigma_{B,\lambda} \subset \bigcup_{n=1}^{\infty} \mathcal{B}_n(B,\lambda), \quad \text{where} \quad \mathcal{B}_n(B,\lambda) = [B_n - \lambda M_1, B_n + \lambda M_2]. \tag{2.8}$$

If the *disjoint bands condition*

$$\lambda(M_1 + M_2) < 2B, \tag{2.9}$$

is satisfied (true at either weak disorder or strong magnetic field), the (disorder-broadened) Landau bands $\mathcal{B}_n(B,\lambda)$ are disjoint, and hence the open intervals

$$\mathcal{G}_n(B,\lambda) =]B_n + \lambda M_2, B_{n+1} - \lambda M_1[, \quad n = 0, 1, 2, \ldots, \tag{2.10}$$

are nonempty spectral gaps for $H_{B,\lambda,\omega}$. Moreover, if $\rho > 0$ a.e. on $[-M_1, M_2]$ and (2.9) holds, then for each $B > 0$, $\lambda > 0$, and $n = 1, 2, \ldots$ we can find

$a_{j,B,\lambda,n} \in [0, \lambda M_j]$, $j = 1, 2$, continuous in λ, such that (using an argument similar to [KM2, Theorem 4])

$$\Sigma_{B,\lambda} = \bigcup_{n=1}^{\infty} \mathcal{I}_n(B,\lambda), \quad \mathcal{I}_n(B,\lambda) = [B_n - a_{1,B,\lambda,n}, B_n + a_{2,B,\lambda,n}] \,. \tag{2.11}$$

Our main result says that under the disjoint bands condition the random Landau Hamiltonian $H_{B,\lambda,\omega}$ exhibits dynamical delocalization in each Landau band $\mathcal{B}_n(B,\lambda)$. To measure "dynamical delocalization" we introduce

$$M_{B,\lambda,\omega}(p,\mathcal{X},t) = \left\| \langle x \rangle^{\frac{p}{2}} \mathrm{e}^{-itH_{B,\lambda,\omega}} \mathcal{X}(H_{B,\lambda,\omega}) \chi_0 \right\|_2^2, \tag{2.12}$$

the random moment of order $p \geq 0$ at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in the square of side one around the origin (with characteristic function χ_0), and "localized" in energy by the function $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$. Its time averaged expectation is given by

$$\mathcal{M}_{B,\lambda}(p,\mathcal{X},T) = \frac{1}{T} \int_0^\infty \mathbb{E}\left\{ M_{B,\lambda,\omega}(p,\mathcal{X},t) \right\} \mathrm{e}^{-\frac{t}{T}} \, \mathrm{d}t. \tag{2.13}$$

Theorem 2.1. *Under the disjoint bands condition the random Landau Hamiltonian $H_{B,\lambda,\omega}$ exhibits dynamical delocalization in each Landau band $\mathcal{B}_n(B,\lambda)$: For each $n = 1, 2, \ldots$ there exists at least one energy $E_n(B,\lambda) \in \mathcal{B}_n(B,\lambda)$, such that for every $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \ni E_n(B,\lambda)$ and $p > 0$, we have*

$$\mathcal{M}_{B,\lambda}(p,\mathcal{X},T) \geq \; C_{p,\mathcal{X}}\, T^{\frac{p}{4}-6} \tag{2.14}$$

for all $T \geq 0$ with $C_{p,\mathcal{X}} > 0$.

The random Landau Hamiltonian $H_{B,\lambda,\omega}$ ($\lambda > 0$) satisfies all the hypotheses in [GK1], [GK5] at all energies (see Section 4). Following [GK5], we introduce the (lower) transport exponent

$$\beta_{B,\lambda}(p,\mathcal{X}) = \liminf_{T\to\infty} \frac{\log_+ \mathcal{M}_{B,\lambda}(p,\mathcal{X},T)}{p \log T} \quad \text{for } p \geq 0,\ \mathcal{X} \in C^\infty_{c,+}(\mathbb{R}), \tag{2.15}$$

where $\log_+ t = \max\{\log t, 0\}$, and define the *p-th local transport exponent* at the energy E by (I denotes an open interval)

$$\beta_{B,\lambda}(p,E) = \inf_{I \ni E} \sup_{\mathcal{X} \in C^\infty_{c,+}(I)} \beta_{B,\lambda}(p,\mathcal{X}). \tag{2.16}$$

The transport exponents $\beta_{B,\lambda}(p,E)$ provide a measure of the rate of transport in wave packets with spectral support near E. They are increasing in p and hence we define the *local* (*lower*) *transport exponent* $\beta_{B,\lambda}(E)$ by

$$\beta_{B,\lambda}(E) = \lim_{p\to\infty} \beta_{B,\lambda}(p,E) = \sup_{p>0} \beta_{B,\lambda}(p,E). \tag{2.17}$$

These transport exponents satisfy the ballistic bound [GK5, Prop. 3.2]: $0 \leq \beta_{B,\lambda}(p,\mathcal{X})$, $\beta_{B,\lambda}(p,E)$, $\beta_{B,\lambda}(E) \leq 1$. Note that $\beta_{B,\lambda}(E) = 0$ if $E \notin \Sigma_{B,\lambda}$.

Using this local transport exponent we define two complementary regions in the energy axis for fixed $B > 0$ and $\lambda > 0$: the *region of dynamical localization*,

$$\Xi^{\mathrm{DL}}_{B,\lambda} = \{E \in \mathbb{R}; \quad \beta_{B,\lambda}(E) = 0\}\,, \tag{2.18}$$

and the region of *dynamical delocalization*,

$$\Xi^{\mathrm{DD}}_{B,\lambda} = \{E \in \mathbb{R}; \quad \beta_{B,\lambda}(E) > 0\}\,. \tag{2.19}$$

It is easily seen that $\Xi^{\mathrm{DD}}_{B,\lambda} \subset \Sigma_{B,\lambda}$. In addition, $\Xi^{DL}_{B,\lambda}$ is an open set (see [GK5]), and hence $\Xi^{\mathrm{DD}}_{B,\lambda}$ is a closed set.

We may now restate Theorem 2.1 in a more general form as

Theorem 2.2. *Consider a random Landau Hamiltonian $H_{B,\lambda,\omega}$ under the disjoint bands condition* (2.9). *Then for all $n = 1, 2, \ldots$ we have*

$$\Xi^{DD}_{B,\lambda} \cap \mathcal{B}_n(B,\lambda) \neq \emptyset. \tag{2.20}$$

In particular, there exists at least one energy $E_n(B,\lambda) \in \mathcal{B}_n(B,\lambda)$ satisfying (2.14) *and*

$$\begin{aligned} \beta_{B,\lambda}(p, E_n(B,\lambda)) &\geq \frac{1}{4} - \frac{6}{p} > 0 \quad \textit{for all} \;\; p > 24, \\ \beta_{B,\lambda}(E_n(B,\lambda)) &\geq \frac{1}{4}. \end{aligned} \tag{2.21}$$

Theorem 2.2 is proved in Section 3. We will prove (2.20), from which (2.21) and (2.14) follows by [GK5, Th.s 2.10 and 2.11]. Note that (2.14) actually holds with $T^{\frac{p}{4}-\frac{11}{2}-\varepsilon}$ for any $\varepsilon > 0$.

Next we investigate the location of the delocalized energy $E_n(B,\lambda)$, and show in two different asymptotic regimes that it converges to the n-th Landau level. We recall that in the physics literature localized and extended states are expected to be separated by an energy called a mobility edge. Similarly, there is a natural definition for a *dynamical mobility edge*: an energy $\tilde{E} \in \Xi^{\mathrm{DD}}_{B,\lambda} \cap \left\{\overline{\Xi^{DL}_{B,\lambda} \cap \Sigma_{B,\lambda}}\right\}$, that is, an energy where the spectrum undergoes a transition from dynamical localization to dynamical delocalization.

In the regime of large magnetic field (and fixed disorder) we have the following rather complete picture for the model studied in [CoH2], [GK3], consistent with the prediction that at very large magnetic field there is only one delocalized energy in each Landau band, located at the Landau level [ChC].

Corollary 2.3. *Consider a random Landau Hamiltonian $H_{B,\lambda,\omega}$ satisfying the following additional conditions on the random potential*: (i) $u \in C^2$ *and* $\operatorname{supp} u \subset D_{\frac{\sqrt{2}}{2}}(0)$, *the open disc of radius* $\frac{\sqrt{2}}{2}$ *centered at* 0. (ii) *The density of the probability distribution ν is an even function $\rho > 0$ a.e. on* $[-M, M]$

($M = M_1 = M_2$). (iii) $\nu([0,s]) \geq c \min\{s, M\}^{\zeta}$ *for some* $c > 0$ *and* $\zeta > 0$. *Fix* $\lambda > 0$ *and let* $B > 0$ *satisfy* (2.9), *in which case the spectrum* $\Sigma_{B,\lambda}$ *is given by* (2.11) *with*

$$0 \leq \lambda M - a_{j,B,\lambda,n} \leq C_n(\lambda) B^{-\frac{1}{2}}, \quad j = 1, 2. \tag{2.22}$$

Then for each $n = 1, 2, \ldots,$ *if* B *is large enough* (*depending on* n) *there exist dynamical mobility edges* $\widetilde{E}_{j,n}(B, \lambda)$, $j = 1, 2$, *with*

$$\max_{j=1,2} \left| \widetilde{E}_{j,n}(B, \lambda) - B_n \right| \leq K_n(\lambda) \frac{\log B}{B} \to 0 \quad \text{as } B \to \infty, \tag{2.23}$$

$$B_n - a_{1,B,\lambda,n} < \widetilde{E}_{1,n}(B, \lambda) \leq \widetilde{E}_{2,n}(B, \lambda) < B_n + a_{2,B,\lambda,n}, \tag{2.24}$$

$$[B_n - a_{1,B,\lambda,n}, \widetilde{E}_{1,n}(B, \lambda)[\cup]\widetilde{E}_{2,n}(B, \lambda), B_n + a_{2,B,\lambda,n}] \subset \Xi^{DL}_{B,\lambda}. \tag{2.25}$$

(*By* $C_n(\lambda), K_n(\lambda)$ *we denote finite constants. It is possible that* $\widetilde{E}_{1,n}(B, \lambda) = \widetilde{E}_{2,n}(B, \lambda)$, *i.e., dynamical delocalization occurs at a single energy.*)

Proof. The estimate (2.22) is proven in [CoH2], the existence of energies $\widetilde{E}_{j,n}(B, \lambda)$, $j = 1, 2$, satisfying (2.24), (2.25) and (2.23) is proven in [GK3, Theorem 4.1]. The fact that we can choose $\widetilde{E}_{j,n}(B, \lambda)$, $j = 1, 2$, that are also dynamical mobility edges follows from Theorem 2.1. □

We now investigate the small disorder regime (at fixed magnetic field) and prove a result in the spirit of Corollary 2.3. It is not too interesting to just let $\lambda \to 0$ in (2.1), since the spectrum of the Hamiltonian would then shrink to the Landau levels (see (2.8)) and the result would be trivial. In order to keep the size of the spectrum constant we rescale the probability distribution ν of the ω_i's by concentrating more and more of the mass of ν around zero as $\lambda \to 0$.

COROLLARY 2.4. *Let* $\rho > 0$ *a.e. on* $\mathbb{R}$ *be the density of a probability distribution* ν *with* $\langle u \rangle^{\gamma} \rho(u)$ *bounded for some* $\gamma > 1$. *Fix* $b > 0$, *and set* ν_λ *to be the probability distribution with density* $\rho_\lambda(u) = c_{b,\lambda} \lambda^{-1} \rho(\lambda^{-1} u) \chi_{[-b,b]}(u)$, *where the constant* $c_{b,\lambda}$ *is chosen so that* $\nu_\lambda(\mathbb{R}) = \nu_\lambda([-b, b]) = 1$. *Define* $H_{\omega,B,\lambda}$ *by* (2.1) *with* $\lambda = 1$ *but with the* λ *dependent common probability distribution* ν_λ *for the random variables* $\{\omega_i;\, i \in \mathbb{Z}^2\}$. *Assuming* $B > b$, (2.9) *holds and the spectrum* $\Sigma_{B,\lambda}$ *given by* (2.11) *is independent of* λ. *Then for each* $n = 1, 2, \ldots,$ *if* λ *is small enough* (*depending on* n) *there exist dynamical mobility edges* $\widetilde{E}_{j,n}(B, \lambda)$, $j = 1, 2$, *satisfying* (2.24) *and* (2.25), *and we have*

$$\max_{j=1,2} \left| \widetilde{E}_{j,n}(B, \lambda) - B_n \right| \leq K_n(B) \lambda^{\frac{\gamma - 1}{\gamma}} \left| \log \lambda \right|^{\frac{2}{\gamma}} \to 0 \quad \text{as } \lambda \to 0, \tag{2.26}$$

with a finite constant $K_n(B)$. *Moreover, if the density* ρ *satisfies the stronger condition of* $\mathrm{e}^{|u|^{\alpha}} \rho(u)$ *being bounded for some* $\alpha > 0$, *the estimate in* (2.26) *holds with* $K_n(B)\lambda\,|\log\lambda|^{\frac{1}{\alpha}}$ *in the right-hand side.* (*It is possible that* $\widetilde{E}_{1,n}(B,\lambda) = \widetilde{E}_{2,n}(B,\lambda)$, *i.e.*, *dynamical delocalization occurs at a single energy.*)

Corollary 2.4 is proven in Section 6.

3. The existence of dynamical delocalization

In this section we prove Theorem 2.2 (and hence Theorem 2.1). For convenience we write $H_{B,0,\omega} = H_B$ and extend (2.18) to $\lambda = 0$ by $\Xi^{\mathrm{DL}}_{B,0} = \mathbb{R}\backslash\sigma(H_B) = \mathbb{R}\backslash\{B_n;\ n = 1,2,\dots\}$; the statements below will hold (trivially) for $\lambda = 0$ unless this case is explicitly excluded. Given a Borel set $\mathcal{J} \subset \mathbb{R}$, we set $P_{B,\lambda,\mathcal{J},\omega} = \chi_{\mathcal{J}}(H_{B,\lambda,\omega})$. If $\mathcal{J} =]-\infty, E]$, we write $P_{B,\lambda,E,\omega}$ for $P_{B,\lambda,]-\infty,E],\omega}$, the Fermi projection corresponding to the Fermi energy E.

The random Landau Hamiltonian $H_{B,\lambda,\omega}$ ($\lambda > 0$) satisfies all the hypotheses in [GK1], [GK5], [GK6] at all energies, as shown in Section 4. The following results, stated below as properties, are relevant to the proof of Theorem 2.2: RDL (region of dynamical localization), RDD (region of dynamical delocalization), DFP (decay of the Fermi projection), and SUDEC (summable uniform decay of eigenfunction correlations). (We refer the reader to Section 4 for a discussion of the multiscale analysis and the relevant results.)

Property RDL. *The region of dynamical localization* $\Xi^{DL}_{B,\lambda}$ (*see* (2.18)) *is exactly the region of applicability of the multiscale analysis*, *that is*, *the conclusions of the multiscale analysis are valid at every energy* $E \in \Xi^{DL}_{B,\lambda}$ [GK5, Theorem 2.8].

Property RDD. *Let* $\lambda > 0$. *If an energy* E *is in the region of dynamical delocalization* $\Xi^{DD}_{B,\lambda}$ (*see* (2.19)) *we must have* $\beta_{B,\lambda}(E) \ge \frac{1}{4}$; *in fact*, $\beta_{B,\lambda}(p,E) \ge \frac{1}{4} - \frac{11}{2p} > 0$ *for all* $p > 22$. *Moreover*, *for each* $\mathcal{X} \in C^{\infty}_{c,+}(\mathbb{R})$ *with* $\mathcal{X} \equiv 1$ *on some open interval* $J \ni E$, *we have*

$$\lim_{T\to\infty} \frac{1}{T^{\alpha}} \mathcal{M}_{B,\lambda}(p,\mathcal{X},T) = \infty \tag{3.1}$$

for all $\alpha \ge 0$ *and* $p > 4\alpha + 22$ [GK5, Theorems 2.10 and 2.11].

Property DFP. *The Fermi projection* $P_{B,\lambda,E,\omega}$ *exhibits fast decay if the Fermi energy* E *is in the region of dynamical localization* $\Xi^{DL}_{B,\lambda}$: *If* $E \in \Xi^{DL}_{B,\lambda}$ *and* $\zeta \in]0,1[$,

$$\mathbb{E}\left\{\|\chi_x P_{B,\lambda,E,\omega}\chi_y\|_2^2\right\} \le C_{\zeta,B,\lambda,E}\ \mathrm{e}^{-|x-y|^{\zeta}} \quad \textit{for all } x,y \in \mathbb{Z}^2, \tag{3.2}$$

with the constant $C_{\zeta,B,\lambda,E}$ *locally bounded in* E . (See [GK6, Theorem 3]– the result is based on [GK1, Theorem 3.8] and [BoGK, Theorem 1.4].) *As a consequence*, *for* $\mathbb{P}$-*a.e.* ω *and each* $\zeta \in]0,1[$ *there exists* $C_{\zeta,B,\lambda,E,\omega} < \infty$, *locally bounded in* E, *such that*

$$\|\chi_x P_{B,\lambda,E,\omega}\chi_y\|_2 \le C_{\zeta,B,\lambda,E,\omega}\langle x\rangle\langle y\rangle\ \mathrm{e}^{-|x-y|^{\zeta}} \quad \textit{for all } x,y \in \mathbb{Z}^2. \tag{3.3}$$

(Sufficiently fast polynomial decay would suffice for our purposes. Note that in the special case when E is in a spectral gap of $H_{B,\lambda,\omega}$ a simple argument based on the Combes-Thomas estimate yields exponential decay, i.e., $\zeta = 1$.)

PROPERTY SUDEC. *For $\mathbb{P}$-a.e. ω the Hamiltonian $H_{B,\lambda,\omega}$ has pure point spectrum in $\Xi_{B,\lambda}^{DL}$ with the following property: Given a closed interval $I \subset \Xi_{B,\lambda}^{DL}$, let $\{\phi_{n,\omega}\}_{n\in\mathbb{N}}$ be a complete orthonormal set of eigenfunctions of $H_{B,\lambda,\omega}$ with eigenvalues $E_{n,\omega} \in I$; for each n we denote by $P_{n,\omega}$ the one-dimensional orthogonal projection on the span of $\phi_{n,\omega}$ and set $\alpha_{n,\omega} = \left\|\langle x\rangle^{-2} P_{n,\omega}\right\|_2^2 = \|\langle x\rangle^{-2}\phi_{n,\omega}\|^2$. Then for each $\zeta \in]0,1[$ there exists $C_{I,\zeta,\omega} < \infty$ such that for all $x, y \in \mathbb{Z}^2$ we have*

$$\|\chi_x P_{n,\omega}\chi_y\|_2 = \|\chi_x\phi_{n,\omega}\|\|\chi_y\phi_{n,\omega}\| \leq C_{I,\zeta,\omega}\alpha_{n,\omega}\langle x\rangle^2\langle y\rangle^2 \ \mathrm{e}^{-|x-y|^\zeta}. \tag{3.4}$$

Moreover,

$$\sum_{n\in\mathbb{N}} \alpha_{n,\omega} = \mu_\omega(I) := \operatorname{tr}\left\{\langle x\rangle^{-2} P_{B,\lambda,I,\omega}\langle x\rangle^{-2}\right\} < \infty. \tag{3.5}$$

(See [GK6, Corollary 3(iii)]. Almost-sure pure point spectrum is well known, e.g., [FrMSS], [DrK], [GK1], [Kl3]. Property SUDEC, namely (3.4) with (3.5), is a modification of Germinet's WULE [G]. It is the almost everywhere consequence of a new characterization of the region of dynamical localization [GK6, Theorem 1].)

Remark. Throughout this work we characterize the localization regime using consequences of the multiscale analysis. If the single site bumps of the Anderson-type potential cover the whole space, i.e. if $\sum_{i\in\mathbb{Z}^d} u(x-i) \geq \delta > 0$, then another option is available, namely the fractional moment method [AENSS], which yields exponential bounds for expectations. However at this time the fractional moment method is not available for potentials which violate the aforementioned "covering condition."

We now turn to the Hall conductance. Consider the switch function $\Lambda(t) = \chi_{[\frac{1}{2},\infty)}(t)$ and let Λ_j denote multiplication by the function $\Lambda_j(x) = \Lambda(x_j)$, $j = 1, 2$. Given an orthogonal projection P on $\mathrm{L}^2(\mathbb{R}^2)$, we set

$$\Theta(P) := \operatorname{tr}\{P\left[[P,\Lambda_1],[P,\Lambda_2]\right]\}, \tag{3.6}$$

defined whenever

$$|\Theta|(P) := \|P\left[[P,\Lambda_1],[P,\Lambda_2]\right]\|_1 < \infty, \tag{3.7}$$

in which case we also have

$$\Theta(P) = \operatorname{tr}\{[P\Lambda_1 P, P\Lambda_2 P]\}. \tag{3.8}$$

Note that although $\Theta(P)$ is the trace of a commutator it need not be zero, because in general the two summands $P\Lambda_1 P\Lambda_2 P$ and $P\Lambda_2 P\Lambda_1 P$ are not separately trace class.

LEMMA 3.1. *Let P be an orthogonal projection on* $\mathrm{L}^2(\mathbb{R}^2)$ *such that for some* $\xi \in]0,1]$, $\kappa > 0$, *and* $K_P < \infty$,

$$\|\chi_x P \chi_y\|_2 \le K_P \langle x\rangle^\kappa \langle y\rangle^\kappa \,\mathrm{e}^{-|x-y|^\xi} \quad \textit{for all } x, y \in \mathbb{Z}^2. \tag{3.9}$$

Then:

(i) $|\Theta|(P) \le C_{\xi,\kappa} K_P^2$ *for some constant* $C_{\xi,\kappa}$ *independent of* P, *and* $\Theta(P)$ *is well defined.*

(ii) *Given* $s \in \mathbb{R}$, *let* $\Lambda^{(s)}(t) = \Lambda(t-s)$ *and* $\Lambda_j^{(s)}(x) = \Lambda^{(s)}(x_j)$, $j = 1,2$. *Set* $\Theta_{r,s}(P) = \operatorname{tr}\left\{P\left[\left[P, \Lambda_1^{(r)}\right], \left[P, \Lambda_2^{(s)}\right]\right]\right\}$, $r, s \in \mathbb{R}$. *Then* $\Theta_{r,s}(P)$ *is well defined as in* (i), *and* $\Theta_{r,s}(P) = \Theta(P)$.

(iii) *Let Q be another orthogonal projection on* $\mathrm{L}^2(\mathbb{R}^2)$ *satisfying* (3.9) *with some constant* K_Q, *such that* $QP = PQ = 0$. *Then* $P + Q$ *is an orthogonal projection satisfying* (3.9) *with constant* $K_{P+Q} = K_P + K_Q$, *and we have*

$$\Theta(P+Q) = \Theta(P) + \Theta(Q). \tag{3.10}$$

Remark. (i) is similar to statements in [AvSS], [AG], (ii) and (iii) are well known [AvSS], [BeES], [AG]. We provide a short proof in our setting; the precise form of the bound in (3.9) is important for Lemma 3.2. Lemma 3.1 remains true if Λ is replaced by any monotone "switch function," with $\Lambda(t) \to 0, 1$ as $t \to -\infty, +\infty$, with essentially the same proof.

Proof. If $x \in \mathbb{Z}^2$ we have $\Lambda_j \chi_x = \Lambda(x_j)\chi_x$, $j = 1,2$, and hence, if $x_1 y_1 > 0$ we get $\chi_x[P, \Lambda_1]\chi_y = (\Lambda(y_1) - \Lambda(x_1))\chi_x P \chi_y = 0$. If $x_1 y_1 \le 0$, we have $|x_1 - y_1|^\xi \ge \frac{1}{2}|x_1|^\xi + \frac{1}{2}|y_1|^\xi$. Thus it follows from (3.9) that for all $x, y \in \mathbb{Z}^2$,

$$\|\chi_x [P, \Lambda_1] \chi_y\|_2 \le K_P \langle x\rangle^\kappa \langle y\rangle^\kappa \,\mathrm{e}^{-\frac{1}{4}|x_1|^\xi - \frac{1}{4}|y_1|^\xi - \frac{1}{2}|x_2 - y_2|^\xi}, \tag{3.11}$$

and, similarly,

$$\|\chi_x [P, \Lambda_2] \chi_y\|_2 \le K_P \langle x\rangle^\kappa \langle y\rangle^\kappa \,\mathrm{e}^{-\frac{1}{4}|x_2|^\xi - \frac{1}{4}|y_2|^\xi - \frac{1}{2}|x_1 - y_1|^\xi}. \tag{3.12}$$

We conclude that

$$\|P[P,\Lambda_1][P,\Lambda_2]\|_1 \le \sum_{x,y,z \in \mathbb{Z}^2} \|\chi_x [P, \Lambda_1] \chi_y\|_2 \, \|\chi_y [P, \Lambda_2] \chi_z\|_2 \le C_1 K_P^2 < \infty, \tag{3.13}$$

where C_1 is a finite constant independent of P, and similarly $\|P[P,\Lambda_2][P,\Lambda_1]\|_1 \le C_1 K_P^2$. Part (i) follows.

The only nontrivial item in (iii) is (3.10). It follows from (3.6), cyclicity of the trace, and the fact that $P[Q, \Lambda_j] = -P\Lambda_j Q$ for $j = 1, 2$.

It remains to prove (ii). The proof of (i) clearly applies also to $\Theta_{r,s}(P)$; we only need to show that $\Theta_{r,s}(P) = \Theta(P)$. This will follow if we can show that

$$\operatorname{tr}\{P\,[[P,F_1]\,,[P,G_2]]\} = \operatorname{tr}\{P\,[[P,G_1]\,,[P,F_2]]\} = 0, \tag{3.14}$$

if $F = \Lambda^{(s)} - \Lambda^{(s')}$ and $G = \Lambda^{(s'')}$ for some $s, s', s'' \in \mathbb{R}$, with $F_j(x) = F(x_j)$, $G_j(x) = G(x_j)$, $j = 1, 2$. Note that $F_j(x)$ has compact support in the x_j direction. If we write a trace without a comment, as in (3.14), we are implicitly stating that it is well defined by the argument in (3.11)–(3.13).

We have

$$\begin{aligned} \operatorname{tr}\{P\,[[P,F_1]\,,[P,G_2]]\} &= \operatorname{tr}\{PF_1(I-P)\,[P,G_2]\} + \operatorname{tr}\{[P,G_2]\,(I-P)F_1P\} \\ &= \operatorname{tr}\{F_1(I-P)\,[P,G_2]\,P + F_1P\,[P,G_2]\,(I-P)\} \\ &= \operatorname{tr}\{F_1\,[P,G_2]\} = \operatorname{tr}\{[F_1P,G_2]\}\,. \end{aligned} \tag{3.15}$$

Here $[F_1P,G_2] = F_1\,[P,G_2] = F_1(I-P)\,[P,G_2]\,P + F_1P\,[P,G_2]\,(I-P)$ is trace class, since the two operators in the sum are trace class by the argument in (3.11)–(3.13). If F_1PG_2 and G_2F_1P were trace class, we could then conclude that $\operatorname{tr}\{[F_1P,G_2]\} = 0$. Since F_1PG_2 and G_2F_1P may not be trace class, we need an extra argument. Since P is a projection satisfying (3.9), using $\|\chi_x P \chi_y\|_1 \le \sum_{z\in\mathbb{Z}^2} \|\chi_x P \chi_z P \chi_y\|_1$ we get

$$\|\chi_x P \chi_y\|_1 \le C K_P^2 \langle x\rangle^{2\kappa} \langle y\rangle^{2\kappa}\, \mathrm{e}^{-\frac{1}{2}|x-y|^{\xi}} \quad \text{for all } x, y \in \mathbb{Z}^2, \tag{3.16}$$

for some constant C. We recall that the function $F_1(x)$ has compact support in the x_1 direction, and introduce a cutoff $Y_n(x) = \chi_{[-n,n]}(x_2)$ in the x_2 direction. Then

$$\operatorname{tr}\{Y_n\,[F_1P,G_2]\} = \operatorname{tr}\{[Y_nF_1P,G_2]\} = 0, \tag{3.17}$$

since $Y_nF_1PG_2$ and $Y_nG_2F_1P$ are then trace class by (3.16). Thus, since $Y_n \to I$ strongly and boundedly ($\|Y_n\| = 1$),

$$\operatorname{tr}\{[F_1P,G_2]\} = \lim_{n\to\infty} \operatorname{tr}\{Y_n\,[F_1P,G_2]\} = 0. \tag{3.18}$$

The other term in (3.14) is treated in the same way, and Part (ii) is proven. □

For a given disorder $\lambda \ge 0$, magnetic field $B > 0$, and energy $E \in \Xi^{\mathrm{DL}}_{B,\lambda}$, we consider the Hall conductance [AvSS], [ES]

$$\sigma_{H,\omega}(B,\lambda,E) = -2\pi i\,\Theta(P_{B,\lambda,E,\omega}), \tag{3.19}$$

well defined for $\mathbb{P}$-a.e. ω in view of Lemma 3.1(i) and DFP, namely (3.3). The covariance relation (2.6) and Lemma 3.1(ii) then imply $\sigma_{H,\omega}(B,\lambda,E) = \sigma_{H,\tau_a\omega}(B,\lambda,E)$ for all $a \in \mathbb{Z}^2$ for $\mathbb{P}$-a.e. ω, and hence ergodicity yields

$$\sigma_H(B,\lambda,E) := \mathbb{E}\{\sigma_{H,\omega}(B,\lambda,E)\} = \sigma_{H,\omega}(B,\lambda,E) \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{3.20}$$

A key ingredient in justifications of the quantum Hall effect is that the Hall conductance should be constant in intervals of localization since localized states do not carry current [L], [H], [Ku], [BeES]. The following lemma makes this precise in a very transparent way: In intervals of dynamical localization the change in the Hall conductance is given by the (infinite) sum of the Hall conductance of the individual localized states. But the conductance of a localized state is equal to $-2\pi i\Theta(P)$, where P is the orthogonal projection on the localized state. But if P is a one-dimensional orthogonal projection, say on the span of unit vector ψ, (3.8) yields

$$\Theta(P) = \langle\psi, \Lambda_1\psi\rangle\langle\psi, \Lambda_2\psi\rangle - \langle\psi, \Lambda_2\psi\rangle\langle\psi, \Lambda_1\psi\rangle = 0. \tag{3.21}$$

LEMMA 3.2. *The Hall conductance $\sigma_H(B, \lambda, E)$ is constant on connected components of $\Xi^{DL}_{B,\lambda}$, that is, if $[E_1, E_2] \subset \Xi^{DL}_{B,\lambda}$ we must have $\sigma_H(B, \lambda, E_1) = \sigma_H(B, \lambda, E_2)$.*

Proof. If $I = [E_1, E_2] \subset \Xi^{\mathrm{DL}}_{B,\lambda}$, we apply property (SUDEC) in I for the $\mathbb{P}$-a.e. ω for which we have (3.4) and (3.5). Given a (finite or infinite) subset M of the index set $\mathbb{N}$, we set $P_{M,\omega} = \sum_{n\in M} P_{n,\omega}$; it follows that we have condition (3.9) for $P_{M,\omega}$ for $\kappa = 2$ and all $\zeta \in]0, 1[$ with constant

$$K_{P_{M,\omega}} = C_{I,\zeta,\omega} \sum_{n\in M} \alpha_{n,\omega} \leq C_{I,\zeta,\omega}\, \mu_\omega(I) < \infty. \tag{3.22}$$

Since $P_{B,\lambda,]E_1,E_2],\omega} = P_{B,\lambda,E_2,\omega} - P_{B,\lambda,E_1,\omega}$, it follows from Lemma 3.1, (i) and (iii), that it suffices to prove that

$$\Theta(P_{B,\lambda,]E_1,E_2],\omega}) = 0. \tag{3.23}$$

But again using Lemma 3.1, (i) and (iii), taking $M = \{1, 2\ldots, m\} \subset \mathbb{N}$, we have

$$\begin{aligned} \Theta(P_{B,\lambda,]E_1,E_2],\omega}) &= \Theta(P_{\mathbb{N},\omega}) = \Theta(P_{M,\omega}) + \Theta(P_{(\mathbb{N}\setminus M),\omega}) \\ &= \sum_{n=1}^{m} \Theta(P_{n,\omega}) + \Theta(P_{(\mathbb{N}\setminus M),\omega}). \end{aligned} \tag{3.24}$$

Since by Lemma 3.1(i), (3.22) and (3.5) we have

$$\left|\Theta(P_{(\mathbb{N}\setminus M),\omega})\right| \leq C_\zeta \left(C_{I,\zeta,\omega} \sum_{n=m+1}^{\infty} \alpha_{n,\omega} \right)^2 \to 0 \quad \text{as } m \to \infty, \tag{3.25}$$

we conclude that

$$\Theta(P_{B,\lambda,]E_1,E_2],\omega}) = \sum_{n=1}^{\infty} \Theta(P_{n,\omega}) = 0 \tag{3.26}$$

in view of (3.21). □

Remark. The constancy of the Hall conductance in intervals of localization is known for lattice Hamiltonians with eigenvalues of finite multiplicity [BeES], [AG], [EGS]. Our proof allows eigenvalues of infinite multiplicity; the crucial estimate is (3.22), a consequence of property (SUDEC).

In the next lemma, we calculate the value of the Hall conductance in the spectral gaps between the bands under the disjoint bands condition.

LEMMA 3.3. *Under the disjoint bands conditions* (2.9) *we have* $\sigma_H(B, \lambda, E) = n$ *if* $E \in \mathcal{G}_n(B, \lambda)$ *for all* $n = 0, 1, 2 \ldots$.

Proof. It is well known that $\sigma_H(B, 0, E) = n$ if $E \in]B_n, B_{n+1}[$ for all $n = 0, 1, 2 \ldots$ [AvSS], [BeES]. Under condition (2.9), if $E \in \mathcal{G}_n(B, \lambda_1)$ for some $n \in \{0, 1, 2 \ldots\}$ we can find $\lambda_E > \lambda_1$ such that $E \in \mathcal{G}_n(B, \lambda)$ for all $\lambda \in I = [0, \lambda_E[$. We take $\omega \in [-M_1, M_2]^{\mathbb{Z}^2}$, a set of probability one, and note that the contour Γ below and all the constants on what follows are independent of ω. We have

$$P_\lambda = -\tfrac{1}{2\pi i} \int_\Gamma R_\lambda(z)\, \mathrm{d}z \quad \text{for all } \lambda \in I, \tag{3.27}$$

where $P_\lambda = P_{B,\lambda,E,\omega}$, $R_\lambda(z) = (H_{B,\lambda,\omega} - z)^{-1}$, and Γ is a bounded contour such that $d(\Gamma, \sigma(H_{B,\lambda,\omega})) \geq \eta > 0$ for all $\lambda \in I$. (Note $H_{B,\lambda,\omega} \geq B - \lambda_E M_1$ for all $\lambda \in I$.) We have ($K_1, K_2, \ldots$ denote constants)

$$\|\chi_x R_\lambda(z) \chi_y\| \leq K_1 \mathrm{e}^{-K_1|x-y|} \quad \text{for all } x, y \in \mathbb{Z}^2, z \in \Gamma, \lambda \in I, \tag{3.28}$$

$$\|R_\lambda(z)\chi_x\|_2 \leq K_2 \quad \text{for all } x \in \mathbb{Z}^2, z \in \Gamma, \lambda \in I, \tag{3.29}$$

where (3.28) is the Combes-Thomas estimate (e.g., [GK2, Cor. 1]) and (3.29) is in [BoGKS, Proposition 2.1]. Combining with (3.27), we get

$$\|\chi_x P_\lambda \chi_y\| \leq \tfrac{K_1|\Gamma|}{2\pi}\, \mathrm{e}^{-K_1|x-y|} \quad \text{for all } x, y \in \mathbb{Z}^2, \lambda \in I, \tag{3.30}$$

$$\|\chi_x P_\lambda \chi_y\|_1 \leq \left(\tfrac{K_2|\Gamma|}{2\pi}\right)^2 \quad \text{for all } x, y \in \mathbb{Z}^2, \lambda \in I, \tag{3.31}$$

$$\|\chi_x P_\lambda \chi_y\|_2 \leq K_3\, \mathrm{e}^{-K_3|x-y|} \quad \text{for all } x, y \in \mathbb{Z}^2, \lambda \in I, \tag{3.32}$$

where (3.32) follows from (3.30) and (3.31).

Given $\lambda, \xi \in I$, it follows from (3.27) and the resolvent identity that

$$Q_{\lambda,\xi} := P_\xi - P_\lambda = \tfrac{(\xi-\lambda)}{2\pi i} \int_\Gamma R_\lambda(z) V R_\xi(z)\, \mathrm{d}z, \tag{3.33}$$

with $V = V_\omega$ (recall $\|V\| \leq \max\{M_1, M_2\}$). Using (3.28) and (3.29), we get

$$\|\chi_x Q_{\lambda,\xi} \chi_y\|_2 \leq K_4\, \mathrm{e}^{-K_4|x-y|} \quad \text{for all } x, y \in \mathbb{Z}^2, \lambda, \xi \in I. \tag{3.34}$$

We now use an idea of Elgart and Schlein [ES]. If the potential V had compact support, it would follow from (3.29) that $Q_{\lambda,\xi}$ is trace class. In this case, using (3.8) and (3.33), we get

$$\begin{aligned}\Theta(P_\xi)-\Theta(P_\lambda) = \operatorname{tr}\{&[Q_{\lambda,\xi}\Lambda_1 P_\xi, P_\xi\Lambda_2 P_\xi] + [P_\lambda\Lambda_1 Q_{\lambda,\xi}, P_\xi\Lambda_2 P_\xi] \\ &+ [P_\lambda\Lambda_1 P_\lambda, Q_{\lambda,\xi}\Lambda_2 P_\xi] + [P_\lambda\Lambda_1 P_\lambda, P_\lambda\Lambda_2 Q_{\lambda,\xi}]\} = 0,\end{aligned} \tag{3.35}$$

since $\operatorname{tr}[A,B]=0$ if either A or B are trace class by centrality of the trace. Our potential V, given in (2.3), does not have compact support, so we will use an approximation argument.

Given $L>0$ and $\omega\in[-M_1,M_2]^{\mathbb{Z}^2}$, we define $\omega^{(L)},\omega^{(>L)}\in[-M_1,M_2]^{\mathbb{Z}^2}$ by $\omega_i^{(L)}=\omega_i$ if $|i|\le L$ and $\omega_i^{(L)}=0$ otherwise, and $\omega_i^{(>L)}=\omega_i-\omega_i^{(L)}$ for all $i\in\mathbb{Z}^2$. Recalling (2.3), we set $V_L=V_{\omega^{(L)}}$, $V_{>L}=V_{\omega^{(>L)}}=V-V_L$, $P_{\lambda,L}=P_{B,\lambda,E,\omega^{(L)}}$, $P_{\lambda,>L}=P_{B,\lambda,E,\omega^{(>L)}}=P_\lambda-P_{\lambda,L}$, etc. We have

$$Q_{\lambda,>L}:=P_\lambda-P_{\lambda,L}=\tfrac{\lambda}{2\pi i}\int_\Gamma R_\lambda(z)V_{>L}R_{\lambda,L}(z)\,\mathrm{d}z. \tag{3.36}$$

Moreover, it follows from (3.36) and (3.28) that

$$\|\chi_x Q_{\lambda,>L}\chi_y\|\le K_5\,\mathrm{e}^{-K_5(\max\{L-|x|,0\}+\max\{L-|y|,0\}}\,\mathrm{e}^{-K_5|x-y|} \tag{3.37}$$

for all $x,y\in\mathbb{Z}^2,\lambda\in I$ and $L>0$, with a similar estimate holding in the Hilbert-Schmidt norm by the argument used for (3.32). Using (3.6) and (3.36), we have

$$\begin{aligned}&\Theta(P_\lambda)-\Theta(P_{\lambda,L}) \\ &\quad=\operatorname{tr}\{Q_{\lambda,>L}\left[[P_\lambda,\Lambda_1],[P_\lambda,\Lambda_2]\right]+P_{\lambda,L}\left[[Q_{\lambda,>L},\Lambda_1],[P_\lambda,\Lambda_2]\right] \\ &\qquad+P_{\lambda,L}\left[[P_{\lambda,L},\Lambda_1],[Q_{\lambda,>L},\Lambda_2]\right]\}\to 0\quad\text{as } L\to\infty,\end{aligned} \tag{3.38}$$

where the convergence to 0 is proved as follows: Since $\|Q_{\lambda,>L}\|\le K_6$ for all $L>0$ and $Q_{\lambda,>L}\to 0$ strongly as $L\to\infty$, the trace of the first term goes to 0 as $L\to\infty$. The traces of the other two terms can be estimated as in (3.13), and converge to 0 as $L\to\infty$ by an argument using (3.37) and dominated convergence.

The lemma now follows from (3.35) and (3.38). □

We may now finish the proof of Theorem 2.2. Since (2.9) holds, if $\mathcal{B}_n(B,\lambda)\subset\Xi^{\mathrm{DL}}_{B,\lambda}$ for some $n\in\{1,2,\dots\}$, we have

$$]B_{n-1}+\lambda M_1, B_{n+1}-\lambda M_2[\,=\mathcal{G}_{n-1}(B,\lambda)\cup\mathcal{B}_n(B,\lambda)\cup\mathcal{G}_n(B,\lambda)\subset\Xi^{\mathrm{DL}}_{B,\lambda},$$

and hence it follows from Lemma 3.2 that the Hall conductance $\sigma_H(B,\lambda,E)$ has the same value on the spectral gaps $\mathcal{G}_{n-1}(B,\lambda)$ and $\mathcal{G}_n(B,\lambda)$, which contradicts Lemma 3.3. Thus we must have $\mathcal{B}_n(B,\lambda)\cap\Xi^{\mathrm{DD}}_{B,\lambda}\ne\emptyset$ for all $n\in\{1,2,\dots\}$, and hence Theorem 2.2 follows from property (RDD).

4. The applicability of the multiscale analysis

In order to use properties RDL, RDD, DFP, and SUDEC, stated in Section 3, we must show that the results in [GK1], [GK5], [GK6] apply to the random Landau Hamiltonian $H_{B,\lambda,\omega}$ as in (2.1). Thus we need to verify that the random Landau Hamiltonian satisfy the requirements for the bootstrap multiscale analysis–the hypotheses in [GK1], [GK5], [GK6]–at all energies, including the Landau levels. To do so, we will define finite volume operators for the multiscale analysis in a nonstandard way, which in turn will require slight changes in the multiscale analysis.

In this context the multiscale analysis is a technique, initially developed by Fröhlich and Spencer [FrS] and Fröhlich, Martinelli, Spencer and Scoppolla [FrMSS], and simplified by von Dreifus [Dr] and von Dreifus and Klein [DrK], for the purpose of proving exponential localization (pure point spectrum and exponential decay of eigenfunctions). Although originally developed for lattice Hamiltonians, it was extended to continuum Hamiltonians by Combes and Hislop [CoH1] and Figotin and Klein [FK2]. It was shown to yield dynamical localization almost-surely by Germinet and De Bièvre [GD], and strong (i.e., in expectation) dynamical localization for moments up to some finite order by Damanik and Stollman [DS]. To go beyond this limitation, Germinet and Klein [GK1] developed the bootstrap multiscale analysis, built out of four different multiscale analyses, which yields exponential localization, strong dynamical localization (up to all orders) in the Hilbert-Schmidt norm, sub-exponential decay of the expectation of the kernel of the evolution operator, semi-uniformly localized eigenfunctions (SULE), and, as shown in [GK6], SUDEC, decay of the Fermi projection, and finite multiplicity of eigenvalues. It also plays a crucial role in the converse to the multiscale analysis of Germinet and Klein [GK5]; previous versions of the multiscale analysis do not suffice for the nontrivial lower bound on the transport exponent outside the region of applicability of the multiscale analysis.

The requirements for the bootstrap multiscale analysis were called assumptions or properties SGEE (*strong generalized eigenfunction expansion*), SLI (*Simon-Lieb inequality*), EDI (*eigenfunction decay inequality*), IAD (*independence at a distance*), NE (*number of eigenvalues*), and W (*Wegner estimate*) in [GK1], [GK3], [GK5], [Kl3]; they will be discussed below in the context of the random Landau Hamiltonian. It is important to note that these properties are also the requirements for the converse to the multiscale analysis given in [GK5]. (Although the results in [GK1], [GK5] are written for random Schrödinger operators without magnetic fields, they hold without change with magnetic fields as long as these properties are satisfied. Note also that these results only require a Wegner estimate as in (4.7) below; see [GK1, Remark 2.4] and [GK5, Remark 2.13].)

The random Landau Hamiltonian satisfies the trace estimate

$$\operatorname{tr}\left\{\langle x\rangle^{-2}\left(H_{B,\lambda,\omega}-(B-\lambda M_1)+1\right)^{-2}\langle x\rangle^{-2}\right\}\leq C_{B,\lambda}<\infty \quad \text{for all } \omega, \tag{4.1}$$

where $C_{B,\lambda}$ is a constant independent of ω, locally bounded in B and λ. (for example, [BoGKS, Prop. 2.1]; recall $H_{B,\lambda,\omega}\geq B-\lambda M_1$.) This estimate guarantees the existence of a generalized eigenfunction expansion as in [KlKS, §3] for the random Landau Hamiltonian, establishing property SGEE.

The multiscale analysis requires the notion of a *finite volume operator*. For the random Landau Hamiltonian the finite volumes may be the squares $\Lambda_L(x)$, with center $x\in\mathbb{Z}^2$ and side $L\in\bar{L}\mathbb{N}$ for a suitable $\bar{L}\geq 1$. The finite volume operator is a "restriction" $H_{B,\lambda,\omega,x,L}$ of $H_{B,\lambda,\omega}$ to the square $\Lambda_L(x)$, where the "randomness based outside the square $\Lambda_L(x)$" is not taken into account. Usually the finite volume operator is defined as an operator on $\mathrm{L}^2(\Lambda_L(x))$ by specifying the boundary condition, most commonly Dirichlet or periodic boundary condition. (In the case of the random Landau Hamiltonian it has also been defined as an operator on the whole space by throwing away the random coefficients "based outside the square $\Lambda_L(x)$" [CoH2], [W1], [GK4].) Properties SLI, EDI, IAD, NE, and W are statements about these finite volume operators.

A key observation for our purposes is that *it is not necessary to use the same boundary condition on all squares*; *what is important are compatibility conditions.* (This observation plays an important role in Theorem 5.1.) For the multiscale analysis it suffices to fix a reference scale $\bar{L}\geq 1$, *not necessarily an integer*, fix some $\delta>0$, and define a random operator $H_{B,\lambda,\omega,x,L}$ on $\mathrm{L}^2(\Lambda_L(x))$ for each $x\in\mathbb{Z}^2$ and $L\in\bar{L}\mathbb{N}$ as follows: First pick a closed densely defined operator $\mathbf{D}_{B,x,L}$ from $\mathrm{L}^2(\Lambda_L(x))$ to $\mathrm{L}^2(\Lambda_L(x);\mathbb{C}^2)$ which is an extension of the differential operator $\mathbf{D}_B=(-i\nabla-\mathbf{A})$ restricted to $C_c^\infty(\Lambda_L(x))$. Second, pick a random potential $V_{x,L,\omega}$ in the square $\Lambda_L(x)$ depending only on the random variables $\{\omega_i;\, i\in\Lambda_L(x)\}$, and set $H_{B,\lambda,\omega,x,L}=\mathbf{D}^*_{B,x,L}\mathbf{D}_{B,x,L}+\lambda V_{x,L,\omega}$ on $\mathrm{L}^2(\Lambda_L(x))$. Require that the resulting operators $H_{B,\lambda,\omega,x,L}$ have compact resolvent and satisfy the following compatibility conditions: If $\varphi\in\mathcal{D}(\mathbf{D}_{B,x,L})$ with $\operatorname{supp}\varphi\subset\Lambda_{L-\delta}(x)$, then $\mathcal{I}_{x,L}\varphi\in\mathcal{D}(\mathbf{D}_B)$, and

$$\mathcal{I}_{x,L}\mathbf{D}_{B,x,L}\varphi=\mathbf{D}_B\mathcal{I}_{x,L}\varphi,\quad \mathcal{I}_{x,L}\chi_{x,L-\delta}V_{x,L,\omega}=\chi_{x,L-\delta}V_\omega, \tag{4.2}$$

where $\mathcal{I}_{x,L}\colon \mathrm{L}^2(\Lambda_L(x))\to\mathrm{L}^2(\mathbb{R}^2)$ is the canonical injection: $(\mathcal{I}_{x,L}\varphi)(y)=\varphi(y)$ if $y\in\Lambda_L(x)$, $(\mathcal{I}_{x,L}\varphi)(y)=0$ otherwise (we will also use $\mathcal{I}_{x,L}$ for $\mathbb{C}^2$ valued functions). Note that in the square centered at $x\in\mathbb{Z}^2$ with side $L-\delta$ the potential $V_{x,L,\omega}$ is just V_ω. Furthermore, require the covariance condition (*but only between* x *and* 0, *not between arbitrary* x *and* y *in* $\mathbb{Z}^2$)

$$H_{B,\lambda,\omega,x,L}=U_x H_{B,\lambda,\tau_{-x}(\omega),0,L}U_x^* \quad \text{for all } x\in\mathbb{Z}^2, \tag{4.3}$$

where the magnetic translation U_x is as in (2.4) but considered as a unitary map from $\mathrm{L}^2(\Lambda_L(0))$ to $\mathrm{L}^2(\Lambda_L(x))$. This is equivalent to fixing the boundary condition for the operators $\mathbf{D}_{B,x,L}$ at the square centered at $x=0$, and using the magnetic translations to define the finite volume operators in all other squares by (4.3). (See Section 5 for specific choices for these finite volume operators.)

The compatibility conditions (4.2) *ensure that the finite volume operators* $H_{B,\lambda,\omega,x,L}$ *agree with* $H_{B,\lambda,\omega}$ "*inside*" *the square* $\Lambda_L(x)$; *that is, we have*

$$\mathcal{I}_{x,L}H_{B,\lambda,\omega,x,L}\varphi = H_{B,\lambda,\omega}\mathcal{I}_{x,L}\varphi \quad \textit{for } \varphi \in C_c^\infty(\Lambda_L(x)),\ \operatorname{supp}\varphi \subset \Lambda_{L-\delta}(x). \tag{4.4}$$

The covariance condition (4.3) *ensures that the probabilities of events based in squares* $\Lambda_L(x)$ (*that is*, *determined by conditions on the finite volume operator* $H_{B,\lambda,\omega,x,L}$) *are translation invariant*; *that is*, *independent of the center* $x \in \mathbb{Z}^2$. Note that given (4.3) it suffices to establish (4.2) for $x=0$. Note also that events based on disjoint squares are independent; this gives property IAD.

Taking (4.2) into account, given a square $\Lambda_L(x)$ we define its "boundary belt" as $\Lambda_{L-\delta-1}(x)\backslash\Lambda_{L-\delta-3}(x)$, with $\Gamma_{x,L}$ denoting its characteristic function. We write $\Lambda_\ell(y) \sqsubset \Lambda_L(x)$ if $\Lambda_\ell(y) \subset \Lambda_{L-\delta-3}(x)$. We let $R_{B,\lambda,\omega,x,L}(z) = (H_{B,\lambda,\omega,x,L}-z)^{-1}$ be the resolvent of the finite volume operator $H_{B,\lambda,\omega,x,L}$.

Properties SLI and EDI follow from (4.2) and (4.3) by the geometric resolvent identity and interior estimates, as in [GK5, Theorem A.1] (see also the discussion in [GK3, §4]). Property SLI here says that for all $E \in \mathbb{R}$ there exists a finite constant $\gamma_{\lambda,E}$, locally bounded in λ and E, such that, given $L, \ell' \in \bar{L}\mathbb{N}$, $\ell''>0$, $x,y,y' \in \mathbb{Z}^2$ with $\Lambda_{\ell''}(y) \sqsubset \Lambda_{\ell'}(y') \sqsubset \Lambda_L(x)$, then for every ω such that $E \notin \sigma(H_{B,\lambda,\omega,x,L}) \cup \sigma(H_{B,\lambda,\omega,y',\ell'})$,

$$\begin{aligned}\|\Gamma_{x,L}R_{B,\lambda,\omega,x,L}(E)\chi_{y,\ell''}\| \\ \le \gamma_{\lambda,E}\,\|\Gamma_{y',\ell'}R_{B,\lambda,\omega,y',\ell'}(E)\chi_{y,\ell''}\|\,\|\Gamma_{x,L}R_{B,\lambda,\omega,x,L}(E)\Gamma_{y',\ell'}\|.\end{aligned} \tag{4.5}$$

Property EDI states that for all $E \in \mathbb{R}$ and every ω, given a generalized eigenfunction ψ of $H_{B,\lambda,\omega}$ with generalized eigenvalue E, we have for any $x \in \mathbb{Z}^2$ and $L \in \bar{L}\mathbb{N}$ with $E \notin \sigma(H_{B,\lambda,\omega,x,L})$ that (with the same $\gamma_{\lambda,E}$ as above)

$$\|\chi_x\psi\| \le \gamma_{\lambda,E}\|\Gamma_{x,L}R_{B,\lambda,\omega,x,L}(E)\chi_x\|\|\Gamma_{x,L}\psi\|. \tag{4.6}$$

We write $P_{B,\lambda,\omega,x,L}(\mathcal{J}) = \chi_{\mathcal{J}}(H_{B,\lambda,\omega,x,L})$ if $\mathcal{J} \subset \mathbb{R}$ is a Borel set. To establish properties NE and W for the random Landau Hamiltonian at all energies, it suffices to prove the following Wegner estimate: given a bounded interval $I \subset \mathbb{R}$ and $q \in]0,1[$, there exist constants $Q_{B,\lambda,I,q} < \infty$ and $\eta_{B,\lambda,I} \in]0,1]$, and a finite scale $L_{B,\lambda,I,q}$, such that for all subintervals $J \subset I$ with $|J| \le \eta_{B,\lambda,I}$, $L \in L_0\mathbb{N}$ with $L \ge L_{B,\lambda,I,q}$, and $x \in \mathbb{Z}^2$,

$$\mathbb{E}\left\{\operatorname{tr} P_{B,\lambda,\omega,x,L}(J)\right\} \le Q_{B,\lambda,I,q}\,\|\rho\|_\infty\,|J|^q L^2, \tag{4.7}$$

where ρ is the bounded density of the common probability distribution of the ω_i's.

If the single bump potential u in (2.3) has $\varepsilon_u \geq 1$, then such a Wegner estimate is proven for appropriate finite dimensional operators in [CoH2], [HuLMW] at all energies. But if ε_u is small (the most interesting case for this paper in view of Corollary 2.3), a Wegner estimate at all energies was only known under the rational flux condition on the unit square, namely $B \in 2\pi\mathbb{Q}$ [CoHK]. Under the hypotheses of Corollary 2.3, without the rational flux condition a Wegner estimate was known only at energies away from the Landau levels [CoH2], [W1].

The Wegner estimate is closely connected to Hölder continuity of the integrated density of states; in fact Combes, Hislop and Klopp [CoHK] proved first a Wegner estimate for random Landau Hamiltonians with $B \in 2\pi\mathbb{Q}$, and from it derived the Hölder continuity of the integrated density of states. Combes, Hislop, Klopp and Raikov [CoHKR] established the Hölder continuity of the integrated density of states for $H_{B,\lambda,\omega}$ as in (2.1) with no extra hypotheses, but they did not obtain estimates on finite volume operators, and hence no Wegner estimate.

In Theorem 5.1 we establish a Wegner estimate for the random Landau Hamiltonian as in (2.1), for an appropriate choice of finite volume operators. Although the Wegner estimate does not follow from Hölder continuity of the integrated density of states, we use some of the key results in [CoHKR] to obtain the crucial estimate [CoHK, Eq. (3.1)], from which the Wegner estimate follows as in [CoHK, Proof of Theorem 1.2].

Thus the random Landau Hamiltonian as in (2.1) satisfies all the requirements for the bootstrap multiscale analysis (BMSA) at all energies, including the Landau levels. For our purposes the BMSA can be thought of a "black box". The input is an "initial estimate", which gives control of the finite volume resolvent at a sufficiently large scale with good probability, the output is control of the finite volume resolvents for scales increasing to ∞, with appropriately increasing probabilities, and its consequences. This initial estimate at an energy E is of the form

$$\mathbb{P}\left\{\|\Gamma_{x,L} R_{B,\lambda,\omega,x,L}(E)\chi_{x,\frac{L}{3}}\| \leq \frac{1}{L^\theta}\right\} > 1 - p_0, \tag{4.8}$$

where $\theta > 2$. It follows from the BMSA [GK1] that there is $p_0 \in]0,1[$, such that if (4.8) holds for some L sufficiently large, then there is an open interval $I \ni E$ such that in I we have exponential localization, strong dynamical localization in the Hilbert-Schmidt norm, sub-exponential decay of the expectation of the kernel of the evolution operator, semi-uniformly localized eigenfunctions (SULE), and, as shown in [GK6], SUDEC, decay of the Fermi projection, and finite multiplicity of eigenvalues. In particular, $I \subset \Xi^{\mathrm{DL}}_{B,\lambda}$, with $\Xi^{\mathrm{DL}}_{B,\lambda}$ as defined in (2.18).

The multiscale analysis yields localization. To obtain delocalization, the main result of this paper, we use the converse to the multiscale analysis given in [GK5]. The main result here is [GK5, Theorem 2.11], which states that slow transport implies the initial estimate of the BMSA. An important consequence is that, if $E \in \Xi^{\mathrm{DL}}_{B,\lambda}$, then the initial estimate (4.8) for the BMSA is satisfied at the energy E with appropriate θ, p_0, L, and hence the BMSA can be performed with all its consequences. This is stated as property RDL in Section 3. Another crucial consequence for this paper is the estimate on the minimal rate of transport at energies $E \in \Xi^{\mathrm{DD}}_{B,\lambda}$ [GK5, Theorem 2.10], given as property RDD in Section 3.

Once we have established properties RDL and RDD, properties DFP and SUDEC in Section 3 are just consequences of the BMSA given in [GK1], [GK6].

5. The Wegner estimate

In this section we establish a Wegner estimate for the random Landau Hamiltonian as in (2.1).

Let $B > 0$ be arbitrary; since we do not assume the rational flux condition on the unit square, we set a reference length scale $\bar{L} = L_B$ corresponding to squares with even (for convenience) integer flux. We take

$$K_B = \min\Big\{k \in \mathbb{N}; k \geq \sqrt{\tfrac{B}{4\pi}}\Big\},$$

and set

$$L_B = K_B\sqrt{\tfrac{4\pi}{B}}, \quad \mathbb{N}_B = L_B\mathbb{N}, \quad \text{and} \quad \mathbb{Z}^2_B = L_B\mathbb{Z}^2. \tag{5.1}$$

Note that $L_B \geq 1$ may not be an integer. We consider squares $\Lambda_L(0)$ with $L \in \mathbb{N}_B$ and identify them with the torii $\mathbb{T}_L := \mathbb{R}^2/(L\mathbb{Z}^2)$ in the usual way. As shown in [CoHK, §4], the magnetic translations $\mathcal{U}_B := \{U_a;\ a \in \mathbb{Z}^2_B\}$ form a unitary representation of the abelian group $\mathbb{Z}^2_B$; we write $\widehat{U}_a$ for the corresponding action on $\mathrm{L}^2(\Lambda_L(0))$, with $\widehat{\mathcal{U}}_B := \{\widehat{U}_a;\ a \in \mathbb{Z}^2_B\}$. If $x \in \Lambda_L(0)$ and $r < L$ we denote by $\widehat{\Lambda}_r(x)$ and $\widehat{\chi}_{x,r}$ the square and characteristic function in the torus $\mathbb{T}_L$.

Given $L \in \mathbb{N}_B$, we define $H_{B,0,L} = \mathbf{D}^*_{B,0,L}\mathbf{D}_{B,0,L}$, with $\mathbf{D}_{B,0,L}$ the restriction of $\mathbf{D}_B$ to $\mathrm{L}^2(\Lambda_L(0))$ with periodic boundary condition with respect to $\widehat{\mathcal{U}}_B$. The spectrum of $H_{B,0,L}$ still consists of the Landau levels: $\sigma(H_{B,0,L}) = \sigma(H_B) = \{B_n; n = 0, 1, \dots\}$, but since $H_{B,0,L}$ has compact resolvent each Landau level has now finite multiplicity. We let $\widetilde{\Lambda}_L(x) = \mathbb{Z}^2 \cap \Lambda_L(x)$. Given $L \in \mathbb{N}_B$ we set

$$\begin{aligned} H_{B,\lambda,0,L,\omega} &= H_{B,0,L} + \lambda V_{0,L,\omega} \quad \text{on} \quad \mathrm{L}^2(\Lambda_L(0)), \\ V_{0,L,\omega}(x) &= \sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} \omega_i\, u(x-i), \end{aligned} \tag{5.2}$$

where $\operatorname{supp} u \subset \Lambda_{\delta_u}(0)$, and then define $H_{B,\lambda,\omega,x,L}$ for all $x \in \mathbb{Z}^2$ by (4.3). (We prescribed periodic boundary condition for the (free) Landau Hamiltonian at the square centered at 0, and used the magnetic translations to define the finite volume operators in all other squares by (4.3); in the square centered at $x \in \mathbb{Z}^2$ the potential $V_{x,L,\omega}$ is exactly as in (5.2) except that the sum is now over $i \in \widetilde{\Lambda}_{L-\delta_u}(x)$.) Note that $H_{B,\lambda,x,L,\omega}$ has compact resolvent and satisfies the compatibility conditions (4.2) with $\delta = 2\delta_u$.

THEOREM 5.1. *Fix $B > 0$ and $\lambda > 0$. Given a bounded interval $I \subset \mathbb{R}$ and $q \in]0,1[$, there exist constants $Q_{B,\lambda,I,q} < \infty$ and $\eta_{B,\lambda,I} \in]0,1]$, and a finite scale $L_{B,\lambda,I,q}$, such that for all subintervals $J \subset I$ with $|J| \leq \eta_{B,\lambda,I}$, $L \in \mathbb{N}_B$ with $L \geq L_{B,\lambda,I,q}$, and $x \in \mathbb{Z}^2$,*

$$\mathbb{E}\left\{\operatorname{tr} P_{B,\lambda,\omega,x,L}(J)\right\} \leq Q_{B,\lambda,I,q} \left\|\rho\right\|_\infty |J|^q L^2. \tag{5.3}$$

Proof. In view of (4.3) it suffices to prove the theorem for $x = 0$.

We start by proving a lemma that will allow us to derive the theorem from the results of [CoHKR], [CoHK]. For each $L \in \mathbb{N}_B$ we set $\Gamma_L = \chi_{\overline{\Lambda}_{L-1}(0)\setminus\Lambda_{L-3}(0)}$ and fix a function $\Phi_L \in C^\infty(\mathbb{R}^2)$ such that $\Phi_L(x) \equiv 1$ on $\Lambda_{L-\frac{5}{2}}(0)$, $\operatorname{supp}\Phi_L \subset \Lambda_{L-\frac{3}{2}}(0)$, and $0 \leq \Phi_L(x) \leq 1$, $|\nabla\Phi_L(x)| \leq 5$ for all $x \in \mathbb{R}^2$. (Such a function always exists.) We use Φ_L, $(\nabla\Phi_L)$, and $\chi_r = \chi_{0,r}$ $(0 < r \leq L)$ to denote the operators given by multiplication by the functions Φ_L, $\nabla\Phi_L$ and χ_r in both $\mathrm{L}^2(\Lambda_L(0))$ and $\mathrm{L}^2(\mathbb{R}^2)$. For convenience we set $H_{B,L} = H_{B,0,L}$, $\mathcal{I}_L = \mathcal{I}_{0,L}$, $\widetilde{\mathbb{N}}_B = \mathbb{N}_B \cup \{\infty\}$, $H_{B,\infty} = H_B$, and so on. By $C_{a,b,\ldots}$ we denote a constant depending only on the parameters $a, b, \ldots$ (we may use the same $C_{a,b,\ldots}$ for different constants), and similarly for constants $m_{a,b,\ldots} > 0$.

LEMMA 5.2. *Fix $B > 0$. Given $n \in \mathbb{N}$ and $L \in \widetilde{\mathbb{N}}_B$, let $\Pi_{n,L} = \Pi_{B,n,L}$ denote the orthogonal projection on the eigenspace corresponding to the n-th Landau level B_n for the Landau Hamiltonian $H_{B,L}$. Then for all $x \in \Lambda_{L_B}(0)$, $r > 0$, and $L \in \mathbb{N}_B$ such that $L \geq 2(L_B + r)$,*

$$\Pi_{n,L}\chi_{x,r}\Pi_{n,L} = \Phi_L\mathcal{I}_L^*\Pi_n\chi_{x,r}\Pi_n\mathcal{I}_L\Phi_L + \mathcal{E}_{x,r,n,L}, \tag{5.4}$$

with the error operator $\mathcal{E}_{x,r,n,L}$ satisfying

$$\|\mathcal{E}_{x,r,n,L}\| \leq C_{n,B}\,\mathrm{e}^{-m_{n,B}L}. \tag{5.5}$$

Proof. Let L, r, and x be as in the lemma. Since all $H_{B,L}$ have the same spectrum, namely the Landau levels, we have

$$\Pi_{n,L} = -\tfrac{1}{2\pi i}\int_{\gamma_n} R_L(z)\,\mathrm{d}z \quad \text{with} \quad R_L(z) = (H_{B,L} - z)^{-1} \text{ if } L \in \widetilde{\mathbb{N}}_B, \tag{5.6}$$

where γ_n denotes the circle centered at B_n with radius B. Let $z \in \gamma_n$, in view of (4.2) we may use the smooth resolvent identity as in [GK5, Eq. (6.13)] to obtain,

$$\begin{aligned}\chi_{x,r}\mathcal{I}_L R_L(z) &= \chi_{x,r}\Phi_L\mathcal{I}_L R_L(z) = \chi_{x,r}R(z)\Phi_L\mathcal{I}_L - \chi_{x,r}Y_L(z),\\ Y_L(z) &:= iR(z)\left\{\mathbf{D}_B^*(\nabla\Phi)\mathcal{I}_L + \mathcal{I}_L(\nabla\Phi)^*\mathbf{D}_{B,L}\right\}R_L(z).\end{aligned} \tag{5.7}$$

Proceeding as in [GK5, Proof of Lemma 6.4], using $L \ge 2(L_B + r)$, $\|R_L(z)\| = \frac{1}{B}$, $|z| \le B_n + B$, and the Combes-Thomas estimate (e.g., [GK2, Theorem 1]), we obtain

$$\begin{aligned}\|\chi_{x,r}Y_L(z)\| &\le \|\chi_{x,r}R(z)\mathbf{D}_B^*\,|\nabla\Phi|\|\,\|R_L(z)\| + \|\chi_{x,r}R(z)\,|\nabla\Phi|\|\,\|\mathbf{D}_{B,L}R_L(z)\|\\ &\le C_{n,B}\,\|\chi_{x,r}R(z)\Gamma_L\| \le C_{n,B}\,\mathrm{e}^{-m_{n,B}L}.\end{aligned} \tag{5.8}$$

Putting together (5.6), (5.7), and (5.8) we get

$$\chi_{x,r}\Pi_{n,L} = \chi_{x,r}\mathcal{I}_L^*\Pi_n\mathcal{I}_L\Phi_L + \mathcal{E}'_{x,r,n,L}, \tag{5.9}$$

with the error operator $\mathcal{E}'_{x,r,n,L}$ satisfying the estimate (5.5). The lemma now follows from (5.9). □

Using Lemma 5.2 we adapt the crucial [CoHKR, Lemma 2] to finite volume.

LEMMA 5.3. *Fix $B > 0$, $n \in \mathbb{N}$, $\varepsilon > 0$, $R > \varepsilon$, and $\eta > 0$. If $\kappa > 1$ and $L \in \mathbb{N}_B$ are such that $L > 2(L_B + \kappa R)$, then for all $x \in \Lambda_L(0)$, we have*

$$\Pi_{n,L}\widehat{\chi}_{x,\varepsilon}\Pi_{n,L} \ge C_0\Pi_{n,L}(\widehat{\chi}_{x,R} - \eta\widehat{\chi}_{x,\kappa R})\Pi_{n,L} + \Pi_{n,L}\mathcal{E}_{n,x,L}\Pi_{n,L}, \tag{5.10}$$

where $C_0 = C_{0;n,B,\varepsilon,R,\eta} > 0$ is a constant and the error operator $\mathcal{E}_{n,x,L} = \mathcal{E}_{n,x,L,B,\varepsilon,R,\eta}$ satisfies

$$\|\mathcal{E}_{n,x,L}\| \le C_{n,B,\varepsilon,R,\eta}\,\mathrm{e}^{-m_{n,B}L}. \tag{5.11}$$

Proof. Given $B, n, \varepsilon, R, \eta$ as in the lemma, it follows from [CoHKR, Lemma 2] that for all $\kappa > 1$ and $x \in \mathbb{R}^2$, we have

$$\Pi_n\chi_{x,\varepsilon}\Pi_n \ge C_0\Pi_n(\chi_{x,R} - \eta\chi_{x,\kappa R})\Pi_n, \quad C_0 = C_{0;B,n,\varepsilon,R,\eta,\kappa} > 0. \tag{5.12}$$

(Although [CoHKR, Eq. 61] is stated for discs instead of squares, (5.12) follows with a small change in the constant C_0.)

Let $\kappa > 1$ and $L \in \mathbb{N}_B$ be such that $L > 2(L_B + \kappa R)$. If $x \in \Lambda_{L_B}(0)$, it follows from Lemma 5.2 and (5.12) that

$$\begin{aligned}\Pi_{n,L}\chi_{x,\varepsilon}\Pi_{n,L} &= \Phi_L\mathcal{I}_L^*\Pi_n\chi_{x,\varepsilon}\Pi_n\mathcal{I}_L\Phi_L + \mathcal{E}_{x,\varepsilon,n,L}\\ &\ge C_0\Phi_L\mathcal{I}_L^*\Pi_n(\chi_{x,R} - \eta\chi_{x,\kappa R})\Pi_n\mathcal{I}_L\Phi_L + \mathcal{E}_{x,\varepsilon,n,L}\\ &= C_0\Pi_{n,L}(\chi_{x,R} - \eta\chi_{x,\kappa R})\Pi_{n,L} + \mathcal{E}_{x,\varepsilon,R,\kappa,n,L},\end{aligned} \tag{5.13}$$

and hence we have (5.10) and (5.11) for $x \in \Lambda_{L_B}(0)$. For arbitrary $x \in \Lambda_L(0)$, we pick $a_x \in \mathbb{Z}_B^2$ such that $x - a_x \in \Lambda_{L_B}(0)$ (such a_x always exists). Since $\widehat{\chi}_{x,\ell} = \widehat{U}_{a_x}\widehat{\chi}_{x-a_x,\ell}\widehat{U}_{a_x}^*$ for $\ell < L$ and $\widehat{U}_{a_x}\Pi_{n,L}\widehat{U}_{a_x}^* = \Pi_{n,L}$, (5.10) and (5.11) follows with $\mathcal{E}_{n,x,L} = \widehat{U}_{a_x}\mathcal{E}_{n,x-a_x,L}\widehat{U}_{a_x}^*$. □

We can now finish the proof of of Theorem 5.1. Let

$$\widetilde{V}_L(x) := \sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} u(x-i) \geq u^- \sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} \chi_{i,\varepsilon_u}. \tag{5.14}$$

We fix $R > 1 + 2\delta_u$, in which case $\sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} \widehat{\chi}_{i,R} \geq \chi_{0,L}$, and $\kappa > 1$, and pick $\eta > 0$ such that $\eta \sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} \widehat{\chi}_{i,\kappa R} \leq \frac{1}{2}\chi_{0,L}$. It follows from (5.14) and Lemma 5.3 that for all $L \in \mathbb{N}_B$ with $L > 2(L_B + \kappa R)$,

$$\Pi_{n,L}\widetilde{V}_L\Pi_{n,L} \geq u^- C_0 \sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} \Pi_{n,L}(\widehat{\chi}_{i,R} - \eta\widehat{\chi}_{i,\kappa R})\Pi_{n,L} + \Pi_{n,L}\mathcal{E}_{n,L}\Pi_{n,L}$$

$$\geq \frac{u^- C_0}{2}\Pi_{n,L} + \Pi_{n,L}\mathcal{E}_{n,L}\Pi_{n,L} \geq C_1\Pi_{n,L} \tag{5.15}$$

for $L \geq L^*$ for some $L^* = L^*_{n,B,\varepsilon,R,\kappa,\eta} < \infty$ and $C_1 = \frac{u^- C_0}{4}$, since the error term $\mathcal{E}_{n,L}$ satisfies

$$\|\mathcal{E}_{n,L}\| \leq 2L^2 C_{n,B,\varepsilon,R,\eta}\, \mathrm{e}^{-m_{n,B}L}. \tag{5.16}$$

Theorem 5.1 now follows by [CoHK, Proof of Theorem 1.2], since (5.15) for all $n = 1, 2, \ldots$ gives the crucial estimate [CoHK, Eq. (3.1)]. ∎

6. The small disorder limit

Proof of Corollary 2.4. Note first that $1 < c_{b,\lambda} \leq 2$ for $\lambda \leq \lambda_1$, which we assume from now on. Fixing $B > b$, we have (2.11) with $\mathcal{I}_n(B,\lambda) = \mathcal{I}_n(B) := \mathcal{I}_n(B,1)$ for all λ and $n = 1, 2, \ldots$. By the hypothesis on the density ρ, for all $\varepsilon > 0$ we have

$$\nu_\lambda(\{|u| \geq \varepsilon\}) \leq C_1 \left(\lambda\varepsilon^{-1}\right)^{\gamma-1}. \tag{6.1}$$

Let $\bar{L} \in \mathbb{N}_B$ (see (5.1)), and let $H_{B,\lambda,0,\bar{L},\omega}$ and $V_{0,\bar{L},\omega}$ be as in (5.2) with $\lambda = 1$ but with ν_λ being the common probability distribution of the random variables $\{\omega_i;\, i \in \mathbb{Z}^2\}$. The spectrum of these finite volume Hamiltonians satisfies (2.8) (appropriately modified) for each ω, and hence

$$\mathbb{P}\left\{\sigma(H_{B,\lambda,0,\bar{L},\omega}) \subset \bigcup_{n=1}^{\infty}[B_n - \varepsilon, B_n + \varepsilon]\right\} \geq \mathbb{P}\left\{|\omega_i| \leq \varepsilon \text{ if } i \in \widetilde{\Lambda}_{\bar{L}-\delta_u}(0)\right\} \tag{6.2}$$

$$\geq \left(1 - C_1\left(\lambda\varepsilon^{-1}\right)^{\gamma-1}\right)^{(\bar{L}-\delta_u)^2} \geq 1 - C_2\left(\lambda\varepsilon^{-1}\right)^{\gamma-1}\bar{L}^2$$

for small $\left(\lambda\varepsilon^{-1}\right)^{\gamma-1}$.

We now apply the finite volume criterion for localization given in [GK3, Theorem 2.4], in the same way as in [GK3, Proof of Theorem 3.1], with parameters (we fix $q \in]0,1]$) $\eta_{I,\lambda} = \frac{1}{2}\eta_{B,\lambda,I,q} = \frac{1}{2}\eta_{B,1,I,q}$ and $Q_{I,\lambda} = Q_{B,\lambda,I,q} \leq 2\lambda^{-1}Q_{I,1}$, where $\eta_{B,\lambda,I}$ and $Q_{B,\lambda,I,q}$ come from Theorem 5.1. (Note that the fact that we work with length scales $L \in \mathbb{N}_B$ instead of $L \in 6\mathbb{N}$ only affects the values of the constants in [GK3, Eqs. (2.16)–(2.18)].) The SLI constant $\gamma_{I,B,\lambda}$ is uniformly bounded in closed intervals I if $\lambda \le B$. Since we are working in spectral gaps, we use the Combes-Thomas estimate of [BCH, Prop. 3.2] (see also [KlK1, Theorem 3.5]–its proof, based on [BCH, Lemma 3.1], also works for Schrödinger operators with magnetic fields), adapted to finite volume as in [GK3, §3].

Now fix $n \in \mathbb{N}$, take $I = \mathcal{I}_n(B)$, and set $\bar{L} = \bar{L}(n, B)$ to be the smallest $L \in \mathbb{N}_B$ satisfying [GK3, Eq. (2.16)]. Let $E \in \mathcal{I}_n(B)$, $|E - B_n| \ge 2\varepsilon$, where $\varepsilon = \varepsilon(n, B, \lambda)) > 0$ will be chosen later. Then, using (6.2) and the Combes-Thomas estimate, we conclude that condition [GK3, Eq. (2.17)] will be satisfied at energy E if

$$\varepsilon \ge C_3\, \lambda \bar{L}^{\frac{2}{\gamma - 1}}, \tag{6.3}$$

$$C_4\, (\lambda\varepsilon)^{-1}\, \bar{L}^{\frac{25}{3}} \mathrm{e}^{-C_5\sqrt{\varepsilon}\bar{L}} < 1, \tag{6.4}$$

for appropriate constants $C_j = C_j(n, B)$, $j = 3, 4, 5$, with $C_5 > 0$. This can be done by choosing

$$\varepsilon = C_6 \lambda^{\frac{\gamma-1}{\gamma}}\, |\log\lambda|^{\frac{2}{\gamma}}, \tag{6.5}$$

with a sufficiently large constant $C_6 = C_6(n, B)$ and taking $\lambda \le \lambda_2$ for some $0 < \lambda_2 = \lambda(n, B, C_6)$. We conclude from [GK3, Theorem 2.4] that

$$\left\{ E \in \mathcal{I}_n(B);\ |E - B_n| \ge 2C_5 \lambda^{\frac{\gamma-1}{\gamma}}\, |\log\lambda|^{\frac{2}{\gamma}} \right\} \subset \Xi^{\mathrm{DL}}_{B,\lambda} \tag{6.6}$$

for all $\lambda \le \lambda_2$.

The existence at small disorder of dynamical mobility edges $\widetilde{E}_{j,n}(B, \lambda)$, $j = 1, 2$, satisfying (2.24), (2.25), and (2.26) now follows from Theorem 2.1 and (6.6).

The case when $\mathrm{e}^{|u|^\alpha}\rho(u)$ is bounded for some $\alpha > 0$ can be treated in a similar way. ◻

Université de Cergy-Pontoise, Cergy-Pontoise, France
E-mail address: germinet@math.u-cergy.fr

University of California, Irvine, California, USA
E-mail address: aklein@uci.edu

ETH Zürich, Zürich, Switzerland
Current address: Michigan State University, East Lansing, MI, USA
E-mail address: jeffrey@math.msu.edu

References

[A] M. Aizenman, Localization at weak disorder: some elementary bounds, *Rev. Math. Phys.* **6** (1994), 1163–1182.

[AENSS] M. Aizenman, A. Elgart, S. Naboko, J. H. Schenker, and G. Stolz, Moment analysis for localization in Random Schrödinger operators, *Invent. Math*. **163** (2006), 343–413.

[AG] M. Aizenman and G. M. Graf, Localization bounds for an electron gas, *J. Physics A*: *Math. Gen*. **31** (1998), 6783–6806.

[AM] M. Aizenman and S. Molchanov, Localization at large disorder and extreme energies: an elementary derivation, *Comm. Math. Phys*. **157** (1993), 245–278.

[ASFH] M. Aizenman, J. Schenker, R. Friedrich, and D. Hundertmark, Finite volume fractional-moment criteria for Anderson localization, *Comm. Math. Phys*. **224** (2001), 219–253.

[An] P. Anderson, Absence of diffusion in certain random lattices, *Phys. Rev*. **109** (1958), 1492–1505.

[AoA] H. Aoki and T. Ando, Effects of localization on the Hall conductivity in the two-dimensional system in strong magnetic field, *Solid State Commun*. **38** (1981), 1079–1082.

[AvSS] J. Avron, R. Seiler, and B. Simon, Charge deficiency, charge transport and comparison of dimensions, *Comm. Math. Phys*. **159** (1994), 399–422.

[BCH] J. M. Barbaroux, J. M. Combes, and P. D. Hislop, Localization near band edges for random Schrödinger operators, *Helv. Phys. Acta* **70** (1997), 16–43.

[Be] J. Bellissard, Ordinary quantum Hall effect and noncommutative cohomology, in *Localization in Disordered Systems* (Bad Schandau, 1986), 61–74, *Teubner-Texte Phys*. **16**, Teubner, Leipzig, 1988.

[BeES] J. Bellissard, A. van Elst, and H. Schulz-Baldes, The noncommutative geometry of the quantum Hall effect, *J. Math. Phys*. **35** (1994), 5373–5451.

[BMR] J. Bellissard, J. Magnen, and V. Rivasseau, Supersymmetric analysis of a simplified two-dimensional Anderson model at small disorder, *Markov Process*. *Related Fields* **9** (2003), 261–278.

[BoGK] J. M. Bouclet, F. Germinet, and A. Klein, Sub-exponential decay of operator kernels for functions of generalized Schrödinger operators, *Proc. Amer. Math. Soc*. **132** (2004), 2703–2712.

[BoGKS] J. M. Bouclet, F. Germinet, A. Klein, and J. Schenker, Linear response theory for magnetic Schrödinger operators in disordered media, *J. Funct. Anal.* **226** (2005), 301–372.

[Bou1] J. Bourgain, New results on the spectrum of lattice Schrödinger operators and applications, *Contemp. Math*. **307** (2002), 27–38.

[Bou2] ———, Random lattice Schrödinger operators with decaying potential: some higher dimensional phenomena, *Lecture Notes in Math*. **1807** (2003), 70–98.

[BouK] J. Bourgain and C. Kenig, On localization in the continuous Anderson-Bernoulli model in higher dimensions, *Invent. Math*. **161** (2005), 389–426.

[CKM] R. Carmona, A. Klein, and F. Martinelli, Anderson localization for Bernoulli and other singular potentials, *Comm. Math. Phys*. **108** (1987), 41–66.

[ChC] J. T. Chalker and P. D. Coddington, Percolation, quantum tunnelling and the integer Hall effect, *J. Phys. C*: *Solid State Phys*. **21** (1988), 2665–2679.

[Che] T. Chen, Localization lengths and Boltzmann limit for the Anderson model at small disorders in dimension 3, *J. Statist. Phys.* **120** (2005), 279–337.

[CoH1] J. M. Combes and P. D. Hislop, Localization for some continuous, random Hamiltonian in d-dimension. *J. Funct. Anal*. **124** (1994), 149–180.

[CoH2] J. M. Combes and P. D. Hislop, Landau Hamiltonians with random potentials: localization and the density of states, *Comm. Math. Phys*. **177** (1996), 603–629.

[CoHK] J. M. Combes, P. D. Hislop, and F. Klopp, Hölder continuity of the integrated density of states for some random operators at all energies, *Internat. Math. Res. Notices* **(2003)**, no. 4, 179–209.

[CoHKR] J. M. Combes, P. D. Hislop, F. Klopp, and G. Raikov, Global continuity of the integrated density of states for random Landau Hamiltonians, *Comm. Partial Differential Equations* **29** (2004), 1187–1213.

[CoHN] J. M. Combes, P. D. Hislop, and S. Nakamura, The L^p-theory of the spectral shift function, the Wegner estimate and the integrated density of states for some random operators, *Comm. Math. Phys*. **218** (2001), 113–130.

[CoHT] J. M. Combes, P. D. Hislop, and A. Tip, Band edge localization and the density of states for acoustic and electromagnetic waves in random media, *Ann. Inst. H. Poincaré Phys. Theor*. **70** (1999), 381–428.

[DSS] D. Damanik, R. Sims, and G. Stolz, Localization for one dimensional, continuum, Bernoulli-Anderson models, *Duke Math. J*. **114** (2002), 59–100.

[DS] D. Damanik and P. Stollmann, Multi-scale analysis implies strong dynamical localization, *Geom. Funct. Anal*. **11** (2001), 11–29.

[DiPS] M. Disertori, H. Pinson, and T. Spencer, Density of states for random band matrices, *Comm. Math. Phys*. **232** (2002), 83–124.

[Dr] H. von Dreifus, On the effects of randomness in ferromagnetic models and Schrödinger operators, Ph. D. thesis, New York University (1987).

[DrK] H. von Dreifus and A. Klein, A new proof of localization in the Anderson tight binding model, *Comm. Math. Phys*. **124** (1989), 295–299.

[EGS] A. Elgart, G. M. Graf, and J. H. Schenker, Equality of the bulk and edge Hall conductances in a mobility gap, *Comm. Math. Phys.* **259** (2005), 185–221.

[ES] A. Elgart and B. Schlein, Adiabatic charge transport and the Kubo formula for Landau-type Hamiltonians, *Comm. Pure Appl. Math*. **57** (2004), 590–615.

[ErSY] L. Erdös, M. Salmhofer, and H.-T. Yau, Quantum diffusion of the random Schrödinger evolution in the scaling limit, *Acta Math.*, to appear.

[ErY] L. Erdös and H.- T. Yau, Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation, *Comm. Pure Appl. Math*. **53** (2000), 667–735.

[FK1] A. Figotin and A. Klein, Localization phenomenon in gaps of the spectrum of random lattice operators, *J. Statist. Phy*. **75** (1994), 997–1021.

[FK2] ———, Localization of classical waves I: Acoustic waves, *Comm. Math. Phys*. **180** (1996), 439–482.

[FK3] ———, Localization of classical waves II: Electromagnetic waves, *Comm. Math. Phys*. **184** (1997), 411–441.

[FLM] W. Fischer, H. Leschke, and P. Müller, Spectral localization by Gaussian random potentials in multi-dimensional continuous space, *J. Statist. Phys*. **101** (2000), 935–985.

[FrMSS] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, Constructive proof of localization in the Anderson tight binding model, *Comm. Math. Phys*. **101** (1985), 21–46.

[FrS] J. Fröhlich and T. Spencer, Absence of diffusion with Anderson tight binding model for large disorder or low energy, *Comm. Math. Phys*. **88** (1983), 151–184.

[G] F. Germinet, Dynamical localization II with an application to the almost Mathieu operator, *J. Statist. Phys.* **95** (1999), 273–286.

[GD] F. Germinet and S. De Bièvre, Dynamical localization for discrete and continuous random Schrödinger operators, *Comm. Math. Phys.* **194** (1998), 323–341.

[GK1] F. Germinet and A. Klein, Bootstrap multiscale analysis and localization in random media, *Comm. Math. Phys.* **222** (2001), 415–448.

[GK2] ———, Operator kernel estimates for functions of generalized Schrödinger operators, *Proc. Amer. Math. Soc.* **131** (2003), 911–920.

[GK3] ———, Explicit finite volume criteria for localization in continuous random media and applications, *Geom. Funct. Anal.* **13** (2003), 1201–1238.

[GK4] ———, The Anderson metal-insulator transport transition, *Contemp. Math.* **339** (2003), 43–57.

[GK5] ———, A characterization of the Anderson metal-insulator transport transition, *Duke Math. J.* **124** (2004), 309–351.

[GK6] ———, New characterizations of the region of complete localization for random Schrödinger operators, *J. Statist. Phys.* **122** (2006), 73–94.

[GoMP] Ya. Gol'dsheid, S. Molchanov, and L. Pastur, Pure point spectrum of stochastic one dimensional Schrödinger operators, *Funct. Anal. Appl.* **11** (1977), 1–10.

[H] B. Halperin, Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential, *Phys. Rev.* B **25** (1982), 2185–2190.

[HiK] P. D. Hislop and F. Klopp, The integrated density of states for some random operators with nonsign definite potentials, *J. Funct. Anal.* **195** (2002), 12–47.

[HuLMW] T. Hupfer, H. Leschke, P. Müller, and S. Warzel, The absolute continuity of the integrated density of states for magnetic Schrödinger operators with certain unbounded potentials, *Comm. Math. Phys.* **221** (2001), 229–254.

[JL] V. Jaksic and Y. Last, Spectral structure of Anderson type Hamiltonians, *Invent. Math.* **141** (2000), 561–577.

[KM1] W. Kirsch and F. Martinelli, On the ergodic properties of the spectrum of general random operators, *J. Reine Angew. Math.* **334** (1982), 141–156.

[KM2] ———, On the spectrum of Schrödinger operators with a random potential, *Comm. Math. Phys.* **85** (1982), 329–350.

[KSS] W. Kirsch, P. Stollman, and G. Stolz, Localization for random perturbations of periodic Schrödinger operators, *Random Oper. Stochastic Equations* **6** (1998), 241–268.

[Kl1] A. Klein, Extended states in the Anderson model on the Bethe lattice, *Adv. Math.* **133** (1998), 163–184.

[Kl2] ———, Spreading of wave packets in the Anderson model on the Bethe lattice, *Comm. Math. Phys.* **177** (1996), 755–773.

[Kl3] ———, Multiscale analysis and localization of random operators, in *Random Schrödinger Operators: Methods, Results, and Perspectives*, Panorama & Synthèse, Société Mathématique de France, Paris, to appear.

[KlK1] A. Klein and A. Koines, A general framework for localization of classical waves: I. Inhomogeneous media and defect eigenmodes, *Math. Phys. Anal. Geom.* **4** (2001), 97–130.

[KlK2] ———, A general framework for localization of classical waves: II. Random media, *Math. Phys. Anal. Geom.* **7** (2004), 151–185.

[KlKS] A. Klein, A. Koines, and M. Seifert, Generalized eigenfunctions for waves in inhomogeneous media, *J. Funct. Anal.* **190** (2002), 255–291.

[KlLS] A. Klein, J. Lacroix, and A. Speis, Localization for the Anderson model on a strip with singular potentials, *J. Funct. Anal.* **94** (1990), 135–155.

[Kli] K. von Klitzing, G. Dorda, and N. Pepper, New method for high-accuracy determination of the fine structure constant based on quantized Hall resistance, *Phys. Rev. Lett.* **45** (1980), 494.

[Klo1] F. Klopp, Localization for continuous random Schrödinger operators, *Comm. Math. Phys.* **167** (1995), 553–569.

[Klo2] ———, Weak disorder localization and Lifshitz tails: continuous Hamiltonians, *Ann.* I.H.P. **3** (2002), 711–737.

[Ku] H. Kunz, The quantum Hall effect for electrons in a random potential, *Comm. Math. Phys.* **112** (1987), 121–145.

[L] R. B. Laughlin, Quantized Hall conductivity in two dimensions, *Phys. Rev. B* **23** (1981), 5632–5633.

[NT] Q. Niu and D. J. Thouless, Quantum Hall effect with realistic boundary conditions, *Phys. Rev. B* **35** (1987), 2188–2197.

[PF] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer-Verlag, New York, 1992.

[S] T. Spencer, Localization for random and quasiperiodic potentials, *J. Statist. Phys.* **51** (1988), 1009–1019.

[St] P. Stollmann, *Caught by Disorder. Bound States in Random Media*, Birkhaüser, Basel, 2001.

[SZ] T. Spencer and M. R. Zirnbauer, Spontaneous symmetry breaking of a hyperbolic sigma model in three dimensions, *Comm. Math. Phys.* **252** (2004), 167–187.

[T] D. J. Thouless, Localisation and the two-dimensional Hall effect, *J. Phys.* C **14** (1981), 3475–3480.

[TKNN] D. J. Thouless, K. Kohmoto, M. P. Nightingale, and M. uden Nijs, Quantized Hall conductance in a two-dimensional periodic potential, *Phys. Rev. Lett.* **49** (1982), 405–408.

[U] N. Ueki, Wegner estimates and localization for Gaussian random potentials, *Publ. Publ. Res. Inst. Math. Sci.* **40** (2004), 29–90.

[W1] W.-M. Wang, Microlocalization, percolation, and Anderson localization for the magnetic Schrödinger operator with a random potential, *J. Funct. Anal.* **146** (1977), 1–26.

[W2] ———, Localization and universality of Poisson statistics for the multidimensional Anderson model at weak disorder, *Invent. Math.* **146** (2001), 365–398.

[We] F. Wegner, Bounds on the density of states in disordered systems, *Z. Phys.* B **44** (1981), 9–15.

(Received December 19, 2004)