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Subelliptic $\mathrm{Spin}_{\mathbb{C}}$ Dirac operators, I

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Dedicated to my parents, Jean and Herbert Epstein, on the occasion of their eightieth birthdays

Abstract

Let X be a compact Kähler manifold with strictly pseudoconvex boundary, Y. In this setting, the $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator is canonically identified with $\bar{\partial}+\bar{\partial}^* : \mathcal{C}^\infty(X;\Lambda^{0,\mathrm{e}}) \to \mathcal{C}^\infty(X;\Lambda^{0,\mathrm{o}})$. We consider modifications of the classical $\bar{\partial}$-Neumann conditions that define Fredholm problems for the $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator. In Part 2, [7], we use boundary layer methods to obtain subelliptic estimates for these boundary value problems. Using these results, we obtain an expression for the finite part of the holomorphic Euler characteristic of a strictly pseudoconvex manifold as the index of a $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator with a subelliptic boundary condition. We also prove an analogue of the Agranovich-Dynin formula expressing the change in the index in terms of a relative index on the boundary. If X is a complex manifold partitioned by a strictly pseudoconvex hypersurface, then we obtain formulæ for the holomorphic Euler characteristic of X as sums of indices of $\mathrm{Spin}_{\mathbb{C}}$ Dirac operators on the components. This is a subelliptic analogue of Bojarski's formula in the elliptic case.

Introduction

Let X be an even dimensional manifold with a $\mathrm{Spin}_{\mathbb{C}}$-structure; see [6], [12]. A compatible choice of metric, g, defines a $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator, $\eth$ which acts on sections of the bundle of complex spinors, $\not{S}$. The metric on X induces a metric on the bundle of spinors. If $\langle\sigma,\sigma\rangle_g$ denotes a pointwise inner product, then we define an inner product of the space of sections of $\not{S}$, by setting:

$$\langle\sigma,\sigma\rangle_X = \int\limits_X \langle\sigma,\sigma\rangle_g dV_g.$$

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If X has an almost complex structure, then this structure defines a $\mathrm{Spin}_{\mathbb{C}}$-structure. If the complex structure is integrable; then the bundle of complex spinors is canonically identified with $\oplus_{q\geq 0}\Lambda^{0,q}$. As we usually work with the chiral operator, we let

$$\Lambda^{\mathrm{e}} = \bigoplus_{q=0}^{\lfloor \frac{n}{2} \rfloor} \Lambda^{0,2q} \quad \Lambda^{\mathrm{o}} = \bigoplus_{q=0}^{\lfloor \frac{n-1}{2} \rfloor} \Lambda^{0,2q+1}. \tag{1}$$

If the metric is Kähler, then the $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator is given by

$$\eth = \bar{\partial} + \bar{\partial}^*.$$

Here $\bar{\partial}^*$ denotes the formal adjoint of $\bar{\partial}$ defined by the metric. This operator is called the Dolbeault-Dirac operator by Duistermaat; see [6]. If the metric is Hermitian, though not Kähler, then

$$\eth = \bar{\partial} + \bar{\partial}^* + \mathcal{M}_0, \tag{2}$$

where $\mathcal{M}_0$ is a homomorphism carrying Λ^{e} to Λ^{o} and vice versa. It vanishes at points where the metric is Kähler. It is customary to write $\eth = \eth^{\mathrm{e}} + \eth^{\mathrm{o}}$ where

$$\eth^{\mathrm{e}} : \mathcal{C}^\infty(X; \Lambda^{\mathrm{e}}) \longrightarrow \mathcal{C}^\infty(X, \Lambda^{\mathrm{o}})$$

and $\eth^{\mathrm{o}}$ is the formal adjoint of $\eth^{\mathrm{e}}$. If X is a compact, complex manifold, then the graph closure of $\eth^{\mathrm{e}}$ is a Fredholm operator. It has the same principal symbol as $\bar{\partial} + \bar{\partial}^*$ and therefore its index is given by

$$\operatorname{Ind}(\eth^{\mathrm{e}}) = \sum_{j=0}^{n} (-1)^j \dim H^{0,j}(X) = \chi_{\mathcal{O}}(X). \tag{3}$$

If X is a manifold with boundary, then the kernels and cokernels of $\eth^{\mathrm{eo}}$ are generally infinite dimensional. To obtain a Fredholm operator we need to impose boundary conditions. In this instance there are no local boundary conditions for $\eth^{\mathrm{eo}}$ that define elliptic problems. Starting with Atiyah, Patodi and Singer, boundary conditions defined by classical pseudodifferential projections have been the focus of most of the work in this field. Such boundary conditions are very useful for studying topological problems, but are not well suited to the analysis of problems connected to the holomorphic structure of X. To that end we begin the study of boundary conditions for $\eth^{\mathrm{eo}}$ obtained by modifying the classical $\bar{\partial}$-Neumann and dual $\bar{\partial}$-Neumann conditions. For a $(0,q)$-form, σ^{0q}, the $\bar{\partial}$-Neumann condition is the requirement that

$$[\overline{\partial}\rho \rfloor \sigma^{0q}]_{bX} = 0.$$

This imposes no condition if $q = 0$, and all square integrable holomorphic functions thereby belong to the domain of the operator, and define elements of the null space of $\eth^{\mathrm{e}}$. Let $\mathcal{S}$ denote the Szegő projector; this is an operator acting on functions on bX with range equal to the null space of the tangential Cauchy-Riemann operator, $\bar{\partial}_b$. We can remove the null space in degree 0 by adding the condition

$$\mathcal{S}[\sigma^{00}]_{bX} = 0. \tag{4}$$

This, in turn, changes the boundary condition in degree 1 to

$$(\mathrm{Id} - \mathcal{S})[\bar{\partial}\rho\rfloor\sigma^{01}]_{bX} = 0. \tag{5}$$

If X is strictly pseudoconvex, then these modifications to the $\bar{\partial}$-Neumann condition produce a Fredholm boundary value problem for $\eth$. Indeed, it is not necessary to use the exact Szegő projector, defined by the induced CR-structure on bX. Any generalized Szegő projector, as defined in [9], suffices to prove the necessary estimates. There are analogous conditions for strictly pseudoconcave manifolds. In [2] and [13], [14] the Spin$_{\mathbb{C}}$ Dirac operator with the $\bar{\partial}$-Neumann condition is considered, though from a very different perspective. The results in these papers are largely orthogonal to those we have obtained.

A pseudoconvex manifold is denoted by X_+ and objects associated with it are labeled with a $+$ subscript, e.g., the Spin$_{\mathbb{C}}$-Dirac operator on X_+ is denoted $\eth_+$. Similarly, a pseudoconcave manifold is denoted by X_- and objects associated with it are labeled with a $-$ subscript. Usually X denotes a compact manifold, partitioned by an embedded, strictly pseudoconvex hypersurface, Y, into two components, $X \setminus Y = X_+ \coprod X_-$.

If $X_\pm$ is either strictly pseudoconvex or strictly pseudoconcave, then the modified boundary conditions are subelliptic and define Fredholm operators. The indices of these operators are connected to the holomorphic Euler characteristics of these manifolds with boundary, with the contributions of the infinite dimensional groups removed. We also consider the Dirac operator acting on the twisted spinor bundles

$$\Lambda^{p,\mathrm{eo}} = \Lambda^{\mathrm{eo}} \otimes \Lambda^{p,0},$$

and more generally $\Lambda^{\mathrm{eo}} \otimes \mathcal{V}$ where $\mathcal{V} \to X$ is a holomorphic vector bundle. When necessary, we use $\eth^{\mathrm{eo}}_{\mathcal{V}\pm}$ to *specify* the twisting bundle. The boundary conditions are defined by projection operators $\mathcal{R}^{\mathrm{eo}}_\pm$ acting on boundary values of sections of $\Lambda^{\mathrm{eo}} \otimes \mathcal{V}$. Among other things we show that the index of $\eth^{\mathrm{e}}_+$ with boundary condition defined by $\mathcal{R}^{\mathrm{e}}_+$ equals the regular part of the holomorphic Euler characteristic:

$$\mathrm{Ind}(\eth^{\mathrm{e}}_+, \mathcal{R}^{\mathrm{e}}_+) = \sum_{q=1}^{n} \dim H^{0,q}(X)(-1)^q. \tag{6}$$

In [7] we show that the pairs $(\eth^{\mathrm{eo}}_\pm, \mathcal{R}^{\mathrm{eo}}_\pm)$ are Fredholm and identify their L^2-adjoints. In each case, the L^2-adjoint is the closure of the formally adjoint boundary value problem, e.g.

$$(\eth^{\mathrm{e}}_+, \mathcal{R}^{\mathrm{e}}_+)^* = \overline{(\eth^{\mathrm{o}}_+, \mathcal{R}^{\mathrm{o}}_+)}.$$

This is proved by using a boundary layer method to reduce to analysis of operators on the boundary. The operators we obtain on the boundary are neither classical, nor Heisenberg pseudodifferential operators, but rather operators belonging to the extended Heisenberg calculus introduced in [9]. Similar classes of operators were also introduced by Beals, Greiner and Stanton as well as Taylor; see [4], [3], [15]. In this paper we apply the analytic results obtained in [7] to obtain Hodge decompositions for each of the boundary conditions and (p, q)-types.

In Section 1 we review some well known facts about the $\bar{\partial}$-Neumann problem and analysis on strictly pseudoconvex CR-manifolds. In the following two sections we introduce the boundary conditions we consider in the remainder of the paper and deduce subelliptic estimates for these boundary value problems from the results in [7]. The fourth section introduces the natural dual boundary conditions. In Section 5 we deduce the Hodge decompositions associated to the various boundary value problems defined in the earlier sections. In Section 6 we identify the nullspaces of the various boundary value problems when the classical Szegő projectors are used. In Section 7 we establish the basic link between the boundary conditions for (p, q)-forms considered in the earlier sections and boundary conditions for $\eth_{\pm}^{\mathrm{eo}}$ and prove an analogue of the Agranovich-Dynin formula. In Section 8 we obtain "regularized" versions of some long exact sequences due to Andreotti and Hill. Using these sequences we prove gluing formulæ for the holomorphic Euler characteristic of a compact complex manifold, X, with a strictly pseudoconvex separating hypersurface. These formulæ are subelliptic analogues of Bojarski's gluing formula for the classical Dirac operator with APS-type boundary conditions.

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1. Some background material

Henceforth X_+ (X_-) denotes a compact complex manifold of complex dimension n with a strictly pseudoconvex (pseudoconcave) boundary. We assume that a Hermitian metric, g is fixed on $X_{\pm}$. For some of our results we make additional assumptions on the nature of g, e. g., that it is Kähler. This metric induces metrics on all the natural bundles defined by the complex structure on $X_{\pm}$. To the extent possible, we treat the two cases in tandem. For example, we sometimes use $bX_{\pm}$ to denote the boundary of either X_+ or X_-. The kernels of $\eth_{\pm}$ are both infinite dimensional. Let $\mathcal{P}_{\pm}$ denote the operators defined on $bX_{\pm}$ which are the projections onto the boundary values of elements in $\ker \eth_{\pm}$; these are the Calderon projections. They are classical pseudodifferential operators of order 0; we use the definitions and analysis of these operators presented in [5].

We often work with the chiral Dirac operators $\eth^{\text{eo}}_{\pm}$ which act on sections of

$$\Lambda^{p,\text{e}} = \bigoplus_{q=0}^{\lfloor \frac{n}{2} \rfloor} \Lambda^{p,2q} X_{\pm}, \quad \Lambda^{p,\text{o}} = \bigoplus_{q=0}^{\lfloor \frac{n-1}{2} \rfloor} \Lambda^{p,2q+1} X_{\pm}, \tag{7}$$

respectively. Here p is an integer between 0 and n; except when entirely necessary it is omitted from the notation for things like $\mathcal{R}^{\text{eo}}_{\pm}, \eth^{\text{eo}}_{\pm}$, etc. The L^2-closure of the operators $\eth^{\text{eo}}_{\pm}$, with domains consisting of smooth spinors such that $\mathcal{P}^{\text{eo}}_{\pm}(\sigma\big|_{bX_{\pm}}) = 0$, are elliptic operators with Fredholm index zero.

Let ρ be a smooth defining function for the boundary of $X_{\pm}$. Usually we take ρ to be *negative* on X_+ and *positive* on X_-, so that $\partial\bar{\partial}\rho$ is positive definite near $bX_{\pm}$. If σ is a section of $\Lambda^{p,q}$, smooth up to $bX_{\pm}$, then the $\bar{\partial}$-Neumann boundary condition is the requirement that

$$\bar{\partial}\rho \rfloor \sigma \restriction_{bX_{\pm}} = 0. \tag{8}$$

If X_+ is strictly pseudoconvex, then there is a constant C such that if σ is a smooth section of $\Lambda^{p,q}$, with $q \geq 1$, satisfying (8), then σ satisfies the *basic estimate*:

$$\|\sigma\|^2_{(1,-\frac{1}{2})} \leq C(\|\bar{\partial}\sigma\|^2_{L^2} + \|\bar{\partial}^*\sigma\|^2_{L^2} + \|\sigma\|^2_{L^2}). \tag{9}$$

If X_- is strictly pseudoconcave, then there is a constant C such that if σ is a smooth section of $\Lambda^{p,q}$, with $q \neq n-1$, satisfying (8), then σ again satisfies the basic estimate (9). The $\Box$-operator is defined formally as

$$\Box\sigma = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\sigma.$$

The $\Box$-operator, with the $\bar{\partial}$-Neumann boundary condition is the graph closure of $\Box$ acting on smooth forms, σ, that satisfy (8), such that $\bar{\partial}\sigma$ also satisfies (8). It has an infinite dimensional nullspace acting on sections of $\Lambda^{p,0}(X_+)$ and $\Lambda^{p,n-1}(X_-)$, respectively. For clarity, we sometimes use the notation $\Box^{p,q}$ to denote the $\Box$-operator acting on sections of $\Lambda^{p,q}$.

Let Y be a compact strictly pseudoconvex CR-manifold of real dimension $2n-1$. Let $T^{0,1}Y$ denote the $(0,1)$-part of $TY \otimes \mathbb{C}$ and $\mathcal{T}Y$ the holomorphic vector bundle $TY \otimes \mathbb{C}/T^{0,1}Y$. The dual bundles are denoted $\Lambda^{0,1}_b$ and $\Lambda^{1,0}_b$ respectively. For $0 \leq p \leq n$, let

$$\mathcal{C}^\infty(Y;\Lambda^{p,0}_b) \xrightarrow{\bar{\partial}_b} \mathcal{C}^\infty(Y;\Lambda^{p,1}_b) \xrightarrow{\bar{\partial}_b} \dots \xrightarrow{\bar{\partial}_b} \mathcal{C}^\infty(Y;\Lambda^{p,n-1}_b) \tag{10}$$

denote the $\bar{\partial}_b$-complex. Fixing a choice of Hermitian metric on Y, we define formal adjoints

$$\bar{\partial}^*_b : \mathcal{C}^\infty(Y;\Lambda^{p,q}_b) \longrightarrow \mathcal{C}^\infty(Y;\Lambda^{p,q-1}_b).$$

The $\Box_b$-operator acting on $\Lambda^{p,q}_b$ is the graph closure of

$$\Box_b = \bar{\partial}_b\bar{\partial}^*_b + \bar{\partial}^*_b\bar{\partial}_b, \tag{11}$$

acting on $\mathcal{C}^\infty(Y;\Lambda_b^{p,q})$. The operator $\Box_b^{p,q}$ is subelliptic if $0 < q < n-1$. If $q=0$, then $\bar\partial_b$ has an infinite dimensional nullspace, while if $q=n-1$, then $\bar\partial_b^*$ has an infinite dimensional nullspace. We let $\mathcal{S}_p$ denote an orthogonal projector onto the nullspace of $\bar\partial_b$ acting on $\mathcal{C}^\infty(Y;\Lambda_b^{p,0})$, and $\bar{\mathcal{S}}_p$ an orthogonal projector onto the nullspace of $\bar\partial_b^*$ acting on $\mathcal{C}^\infty(Y;\Lambda_b^{p,n-1})$. The operator $\mathcal{S}_p$ is usually called "the" Szegő projector; we call $\bar{\mathcal{S}}_p$ the conjugate Szegő projector. These projectors are only defined once a metric is selected, but this ambiguity has no bearing on our results. As is well known, these operators are *not* classical pseudodifferential operators, but belong to the Heisenberg calculus. Generalizations of these projectors are introduced in [9] and play a role in the definition of subelliptic boundary value problems for $\eth$. For $0 < q < n-1$, the Kohn-Rossi cohomology groups

$$H_b^{p,q}(Y) = \frac{\ker\{\bar\partial_b : \mathcal{C}^\infty(Y;\Lambda_b^{p,q}) \to \mathcal{C}^\infty(Y;\Lambda_b^{p,q+1})\}}{\bar\partial_b\mathcal{C}^\infty(Y;\Lambda_b^{p,q-1})}$$

are finite dimensional. The regularized $\bar\partial_b$-Euler characteristics of Y are defined to be

$$\chi'_{pb}(Y) = \sum_{q=1}^{n-2}(-1)^q \dim H_b^{p,q}(Y), \text{ for } 0 \le p \le n. \tag{12}$$

Very often we use Y to denote the boundary of $X_\pm$.

The Hodge star operator on $X_\pm$ defines an isomorphism

$$\star : \Lambda^{p,q}(X_\pm) \longrightarrow \Lambda^{n-p,n-q}(X_\pm). \tag{13}$$

Note that we have incorporated complex conjugation into the definition of the Hodge star operator. The usual identities continue to hold, i.e.,

$$\star\star = (-1)^{p+q}, \quad \bar\partial^* = -\star\bar\partial\star. \tag{14}$$

There is also a Hodge star operator on Y that defines an isomorphism:

$$\star_b : \Lambda_b^{p,q}(Y) \longrightarrow \Lambda_b^{n-p,n-q-1}(Y), \quad [\bar\partial_b^{p,q}]^* = (-1)^{p+q+1}\star_b\bar\partial_b\star_b. \tag{15}$$

There is a canonical boundary condition dual to the $\bar\partial$-Neumann condition. The dual $\bar\partial$-Neumann condition is the requirement that

$$\bar\partial\rho\wedge\sigma \restriction_{bX_\pm} = 0. \tag{16}$$

If σ is a (p,q)-form defined on $X_\pm$, then, along the boundary we can write

$$\sigma\restriction_{bX_\pm} = \bar\partial\rho\wedge(\bar\partial\rho\rfloor\sigma) + \sigma_b. \tag{17}$$

Here $\sigma_b \in \mathcal{C}^\infty(Y;\Lambda_b^{p,q})$ is a representative of $\sigma\restriction_{(TY)^p\otimes(T^{0,1}Y)^q}$. The dual $\bar\partial$-Neumann condition is equivalent to the condition

$$\sigma_b = 0. \tag{18}$$

For later applications we note the following well known relations: For sections $\sigma \in \mathcal{C}^\infty(\overline{X}_\pm, \Lambda^{p,q})$, we have

$$(\bar\partial\rho\rfloor\sigma)^{\star_b} = (\sigma^\star)_b, \quad \bar\partial\rho\rfloor(\sigma^\star) = \sigma_b^{\star_b}, \quad (\bar\partial\sigma)_b = \bar\partial_b\sigma_b. \tag{19}$$

The dual $\bar\partial$-Neumann operator on $\Lambda^{p,q}$ is the graph closure of $\Box^{p,q}$ on smooth sections, σ of $\Lambda^{p,q}$ satisfying (16), such that $\bar\partial^*\sigma$ also satisfies (16). For a strictly pseudoconvex manifold, the basic estimate holds for (p,q)-forms satisfying (16), provided $0 \le q \le n-1$. For a strictly pseudoconcave manifold, the basic estimate holds for (p,q)-forms satisfying (16), provided $q \neq 1$.

As we consider many different boundary conditions, it is useful to have notation that specifies the boundary condition under consideration. If $\mathcal{D}$ denotes an operator acting on sections of a complex vector bundle, $E \to X$, and $\mathcal{B}$ denotes a boundary operator acting on sections of $E \restriction_{bX}$, then the pair $(\mathcal{D}, \mathcal{B})$ is the operator $\mathcal{D}$ acting on smooth sections s that satisfy

$$\mathcal{B}s \restriction_{bX} = 0.$$

The notation $s \restriction_{bX}$ refers to the section of $E \restriction_{bX}$ obtained by restricting a section s of $E \to X$ to the boundary. The operator $\mathcal{B}$ is a pseudodifferential operator acting on sections of $E \restriction_{bX}$. Some of the boundary conditions we consider are defined by Heisenberg pseudodifferential operators. We often denote objects connected to $(\mathcal{D}, \mathcal{B})$ with a subscripted $\mathcal{B}$. For example, the nullspace of $(\mathcal{D}, \mathcal{B})$ (or harmonic sections) might be denoted $\mathcal{H}_\mathcal{B}$. We denote objects connected to the $\bar\partial$-Neumann operator with a subscripted $\bar\partial$, e. g., $\Box^{p,q}_{\bar\partial}$. Objects connected to the dual $\bar\partial$-Neumann problem are denoted by a subscripted $\bar\partial^*$, e.g., $\Box^{p,q}_{\bar\partial^*}$.

Let $\mathcal{H}^{p,q}_{\bar\partial}(X_\pm)$ denote the nullspace of $\Box^{p,q}_{\bar\partial}$ and $\mathcal{H}^{p,q}_{\bar\partial^*}(X_\pm)$ the nullspace of $\Box^{p,q}_{\bar\partial^*}$. In [11] it is shown that

$$\begin{aligned} \mathcal{H}^{p,q}_{\bar\partial}(X_+) &\simeq [\mathcal{H}^{n-p,n-q}_{\bar\partial^*}(X_+)]^*, \text{ if } q \neq 0, \\ \mathcal{H}^{p,q}_{\bar\partial}(X_-) &\simeq [\mathcal{H}^{n-p,n-q}_{\bar\partial^*}(X_-)]^*, \text{ if } q \neq n-1. \end{aligned} \tag{20}$$

Remark 1. In this paper C is used to denote a variety of *positive* constants which depend only on the geometry of X. If M is a manifold with a volume form dV and f_1, f_2 are sections of a bundle with a Hermitian metric $\langle\cdot,\cdot\rangle_g$, then the L^2-inner product over M is denoted by

$$\langle f_1, f_2\rangle_M = \int\limits_M \langle f_1, f_2\rangle_g \, \mathrm{dV}\,. \tag{21}$$

2. Subelliptic boundary conditions for pseudoconvex manifolds

In this section we define a modification of the classical $\bar\partial$-Neumann condition for sections belonging to $\mathcal{C}^\infty(\bar X_+; \Lambda^{p,q})$, for $0 \le p \le n$ and $0 \le q \le n$.

The bundles $\Lambda^{p,0}$ are holomorphic, and so, as in the classical case they do not not really have any effect on the estimates. As above, $\mathcal{S}_p$ denotes an orthogonal projection acting on sections of $\Lambda_b^{p,0}$ with range equal to the null space of $\bar{\partial}_b$ acting sections of $\Lambda_b^{p,0}$. The range of $\mathcal{S}_p$ includes the boundary values of holomorphic $(p,0)$-forms, but may in general be somewhat larger. If σ^{p0} is a holomorphic section, then $\sigma_b^{p0} = \mathcal{S}_p\sigma_b^{p0}$. On the other hand, if σ^{p0} is any smooth section of $\Lambda^{p,0}$, then $\bar{\partial}\rho\rfloor\sigma^{p0} = 0$ and therefore, the L^2-holomorphic sections belong to the nullspace of $\Box_{\bar{\partial}}^{p0}$.

To obtain a subelliptic boundary value problem for $\Box^{pq}$ in all degrees, we modify the $\bar{\partial}$-Neumann condition in degrees 0 and 1. The modified boundary condition is denoted by $\mathcal{R}_+$. A smooth form $\sigma^{p0} \in \operatorname{Dom}(\bar{\partial}_{\mathcal{R}_+}^{p,0})$ provided

$$\mathcal{S}_p\sigma_b^{p0} = 0. \tag{22}$$

There is no boundary condition if $q > 0$. A smooth form belongs to $\operatorname{Dom}([\bar{\partial}_{\mathcal{R}_+}^{p,q}]^*)$ provided

$$\begin{aligned} (\operatorname{Id} - \mathcal{S}_p)[\bar{\partial}\rho\rfloor\sigma^{p1}]_b &= 0, \\ [\bar{\partial}\rho\rfloor\sigma^{pq}]_b &= 0 \quad \text{if } 1 < q. \end{aligned} \tag{23}$$

For each (p,q) we define the quadratic form

$$\mathcal{Q}^{p,q}(\sigma^{pq}) = \langle\bar{\partial}\sigma^{pq}, \bar{\partial}\sigma^{pq}\rangle_{L^2} + \langle\bar{\partial}^*\sigma^{pq}, \bar{\partial}^*\sigma^{pq}\rangle_{L^2}. \tag{24}$$

We can consider more general conditions than these by replacing the classical Szegő projector $\mathcal{S}_p$ by a generalized Szegő projector acting on sections of $\Lambda_b^{p,0}$. Recall that an order-zero operator, S_E in the Heisenberg calculus, acting on sections of a complex vector bundle $E \to Y$, is a generalized Szegő projector if

1. $S_E^2 = S_E$ and $S_E^* = S_E$.
2. $\sigma_0^H(S_E) = s \otimes \operatorname{Id}_E$ where s is the symbol of a field of vacuum state projectors defined by a choice of compatible almost complex structure on the contact field of Y.

This class of projectors is defined in [8] and analyzed in detail in [9]. Among other things we show that, given a generalized Szegő projector, there is a $\bar{\partial}_b$-like operator, D_E so that the range of S_E is precisely the null space of D_E. The operator D_E is $\bar{\partial}_b$-like in the following sense: If $\overline{Z}_j'$ is a local frame field for the almost complex structure defined by the principal symbol of S_E, then there are order-zero Heisenberg operators μ_j, so that, locally

$$D_E\sigma = 0 \text{ if and only if } (\overline{Z}_j' + \mu_j)\sigma = 0 \text{ for } j = 1, \dots, n-1. \tag{25}$$

Similar remarks apply to define generalized conjugate Szegő projectors. We use the notation $\mathcal{S}_p'$ to denote a generalized Szegő projector acting on sections of $\Lambda_b^{p,0}$.

We can view these boundary conditions as boundary conditions for the operator $\eth_+$ acting on sections of $\oplus_q \Lambda^{p,q}$. Let σ be a such a section. The boundary condition is expressed as a projection operator acting on $\sigma \restriction_{bX_+}$. We write

$$\begin{gathered} \sigma \restriction_{bX_+} = \sigma_b + \bar\partial\rho \wedge \sigma_\nu, \text{ with} \\ \sigma_b = (\sigma_b^{p0}, \tilde\sigma_b^p) \text{ and } \sigma_\nu = (\sigma_\nu^{p1}, \tilde\sigma_\nu^p). \end{gathered} \tag{26}$$

Recall that σ_b^{pn} and σ_ν^{p0} always vanish. With this notation we have, in block form, that

$$\mathcal{R}'_+\sigma \restriction_{bX_+} = \begin{pmatrix} \mathcal{S}'_p & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & \mathbf{0} \\ 0 & 0 & \mathrm{Id} - \mathcal{S}'_p & 0 \\ 0 & \mathbf{0} & 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \sigma_b^{p0} \\ \tilde\sigma_b^p \\ \sigma_\nu^{p1} \\ \tilde\sigma_\nu^p \end{pmatrix}. \tag{27}$$

Here $\mathbf{0}$ denotes an $(n-1)\times(n-1)$ matrix of zeros. The boundary condition for $\eth_+$ is $\mathcal{R}'_+\sigma \restriction_{bX_+} = 0$. These can of course be split into boundary conditions for $\eth_+^{\mathrm{eo}}$, which we denote by $\mathcal{R}'^{\mathrm{eo}}_+$. The formal adjoint of $(\eth^{\mathrm{e}}_+, \mathcal{R}'^{\mathrm{e}}_+)$ is $(\eth^{\mathrm{o}}_+, \mathcal{R}'^{\mathrm{o}}_+)$. In Section 7 we show that the L^2-adjoint of $(\eth^{\mathrm{e}}_+, \mathcal{R}'^{\mathrm{e}}_+)$ is the graph closure of $(\eth^{\mathrm{o}}_+, \mathcal{R}'^{\mathrm{o}}_+)$. When the distinction is important, we explicitly indicate the dependence on p by using $\mathcal{R}'_{p+}$ to denote the projector acting on sections of $\oplus_q \Lambda^{p,q} \restriction_{bX_+}$ and $\eth_{p+}$ to denote the operator acting on sections of $\oplus_q \Lambda^{p,q}$.

We use $\mathcal{R}_+$ (without the $'$) to denote the boundary condition defined by the matrix in (27), with $\mathcal{S}'_p = \mathcal{S}_p$, the classical Szegő projector. In [7], we prove estimates for the Spin$_{\mathbb{C}}$ Dirac operator with these sorts of boundary conditions. We first state a direct consequence of Corollary 13.9 in [5].

LEMMA 1. *Let X be a complex manifold with boundary and $\sigma^{pq} \in L^2(X; \Lambda^{p,q})$. Suppose that $\bar\partial\sigma^{pq}, \bar\partial^*\sigma^{pq}$ are also square integrable; then $\sigma^{pq} \restriction_{bX}$ is well defined as an element of $H^{-\frac{1}{2}}(bX; \Lambda^{p,q}_{bX})$.*

Proof. Because X is a complex manifold, the twisted Spin$_{\mathbb{C}}$ Dirac operator acting on sections of $\Lambda^{p,*}$ is given by (2). The hypotheses of the lemma therefore imply that $\eth\sigma^{pq}$ is square integrable and the lemma follows directly from Corollary 13.9 in [5]. □

Remark 2. If the restriction of a section of a vector bundle to the boundary is well defined in the sense of distributions then we say that the section has distributional boundary values. Under the hypotheses of the lemma, σ^{pq} has distributional boundary values.

Theorem 3 in [7] implies the following estimates for the individual form degrees:

PROPOSITION 1. *Suppose that X is a strictly pseudoconvex manifold, $\mathcal{S}'_p$ is a generalized Szegő projector acting on sections of $\Lambda_b^{p,0}$, and let $s \in [0,\infty)$. There is a constant C_s such that if σ^{pq} is an L^2-section of $\Lambda^{p,q}$ with $\bar{\partial}\sigma^{pq}$, $\bar{\partial}^*\sigma^{pq} \in H^s$ and*

$$\begin{aligned} \mathcal{S}'_p[\sigma^{pq}]_b &= 0 \quad \text{if } q=0,\\ (\mathrm{Id}-\mathcal{S}'_p)[\bar{\partial}\rho\rfloor\sigma^{pq}]_b &= 0 \quad \text{if } q=1,\\ [\bar{\partial}\rho\rfloor\sigma^{pq}]_b &= 0 \quad \text{if } q>1, \end{aligned} \tag{28}$$

then

$$\|\sigma^{pq}\|_{H^{s+\frac{1}{2}}} \leq C_s[\|\bar{\partial}\sigma^{pq}\|_{H^s} + \|\bar{\partial}^*\sigma^{pq}\|_{H^s} + \|\sigma^{pq}\|_{L^2}]. \tag{29}$$

Remark 3. As noted in [7], the hypotheses of the proposition imply that σ^{pq} has a well defined restriction to bX_+ as an L^2-section of $\Lambda^{pq}\restriction_{bX_+}$. The boundary conditions in (28) can therefore be interpreted in the sense of distributions. If $s=0$ then the norm on the left-hand side of (29) can be replaced by the slightly stronger $H_{(1,-\frac{1}{2})}$-norm.

Proof. These estimates follow immediately from Theorem 3 in [7] when we observe that the hypotheses imply that

$$\begin{aligned} &\eth_{\Lambda^{p,0}+}\sigma^{pq} \in H^s(X_+) \text{ and}\\ &\mathcal{R}'_{\Lambda^{p,0}+}[\sigma^{pq}]_{bX_+} = 0. \end{aligned} \tag{30}$$

□

These estimates show that, for all $0 \leq p,q \leq n$, the form domain for $\bar{\mathcal{Q}}^{p,q}_{\mathcal{R}_+}$, the closure of $\mathcal{Q}^{p,q}_{\mathcal{R}_+}$, lies in $H_{(1,-\frac{1}{2})}(X_+;\Lambda^{p,q})$. This implies that the self-adjoint operator, $\Box^{p,q}_{\mathcal{R}_+}$, defined by the Friedrichs extension process, has a compact resolvent and therefore a finite dimensional null space $\mathcal{H}^{p,q}_{\mathcal{R}_+}(X_+)$. We define closed, unbounded operators on $L^2(X_+;\Lambda^{p,q})$ denoted $\bar{\partial}^{p,q}_{\mathcal{R}_+}$ and $[\bar{\partial}^{p,q-1}_{\mathcal{R}_+}]^*$ as the graph closures of $\bar{\partial}$ and $\bar{\partial}^*$ acting on smooth sections with domains given by the appropriate condition in (22), (23). The domains of these operators are denoted $\mathrm{Dom}_{L^2}(\bar{\partial}^{p,q}_{\mathcal{R}_+}), \mathrm{Dom}_{L^2}([\bar{\partial}^{p,q-1}_{\mathcal{R}_+}]^*)$, respectively. It is clear that

$$\mathrm{Dom}(\bar{\mathcal{Q}}^{p,q}_{\mathcal{R}_+}) = \mathrm{Dom}_{L^2}(\bar{\partial}^{p,q}_{\mathcal{R}_+}) \cap \mathrm{Dom}_{L^2}([\bar{\partial}^{p,q-1}_{\mathcal{R}_+}]^*).$$

3. Subelliptic boundary conditions for pseudoconcave manifolds

We now repeat the considerations of the previous section for X_-, a strictly pseudoconcave manifold. In this case the $\bar{\partial}$-Neumann condition fails to define a subelliptic boundary value problem on sections of $\Lambda^{p,n-1}$. We let $\bar{\mathcal{S}}_p$ denote an orthogonal projection onto the nullspace of $[\bar{\partial}_b^{p(n-1)}]^*$. The projector acts on sections of $\Lambda_b^{p(n-1)}$. From this observation, and equation (15), it follows immediately that

$$\bar{\mathcal{S}}_p = \star_b \mathcal{S}_{n-p} \star_b . \tag{31}$$

If instead we let $\mathcal{S}'_{n-p}$ denote a generalized Szegő projector acting on $(n-p,0)$-forms, then (31), with $\mathcal{S}_{n-p}$ replaced by $\mathcal{S}'_{n-p}$, defines a generalized conjugate Szegő projector acting on $(p,n-1)$-forms, $\bar{\mathcal{S}}'_p$.

Recall that the defining function, ρ, is positive on the interior of X_-. We now define a modified $\bar{\partial}$-Neumann condition for X_-, which we denote by $\mathcal{R}'_-$. The $\operatorname{Dom}(\bar{\partial}^{p,q}_{\mathcal{R}'_-})$ requires no boundary condition for $q \neq n-1$ and is specified for $q = n-1$ by

$$\bar{\mathcal{S}}'_p \sigma_b^{p(n-1)} = 0. \tag{32}$$

The $\operatorname{Dom}([\bar{\partial}^{p,q}_{\mathcal{R}'_-}]^*)$ is given by

$$\bar{\partial}\rho \rfloor \sigma^{pq} = 0 \quad \text{ if } q \neq n, \tag{33}$$

$$(\operatorname{Id} - \bar{\mathcal{S}}'_p)(\bar{\partial}\rho \rfloor \sigma^{pn})_b = 0. \tag{34}$$

As before we assemble the individual boundary conditions into a boundary condition for $\eth_-$. The boundary condition is expressed as a projection operator acting on $\sigma \restriction_{bX_-}$. We write

$$\begin{gathered} \sigma \restriction_{bX_-} = \sigma_b + \bar{\partial}\rho \wedge \sigma_\nu, \text{ with} \\ \sigma_b = (\tilde{\sigma}_b^p, \sigma_b^{p(n-1)}) \text{ and } \sigma_\nu = (\tilde{\sigma}_\nu^p, \sigma_\nu^{pn}). \end{gathered} \tag{35}$$

Recall that σ_b^{pn} and σ_ν^{p0} always vanish. With this notation we have, in block form that

$$\mathcal{R}'_- \sigma \restriction_{bX_-} = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \bar{\mathcal{S}}'_p & 0 & \mathbf{0} \\ 0 & 0 & \operatorname{Id} & 0 \\ 0 & \mathbf{0} & 0 & \operatorname{Id} - \bar{\mathcal{S}}'_p \end{pmatrix} \begin{pmatrix} \tilde{\sigma}_b^p \\ \sigma_b^{p(n-1)} \\ \tilde{\sigma}_\nu^p \\ \sigma_\nu^{pn} \end{pmatrix} . \tag{36}$$

Here $\mathbf{0}$ denotes an $(n-1) \times (n-1)$ matrix of zeros. The boundary condition for $\eth_-$ is $\mathcal{R}'_- \sigma \restriction_{bX_-} = 0$. These can of course be split into boundary conditions for $\eth_-^{\mathrm{eo}}$, which we denote by $\mathcal{R}'^{\mathrm{eo}}_-$. The formal adjoint of $(\eth_-^{\mathrm{e}}, \mathcal{R}'^{\mathrm{e}}_-)$ is $(\eth_-^{\mathrm{o}}, \mathcal{R}'^{\mathrm{o}}_-)$. In Section 7 we show that the L^2-adjoint of $(\eth_-^{\mathrm{eo}}, \mathcal{R}'^{\mathrm{eo}}_-)$ is the graph closure of $(\eth_-^{\mathrm{oe}}, \mathcal{R}'^{\mathrm{oe}}_-)$. When the distinction is important, we explicitly indicate the dependence on p by using $\mathcal{R}'_{p-}$ to denote this projector acting on sections of $\oplus_q \Lambda^{p,q} \restriction_{bX_-}$ and $\eth_{p-}$ to denote the operator acting on sections of $\oplus_q \Lambda^{p,q}$. If we are using the classical conjugate Szegő projector, then we omit the prime, i.e., the notation $\mathcal{R}_-$ refers to the boundary condition defined by the matrix in (36) with $\bar{\mathcal{S}}'_p = \bar{\mathcal{S}}_p$, the classical conjugate Szegő projector.

Theorem 3 in [7] also provides subelliptic estimates in this case.

Proposition 2. *Suppose that X is a strictly pseudoconcave manifold, $\bar{\mathcal{S}}'_p$ is a generalized Szegő projector acting on sections of $\Lambda_b^{p,n-1}$, and let $s \in [0,\infty)$. There is a constant C_s such that if σ^{pq} is an L^2-section of $\Lambda^{p,q}$ with $\bar{\partial}\sigma^{pq}, \bar{\partial}^*\sigma^{pq} \in H^s$ and*

$$\begin{aligned} \bar{\mathcal{S}}'_p[\sigma^{pq}]_b &= 0 \quad \text{if } q = n-1, \\ (\mathrm{Id} - \bar{\mathcal{S}}'_p)[\bar{\partial}\rho\rfloor\sigma^{pq}]_b &= 0 \quad \text{if } q = n, \\ [\bar{\partial}\rho\rfloor\sigma^{pq}]_{bX_-} &= 0 \quad \text{if } q \neq n-1, n, \end{aligned} \tag{37}$$

then

$$\|\sigma^{pq}\|_{H^{s+\frac{1}{2}}} \leq C_s[\|\bar{\partial}\sigma^{pq}\|_{H^s} + \|\bar{\partial}^*\sigma^{pq}\|_{H^s} + \|\sigma^{pq}\|_{L^2}]. \tag{38}$$

Proof. The hypotheses imply that

$$\begin{aligned} &\eth_{\Lambda^{p,0}-}\sigma^{pq} \in H^s(X_-) \text{ and} \\ &\mathcal{R}'_{\Lambda^{p,0}-}[\sigma^{pq}]_{bX_-} = 0. \end{aligned} \tag{39}$$

Thus σ^{pq} satisfies the hypotheses of Theorem 3 in [7]. □

4. The dual boundary conditions

In the two previous sections we have established the basic estimates for L^2 forms on X_+ (resp. X_-) that satisfy $\mathcal{R}'_+$ (resp. $\mathcal{R}'_-$). The Hodge star operator defines isomorphisms

$$\star : L^2(X_\pm; \oplus_q\Lambda^{p,q}) \longrightarrow L^2(X_\pm; \oplus_q\Lambda^{n-p,n-q}). \tag{40}$$

Under this isomorphism, a form satisfying $\mathcal{R}'_\pm\sigma \restriction_{bX_\pm} = 0$ is carried to a form, ${}^\star\sigma$, satisfying $(\mathrm{Id} - \mathcal{R}'_\mp)\star\sigma \restriction_{bX_\pm} = 0$, and vice versa. Here of course the generalized Szegő and conjugate Szegő projectors must be related as in (31). In form degrees where $\mathcal{R}'_\pm$ coincides with the usual $\bar{\partial}$-Neumann conditions, this statement is proved in [10]. In the degrees where the boundary condition has been modified, it follows from the identities in (19) and (31). Applying Hodge star, we immediately deduce the basic estimates for the dual boundary conditions, $\mathrm{Id} - \mathcal{R}'_\mp$.

Lemma 2. *Suppose that X_+ is strictly pseudoconvex and $\sigma^{pq} \in L^2(X_+; \Lambda^{p,q})$. For $s \in [0,\infty)$, there is a constant C_s so that, if $\bar{\partial}\sigma^{pq}, \bar{\partial}^*\sigma^{pq} \in H^s$, and*

$$\begin{aligned} \sigma_b^{pq} &= 0 \quad \text{if } q < n-1, \\ (\mathrm{Id} - \bar{\mathcal{S}}'_p)\sigma_b^{pq} &= 0 \quad \text{if } q = n-1, \\ \bar{\mathcal{S}}'_p(\bar{\partial}\rho\rfloor\sigma^{pq})_b &= 0 \quad \text{if } q = n, \end{aligned} \tag{41}$$

then

$$\|\sigma^{pq}\|_{H^{s+\frac{1}{2}}} \leq C_s\left[\|\bar{\partial}\sigma^{pq}\|_{H^s} + \|\bar{\partial}^*\sigma^{pq}\|_{H^s} + \|\sigma^{pq}\|^2_{L^2}\right]. \tag{42}$$

LEMMA 3. *Suppose that X_- is strictly pseudoconcave and $\sigma^{pq} \in L^2(X_-;\Lambda^{p,q})$. For $s \in [0,\infty)$, there is a constant C_s so that, if $\bar{\partial}\sigma^{pq}, \bar{\partial}^*\sigma^{pq} \in H^s$, and*

$$\begin{aligned} \sigma_b^{pq} &= 0 \quad \text{if } q > 1, \\ \mathcal{S}'_p(\bar{\partial}\rho \rfloor \sigma^{pq})_b = 0 \text{ and } \sigma_b^{pq} &= 0 \quad \text{if } q = 1, \\ (\mathrm{Id} - \mathcal{S}'_p)\sigma_b^{pq} &= 0 \quad \text{if } q = 0, \end{aligned} \tag{43}$$

then

$$\|\sigma^{pq}\|_{H^{s+\frac{1}{2}}} \leq C_s \left[\|\bar{\partial}\sigma^{pq}\|_{H^s} + \|\bar{\partial}^*\sigma^{pq}\|_{H^s} + \|\sigma^{pq}\|^2_{L^2}\right]. \tag{44}$$

5. Hodge decompositions

The basic analytic ingredient that is needed to proceed is the higher norm estimates for the $\square$-operator. Because the boundary conditions $\mathcal{R}'_\pm$ are non-local, the standard elliptic regularization and approximation arguments employed, e.g., by Folland and Kohn, do not directly apply. Instead of trying to adapt these results and treat each degree (p,q) separately, we instead consider the operators $\eth^{\mathrm{eo}}_\pm$ with boundary conditions defined by $\mathcal{R}'^{\mathrm{eo}}_\pm$. In [7] we use a boundary layer technique to obtain estimates for the inverses of the operators $[\eth^{\mathrm{eo}}_\pm]^*\eth^{\mathrm{eo}}_\pm + \mu^2$. On a Kähler manifold the operators $[\eth^{\mathrm{eo}}_\pm]^*\eth^{\mathrm{eo}}_\pm$ preserve form degree, which leads to estimates for the inverses of $\square^{p,q}_{\mathcal{R}_\pm} + \mu^2$. For our purposes the following consequence of Corollary 3 in [7] suffices.

THEOREM 1. *Suppose that $X_\pm$ is a strictly pseudoconvex (pseudoconcave) compact, complex Kähler manifold with boundary. Fix $\mu > 0$, and $s \geq 0$. There is a positive constant C_s such that for $\beta \in H^s(X_\pm;\Lambda^{p,q})$, there exists a unique section $\alpha \in H^{s+1}(X_\pm;\Lambda^{p,q})$ satisfying $[\square^{p,q} + \mu^2]\alpha = \beta$ with*

$$\alpha \in \mathrm{Dom}(\bar{\partial}^{p,q}_{\mathcal{R}'_\pm}) \cap \mathrm{Dom}([\bar{\partial}^{p,q-1}_{\mathcal{R}'_\pm}]^*) \text{ and } \bar{\partial}\alpha \in \mathrm{Dom}([\bar{\partial}^{p,q}_{\mathcal{R}'_\pm}]^*),\ \bar{\partial}^*\alpha \in \mathrm{Dom}(\bar{\partial}^{p,q-1}_{\mathcal{R}'_\pm}) \tag{45}$$

such that

$$\|\alpha\|_{H^{s+1}} \leq C_s\|\beta\|_{H^s}. \tag{46}$$

The boundary conditions in (45) are in the sense of distributions. If s is sufficiently large, then we see that this boundary value problem has a classical solution.

As in the classical case, these estimates imply that each operator $\square^{p,q}_{\mathcal{R}'_\pm}$ has a complete basis of eigenvectors composed of smooth forms. Moreover the orthocomplement of the nullspace is the range. This implies that each operator has an associated Hodge decomposition. If $G^{p,q}_{\mathcal{R}'_\pm}$, $H^{p,q}_{\mathcal{R}'_\pm}$ are the partial inverse and projector onto the nullspace, then we have that

$$\square^{p,q}_{\mathcal{R}'_\pm} G^{p,q}_{\mathcal{R}'_\pm} = G^{p,q}_{\mathcal{R}'_\pm} \square^{p,q}_{\mathcal{R}'_\pm} = \operatorname{Id} - H^{p,q}_{\mathcal{R}'_\pm}. \tag{47}$$

To get the usual and more useful Hodge decomposition, we use boundary conditions defined by the classical Szegő projectors. The basic property needed to obtain these results is contained in the following two lemmas.

Lemma 4. *If* $\alpha \in \operatorname{Dom}_{L^2}(\bar\partial^{p,q}_{\mathcal{R}_\pm})$, *then* $\bar\partial\alpha \in \operatorname{Dom}_{L^2}(\bar\partial^{p,q+1}_{\mathcal{R}_\pm})$.

Proof. The L^2-domain of $\bar\partial^{p,q}_{\mathcal{R}_\pm}$ is defined as the graph closure of smooth forms satisfying the appropriate boundary conditions, defined by (22) and (32). Hence, if $\alpha \in \operatorname{Dom}_{L^2}(\bar\partial^{p,q}_{\mathcal{R}_\pm})$, then there is a sequence of smooth (p, q)-forms $< \alpha_n >$ such that

$$\lim_{n\to\infty} \|\bar\partial\alpha_n - \bar\partial\alpha\|_{L^2} + \|\alpha_n - \alpha\|_{L^2} = 0, \tag{48}$$

and each α_n satisfies the appropriate boundary condition. First we consider $\mathcal{R}_+$. If $q = 0$, then $\mathcal{S}_p(\alpha_n)_b = 0$. The operator $\bar\partial^{p,1}_{\mathcal{R}_+}$ has no boundary condition, so $\bar\partial\alpha_n$ belongs to $\operatorname{Dom}(\bar\partial^{p,1}_{\mathcal{R}_+})$. Since $\bar\partial^2\alpha_n = 0$. we see that $\bar\partial\alpha \in \operatorname{Dom}_{L^2}(\bar\partial^{p,1}_{\mathcal{R}_+})$. In all other cases $\bar\partial^{p,q}_{\mathcal{R}_+}$ has no boundary condition.

We now turn to $\mathcal{R}_-$. In this case there is only a boundary condition if $q = n-1$, so we only need to consider $\alpha \in \operatorname{Dom}_{L^2}(\bar\partial^{p,n-2}_{\mathcal{R}_-})$. Let $< \alpha_n >$ be smooth forms converging to α in the graph norm. Because $\bar{\mathcal{S}}_p\bar\partial_b = 0$, it follows that

$$\bar{\mathcal{S}}_p(\bar\partial\alpha_n)_b = \bar{\mathcal{S}}_p(\bar\partial_b(\alpha_n)_b) = 0.$$

Hence $\bar\partial\alpha_n \in \operatorname{Dom}(\bar\partial^{p,n-1}_{\mathcal{R}_-})$. Again $\bar\partial^2\alpha_n = 0$ implies that $\bar\partial\alpha \in \operatorname{Dom}_{L^2}(\bar\partial^{p,n-1}_{\mathcal{R}_-})$. □

Remark 4. The same argument applies to show that the lemma holds for the boundary condition defined by $\mathcal{R}'_+$.

We have a similar result for the adjoint. The domains of $[\bar\partial^{p,q}_{\mathcal{R}_\pm}]^*$ are defined as the graph closures of $[\bar\partial^{p,q}]^*$ with boundary conditions defined by (23), (33) and (34).

Lemma 5. *If* $\alpha \in \operatorname{Dom}_{L^2}([\bar\partial^{p,q}_{\mathcal{R}_\pm}]^*)$ *then* $\bar\partial^*\alpha \in \operatorname{Dom}_{L^2}([\bar\partial^{p,q-1}_{\mathcal{R}_\pm}]^*)$.

Proof. Let $\alpha \in \operatorname{Dom}_{L^2}([\bar\partial^{p,q}_{\mathcal{R}_\pm}]^*)$. As before there is a sequence $\langle\alpha_n\rangle$ of smooth forms in $\operatorname{Dom}([\bar\partial^{p,q}_{\mathcal{R}_\pm}]^*)$, converging to α in the graph norm. We need to consider the individual cases. We begin with $\mathcal{R}_+$. The only case that is not classical is that of $q = 1$. We suppose that $\langle\alpha_n\rangle$ is a sequence of forms in $\mathcal{C}^\infty(\overline{X}_+; \Lambda^{p,2})$ with $\bar\partial\rho\rfloor\alpha_n = 0$. Using the identities in (19) we see that

$$[\bar\partial\rho\rfloor\bar\partial^*\alpha_n]_b = [(\bar\partial^\star\alpha_n)_b]^{\star_b}. \tag{49}$$

On the other hand, as $(\bar\partial\rho\rfloor\alpha_n)_b = 0$ it follows that $(^\star\alpha_n)_b = 0$ and therefore

$$(\bar\partial^\star\alpha_n)_b = \bar\partial_b(^\star\alpha_n)_b = 0.$$

This shows that $(\mathrm{Id} - \mathcal{S}_p)\bar\partial\rho\rfloor\bar\partial^*\alpha_n = 0$ and therefore $\bar\partial^*\alpha_n$ is in the domain of $[\bar\partial^{p,0}_{\mathcal{R}_+}]^*$. As $[\bar\partial^*]^2 = 0$ this shows that $\bar\partial^*\alpha \in \mathrm{Dom}_{L^2}([\bar\partial^{p,0}_{\mathcal{R}_+}]^*)$.

On the pseudoconcave side we only need to consider $q = n-1$. The boundary condition implies that $\bar\partial_b^*(\bar\partial\rho\rfloor\alpha_n)_b = 0$. Using the identities in (19) we see that

$$\bar\partial\rho\rfloor\bar\partial^*\alpha_n = {}^{\star_b}(\bar\partial^\star\alpha_n)_b = \bar\partial_b^*(\bar\partial\rho\rfloor\alpha_n)_b = 0. \tag{50}$$

Thus $\bar\partial^*\alpha_n \in \mathrm{Dom}([\bar\partial^{p,n-2}_{\mathcal{R}_-}]^*)$. □

Remark 5. Again, the same argument applies to show that the lemma holds for the boundary condition defined by $\mathcal{R}_+'$.

These lemmas show that, in the sense of closed operators, $\bar\partial^2_{\mathcal{R}_\pm}$ and $[\bar\partial^*_{\mathcal{R}_\pm}]^2$ vanish. This, along with the higher norm estimates, gives the strong form of the Hodge decomposition, as well as the important commutativity results, (52) and (53).

THEOREM 2. *Suppose that $X_\pm$ is a strictly pseudoconvex (pseudoconcave) compact, Kähler complex manifold with boundary. For $0 \le p, q \le n$, we have the strong orthogonal decompositions*

$$\alpha = \bar\partial\bar\partial^* G^{p,q}_{\mathcal{R}_\pm}\alpha + \bar\partial^*\bar\partial G^{p,q}_{\mathcal{R}_\pm}\alpha + H^{p,q}_{\mathcal{R}_\pm}\alpha. \tag{51}$$

If $\alpha \in \mathrm{Dom}_{L^2}(\bar\partial^{p,q}_{\mathcal{R}_\pm})$ then

$$\bar\partial G^{p,q}_{\mathcal{R}_\pm}\alpha = G^{p,q+1}_{\mathcal{R}_\pm}\bar\partial\alpha. \tag{52}$$

If $\alpha \in \mathrm{Dom}_{L^2}([\bar\partial^{p,q}_{\mathcal{R}_\pm}]^*)$ then

$$\bar\partial^* G^{p,q}_{\mathcal{R}_\pm}\alpha = G^{p,q-1}_{\mathcal{R}_\pm}\bar\partial^*\alpha. \tag{53}$$

Given Theorem 1 and Lemmas 4 and 5 the proof of this theorem is exactly the same as the proof of Theorem 3.1.14 in [10]. Similar decompositions also hold for the dual boundary value problems defined by $\mathrm{Id} - \mathcal{R}_+$ on X_- and $\mathrm{Id} - \mathcal{R}_-$ on X_+. We leave the explicit statements to the reader.

As in the case of the standard $\bar\partial$-Neumann problems these estimates show that the domains of the self-adjoint operators defined by the quadratic forms $\mathcal{Q}^{p,q}$ with form domains specified as the intersection of $\mathrm{Dom}(\bar\partial^{p,q}_{\mathcal{R}_\pm}) \cap \mathrm{Dom}([\bar\partial^{p,q-1}_{\mathcal{R}_\pm}]^*)$ are exactly as one would expect. As in [10] one easily deduces the following descriptions of the unbounded self-adjoint operators $\Box^{p,q}_{\mathcal{R}_\pm}$.

PROPOSITION 3. *Suppose that* X_+ *is strictly pseudoconvex, then the operator* $\Box^{p,q}_{\mathcal{R}_+}$ *with domain specified by*

$$\begin{aligned}\sigma^{pq} &\in \mathrm{Dom}_{L^2}(\bar\partial^{p,q}_{\mathcal{R}_+}) \cap \mathrm{Dom}_{L^2}([\bar\partial^{p,q-1}_{\mathcal{R}_+}]^*) \text{ such that}\\ \bar\partial^*\sigma^{pq} &\in \mathrm{Dom}_{L^2}(\bar\partial^{p,q-1}_{\mathcal{R}_+}) \text{ and } \bar\partial\sigma^{pq} \in \mathrm{Dom}_{L^2}([\bar\partial^{p,q}_{\mathcal{R}_+}]^*)\end{aligned} \tag{54}$$

is a self-adjoint operator. It coincides with the Friedrichs extension defined by $\mathcal{Q}^{pq}$ *with form domain given by the first condition in* (54).

PROPOSITION 4. *Suppose that* X_- *is strictly pseudoconcave, then the operator* $\Box^{p,q}_{\mathcal{R}_-}$ *with domain specified by*

$$\begin{aligned}\sigma^{pq} &\in \mathrm{Dom}_{L^2}(\bar\partial^{p,q}_{\mathcal{R}_-}) \cap \mathrm{Dom}_{L^2}([\bar\partial^{p,q-1}_{\mathcal{R}_-}]^*) \text{ such that}\\ \bar\partial^*\sigma^{pq} &\in \mathrm{Dom}_{L^2}(\bar\partial^{p,q-1}_{\mathcal{R}_-}) \text{ and } \bar\partial\sigma^{pq} \in Dom_{L^2}([\bar\partial^{p,q}_{\mathcal{R}_-}]^*)\end{aligned} \tag{55}$$

is a self-adjoint operator. It coincides with the Friedrichs extension defined by $\mathcal{Q}^{pq}$ *with form domain given by the first condition in* (55).

6. The nullspaces of the modified $\bar\partial$-Neumann problems

As noted above $\Box^{p,q}_{\mathcal{R}_\pm}$ has a compact resolvent in all form degrees and therefore the harmonic spaces $\mathcal{H}^{p,q}_{\mathcal{R}_\pm}(X_\pm)$ are finite dimensional. The boundary conditions easily imply that

$$\mathcal{H}^{p,0}_{\mathcal{R}_+}(X_+) = 0 \text{ for all } p \text{ and } \mathcal{H}^{p,q}_{\mathcal{R}_+}(X_+) = \mathcal{H}^{p,q}_{\bar\partial}(X_+) \text{ for } q > 1. \tag{56}$$

$$\mathcal{H}^{p,q}_{\mathcal{R}_-}(X_-) = \mathcal{H}^{p,q}_{\bar\partial}(X_-) \text{ for } q < n-1. \tag{57}$$

We now identify $\mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+)$, and $\mathcal{H}^{p,n}_{\mathcal{R}_-}(X_-)$, but leave $\mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-)$ to the next section.

We begin with the pseudoconvex case. To identify the null space of $\Box^{p,1}_{\mathcal{R}_+}$ we need to define the following vector space:

$$E^{p,1}_0(\overline{X}_+) = \frac{\{\bar\partial\alpha:\ \alpha \in \mathcal{C}^\infty(\overline{X}_+;\Lambda^{p,0}) \text{ and } \bar\partial_b\alpha_b = 0\}}{\{\bar\partial\alpha:\ \alpha \in \mathcal{C}^\infty(\overline{X}_+;\Lambda^{p,0}) \text{ and } \alpha_b = 0\}}. \tag{58}$$

It is clear that $E^{p,1}_0(\overline{X}_+)$ is a subspace of the "zero"-cohomology group $H^{p,1}_0(\overline{X}_+) \simeq \mathcal{H}^{p,1}_{\bar\partial^*}(X_+) \simeq [\mathcal{H}^{n-p,n-1}_{\bar\partial}]^*(X_+)$ and is therefore finite dimensional. If X_+ is a Stein manifold, then this vector space is trivial. It is also not difficult to show that

$$E^{p,1}_0(\overline{X}_+) \simeq \frac{H^{p,0}_b(Y)}{[H^{p,0}(\overline{X}_+)]_b}. \tag{59}$$

Thus $E^{p,1}_0$ measures the extent of the failure of closed $(p,0)$ forms on bX_+ to have holomorphic extensions to X_+.

LEMMA 6. *If X_+ is strictly pseudoconvex, then*

$$\mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+) \simeq \mathcal{H}^{p,1}_{\bar{\partial}}(X_+) \oplus E_0^{p,1}.$$

Proof. Clearly $\mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+) \supset \mathcal{H}^{p,1}_{\bar{\partial}}(X_+)$. If $\sigma^{p1} \in \mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+)$, then

$$(\mathrm{Id} - \mathcal{S}_p)(\bar{\partial}\rho \rfloor \sigma^{p1})_b = 0.$$

If $\beta \in \mathcal{H}^{p,0}_{\bar{\partial}}(X_+)$, then

$$0 = \langle \bar{\partial}\beta, \sigma^{p1}\rangle_{X_+} = \langle \beta, \bar{\partial}\rho \rfloor \sigma^{p1}\rangle_{bX_+}. \tag{60}$$

Thus, we see that $\bar{\partial}\rho \rfloor \sigma^{p1}$ is orthogonal to $\mathcal{H}^{p,0}_{\bar{\partial}}(X_+) \restriction_{bX_+}$.

Let $a \in \operatorname{Im} \mathcal{S}_p \ominus \mathcal{H}^{p,0}_{\bar{\partial}}(X_+) \restriction_{bX_+}$. We now show that there is an element $\alpha \in \mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+)$ with $\bar{\partial}\rho \rfloor \alpha = a$. Let $\widetilde{a}$ denote a smooth extension of a to X_+. If $\xi \in \mathcal{H}^{p,0}_{\bar{\partial}}(X_+)$, then

$$\langle \bar{\partial}^* \bar{\partial}(\rho \widetilde{a}), \xi\rangle_{X_+} = -\langle a, \xi\rangle_{bX_+}. \tag{61}$$

By assumption, a is orthogonal to $\mathcal{H}^{p,0}_{\bar{\partial}}(X_+) \restriction_{bX_+}$; thus $H^{p,0}_{\bar{\partial}}(\bar{\partial}^* \bar{\partial}(\rho\widetilde{a})) = 0$. With $b = G^{p,0}_{\bar{\partial}} \bar{\partial}^* \bar{\partial}(\rho\widetilde{a})$, we see that

$$\begin{aligned} \bar{\partial}^* \bar{\partial} b = (\mathrm{Id} - H^{p,0}_{\bar{\partial}})\bar{\partial}^* \bar{\partial} a = \bar{\partial}^* \bar{\partial} a, \\ \bar{\partial}\rho \rfloor \bar{\partial} b = 0. \end{aligned} \tag{62}$$

Hence if $\alpha = \bar{\partial}(\rho\widetilde{a} - b)$, then $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$, and $\bar{\partial}\rho \rfloor \alpha = a$. If $\alpha_1, \alpha_2 \in \mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+)$ both satisfy $\bar{\partial}\rho \rfloor \alpha_1 = \bar{\partial}\rho \rfloor \alpha_2 = a$, then $\alpha_1 - \alpha_2 \in \mathcal{H}^{p,1}_{\bar{\partial}}(X_+)$. Together with the existence result, this shows that

$$\frac{\mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+)}{\mathcal{H}^{p,1}_{\bar{\partial}}(X_+)} \simeq E_0^{p,1}, \tag{63}$$

which completes the proof of the lemma. □

For the pseudoconcave side we have

LEMMA 7. *If X_- is strictly pseudoconcave then*

$$\mathcal{H}^{p,n}_{\mathcal{R}_-}(X_-) \simeq [H^{n-p,0}(X_-)]^\star \simeq \mathcal{H}^{p,n}_{\mathrm{Id} - \mathcal{R}_+}(X_-).$$

Proof. A (p, n)-form σ^{pn} belongs to $\mathcal{H}^{p,n}_{\mathcal{R}_-}(X_-)$ provided that

$$\bar{\partial}^* \sigma^{pn} = 0, \text{ and } (\mathrm{Id} - \bar{\mathcal{S}}_p)(\bar{\partial}\rho \rfloor \sigma^{pn})_b = 0.$$

The identities in (14) imply that ${}^\star\sigma^{pn} \in H^{n-p,0}(X_-)$.

On the other hand, if $\eta \in H^{n-p,0}(X_-)$, then $\bar{\partial}^{*\star}\eta = 0$, and $(\mathrm{Id} - \mathcal{S}_{n-p})\eta_b = 0$. The identities in (19) and (31) imply that $(\mathrm{Id} - \bar{\mathcal{S}}_p)(\bar{\partial}\rho \rfloor^\star \eta)_b = 0$. Since this shows that $\star\eta \in \mathcal{H}^{pn}_{\mathcal{R}_-}(X_-)$, completing the proof of the first isomorphism. A form $\eta \in \mathcal{H}^{p,n}_{\mathrm{Id}-\mathcal{R}_+}(X_-)$ provided that $\bar\partial^*\eta = 0$. The boundary condition $\eta_b = 0$ is vacuous for a (p,n)-form. This shows that $\star\eta \in H^{n-p,0}(X_-)$, the converse is immediate. □

All that remains is $\mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-)$. This space does not have as simple a description as the others. We return to this question in the next section. We finish this section with the observation that the results in Section (4) imply the following duality statements, for $0 \le q, p \le n$:

$$[\mathcal{H}^{p,q}_{\mathcal{R}_+}(X_+)]^* \simeq \mathcal{H}^{n-p,n-q}_{\mathrm{Id}-\mathcal{R}_-}(X_+), \quad [\mathcal{H}^{p,q}_{\mathcal{R}_-}(X_-)]^* \simeq \mathcal{H}^{n-p,n-q}_{\mathrm{Id}-\mathcal{R}_+}(X_-). \tag{64}$$

The isomorphisms are realized by applying the Hodge star operator.

7. Connection to $\eth_\pm$ and the Agranovich-Dynin formula

Thus far we have largely considered one (p,q)-type at a time. As noted in the introduction, by grouping together the even, or odd, forms we obtain bundles of complex spinors on which the $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator acts. We let

$$\Lambda^{p,\mathrm{e}} = \bigoplus_{q=0}^{\lfloor \frac{n}{2} \rfloor} \Lambda^{p,2q}, \quad \Lambda^{p,\mathrm{o}} = \bigoplus_{q=0}^{\lfloor \frac{n-1}{2} \rfloor} \Lambda^{p,2q+1}. \tag{65}$$

The bundles $\Lambda^{p,\mathrm{e}}, \Lambda^{p,\mathrm{o}}$ are the basic complex spinor bundles, $\Lambda^{\mathrm{e}}, \Lambda^{\mathrm{o}}$, twisted with the holomorphic vector bundles $\Lambda^{p,0}$. Unless it is needed for clarity, we do not include the value of p in the notation.

When we assume that the underlying manifold is a Kähler manifold, the $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator is $\eth = \bar\partial + \bar\partial^*$. It maps even forms to odd forms and we denote this by

(66)
$$\eth^{\mathrm{e}}_\pm : \mathcal{C}^\infty(X_\pm; \Lambda^{p,\mathrm{e}}) \longrightarrow \mathcal{C}^\infty(X_\pm; \Lambda^{p,\mathrm{o}}), \, \eth^{\mathrm{o}}_\pm : \mathcal{C}^\infty(X_\pm; \Lambda^{p,\mathrm{o}}) \longrightarrow \mathcal{C}^\infty(X_\pm; \Lambda^{p,\mathrm{e}}).$$

As noted above, the boundary projection operators $\mathcal{R}_\pm$ (or $\mathcal{R}'_\pm$) can be divided into operators acting separately on even and odd forms, $\mathcal{R}^{\mathrm{eo}}_\pm$, ($\mathcal{R}'^{\mathrm{eo}}_\pm$). These boundary conditions define subelliptic boundary value problems for $\eth^{\mathrm{eo}}_\pm$ that are closely connected to the individual (p,q)-types. The connection is via the basic integration-by-parts formulæ for $\eth^{\mathrm{eo}}_\pm$. There are several cases, which we present in a series of lemmas.

Lemma 8. *If* $\sigma \in \mathcal{C}^\infty(\overline{X}_\pm; \Lambda^{p,\mathrm{eo}})$ *satisfies either* $\mathcal{R}'^{\mathrm{eo}}_+\sigma \restriction_{bX_\pm} = 0$ *or* $(\mathrm{Id} - \mathcal{R}'^{\mathrm{eo}}_-)\sigma \restriction_{bX_\pm} = 0$, *then*

$$\langle \eth_\pm\sigma, \eth_\pm\sigma\rangle_{X_\pm} = \langle \bar\partial\sigma, \bar\partial\sigma\rangle_{X_\pm} + \langle \bar\partial^*\sigma, \bar\partial^*\sigma\rangle_{X_\pm}. \tag{67}$$

Remark 6. Note that when using the boundary conditions defined by $\mathcal{R}_+$ and $\mathrm{Id}-\mathcal{R}_-$, we are able to use a generalized Szegő projector, unconnected to the complex structure on $X_\pm$. This is not always true for $\mathcal{R}_-$ and $\mathrm{Id}-\mathcal{R}_+$. See Lemmas 9 and 10.

Proof. The proof for $\mathcal{R}'^{\mathrm{eo}}_\pm$ is a consequence of the facts that

(a) $\bar\partial^2 = 0$.

(b) If η is a (p,j)-form satisfying $\bar\partial\rho\rfloor\eta\restriction_{bX_\pm}=0$, then, for β any smooth $(p,j-1)$-form,

$$\langle\beta,\bar\partial^*\eta\rangle_{X_\pm}=\langle\bar\partial\beta,\eta\rangle_{X_\pm}. \tag{68}$$

We need to show that

$$\langle\bar\partial\sigma^{pq},\bar\partial^*\sigma^{p(q+2)}\rangle_{X_\pm}=0. \tag{69}$$

This follows immediately from (a), (b), and the fact that $\sigma^{p(q+2)}$ satisfies

$$\bar\partial\rho\rfloor\sigma^{p(q+2)}=0,\ \text{for all } q\geq 0.$$

In the proof for $\mathrm{Id}-\mathcal{R}'^{\mathrm{eo}}_-$, we replace (a) and (b) above with

(a′) $[\bar\partial^*]^2=0$.

(b′) If η is a (p,j)-form satisfying $\bar\partial\rho\wedge\eta\restriction_{bX_\pm}=0$, then, for β any smooth $(p,j+1)$-form we have

$$\langle\beta,\bar\partial\eta\rangle_{X_\pm}=\langle\bar\partial^*\beta,\eta\rangle_{X_\pm}. \tag{70}$$

Since $(\mathrm{Id}-\mathcal{R}'^{\mathrm{eo}}_-)\sigma\restriction_{bX_\pm}=0$ implies that $\bar\partial\rho\wedge\sigma^{pq}\restriction_{bX_\pm}=0$ holds for $q<n-1$, the relation in (69) holds for all q of interest. This case could also be treated by observing that it is dual to $\mathcal{R}'_+$. □

Now we consider $\mathcal{R}_-$ and $\mathrm{Id}-\mathcal{R}_+$. Let b_n denote the parity (even or odd) of n, and $\tilde b_n$ the opposite parity.

LEMMA 9. *If a section $\sigma\in\mathcal{C}^\infty(\overline{X}_\pm;\Lambda^{p,\mathrm{o}})$ satisfies $(\mathrm{Id}-\mathcal{R}'^{\mathrm{o}}_+)\sigma\restriction_{bX_\pm}=0$, or $\sigma\in\mathcal{C}^\infty(\overline{X}_\pm;\Lambda^{p,\tilde b_n})$ satisfies $\mathcal{R}'^{\tilde b_n}_-\sigma\restriction_{bX_\pm}=0$, then (67) holds.*

Remark 7. In these cases we can again use generalized Szegő projectors.

Proof. The proofs here are very much as before. For $\mathrm{Id}-\mathcal{R}'^{\mathrm{o}}_+$ we use the fact that

$$\langle\bar\partial\sigma^{pq},\bar\partial^*\sigma^{p(q+2)}\rangle_{X_\pm}=\langle\bar\partial\rho\wedge\sigma^{pq},\bar\partial^*\sigma^{p(q+2)}\rangle_{bX_\pm}, \tag{71}$$

and this vanishes if $q\geq 1$. For $\mathcal{R}'^{\tilde b_n}_-$ we use the fact that

$$\langle\bar\partial\sigma^{pq},\bar\partial^*\sigma^{p(q+2)}\rangle_{X_\pm}=-\langle\bar\partial\sigma^{pq},\bar\partial\rho\rfloor\sigma^{p(q+2)}\rangle_{bX_\pm}, \tag{72}$$

and this vanishes if $q<n-2$. □

In the final cases we are restricted to the boundary conditions which employ the classical Szegő projector defined by the complex structure on $X_\pm$.

LEMMA 10. *If a section* $\sigma \in \mathcal{C}^\infty(\overline{X}_\pm; \Lambda^{p,\mathrm{e}})$ *satisfies* $(\mathrm{Id} - \mathcal{R}_+^{\mathrm{e}})\sigma \restriction_{bX_\pm} = 0$, *or* $\sigma \in \mathcal{C}^\infty(\overline{X}_\pm; \Lambda^{p,b_n})$ *satisfies* $\mathcal{R}_-^{b_n}\sigma \restriction_{bX_\pm} = 0$, *then* (67) *holds.*

Proof. First we consider $\mathrm{Id} - \mathcal{R}_+^{\mathrm{e}}$. For even $q \geq 2$, the proof given above shows that (69) holds; so we are left to consider $q = 0$. The boundary condition satisfied by σ^{p0} is $(\mathrm{Id} - \mathcal{S}_p)\sigma_b^{p0} = 0$. Hence, we have

$$\begin{aligned} \langle \bar\partial \sigma^{p0}, \bar\partial^* \sigma^{p2} \rangle_{X_\pm} &= -\langle \bar\partial \sigma_b^{p0}, \bar\partial\rho \rfloor \sigma^{p2} \rangle_{bX_\pm} \\ &= -\langle \bar\partial\rho \wedge \bar\partial \sigma_b^{p0}, \sigma^{p2} \rangle_{bX_\pm} = 0. \end{aligned} \tag{73}$$

The last equality follows because $\bar\partial\rho \wedge \bar\partial\sigma^{p0} = 0$ if $\bar\partial_b \sigma_b^{p0} = 0$.

Finally we consider $\mathcal{R}_-$. The proof given above suffices for $q < n$. We need to consider $q = n$; in this case $(\mathrm{Id} - \bar{\mathcal{S}}_p)(\bar\partial\rho \rfloor \sigma^{pn})_b = 0$. We begin by observing that

$$\begin{aligned} \langle \bar\partial \sigma^{p(n-2)}, \bar\partial^* \sigma^{pn} \rangle_{X_\pm} &= -\langle \bar\partial_b \sigma_b^{p(n-2)}, (\bar\partial\rho \rfloor \sigma^{pn})_b \rangle_{bX_\pm} \\ &= -\langle \sigma_b^{p(n-2)}, \bar\partial_b^* (\bar\partial\rho \rfloor \sigma^{pn})_b \rangle_{bX_\pm} = 0. \end{aligned} \tag{74}$$

The last equality follows from the fact that$(\bar\partial\rho \rfloor \sigma^{pn})_b = \bar{\mathcal{S}}_p(\bar\partial\rho \rfloor \sigma^{pn})_b$. □

In all cases where (67) holds we can identify the null spaces of the operators $\eth_\pm^{\mathrm{eo}}$. Here we stick to the pseudoconvex side and boundary conditions defined by the classical Szegő projectors. It follows from (67) that

$$\begin{aligned} \ker(\eth_{p+}^{\mathrm{e}}, \mathcal{R}_+^{\mathrm{e}}) &= \bigoplus_{j=1}^{\lfloor \frac{n}{2} \rfloor} \mathcal{H}_{\bar\partial}^{p,2j}(X_+), \\ \ker(\eth_{p+}^{\mathrm{o}}, \mathcal{R}_+^{\mathrm{o}}) &= E_0^{p,1} \oplus \bigoplus_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{H}_{\bar\partial}^{p,2j+1}(X_+). \end{aligned} \tag{75}$$

In [7] we identify the L^2-adjoints of the operators $(\eth_\pm^{\mathrm{eo}}, \mathcal{R}_\pm'^{\mathrm{eo}})$with the graph closures of the formal adjoints, e.g.,

$$\begin{aligned} (\eth_+^{\mathrm{eo}}, \mathcal{R}_+'^{\mathrm{eo}})^* &= \overline{(\eth_+^{\mathrm{oe}}, \mathcal{R}_+'^{\mathrm{oe}})}, \\ (\eth_-^{\mathrm{eo}}, \mathcal{R}_-'^{\mathrm{eo}})^* &= \overline{(\eth_-^{\mathrm{oe}}, \mathcal{R}_-'^{\mathrm{oe}})}. \end{aligned} \tag{76}$$

Using these identities, the Dolbeault isomorphism and standard facts about the $\bar\partial$-Neumann problem on a strictly pseudoconvex domain, we obtain

$$\operatorname{Ind}(\eth_{p+}^{\mathrm{e}}, \mathcal{R}_+^{\mathrm{e}}) = -\dim E_0^{p,1} + \sum_{q=1}^{n} (-1)^q \dim H^{p,q}(X_+). \tag{77}$$

Recall that if $\mathcal{S}_p'$ and $\mathcal{S}_p''$ are generalized Szegő projectors, then their relative index R-Ind$(\mathcal{S}_p', \mathcal{S}_p'')$ is defined to be the Fredholm index of the restriction

$$\mathcal{S}_p'' : \operatorname{Im} \mathcal{S}_p' \longrightarrow \operatorname{Im} \mathcal{S}_p''. \tag{78}$$

For the pseudoconvex side we now prove an Agranovich-Dynin type formula.

THEOREM 3. *Let X_+ be a compact strictly pseudoconvex Kähler manifold, with $\mathcal{S}_p$ the classical Szegő projector, defined as the projector onto the null space of $\bar\partial_b$ acting on $\mathcal{C}^\infty(bX_+; \Lambda_b^{p,0})$. If $\mathcal{S}_p'$ is a generalized Szegő projector, then*

$$\operatorname{Ind}(\eth_+^{\mathrm{e}}, \mathcal{R}_+'^{\mathrm{e}}) - \operatorname{Ind}(\eth_+^{\mathrm{e}}, \mathcal{R}_+^{\mathrm{e}}) = \text{R-Ind}(\mathcal{S}_p, \mathcal{S}_p'). \tag{79}$$

Proof. It follows from Lemma 8 that all other groups are the same, so we only need to compare $\mathcal{H}^{p,0}_{\mathcal{R}_+'}(X_+)$ to $\mathcal{H}^{p,0}_{\mathcal{R}^+}(X_+)$ and $\mathcal{H}^{p,1}_{\mathcal{R}_+'}(X_+)$ to $\mathcal{H}^{p,1}_{\mathcal{R}^+}(X_+)$. For this purpose we introduce the subprojector $\widehat{\mathcal{S}}_p$ of $\mathcal{S}_p$, defined to be the orthogonal projection onto $\mathcal{H}^{p,0}_{\bar\partial}(X_+) \restriction_{bX_+}$. Note that

$$\text{R-Ind}(\mathcal{S}_p, \widehat{\mathcal{S}}_p) = \dim E_0^{p,1}. \tag{80}$$

The $q = 0$ case is quite easy. The group $\mathcal{H}^{p,0}_{\mathcal{R}^+}(X_+) = 0$. A section $\sigma^{p0} \in \mathcal{H}^{p,0}_{\mathcal{R}_+'}(X_+)$, if and only if $\bar\partial\sigma^{p0} = 0$ and $\mathcal{S}_p'\sigma_b^{p0} = 0$. The first condition implies that $\sigma_b^{p0} \in \operatorname{Im} \widehat{\mathcal{S}}_p$. Conversely, if $\eta \in \ker[\mathcal{S}_p' : \operatorname{Im} \widehat{\mathcal{S}}_p \to \operatorname{Im} \mathcal{S}_p']$, then there is a unique holomorphic $(p, 0)$-form σ^{p0} with $\sigma_b^{p0} = \eta$. This shows that

$$\mathcal{H}^{p,0}_{\mathcal{R}_+'}(X_+) \simeq \ker[\mathcal{S}_p' : \operatorname{Im} \widehat{\mathcal{S}}_p \to \operatorname{Im} \mathcal{S}_p']. \tag{81}$$

Now we turn to the $q = 1$ case. No matter which boundary projection is used

$$\mathcal{H}^{p,1}_{\bar\partial}(X_+) \subset \mathcal{H}^{p,1}_{\mathcal{R}_+'}(X_+). \tag{82}$$

As shown in Lemma 6

$$\frac{\mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+)}{\mathcal{H}^{p,1}_{\bar\partial}(X_+)} \simeq E_0^{p,1}. \tag{83}$$

Now suppose that $\sigma^{p1} \in \mathcal{H}^{p,1}_{\mathcal{R}_+'}(X_+)$ and $\eta \in \mathcal{H}^{p,0}_{\bar\partial}(X_+)$; then

$$0 = \langle \bar\partial\eta, \sigma^{p1}\rangle_{X_+} = \langle \eta, (\bar\partial\rho \rfloor \sigma^{p1})_b\rangle_{bX_+}. \tag{84}$$

Hence $(\bar\partial\rho \rfloor \sigma^{p1})_b \in \ker[\widehat{\mathcal{S}}_p : \operatorname{Im} \mathcal{S}_p' \to \operatorname{Im} \widehat{\mathcal{S}}_p]$.

To complete the proof we need to show that for $\eta_b \in \ker[\widehat{\mathcal{S}}_p : \operatorname{Im} \mathcal{S}_p' \to \operatorname{Im} \widehat{\mathcal{S}}_p]$ there is a harmonic $(p, 1)$-form, σ^{p1} with $(\bar\partial\rho \rfloor \sigma^{p1})_b = \eta_b$. Let η denote a smooth extension of η_b to X_+. We need to show that there is a $(p, 0)$ form β such that

$$\bar\partial^*\bar\partial(\rho\eta) = \bar\partial^*\bar\partial\beta \text{ and } (\bar\partial\rho \rfloor \bar\partial\beta)_b = 0. \tag{85}$$

This follows from the fact that $\widehat{\mathcal{S}}_p\eta_b = 0$, exactly as in the proof of Lemma 6. Hence $\sigma^{p1} = \bar{\partial}(\rho\eta - \beta)$ is an element of $\mathcal{H}^{p,1}_{\mathcal{R}'_+}(X_+)$ such that $(\bar{\partial}\rho\rfloor\sigma^{p1})_b = \eta_b$. This shows that

$$\frac{\mathcal{H}^{p,1}_{\mathcal{R}'_+}(X_+)}{\mathcal{H}^{p,1}_{\bar{\partial}}(X_+)} \simeq \ker[\widehat{\mathcal{S}}_p : \operatorname{Im}\mathcal{S}'_p \to \operatorname{Im}\widehat{\mathcal{S}}_p]. \tag{86}$$

Combining (83) with (86) we obtain that

$$\dim \mathcal{H}^{p,1}_{\mathcal{R}'_+}(X_+) - \dim \mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+) = \dim\ker[\widehat{\mathcal{S}}_p : \operatorname{Im}\mathcal{S}'_p \to \operatorname{Im}\widehat{\mathcal{S}}_p] - \dim E^{p,1}_0. \tag{87}$$

Combining this with (81) and (80) gives

$$\operatorname{Ind}(\eth^{\mathrm{e}}_+, \mathcal{R}'_+) - \operatorname{Ind}(\eth^{\mathrm{e}}_+, \mathcal{R}_+) = \text{R-Ind}(\widehat{\mathcal{S}}_p, \mathcal{S}'_p) + \text{R-Ind}(\mathcal{S}_p, \widehat{\mathcal{S}}_p) = \text{R-Ind}(\mathcal{S}_p, \mathcal{S}'_p). \tag{88}$$

The last equality follows from the cocycle formula for the relative index. □

8. Long exact sequences and gluing formulæ

Suppose that X is a compact complex manifold with a separating, strictly pseudoconvex hypersurface Y. Let $X \setminus Y = X_+ \coprod X_-$, with X_+ strictly pseudoconvex and X_- strictly pseudoconcave. A principal goal of this paper is to express

$$\chi^p_{\mathcal{O}}(X) = \sum_{q=0}^{n} (-1)^q \dim H^{p,q}(X),$$

in terms of indices of operators on $X_\pm$. Such results are classical for the topological Euler characteristic and Dirac operators with elliptic boundary conditions; see for example Chapter 24 of [5]. In this section we modify long exact sequences given by Andreotti and Hill in order to prove such results for subelliptic boundary conditions.

The Andreotti-Hill sequences relate the smooth cohomology groups

$$H^{p,q}(\overline{X}_\pm, \mathcal{I}), \quad H^{p,q}(\overline{X}_\pm), \quad \text{and } H^{p,q}_b(Y).$$

The notation $\overline{X}_\pm$ is intended to remind the reader that these are cohomology groups defined by the $\bar{\partial}$-operator acting on forms that are smooth on the closed manifolds with boundary, $\overline{X}_\pm$. The differential ideal $\mathcal{I}$ is composed of forms, σ, so that near Y, we have

$$\sigma = \bar{\partial}\rho \wedge \alpha + \rho\beta. \tag{89}$$

These are precisely the forms that satisfy the dual $\bar{\partial}$-Neumann condition (16). If ξ is a form defined on all of X, then we use the shorthand notation

$$\xi_\pm \overset{d}{=} \xi \restriction_{X_\pm}.$$

For a strictly pseudoconvex manifold, it follows from the Hodge decomposition and the results in Section 6 that

$$\begin{aligned} H^{p,q}(\overline{X}_+) &\simeq \mathcal{H}^{p,q}_{\bar\partial}(X_+) \text{ for } q \neq 0, \text{ and} \\ H^{p,q}(\overline{X}_+) &\simeq \mathcal{H}^{p,q}_{\mathcal{R}_+}(X_+) \text{ for } q \neq 0, 1, \end{aligned} \tag{90}$$

and for a strictly pseudoconcave manifold

$$\begin{aligned} H^{p,q}(\overline{X}_-) \simeq \mathcal{H}^{p,q}_{\bar\partial}(X_-) &= \mathcal{H}^{p,q}_{\mathcal{R}_-}(X_-) \text{ for } q \neq n-1, n \text{ and} \\ [H^{n-p,0}(X_-)]^\star &= \mathcal{H}^{p,n}_{\mathcal{R}_-}(X_-). \end{aligned} \tag{91}$$

By duality we also have the isomorphisms

$$\begin{aligned} H^{p,q}(\overline{X}_+, \mathcal{I}) &\simeq \mathcal{H}^{p,q}_{\bar\partial^*}(X_+) \text{ for } q \neq n, \text{ and} \\ H^{p,q}(\overline{X}_+, \mathcal{I}) &\simeq \mathcal{H}^{p,q}_{\mathrm{Id}-\mathcal{R}_-}(X_+) \text{ for } q \neq n, n-1, \end{aligned} \tag{92}$$

and for a strictly pseudoconcave manifold

$$\begin{aligned} H^{p,q}(\overline{X}_-, \mathcal{I}) \simeq \mathcal{H}^{p,q}_{\bar\partial^*}(X_-) &= \mathcal{H}^{p,q}_{\mathrm{Id}-\mathcal{R}_+}(X_-) \text{ for } q \neq 0, 1 \text{ and} \\ H^{p,0}(X_-) &= \mathcal{H}^{p,0}_{\mathrm{Id}-\mathcal{R}_+}(X_-). \end{aligned} \tag{93}$$

We recall the definitions of various maps introduced in [1]:

$$\begin{aligned} &\alpha_q : H^{p,q}(X) \longrightarrow H^{p,q}(\overline{X}_+) \oplus H^{p,q}(\overline{X}_-), \\ &\beta_q : H^{p,q}(\overline{X}_+) \oplus H^{p,q}(\overline{X}_-) \longrightarrow H^{p,q}_b(Y). \\ &\gamma_q : H^{p,q}_b(Y) \longrightarrow H^{p,q+1}(X). \end{aligned} \tag{94}$$

The first two are simple

$$\alpha_q(\sigma^{pq}) \overset{d}{=} \sigma^{pq} \restriction_{\overline{X}_+} \oplus \sigma^{pq} \restriction_{\overline{X}_-}, \quad \beta_q(\sigma^{pq}_+, \sigma^{pq}_-) \overset{d}{=} [\sigma^{pq}_+ - \sigma^{pq}_-]_b. \tag{95}$$

To define γ_q we recall the notion of *distinguished representative* defined in [1]: If $\eta \in H^{p,q}_b(Y)$ then there is a (p,q)-form ξ defined on X so that

1. ξ_b represents η in $H^{p,q}_b(Y)$.
2. $\bar\partial\xi$ vanishes to infinite order along Y.

The map γ_q is defined in terms of a distinguished representative ξ for η by

$$\gamma_q(\eta) \overset{d}{=} \begin{cases} \bar\partial\xi & \text{on } \overline{X}_+ \\ -\bar\partial\xi & \text{on } \overline{X}_-. \end{cases} \tag{96}$$

As $\bar\partial\xi$ vanishes to infinite order along Y, this defines a smooth form.

The map $\widetilde{\alpha}_0 : H^{p,0}(X) \to H^{p,0}(\overline{X}_-)$ is defined by restriction. To define $\widetilde{\beta}_0 : H^{p,0}(\overline{X}_-) \to E^{p,1}_0(\overline{X}_+)$, we extend $\xi \in H^{p,0}(\overline{X}_-)$ to a smooth form, $\widetilde{\xi}$ on all of X and set

$$\widetilde{\beta}_0(\xi) = \bar\partial\widetilde{\xi} \restriction_{\overline{X}_+}. \tag{97}$$

It is easy to see that $\widetilde{\beta}_0(\xi)$ is a well defined element of the quotient, $E_0^{p,1}(\overline{X}_+)$. To define $\widetilde{\gamma}_0 : E_0^{p,1}(\overline{X}_+) \to H^{p,1}(X)$ we observe that an element $[\xi] \in E_0^{p,1}(\overline{X}_+)$ has a representative, ξ which vanishes on bX_+. The class $\widetilde{\gamma}_0([\xi])$ is defined by extending such a representative by zero to X_-. As noted in [1], one can in fact choose a representative so that ξ vanishes to infinite order along bX_+.

We can now state our modification to the Mayer-Vietoris sequence in Theorem 1 in [1].

THEOREM 4. *Let X, X_+, X_-, Y be as above. Then the following sequence is exact*

(98)

$$\begin{array}{ccccc}
 & & 0 & \longrightarrow & \\
H^{p,0}(X) & \xrightarrow{\widetilde{\alpha}_0} & H^{p,0}(\overline{X}_-) & \xrightarrow{\widetilde{\beta}_0} & E_0^{p,1}(\overline{X}_+) \\
 & \xrightarrow{\widetilde{\gamma}_0} & H^{p,1}(X) & \xrightarrow{\alpha_1} & H^{p,1}(\overline{X}_+) \oplus H^{p,1}(\overline{X}_-) \\
 & \xrightarrow{\beta_1} & H_b^{p,1}(Y) & \xrightarrow{\gamma_1} & \cdots \\
 & \xrightarrow{\beta_{n-2}} & H_b^{p,n-2}(Y) & \xrightarrow{\gamma_{n-2}} & H^{p,n-1}(X) \\
 & \xrightarrow{r_+ \oplus H^{p,n-1}_{\mathcal{R}_-}} & H^{p,n-1}(\overline{X}_+) \oplus \mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-) & \longrightarrow & \dfrac{H^{p,n-1}(X_+)}{K_+^{p,n-1}} \\
 & \longrightarrow & 0. & &
\end{array}$$

Here r_+ denotes restriction to X_+ and

$$K_+^{p,n-1} = \{\alpha \in H^{p,n-1}(\overline{X}_+) : \int\limits_Y \xi \wedge \alpha_b = 0 \text{ for all } \xi \in H^{n-p,0}(\overline{X}_-)\}. \tag{99}$$

The last nontrivial map in (98) *is the canonical quotient by the subspace* $K_+^{p,n-1} \oplus \mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-)$.

Remark 8. Note that if $p = 0$, then $E_0^{0,1} = 0$. This follows from (59) and the fact that, on a strictly pseudoconvex manifold, all CR-functions on the boundary extend as holomorphic functions. The proof given below works for all $n \geq 2$. If $n = 2$, then one skips in (98) from $H^{p,1}(X)$ to $H^{p,1}(\overline{X}_+) \oplus \mathcal{H}^{p,1}_{\mathcal{R}_-}(X_-)$.

Proof. It is clear that $\widetilde{\alpha}_0$ is injective as $H^{p,0}(X)$ consists of holomorphic forms. We now establish exactness at $H^{p,0}(\overline{X}_-)$. That $\operatorname{Im} \widetilde{\alpha}_0 \subset \ker \widetilde{\beta}_0$ is clear. Now suppose that on $\overline{X}_+$ we have $\widetilde{\beta}_0(\xi) = 0$; this means that

$$\bar{\partial}\widetilde{\xi} \restriction_{\overline{X}_+} = \bar{\partial}\theta \text{ where } \theta_b = 0. \tag{100}$$

This implies that $\widetilde{\xi}_+ - \theta$ defines a holomorphic extension of ξ to all of X and therefore $\xi \in \operatorname{Im} \widetilde{\alpha}_0$. That $\operatorname{Im} \widetilde{\beta}_0 \subset \ker \widetilde{\gamma}_0$ is again clear. Suppose on the other hand that $\widetilde{\gamma}_0(\xi) = 0$. This means that there is a $(p,0)$-form, β, defined on all of X so that $\bar{\partial}\beta = \xi$ on X_+ and $\bar{\partial}\beta = 0$ on X_-. This shows that $\xi = \widetilde{\beta}_0(\beta_-)$.

It is once again clear that $\operatorname{Im}\widetilde{\gamma}_0 \subset \ker \alpha_1$. If $\alpha_1(\xi) = 0$, then there are forms $\beta_\pm$ so that

$$\bar{\partial}\beta_\pm = \xi_\pm. \tag{101}$$

Let $\widetilde{\beta}$ be a smooth extension of β_- to all of X. The form $\xi - \bar{\partial}\widetilde{\beta}$ represents the same class in $H^{p,1}(X)$ as ξ. Since

$$(\xi - \bar{\partial}\widetilde{\beta}) \restriction_{X_-} = 0 \text{ and } (\xi - \bar{\partial}\widetilde{\beta}) \restriction_{X_+} = \bar{\partial}(\beta_+ - \widetilde{\beta}_-), \tag{102}$$

we see that $\xi \in \operatorname{Im}\widetilde{\gamma}_0$.

Exactness through $H_b^{p,n-2}(Y)$ is proved in [1]. We now show exactness at $H^{p,n-1}(X)$. The $\bar{\partial}$-Neumann condition, satisfied by elements of $\mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-)$, implies that $H^{p,n-1}_{\mathcal{R}_-}(\bar{\partial}\alpha_-) = 0$; that $r_+(\bar{\partial}\alpha_+) = 0$ is obvious. Hence

$$\operatorname{Im}\gamma_{n-2} \subset \left[\ker r_+ \oplus H^{p,n-1}_{\mathcal{R}_-}\right].$$

Now suppose that $\beta \in H^{p,n-1}(X)$ satisfies $H^{p,n-1}_{\mathcal{R}_-}\beta_- = 0$, $r_+(\beta_+) = 0$. The second condition implies that

$$\beta_+ = \bar{\partial}\gamma_+. \tag{103}$$

Let γ_- denote a smooth extension of γ_+ to X_-. Then $\beta_- - \bar{\partial}\gamma_-$ vanishes along Y and therefore Theorem 2 gives

$$\beta_- - \bar{\partial}\gamma_- = \bar{\partial}\bar{\partial}^* G^{p,n-1}_{\mathcal{R}_-}(\beta_- - \bar{\partial}\gamma_-) = \bar{\partial}\chi_-. \tag{104}$$

Putting these equations together, we have shown that

$$\beta_+ = \bar{\partial}\gamma_+, \quad \beta_- = \bar{\partial}(\gamma_- + \chi_-). \tag{105}$$

Andreotti and Hill show that this implies that $\beta \in \operatorname{Im}\gamma_{n-2}$, thus establishing exactness at $H^{p,n-1}(X)$.

To show exactness at $H^{p,n-1}(X_+) \oplus \mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-)$ we need to show that

$$\operatorname{Im}\left[r_+ \oplus H^{p,n-1}_{\mathcal{R}_-}\right] = K_+^{p,n-1} \oplus \mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-). \tag{106}$$

Let $\alpha \in \mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-)$; then $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$ and $(\bar{\partial}\rho \rfloor \alpha)_b = \bar{\mathcal{S}}_p\alpha_b = 0$. The last condition implies that

$$\alpha_b = \bar{\partial}_b\beta.$$

We can extend β to β_+ on X_+ so that $\bar{\partial}\rho \rfloor \bar{\partial}\beta_+ = 0$. Defining

$$\widetilde{\alpha} = \begin{cases} \alpha & \text{on } X_- \\ \bar{\partial}\beta_+ & \text{on } X_+, \end{cases} \tag{107}$$

gives a $\bar{\partial}$-closed form that defines a class in $H^{p,n-1}(X)$. It is clear that

$$r_+(\widetilde{\alpha}_+) = 0 \text{ and } H^{p,n-1}_{\mathcal{R}_-}(\widetilde{\alpha}_-) = \alpha.$$

To finish the argument we only need to describe $I_{+}^{p,n-1} = \{r_{+}(\theta) : \theta \in H^{p,n-1}(X)\}$. If α_{+} belongs to $I_{+}^{p,n-1}$, then evidently α_{+} has a closed extension to X_{-}; call it α_{-}. If $\xi \in H^{n-p,0}(X_{-})$, then

$$0 = \int\limits_{X_{-}} \bar{\partial}(\alpha_{-} \wedge \xi) = \int_{Y} \alpha_{+b} \wedge \xi. \tag{108}$$

Hence $I_{+}^{p,n-1} \subset K_{+}^{p,n-1}$. If $\alpha_{+} \in K_{+}^{p,n-1}$, then α_{+} has a closed extension to X_{-}. This follows from Theorem 5.3.1 in [10] and establishes (106). □

We now identify $\mathcal{H}^{p,n}_{\mathcal{R}_{-}}(X_{-})$.

Proposition 5. *With X, X_{+}, X_{-} as above, we have the isomorphism*

$$\mathcal{H}^{p,n}_{\mathcal{R}_{-}}(X_{-}) \simeq \mathcal{H}^{p,n}(X) \oplus \frac{H^{p,n-1}(\overline{X}_{+})}{K_{+}^{p,n-1}}. \tag{109}$$

Remark 9. If X_{+} is a Stein manifold then the groups $H^{p,q}(X_{+})$ vanish for $q > 0$, as do the groups $H_{b}^{p,q}(Y)$ for $1 < q < n-1$. This proposition and Theorem 4, then imply that

$$H^{p,q}(X) \simeq \mathcal{H}^{p,q}_{\mathcal{R}_{-}}(X_{-}) \tag{110}$$

for all $0 \leq p, q \leq n$.

Proof. The group $\mathcal{H}^{p,n}_{\mathcal{R}_{-}}(X_{-})$ consists of (p,n)-forms α_{-} on X_{-} that satisfy:

$$\bar{\partial}^{*}\alpha_{-} = 0 \text{ and } \bar{\mathcal{S}}_{p}(\bar{\partial}\rho \rfloor \alpha_{-})_{b} = (\bar{\partial}\rho \rfloor \alpha_{-})_{b}. \tag{111}$$

It is a simple matter to show that the first condition implies the second. Hence if $\beta_{-} \in H^{n-p,0}(X_{-})$, then $\bar{\partial}^{*}\star\beta_{-} = 0$ and therefore $\star\beta_{-} \in \mathcal{H}^{p,n}_{\mathcal{R}_{-}}(X_{-})$. From this we conclude that the inclusion of $\mathcal{H}^{p,n}(X)$ into $\mathcal{H}^{p,n}_{\mathcal{R}_{-}}(X_{-})$ is injective. The range consists of exactly those forms α_{-} such that $\star\alpha_{-}$ has a holomorphic extension to X_{+}. Again applying Theorem 5.3.1 of [10], we see that the obstruction to having such an extension is precisely $\frac{H^{p,n-1}(X_{+})}{K_{+}^{p,n-1}}$, thus proving the proposition. □

Putting together Proposition 5 with Theorem 4 and the results of Section 6, we have our first gluing formula.

COROLLARY 1. *Suppose that* X, X_+, X_- *are as above; then, for* $0 \leq p \leq n$, *there are the following identities*:

(112)
$$\begin{aligned}\chi^p_{\mathcal{O}}(X) &= \sum_{q=0}^{n} \dim H^{p,q}(X)(-1)^q \\ &= \sum_{q=0}^{n} [\dim \mathcal{H}^{p,q}_{\mathcal{R}_+}(X_+) + \dim \mathcal{H}^{p,q}_{\mathcal{R}_-}(X_-)](-1)^q - \sum_{q=1}^{n-2} (-1)^q \dim H^{p,q}_b(Y).\end{aligned}$$

The last term is absent if $\dim X = 2$.

Proof. The identity in (112) follows from the fact that the alternating sum of the dimensions in a long exact sequence is zero, along with the consequence of Proposition 5:

$$\dim \mathcal{H}^{p,n}_{\mathcal{R}_-}(X_-) = \dim H^{p,n}(X) + \dim \mathcal{H}^{p,n-1}_{\mathcal{R}_+}(\overline{X}_+) - \dim K^{p,n-1}_+. \tag{113}$$

We also use that

$$\begin{gathered} H^{0,0}(X) \simeq \mathcal{H}^{0,0}_{\mathcal{R}_-}(X_-) \text{ and } \mathcal{H}^{p,0}_{\mathcal{R}_+}(X_+) = 0 \text{ for all } p \geq 0, \\ \mathcal{H}^{p,1}_{\mathcal{R}_+}(X_+) \simeq \mathcal{H}^{p,1}_{\bar{\partial}}(X_+) \oplus E^{p,1}_+ \simeq H^{p,1}(\overline{X}_+) \oplus E^{p,1}_+. \end{gathered} \tag{114}$$

□

We modify a second exact sequence in [1] in order to obtain an expression for $\chi^p_{\mathcal{O}}(X)$ in terms of $\mathcal{H}^{p,q}_{\mathcal{R}_+}(X_+)$ and $\mathcal{H}^{p,q}_{\mathrm{Id}-\mathcal{R}_+}(X_-)$. This formula is a subelliptic analogue of Bojarski's formula expressing the index of a Dirac operator on a partitioned manifold in terms of the indices of boundary value problems on the pieces. First we state the modification of the exact sequence from Proposition 4.3 in [1].

THEOREM 5. *Let* X, X_+, X_-, Y *be as above. Then the following sequence is exact*

$$\begin{array}{llllll} 0 \longrightarrow & \mathcal{H}^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(X_-) & \xrightarrow{\widetilde{\alpha}_1} & H^{p,1}(\overline{X}_-) & & \\ \xrightarrow{\beta_1} & H^{p,1}_b(Y) & \xrightarrow{\gamma_1} & H^{p,2}(X_-,\mathcal{I}) & \xrightarrow{\alpha_2} & H^{p,2}(\overline{X}_-) \\ \xrightarrow{\beta_2} & \cdots & \cdots & \xrightarrow{\alpha_{n-2}} & H^{p,n-2}(\overline{X}_-) & \\ \xrightarrow{\beta_{n-2}} & H^{p,n-2}_b(Y) & \xrightarrow{\gamma_{n-2}} & H^{p,n-1}(X_-,\mathcal{I}) & & \\ \xrightarrow{H^{p,n-1}_{\mathcal{R}_-}} & \mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-) & \longrightarrow & 0. & & \end{array} \tag{115}$$

The map γ_q *is defined here by following the map* γ_q, *defined above, by restriction to* X_-.

Remark 10. If $n = 2$, then this sequence degenerates to

$$0 \longrightarrow \mathcal{H}^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(X_-) \xrightarrow{H^{p,1}_{\mathcal{R}_-}} \mathcal{H}^{p,1}_{\mathcal{R}_-}(X_-) \longrightarrow 0. \tag{116}$$

In this case $H^{p,1}(\overline{X}_-)$ is not isomorphic to $\mathcal{H}^{p,1}_{\mathcal{R}_-}(X_-)$, nor is $H^{p,1}(X_-, \mathcal{I})$ isomorphic to $\mathcal{H}^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(X_-)$. The argument given below shows that $H^{p,1}_{\mathcal{R}_-}$ is injective for all p. The duality argument used at the end of the proof allows us to use the injectivity of $H^{2-p,1}_{\mathcal{R}_-}$ to deduce that it is also surjective.

Proof. We first need to show that $\mathcal{H}^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(X_-)$ injects into $H^{p,1}(\overline{X}_-)$. A form α belongs to $\mathcal{H}^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(X_-)$ provided that $\bar\partial\alpha = \bar\partial^*\alpha = 0$, $\alpha_b = 0$, and $\mathcal{S}_p(\bar\partial\rho\rfloor\alpha)_b = 0$. As $H^{p,1}(\overline{X}_-) \simeq \mathcal{H}^{p,1}_{\mathcal{R}_-}(X_-)$, it suffices to show that $H^{p,1}_{\mathcal{R}_-}(\alpha) = 0$ if and only if $\alpha = 0$. A form in $\mathcal{H}^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(X_-)$ belongs to $\mathrm{Dom}_{L^2}(\bar\partial^{p,1}_{\mathcal{R}_-})$; hence, if $H^{p,1}_{\mathcal{R}_-}(\alpha) = 0$, then

$$\alpha = \bar\partial\bar\partial^* G^{p,1}_{\mathcal{R}_-}(\alpha) = \bar\partial\beta. \tag{117}$$

Observe that $0 = \alpha_b = \bar\partial_b\beta_b$. We can now show that $\alpha = 0$:

$$\begin{aligned} \langle\alpha,\alpha\rangle_{X_-} &= \langle\bar\partial\beta,\alpha\rangle_{X_-} \\ &= \langle\beta,(\bar\partial\rho\rfloor\alpha)_b\rangle_Y. \end{aligned} \tag{118}$$

On the one hand $\mathcal{S}_p(\bar\partial\rho\rfloor\alpha)_b = 0$, while, on the other hand $\mathcal{S}_p(\beta_b) = \beta_b$. This shows that $\langle\alpha,\alpha\rangle_{X_-} = 0$.

Now we show that $\operatorname{Im}\widetilde{\alpha}_1 = \ker\beta_1$. The containment $\operatorname{Im}\widetilde{\alpha}_1 \subset \ker\beta_1$ is clear because $\alpha_b = 0$ for $\alpha \in \mathcal{H}^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(X_-)$. If $\xi \in \ker\beta_1$, then there is a $(p,0)$-form, ψ on Y so that

$$\bar\partial_b\psi = \xi_b. \tag{119}$$

Let Ψ_0 denote a smooth extension of ξ to X_-; the form $\xi - \bar\partial\Psi_0$ satisfies $(\xi - \bar\partial\Psi_0)_b = 0$, and therefore belongs to $\mathrm{Dom}_{L^2}(\bar\partial^{p,1}_{\mathrm{Id}-\mathcal{R}_+})$. Hence we have the expression

$$\xi - \bar\partial\Psi_0 = H^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(\xi - \bar\partial\Psi_0) + \bar\partial\bar\partial^* G^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(\xi - \bar\partial\Psi_0). \tag{120}$$

If we let $\Psi_1 = \bar\partial^* G^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(\xi - \bar\partial\Psi_0)$, then

$$\xi - \bar\partial(\Psi_0 + \Psi_1) = H^{p,1}_{\mathrm{Id}-\mathcal{R}_+}(\xi - \bar\partial\Psi_0). \tag{121}$$

As $\xi - \bar\partial(\Psi_0 + \Psi_1)$ and ξ represent the same class $\xi \in H^{p,1}(\overline{X}_-)$, we see that $[\xi] \in \operatorname{Im}\widetilde{\alpha}_1$. This shows the exactness at $H^{p,1}(\overline{X}_-)$. The exactness through $H^{p,n-2}_b(Y)$ follows from Proposition 4.3 in [1].

The next case we need to consider is $H^{p,n-1}(X_-, \mathcal{I})$. The range of γ_{n-2} consists of equivalence classes of exact $(p, n-1)$-forms, $\bar\partial\widetilde{\xi}$, such that $\bar\partial_b\xi_b = 0$. Such a form is evidently in $\mathrm{Dom}_{L^2}(\bar\partial^{p,n-1}_{\mathcal{R}_-})$, and therefore $H^{p,n-1}_{\mathcal{R}_-}(\bar\partial\widetilde{\xi}) = 0$. Now suppose that $H^{p,n-1}_{\mathcal{R}_-}(\xi)=0$, for a ξ with $\bar\partial\xi=\xi_b=0$. As $\xi\in\operatorname{Dom}_{L^2}(\bar\partial^{p,n-1}_{\mathcal{R}_-})$ it follows that

$$\xi=\bar\partial\bar\partial^* G^{p,n-1}_{\mathcal{R}_-}(\xi). \tag{122}$$

If we let $\theta=\bar\partial^* G^{p,n-1}_{\mathcal{R}_-}(\xi)$, then clearly

$$0=\xi_b=\bar\partial_b\theta_b, \tag{123}$$

and therefore $\xi\in\operatorname{Im}\gamma_{n-2}$.

To complete the proof of this theorem, we need to show that $H^{p,n-1}_{\mathcal{R}_-}$ is surjective. We use the isomorphism $H^{p,n-1}(X_-,\mathcal{I})\simeq\mathcal{H}^{p,n-1}_{\operatorname{Id}-\mathcal{R}_+}(X_-)$. If $\xi\in\mathcal{H}^{p,n-1}_{\mathcal{R}_-}(X_-)$ and $\theta\in\mathcal{H}^{p,n-1}_{\operatorname{Id}-\mathcal{R}_+}(X_-)$, then

$$\langle\xi,\theta\rangle_{X_-}=\langle\xi,H^{p,n-1}_{\mathcal{R}_-}\theta\rangle_{X_-}=\langle H^{p,n-1}_{\operatorname{Id}-\mathcal{R}_+}\xi,\theta\rangle_{X_-}. \tag{124}$$

Using the relations in (124) we see, by duality, that $H^{p,n-1}_{\mathcal{R}_-}$ is surjective if and only if $H^{p,n-1}_{\operatorname{Id}-\mathcal{R}_+}$ is injective. As $H^{p,n-1}_{\operatorname{Id}-\mathcal{R}_+}={}^\star H^{n-p,1\star}_{\mathcal{R}_-}$, this injectivity follows from the proof of exactness at $\mathcal{H}^{r,1}_{\operatorname{Id}-\mathcal{R}_+}(X_-)$ for the case $r=n-p$. □

We get a second gluing formula for $\chi^p_{\mathcal{O}}(X)$.

Corollary 2. *Suppose that X, X_+, X_- are as above; then for $0\le p\le n$, there are the following identities:*

$$\sum_{q=0}^{n}\dim H^{p,q}(X)(-1)^q=\sum_{q=0}^{n}[\dim\mathcal{H}^{p,q}_{\mathcal{R}_+}(X_+)+\dim\mathcal{H}^{p,q}_{\operatorname{Id}-\mathcal{R}_+}(X_-)](-1)^q, \tag{125}$$

that is,

$$\operatorname{Ind}(\eth^{\mathrm{e}}_X)=\operatorname{Ind}(\eth^{\mathrm{e}}_+,\mathcal{R}^{\mathrm{e}}_+)+\operatorname{Ind}(\eth^{\mathrm{e}}_-,\operatorname{Id}-\mathcal{R}^{\mathrm{e}}_+). \tag{126}$$

Proof. These formulæ follow from those in Corollary 1 as a consequence of the previous theorem that

$$\sum_{q=1}^{n-1}\dim\mathcal{H}^{p,q}_{\operatorname{Id}-\mathcal{R}_+}(X_-)(-1)^q=\sum_{q=1}^{n-1}\dim\mathcal{H}^{p,q}_{\mathcal{R}_-}(X_-)(-1)^q+\sum_{q=1}^{n-2}\dim H^{p,q}_b(Y)(-1)^q. \tag{127}$$

If $n=2$ the last sum is absent. To complete the proof we use the isomorphisms

$$\begin{aligned}\mathcal{H}^{p,0}_{\mathcal{R}_-}(X_-)&=\mathcal{H}^{p,0}_{\operatorname{Id}-\mathcal{R}_+}(X_-)=H^{p,0}(X_-),\\ \mathcal{H}^{p,n}_{\mathcal{R}_-}(X_-)&=\mathcal{H}^{p,n}_{\operatorname{Id}-\mathcal{R}_+}(X_-)\simeq[H^{n-p,0}(X_-)]^\star.\end{aligned} \tag{128}$$

□

Remark 11. These formulæ are exactly what would be predicted, in the elliptic case, from Bojarski's formula: Let $\mathcal{P}_{\pm}^{\mathrm{eo}}$ denote the Calderon projectors for $\bar{\partial}+\bar{\partial}^*$ acting on $\Lambda^{p,\mathrm{eo}}X_{\pm}$. Bojarski proved that,

$$\operatorname{Ind}(\eth_X^{\mathrm{e}}) = \text{R-Ind}(\mathrm{Id}-\mathcal{P}_-^{\mathrm{e}}, \mathcal{P}_+^{\mathrm{e}}). \tag{129}$$

Let P be a projection in the Grassmanian of $\mathcal{P}_+^{\mathrm{e}}$. From Bojarski's formula we easily deduce the following identity

$$\operatorname{Ind}(\eth_X^{\mathrm{e}}) = \operatorname{Ind}(\eth_+^{\mathrm{e}}, P) + \operatorname{Ind}(\eth_-^{\mathrm{e}}, \mathrm{Id}-P). \tag{130}$$

The proof uses elementary properties of the relative index:

$$\begin{gathered} -\text{R-Ind}(P_2,P_1) = \text{R-Ind}(P_1,P_2) = -\text{R-Ind}(\mathrm{Id}-P_1, \mathrm{Id}-P_2) \\ \text{R-Ind}(P_1,P_3) = \text{R-Ind}(P_1,P_2) + \text{R-Ind}(P_2,P_3). \end{gathered} \tag{131}$$

To deduce (130) we use the observation that

$$\operatorname{Ind}(\eth_+^{\mathrm{e}}, P) = \text{R-Ind}(\mathcal{P}_+^{\mathrm{e}}, P), \quad \operatorname{Ind}(\eth_-^{\mathrm{e}}, \mathrm{Id}-P) = \text{R-Ind}(\mathcal{P}_-^{\mathrm{e}}, \mathrm{Id}-P). \tag{132}$$

Hence, we see that

$$\begin{aligned} \operatorname{Ind}(\eth_+^{\mathrm{e}}, P) + \operatorname{Ind}(\eth_-^{\mathrm{e}}, \mathrm{Id}-P) &= \text{R-Ind}(\mathcal{P}_+^{\mathrm{e}}, P) + \text{R-Ind}(\mathcal{P}_-^{\mathrm{e}}, \mathrm{Id}-P) \\ &= \text{R-Ind}(\mathcal{P}_+^{\mathrm{e}}, P) - \text{R-Ind}(\mathrm{Id}-\mathcal{P}_-^{\mathrm{e}}, P) \\ &= \text{R-Ind}(\mathcal{P}_+^{\mathrm{e}}, \mathrm{Id}-\mathcal{P}_-^{\mathrm{e}}). \end{aligned} \tag{133}$$

The proofs of the identities in (131) use the theory of Fredholm pairs. If H is a Hilbert space, then a pair of subspaces H_1, H_2 of H is a Fredholm pair if $H_1 \cap H_2$ is finite dimensional, $H_1 + H_2$ is closed and $H/(H_1+H_2) \simeq H_1^{\perp} \cap H_2^{\perp}$ is finite dimensional. One uses that, for two admissible projectors P_1, P_2, the subspaces of $L^2(Y;E)$ given by $H_1 = \operatorname{Im} P_1, H_2 = \operatorname{Im}(\mathrm{Id}-P_2)$ are a Fredholm pair and

$$\text{R-Ind}(P_1,P_2) = \dim H_1 \cap H_2 - \dim H_1^{\perp} \cap H_2^{\perp}. \tag{134}$$

In our case the projectors are $\mathcal{P}_{\pm}^{\mathrm{e}}$ and $\mathcal{R}_{\pm}^{\mathrm{e}}$. While it is true that, e.g. $\operatorname{Im}\mathcal{P}_+^{\mathrm{e}} \cap \operatorname{Im}(\mathrm{Id}-\mathcal{R}_+^{\mathrm{e}})$ is finite dimensional, it is not true that $\operatorname{Im}\mathcal{P}_+^{\mathrm{e}} + \operatorname{Im}(\mathrm{Id}-\mathcal{R}_+^{\mathrm{e}})$ is a closed subspace of L^2. So these projectors do not define a traditional Fredholm pair. If we instead consider these operators as acting on smooth forms, then the $\operatorname{Im}\mathcal{P}_+^{\mathrm{e}}$ and $\operatorname{Im}(\mathrm{Id}-\mathcal{R}_+^{\mathrm{e}})$ are a "Fréchet" Fredholm pair. As the result predicted by Bojarski's theorem remains true, this indicates that perhaps there is a generalization of the theory of Fredholm pairs that includes both the elliptic and subelliptic cases.

It seems a natural question whether the Agranovich-Dynin formula holds on the pseudoconcave side as well, that is, if

$$\operatorname{Ind}(\eth_-^{\mathrm{e}}, \mathrm{Id}-\mathcal{R}_+'^{\mathrm{e}}) + \operatorname{Ind}(\eth_-^{\mathrm{e}}, \mathrm{Id}-\mathcal{R}_+^{\mathrm{e}}) \stackrel{?}{=} \text{R-Ind}(\mathcal{S}_p', \mathcal{S}_p). \tag{135}$$

If this were the case, then (126) would also hold for boundary conditions defined by generalized Szegő projectors. Because the null space of $(\eth^{\mathrm{e}}_{-}, \mathrm{Id} - \mathcal{R}'^{\mathrm{e}}_{+})$ does not seem to split as a direct sum over form degrees, the argument used to prove Theorem 3 does not directly apply to this case.

9. General holomorphic coefficients

Thus far we have considered the Dirac operator acting on sections of $\Lambda^{p,\mathrm{eo}}$. Essentially everything we have proved for cases where $p > 0$ remains true if the bundles $\Lambda^{p,\mathrm{eo}}$ are replaced by $\Lambda^{\mathrm{eo}} \otimes \mathcal{V}$, where $\mathcal{V} \to X$ is a holomorphic vector bundle. In [7] we proved the necessary estimates for the twisted Dirac operator acting on sections of $\Lambda^{\mathrm{eo}} \otimes \mathcal{V}$. For example, suppose that X_+ is strictly pseudoconvex, then defining

$$E_0^{\mathcal{V},1}(\overline{X}_+) = \frac{\{\bar{\partial}\alpha : \ \alpha \in \mathcal{C}^\infty(\overline{X}_+; \mathcal{V}) \text{ and } \bar{\partial}_b \alpha_b = 0\}}{\{\bar{\partial}\alpha : \ \alpha \in \mathcal{C}^\infty(\overline{X}_+; \mathcal{V}) \text{ and } \alpha_b = 0\}}, \tag{136}$$

we can easily show that

$$\operatorname{Ind}(\eth^{\mathrm{e}}_{\mathcal{V}+}, \mathcal{R}^{\mathrm{e}}_{+}) = -\dim E_0^{\mathcal{V},1} + \sum_{q=1}^{n} H^q(X_+; \mathcal{V}). \tag{137}$$

The vector space $E_0^{\mathcal{V},1}$ is the obstruction to extending $\bar{\partial}_b$-closed sections of $\mathcal{V} \restriction_{bX_+}$ as holomorphic sections of $\mathcal{V}$. Hence it is isomorphic to $H^{n-1}_{\bar{\partial}}(X_+; \Lambda^{n,0} \otimes \mathcal{V}')$, see Proposition 5.13 in [11]. It is therefore finite dimensional, and vanishes if X_+ is a Stein manifold.

The Agranovich-Dynin formula and the Bojarski formula also hold for general holomorphic coefficients.

THEOREM 6. *Let X_+ be a compact strictly pseudoconvex Kähler manifold and $\mathcal{V} \to X_+$ a holomorphic vector bundle. If the classical Szegő projector onto the null space of $\bar{\partial}_b$, acting on sections of $\mathcal{V} \restriction_{bX_+}$ is denoted $\mathcal{S}_{\mathcal{V}}$, and $\mathcal{S}'_{\mathcal{V}}$ is a generalized Szegő projector, then*

$$\operatorname{Ind}(\eth^{\mathrm{e}}_{\mathcal{V}+}, \mathcal{R}'^{\mathrm{e}}_{+}) - \operatorname{Ind}(\eth^{\mathrm{e}}_{\mathcal{V}+}, \mathcal{R}^{\mathrm{e}}_{+}) = \text{R-Ind}(\mathcal{S}_{\mathcal{V}}, \mathcal{S}'_{\mathcal{V}}). \tag{138}$$

COROLLARY 3. *Suppose that X, X_+, X_- are as above and $\mathcal{V} \to X$ is a holomorphic vector bundle, then the following identity holds:*

$$\sum_{q=0}^{n} \dim H^q(X; \mathcal{V})(-1)^q = \sum_{q=0}^{n} [\dim \mathcal{H}^q_{\mathcal{R}_+}(X_+; \mathcal{V}) + \dim \mathcal{H}^q_{\mathrm{Id} - \mathcal{R}_+}(X_-; \mathcal{V})](-1)^q; \tag{139}$$

that is,

$$\operatorname{Ind}(\eth^{\mathrm{e}}_{\mathcal{V}+}) = \operatorname{Ind}(\eth^{\mathrm{e}}_{\mathcal{V}+}, \mathcal{R}^{\mathrm{e}}_{+}) + \operatorname{Ind}(\eth^{\mathrm{e}}_{\mathcal{V}-}, \mathrm{Id} - \mathcal{R}^{\mathrm{e}}_{+}). \tag{140}$$

The proofs of these statements are essentially identical to those given above and are left to the interested reader.

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