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Proof of the Lovász conjecture

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Abstract

To any two graphs G and H one can associate a cell complex $\mathtt{Hom}\,(G,H)$ by taking all graph multihomomorphisms from G to H as cells.

In this paper we prove the Lovász conjecture which states that

$$\textit{if } \mathtt{Hom}\,(C_{2r+1},G) \textit{ is } k\textit{-connected, then } \chi(G)\geq k+4,$$

where $r,k\in\mathbb{Z}$, $r\geq 1$, $k\geq -1$, and C_{2r+1} denotes the cycle with $2r+1$ vertices.

The proof requires analysis of the complexes $\mathtt{Hom}\,(C_{2r+1},K_n)$. For even n, the obstructions to graph colorings are provided by the presence of torsion in $H^*(\mathtt{Hom}\,(C_{2r+1},K_n);\mathbb{Z})$. For odd n, the obstructions are expressed as vanishing of certain powers of Stiefel-Whitney characteristic classes of $\mathtt{Hom}\,(C_{2r+1},K_n)$, where the latter are viewed as $\mathbb{Z}_2$-spaces with the involution induced by the reflection of C_{2r+1}.

1. Introduction

The main idea of this paper is to look for obstructions to graph colorings in the following indirect way: take a graph, associate to it a topological space, and then look for obstructions to colorings of the graph by studying the algebraic invariants of this space.

The construction of such a space, which is of interest here, has been suggested by L. Lovász. The obtained complex $\mathtt{Hom}\,(G,H)$ depends on two graph parameters. The algebraic invariants of this space, which we proceed to study, are its cohomology groups, and, when it can be viewed as a $\mathbb{Z}_2$-space, its Stiefel-Whitney characteristic classes.

1.1. *The vertex colorings and the category of graphs.* All graphs in this paper are undirected. The following definition is a key in turning the set of all undirected graphs into a category.

Definition 1.1. For two graphs G and H, a *graph homomorphism* from G to H is a map $\phi: V(G)\to V(H)$, such that if $(x,y)\in E(G)$, then $(\phi(x),\phi(y))\in E(H)$.

Here, $V(G)$ denotes the set of vertices of G, and $E(G)$ denotes the set of its edges.

For a graph G the *vertex coloring* is an assignment of colors to vertices such that no two vertices which are connected by an edge get the same color. The minimal needed number of colors is denoted by $\chi(G)$, and is called the *chromatic number* of G.

Deciding whether or not there exists a graph homomorphism between two graphs is in general at least as difficult as bounding the chromatic numbers of graphs because of the following observation: a vertex coloring of G with n colors is the same as a graph homomorphism from G to the complete graph on n vertices K_n. Because of this, one can also think of graph homomorphisms from G to H as vertex colorings of G with colors from $V(H)$ subject to the natural condition.

Since an identity map is a graph homomorphism, and a composition of two graph homomorphisms is again a graph homomorphism, we can consider the category *Graphs* whose objects are all undirected graphs, and morphisms are all the graph homomorphisms.

We denote the set of all graph homomorphisms from G to H by $\mathtt{Hom}_0(G,H)$. Lovász has suggested the following way of turning this set into a topological space.

Definition 1.2. We define $\mathtt{Hom}\,(G,H)$ to be a *polyhedral complex* whose cells are indexed by all functions $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that if $(x,y) \in E(G)$, for any $\tilde{x} \in \eta(x)$ and $\tilde{y} \in \eta(y)$ we have $(\tilde{x},\tilde{y}) \in E(H)$.

The *closure of a cell* η consists of all cells indexed by $\tilde{\eta} : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, which satisfy $\tilde{\eta}(v) \subseteq \eta(v)$, for all $v \in V(G)$.

We think of a cell in $\mathtt{Hom}\,(G,H)$ as a collection of nonempty lists of vertices of H, one for each vertex of G, with the condition that any choice of one vertex from each list will yield a graph homomorphism from G to H. A geometric realization of $\mathtt{Hom}\,(G,H)$ can be described as follows: number the vertices of G with $1,\dots,|V(G)|$, the cell indexed with $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ is realized as a direct product of simplices $\Delta^1,\dots,\Delta^{|V(G)|}$, where Δ^i has $|\eta(i)|$ vertices and is realized as the standard simplex in $\mathbb{R}^{|\eta(i)|}$. In particular, the set of vertices of $\mathtt{Hom}\,(G,H)$ is precisely $\mathtt{Hom}_0(G,H)$.

The barycentric subdivision of $\mathtt{Hom}\,(G,H)$ is isomorphic as a simplicial complex to the geometric realization of its face poset. So, alternatively, it could be described by first defining a poset of all η satisfying conditions of Definition 1.2, with $\eta \geq \tilde{\eta}$ if and only if $\eta(v) \supseteq \tilde{\eta}(v)$, for all $v \in V(G)$, and then taking the geometric realization.

The $\mathtt{Hom}$ complexes are functorial in the following sense: $\mathtt{Hom}\,(H,-)$ is a covariant, while $\mathtt{Hom}\,(-,H)$ is a contravariant functor from *Graphs* to *Top*.

If $\phi \in \mathtt{Hom}_0(G, G')$, then we shall denote the induced cellular maps as ϕ^H : $\mathtt{Hom}\,(H, G) \to \mathtt{Hom}\,(H, G')$ and $\phi_H : \mathtt{Hom}\,(G', H) \to \mathtt{Hom}\,(G, H)$.

1.2. *The statement of the Lovász conjecture.* Lovász has stated the following conjecture, which we prove in this paper.

THEOREM 1.3 (Lovász conjecture). *Let G be a graph, such that $\mathtt{Hom}\,(C_{2r+1}, G)$ is k-connected for some $r, k \in \mathbb{Z}$, $r \geq 1$, $k \geq -1$; then $\chi(G) \geq k + 4$.*

Here C_{2r+1} is a cycle with $2r+1$ vertices: $V(C_{2r+1}) = \mathbb{Z}_{2r+1}$, $E(C_{2r+1}) = \{(x, x+1), (x+1, x) \,|\, x \in \mathbb{Z}_{2r+1}\}$.

The motivation for this conjecture stems from the following theorem which Lovász proved in 1978.

THEOREM 1.4 (Lovász, [16]). *Let H be a graph, such that $\mathtt{Hom}\,(K_2, H)$ is k-connected for some $k \in \mathbb{Z}$, $k \geq -1$; then $\chi(H) \geq k + 3$.*

One corollary of Theorem 1.4 is the Kneser conjecture from 1955; see [8].

Remark 1.5. The actual theorem from [16] is stated using the neighborhood complexes $\mathcal{N}(H)$. However, it is well known that $\mathcal{N}(H)$ is homotopy equivalent to $\mathtt{Hom}\,(K_2, H)$ for any graph H; see, e.g., [2] for an argument. In fact, these two spaces are known to be simple-homotopy equivalent; see [14].

We note here that Theorem 1.3 is trivially true for $k = -1$: $\mathtt{Hom}\,(C_{2r+1}, G)$ is (-1)-connected if and only if it is nonempty, and since there are no homomorphisms from odd cycles to bipartite graphs, we conclude that $\chi(G) \geq 3$. It is also not difficult to show that Theorem 1.3 holds for $k = 0$ by using the winding number. A short argument for a more general statement can be found in subsection 2.2.

1.3. *Plan of the paper.* In Section 2, we formulate the main theorems and describe the general framework of finding obstructions to graph colorings via vanishing of powers of Stiefel-Whitney characteristic classes.

In Section 3, we introduce auxiliary simplicial complexes, which we call $\mathtt{Hom}_+(-, -)$. For any two graphs G and H, there is a canonical support map $\mathrm{supp} : \mathtt{Hom}_+(G, H) \to \Delta_{|V(G)|-1}$, and the preimage of the barycenter is precisely $\mathtt{Hom}\,(G, H)$. This allows us to set up a useful spectral sequence, filtering by the preimages of the i-skeletons.

In Section 4, we compute the cohomology groups $H^*(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z})$ up to dimension $n - 2$, and we find the $\mathbb{Z}_2$-action on these groups. These computations allow us to prove the Lovász conjecture for the case of odd k, $k \geq 1$.

In Section 5, we study a different spectral sequence, this one converging to $H^*(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2)$. Understanding certain entries and differentials leads to the proof of the Lovász conjecture for the case of even k as well.

The results of this paper were announced in [1], where no complete proofs were given. The reader is referred to [13] for a survey on Hom complexes, which also includes a lot of background material which is omitted in this paper. As the general reference in Combinatorial Algebraic Topology we recommend [10].

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2. The idea of the proof of the Lovász conjecture

2.1. *Group actions on* Hom *complexes and Stiefel-Whitney classes.* Consider an arbitrary CW complex X on which a finite group Γ acts freely. By the general theory of principal Γ-bundles, there exists a Γ-equivariant map $\tilde{w} : X \to \mathbf{E}\Gamma$, and the induced map $w : X/\Gamma \to \mathbf{B}\Gamma = \mathbf{E}\Gamma/\Gamma$ is unique up to homotopy.

Specifying $\Gamma = \mathbb{Z}_2$, we get a map $\tilde{w} : X \to S^\infty = \mathbf{E}\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on S^∞ by the antipodal map, and the induced map $w : X/\mathbb{Z}_2 \to \mathbb{RP}^\infty = \mathbf{B}\mathbb{Z}_2$. We denote the induced $\mathbb{Z}_2$-algebra homomorphism $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$ by w^*. Let z denote the nontrivial cohomology class in $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Then $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \simeq \mathbb{Z}_2[z]$ as a graded $\mathbb{Z}_2$-algebra, with z having degree 1. We denote the image $w^*(z) \in H^1(X/\mathbb{Z}_2; \mathbb{Z}_2)$ by $\varpi_1(X)$. This is the *first Stiefel-Whitney class* of the $\mathbb{Z}_2$-space X. Clearly, $\varpi_1^k(X) = w^*(z^k)$, since w^* is a $\mathbb{Z}_2$-algebra homomorphism. We will be mainly interested in the *height* of the Stiefel-Whitney class, i.e., largest k, such that $\varpi_1^k(X) \neq 0$; it was called cohomology co-index in [3].

Turning to graphs, let G be a graph with $\mathbb{Z}_2$-action given by $\phi : G \to G$, $\phi \in \mathtt{Hom}_0(G, G)$, such that ϕ flips an edge, that is, there exist $a, b \in V(G)$, $a \neq b$, $(a, b) \in E(G)$, such that $\phi(a) = b$ (which implies $\phi(b) = a$). For any graph H we have the induced $\mathbb{Z}_2$-action $\phi_H : \mathtt{Hom}\,(G, H) \to \mathtt{Hom}\,(G, H)$. In case H has no loops, it follows from the fact that ϕ flips an edge that this $\mathbb{Z}_2$-action is free.

Indeed, since ϕ_H is a cellular map, if it fixes a point from some cell $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, then it maps η onto itself. By definition, ϕ maps η to $\eta \circ \phi$, and so this means that $\eta = \eta \circ \phi$. In particular, $\eta(a) = \eta \circ \phi(a) = \eta(b)$. Since $\eta(a) \neq \emptyset$, we can take $v \in V(H)$, such that $v \in \eta(a)$. Now, $(a, b) \in E(G)$, but $(v, v) \notin E(H)$, since H has no loops, which contradicts the fact that $\eta \in \mathtt{Hom}\,(G, H)$.

Therefore, in this situation, $\mathtt{Hom}\,(G, -)$ is a covariant functor from the induced subcategory of *Graphs*, consisting of all loopfree graphs, to $\mathbb{Z}_2$*-spaces* (the category whose objects are $\mathbb{Z}_2$-spaces and morphisms are $\mathbb{Z}_2$-maps).

We order $V(C_{2r+1})$ by identifying it with $[1, 2r+1]$ by the map $q : \mathbb{Z} \to \mathbb{Z}_{2r+1}$, taking $x \mapsto [x]_{2r+1}$. With this notation $\mathbb{Z}_2$ acts on C_{2r+1} by mapping $[x]_{2r+1}$ to $[-x]_{2r+1}$, for $x \in V(C_{2r+1})$. Let $\gamma \in \mathtt{Hom}_0(C_{2r+1}, C_{2r+1})$ denote the corresponding graph homomorphism. This action has a fixed point $2r+1$, and it flips one edge $(r, r+1)$.

Furthermore, let $\mathbb{Z}_2$ act on K_m for $m \geq 2$, by swapping the vertices 1 and 2 and fixing the vertices $3, \dots, m$; here, K_m is the graph defined by $V(K_m) = [1, m]$, $E(K_m) = \{(x, y) \,|\, x, y \in V(K_m), x \neq y\}$. Since in both cases the graph homomorphism flips an edge, they induce free $\mathbb{Z}_2$-actions on $\mathtt{Hom}\,(C_{2r+1}, G)$ and $\mathtt{Hom}\,(K_m, G)$, for an arbitrary graph G without loops.

2.2. *Nonvanishing of powers of Stiefel-Whitney classes as obstructions to graph colorings.* The connection between the nonnullity of the powers of Stiefel-Whitney characteristic classes and the lower bounds for graph colorings is provided by the following general observation.

THEOREM 2.1. *Let G be a graph without loops, and let T be a graph with $\mathbb{Z}_2$-action which flips some edge in T. If, for some integers $k \geq 0$, $m \geq 1$, we have $\varpi_1^k(\mathtt{Hom}\,(T, G)) \neq 0$, and $\varpi_1^k(\mathtt{Hom}\,(T, K_m)) = 0$, then $\chi(G) \geq m+1$.*

Proof. We have already shown that, under the assumptions of the theorem, $\mathtt{Hom}\,(T, H)$ is a $\mathbb{Z}_2$-space for any loopfree graph H. Assume now that the graph G is m-colorable, i.e., there exists a homomorphism $\phi : G \to K_m$. It induces a $\mathbb{Z}_2$-map $\phi^T : \mathtt{Hom}\,(T, G) \to \mathtt{Hom}\,(T, K_m)$. Since the Stiefel-Whitney classes are functorial and $\varpi_1^k(\mathtt{Hom}\,(T, K_m)) = 0$, the existence of the $\mathbb{Z}_2$-map ϕ^T implies that $\varpi_1^k(\mathtt{Hom}\,(T, G)) = 0$, which is a contradiction to the assumption of the theorem. □

LEMMA 2.2. *If a $\mathbb{Z}_2$-space X is k-connected, then there exists a $\mathbb{Z}_2$-map $\phi : S_a^{k+1} \to X$; in particular, $\varpi_1^{k+1}(X) \neq 0$.*

Proof. To construct ϕ, subdivide S_a^{k+1} simplicially as a join of $k+2$ copies of S^0, and then define ϕ on the join of the first i factors, starting with $i = 1$, and increasing i by 1 at the time. To define ϕ on the first factor $\{a, b\}$, simply map a to an arbitrary point $x \in X$, and then map b to $\gamma(x)$, where γ is the free involution of X. Assume ϕ is defined on Y - the join of the first i factors. Extend ϕ to $Y * \{a, b\}$ by extending it first to $Y * \{a\}$, which we can do, since X is k-connected, and then extending ϕ to the second hemisphere $Y * \{b\}$, by applying the involution γ.

Since the Stiefel-Whitney classes are functorial, we have $\phi^*(\varpi_1^{k+1}(X)) = \varpi_1^{k+1}(S_a^{k+1})$, and the latter is clearly nontrivial. □

Let T be any graph and consider the following equation

$$\varpi_1^{n-\chi(T)+1}(\mathtt{Hom}\,(T, K_n)) = 0, \text{ for all } n \geq \chi(T) - 1. \tag{2.1}$$

THEOREM 2.3.

(a) *The equation* (2.1) *is true for* $T = K_m$, $m \geq 2$.

(b) *The equation* (2.1) *is true for* $T = C_{2r+1}$, $r \geq 1$, *and odd* n.

Proof. The case $T = K_m$ is from [2, Th. 1.6] and has been proved there. The case $T = C_{2r+1}$ will be proved in Section 6. $\square$

LEMMA 2.4. *For a fixed value of* n, *if equation* (2.1) *is true for* $T = C_{2r+1}$, *then it is true for any* $T = C_{2\tilde{r}+1}$, *if* $r \geq \tilde{r}$.

Proof. If $r \geq \tilde{r}$, there exists a graph homomorphism $\phi : C_{2r+1} \to C_{2\tilde{r}+1}$ which respects the $\mathbb{Z}_2$-action. This induces a $\mathbb{Z}_2$-map

$$\phi_{K_n} : H^*(\texttt{Hom}\,(C_{2r+1}, K_n)) \to H^*(\texttt{Hom}\,(C_{2\tilde{r}+1}, K_n)),$$

yielding

$$\tilde{\phi}_{K_n} : H^*(\texttt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \to H^*(\texttt{Hom}\,(C_{2\tilde{r}+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2).$$

Clearly, $\tilde{\phi}_{K_n}(\varpi_1(\texttt{Hom}\,(C_{2r+1}, K_n))) = \varpi_1(\texttt{Hom}\,(C_{2\tilde{r}+1}, K_n))$. In particular, $\varpi_1^i(\texttt{Hom}\,(C_{2r+1}, K_n)) = 0$, implies $\varpi_1^i(\texttt{Hom}\,(C_{2\tilde{r}+1}, K_n)) = 0$. $\square$

Note that for $T = C_{2r+1}$ and $n = 2$, the equation (2.1) is obvious, since $\texttt{Hom}\,(C_{2r+1}, K_2) = \emptyset$. We give a quick argument for the next case $n = 3$. One can see by inspection that the connected components of $\texttt{Hom}\,(C_{2r+1}, K_3)$ can be indexed by the winding numbers α. These numbers must be odd, so that $\alpha = \pm 1, \pm 3, \ldots, \pm(2s+1)$, where

$$s = \begin{cases} (r-1)/3, & \text{if } r \equiv 1 \mod 3, \\ \lfloor (r-2)/3 \rfloor, & \text{otherwise;} \end{cases}$$

in particular $s \geq 0$. Let $\phi : \texttt{Hom}\,(C_{2r+1}, K_3) \to \{\pm 1, \pm 3, \ldots, \pm(2s+1)\}$ map each point $x \in \texttt{Hom}\,(C_{2r+1}, K_3)$ to the point on the real line, indexing the connected component of x. Clearly, ϕ is a $\mathbb{Z}_2$-map. Since

$$H^1(\{\pm 1, \pm 3, \ldots, \pm(2s+1)\}/\mathbb{Z}_2; \mathbb{Z}_2) = 0,$$

the functoriality of the characteristic classes implies $\varpi_1(\texttt{Hom}\,(C_{2r+1}, K_3)) = 0$.

CONJECTURE 2.5. *Equation* (2.1) *is true for* $T = C_{2r+1}$, $r \geq 1$, *and all* n.

2.3. *Completion of the sketch of the proof of the Lovász conjecture.* Consider one of the two maps $\iota : K_2 \to C_{2r+1}$ mapping the edge to the $\mathbb{Z}_2$-invariant edge of C_{2r+1}. Clearly, ι is $\mathbb{Z}_2$-equivariant. Since $\texttt{Hom}\,(-, H)$ is a contravariant functor, ι induces a map of $\mathbb{Z}_2$-spaces $\iota_{K_n} : \texttt{Hom}\,(C_{2r+1}, K_n) \to \texttt{Hom}\,(K_2, K_n)$, which in turn induces a $\mathbb{Z}$-algebra homomorphism $\iota_{K_n}^* : H^*(\texttt{Hom}\,(K_2, K_n); \mathbb{Z}) \to H^*(\texttt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z})$.

THEOREM 2.6. *Assume n is even; then $2 \cdot \iota_{K_n}^*$ is a 0-map.*

Theorem 2.6 is proved in Section 4. The results of this paper were announced in [1], and the preprint of this paper has been available since February 2004. In the summer 2005 an alternative proof of Theorem 2.6 appeared in the preprint [19], and a proof of Conjecture 2.5 was announced by C. Schultz.

Proof of Theorem 1.3 (Lovász conjecture).The case $k = -1$ is trivial, so take $k \geq 0$. Assume first that k is even. By the Remark 2.2, we have $\varpi_1^{k+1}(\mathtt{Hom}\,(C_{2r+1}, G)) \neq 0$. By Theorem 2.3(b), we have

$$\varpi_1^{k+1}(\mathtt{Hom}\,(C_{2r+1}, K_{k+3})) = 0.$$

Hence, applying Theorem 2.1 for $T = C_{2r+1}$ we get $\chi(G) \geq k+4$.

Assume now that k is odd, and that $\chi(G) \leq k+3$. Let $\phi : G \to K_{k+3}$ be a vertex-coloring map. Combining the Remark 2.2, the fact that $\mathtt{Hom}\,(C_{2r+1}, -)$ is a covariant functor from loopfree graphs to $\mathbb{Z}_2$-spaces, and the map $\iota : K_2 \to C_{2r+1}$, we get the following diagram of $\mathbb{Z}_2$-spaces and $\mathbb{Z}_2$-maps:

$$\begin{aligned} S_a^{k+1} \xrightarrow{f} \mathtt{Hom}\,(C_{2r+1}, G) &\xrightarrow{\phi^{C_{2r+1}}} \mathtt{Hom}\,(C_{2r+1}, K_{k+3}) \\ &\xrightarrow{\iota_{K_{k+3}}} \mathtt{Hom}\,(K_2, K_{k+3}) \cong S_a^{k+1}. \end{aligned}$$

This gives a homomorphism on the corresponding cohomology groups in dimension $k+1$, $h^* = f^* \circ (\phi^{C_{2r+1}})^* \circ (\iota_{K_{k+3}})^* : \mathbb{Z} \to \mathbb{Z}$. It is well-known, see, e.g., [7, Prop. 2B.6, p. 174], that a $\mathbb{Z}_2$-map $S_a^n \to S_a^n$ cannot induce a 0-map on the nth cohomology groups (in fact it must be of odd degree). Hence, we have a contradiction, and so $\chi(G) \geq k+4$. □

Let us make a couple of remarks.

Remark 2.7. As is apparent from our argument, we are actually proving a sharper statement than the original Lovász conjecture. First of all, the condition "$\mathtt{Hom}\,(C_{2r+1}, G)$ is k-connected" can be replaced by a weaker condition "the coindex of $\mathtt{Hom}\,(C_{2r+1}, G)$ is at least $k+1$". Furthermore, for even k, that condition can be weakened even further to "$\varpi_1^{k+1}(\mathtt{Hom}\,(C_{2r+1}, G)) \neq 0$". Conjecture 2.5 would imply that this weakening can be done for odd k as well.

Remark 2.8. It follows from [2, Prop. 5.1] that the Lovász conjecture is true if C_{2r+1} is replaced by any graph T, such that T can be reduced to C_{2r+1}, by a sequence of folds.

3. $\mathtt{Hom}_+$ and filtrations

3.1. *The + construction.* For a finite graph H, let H_+ be the graph obtained from H by adding an extra vertex b, called the base vertex, and connecting it by edges to all the vertices of H_+ including itself, i.e., $V(H_+) = V(H) \cup \{b\}$, and $E(H_+) = E(H) \cup \{(v,b),(b,v) \,|\, v \in V(H_+)\}$.

Definition 3.1. Let G and H be two graphs. The *simplicial complex* $\mathtt{Hom}_+(G,H)$ is defined to be the link in $\mathtt{Hom}\,(G,H_+)$ of the homomorphism mapping every vertex of G to the base vertex in H_+.

So the cells in $\mathtt{Hom}_+(G,H)$ are indexed by all $\eta : V(G) \to 2^{V(H)}$ satisfying the same condition as in the Definition 1.2. The closure of η is also defined identically to how it was defined for $\mathtt{Hom}$. Note, that $\mathtt{Hom}_+(G,H)$ is simplicial, and that $\mathtt{Hom}_+(G,-)$ is a covariant functor from *Graphs* to *Top*. One can think of $\mathtt{Hom}_+(G,H)$ as a cell structure imposed on the set of all partial homomorphisms from G to H.

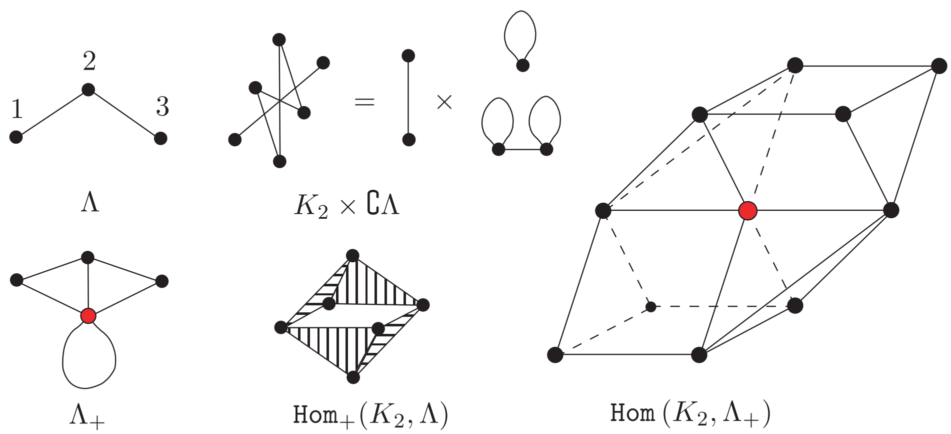

Figure 3.1: The hom plus construction.

For an arbitrary graph G, let $\mathrm{Ind}\,(G)$ denote the independence complex of G, i.e., the vertices of $\mathrm{Ind}\,(G)$ are all vertices of G, and simplices are all the independent sets of G. The dimension of $\mathtt{Hom}_+(G,H)$, unlike that of $\mathtt{Hom}\,(G,H)$ is easy to find:

$$\dim(\mathtt{Hom}_+(G,H)) = |V(H)| \cdot (\dim \mathrm{Ind}\,(G) + 1) - 1.$$

Recall that for any graph G, the strong complement $\complement G$ is defined by $V(\complement G) = V(G)$, $E(\complement G) = V(G) \times V(G) \setminus E(G)$. Also, for any two graphs G and H, the direct product $G \times H$ is defined by $V(G \times H) = V(G) \times V(H)$, $E(G \times H) = \{((x,y),(x',y'))\,|\,(x,x') \in E(G), (y,y') \in E(H)\}$.

Sometimes, it is convenient to view $\mathtt{Hom}_+(G,H)$ as an independent complex of a certain graph.

Proposition 3.2. *The complex* $\mathtt{Hom}_+(G,H)$ *is isomorphic to* $\mathrm{Ind}\,(G \times \complement H)$. *In particular*, $\mathtt{Hom}_+(G,K_n)$ *is isomorphic to* $\mathrm{Ind}\,(G)^{*n}$, *where* $*$ *denotes the simplicial join.*

Proof. By the definition, $V(G \times \complement H) = V(G) \times V(H)$. Let $S \subseteq V(G) \times V(H)$, $S = \{(x_i, y_i) \mid i \in I, x_i \in V(G), y_i \in V(H)\}$. Then $S \in \operatorname{Ind}(G \times \complement H)$ if and only if, for any $i, j \in I$, we have either $(x_i, x_j) \notin E(G)$ or $(y_i, y_j) \in E(H)$, since the forbidden constellation occurs when $(x_i, x_j) \in E(G)$ and $(y_i, y_j) \notin E(H)$.

Identify S with $\eta_S : V(G) \to 2^{V(H)}$ defined by: for $v \in V(G)$, set $\eta_S(v) := \{w \in V(H) \mid (v, w) \in S\}$. The condition for $\eta_S \in \texttt{Hom}_+(G, H)$ is that, if $(v_1, v_2) \in E(G)$, and $w_1 \in \eta_S(v_1)$, $w_2 \in \eta_S(v_2)$, then $(w_1, w_2) \in E(H)$, which is visibly identical to the condition for $S \in \operatorname{Ind}(G \times \complement H)$. Hence $\texttt{Hom}_+(G, H) = \operatorname{Ind}(G \times \complement H)$.

To see the second statement note first that $\complement K_n$ is the disjoint union of n looped vertices. Since taking direct products is distributive with respect to disjoint unions, and a direct product of G with a loop is again G, we see that $G \times \complement K_n$ is a disjoint union of n copies of G. Clearly, its independent complex is precisely the n-fold join of $\operatorname{Ind}(G)$. □

3.2. *Cochain complexes for* $\texttt{Hom}(G, H)$ *and* $\texttt{Hom}_+(G, H)$. For any CW complex K, let $K^{(i)}$ denote the i-th skeleton of K. Let R be a commutative ring with a unit. In this paper we will have two cases: $R = \mathbb{Z}$ and $R = \mathbb{Z}_2$. For any $\eta \in K^{(i)}$, we fix an orientation on η, and let $C_i(K; R) := R[\eta \mid \eta \in K^{(i)}]$, where $R[\alpha \mid \alpha \in I]$ denotes the free R-module generated by $\alpha \in I$. Furthermore, let $C^i(K; R)$ be the dual R-module to $C_i(K; R)$. For arbitrary $\alpha \in C_i(K; R)$ let α^* denote the element of $C^i(K; R)$ which is dual to α. Clearly, $C^i(K; R) = R[\eta^* \mid \eta \in K^{(i)}]$, and the cochain complex of K is

$$\cdots \xrightarrow{\partial^{i-1}} C^i(K; R) \xrightarrow{\partial^i} C^{i+1}(K; R) \xrightarrow{\partial^{i+1}} \cdots .$$

For $\eta \in K^{(i)}$, $\tilde{\eta} \in K^{(i+1)}$, we have the incidence number $[\eta : \tilde{\eta}]$, which is 0 if $\eta \notin \tilde{\eta}$. In this notation $\partial^i(\eta^*) = \sum_{\tilde{\eta} \in K^{(i+1)}} [\eta : \tilde{\eta}]\, \tilde{\eta}^*$. For arbitrary $\alpha \in C_i(K; R)$, resp. $\alpha^* \in C^i(K; R)$, we let $[\alpha]$, resp. $[\alpha^*]$, denote the corresponding element of $H_i(K; R)$, resp. $H^i(K; R)$.

When coming after the name of a cochain complex, the brackets $[-]$ will denote the index shifting (to the left); that is for the cochain complex C^*, the cochain complex $C^*[s]$ is defined by $C^i[s] := C^{i+s}$, and the differential is the same (we choose not to change the sign of the differential).

We now return to our context. Let G and H be two graphs, and let us choose some orders on $V(G) = \{v_1, \dots, v_{|V(G)|}\}$ and on $V(H) = \{w_1, \dots, w_{|V(H)|}\}$. Through the end of this subsection we assume the coefficient ring to be $\mathbb{Z}$; the situation over $\mathbb{Z}_2$ is simpler and can be described by tensoring with $\mathbb{Z}_2$.

Vertices of $\texttt{Hom}_+(G, H)$ are indexed with pairs (x, y), where $x \in V(G)$, $y \in V(H)$, such that if x is looped, then so is y. We order these pairs lexicographically: $(v_{i_1}, w_{j_1}) \prec (v_{i_2}, w_{j_2})$ if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$. Orient each simplex of $\texttt{Hom}_+(G, H)$ according to this order on the vertices. We call this orientation *standard*, and call the oriented simplex η_+. If $\tilde{\eta}_+ \in \mathtt{Hom}_+^{(i+1)}(G,H)$ is obtained from $\eta_+ \in \mathtt{Hom}_+^{(i)}(G,H)$ by adding a vertex v, then $[\eta_+ : \tilde{\eta}_+]$ is $(-1)^{k-1}$, where k is the position of v in the order of the vertices of $\tilde{\eta}_+$.

Let us now turn to $C^*(\mathtt{Hom}\,(G,H))$. We can fix an orientation, which we also call standard, on each cell $\eta \in \mathtt{Hom}\,(G,H)$ as follows: orient each simplex $\eta(i)$ according to the chosen order on the vertices of H; then, order these simplices in the direct product according to the chosen order on the vertices of G. To simplify our notation, we still call this oriented cell η, even though a choice of orders on the vertex sets of G and H is implicit.

We remark for later use, that permuting the vertices of the simplex $\eta(i)$ by some $\sigma \in \mathcal{S}_{|\eta(i)|}$ changes the orientation of the cell η by $\operatorname{sgn}(\sigma)$, whereas swapping the simplices with vertex sets $\eta(i)$ and $\eta(i+1)$ in the direct product changes the orientation by $(-1)^{(|\eta(i)|-1)(|\eta(i+1)|-1)} = (-1)^{\dim \eta(i) \cdot \dim \eta(i+1)}$.

If $\tilde{\eta} \in \mathtt{Hom}^{\,(i+1)}(G,H)$ is obtained from $\eta \in \mathtt{Hom}^{\,(i)}(G,H)$ by adding a vertex v to the list $\eta(t)$, then $[\eta : \tilde{\eta}]$ is $(-1)^{k+d-1}$, where k is the position of v in $\tilde{\eta}(t)$, and d is the dimension of the product of the simplices with the vertex sets $\eta(1), \dots, \eta(t-1)$; that is, $d = 1 - t + \sum_{j=1}^{t-1} |\eta(j)|$. To see this, note that $[\eta : \tilde{\eta}] = 1$ if the first vertex in the first simplex is inserted. The general case follows from the previously described rules for changing the sign of the orientation under permuting simplices in the product and permuting vertices within simplices.

3.3. *The support map and the relation between* $\mathtt{Hom}\,(G,H)$ *and* $\mathtt{Hom}_+(G,H)$. For each simplex of $\mathtt{Hom}_+(G,H)$, $\eta : V(G) \to 2^{V(H)}$, define the support of η to be $\operatorname{supp} \eta := V(G) \setminus \eta^{-1}(\emptyset)$. A concise way to phrase the definition of supp differently is to consider the map $t^G : \mathtt{Hom}_+(G,H) \to \mathtt{Hom}_+(G, \complement K_1) \simeq \Delta_{|V(G)|-1}$ induced by the homomorphism $t : H \to \complement K_1$. Then, for each $\eta \in \mathtt{Hom}_+(G,H)$ we have $\operatorname{supp} \eta = t^G(\eta)$, where the simplices in $\Delta_{|V(G)|-1}$ are identified with the subsets of $V(G)$.

Let $\widetilde{C}^*$ be the subcomplex of $C^*(\mathtt{Hom}_+(G,H))$ generated by all η_+^*, for $\eta : V(G) \to 2^{V(H)}$, such that $\operatorname{supp} \eta = V(G)$ (cf. filtration in subsection 3.5). Set

$$X^*(G,H) := \widetilde{C}^*[|V(G)|-1]. \tag{3.1}$$

Note that both $C^i(\mathtt{Hom}\,(G,H))$ and $X^i(G,H)$ are free $\mathbb{Z}$-modules with the bases $\{\eta^*\}_\eta$ and $\{\eta_+^*\}_\eta$ indexed by $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that $\sum_{j=1}^{|V(G)|} |\eta(j)| = |V(G)| + i$.

At this point we introduce the following notation: for $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, set

$$c(\eta) := \sum_{\substack{i \text{ is even} \\ 1 \le i \le |V(G)|}} |\eta(i)|.$$

For any $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, set $\rho(\eta_+) := (-1)^{c(\eta)}\eta$. Obviously, the induced map $\rho^* : X^i(G,H) \to C^i(\mathtt{Hom}\,(G,H))$ is a $\mathbb{Z}$-module isomorphism for any i.

Proposition 3.3. *The map $\rho^* : X^*(G,H) \to C^*(\mathtt{Hom}\,(G,H))$ is an isomorphism of the cochain complexes.*

Proof. Indeed, let $\tilde{\eta} : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ be obtained from η by adding a vertex v to the list $\eta(t)$, and let k be the position of v in $\tilde{\eta}(t)$. By our previous computation: $[\eta_+ : \tilde{\eta}_+] = (-1)^{k+d+t}$, whereas $[\eta : \tilde{\eta}] = (-1)^{k+d-1}$, where $d = 1 - t + \sum_{j=1}^{t-1} |\eta(j)|$. This shows that

$$[\rho(\eta_+) : \rho(\tilde{\eta}_+)] = (-1)^{c(\eta)+c(\tilde{\eta})}[\eta : \tilde{\eta}] = (-1)^{c(\eta)+c(\tilde{\eta})+t+1}[\eta_+ : \tilde{\eta}_+],$$

but

$$c(\eta) + c(\tilde{\eta}) + t + 1 = \sum_{i \text{ is even}} |\eta(i)| + \sum_{i \text{ is even}} |\tilde{\eta}(i)| + t + 1 \equiv 0 \pmod 2$$

for any t; hence $[\rho(\eta_+) : \rho(\tilde{\eta}_+)] = [\eta_+ : \tilde{\eta}_+]$. □

3.4. *Relating $\mathbb{Z}_2$-actions on $\mathtt{Hom}\,(G,H)$ and $\mathtt{Hom}_+(G,H)$.* Assume that we have $\gamma \in \mathtt{Hom}_0(G,G)$, and $0 \le r \le |V(G)|/2$, such that

$$\gamma(i) = \begin{cases} 2r + 1 - i, & \text{if } 1 \le i \le 2r; \\ i, & \text{if } 2r + 1 \le i \le |V(G)|, \end{cases}$$

where $V(G)$ is identified with the numbering $[1, V(G)]$. In particular, we have $\gamma^2 = 1$.

The homomorphism γ induces $\mathbb{Z}_2$-action both on $\mathtt{Hom}\,(G,H)$, and on $\mathtt{Hom}_+(G,H)$. We shall see how ρ^* behaves with respect to this $\mathbb{Z}_2$-action. For any $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, γ takes η to $\eta \circ \gamma$. By a slight abuse of notation we let γ denote the induced actions both on $C^*(\mathtt{Hom}\,(G,H))$ and on $X^*(G,H)$.

Let $(u_1, \dots, u_q)$ be the vertices of the simplex η_+ listed in increasing order. By definition, $\gamma(\eta_+) = (\gamma(u_1), \dots, \gamma(u_q))$, where $\gamma((v,w)) := (\gamma(v), w)$, for $v \in V(G)$, $w \in V(H)$. Clearly, $\gamma(\eta_+)$ has the same set of vertices as $(\eta \circ \gamma)_+$, so we just need to see how their orientations relate. To order the vertices of $\gamma(\eta_+)$ we need to invert the order of the blocks with cardinalities $|\eta(1)|, \dots, |\eta(2r)|$ without changing the vertex orders within the blocks. The sign of this permutation is $(-1)^c$, where $c = \sum_{1 \le i < j \le 2r} |\eta(i)| \cdot |\eta(j)|$, and so we conclude that

$$\gamma(\eta_+^*) = (-1)^c (\eta \circ \gamma)_+^*. \tag{3.2}$$

Consider now the oriented cell η. It is a direct product of simplices $\Delta^1, \dots, \Delta^{|V(G)|}$ of dimensions $|\eta(1)| - 1, \dots, |\eta(|V(G)|)| - 1$, with the standard orientation as defined above. The cell $\gamma(\eta)$ is the direct product of $\gamma(\Delta^1) =$

$\Delta^{2r}, \gamma(\Delta^2) = \Delta^{2r-1}, \ldots, \gamma(\Delta^{2r}) = \Delta^1, \gamma(\Delta^{2r+1}) = \Delta^{2r+1}, \ldots, \gamma(\Delta^{|V(G)|}) = \Delta^{|V(G)|}$, with the order of the vertices (hence the orientation) within each simplex being the same as in η.

We see that $\gamma(\eta)$ is, up to the orientation, the same cell as $\eta \circ \gamma$. To relate their orientations, we need to permute the simplices $\Delta^{2r}, \ldots, \Delta^1$ back in order, which, by the previous observations, changes the orientation by $(-1)^{\tilde{d}}$, where

$$\begin{aligned}\tilde{d} = \sum_{1 \le i < j \le 2r} \dim \Delta^i \cdot \dim \Delta^j &= \sum_{1 \le i < j \le 2r} (|\eta(i)| - 1)(|\eta(j)| - 1) \\ &= c - (2r-1) \sum_{i=1}^{2r} |\eta(i)| + \binom{2r}{2}.\end{aligned}$$

Reducing modulo 2, we conclude that

$$\gamma(\eta^*) = (-1)^d (\eta \circ \gamma)^*, \tag{3.3}$$

where $d = c + \sum_{i=1}^{2r} |\eta(i)| + r$.

Let us now see how ρ^* interacts with γ. We have

$$\rho^*(\gamma(\eta_+^*)) = (-1)^c \rho^*((\eta \circ \gamma)_+^*) = (-1)^{c + c(\eta \circ \gamma)} (\eta \circ \gamma)^*, \tag{3.4}$$

where the first equality is by (3.2) and the second one is by definition of ρ, and

$$\gamma(\rho^*(\eta_+^*)) = (-1)^{c(\eta)} \gamma(\eta^*) = (-1)^{d + c(\eta)} (\eta \circ \gamma)^*, \tag{3.5}$$

where the first equality is by definition of ρ and second one is by (3.3). Comparing (3.4) with (3.5), and using the computation

$$\begin{aligned}c(\eta) + c(\eta \circ \gamma) &= \sum_{\substack{i \text{ is even} \\ 1 \le i \le |V(G)|}} |\eta(i)| + \sum_{\substack{i \text{ is even} \\ 1 \le i \le |V(G)|}} |\eta \circ \gamma(i)| \\ &= \sum_{i=1}^{2r} |\eta(i)| + 2 \cdot \sum_{\substack{i \text{ is even} \\ 2r+1 \le i \le |V(G)|}} |\eta(i)|,\end{aligned}$$

we see that, for any η

$$\rho^*(\gamma(\eta_+^*)) = (-1)^r \gamma(\rho^*(\eta_+^*)). \tag{3.6}$$

3.5. *The filtration of $C^*(\mathtt{Hom}_+(G,H);\mathbb{Z})$ and the $E_0^{*,*}$-tableau.* We shall now filter $C^*(\mathtt{Hom}_+(G,H);\mathbb{Z})$. Define the subcomplexes of $C^*(\mathtt{Hom}_+(G,H);\mathbb{Z})$, $F^p = F^p C^*(\mathtt{Hom}_+(G,H);\mathbb{Z})$, as follows:

$$F^p : \cdots \xrightarrow{\partial^{q-1}} F^{p,q} \xrightarrow{\partial^q} F^{p,q+1} \xrightarrow{\partial^{q+1}} \ldots,$$

where

$$F^{p,q} = F^p C^q(\mathtt{Hom}_+(G,H);\mathbb{Z}) = \mathbb{Z}\left[\eta_+^* \;\middle|\; \eta_+ \in \mathtt{Hom}_+^{(q)}(G,H), |\operatorname{supp} \eta| \ge p+1\right],$$

and ∂^* is the restriction of the differential in $C^*(\mathtt{Hom}_+(G,H);\mathbb{Z})$. Then,

$$C^q(\mathtt{Hom}_+(G,H);\mathbb{Z}) = F^{0,q} \supseteq F^{1,q} \supseteq \cdots \supseteq F^{|V(G)|-1,q} \supseteq F^{|V(G)|,q} = 0.$$

PROPOSITION 3.4. *For any* p,

$$F^p/F^{p+1} = \bigoplus_{\substack{S\subseteq V(G)\\ |S|=p+1}} C^*(\mathtt{Hom}\,(G[S],H);\mathbb{Z})[-p]. \tag{3.7}$$

Hence, the 0^{th} *tableau of the spectral sequence associated to the cochain complex filtration* F^* *is given by*

$$E_0^{p,q} = C^{p+q}(F^p,F^{p+1}) = \bigoplus_{\substack{S\subseteq V(G)\\ |S|=p+1}} C^q(\mathtt{Hom}\,(G[S],H);\mathbb{Z}). \tag{3.8}$$

Proof. By construction

$$\begin{aligned} F^{p,q}/F^{p+1,q} &= \mathbb{Z}\left[\eta_+^* \,\middle|\, \eta_+\in \mathtt{Hom}_+^{(q)}(G,H), |\operatorname{supp}\eta| = p+1\right] \\ &= \bigoplus_{\substack{S\subseteq V(G)\\ |S|=p+1}} X^{q-p}(G[S],H;\mathbb{Z}) \overset{\rho^*}{=} \bigoplus_{\substack{S\subseteq V(G)\\ |S|=p+1}} C^{q-p}(\mathtt{Hom}\,(G[S],H);\mathbb{Z}), \end{aligned}$$

where X^* is as defined in (3.1), and ρ^* is the map defined in subsection 3.3. □

Note, that in particular we have $F^{|V(G)|-1,q}/F^{|V(G)|,q} = F^{|V(G)|-1,q} = C^q(\mathtt{Hom}\,(G,H);\mathbb{Z})[1-|V(G)|]$.

4. $\mathbb{Z}_2$-action on $H^*(\mathtt{Hom}_+(C_{2r+1},K_n);\mathbb{Z})$

In this section we shall derive some information about the $\mathbb{Z}[\mathbb{Z}_2]$-modules $H^*(\mathtt{Hom}\,(C_{2r+1},K_n);\mathbb{Z})$, for $r\geq 2$, $n\geq 4$. For $r=2$ our computation will be complete.

We adopt the following convention: we think of C_{2r+1} as a unit circle on the plane with $2r+1$ marked points with numbers $1,\dots,2r+1$ following each other in the clockwise increasing order. The directions *left*, resp. *right* on this circle will denote counterclockwise, resp. clockwise.

Furthermore, before we start our computation, we introduce the following terminology. For $S\subset V(C_{2r+1})$, we call those connected components of $C_{2r+1}[S]$ which have at least 2 vertices the *arcs*. For $x,y\in\mathbb{Z}$, we let $[x,y]_{2r+1}$ denote the arc starting from x and going clockwise to y, that is $[x,y]_{2r+1} = \{[x]_{2r+1},[x+1]_{2r+1},\dots,[y-1]_{2r+1},[y]_{2r+1}\}$.

4.1. *The simplicial complex of partial homomorphisms from a cycle to a complete graph.* Here and in the next subsection we summarize some previously published results which are necessary for our present computations. To start with, recall that the homotopy type of the independence complexes of cycles was computed in [9].

PROPOSITION 4.1 ([9, Prop. 5.2]). *For any integer $m \geq 2$,*

$$\operatorname{Ind}(C_m) \simeq \begin{cases} S^{k-1} \vee S^{k-1}, & \textit{if } m = 3k; \\ S^{k-1}, & \textit{if } m = 3k \pm 1. \end{cases}$$

Combining Propositions 3.2 and 4.1 we get the following formula.

COROLLARY 4.2. *For any integers $m \geq 2$, $n \geq 1$,*

$$\texttt{Hom}_+(C_m, K_n) \simeq \begin{cases} \bigvee_{2^n \textit{ copies}} S^{nk-1}, & \textit{if } m = 3k; \\ S^{nk-1}, & \textit{if } m = 3k \pm 1. \end{cases} \tag{4.1}$$

The following estimates will be needed later for our spectral sequence computations.

COROLLARY 4.3. $\widetilde{H}^i(\texttt{Hom}_+(C_{2r+1}, K_n)) = 0$ *for $r \geq 2$, $n \geq 4$, and $i \leq n + 2r - 2$, except for the two cases $(n, r) = (4, 3)$ and $(5, 3)$.*

Proof. Note, that if $2r + 1 = 3k + \varepsilon$, with $\varepsilon \in \{-1, 0, 1\}$, then

$$\widetilde{H}^i(\texttt{Hom}_+(C_{2r+1}, K_n)) = 0, \text{ for } i \leq nk - 2.$$

Assume first $2r + 1 = 3k$. The inequality $nk - 2 \geq n + 2r - 2$ is equivalent to $n \geq 3 + 2/(k - 1)$, and the latter is always true since $k \geq 3$ and $n \geq 4$.

Assume now $2r + 1 = 3k + 1$. This time, $nk - 2 \geq n + 2r - 2$ is equivalent to $n \geq 3 + 3/(k - 1)$. If $k \geq 4$, this is always true, since $n \geq 4$. If $k = 2$, this reduces to saying that $n \geq 6$. This yields the two exceptional cases: $r = 3$ and $n = 4, 5$.

Finally, assume $2r + 1 = 3k - 1$. Here, $nk - 2 \geq n + 2r - 2$ is equivalent to $n \geq 3 + 1/(k - 1)$, which is always true, since $k \geq 2$, $n \geq 4$. □

Corollary 4.2 can be strengthened to include the information on the $\mathbb{Z}_2$-action.

PROPOSITION 4.4. *For any positive integers r and n,*

$$\texttt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2 \simeq \begin{cases} \bigvee_{2^{n-1} \textit{ copies}} S^{nk-1}, & \textit{if } 2r + 1 = 3k; \\ S^{kn/2-1} * \mathbb{RP}^{kn/2-1}, & \textit{if } 2r + 1 = 3k \pm 1. \end{cases} \tag{4.2}$$

Proof. By Proposition 3.2 we know that $\texttt{Hom}_+(C_{2r+1}, K_n)$ is isomorphic to $\operatorname{Ind}(C_{2r+1})^{*n}$. We analyze $\mathbb{Z}_2$-action on $\operatorname{Ind}(C_{2r+1})$ in more detail.

Assume first $2r + 1 = 3k - 1$; in particular, k is even. It was shown in [9, Prop. 5.2] that $X = \operatorname{Ind}(C_{2r+1}) \setminus \{1, 4, \ldots, 2r - 3, 2r\}$ is contractible (here "$\setminus$" just means the removal of an open maximal simplex). It follows from the standard fact in the theory of transformation groups, see e.g., [5, Th. 5.16, p. 222], that $X/\mathbb{Z}_2$ is contractible as well. Hence $\operatorname{Ind}(C_{2r+1})$ is $\mathbb{Z}_2$-homotopy equivalent to the unit sphere $S^{k-1} \subset \mathbb{R}^k$ with the $\mathbb{Z}_2$ acting by fixing $k/2$ coordinates and multiplying the other $k/2$ coordinates by -1.

Assume $2r+1 = 3k+1$. The link of the vertex $2r+1$ is $\mathbb{Z}_2$-homotopy equivalent to a point. Hence, deleting the open star of the vertex $2r+1$ produces a complex X, which is $\mathbb{Z}_2$-homotopy equivalent to $\operatorname{Ind}(C_{2r+1})$. It was shown in [9, Prop. 5.2], that $X \setminus \{2, 5, \ldots, 2r-4, 2r-1\}$ is contractible. By an argument, similar to the previous case, we conclude that $\operatorname{Ind}(C_{2r+1})$ is $\mathbb{Z}_2$-homotopy equivalent to the unit sphere $S^{k-1} \subset \mathbb{R}^k$ with the $\mathbb{Z}_2$ acting by fixing $k/2$ coordinates and multiplying the other $k/2$ coordinates by -1.

In both cases we see that $\mathtt{Hom}_+(C_{2r+1}, K_n)$ is $\mathbb{Z}_2$-homotopy equivalent to $\operatorname{susp}^{kn/2} S^{kn/2-1}$, with the $\mathbb{Z}_2$-action and the latter space being induced by the antipodal action on $S^{kn/2-1}$. It follows that $\mathtt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2$ is homotopy equivalent to $\operatorname{susp}^{kn/2}\mathbb{RP}^{kn/2-1}$.

Consider the remaining case $2r+1 = 3k$. It was shown in [9, Prop. 5.2] that $\operatorname{Ind}(C_{2r+1})$ becomes contractible if one removes the simplices $\{1, 4, \ldots, 2r-1\}$ and $\{2, 5, \ldots, 2r\}$. It follows that $\operatorname{Ind}(C_{2r+1})$ is $\mathbb{Z}_2$-homotopy equivalent to the wedge of two unit spheres S^{k-1} with the $\mathbb{Z}_2$ acting by swapping the spheres. Thus $\mathtt{Hom}_+(C_{2r+1}, K_n)$ is $\mathbb{Z}_2$-homotopy equivalent to a wedge of 2^n $(nk-1)$-dimensional spheres, with the $\mathbb{Z}_2$-action swapping them in pairs. Thus, $\mathtt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2$ is homotopy equivalent to a wedge of 2^{n-1} $(nk-1)$-dimensional spheres. □

We summarize the estimates needed later.

COROLLARY 4.5. *$\widetilde{H}^i(\mathtt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2) = 0$ for $r \geq 2$, $n \geq 5$, and $i \leq n+r-2$, except for the case $r = 3$.*

Proof. If $2r+1 = 3k$, the inequality $nk-2 \geq n+r-2$ is equivalent to $n \geq 3r/(2r-2)$, which is true for $n \geq 3$, $r \geq 2$. If $2r+1 = 3k-1$, then $nk/2 \geq n+r-2$ is equivalent to $(n-3)(k-2) \geq 0$, again true for our parameters.

If $2r+1 = 3k+1$, then $nk/2 \geq n+r-2$ is equivalent to $(n-3)(k-2) \geq 2$. This is true for all parameters $n \geq 5$, $k \geq 2$, except for $k = 2$. □

4.2. *The cell complex of homomorphisms from a tree to a complete graph.* In the next proposition we summarize several results proved in [2], [12].

PROPOSITION 4.6 ([2, Props. 4.3, 5.4, and 5.5], [12]). *Let T be a tree with at least one edge.*

(i) *The map $i_{K_n} : \mathtt{Hom}\,(T, K_n) \to \mathtt{Hom}\,(K_2, K_n)$ induced by any inclusion $i : K_2 \hookrightarrow T$ is a homotopy equivalence.*

(ii) *$\mathtt{Hom}\,(K_2, K_n)$ is a boundary complex of a polytope of dimension $n-2$, in particular $\mathtt{Hom}\,(T, K_n) \simeq S^{n-2}$.*

(iii) *Given a $\mathbb{Z}_2$-action determined by an invertible graph homomorphism $\gamma : T \to T$, if γ flips an edge in T, then* $\mathtt{Hom}\,(T, K_n) \simeq_{\mathbb{Z}_2} S_a^{n-2}$; *otherwise* $\mathtt{Hom}\,(T, K_n) \simeq_{\mathbb{Z}_2} S_t^{n-2}$.

Here S_a^m denotes the m-sphere equipped with an antipodal $\mathbb{Z}_2$-action, whereas S_t^m is the m-sphere equipped with the trivial one.

Let F be any graph, with $F_1, \dots, F_t$ being the list of all those connected components of F which have at least two vertices. For any $\emptyset \neq S \subseteq [1,t]$, and $V = \{v_i\}_{i \in S}$, such that $v_i \in V(F_i)$, for any $i \in S$, set

$$\alpha_+(F, V) := \sum_\eta \eta_+^*, \quad \alpha(F, V) := \sum_\eta \eta^*,$$

where both sums are taken over all $\eta : V(F) \to 2^{[1,n]} \setminus \{\emptyset\}$, such that

- $\eta(v_i) = [1, n-1]$, for all $i \in S$;
- $|\eta(w)| = 1$, for all $w \in V(F) \setminus V$.

Note that, for fixed S and V, $(-1)^{c(\eta)}$ does not depend on the choice of η as long as η satisfies these two conditions. From our previous notation we have $\alpha_+(F, V) \in X^{|S|(n-2)}(F, K_n)$, and $\alpha(F, V) \in C^{|S|(n-2)}(\mathtt{Hom}\,(F, K_n))$. When $|S| = 1$, $V = \{v\}$, we shall simply write $\alpha_+(F, v)$ and $\alpha(F, v)$.

Assume now that F is a forest. For $w \in V(F_i)$, such that $(v_i, w) \in E(F)$, set $W := \{v_1, \dots, v_{i-1}, w, v_{i+1}, \dots, v_t\} = (V \cup \{w\}) \setminus \{v_i\}$. We have a graph homomorphism $K_2 \to (v, w)$, which induces a $\mathbb{Z}_2$-equivariant map $\varphi^* : H^*(\mathtt{Hom}\,(K_2, K_n)) \to H^*(\mathtt{Hom}\,(F, K_n))$. We know that $\mathtt{Hom}\,(K_2, K_n) \cong_{\mathbb{Z}_2} S_a^{n-2}$, and that the dual of any $(n-2)$-dimensional cell of $\mathtt{Hom}\,(K_2, K_n)$ is a generator of $H^{n-2}(\mathtt{Hom}\,(K_2, K_n); \mathbb{Z})$. Comparing orientations of the cells of $\mathtt{Hom}\,(K_2, K_n)$ we see that $[\alpha(K_2, 1)] = (-1)^{n-1}[\alpha(K_2, 2)]$, where 1 and 2 denote the vertices of K_2. Applying φ^* we conclude that

$$[\alpha(F, V)] = (-1)^{n-1}[\alpha(F, W)].$$

Since ρ^* is a cochain isomorphism and $\rho^*(\alpha_+(F, V)) = (-1)^{c(\eta)}\alpha(F, V)$, we have

$$[\alpha_+(F, V)] = \begin{cases} -[\alpha_+(F, W)], & \text{if } v \text{ and } w \text{ have different} \\ & \text{parity in the order on } V(F); \\ (-1)^{n-1}[\alpha_+(F, W)], & \text{if they have the same parity.} \end{cases} \tag{4.3}$$

4.3. *The $E_1^{*,*}$-tableau for $E_1^{p,q} \Rightarrow H^{p+q}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$.*[1] We fix integers $r \geq 2$ and $n \geq 4$. Let $(F^p)_{p=0,\dots,|V(G)|-1}$ be the filtration on

[1]The calculations performed in the subsections 4.3–4.8 have been verified and generalized in [15].

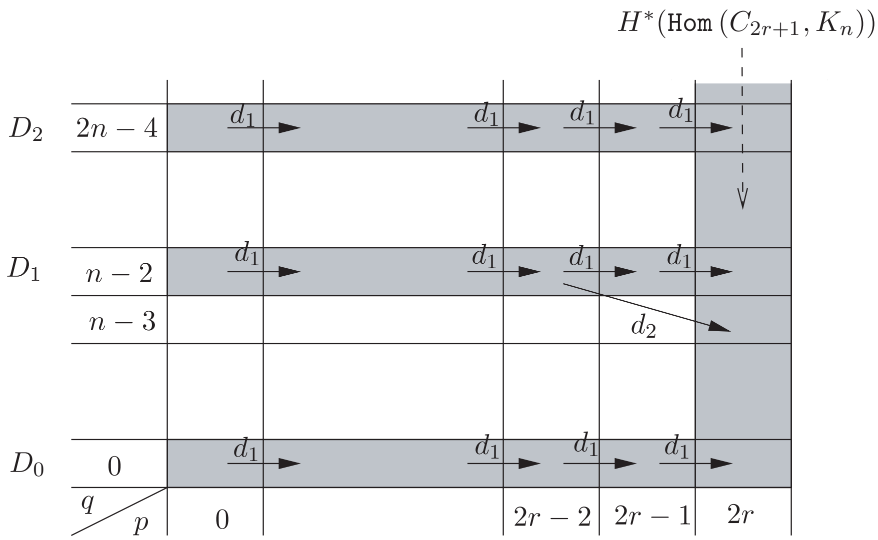

Figure 4.1: The $E_1^{*,*}$-tableau, for $E_1^{p,q} \Rightarrow H^{p+q}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$.

$C^*(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$ defined in subsection 3.5, and consider the corresponding spectral sequence. The entries of the E_1-tableau are given by $E_1^{p,q} = H^{p+q}(F^p, F^{p+1})$. Since all proper subgraphs of C_{2r+1} are forests, we can now use the formula (3.8) to obtain almost complete information about the E_1-tableau. See Figure 4.1, where all the entries outside of the shaded area are equal to 0

Let $\emptyset \neq S \subset V(C_{2r+1})$, and let $S_1, \dots, S_{l(S)}$ be the connected components of $C_{2r+1}[S]$, with $|S_1| \geq |S_2| \geq \dots \geq |S_{d(S)}| > |S_{d(S)+1}| = \dots = |S_{l(S)}| = 1$, where $l(S) \geq 1$, but possibly $d(S) = 0$ or $d(S) = l(S)$. By Proposition 4.6 together with property (3) from [2, §2.4] we see that

$$\mathtt{Hom}\,(C_{2r+1}[S], K_n) \simeq \prod_{i=1}^{d(S)} S^{n-2}. \tag{4.4}$$

Combining this with the formula (3.8) we conclude

$$H^{p+q}(F^p, F^{p+1}) = \bigoplus_{\substack{S \subset V(C_{2r+1}) \\ |S| = p+1}} H^q\left(\prod_{i=1}^{d(S)} S^{n-2}; \mathbb{Z}\right), \tag{4.5}$$

for $p \leq 2r-1$.

Since the spectral sequence converges to $\widetilde{H}^*(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$, and, since by Corollary 4.3, $\widetilde{H}^i(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}) = 0$ for $i \leq n+2r-2$, we know that the entries on the diagonals $p+q = n+2r-2$, and $p+q = n+2r-3$, should eventually all become 0.

4.4. *The cochain complex* $(D_0^*, d_1) = (E_1^{*,0}, d_1)$, *for* $E_1^{p,q} \Rightarrow H^{p+q}(\texttt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$. Let (D_i^*, d_1) denote the cochain complex in the $i(n-2)$-th row of $E_1^{*,*}$, for any $i = 0, \dots, \left\lfloor \dfrac{2r+1}{3} \right\rfloor$. Next we show that (D_0^*, d_1) is isomorphic to the cochain complex of a simplex.

Lemma 4.7. $E_2^{0,0} = \mathbb{Z}$, *and* $E_2^{1,0} = E_2^{2,0} = \cdots = E_2^{2r,0} = 0$.

Proof. Let Δ_{2r} denote an abstract simplex with $2r+1$ vertices indexed by $[1, 2r+1]$, and identify simplices of Δ_{2r} with the subsets of $[1, 2r+1]$. Let $(C^*(\Delta_{2r}; \mathbb{Z}), d^*)$ be the cochain complex of Δ_{2r} corresponding to the order on the vertices given by this indexing. By (4.5), each $S \subseteq V(C_{2r+1})$, $|S| = p+1$, contributes one independent generator (over $\mathbb{Z}$) to $E_1^{p,0}$. Identifying these with the generator in $C^*(\Delta_{2r}; \mathbb{Z})$ of the corresponding p-simplex in Δ_{2r}, we see that (D_0^*, d_1) and $(C^*(\Delta_{2r}; \mathbb{Z}), d^*)$ are isomorphic as cochain complexes.

Indeed, for such an S, $\tau_S := \sum_{\varphi \in \texttt{Hom}_0(C_{2r+1}[S], K_n)} \varphi_+^*$ is a representative of the corresponding generator in $E_1^{p,0}$. This is true even for $S = V(C_{2r+1})$, since $\texttt{Hom}\,(C_{2r+1}, K_n)$ is connected for $n \geq 4$, as was shown in [2, Prop. 2.1]. Clearly,

$$\begin{aligned} d_1(\tau_S) &= \sum_{\varphi \in \texttt{Hom}_0(C_{2r+1}[S], K_n)} \sum_{v \notin S} \sum_{\psi | S = \varphi} [\varphi_+ : \psi_+] \psi_+^* \\ &= \sum_{v \notin S} [S : S \cup \{v\}] \sum_{\psi \in \texttt{Hom}_0(C_{2r+1}[S \cup \{v\}], K_n)} \psi_+^* = \sum_{v \notin S} [S : S \cup \{v\}] \tau_{S \cup \{v\}}, \end{aligned}$$

where the second equality is true since $[\varphi_+ : \psi_+]$ only depends on S and v, not on the specific choice of φ and ψ. This shows that the following diagram commutes:

$$\begin{array}{ccc} C^p(\Delta_{2r}; \mathbb{Z}) & \xrightarrow{d^p} & C^{p+1}(\Delta_{2r}; \mathbb{Z}) \\ {\scriptstyle \tau.}\downarrow & & \downarrow{\scriptstyle \tau.} \\ E_1^{p,0} & \xrightarrow{d_1} & E_1^{p+1,0}, \end{array}$$

where $\tau. : C^*(\Delta_{2r}; \mathbb{Z}) \to E_1^{*,0}$ is the linear extension of the map taking S to τ_S, for $S \subseteq V(C_{2r+1})$. It follows that (D_0^*, d_1) is isomorphic to $(C^*(\Delta_{2r}; \mathbb{Z}), d^*)$; therefore $E_2^{0,0} = \mathbb{Z}$, and $E_2^{1,0} = E_2^{2,0} = \cdots = E_2^{2r,0} = 0$. $\square$

4.5. *The cochain complexes* $(D_t^*, d_1) = (E_1^{*,(n-2)t}, d_1)$, *for* $\lfloor (2r+1)/3 \rfloor \geq t \geq 2$, *and* $E_1^{p,q} \Rightarrow H^{p+q}(\texttt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$. We shall perform only a partial analysis of the cohomology groups of (D_t^*, d_1), which will however be sufficient for our purpose.

For $S \subset V(C_{2r+1})$ and $v \in S$, let $a(S, v)$ denote the arc of S to which v belongs (assuming this arc exists). Furthermore, for an arbitrary arc a

of S, let $a = [a_\bullet, a^\bullet]_{2r+1}$. Let $|a|$ denote the number of vertices on a, and set $\widehat{a} := [a_\bullet - 1, a^\bullet + 1]_{2r+1}$ (so $|\widehat{a}| = |a| + 2$, if $|a| \leq 2r - 1$).

For any $V \subseteq S \subseteq V(C_{2r+1})$, as in section 4.2, set $\sigma_{S,V} := \alpha_+(C_{2r+1}[S], V)$. By our previous observations, $E_1^{0,t(n-2)} = 0$. Furthermore, for any $1 \leq i \leq 2r-1$, $E_1^{i,t(n-2)}$ is a free $\mathbb{Z}$-module with the basis $\{[\sigma_{S,V}]\}$, where $S \subset V(C_{2r+1})$, $|S| = i+1$, $|V| = t$, and $v = a_\bullet(S, v)$ (i.e., $[v-1]_{2r+1} \notin S$) for all $v \in V$. Since $\sigma_{S,v}$ is a cocycle in $X^{n-2}(C_{2r+1}[S], K_n; \mathbb{Z})$, we have

$$d_1([\sigma_{S,v}]) = \sum_{w \notin S} (-1)^{z(w)} [\sigma_{S \cup \{w\}, v}], \tag{4.6}$$

where

$$z(w) = \begin{cases} |S \cap [1, w-1]|, & \text{if } v \notin [1, w-1]; \\ n + |S \cap [1, w-1]|, & \text{if } v \in [1, w-1]. \end{cases}$$

Note, that if $i \leq 2r - 2$ and $[w]_{2r+1} = [v-1]_{2r+1}$, then $v \neq a_\bullet(S \cup \{w\}, w)$, so $[\sigma_{S \cup \{w\}, v}]$ may differ by a sign from one of the elements in our chosen basis. We shall not need the analog of the equation (4.6) for the case $|V| \geq 2$.

Let A_1^* be the subcomplex of D_1^* defined by:

$$A_1^* : 0 \longrightarrow \widetilde{E}_1^{2r-2,n-2} \xrightarrow{d_1} E_1^{2r-1,n-2} \xrightarrow{d_1} E_1^{2r,n-2} \longrightarrow 0,$$

where the $\mathbb{Z}$-modules indexed with $0, \ldots, 2r-3$ are equal to 0, and $\widetilde{E}_1^{2r-2,n-2}$ is generated by $\{[\sigma_{S,v}]\}$, such that S and v satisfy all the previously required conditions and, in addition, $C_{2r+1}[S]$ is connected.

In general, let A_t^* be the subcomplex of D_t^* generated by all $\{[\sigma_{S,V}]\}$, such that

$$\bigcup_{v \in V} \widehat{a(S, v)} = V(C_{2r+1}). \tag{4.7}$$

In words: the gaps between those arcs of S which have points in V are of length at most 2. For future reference, we note, that (4.7) implies that $|S| + 2|V| \geq 2r + 1$, i.e., $|S| - 1 \geq 2r - 2t$; hence $A_t^j = 0$ for $j < 2r - 2t$.

Lemma 4.8. $H^*(D_t^*) = H^*(A_t^*)$.

Proof. Let us set up another spectral sequence for computing the cohomology of the relative complex (D_t^*, A_t^*). We filter by $\sum_{v \in V} |a(S, v)|$. More precisely, $F^p(D_t^*, A_t^*) = \mathbb{Z}[[\sigma_{S,V}] \mid \sum_{v \in V} |a(S, v)| \geq p]$. We see that

$$F^p(D_t^*, A_t^*)/F^{p+1}(D_t^*, A_t^*) = \mathbb{Z}[[\sigma_{S,v}] \mid \sum_{v \in V} |a(S, v)| = p];$$

hence

$$E_1^{p,q}(D_t^*, A_t^*) = H^{p+q}(F^p(D_t^*, A_t^*)/F^{p+1}(D_t^*, A_t^*)) = \bigoplus_{a_1, \ldots, a_t} H^{p+q}(M^*_{a_1, \ldots, a_t}),$$

where the sum is taken over all possible t-tuples of arcs $a_1, \ldots, a_t$ such that

(1) $a_i \cap \widehat{a_j} = \emptyset$, for any $i \neq j$, $i, j \in [1, t]$;

(2) $|a_1| + \cdots + |a_t| = p$;

(3) $\bigcup_{v \in V} \widehat{a(S, v)} \neq V(C_{2r+1})$,

and $M^*_{a_1,\dots,a_t}$ is the cochain subcomplex generated by all $\{[\sigma_{S,v}]\}$, such that the arcs with vertices in V are precisely $a_1, \dots, a_t$, i.e., $\{a(S, v) \,|\, v \in V\} = \{a_1, \dots, a_t\}$.

Restricting the formula (4.6) to M^*_a, we see that M^*_a is isomorphic to the cochain complex $C^*(\Delta_{2r-p-2}; \mathbb{Z})$. More generally, we see that $M^*_{a_1,\dots,a_t}$ is isomorphic to $C^*(\Delta_{2r-\tilde{p}}; \mathbb{Z})$, where $\tilde{p} = \left|\bigcup_{v \in V} \widehat{a(S, v)}\right|$.

As mentioned, $\tilde{p} \leq 2r$; hence $M^*_{a_1,\dots,a_t}$ is acyclic for any $a_1, \dots, a_t$ satisfying the above conditions. We conclude that (D^*_t, A^*_t) is acyclic. The long exact sequence for the relative cohomology implies that $H^*(A^*_t) = H^*(D^*_t)$. □

Now, we can show that $E_2^{i,t(n-2)} = 0$ for $t \geq 2$, $i < 2r - (t-1)(n-2)$, that is $E_2^{*,*}$ is 0 in the region strictly above row $n-2$ and strictly below the diagonal $x + y = 2r + n - 2$; see Figure 4.2. Indeed, this is immediate when $2r - 2t \geq 2r - (t-1)(n-2)$, which after cancellations reduces to $(n-4)(t-1) \geq 2$. The only cases when this inequality is false are $(t, n) = (2, 5)$, and $n = 4$.

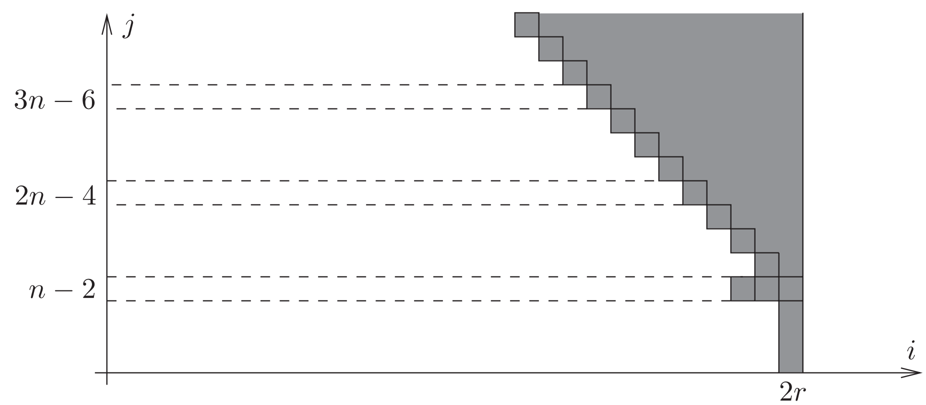

Figure 4.2: The possibly nonzero entries in the $E_2^{*,*}$-tableau, for $E_2^{p,q} \Rightarrow H^{p+q}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$.

4.6. *The case $n = 5$, for $E_1^{p,q} \Rightarrow H^{p+q}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$.* Assume now that $n = 5$, $t = 2$.

Lemma 4.9. $E_2^{2r-4,6} = 0$.

Proof. By a dimensional argument, this is true if $2r + 1 < 8$, and so we can assume that $r \geq 4$. By our previous arguments we need to see that

$d_1 : A_1^{2r-4,6} \to A_1^{2r-3,6}$ is an injective map. The generators of $A_1^{2r-4,6}$ can be indexed with unordered pairs $\{v,w\}$, $v,w \in V(C_{2r+1})$, such that

$$[v-1, v+2]_{2r+1} \cap [w-1, w+2]_{2r+1} = \emptyset,$$

whereas the generators of $A_1^{2r-3,6}$ can be indexed with ordered pairs (v,w), $v,w \in V(C_{2r+1})$, such that

$$[v-1, v+2]_{2r+1} \cap [w-1, w+1]_{2r+1} = \emptyset.$$

With this notation, we have

(4.8)
$$d_1(\{v,w\}) = \varepsilon_1(v,w) + \varepsilon_2(v,[w+1]_{2r+1}) + \varepsilon_3(w,v) + \varepsilon_4(w,[v+1]_{2r+1}),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}$.

Take $0 \neq \sum_{v,w} \alpha_{v,w}\{v,w\} \in \ker d_1$. Choose v,w such that $\alpha_{v,w} \neq 0$, and the minimum of the two distances between the arcs $\{v,[v+1]_{2r+1}\}$ and $\{w,[w+1]_{2r+1}\}$ is minimized. By symmetry we may assume $[w-v-1]_{2r+1} \leq [v-w-1]_{2r+1}$. Then, it follows from (4.8) that $\alpha_{[v+1]_{2r+1},w} \neq 0$ as well.

Either $\{[v+1]_{2r+1}, w\}$ is not a well-defined pair or the minimal distance between the two arcs is smaller for this pair, than for $\{v,w\}$: $[w-v-1]_{2r+1} \geq [w-v-2]_{2r+1}$. Both ways we get a contradiction to the assumption that there exists $\{v,w\}$, such that $\alpha_{v,w} \neq 0$. We conclude that $d_1 : A_1^{2r-4,6} \to A_1^{2r-3,6}$ is injective, hence $E_2^{2r-4,6} = 0$. □

This shows, that when $n \geq 5$, there are no higher differentials d_i, $i \geq 2$, in our spectral sequence, originating in the region above row $n-2$ and below diagonal $x + y = 2r + n - 2$. Hence, to figure out what happens to the entries $E_\infty^{2r,n-2}$ and $E_\infty^{2r,n-3}$, it is sufficient to consider rows $n-2$ and $n-3$.

4.7. *The case $n = 4$, for $E_1^{p,q} \Rightarrow H^{p+q}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$.* For $n = 4$ the nonzero rows of $E_1^{*,*}$ are too close to each other, so we are to do the computation by hand in a somewhat detailed way. Since the reduction from (D_t^*, d_1) to (A_t^*, d_1) described in subsection 4.5 was valid when $n = 4$, we may concentrate on the study of the latter complex. Let us first deal with (A_2^*, d_1).

Lemma 4.10. *$H^{2r-2}(A_2^*) = H^{2r-3}(A_2^*) = \mathbb{Z}$, and $H^i(A_2^*) = 0$, for $i \neq 2r-2, 2r-3$.*

Proof. We filter A_2^* by

$$F^p A_2^* = \mathbb{Z}\left[[\sigma_{S,\{v_1,v_2\}}] \,\middle|\, \min(|a(S,v_1)|, |a(S,v_2)|) \geq p\right].$$

Clearly, $\dots \subseteq F^p \subseteq F^{p+1} \subseteq \dots$. Inspecting the case $p \leq r-2$, we see that in this situation

$$C^*(F^p A_2^*/F^{p-1}A_2^*) = \bigoplus_{i=1}^{2r+1} B_i,$$

where each B_i is isomorphic to $C^*(\Delta_1)$, hence is acyclic.

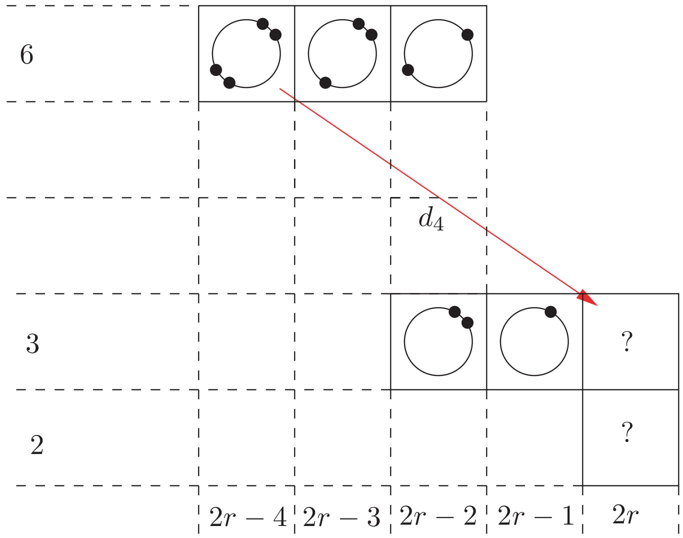

Figure 4.3: A part of the $E_1^{*,*}$-tableau, for $n = 5$, and $E_1^{p,q} \Rightarrow H^{p+q}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$.

It follows that $H^*(A_2^*) = H^*(F^{r-1}A_2^*/F^{r-2}A_2^*)$. Let $\sigma_i = \sigma_{S_i,V_i}$, where $S_i = [i+1, i+r-1]_{2r+1} \cup [i+r+2, i-1]_{2r+1}$, $V_i = \{i+1, i+r+2\}$, and let $\tau_i = \sigma_{\widetilde{S}_i,V_i}$, where $\widetilde{S}_i = S_i \cup \{[i+r]_{2r+1}\}$. Clearly, $d_1(\sigma_i) = \pm\tau_i \pm \tau_{i+r}$, and to verify the statement of the lemma we need to show that the number of those σ_i, for which $d_1(\sigma_i) = \pm(\tau_i + \tau_{i+r})$, is even. By (4.3) and (4.6) we see that $d_1(\sigma_i) = \pm(\tau_i - \tau_{i+r})$ if $i + r \neq 2r, 2r+1$; i.e., the only cases we need to consider are $i = r$ and $i = r+1$.

If $i = 2r+1$, the different sign comes from (4.6), and the sign contribution is $2r+2$. This is an even number, hence again $d_1(\sigma_i) = \pm(\tau_i - \tau_{i+r})$. If $i = 2r$, the different sign comes from (4.3), but since $n = 4$ is even, the sign remains the same. $\square$

Next, we consider (A_t^*, d_1), for $t \geq 3$. First, we introduce some additional notation. Since the sign will not matter in our argument, we write σ_S instead of $\sigma_{S,V}$, it is then defined only up to a sign. For $S \subset V(C_{2r+1})$, $\bar{S} = V(C_{2r+1}) \setminus S$; the connected components of $C_{2r+1}[\bar{S}]$ are called *gaps*. Each gap consists of either one or two elements; we call the first ones *singletons*, and the second ones *double gaps*. Let $m(S)$ be the leftmost element of the gap which contains $\min(\bar{S} \cap [2, 2r]_{2r+1})$. For $s \in \bar{S}$, let $\overleftarrow{s}$ be the leftmost element of the first gap to the left of the gap containing s, and let $\overrightarrow{s}$ be the leftmost element of the first gap to the right of the gap containing s. For $x, y \in V(C_{2r+1})$, let $d(x,y)$ denote $|[x,y]_{2r+1}| - 1$.

Lemma 4.11. *$E_2^{2r-2t,2t} = E_2^{2r-2t+1,2t} = 0$.*

Proof.[2] Clearly $A_t^i = 0$, unless $2r - 2t \le i \le 2r - t$. Note that if σ_S is a generator of $A_t^{2r-2t+1}$, then S has exactly one singleton. We decompose $A_t^{2r-2t+1} = B_1 \oplus B_2 \oplus B_3 \oplus B_4$, where each B_i is spanned by the generators σ_S, for which certain conditions are satisfied; see Figure 4.4:

(B_1) $\overleftarrow{m(S)}$ is the singleton and $d(\overleftarrow{m(S)}, m(S)) = 3$, or $m(S)$ is the singleton and $d(\overleftarrow{m(S)}, m(S)) = 4$;

(B_2) $\overleftarrow{m(S)}$ is the singleton, and $d(\overleftarrow{m(S)}, m(S)) \ge 4$;

(B_3) $m(S)$ and $\overleftarrow{m(S)}$ are in double gaps;

(B_4) $m(S)$ is the singleton, and $d(\overleftarrow{m(S)}, m(S)) \ge 5$.

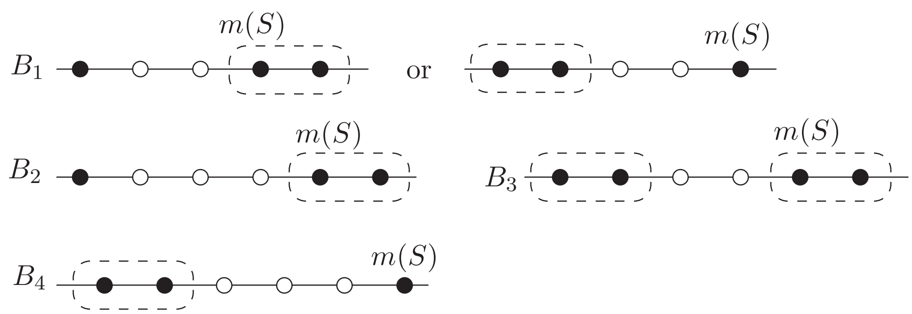

Figure 4.4: The four cases in the proof of Lemma 4.11.

Let $(\widetilde{A}_t^*, d_1)$ be the complex spanned by B_1, B_2, B_3, and A_t^i, for $2r - 2t + 2 \le i \le 2r - t$. The relative complex $(A_t^*/\widetilde{A}_t^*, d_1)$ has only cochains in dimensions $2r - 2t$ and $2r - 2t + 1$. It is easy to see that the projection $d_1 : A_t^{2r-2t} \to A_t^{2r-2t+1}/\widetilde{A}_t^{2r-2t+1} = B_4$ is an isomorphism, with the inverse given by $\sigma_S \mapsto \sigma_{S \setminus \{m(S)-1\}}$, where σ_S is a generator from B_4. Hence $(A_t^*/\widetilde{A}_t^*, d_1)$ is acyclic, and we are led to study the complex $(\widetilde{A}_t^*, d_1)$.

Next, we show that $H^{2r-2t+1}(\widetilde{A}_t^*) = 0$, which is the same as saying that d_1 is injective on $B = B_1 \oplus B_2 \oplus B_3$. Let $\sigma \ne 0$ be in $\ker d_1(B)$. We think of σ as a linear combination of the generators from the descriptions of B_1, B_2, and B_3, and let M be the set of generators which have a nonzero coefficient in σ.

Assume M contains a generator σ_S from B_3, and choose σ_S so that $d(m(S), \overrightarrow{m(S)})$ is minimized. The coboundary $d_1(\sigma_S)$ contains a copy of $\sigma_{S \cup \{m(S)\}}$. At most three other generators will contain $\sigma_{S \cup \{m(S)\}}$ in the coboundary, depending on which element we remove from $S \cup \{m(S)\}$ instead of $m(S)$. Since we have chosen $d(m(S), \overrightarrow{m(S)})$ to be minimal, we cannot remove

[2]It is possible to rephrase this argument in terms of matchings on chain complexes; see [11].

$m(S)+2$. Hence, we must remove an element extending the singleton gap which is not $m(S)+1$. This gives a generator of B_4, yielding a contradiction.

From now on we may presume that M contains no generators from B_4 or B_3. Assume now that M contains a generator σ_S from B_2. Again, choose σ_S so that $d(m(S), \overrightarrow{m(S)})$ is minimized, and note that $d_1(\sigma_S)$ contains a copy of $\sigma_{S\cup\{m(S)\}}$. Examining the generators which contain $\sigma_{S\cup\{m(S)\}}$ in the coboundary, we see again that, since removing $m(S)+2$ from $S\cup\{m(S)\}$ would contradict the minimality; we must remove an element extending the singleton gap $\overleftarrow{m(S)}$. This way we will produce a generator of B_4, except for one case: when $\overleftarrow{m(S)}=1$, and we remove vertex 2. In this case we produce a generator from B_3, hence again a contradiction.

Finally, assume M consists only of generators from B_1. Choose σ_S so that $m(S)$ is maximized. Assume first that $m(S)$ is in a double gap, and consider the copy of $\sigma_{S\cup\{m(S)\}}$ in $d_1(\sigma_S)$. There are three possibilities. Removing $m(S)+2$ gives a generator of B_2, whereas removing $m(S)-4$ gives a generator of B_4. Removing $m(S)-2$ gives either a generator of B_3, if $m(S)=4$, or a generator σ_T of B_1, such that $m(T)=m(S)+1$. In either case we get a contradiction.

Assume now that $m(S)$ is a singleton, and examine the copy of $\sigma_{S\cup\{\overleftarrow{m(S)}\}}$ in $d_1(\sigma_S)$. There is only one possibility for deletion: remove $m(S)+1$. This will produce a generator σ_T of B_1, with $m(T)=m(S)$, but such that $m(T)$ is in a double gap, the case already dealt with.

This finishes the proof that $H^{2r-2t+1}(\widetilde{A}_t^*)=0$, which, combined with the acyclicity of $(A_t^*/\widetilde{A}_t^*, d_1)$, and the fact that $H^*(A_t^*)=H^*(D_t^*)$, yields $E_2^{2r-2t,2t}=E_2^{2r-2t+1,2t}=0$. ∎

4.8. *Finishing the computation of* $H^{n-2}(\mathtt{Hom}\,(C_{2r+1},K_n);R)$ *and of* $H^{n-3}(\mathtt{Hom}\,(C_{2r+1},K_n);R)$, *for* $R=\mathbb{Z}_2$ *or* $\mathbb{Z}$. Let us now turn our attention to the cochain complex A_1^*. The generators of $\widetilde{E}_1^{2r-2,n-2}$ correspond to arcs of length $2r-1$ and can be indexed with the elements of $V(C_{2r+1})$: we set $\tau_{v,2}:=\sigma_{V(C_{2r+1})\setminus\{v-2,v-1\},v}$, for any $v\in V(C_{2r+1})$. In the same way the generators of $E_1^{2r-1,n-2}$ correspond to arcs of length $2r$, we denote them by setting $\tau_{v,1}:=\sigma_{V(C_{2r+1})\setminus\{v-1\},v}$, for any $v\in V(C_{2r+1})$. It follows from (4.6) that

$$d_1([\tau_{v,2}])=(-1)^{v+1}[\tau_{v,1}]+(-1)^v[\tau_{v-1,1}],$$

for $v=3,\dots,2r+1$, where the second sign follows from (4.3);

$$d_1([\tau_{2,2}])=(-1)^{n+1}[\tau_{2,1}]-[\tau_{1,1}],$$

where the first sign is determined by the fact that there are $n+2r-3$ vertices before the one inserted at position $2r+1$;

$$d_1([\tau_{1,2}])=(-1)^{n+1}[\tau_{1,1}]+(-1)^n[\tau_{2r+1,1}],$$

where we use again that there are $n+2r-3$ vertices before the inserted one, and, for determining the second sign, we use the fact that positions 1 and $2r+1$ have different parity in $[1, 2r+1] \setminus \{2r\}$.

Summarizing, we have the following matrix for the first differential in A_1^*:

$$M = \begin{bmatrix} (-1)^{n+1} & 0 & 0 & 0 & \dots & 0 & (-1)^n \\ -1 & (-1)^{n+1} & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}$$

Assume first that $n \geq 5$, and $(n, r) \neq (5, 3)$.

Case 1: *n is odd*. It is easy to see that the kernel of the differential $d_1 : \widetilde{E}_1^{2r-2,n-2} \to E_1^{2r-1,n-2}$ is one-dimensional and is spanned by

$$[\tau_{1,2}] + [\tau_{2,2}] + [\tau_{3,2}] - [\tau_{4,2}] + [\tau_{5,2}] - [\tau_{6,2}] + \cdots + [\tau_{2r+1,2}],$$

while the image is

$$\left\{ \left[\sum_{i=1}^{2r+1} c_i \tau_{i,1} \right] \,\middle|\, \sum_{i=1}^{2r+1} c_i = 0 \right\}.$$

It follows that $E_2^{2r-2,n-2} = \mathbb{Z}$. Recall that, by Corollary 4.3, the cohomology groups of $\mathtt{Hom}_+(C_{2r+1}, K_n)$ vanish in dimension $n+2r-2$ and less. Hence, since $d_2 : E_2^{2r-2,n-2} \to E_2^{2r,n-3}$ must be an isomorphism, we have $E_1^{2r,n-3} = E_2^{2r,n-3} = \mathbb{Z}$. On the other hand, the map $d_1 : E_1^{2r-1,n-2} \to E_1^{2r,n-2}$ is surjective, and $E_2^{2r-1,n-2} = E_2^{2r,n-2} = 0$, so that $E_1^{2r,n-2} = \mathbb{Z}$.

Case 2: *n is even*. In this case the map $d_1 : \widetilde{E}_1^{2r-2,n-2} \to E_1^{2r-1,n-2}$ is injective. It follows that $E_2^{2r-2,n-2} = 0$, and hence $E_1^{2r,n-3} = E_2^{2r,n-3} = 0$. The image on the other hand is not the whole $E_1^{2r-1,n-2}$, but only

$$\left\{ \left[\sum_{i=1}^{2r+1} c_i \tau_{i,1} \right] \,\middle|\, \sum_{i=1}^{2r+1} c_i \equiv 0 \,(\mathrm{mod}\ 2) \right\}.$$

The fact that $E_2^{2r-1,n-2} = E_2^{2r,n-2} = 0$ and the surjectivity of the map $d_1 : E_1^{2r-1,n-2} \to E_1^{2r,n-2}$ imply that $E_1^{2r,n-2} = \mathbb{Z}_2$. Again, we used the fact that $\widetilde{H}^i(\mathtt{Hom}_+(C_{2r+1}, K_n))$ vanish in dimension $n+2r-2$ and less.

If $(n, r) = (5, 3)$, then the argument above essentially holds, with the exception that $d_1 : E_1^{2r-1,n-2} \longrightarrow E_1^{2r,n-2}$ does not have to be surjective. Instead, $\mathrm{Im}\, d_1 = \mathbb{Z}$ and $E_1^{2r,n-2}/\mathrm{Im}\, d_1 = \mathbb{Z}$. Thus $H^{n-2}(\mathtt{Hom}\,(C_7, K_5); \mathbb{Z}) = E_1^{2r,n-2} = \mathbb{Z}^2$.

Assume, finally, that $n = 4$. If $2r+1 = 5$, then the computations above hold. If $2r+1 \geq 7$, the argument above still shows that the image of the map

(n,r)	R	H^{n-2}	H^{n-3}
$2 \not\mid n, n \geq 5, (n,r) \neq (5,3)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$(n,r) = (5,3)$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
$2 \mid n, n \geq 6$, or $n = 4, r \leq 3$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
$n = 4, r \geq 4$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0
$n \geq 5, (n,r) \neq (5,3)$, or $n = 4, r \leq 3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$(n,r) = (5,3)$, or $n = 4, r \geq 4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$

Table 4.1.

$d_1 : E_1^{2r-1,2} \to E_1^{2r,2}$ is $\mathbb{Z}[d_1(\tau_{1,r})]$, and $2d_1(\tau_{1,r}) = d_1(2\tau_{1,r}) = 0$. If $2r+1 \geq 9$, we can compute $E_1^{2r,2}$ and $E_1^{2r,1}$ completely, since $H^i(\texttt{Hom}_+(C_{2r+1}, K_n))$ vanish in dimension $n + 2r - 2$ and less. In this case, the image of the map $d_1 : E_1^{2r-1,2} \to E_1^{2r,2}$ is $\mathbb{Z}_2$, and the map $d_2 : E_2^{2r-3,4} \to E_2^{2r,2}$ is an isomorphism. Since, as we have shown earlier, $E_2^{2r-3,4} = \mathbb{Z}$, we conclude that $E_1^{2r,2} = \mathbb{Z} \oplus \mathbb{Z}_2$, and $E_1^{2r,1} = 0$.

It follows that, for all (n,r), $H^i(\texttt{Hom}\,(C_{2r+1}, K_n); R) = 0$, if $i \in [1, n-4]$, and $R = \mathbb{Z}$ or $\mathbb{Z}_2$.[3] We summarize our computations of the next two cohomology groups in Table 4.1, where $H^i = H^i(\texttt{Hom}\,(C_{2r+1}, K_n); R)$, and the case $(n,r) = (4,3)$ is conjectural.

Proof of Theorem 2.6. For $n \geq 6$, and $n = 4$, $r \leq 3$, this follows from the fact that the target group of the map is $\mathbb{Z}_2$. For $n = 4$, $r \geq 4$, we have shown above that $2d_1(\tau_{1,r}) = 0$. By the construction, $\tau_{1,r} = \sigma_{V(C_{2r+1}) \setminus \{r-1\}, r}$, so that $d_1(\tau_{1,r}) = \pm\sigma_{V(C_{2r+1}),r}$. Let $V(K_2) = \{1,2\}$, and pick a nontrivial element $\alpha \in H^{n-2}(\texttt{Hom}\,(K_2, K_n); \mathbb{Z})$ by setting $\alpha := \eta^*$, $\eta(1) := [1, n-1]$, $\eta(2) := \{n\}$. Clearly, $\iota^*_{K_n}(\alpha) = \pm\sigma_{V(C_{2r+1}),r}$, where $\iota(1) = r$, $\iota(2) = r+1$. Thus, we see that $2 \cdot \iota^*_{K_n}(\alpha) = \pm 2d_1(\tau_{1,r}) = 0$. □

4.9. *The $\mathbb{Z}_2$-action on the cohomology groups of* $\texttt{Hom}\,(C_{2r+1}, K_n)$ *for odd n.* Throughout this subsection we assume that n is odd, and that $(n,r) \neq (5,3)$. We tensor all our groups with $\mathbb{C}$ to simplify the representations. We denote by χ_i the one-dimensional representation of $\mathbb{Z}_2$ given by the multiplication by $(-1)^i$.

LEMMA 4.12. $E_1^{2r,n-2} = \chi_r$, *as a $\mathbb{Z}_2$-module.*

[3]This has been strengthened to yield connectivity in [4]; later a shorter proof appeared in [6].

Proof. Recall that $\sigma_{V(C_{2r+1}),2r+1} := \sum_\eta \eta_+^*$, where the sum is taken over all η, such that $\eta(2r+1) = [1, n-1]$, and $|\eta(i)| = 1$, for all $i = 1, \dots, 2r$. Note that $\sigma_{V(C_{2r+1}),2r+1}$ is a representative of the generator of $E_1^{2r,n-2}$. Clearly, $\{\eta \circ \gamma\} = \{\eta\}$ as a collection of cells. To orient the cells in the standard way we need to reverse γ as the permutation of $V(C_{2r+1})$. The sign of this is $(-1)^r$; hence $\gamma([\sigma_{V(C_{2r+1}),2r+1}]) = (-1)^r[\sigma_{V(C_{2r+1}),2r+1}]$. $\square$

LEMMA 4.13. $E_1^{2r-1,n-2} = r\chi_0 + r\chi_1 + \chi_{n+r+1}$, *as a* $\mathbb{Z}_2$*-module.*

Proof. $\tau_{1,1}, \dots, \tau_{2r+1,1}$ can be taken as the representatives of the generators of $E_1^{2r-1,n-2}$. We see first that

$$\gamma([\tau_{1,1}]) = (-1)^{n+r+1}[\tau_{1,1}]. \tag{4.9}$$

Indeed, $\gamma([\tau_{1,1}]) = \operatorname{sgn}\pi \cdot [\sigma_{[1,2r],2r}]$, where π is the permutation induced by γ on the vertices of each support simplex of $\tau_{1,1}$; i.e.,

$$\pi = (n+2r-2, n+2r-3, \dots, n+1, n, 1, \dots, n-1).$$

Since π consists of inverting the sequence $(1, \dots, n+2r-2)$, and then inverting the subsequence $(1, \dots, n-1)$, we see that

$$\operatorname{sgn}\pi = (-1)^{\lfloor\frac{n+2r-2}{2}\rfloor + \lfloor\frac{n-1}{2}\rfloor} = (-1)^{r-1+\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n-1}{2}\rfloor} = (-1)^{r+n},$$

where we used the fact that the sign of inverting a sequence $[1, \dots, m]$ is $(-1)^{\lfloor\frac{m}{2}\rfloor}$, and that, for any natural number m, we have $\lfloor\frac{m}{2}\rfloor + \lfloor\frac{m-1}{2}\rfloor = m-1$. Additionally, $[\sigma_{[1,2r],2r}] = -[\sigma_{[1,2r],1}]$ by (4.3); hence 4.9 follows.

Next, we shall see that

$$\gamma([\tau_{2r+2-i,1}]) = (-1)^r[\tau_{i+1,1}], \tag{4.10}$$

for $i = 1, \dots, 2r$. Again, $\gamma([\tau_{2r+2-i,1}]) = \operatorname{sgn}\pi \cdot [\sigma_{V(C_{2r+1})\setminus\{i\},i-1}]$, where π consists of inverting the sequence $(1, \dots, n+2r-3)$, and then inverting some subsequence of length $n-1$ back. It follows that

$$\operatorname{sgn}\pi = (-1)^{\lfloor\frac{n+2r-3}{2}\rfloor + \lfloor\frac{n-1}{2}\rfloor} = (-1)^{r-1+\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n-1}{2}\rfloor} = (-1)^{r+1}.$$

On the other hand, by (4.3), $[\sigma_{V(C_{2r+1})\setminus\{i\},i-1}] = -[\sigma_{V(C_{2r+1})\setminus\{i\},i+1}]$; hence we get (4.10). The actual sign has no bearing on our final conclusion.

Since the permutation action of $\mathbb{Z}_2$ on a 2-dimensional space decomposes as $\chi_0+\chi_1$, the formulae (4.9) and (4.10) together yield the claim of the lemma. $\square$

LEMMA 4.14. $E_1^{2r-2,n-2} = r\chi_0 + r\chi_1 + \chi_{r+1}$, *as a* $\mathbb{Z}_2$*-module.*

Proof. $\tau_{1,2}, \dots, \tau_{2r+1,2}$ can be taken as the representatives of the generators of $E_1^{2r-2,n-2}$. We see first that

$$\gamma([\tau_{r+2,2}]) = (-1)^{r+1}[\tau_{r+2,2}]. \tag{4.11}$$

We have $\gamma([\tau_{r+2,2}]) = \operatorname{sgn}\pi \cdot [\sigma_{V(C_{2r+1})\setminus\{r,r+1\},r-1}]$, where π consists of inverting the sequence of length $n+2r-4$, and then inverting some subsequence of length $n-1$ back. It follows that

$$\operatorname{sgn}\pi = (-1)^{\lfloor \frac{n+2r-4}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor} = (-1)^{r-2+\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor} = (-1)^{r+n+1}.$$

Furthermore, by (4.3) $[\sigma_{V(C_{2r+1})\setminus\{r,r+1\},r-1}] = (-1)^n[\sigma_{V(C_{2r+1})\setminus\{r,r+1\},r+2}]$, where $(-1)^n$ is composed of $2r-3$ steps changing the sign, and one step changing the sign by $(-1)^{n+1}$, since 1 and $2r+1$ have the same parity in $V(C_{2r+1})\setminus\{r,r+1\}$. Summarizing we get (4.11).

Second we note that

$$\gamma([\tau_{2r+2-i,1}]) = \pm[\tau_{i+2,1}], \tag{4.12}$$

for $i \in V(C_{2r+1}) \setminus \{r+2\}$. Indeed, as before we see that $\gamma([\tau_{2r+2-i,1}]) = \pm[\sigma_{V(C_{2r+1})\setminus\{i,i+1\},i-1}] = \pm[\sigma_{V(C_{2r+1})\setminus\{i,i+1\},i+2}]$.

Equations (4.11) and (4.12) show that the $\mathbb{Z}_2$-representation splits into χ_{r+1} and the r-fold permutation action, yielding the claim of the lemma. $\square$

COROLLARY 4.15. *The group $\mathbb{Z}_2$ acts trivially on $H^{n-2}(\mathtt{Hom}\,(C_{2r+1}, K_n);\mathbb{Z}) = \mathbb{Z}$, and as a multiplication by -1 on $H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n);\mathbb{Z}) = \mathbb{Z}$.*

Proof. It follows from Lemmas 4.12, 4.13, and 4.14 that $E_1^{2r,n-3} = \chi_{r+1}$, as a $\mathbb{Z}_2$-module. The result follows now from equation (3.6). $\square$

5. Cohomology groups of $\mathbb{Z}_2$-quotients of products of spheres

From now on, unless explicitely stated otherwise, we shall only work with $\mathbb{Z}_2$-coefficients.

We begin by introducing another piece of terminology: for a positive integer d, let *d-symbols* be elements of the set $\{*, \infty\}$, where $*$ will denote an open d-cell, and ∞ denote a 0-cell. We assume throughout this section that $d \geq 2$. For example, S^d is decomposed into $*$ and ∞, whereas a direct product of t d-dimensional spheres decomposes into cells, indexed by all possible t-tuples of d-symbols. We let $\dim * = d$, $\dim \infty = 0$, and we set the dimension of a tuple of d-symbols as the sum of the dimensions of the constituting symbols.

5.1. *Cohomology groups of $\mathbb{Z}_2$-quotients of products of an odd number of spheres.* Let X be a direct product of $2t+1$ d-dimensional spheres, and let $\mathbb{Z}_2$ act on X by swapping spheres numbered $2i+1$ and $2i$, for $i \in [1,t]$, and acting on the first sphere by an antipodal map. We shall decompose $X/\mathbb{Z}_2$ into cells, and describe its cohomology groups.

Clearly, $X/\mathbb{Z}_2$ is a total space of a fiber bundle over $\mathbb{RP}^d$ with fiber homeomorphic to a direct product of $2t$ d-dimensional spheres. Consider the standard cell decomposition of $\mathbb{RP}^d$ with one cell in each dimension $i \in [0,d]$.

PROPOSITION 5.1. *The space $X/\mathbb{Z}_2$ can be decomposed into cells indexed with (i, x, y), where x and y are t-tuples of d-symbols, $0 \leq i \leq d$. The dimension of this cell is* $\dim(i, x, y) = i + \dim x + \dim y$.

The coboundary is given by the equation

$$d^{i+\dim x+\dim y}((i, x, y)^*) = (i + 1, x, y)^* + (i + 1, y, x)^*, \tag{5.1}$$

where the cochains are considered with $\mathbb{Z}_2$ coefficients.

Proof. Divide X into cells, by taking the product cell structure, where spheres 2 to $2t + 1$ have one 0-cell and one d-cell, whereas the first sphere is subdivided as a join of $d + 1$ 0-spheres, with $\mathbb{Z}_2$ acting antipodally on each of these 0-spheres. The cells can then be indexed with triples $(i, x, y)_+$ and $(i, x, y)_-$. The coboundary is given by

$$d((i, x, y)^*_+) = (i + 1, x, y)^*_+ + (i + 1, x, y)^*_-. \tag{5.2}$$

This cell structure is $\mathbb{Z}_2$-equivariant, and no cells are preserved by the involution. This means that it induces a cell structure on $X/\mathbb{Z}_2$. Let (i, x, y) denote the orbit $\{(i, x, y)_+, (i, y, x)_-\}$. After taking the quotient, (5.2) becomes (5.1). □

From Proposition 5.1, the generators of $H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$ are indexed with

- (i, x, x), for any $0 \leq i \leq d$, and a t-tuple of d-symbols x, here $(i, x, x)^*$ is the cocycle;
- $(0, x, y)$, for any t-tuples of d-symbols $x \neq y$, here $(0, x, y)^* + (0, y, x)^*$ is the cocycle; $(0, x, y)$ and $(0, y, x)$ index the same generator;
- (d, x, y), for any t-tuples of d-symbols $x \neq y$, here $(d, x, y)^*$ is the cocycle; (d, x, y) and (d, y, x) index the same generator.

In other words, the cohomology generators are indexed by pairs $(\langle A\rangle, i)$, where A is a $2 \times t$ array of d-symbols, and $i \in [0, d]$, if A is fixed by $\mathbb{Z}_2$, while $i \in \{0, d\}$, if A is not fixed by $\mathbb{Z}_2$. Here $\mathbb{Z}_2$ acts on the set of all $2 \times t$ arrays of d-symbols by swapping the two rows, and $\langle - \rangle$ denotes an orbit of this action.

For future reference, we note the following property: these generators behave functorially, under the maps which insert additional pairs of spheres. More specifically, assume $q \geq t$, and let $f : [1, t] \hookrightarrow [1, q]$ be an injection. Let $\tilde{f} : \underbrace{S^d \times \cdots \times S^d}_{2q+1} \to \underbrace{S^d \times \cdots \times S^d}_{2t+1}$ be the following map: $\tilde{f}$ is the identity on the first sphere, it maps isomorphically the spheres indexed $2i$ and $2i + 1$, for $i \in \operatorname{Im} f$, to the spheres indexed by $2f^{-1}(i)$ and $2f^{-1}(i) + 1$, and it maps the remaining spheres to the base point. Then, the induced map on the cohomology $\tilde{f}^*$ maps the generator $(\langle A\rangle, i)$ to the generator $(\langle \widetilde{A}\rangle, i)$, where $\widetilde{A}$ is the $2 \times q$

array obtained from A as follows: the column $f(i)$ in $\widetilde{A}$ is equal to the column i in A, and, for $j \notin \operatorname{Im} f$, the column j in $\widetilde{A}$ consists of two ∞'s.

5.2. *Cohomology groups of $\mathbb{Z}_2$-quotients of products of an even number of spheres.* Let X be a direct product of $2t$ d-dimensional spheres, and let $\mathbb{Z}_2$ acting on X be swapping spheres $2i-1$ and $2i$, for $i \in [1,t]$. A customary notation for $X/\mathbb{Z}_2$ is $SP^2(\underbrace{S^d \times \cdots \times S^d}_{t})$. Again, we shall decompose $X/\mathbb{Z}_2$ into cells, and describe its cohomology groups.

PROPOSITION 5.2. *The space $X/\mathbb{Z}_2$ can be decomposed into cells indexed with two types of labels*:

Type 1. *the unordered pairs $\{x,y\}$, where x and y are t-tuples of d-symbols, $x \neq y$; the dimension is* $\dim x + \dim y$;

Type 2. (x,x,k), *where x is a t-tuple of d-symbols, and* $0 \leq k \leq \dim x$; *the dimension is* $\dim x + k$.

With $\mathbb{Z}_2$ coefficients, the coboundary is equal to 0 *for all generators, except for $(x,x,0)^*$, when* $\dim x \geq 1$, *in which case* $d^{\dim x}(x,x,0)^* = (x,x,1)^*$. *In particular, the generators of $\widetilde{H}^i(X/\mathbb{Z}_2;\mathbb{Z}_2)$ are indexed with the same symbols as the cells in our decomposition, except for $(x,x,0)$ and $(x,x,1)$.*

Proof. Start with a usual subdivision of a direct product of $2t$ d-spheres, with the cells indexed by pairs (x,y) of t-tuples of d-symbols. For $x \neq y$, the set $(x,y) \cup (y,x)/\mathbb{Z}_2$ is a cell in $X/\mathbb{Z}_2$, which we label $\{x,y\}$.

To do the same for $x = y$, we need to take a finer subdivision of (x,x). Let $(x,x,k)^+$, resp. $(x,x,k)^-$, be the set of all points $\bar{\alpha} \in \mathbb{R}^{2\dim x}$, $\bar{\alpha} = (\alpha_i)_{i \in [2\dim x]}$, such that $\alpha_j = \alpha_{j+\dim x}$, for $k+1 \leq j \leq \dim x$, and $\alpha_k > \alpha_{k+\dim x}$, resp. $\alpha_k < \alpha_{k+\dim x}$. Obviously, $(x,x,k)^+$ and $(x,x,k)^-$ are cells, which are mapped to each other by the $\mathbb{Z}_2$-action. These cells are different for $k \geq 1$, whereas $(x,x,0)^+ = (x,x,0)^-$ is fixed pointwise.

Set $(x,x,0) := (x,x,0)^+$, and $(x,x,k) := (x,x,k)^+ \cup (x,x,k)^-/\mathbb{Z}_2$, for $k \geq 1$. The statements about the coboundary map and the indexing of the cohomology generators follow immediately from our construction. □

When we rephrase Proposition 5.2 in the language of arrays, the generators of $H^*(X/\mathbb{Z}_2;\mathbb{Z}_2)$ are indexed with $\mathbb{Z}_2$-orbits $\langle A \rangle$ of $2 \times t$ arrays of d-symbols, with an additional index $2 \leq i \leq \dim A/2$, if A is fixed by the $\mathbb{Z}_2$-action. Here $\dim A$ is the sum of the dimensions of all entries of A.

Again, we have functoriality in the following sense: if $q \geq t$, and $f : [1,t] \hookrightarrow [1,q]$ is an injection, define $\tilde{f} : \underbrace{S^d \times \cdots \times S^d}_{2q} \to \underbrace{S^d \times \cdots \times S^d}_{2t}$ analogously to the one in subsection 5.1. Then $\tilde{f}^*$ maps $\langle A \rangle$, resp. $(\langle A \rangle, i)$, to $\langle \widetilde{A} \rangle$, resp.

$(\langle \widetilde{A} \rangle, i)$, where $\widetilde{A}$ is a $2 \times q$ array of d-symbols obtained from A by inserting the columns consisting entirely of ∞'s in the places indexed by $[1, q] \setminus \operatorname{Im} f$.

6. Spectral sequence for $H^*(\mathtt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2)$

Next, we would like to show Theorem 2.3(b). We assume that

$$\varpi_1^{n-2}(\mathtt{Hom}\,(C_{2r+1}, K_n)) \neq 0,$$

and arrive at a contradiction by doing computations in a spectral sequence, which we now proceed to set up.

6.1. *$\mathbb{Z}_2$-equivariant cell decomposition of* $\mathtt{Hom}_+(C_{2r+1}, K_n)$. For convenience, we give following names to the vertices of C_{2r+1}: $c := [0]_{2r+1}$, $a_i := [r+i]_{2r+1}$, and $b_i := [r+1-i]_{2r+1}$, for $i \in [1, r]$. That is, $\gamma : C_{2r+1} \to C_{2r+1}$ fixes c, and $\gamma(a_i) = b_i$, for any $i \in [1, r]$. Identify $V(C_{2r+1})$ with the vertices of an abstract simplex Δ_{2r} of dimension $2r$. It is also convenient to have optional notation for c, namely $a_{r+1}, b_{r+1} := c$; see Figure 6.1.

We subdivide the simplex Δ_{2r} by adding r more vertices, which we denote $c_1, c_2, \ldots, c_r$, and defining a new abstract simplicial complex $\tilde{\Delta}_{2r}$ on the set $\{c, a_1, \ldots, a_r, b_1, \ldots, b_r, c_1, \ldots, c_r\} = V(\tilde{\Delta}_{2r})$. The simplices of $\tilde{\Delta}_{2r}$ are all the subsets of $V(\tilde{\Delta}_{2r})$ which do not contain the subset $\{a_i, b_i\}$, for any $i \in [1, r]$. We set $\mathcal{C} = \{c, c_1, \ldots, c_r\}$. The complex $\tilde{\Delta}_{2r}$ comes equipped with a simplicial $\mathbb{Z}_2$-action, which fixes $\mathcal{C}$ and swaps a_i and b_i, for all $i \in [1, r]$. For $S \subseteq V(\tilde{\Delta}_{2r})$ we let $\langle S \rangle$ denote the $\mathbb{Z}_2$-orbit of S.

One can think of this new complex $\tilde{\Delta}_{2r}$ as the one obtained from Δ_{2r} by representing it as a topological join $\{c\} * [a_1, b_1] * \cdots * [a_r, b_r]$, with the additional simplicial structure defined by inserting an extra vertex c_i into the middle of each $[a_i, b_i]$, and then taking the join of $\{c\}$ and the subdivided intervals. For $\tilde{\sigma} \in \tilde{\Delta}_{2r}$ we obtain $\vartheta(\tilde{\sigma}) \in \Delta_{2r}$ by replacing every c_i in $\tilde{\sigma}$ by $\{a_i, b_i\}$, i.e., $\vartheta(\tilde{\sigma}) = (\tilde{\sigma} \setminus \{c_1, \ldots, c_r\}) \cup \bigcup_{c_i \in \tilde{\sigma}} \{a_i, b_i\}$.

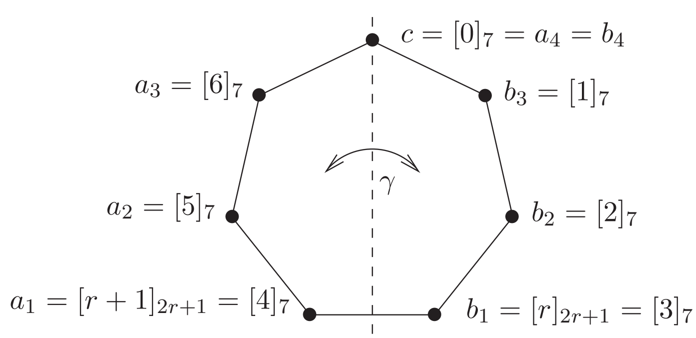

Figure 6.1: Summary of notations.

The simplicial complex $\tilde{\Delta}_{2r}$ has an additional property: if a simplex of $\tilde{\Delta}_{2r}$ is γ-invariant, then it is fixed pointwise. This allows us to introduce a simplicial structure (strictly speaking - a structure of triangulated space) on $\tilde{\Delta}_{2r}/\mathbb{Z}_2$ by taking the orbits of the simplices of $\tilde{\Delta}_{2r}$ as the simplices of $\tilde{\Delta}_{2r}/\mathbb{Z}_2$.

6.2. *The chain complex of the subdivision of* $\mathtt{Hom}_+(C_{2r+1}, K_n)$. Since we are working over $\mathbb{Z}_2$, from now on we shall drop the $+$ notation for the simplices of $\mathtt{Hom}_+(C_{2r+1}, K_n)$; e.g., we shall write η^* instead of η^*_+ (here we refer to notation introduced in §3.2).

Let us now describe a cochain complex $\widetilde{C}^*(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2)$, which comes from a triangulation of the simplicial complex $\mathtt{Hom}_+(C_{2r+1}, K_n)$. The cochain complex consists of vector spaces over $\mathbb{Z}_2$, whose generators are pairs $(\eta, \sigma)^*$, where $\eta \in \mathtt{Hom}_+(C_{2r+1}, K_n)$, and $\sigma \in \tilde{\Delta}_{2r}$, such that $\vartheta(\sigma) = \operatorname{supp}\eta$. Such a pair indexes the cochain which is dual to the cell $\eta \cap \operatorname{supp}^{-1}(\sigma)$. The coboundary of $(\eta, \sigma)^*$ is the sum of the following generators:

(1) $(\tilde{\eta}, \sigma)^*$, if $\operatorname{supp}\tilde{\eta} = \operatorname{supp}\eta$, $\eta \in \partial\tilde{\eta}$, and $\dim\tilde{\eta} = \dim\eta + 1$;

(2) $(\eta, \sigma \cup \{x\})^*$, if $x \in V(\tilde{\Delta}_{2r}) \setminus \sigma$, and $\vartheta(\sigma) = \vartheta(\sigma \cup \{x\})$;

(3) $(\tilde{\eta}, \sigma \cup \{x\})^*$, if $x \in V(\tilde{\Delta}_{2r}) \setminus \sigma$, $\tilde{\eta}|_{\vartheta(\sigma)} = \eta$, and all the values of $\tilde{\eta}$ on $\vartheta(\sigma \cup \{x\}) \setminus \vartheta(\sigma)$ have cardinality 1.

The degree of $(\eta, \sigma)^*$ in $\widetilde{C}^*(\mathtt{Hom}_+(C_{2r+1}, K_n))$ is given by

$$\deg(\eta, \sigma)^* = |\sigma| - 1 + \sum_{v \in \operatorname{supp}\eta} (|\eta(v)| - 1) = \deg\eta + |\sigma| - |\vartheta(\sigma)|.$$

$\mathbb{Z}_2$ acts on $\widetilde{C}^*(\mathtt{Hom}_+(C_{2r+1}, K_n))$ and we let $\widetilde{C}^*_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n))$ denote its subcomplex consisting of the invariant cochains. By construction of the subdivision, $\widetilde{C}^*_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n))$ is a cochain complex for a triangulation of the space $\mathtt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2$.

6.3. *The filtration of* $\widetilde{C}^*_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2)$. This time, we consider the natural filtration $(\widetilde{F}^0 \supseteq \widetilde{F}^1 \supseteq \dots)$ on the cochain complex

$$\widetilde{C}^*_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2)$$

by the cardinality of σ. Namely, $\widetilde{F}^p = \widetilde{F}^p C^*_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2)$, is a cochain subcomplex of $C^*_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2)$ defined by:

$$\widetilde{F}^p : \dots \xrightarrow{\partial^{q-1}} \widetilde{F}^{p,q} \xrightarrow{\partial^q} \widetilde{F}^{p,q+1} \xrightarrow{\partial^{q+1}} \dots,$$

where

$$\widetilde{F}^{p,q} = \mathbb{Z}_2 \left[(\eta, \sigma)^* \;\middle|\; (\eta, \sigma) \in C^q_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2), |\sigma| \geq p + 1 \right],$$

and ∂^* is the restriction of the differential in $C^*_{\mathbb{Z}_2}(\mathtt{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2)$.

The following formula is the analog of (3.7).

Proposition 6.1. *For any* p,

$$\widetilde{F}^p/\widetilde{F}^{p+1} = \bigoplus_{\sigma} C^*(\mathtt{Hom}\,(C_{2r+1}[\vartheta(\sigma)], K_n)/\mathbb{Z}_2;\mathbb{Z}_2)[-p] \tag{6.1}$$
$$\bigoplus_{\langle\tau\rangle} C^*(\mathtt{Hom}\,(C_{2r+1}[\vartheta(\tau)], K_n);\mathbb{Z}_2)[-p],$$

where the first sum is taken over all $\sigma \subseteq \mathcal{C}$, $|\sigma| = p+1$, *and the second sum is taken over all orbits* $\langle\tau\rangle$, *such that* $\tau \subseteq V(\tilde{\Delta}_{2r})$, $|\tau| = p+1$, $\tau \setminus \mathcal{C} \neq \emptyset$.

Hence, the 0^{th} *tableau of the spectral sequence associated to the cochain complex filtration* $\widetilde{F}^*$ *is given by*

$$E_0^{p,q} = \bigoplus_{\sigma} C^q(\mathtt{Hom}\,(C_{2r+1}[\vartheta(\sigma)], K_n)/\mathbb{Z}_2;\mathbb{Z}_2) \tag{6.2}$$
$$\bigoplus_{\langle\tau\rangle} C^q(\mathtt{Hom}\,(C_{2r+1}[\vartheta(\tau)], K_n);\mathbb{Z}_2),$$

with the summations over the same sets as in (6.1).

6.4. *The analysis of the spectral sequence converging to* $H^*(\mathtt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2;\mathbb{Z}_2)$.

The $E_1^{*,*}$-tableau of this spectral sequence is given by

$$E_1^{p,q} = H^{p+q}(\widetilde{F}^p, \widetilde{F}^{p+1}).$$

It follows immediately from the formula (6.2) that each $E_1^{p,q}$ splits as a vector space over $\mathbb{Z}_2$ into direct sums of $H^q(\mathtt{Hom}\,(C_{2r+1}[S], K_n);\mathbb{Z}_2)$, and of $H^q(\mathtt{Hom}\,(C_{2r+1}[S], K_n)/\mathbb{Z}_2;\mathbb{Z}_2)$. More precisely,

$$E_1^{p,q} = \bigoplus_{\sigma\subseteq\mathcal{C}} H^q(\mathtt{Hom}\,(C_{2r+1}[\vartheta(\sigma)], K_n)/\mathbb{Z}_2;\mathbb{Z}_2) \tag{6.3}$$
$$\bigoplus_{\langle\tau\rangle,\,\tau\not\subseteq\mathcal{C}} H^q(\mathtt{Hom}\,(C_{2r+1}[\vartheta(\tau)], K_n);\mathbb{Z}_2).$$

The generators of $E_1^{p,q}$ stemming from $\sigma \subseteq \mathcal{C}$ will be called *symmetric*, whereas the generators stemming from $\langle\tau\rangle$ for $\tau \not\subseteq \mathcal{C}$ will be called *asymmetric*.

For $i \in [1, r]$, we shall denote the arc $\{a_i, a_{i-1}, \dots, a_1, b_1, b_2, \dots, b_i\}$ by $\smile_i$. For $2 \leq i \leq r$, we denote the arc $\{a_i, a_{i+1}, \dots, a_r, c, b_r, b_{r-1}, \dots, b_i\}$ by $\frown_i$. For $2 \leq i < j \leq r$, let $\big(_{i,j}$ denote the arc $\{a_i, a_{i+1}, \dots, a_j\}$, let $\big)_{i,j}$ denote the arc $\{b_j, b_{j-1}, \dots, b_i\}$, and let $\big(\big)_{i,j}$ denote the symmetric pair of arcs $\big(_{i,j}$ and $\big)_{i,j}$.

Proposition 6.2. *The map*

$$q^{n-3} : H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2;\mathbb{Z}_2) \to H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n);\mathbb{Z}_2), \tag{6.4}$$

is a 0-*map.*

Proof. First of all, since we are working over the field $\mathbb{Z}_2$, the map q^{n-3} is dual to the map on homology

$$q_{n-3} : H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}_2) \longrightarrow H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2);$$

hence it is enough to prove that q_{n-3} is a 0-map.

We start by proving that $q_{n-3} = 0$ over integers. The map q_{n-3} commutes with the $\mathbb{Z}_2$-action. Recall that we have proven that

$$H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}) = H^{n-2}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}) = \mathbb{Z},$$

and so it follows that $H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}) = \mathbb{Z}$. Let ξ be a generator of the group $H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z})$. By our previous computations $\gamma^{K_n}(\xi) = -\xi$, since $H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{C}) = \chi_1$ as a $\mathbb{Z}_2$-module (it is a dual $\mathbb{Z}_2$-module to $H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{C})$), and since $H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z})$ is torsion-free. On the other hand, the $\mathbb{Z}_2$-action on $H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z})$ is trivial; hence

$$-q_{n-3}(\xi) = q_{n-3}(-\xi) = q_{n-3}(\gamma^{K_n}(\xi)) = \gamma^{K_n}(q_{n-3}(\xi)) = q_{n-3}(\xi).$$

We conclude that $q_{n-3}(\xi) = 0$.

Second, by the universal coefficient theorem the map

$$\tau : H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}) \otimes \mathbb{Z}_2 \longrightarrow H_{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}_2)$$

is injective and functorial. In our concrete situation, this map is also surjective; hence the claim results from the following commutative diagram:

$$\begin{tikzcd}
H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}) \otimes \mathbb{Z}_2 \arrow[r, "\text{0-map}"] \arrow[d, "\tau"', "\text{iso}"] & H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}) \otimes \mathbb{Z}_2 \arrow[d] \\
H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}_2) \arrow[r, "q^{n-3}"] & H^{n-3}(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2).
\end{tikzcd}$$ □

LEMMA 6.3. $E_2^{r+1,n-3} = \mathbb{Z}_2$.

Proof. To start with, the only contribution to $E_1^{r,n-3}$ comes from $\sigma = \mathcal{C}$, so the fact that q^{n-3} in (6.4) is a 0-map implies that the differential $d_1 : E_1^{r,n-3} \to E_1^{r+1,n-3}$ is a 0-map as well.

Consider the cochain complex

$$A^* : E_1^{r+1,n-3} \xrightarrow{d_1} E_1^{r+2,n-3} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E_1^{2r,n-3}.$$

The generators of $E_1^{r+i,n-3}$ come from $\tau = \mathcal{C} \cup I$, for $I \subseteq \{a_1, \dots, a_r, b_1, \dots, b_r\}$, $|I| = i$. We can identify the generator indexed by $\langle \tau \rangle$ with the simplex of $\mathbb{RP}^{r-1} \cong \{a_1, b_1\} * \cdots * \{a_r, b_r\}/\mathbb{Z}_2$, indexed by $\langle I \rangle$, where the $\mathbb{Z}_2$-action swaps a_i and b_i, for $i \in [1, r]$.

By inspecting the description of the differential d_1 we see that A^* is isomorphic to the chain complex $C^*(\mathbb{RP}^{r-1}; \mathbb{Z}_2)$. It follows that $E_2^{r+1,n-3} = H^0(\mathbb{RP}^{r-1}; \mathbb{Z}_2) = \mathbb{Z}_2$. □

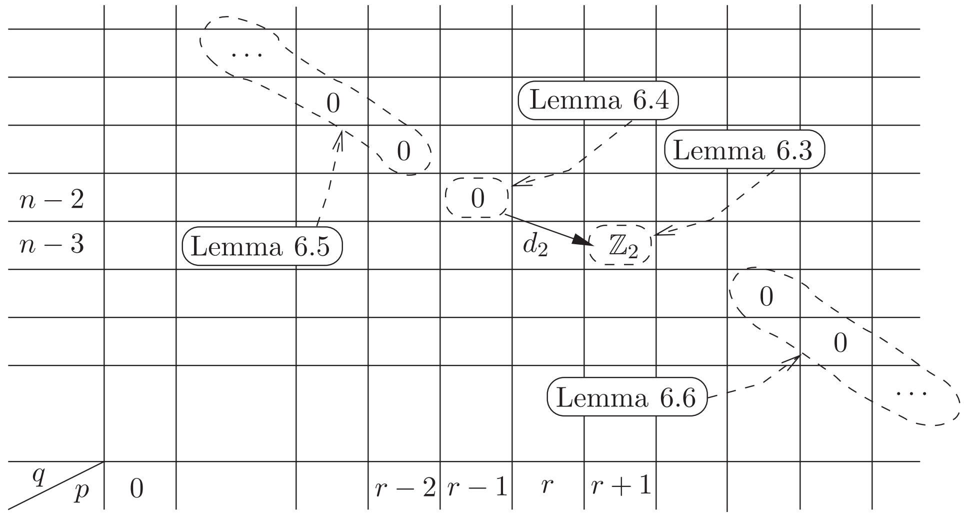

Figure 6.2: The $E_2^{*,*}$-tableau, $E_2^{p,q} \Rightarrow H^{p+q}(\mathtt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2)$.

In the proof of the next lemma we shall often use the chain homotopy between 0 and the identity.

Let (C^*, d) *be a cochain complex, and assume there exist linear maps* $\phi^n : C^n \to C^{n-1}$, $\forall n$, *such that*

$$\phi^{n+1}(d(\alpha)) + d(\phi^n(\alpha)) = \alpha, \text{ for all } \alpha \in C^n. \tag{6.5}$$

Then C^* *is acyclic.*

The proof is immediate, since modulo the coboundaries, every $\alpha \in C^n$ is equal to $\phi^{n+1}(d(\alpha))$; hence $d(\alpha) = 0$ implies $\alpha = 0$ modulo the coboundaries.

Lemma 6.4. $E_2^{r-1,n-2} = 0$.

Proof. Set

$$A^* : E_1^{0,n-2} \xrightarrow{d_1} E_1^{1,n-2} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E_1^{2r,n-2}.$$

Clearly, to show $E_2^{r-1,n-2} = 0$ is the same as to show that $H^{r-1}(A^*) = 0$.

For dimensional reasons, every generator in A^* is indexed either by $\sigma \subset V(\tilde{\Delta}_{2r})$ with an arc selected in $\vartheta(\sigma)$ (which we call the *indexing arc*), or the whole set $V(C_{2r+1})$ (namely, those coming from $H^{n-2}(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2)$ and from $H^{n-2}(\mathtt{Hom}\,(C_{2r+1}, K_n))$). To simplify the terminology, we shall call the set $V(C_{2r+1})$ an arc as well. Filter the cochain complex $A^* = G^{2r+1} \supseteq G^{2r} \supseteq \cdots \supseteq G^2 \supseteq G^1 = 0$, where G^l is spanned by the generators whose indexing arc has length at least l. We shall compute $H^{r-1}(A^*)$ by considering the corresponding spectral sequence $\widetilde{E}_0^{p,q} := C^{p+q}(G^p/G^{p-1})$.

In the same pattern already encountered, the cochain complex $(G^p/G^{p-1}, d_0)$ splits into a direct sum of subcomplexes which are indexed by different arcs. For an arc a, let B_a^* denote the corresponding summand. Hence $\widetilde{E}_1^{p,q} = \bigoplus_a H^{p+q}(B_a^*)$, where the sum is taken over all arcs a of length p.

Next, by considering all possible arcs case-by-case, we compute the entries $\widetilde{E}_1^{p,r-1-p}$, for $p = 2, \dots, 2r+1$. To start with, since $\vartheta(\sigma) = V(C_{2r+1})$ implies $|\sigma| \geq r+1$, $\widetilde{E}_0^{2r+1,-r-2} = C^{r-1}(G^{2r+1}/G^{2r}) = 0$ for dimensional reasons; hence $\widetilde{E}_1^{2r+1,-r-2} = 0$.

We shall only consider the cases where we cannot use dimensional reasons to immediately conclude that $B_a^{r-1} = 0$.

Case 1. Let $a = \smile_r$. Then, $B_a^{r-2} = 0$ for dimensional reasons, and $B_a^{r-1} = \mathbb{Z}_2$ coming from $\sigma = \{c_1, \dots, c_r\}$. The differential $d : B_a^{r-1} \to B_a^r$ is a 0-map since $f^{n-2} : H^{n-2}(\mathbb{RP}^{n-2}; \mathbb{Z}_2) \to H^{n-2}(S^{n-2}; \mathbb{Z}_2)$ is a 0-map, where $f : S^{n-2} \to S^{n-2}/\mathbb{Z}_2 = \mathbb{RP}^{n-2}$ denotes the covering map. Hence, in this case, $H^{r-1}(B_a^*) = \mathbb{Z}_2$.

Case 2. Let $a = \frown_2$. Again, $B_a^{r-2} = 0$ for dimensional reasons, and $B_a^{r-1} = \mathbb{Z}_2$ coming from $\sigma = \{c_2, \dots, c_r, c\}$. However, this time $d(B_a^{r-1}) \neq 0$, since it is induced by the map $f^{n-2} : H^{n-2}(\mathtt{Hom}\,(G, K_n)/\mathbb{Z}_2) \to H^{n-2}(\mathtt{Hom}\,(G, K_n))$, which, as we have seen, is not a 0-map; here G is the tree on three vertices and $\mathbb{Z}_2$ action is swapping the leaves. Hence $H^{r-1}(B_a^*) = 0$.

Case 3. Let $a = \smile_k$, for $1 \leq k \leq r-1$. Let $\alpha \in B_a^m$ be a generator indexed by $\sigma \subset V(\tilde{\Delta}_{2r})$. If $\sigma \subset \mathcal{C}$, and $x \in \{a_1, \dots, a_r, b_1, \dots, b_r\} \setminus \{a_{k+1}, b_{k+1}\}$, then the differential maps α to the generator indexed by $\langle \sigma \cup \{x\} \rangle$ (again $\smile_k$ is selected) as a 0-map, for the reason described in Case 1. This means that the complex B_a^* splits into two direct summands, one containing all generators indexed by $\sigma \subset \mathcal{C}$, and the other those indexed by $\langle \sigma \rangle$, such that $\sigma \setminus \mathcal{C} \neq \emptyset$.

In both summands, define $\phi^m(\alpha)$ to be the generator indexed by $\langle \sigma \setminus \{c\} \rangle$, if $c \in \sigma$, and $\phi^m(\alpha) = 0$ otherwise. The equation (6.5) is satisfied, and so both summands are acyclic, hence so is B_a^*, in particular $H^{r-1}(B_a^*) = 0$.

Case 4. Let $a = \frown_k$, for $3 \leq k \leq r$. We do the same as in Case 3 with c, replaced with c_{k-2}. However, in this complex, if $\sigma \subset \mathcal{C}$, and $x \in \{a_1, \dots, a_r, b_1, \dots, b_r\} \setminus \{a_{k-1}, b_{k-1}\}$, then the differential maps α to the generator indexed with $\langle \sigma \cup \{x\} \rangle$ (again $\frown_k$ is selected) as an identity; hence the complex does not split and equation (6.5) can be applied to the whole complex, yielding $H^{r-1}(B_a^*) = 0$.

Case 5. Let $a = \big(_{2,r}$. For each indexing orbit $\langle \sigma \rangle$ choose the representative σ such that $a_1 \notin \sigma$. Define ϕ^* as in Case 3, taking b_1 instead of c. Equation (6.5) is rather straightforward. We just need to pay attention to what the differential does to the generator indexed by $\sigma = \mathcal{C} \setminus \{c_1, c\}$.

It follows from the description of the cell decomposition and the cohomology of $SP^2(S^{n-2})$, given as a special case in subsection 5.2, that the map $f^{n-2} : H^{n-2}(SP^2(S^{n-2}); \mathbb{Z}_2) \to H^{n-2}(S^{n-2} \times S^{n-2}; \mathbb{Z}_2)$, induced by the quotient map, takes the nonzero element to the sum of the two generators of $H^{n-2}(S^{n-2} \times S^{n-2}; \mathbb{Z}_2)$ corresponding to each of the two spheres. In B_a^* this means that the differential of σ will contain the generator of $\langle \sigma \cup \{b_1\} \rangle$, with the indexing arc $\big(_{2,r}$, but not the generator of $\langle \sigma \cup \{b_1\} \rangle$, with the indexing arc $\big)_{1,r}$. Thus we conclude again that B_a^* is acyclic, and $H^{r-1}(B_a^*) = 0$.

Case 6. Let $a = \big(_{1,r}$. $B_a^{r-2} = 0$ for dimensional reasons. The space B_a^{r-1} is spanned by the 2^{r-1} generators which we can index with sets $\{a_1, \xi_2, \ldots, \xi_r\}$, where $\xi_i \in \{a_i, c_i\}$, for $2 \le i \le r$. Denote by $\bar{a}$ the generator indexed by the set $\{a_1, a_2, \ldots, a_r\}$. The coboundary of every generator $\alpha \neq \bar{a}$ contains some generator β indexed by $\{b_i, a_1, \xi_2, \ldots, \xi_r\}$. Since α is uniquely reconstructible from β, and the coboundary of $\bar{a}$ does not contain such generators as β, we see that an element in $\ker(d : B_a^{r-1} \to B_a^r)$ cannot contain α with a nonzero coefficient. Thus, the only chance for this kernel to be nontrivial would be when $\bar{a}$ lies in it, but, obviously, $d(\bar{a}) \neq 0$. Hence, once again, B_a^* is acyclic, and $H^{r-1}(B_a^*) = 0$.

Case 7. Let a be an assymetric arc, such that $a \cap \{a_r, b_r, c\} = \emptyset$. The complex B_a^* is isomorphic to a simplicial complex of a cone with an apex in the vertex c. Hence B_a^* is acyclic, and $H^{r-1}(B_a^*) = 0$.

Case 8. Let a be such that $a \cap \{a_1, a_2, b_1, b_2\} = \emptyset$. The complex B_a^* is isomorphic to a simplicial complex of a cone with an apex in the vertex c_1. Hence B_a^* is acyclic, and $H^{r-1}(B_a^*) = 0$.

Case 9. Let $a = \{c, a_i, a_{i+1}, \ldots, a_r, b_r, b_{r-1}, \ldots, b_j\}$, for $2 \le i < j$, where possibly $j = r+1$, which means a does not contain any b_i's. The complex B_a^* is isomorphic to a simplicial complex of a cone with an apex in the vertex b_1. Hence B_a^* is acyclic, and $H^{r-1}(B_a^*) = 0$.

Case 10. Let $a = \{a_i, a_{i-1}, \ldots, a_1, b_1, b_2, \ldots, b_j\}$, for $r \ge i > j \ge 1$. The complex B_a^* is isomorphic to a simplicial complex of a cone with an apex in the vertex a_1. Hence B_a^* is acyclic, and $H^{r-1}(B_a^*) = 0$.

We can now summarize our computations as follows: $\widetilde{E}_1^{p,r-1-p} = 0$, for $p = 2, \ldots, 2r-1$, whereas $\widetilde{E}_1^{2r,-r-1} = \mathbb{Z}_2$. The generator of $\widetilde{E}_1^{2r,-r-1}$ comes from $a = \smile_r$, which in turn comes from $\varpi_1^{n-2}(\mathtt{Hom}\,(K_2, K_n))$. The map $d_1 : \widetilde{E}_1^{2r,-r-1} \to \widetilde{E}_1^{2r+1,-r-1}$ is the same as

$$(\iota_{K_n})^{n-2} : H^{n-2}(\mathtt{Hom}\,(K_2, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \to H^{n-2}(\mathtt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2),$$

where $\iota : K_2 \hookrightarrow C_{2r+1}$ is either of the two $\mathbb{Z}_2$-equivariant inclusion maps which take the vertices of K_2 to $\{a_1, b_1\}$.

Since we assumed that $\varpi_1^{n-2}(\texttt{Hom}\,(C_{2r+1},K_n))\neq 0$, and the Stiefel-Whitney characteristic classes are functorial, we see that $d_1 : \widetilde{E}_1^{2r,-r-1} \to \widetilde{E}_1^{2r+1,-r-1}$ has rank 1, hence $\widetilde{E}_2^{2r,-r-1} = 0$. Thus $\widetilde{E}_2^{p,r-1-p} = 0$, for $p = 2,\dots,2r+1$, and we conclude that $E_2^{r-1,n-2} = 0$. $\square$

LEMMA 6.5. $E_2^{r-i,n-3+i} = 0$, *for all* $i = 2,3,\dots,r$.

Proof. First of all, we note that the generators in the columns indexed by $r-1$ and less come from $H^*(\texttt{Hom}\,(C_{2r+1}[S],K_n)/\mathbb{Z}_2;\mathbb{Z}_2)$ and from $H^*(\texttt{Hom}\,(C_{2r+1}[S],K_n);\mathbb{Z}_2)$, with $S \neq V(C_{2r+1})$ in both cases.

For each row q, $q > n-2$, we shall show that the subcomplex $A_q^* = (E_1^{*,q}, d_1)$ is acyclic in the entry $n+r-q-3$.

We begin by dealing with the case $q = n-1$ separately; that is we analyze the entry $E_2^{r-2,n-1}$. It follows from Propositions 5.1, 5.2, and for dimensional reasons, that the entries $E_1^{0,n-1}, E_1^{1,n-1},\dots,E_1^{r-1,n-1}$ are generated by the contributions whose indexing collections of arcs are $(\smile_i,\frown_j)$, for $1 \leq i < j-1 \leq r-1$.

The contributing spaces are homotopy equivalent to $S^{n-2}\times X/\mathbb{Z}_2$, where X is a direct product of $2t+1$ $(n-2)$-dimensional spheres, and $\mathbb{Z}_2$-action is as in Section 5. The generators appearing in the first r entries of the $(n-1)$th row are coming from the $(n-2)$-cocycle of S^{n-2} and the 1-cocycle of $\mathbb{RP}^{n-2}$. The analysis of the differentials shows that the complex $E_1^{0,n-1} \xrightarrow{d_1} E_1^{1,n-1} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E_1^{r-1,n-1}$ computes the nonreduced homology of a simplex with $r-2$ vertices (which could be identified with the set $\{c_2,\dots,c_{r-1}\}$). It follows that the entry $E_2^{r-2,n-1}$, which computes the first homology group is equal to 0.

We assume from now on that $q \geq n$. Similar to subsection 4.5 we filter the complexes A_q^*. To describe the filtration, we sort all generators into five groups. The first group (Gr1) contains all asymmetric generators, i.e., those coming from $\langle\sigma\rangle$, for $\sigma \not\subseteq \mathcal{C}$. The symmetric generators, coming from $\sigma \subseteq \mathcal{C}$, are divided into four groups, depending on whether the indexing collection of arcs

(Gr2) contains both an $\frown$-arc, and an $\smile$-arc,

(Gr3) contains an $\smile$-arc, but not an $\frown$-arc,

(Gr4) contains an $\frown$-arc, but not an $\smile$-arc,

(Gr5) contains no $\frown$-arc, and no $\smile$-arc.

The groups are ordered as above. We filter the complex A_q^* by first sorting the generators by the groups, and then, within each group we filter additionally by the total length of the indexing arcs.

Let $\widetilde{E}_*^{*,*}$ denote the tableaux of the spectral sequence computing the cohomology of A_q^*. In complete analogy to the situation in subsection 4.5, $\widetilde{E}_0^{*,*}$

splits into pieces indexed by various collections of arcs, which we shall call *layers*.

We start by analyzing the contributions of the asymmetric generators. Consider the subcomplex B^* in the splitting indexed by a collection A of t arcs of total length l. Since the asymmetric generators come from the direct products of $(n-2)$-spheres, the only nontrivial cases are $q = t(n-2)$, for $t \geq 2$.

Assume first there is a gap between some pair of arcs of length at least 3, and let $x \in V(C_{2r+1})$ be one of the internal points of a gap. If $x = c$, then B^* is isomorphic to the chain complex of a cone with apex in c. Without loss of generality, we can assume that $x = b_i$, for some i. By the previous assumption, $b_{i-1}, b_{i+1} \notin A$. If $a_i \notin A$, but either a_{i-1}, or a_{i+1} (or both) is in A, then B^* is isomorphic to a chain complex of a cone with apex b_i. If $a_{i-1}, a_i, a_{i+1} \notin A$, then B^* is isomorphic to a chain complex of a cone with apex c_i. Finally, assume $a_i \in A$. Define $\phi^k : B^k \to B^{k-1}$ as follows: for a generator $\sigma \in B^k$:

$$\phi^k(\sigma) = \begin{cases} \sigma \setminus \{b_i\}, & \text{if } b_i \in \sigma, \text{ i.e., if } \sigma \cap \{a_i, b_i, c_i\} \text{ is } \{b_i\}, \text{ or } \{b_i, c_i\}; \\ \sigma \setminus \{c_i\}, & \text{if } \sigma \cap \{a_i, b_i, c_i\} = \{a_i, c_i\}; \\ 0, & \text{if } \sigma \cap \{a_i, b_i, c_i\} \text{ is } \emptyset, \text{ or } \{c_i\}, \text{ or } \{a_i\}. \end{cases}$$

Let $\widetilde{B}^*$ be the subcomplex of B^* generated by all σ, such that $a_i \in \sigma$. Clearly, (6.5) is fulfilled both for $\widetilde{B}^*$ and for $B^*/\widetilde{B}^*$. It implies that they are both acyclic, hence so is B^*.

If all gaps are of length at most 2, then $l + 2t \geq 2r + 1$. On the other hand, $B^p = 0$ for $p \leq l/2 - 1$, since $|\vartheta(\sigma)| \leq 2|\sigma| - 1$, for $\sigma \not\subseteq \mathcal{C}$. Recall that $q = t(n-2)$; it follows that the entry $n + r - q - 3$ is 0, since

$$\begin{aligned} l/2 - 1 - (n + r - t(n-2) - 3) &> r - t - 1 - n - r + tn - 2t + 3 \\ &= tn - 3t - n + 2 = (t-1)(n-3) - 1 \geq 1. \end{aligned}$$

Hence B^* is acyclic in the required entry, for all B^* in the group (Gr1).

Next, we move on to the symmetric generators. For $\sigma \subseteq \mathcal{C}$ we call $|\mathcal{C} \setminus \sigma|$ the *total length of gaps*. Let B^* be a subcomplex in the splitting corresponding to a layer from the group (Gr2). The contributing space here is $S^{n-2} \times X/\mathbb{Z}_2$, where X is a direct product of $2t + 1$ $(n-2)$-spheres and $\mathbb{Z}_2$-action is as in Section 5.

If $t = 0$, since the column number is at most $r - 3$, the gap between the $\smile$-arc and the $\frown$-arc is at least 3. This means that B^* is isomorphic to a cochain complex of the simplex; hence is acyclic.

Assume now $t \geq 1$. By examining the cohomology groups of $S^{n-2} \times X/\mathbb{Z}_2$, and taking into account that each of the t pairs of spheres must contribute nontrivially, we see that the dimension of the contributing cocycle of $S^{n-2} \times X/\mathbb{Z}_2$ is at least $n - 2 + t(n-2) = (t+1)(n-2)$; hence the total length of gaps is at least $t(n-2) + 1$. Assume the total length of gaps is at most

$2(t+1)$, as otherwise B^* is isomorphic to a cochain complex of the simplex. By assumptions, $t \geq 1$ and $n \geq 5$, and so unless $(t,n) = (1,5)$, we have

$$t(n-2) + 1 - (2t+2) = t(n-4) - 1 > 0,$$

yielding a contradiction.

Consider the remaining case $(t,n) = (1,5)$. This is the first situation in which we need to analyze the particular entries of $\widetilde{E}_1^{*,*}$. Since we must have a precise equality, the total length of gaps is 4, and the only nontrivial case is provided by generators indexed with $\smile_i$, $\frown_j$, and $\big(_{i+3,j-3}$. The contributing cohomology generator must be indexed $(0,*,\infty)$, so just the set of arcs determines everything.

Let $\alpha_{i,j}$ denote such a generator, and let $\beta_{i,j}$ denote the generator whose indexing set of arcs is $\smile_i$, $\frown_j$, and $\big(_{i+2,j-3}$, and which is also indexed by $(0,*,\infty)$. Both $\alpha_{i,j}$'s, and $\beta_{i,j}$'s are generators in $\widetilde{E}_1^{*,*}$. Consider a linear combination $\sum_{i,j} p_{i,j}\alpha_{i,j}$ lying in the kernel of d_1. Since $d_1(\alpha_{i,j})$ contains $\beta_{i,j}$, $\beta_{i+1,j}$, and no other $\beta_{i',j'}$'s we see that $p_{i,j} \neq 0$ implies $p_{i-1,j} \neq 0$. This leads obviously to $p_{i,j} = 0$ for all i,j; hence B^* is acyclic in the required entry.

Now consider B^* corresponding to a layer from group (Gr3). The contributing space here is $X/\mathbb{Z}_2$, where X is a direct product of $2t+1$ $(n-2)$-spheres and the $\mathbb{Z}_2$-action is as above. Since we are in the row n or higher, we must have $t \geq 1$. The total length of gaps cannot be larger than $2t+1$, since otherwise B^* is isomorphic to a cochain complex of the simplex. On the other hand, since the dimension of the contributing cohomology generator is at least $t(n-2)$, the total length of gaps must be at least $(t-1)(n-2)+1$. Comparing these two we see that

$$(t-1)(n-2) + 1 - (2t+1) = (t-1)(n-4) - 2 > 0,$$

with exceptions: $t = 1$, n is any, $t = 2$, $n = 5, 6$, and $(t,n) = (3,5)$.

Consider first $t = 1$. Since we can have at most 3 gaps, we must have precisely 3 gaps, so the contributing cohomology generators of $S^{n-2} \times S^{n-2} \times S^{n-2}/\mathbb{Z}_2$ must have dimension n. Inspecting the cohomology description of this space from Section 5 we see that there are no generators in dimensions between $n-2$ and $2n-4$. Since $2n-4 > n$ we verify this case.

Assume now $(t,n) = (2,5)$. The only nontrivial case is when the total length of gaps is 4 or 5, and c is in the gaps. Let $\alpha_{i,j}$ denote the generator where the gaps are $\{c, i, i+1, j\}$, $r-2 \geq j \geq i+4$, $i \geq 2$, and $\beta_{i,j}$ denote the generator where the gaps are $\{c, i, j, j+1\}$, $r-3 \geq j \geq i+3$, $i \geq 2$. Let $\gamma_{i,j}$ denote the generator where the gaps are $\{c, i, j\}$, $r-2 \geq j \geq i+3$, $i \geq 2$. Clearly $d_1(\alpha_{i,j}) = \gamma_{i,j} + \gamma_{i+1,j}$, and $d_1(\beta_{i,j}) = \gamma_{i,j} + \gamma_{i,j+1}$. We see that, restricted to the generators $\alpha_{i,j}$, $\beta_{i,j}$, and $\gamma_{i,j}$, we have a chain complex of the graph in Figure 6.3.

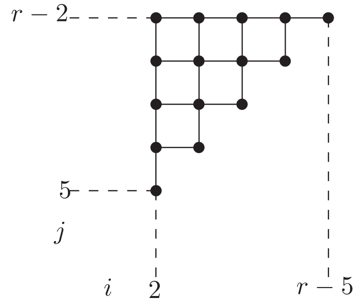

Figure 6.3

The kernel is generated by the elementary squares, to it is enough to see that each square is a coboundary. Indeed, the elementary square with the lower left corner (i,j) is a coboundary of the generator with gaps $\{c,i,i+1,j,j+1\}$.

Finally, assume $(t,n)=(2,6)$ or $(3,5)$. These are the tight cases, in the sense that the lengths of all gaps are predetermined: the top gap consists of just c, and the other 2, resp. 3, gaps are of length 2. Assume that the kernel of d_1 is not zero, and let α be an element in $\ker d_1$. Let g be a generator, which is contained in α with a nonzero coefficient, such that this g maximizes the height of the top gap over all generators appearing with a nonzero coefficient in α. Removing the lower element of the top gap of g gives a generator which cannot be cancelled out by the coboundaries of other elements in α, due to the assumed maximality property. This yields a contradiction, and hence $\ker d_1 = 0$.

We move on to group (Gr4), and let B^* correspond to a generator indexed by $\frown_j$, and t side arcs. We can have at most $2t+2$ gaps. The dimension of the contributing cohomology generator is at least $t(n-2)+n-2$. Thus the total length of the gaps is at least $t(n-2)+1$. Comparing these inequalities we get

$$t(n-2)+1-(2t+2)=t(n-4)-1>0,$$

with the only exception $t=1, n=5$.

Let $(t,n)=(1,5)$. The interesting dimension here is 6; thus the total length of gaps must be exactly 4. The generators α_i are indexed with the collection of arcs $\{(_{3,i}, \frown_{i+3}\}$, for $4 \leq i \leq r-3$. Since $d_1(\alpha_i)$ contains the generator indexed with $\{(_{2,i}, \frown_{i+3}\}$, and this generator is different for different α_i, we see that the only linear combination of α_i's in the kernel of d_1 is the trivial one. Hence, we conclude that the contribution to $\widetilde{E}_2^{*,*}$ is 0.

Finally, we consider the case of generators indexed with collections of arcs avoiding all $\smile$- and $\frown$-arcs. Let us assume there are t such arcs. To avoid a cochain complex of a simplex, the total length of the gaps must be at most $2t+1$. On the other hand, since the dimension of the generator is at least $t(n-2)$, the total length of the gaps must be at least $(t-1)(n-2)+1$.

Comparing, we see that

$$(t-1)(n-2)+1-(2t+1)=(t-1)(n-4)-2>0,$$

with the exceptions $t=1$, any n, $n=5$, $t\le 3$, and $n=6$, $t=2$.

If $t=1$, the only nontrivial case occurs in the row n. Then, in the entry of interest we have only one generator: the one indexed by the arcs $()_{3,r}$. Its coboundary will contain the generator $()_{2,r}$; hence it is different from 0.

Let $(t,n)=(2,5)$. Since we are in the row 6, for dimensional reasons, the total length of gaps in the contributing generator is 4. Thus, we have two types of generators: α_i^1 indexed with arc collections $\{()_{2,i},()_{i+3,r}\}$, $3\le i\le r-4$, and α_i^2 indexed with arc collections $\{()_{3,i},()_{i+2,r}\}$, $4\le i\le r-3$. Considering the value of d_1 on the generator indexed with $\{()_{3,i},()_{i+3,r}\}$, we see that for $i>3$, modulo coboundaries, any generator α_i^1 is a linear combination of the generators α_j^2. The coboundary of α_3^1 contains the generator indexed with $\{()_{2,3},()_{5,r}\}$; hence no element in the kernel of d_1 can contain α_3^1 with a nonzero coefficient. Finally, a nonzero linear combination of α_j^2's cannot lie in the kernel of d_1, since $d_1(\alpha_j^2)$ contains the generator indexed with $\{()_{2,j},()_{j+2,r}\}$, which is different for different j. Again, we conclude that the contribution to $\widetilde{E}_2^{*,*}$ is 0.

Let $(t,n)=(3,5)$. For dimensional reasons, the total length of the gaps is precisely 7, thus we have the generators $\alpha_{i,j}$ indexed with arc collections $\{()_{3,i},()_{i+3,j},()_{j+3,r}\}$, for $4\le i$, $i+4\le j\le r-4$. Since $d_1(\alpha_{i,j})$ contains the generator indexed with $\{()_{2,i},()_{i+3,j},()_{j+3,r}\}$, and these generators are different for different $\alpha_{i,j}$'s, we see that d_1 is injective on the space spanned by $\alpha_{i,j}$'s. Therefore, in this case the contribution to $\widetilde{E}_2^{*,*}$ is 0. The case $(t,n)=(2,6)$ is completely analogous. □

LEMMA 6.6. $E_2^{r+i,n-i-1}=0$, *for all* $i=3,\dots,n-1$.

Proof. Since $|\mathcal{C}|=r+1$, the entries $E_1^{r+i,n-i-1}$, for $i=3,\dots,n-1$, come from $H^{n-i-1}(\mathtt{Hom}\,(C_{2r+1}[\vartheta(\tau)],K_n))$, for $\tau\not\subseteq\mathcal{C}$. We have shown before that these cohomology groups vanish in dimension $n-4$ and less, which implies $E_1^{r+i,n-i-1}=0$; hence $E_2^{r+i,n-i-1}=0$. □

We conclude that $E_\infty^{r+1,n-3}=\mathbb{Z}_2$, contradicting the fact that

$$H^{r+n-2}(\mathtt{Hom}_+(C_{2r+1},K_n)/\mathbb{Z}_2;\mathbb{Z}_2)=0.$$

Therefore, our original assumption that $\varpi_1^{n-2}(\mathtt{Hom}\,(C_{2r+1},K_n))\neq 0$ is wrong, and Theorem 2.3(b) is proved.

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