Projective structures with degenerate holonomy and the Bers density conjecture

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Abstract

We prove the Bers density conjecture for singly degenerate Kleinian surface groups without parabolics.

1. Introduction

In this paper we address a conjecture of Bers about singly degenerate Kleinian groups. These are discrete subgroups of $\mathrm{PSL}_2\mathbb{C}$ that exhibit some unusual behavior:

- As groups of projective transformations of the Riemann sphere $\widehat{\mathbb{C}}$ they act properly discontinuously on a topological disk whose closure is all of $\widehat{\mathbb{C}}$.

- As groups of hyperbolic isometries their action on $\mathbb{H}^3$ is not convex co-compact.

- Viewed as dynamical systems they are not structurally stable.

These groups were first discovered by Bers ([Bers2]) where he made the conjecture that will be the focus of our work here.

Let $M = S \times [-1, 1]$ be an I-bundle over a closed surface S of genus > 1. We will be interested in the space $AH(S)$ of all Kleinian groups isomorphic to $\pi_1(S)$. By a theorem of Bonahon, this is equivalent to studying complete hyperbolic structures on the interior of M. A generic hyperbolic structure on M is *quasi-fuchsian* and the geometry is well understood outside of a compact set. In particular, although the geometry of the surfaces $S \times \{t\}$ will grow exponentially as t limits to -1 or 1, the conformal structures will stabilize and limit to Riemann surfaces X and Y. Then M can be conformally compactified by viewing X and Y as conformal structures on $S \times \{-1\}$ and $S \times \{1\}$,

respectively. Bers showed that X and Y parametrize the space QF(S) of all quasi-fuchsian structures. In other words QF(S) is isomorphic to $T(S) \times T(S)$ where $T(S)$ is the Teichmüller space of marked conformal structures on S. Let the *Bers slice* B_X be the slice of QF(S) obtained by fixing X and letting Y vary in $T(S)$.

This gives an interesting model of $T(S)$ because B_X naturally embeds as a bounded domain in the space $P(X)$ of projective structures on S with conformal structure X. The closure $\overline{B}_X$ of B_X in $P(X)$ is then a compactification of Teichmüller space. A point in $\partial B_X = \overline{B}_X - B_X$ will again correspond to a complete hyperbolic structure on M. As with structures in B_X, the surfaces $S \times \{t\}$ will converge to the conformal structure X as $t \to -1$. However, as $t \to 1$ the structures will not converge.

There are three possibilities for the limiting geometry of the $S \times \{t\}$. In the simplest case there will be an essential simple closed curve (or a collection of curves) c on S such that the length of c on $S \times \{t\}$ limits to zero, while on the complement of c the surfaces grow exponentially but converge to a cusped conformal structure. In this case M is *geometrically finite*. In the other case there will be a sequence $t_i \to 1$ such that $S \times \{t_i\}$ has bounded area yet for any simple closed curve c on S the length of c on $S \times \{t_i\}$ will go to infinity as $t_i \to 1$. In other words, the geometry of the $S \times \{t_i\}$ is bounded but still changing radically. Such manifolds are *singly degenerate*. The final possibility is that M may have a combination of the first two behaviors.

Understanding such structures is a motivating problem in hyperbolic 3-manifolds and Kleinian groups. Bers made the following conjecture:

CONJECTURE 1.1 (Bers Density Conjecture [Bers2]). *Let* $\Gamma \in AH(S)$ *be a Kleinian group. If* $M = \mathbb{H}^3/\Gamma$ *is singly degenerate then* $\Gamma \in \overline{B}_X$ *where* X *is the conformal boundary of* M.

There are some special cases where the conjecture is known. Abikoff [Ab] proved the conjecture when M is geometrically finite. Recently Minsky [Min3] has proved the conjecture in the case where there is a lower bound on the length of any closed geodesic in M and Γ has no parabolics (M has *bounded geometry*). In a separate, earlier paper ([Min2]), Minsky also proved the conjecture if S is a punctured torus. In this paper we prove the conjecture when M has a sequence of closed geodesics c_i whose length limits to zero (M has *unbounded geometry*). Combined with Minsky's result we have an almost complete resolution of Bers's conjecture:

THEOREM 5.4. *Assume that* $\Gamma \in AH(S)$ *has no parabolics. If* $M = \mathbb{H}^3/\Gamma$ *is singly degenerate then* $\Gamma \in \overline{B}_X$ *where* X *is the conformal boundary of* M.

There is a more general version of the density conjecture due to Sullivan and Thurston. It states that every finitely generated Kleinian group is an algebraic limit of geometrically finite Kleinian groups. In joint work with

Brock ([BB]) we use some of the ideas of this paper to prove this more general conjecture for freely indecomposable Kleinian groups without parabolics.

The condition that Γ has no parabolics is a technical one and we believe with more work, present techniques could be used to prove the complete conjecture. More precisely, if the surface S has punctures then instead of studying all Kleinian groups isomorphic to $\pi_1(S)$, we study $AH(S)$, the space of Kleinian groups in which all of the punctures are parabolic. If one could prove Conjecture 1.1 for all $\Gamma \in AH(S)$ such that all parabolics in Γ correspond to punctures then the entire conjecture would follow. If $M = \mathbb{H}^3/\Gamma$ have unbounded geometry most of the work in this paper would generalize easily. If M has bounded geometry then one needs to generalize Minsky's work. In particular, most of Minsky's work applies in this setting; it is only his earliest paper on the problem ([Min1]) that needs to be generalized.

We also remark that the density conjecture is a consequence of the ending lamination conjecture. In fact, Minsky's results on the density conjecture are a consequence of his work on the ending lamination conjecture. More recently Brock, Canary and Minsky have annouced work that completes Minsky's program to prove the full ending lamination conjecture ([Min4], [BCM]).

We now outline our results.

Our approach to Conjecture 1.1 is to understand projective structures with singly degenerate holonomy. Our study will be guided by Goldman's classification of all projective structures with quasi-fuchsian holonomy ([Gol]). In particular, the two conformal structures X and Y that compactify a quasi-fuchsian manifold also have projective structures Σ^- and Σ^+. Goldman showed that all projective structures with quasi-fuchsian holonomy are obtained by *grafting* on Σ^- or Σ^+. For a singly degenerate group we still have the projective structure Σ^- and all of its graftings. On the other hand, while the projective structure Σ^+ is gone we will show that its graftings still exist.

We will use these projective structures to construct a family of quasi-fuchsian hyperbolic cone-manifolds that converge to the singly degenerate manifold M. Here is our main construction. By a theorem of Otal [Ot], any sufficiently short geodesic c will be unknotted. That is, the product structure can be chosen such that c is a simple closed curve on $S \times \{0\}$. Let A be the annulus $c \times [0,1)$ and let $A_{\mathbb{Z}}$ be a lift of A to the $\mathbb{Z}$-cover $M_{\mathbb{Z}}$ of M associated to c. Now remove A from M and $A_{\mathbb{Z}}$ from $M_{\mathbb{Z}}$ and take the metric completion of both spaces. Both of these spaces will be manifolds with boundary isometric to two copies of A meeting at the geodesic c. Next, glue the two manifolds together along their isometric boundary to form a new manifold M_c. This new manifold will be homeomorphic to M but the hyperbolic structure will be singular along the geodesic c. In particular M_c will be a hyperbolic *cone-manifold*, for a cross-section of a tubular neighborhood of c will be a cone of cone angle 4π. We will show:

THEOREM 4.2. *The hyperbolic cone-manifold M_c is a quasi-fuchsian cone-manifold with projective boundary Σ and Σ_c.*

The lower half of M_c is isometric to the lower half of M and is therefore compactified by the same projective structure Σ on the conformal structure X. The upper half of M_c will be compactified by the new projective structure Σ_c which will have conformal structure Y_c. Then there is a unique quasi-fuchsian group $\Gamma_c \in B_X$ such that $M'_c = \mathbb{H}^3/\Gamma_c$ has conformal boundary X and Y_c.

If M has unbounded geometry there will be a sequence of closed geodesics c_i with $\text{length}(c_i) \to 0$. Repeating the above construction for each c_i we obtain cone-manifolds M_i and quasi-fuchsian manifolds $M'_i = \mathbb{H}^3/\Gamma_i$. Let Σ_i be the component of the projective boundary of M'_i corresponding to X. The final step is to bound the distance between Σ_i and Σ in terms of $\text{length}(c_i)$.

This is done using the deformation theory of hyperbolic cone-manifolds developed by Hodgson and Kerckhoff for closed manifolds and extended by the author to geometrically finite cone-manifolds. For each M_i we can use this deformation theory to find a smooth one-parameter family of cone-manifolds that interpolates between M_i and M'_i. Furthermore this deformation theory allows us to control how the projective structure Σ deforms to the projective structure Σ_i. As we will discuss below, there is a canonical way to define a metric on $P(X)$ and in this metric we have:

$$d(\Sigma, \Sigma_i) \leq K \, \text{length}(c_i).$$

Therefore $\Sigma_i \to \Sigma$ in $P(X)$ which implies that $\Gamma_i \to \Gamma$ in $AH(S)$ and $\Gamma \in \overline{B}_X$. A novel feature of the above estimate is its use of the analytic theory of cone-manifolds to obtain results about infinite volume, hyperbolic 3-manifolds. This approach has turned out to be fruitful in other problems (see [Br1], [BB], [BBES]) and we expect it will have further applications as well.

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2. Preliminaries

2.1. *Kleinian groups.* A Kleinian group Γ is a discrete subgroup of $\text{PSL}_2\mathbb{C}$. In this paper we will assume that all Kleinian groups are torsion-free. The Lie group $\text{PSL}_2\mathbb{C}$ acts as both projective transformations of the Riemann sphere $\widehat{\mathbb{C}}$ and as isometries on hyperbolic 3-space $\mathbb{H}^3$. The union $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ is naturally topologized as a closed 3-ball such that the action of $\text{PSL}_2\mathbb{C}$ on $\mathbb{H}^3$ extends continuously to the action on $\widehat{\mathbb{C}}$.

The domain of discontinuity $\Omega \subset \widehat{\mathbb{C}}$ for Γ is the largest subset of $\widehat{\mathbb{C}}$ such that Γ acts properly discontinuously. The limit set $\Lambda = \widehat{\mathbb{C}} - \Omega$ is the complement

of Ω in $\widehat{\mathbb{C}}$. The group Γ will act properly discontinuously on all of $\mathbb{H}^3$ so that the quotient $\mathbb{H}^3/\Gamma$ will be a 3-manifold. The quotient $(\mathbb{H}^3 \cup \Omega)/\Gamma$ will be a 3-manifold with boundary.

2.2. *Projective structures.* Let S be a surface. A *projective structure* Σ on S is an atlas of charts to $\widehat{\mathbb{C}}$ with transition maps elements of $\mathrm{PSL}_2\mathbb{C}$, the group of projective transformations of $\widehat{\mathbb{C}}$. If Γ is a Kleinian group isomorphic to $\pi_1(S)$ and Ω' is a connected component of Ω that is fixed by Γ then the quotient Ω'/Γ will be a projective structure on S.

As projective transformations are conformal maps, a projective structure Σ also defines a conformal structure X on S. If $T(S)$ is the Teichmüller space of marked conformal structures on S and $P(S)$ is the space of projective structures, then there is a map $P(S) \longrightarrow T(S)$ defined by $\Sigma \mapsto X$.

Let $P(X)$ be the pre-image of X in $P(S)$ under this map. We now define a metric on $P(X)$. Given two projective structures Σ and Σ' in $P(X)$ there is a unique conformal map f between them that is isotopic to the identity on S. The Schwarzian derivative of f is a holomorphic, quadratic differential ϕ on X. (See [Le] for the definition of the Schwarzian derivative.) If ρ is the hyperbolic metric on X then $\phi\rho^{-2}$ is a function on X. We let $\|\phi\|_\infty$ be the sup norm of this function. We define our metric on $P(X)$ by setting

$$d(\Sigma, \Sigma') = \|\phi\|_\infty.$$

A projective structure is *Fuchsian* if it is the quotient of a round disk in $\widehat{\mathbb{C}}$. There is a unique Fuchsian element Σ_F in $P(X)$ and we let $\|\Sigma\|_\infty = d(\Sigma, \Sigma_F)$.

2.3. *Hyperbolic structures.* A *hyperbolic structure* on a 3-manifold M is a Riemannian metric with constant sectional curvature equal to -1. Equivalently, a hyperbolic structure can be defined as an atlas of local charts to $\mathbb{H}^3$ with transition maps that are hyperbolic isometries.

We will also be interested in certain singular hyperbolic structures. We let $\mathbb{H}^3_\alpha$ be $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and the Riemannian metric $dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$ where θ is measure modulo α. The metric on $\mathbb{H}^3_\alpha$ is a smooth metric of constant sectional curvature $\equiv -1$ when $r \neq 0$. It extends to a complete, singular metric on all of $\mathbb{H}^3_\alpha$. The sub-surfaces where z is constant are hyperbolic planes away from $r = 0$. At $r = 0$ there is a cone-singularity with cone angle α.

If $\alpha = 2\pi$ then $\mathbb{H}^3_\alpha$ is isometric to $\mathbb{H}^3$. If $\alpha = 2\pi n$ where n is a positive integer then there is an obvious map from $\mathbb{H}^3_\alpha$ to $\mathbb{H}^3$ that is a local isometry when $r \neq 0$ and has an order n branch locus at $r = 0$.

A metric on M is a hyperbolic *cone-metric* if all points in M are either modeled on $\mathbb{H}^3$ or the point $(0, 0, 0)$ in $\mathbb{H}^3_\alpha$ for some α. All points of the second type are the singular locus $\mathcal{C}$ for M. Clearly $\mathcal{C}$ will consist of a collection of disjoint, simple curves and all points in a component c of $\mathcal{C}$ will be modeled

on $\mathbb{H}^3_\alpha$ for some fixed α. Then α is the *cone-angle* for c. In this paper we will assume that the singular locus consists of a finite collection of simple closed curves.

2.4. *Kleinian surface groups.* The space of representations of $\pi_1(S)$ in $\mathrm{PSL}_2\mathbb{C}$ has a natural topology given by convergence on generators. Let $AH(S)$ be the space of conjugacy classes of discrete, faithful representations of $\pi_1(S)$ in $\mathrm{PSL}_2\mathbb{C}$ with the quotient topology. The image of each representation is a marked Kleinian group so we can view $AH(S)$ as a space of Kleinian groups. A group $\Gamma \in AH(S)$ is *quasi-fuchsian* if the limit set of Γ is a Jordan curve. The domain of discontinuity is then two topological disks Ω^- and Ω^+. Let $X = \Omega^-/\Gamma$ and $Y = \Omega^+/\Gamma$ be the quotient conformal structures on S. The assignment

$$\Gamma \mapsto (X, Y)$$

defines a map from the space of quasi-fuchsian structures $\mathrm{QF}(S)$ to $T(S) \times T(S)$.

THEOREM 2.1 (Bers [Bers1]). *The above map from* $\mathrm{QF}(S)$ *to* $T(S) \times T(S)$ *is a homeomorphism.*

We define a *Bers' slice* by $B_X = \{X\} \times T(S) \subset \mathrm{QF}(S)$. This set of quasi-fuchsian groups is isomorphic to $T(S)$. Bers observed that B_X embeds as a bounded domain in $P(X)$ and therefore the closure $\overline{B}_X$ is a compactification of Teichmüller space ([Bers2]).

To understand a general $\Gamma \in AH(S)$, we need the following important theorem:

THEOREM 2.2 (Bonahon [Bon]). *If* Γ *is in* $AH(S)$ *then the quotient 3-manifold* $\mathbb{H}^3/\Gamma$ *is homeomorphic to* $S \times (-1, 1)$.

Bers original study was of groups $\Gamma \in AH(S)$ such that the hyperbolic structure $\mathbb{H}^3/\Gamma$ on $S \times (-1, 1)$ extends to a projective structure Σ on $S \times \{-1\}$. If such a Γ is not quasi-fuchsian and has no parabolics, then Γ is *singly degenerate*. For a singly degenerate group the domain of discontinuity will be a single topological disk. On the other hand, if Γ has parabolics then they will correspond to a collection of disjoint, essential, simple closed curves on S. The subgroups of Γ corresponding to the components of the complement of the simple closed curves will either be quasi-fuchsian groups or singly degenerate groups. We will not investigate groups with parabolics in this paper.

There is a further dichotomy for hyperbolic 3-manifolds with degenerate ends. Namely, M has *bounded geometry* if there is a lower bound on the length of any closed geodesic in M. Otherwise M has *unbounded geometry*. As mentioned in the introduction, Minsky has proved Bers' conjecture (Conjec-

ture 1.1) if M has bounded geometry. In fact he has proved a much stronger result which we only partially state here:

THEOREM 2.3 (Minsky [Min3]). *Suppose $\Gamma \in AH(S)$ has no parabolics. Then if $M = \mathbb{H}^3/\Gamma$ has bounded geometry, $\Gamma \in \overline{QF(S)}$. Furthermore if M is singly degenerate with conformal boundary X then $\Gamma \in \overline{B}_X$.*

2.5. *Quasi-fuchsian cone-manifolds.* There is an alternate definition of a quasi-fuchsian manifold that extends naturally to cone-manifolds. A hyperbolic structure on the interior of $S \times [-1, 1]$ is quasi-fuchsian if it extends to a projective structure on $S \times \{-1\}$ and $S \times \{1\}$. More explicitly, for each point x in $S \times \{-1\}$ or $S \times \{1\}$ there exists a local chart from a neighborhood of x in $S \times [-1, 1]$ (not simply a neighborhood in $S \times \{\pm 1\}$) to $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$. The transition maps will again be elements of $PSL_2\mathbb{C}$ which act as automorphisms of $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$. This definition agrees with our previous definition of a quasi-fuchsian structure and extends to a definition of quasi-fuchsian hyperbolic cone-metrics on $S \times (-1, 1)$.

2.6. *Handlebodies and Schottky groups.* A Kleinian group Γ is a *Schottky group* if $H = (\mathbb{H}^3 \cup \Omega)/\Gamma$ is a closed handlebody with boundary. A handlebody has many distinct product structures. In particular if Y is a properly embedded surface in H such that the inclusion map is a homotopy equivalence then H is homeomorphic to a product $S \times [-1, 1]$ with $S \times \{0\} = Y$.

2.7. *Grafting.* A projective structure Σ on a closed surface S defines a *holonomy representation* of $\pi_1(S)$ via a *developing map*. In particular, Σ lifts to a projective structure $\tilde{\Sigma}$ on the universal cover $\tilde{S}$. Any chart for Σ will lift to a chart for $\tilde{\Sigma}$. Since $\tilde{\Sigma}$ is simply connected, this chart will extend to a projective map $D : \tilde{S} \longrightarrow \widehat{\mathbb{C}}$ on all of $\tilde{S}$. Furthermore there will be a representation $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{C}$ such that

$$D(g(x)) = \rho(g)D(x)$$

for all $g \in \pi_1(S)$ and all $x \in \tilde{S}$. Then D is a developing map with holonomy ρ. Note that D is unique up to post-composition with elements of $PSL_2\mathbb{C}$ while ρ is unique up to conjugacy.

Now let c be an essential, simple closed curve on S and $\tilde{c}$ a component of the pre-image of c in $\tilde{S}$. Let $g \in \pi_1(S)$ generate the $\mathbb{Z}$-subgroup that preserves $\tilde{c}$. We also assume that $\rho(g)$ is hyperbolic and that $D(\tilde{c})$ is a simple arc in $\widehat{\mathbb{C}}$. Then the quotient of $\widehat{\mathbb{C}}$ minus the fixed points of $\rho(g)$ is a torus T, $D(\tilde{c})$ descends to an essential simple closed curve c' on T and $A = T - c'$ is a projective structure on an annulus. We can form a new projective structure on S by removing the curve c from the projective structure Σ and gluing in n copies of A. The new projective structure is then a *grafting* of Σ along the

curve c. Most importantly for our purposes the grafted projective structure has the same holonomy as Σ.

Goldman used grafting to classify projective structures with quasi-fuchsian holonomy. Let Γ be a quasi-fuchsian group with Ω^- and Ω^+ the two components of the domain of discontinuity. Then $\Sigma^\pm = \Omega^\pm/\Gamma$ are projective structures on S.

THEOREM 2.4 (Goldman [Gol]). *All projective structures with holonomy Γ are obtained by grafting on either Σ^- or Σ^+.*

In the next section, we will conjecture that a similar classification holds for singly degenerate Kleinian groups.

3. Projective structures

Let S be a closed surface of genus $g > 1$ and Γ a singly degenerate Kleinian group isomorphic to $\pi_1(S)$. Let $\Sigma = \Omega/\Gamma$ be the quotient projective structure on S.

Let $D : \tilde{S} \longrightarrow \Omega \subset \widehat{\mathbb{C}}$ be a developing map for Σ with holonomy representation $\rho : \pi_1(S) \longrightarrow \Gamma$. Choose an essential simple closed curve c on S and let $\tilde{c}$ be the pre-image of c in the universal cover $\tilde{S}$. We will begin by assuming that c is nonseparating and deal with the general case at the end of the section. We also choose a component $\tilde{K}$ of $\tilde{S} - \tilde{c}$. Note that since c is nonseparating the action of $\pi_1(S)$ on the components of $\tilde{S} - \tilde{c}$ has a single orbit. Let $\tilde{c}_K$ be the components of $\tilde{c}$ which lie on the boundary of $\tilde{K}$.

Let Γ_K be the subgroup of Γ which fixes $D(\tilde{K})$ setwise. Then Ω/Γ_K will be a cover of Σ corresponding to the restriction of $\pi_1(S)$ to $S - c$. In particular Γ_K will be isomorphic to $\pi_1(S - c)$, a free group on $2g - 1$ generators. We also note that $D(\tilde{c}_K)$ will descend to two simple closed curves c_1 and c_2 on the cover Ω/Γ_K.

Let Ω_K be the domain of discontinuity for Γ_K and let $\Sigma_K = \Omega_K/\Gamma_K$ be the quotient projective structure. Since $\Omega_K \supseteq \Omega$, $D(\tilde{c}_K)$ will also descend to two simple closed curves on Σ_K. We abuse notation by also referring to these curves as c_1 and c_2.

LEMMA 3.1. *The group Γ_K is a Schottky group and the projective structure Σ_K is homeomorphic to a surface of genus $2g - 1$. Furthermore there is an orientation reversing involution $\phi : \Sigma_K \longrightarrow \Sigma_K$ that fixes c_1 and c_2 pointwise and lifts to an orientation reversing, Γ_K-invariant involution $\tilde{\phi} : \Omega_K \longrightarrow \Omega_K$ which fixes $D(\tilde{c}_K)$ pointwise.*

Proof. We postpone the 3-dimensional proof of this lemma to the next section where we will prove the stronger Lemma 4.1. $\boxed{3.1}$

Since $D(\tilde{K})$ is contained in Ω_K, $D(\tilde{K})/\Gamma_K$ is a subsurface of Σ_K. Let Σ_K^- be the closure of $D(\tilde{K})/\Gamma_K$ in Σ_K. Then Σ_K^- is homeomorphic to a genus $g-1$ surface with two boundary components c_1 and c_2. Let Σ_K^+ be the closure of the complement of Σ_K^- in Σ_K. The involution ϕ from Lemma 3.1 will then restrict to a homeomorphism from Σ_K^- to Σ_K^+ and so Σ_K^+ is also a genus $g-1$ surface with two boundary components. (See Figure 1.)

We also know that $D(\tilde{K})$ is contained in Ω so that Σ_K^- is also a subsurface of the cover Ω/Γ_K of Σ. In fact the covering map $\pi : \Omega/\Gamma_K \longrightarrow \Sigma$ restricts to a one-to-one map from the interior of Σ_K^- to $\Sigma - c$ and is a two-to-one map from $c_1 \cup c_2$ to c. We use π to define an equivalence relation for points $p_1 \in c_1$ and $p_2 \in c_2$ with $p_1 \sim p_2$ if $\pi(p_1) = \pi(p_2)$. Then the quotient $\Sigma_K^-/\sim$ is exactly the original projective structure Σ. More importantly, the quotient $\Sigma_c = \Sigma_K^+/\sim$ will also be a projective structure on S.
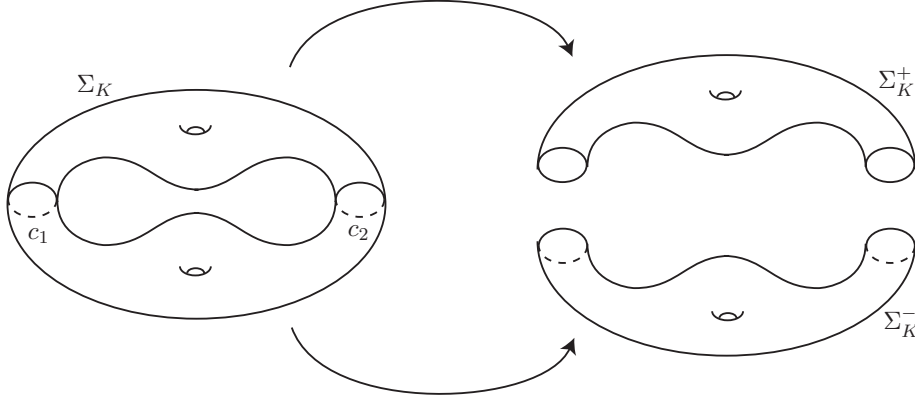

Figure 1: Cutting Σ_K along c_1 and c_2 produces Σ_K^+ and Σ_K^-.

THEOREM 3.2. *Σ_c is a projective structure on S with holonomy ρ.*

Proof. We can explicitly write down a formula for a developing map for Σ_c by modifying the developing map D for Σ. Namely define $D_c : \tilde{S} \longrightarrow \widehat{\mathbb{C}}$ by the formula

$$D_c(x) = \rho(g^{-1}) \circ \tilde{\phi} \circ D(g(x)) \text{ if } g(x) \text{ is in the closure of } \tilde{K} \text{ and } g \in \pi_1(S).$$

It is a simple matter of retracing definitions to see that D_c is well defined, a developing map for Σ_c, and has holonomy ρ. $\boxed{3.2}$

COROLLARY 3.3. *The projective structure Σ_c is not obtained by grafting Σ.*

Proof. The developing map D_c has the opposite orientation to that of D so that Σ_c cannot be a grafting of Σ. $\boxed{3.3}$

In the above work we have assumed that c is nonseparating. This is not essential. In fact, after minor modifications, the construction works for any collection $\mathcal{C}$ of n disjoint, homotopically distinct and essential simple closed curves. If $\tilde{\mathcal{C}}$ is the pre-image of $\mathcal{C}$ in $\tilde{S}$ then the action of $\pi_1(S)$ on $\tilde{S} - \tilde{\mathcal{C}}$ will have k orbits where k is the number of components of $S - \mathcal{C}$. We choose a component $\tilde{K}_i$ corresponding to each orbit and let Γ_{K_i} be the subgroup of Γ that fixes $D(\tilde{K}_i)$ setwise with Ω_{K_i} the domain of discontinuity of Γ_{K_i}. Each projective structure $\Sigma_{K_i} = \Omega_{K_i}/\Gamma_{K_i}$ can then be cut into two pieces $\Sigma_{K_i}^-$ and $\Sigma_{K_i}^+$ and there is an involution ϕ_i of Σ_{K_i} swapping the two pieces. Then the $\Sigma_{K_i}^-$ can be glued together to reform Σ. The $\Sigma_{K_i}^+$ can also be glued together to form a new projective structure $\Sigma_{\mathcal{C}}$. As before we can explicitly define a developing map $D_{\mathcal{C}} : \tilde{S} \longrightarrow \widehat{\mathbb{C}}$ for $\Sigma_{\mathcal{C}}$ by the formula

$$D_{\mathcal{C}}(x) = \rho(g^{-1}) \circ \tilde{\phi}_i \circ D(g(x)) \text{ if } g(x) \text{ is in the closure of } \tilde{K}_i.$$

Again, it is a simple matter of tracing through the definitions to see that $D_{\mathcal{C}}$ is a developing map for a projective structure on S and that the holonomy of $D_{\mathcal{C}}$ is ρ.

We also remark that if c is a component of $\mathcal{C}$ and $\mathcal{C}' = \mathcal{C} - c$, then $\Sigma_{\mathcal{C}}$ can also be obtained by either grafting $\Sigma_{\mathcal{C}'}$ along c or grafting Σ_c along $\mathcal{C}'$.

This construction also works if Γ is quasi-fuchsian. In this case we have two initial projective structures Σ^- and Σ^+ corresponding to the two components of the domain of discontinuity. We leave the following theorem as an exercise for the reader.

THEOREM 3.4. *The projective structure $\Sigma_{\mathcal{C}}^-$ is equivalent to grafting Σ^+ along $\mathcal{C}$.*

This leads us to make the following conjecture for projective structures with singly degenerate holonomy:

CONJECTURE 3.5. *Let S be a closed surface and Γ a singly degenerate group in $AH(S)$. Let $\Sigma = \Omega/\Gamma$ be the quotient projective structure where Ω is the domain of discontinuity for Γ. Every projective structure with holonomy Γ is either*:

  1. Σ,

  2. $\Sigma_{\mathcal{C}}$ *for some collection $\mathcal{C}$,*

  3. *grafting of Σ,*

  4. *grafting of $\Sigma_{\mathcal{C}}$ along $\mathcal{C}$.*

4. Cone-manifolds

We carry over our notation from the previous section. Let $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$ be the quotient 3-manifold with boundary. By Bonahon's theorem (Theorem 2.2), we can fix an identification of M with the product $S \times [-1, 1)$ which we will use throughout this section. The interior of M will have a complete hyperbolic structure while the boundary $S \times \{-1\}$ is the projective structure Σ.

We recall the construction described in the introduction, adding more details. Let c be an essential simple closed curve on S and make the further assumption that $c \times \{0\}$ is a geodesic in M. Let $A = c \times [0, 1)$ be an annulus in M. Then A lifts homeomorphically to an annulus $A_\mathbb{Z}$ in the $\mathbb{Z}$-cover $M_\mathbb{Z}$ of M associated to c. Let $\overline{M - A}$ and $\overline{M_\mathbb{Z} - A_\mathbb{Z}}$ be the metric completions of $M - A$ and $M_\mathbb{Z} - A_\mathbb{Z}$, respectively.

The boundaries of both $\overline{M - A}$ and $\overline{M_\mathbb{Z} - A_\mathbb{Z}}$ are isometric to two copies of A glued at $c \times \{0\}$. Orient A by choosing a normal for A in M. We then distinguish between the two copies of A in the boundary of $\overline{M - A}$ by labeling A^+ the copy of A where the normal points outward and A^- the copy of A where the normal points inward. Similarly label the two copies of A in the boundary of $\overline{M_\mathbb{Z} - A_\mathbb{Z}}$, $A_\mathbb{Z}^+$ and $A_\mathbb{Z}^-$. All four of these annuli are isometric to A and we use this isometry to define an equivalence relation between points on A^+ and $A_\mathbb{Z}^-$ and between A^- and $A_\mathbb{Z}^+$. Namely, if $p_1 \in A^+$ and $p_2 \in A_\mathbb{Z}^-$ then $p_1 \sim p_2$ if they are mapped to the same point by the isometry to A. Similarly define an equivalence relation for points in A^- and $A_\mathbb{Z}^+$. Then

$$M_c = (\overline{M - A} \cup \overline{M_\mathbb{Z} - A_\mathbb{Z}})/ \sim .$$

The hyperbolic structures on $M - A$ and on $M_\mathbb{Z} - A_\mathbb{Z}$ will extend to a smooth hyperbolic structure in M_c except at $c \times \{0\}$. At $c \times \{0\}$ the metric has a cone singularity of cone angle 4π. Furthermore M_c is homeomorphic to $S \times [-1, 1)$ with $S \times \{-1\}$ the projective structure Σ. Our goal for the remainder of this section is to show that M_c is a quasi-fuchsian cone-manifold. That is, we will show that M_c extends to the projective structure Σ_c on $S \times \{1\}$.

As in the previous section we assume for simplicity that c is nonseparating. The general case is the same with more notation. Let $B = c \times [-1, 0]$ be an annulus in M and let $\tilde{B}_K = \tilde{c}_K \times [-1, 0]$ be the components of the pre-image of B that bound $\tilde{K} \times [-1, 0]$ in $\tilde{M}$. Let $\tilde{L} = (\tilde{K} \times \{0\}) \cup \tilde{B}_K$.

Let $H = (\mathbb{H}^3 \cup \Omega_K)/\Gamma_K$. Since Γ_K restricts to an action on $\tilde{L}$, the quotient $L = \tilde{L}/\Gamma_K$ is a surface in H.

LEMMA 4.1. *H is a genus $2g - 1$ handlebody with boundary. Furthermore, there is an orientation reversing involution $\phi : H \longrightarrow H$ with $\phi|_L \equiv$ id which lifts to an orientation reversing, Γ_K-equivariant involution $\tilde{\phi} : (\mathbb{H}^3 \cup \Omega_K) \longrightarrow (\mathbb{H}^3 \cup \Omega_K)$ with $\tilde{\phi}|_{\tilde{L}} \equiv$ id.*

Proof. The interior of H is a genus $2g - 1$ handlebody since Γ_K is a free group on $2g - 1$ generators and $\operatorname{int} H$ covers M which is homeomorphic to the product $S \times (-1, 1)$. The covering map $\operatorname{int} H \longrightarrow M$ is infinite-to-one so on the single end of H it is infinite-to-one. By the covering theorem ([Can]), either Γ_K is geometrically finite or M is covered by a finite volume manifold that fibers over the circle. Since M has infinite volume Γ_K must be geometrically finite. Furthermore, Γ_K does not contain parabolics. A geometrically finite Kleinian group without parabolics is convex co-compact and a convex co-compact Kleinian group that is also free is a Schottky group. Therefore Γ_K is a Schottky group with $2g - 1$ generators and $H = (\mathbb{H}^3 \cup \Omega_K)/\Gamma_K$ is a genus $2g - 1$ handlebody with boundary.

The inclusion of L in H is a homotopy equivalence. Therefore H is homeomorphic to $S' \times [-1, 1]$ where S' is a genus $g - 1$ surface with two boundary components and $S' \times \{0\} = L$. This product structure defines an obvious involution of H which lifts to the universal cover to obtain the desired involution $\tilde{\phi}$ of $\mathbb{H}^3 \cup \Omega_K$. $\boxed{4.1}$

Remark. Note that although the handlebody H covers M the product structure we have chosen for H is not equivariant and does not descend to the product structure on M.

THEOREM 4.2. *The hyperbolic cone-manifold M_c is quasi-fuchsian with projective boundary Σ and Σ_c.*

Proof. To prove the theorem we make an alternative construction of M_c.

We begin with an observation about the surface S. Let S' be the cover of S corresponding to $\pi_1(S - c)$. As we have already noted, in S', c has two homeomorphic lifts c_1 and c_2. Next we divide S' into three subsurfaces S_0, S_1 and S_2 with S_0 a compact genus $g - 1$ surface with two boundary components and S_1 and S_2 both homeomorphic to the annulus $S^1 \times [0, 1)$. We also assume that $S_0 \cap S_1 = c_1$ and $S_0 \cap S_2 = c_2$. Note that the covering map $\pi : S' \longrightarrow S$ defines an equivalence relation on points $p_1 \in c_1$ and $p_2 \in c_2$ by $p_1 \sim p_2$ if $\pi(p_1) = \pi(p_2)$. Then π restricts to a homeomorphism from the quotient $S_0/\sim$ to S. On the quotient $(S_1 \cup S_2)/\sim$, π becomes the covering map for the $\mathbb{Z}$-cover of S associated to c. (See Figure 2.)

We now repeat the above construction with the product $M = S \times (-1, 1)$. We again have a cover $\pi : S' \times (-1, 1) \longrightarrow M$ where the product structure $S' \times (-1, 1)$ is the pre-image of the product structure on M. Let $X_i = S_i \times (-1, 1)$ be submanifolds of $S' \times (-1, 1)$. As above we define an equivalence relation for points in $p_1 \in c_1 \times (-1, 1)$ and $p_2 \in c_2 \times (-1, 1)$ by setting $p_1 \sim p_2$ if $\pi(p_1) = \pi(p_2)$. Then $X_0/\sim$ is homeomorphic and isometric to M while $(X_1 \cup X_2)/\sim$ is the $\mathbb{Z}$-cover $M_\mathbb{Z}$ of M associated to $c \times \{0\}$. (See Figure 3.)
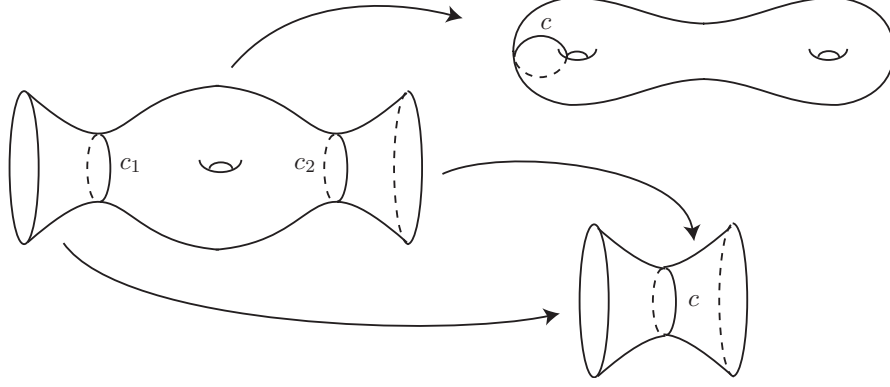
Figure 2: If we cut S' along c_1 and c_2 we have three pieces which can be reglued to form the original surface S and the cover of S associated to c.
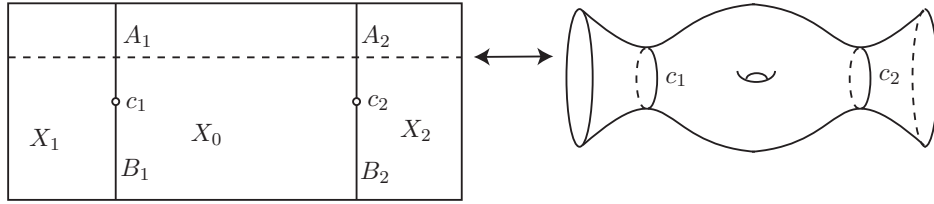

Figure 3: The rectangle gives a schematic picture of the product structure on H. The horizontal lines represent the cover S' of S

To construct M_c we subdivide the annuli that bound the X_i. The boundary of X_1 is the annulus $c_1 \times (-1, 1)$. Let $A_1^+ = c_1 \times [0, 1)$ and $B_1^+ = c_1 \times (-1, 0]$. Similarly divide the boundary of X_2 into two annuli A_2^- and B_2^-. We also divide each of the two annuli that bound X_0 into two sub-annuli A_1^-, B_1^-, A_2^+ and B_2^+. To construct $\overline{M - A}$ we start with X_0 and glue B_1^- to B_2^+. To construct $\overline{M_{\mathbb{Z}} - A_{\mathbb{Z}}}$ we glue X_1 to X_2 by attaching B_1^+ to B_2^-. Finally, to construct M_c we glue the A annuli together. Namely we glue A_1^+ to A_1^- and A_2^+ to A_2^-.

Of course this is simply restating our original construction of M_c. As an alternative we first glue the A annuli and then glue the B annuli. In both cases we use the same gluing pattern and so we get the same hyperbolic structure M_c. To see the advantage of gluing in this order we recall that the cover $S' \times (-1, 1)$ of M is the interior of the handlebody H. The boundary of H is the projective structure Σ_K. The annulus B lifts to two annuli B_1 and B_2 in H which extend to closed curves c_1 and c_2 on Σ_K. Next we note that when we glue X_1 and X_2 to X_0 along the A annuli we get the metric completion of $H - (B_1 \cup B_2)$. This compact manifold has boundary consisting of the B annuli and the projective structures Σ_K^+ and Σ_K^-. When we glue the B

annuli the two boundary curves of Σ_K^+ are identified to form the projective
structure Σ_c. Similarly the boundary curves of Σ_K^- are identified to form the
original projective structure Σ. Therefore M_c is compactified by its projective
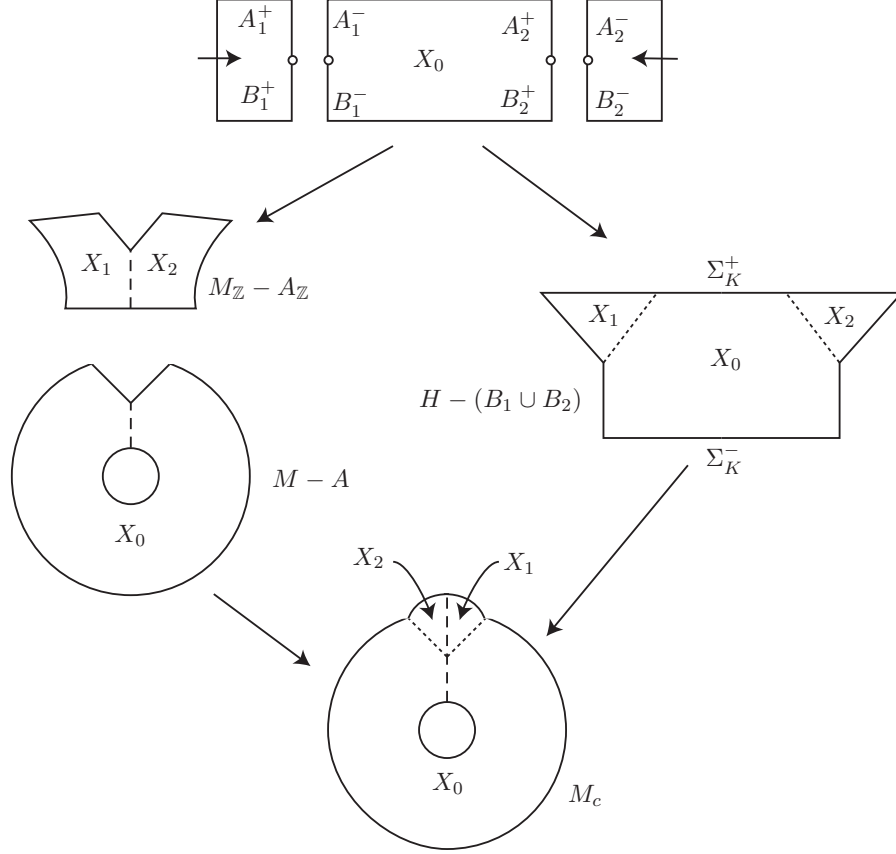boundary and is a quasi-fuchsian cone-manifold. (See Figure 4.)   4.2

Figure 4: The figure gives a schematic description of the two constructions of
M_c. On the left is the original construction while on the right is the alternative
construction.

5. The Bers conjecture

In the previous section we constructed quasi-fuchsian hyperbolic cone-
manifolds. We now use the deformation theory of hyperbolic cone-manifolds
to show that these cone structures are geometrically close to a smooth quasi-
fuchsian structure. The analytic deformation theory of hyperbolic cone-man-
ifolds was developed by Hodgson and Kerckhoff in a series of papers ([HK1],

[HK2], [HK3]) and extended to the geometrically finite setting in [Br2], [Br1]. The basic idea is that if the cone singularity is short and has a large tube radius then there is a one-parameter family of cone-manifolds decreasing the cone angle from 4π to a cone-manifold with cone angle 2π. When the cone angle is 2π the hyperbolic structure is nonsingular.

Although the theory applies in greater generality, we will confine ourselves to quasi-fuchsian cone-manifolds. The following result is essentially Theorems 1.2 and 1.3 of [Br1].

THEOREM 5.1. *Suppose M_α is a quasi-fuchsian cone manifold with cone singularity c, cone angle α and conformal boundary X and Y. Also assume the tube radius of c is greater than $\sinh^{-1}\sqrt{2}$. Then:*

1. *There exists an $\ell_0 > 0$ depending only on α such that for all $t \leq \alpha$ there exists a quasi-fuchsian cone-manifold M_t with cone singularity c, cone angle t and conformal boundary X and Y.*

2. *Furthermore if Σ_α and Σ_t are the projective boundaries corresponding to X for M_α and M_t, respectively, there exists a K depending only on α, $\|\Sigma_\alpha\|_\infty$ and the injectivity radius of the hyperbolic metric on X such that*

$$d(\Sigma_\alpha, \Sigma_t) \leq K \operatorname{length}(c)$$

where the length is measured in the M_α-metric.

We can now prove our main theorem:

THEOREM 5.2. *Assume that $\Gamma \in AH(S)$ has no parabolics. If $M = \mathbb{H}^3/\Gamma$ is singly degenerate and has unbounded geometry then $\Gamma \in \overline{B}_X$ where X is the conformal boundary of M.*

Proof. By the Margulis lemma there exists an ℓ_1 such that if c is a closed geodesic in M with $\operatorname{length}(c) < \ell_1$ then c has an embedded tubular neighborhood of radius $\sinh^{-1}\sqrt{2}$. We need the following theorem of Otal:

THEOREM 5.3 (Otal [Ot]). *Let c be a simple closed geodesic in M. There exists an $\ell_2 > 0$ such that if $\operatorname{length}(c) < \ell_2$ then c is isotopic to a simple closed curve on $S \times \{0\}$ in M.*

Let $\ell = \min(\ell_0, \ell_1, \ell_2)$ where ℓ_0 is the constant from Theorem 5.1. Since M has unbounded geometry there are a sequence of closed geodesics c_i in M with $\operatorname{length}(c_i) \to 0$. Therefore we can assume that $\operatorname{length}(c_i) < \ell$ for all i. We can then apply Theorem 4.2 to construct a sequence of cone-manifolds M_i with cone-singularity c_i and cone-angle 4π. Furthermore, an embedded tubular neighborhood of c_i in M will lift to an embedded tubular neighborhood

of c_i in M_i of the same radius. Therefore c_i will have an embedded tubular neighborhood of radius $\sinh^{-1}\sqrt{2}$ in M_i.

We can now apply Theorem 5.1 to the M_i. If X and Y_i are the components of conformal boundary of M_i let M_i' be the quasi-fuchsian cone manifold with cone singularity c_i, cone angle 2π and conformal boundary X and Y_i given by (a) of Theorem 5.1. Since the cone angle is 2π the hyperbolic structure on M_i' will be smooth so there will be a unique Kleinian group $\Gamma_i \in B_X$ such that $M_i' = \mathbb{H}^3/\Gamma_i$. Note that for each M_i the component of the projective boundary associated to X will be Σ, the projective boundary of the original hyperbolic structure M. Let Σ_i be the component of the projective boundary of M_i' associated to X. By Theorem 5.1,

$$d(\Sigma, \Sigma_i) \leq K \operatorname{length}(c_i).$$

Therefore we have $\Sigma_i \to \Sigma$ in $P(X)$ which implies that $\Gamma_i \to \Gamma$ in $AH(S)$. Since each Γ_i is contained in B_X, we conclude $\Gamma \in \overline{B}_X$. $\boxed{5.4}$

Combining Theorem 5.2 with Theorem 2.3 we have:

THEOREM 5.4. *Assume that $\Gamma \in AH(S)$ has no parabolics. If $M = \mathbb{H}^3/\Gamma$ is singly degenerate then $\Gamma \in \overline{B}_X$ where X is the conformal boundary of M.*

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