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The Hopf condition for bilinear forms over arbitrary fields

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Abstract

We settle an old question about the existence of certain 'sums-of-squares' formulas over a field F, related to the composition problem for quadratic forms. A classical theorem says that if such a formula exists over a field of characteristic 0, then certain binomial coefficients must vanish. We prove that this result also holds over fields of characteristic $p > 2$.

1. Introduction

Fix a field F. A classical problem asks for what values of r, s, and n do there exist identities of the form

$$\Big(\sum_{i=1}^{r} x_i^2\Big)\cdot\Big(\sum_{i=1}^{s} y_i^2\Big)=\sum_{i=1}^{n} z_i^2 \tag{1.1}$$

where the z_i's are bilinear expressions in the x's and y's. Equation (1.1) is to be interpreted as a formula in the polynomial ring $F[x_1,\dots,x_r,y_1,\dots,y_s]$; we call it a *sums-of-squares formula of type* $[r,s,n]$.

The question of when such formulas exist has been extensively studied: [L] and [S1] are excellent survey articles, and [S2] is a detailed sourcebook. In this paper we prove the following result, solving Problem C of [L]:

Theorem 1.2. *If F is a field of characteristic not equal to 2, and a sums-of-squares formula of type $[r,s,n]$ exists over F, then $\binom{n}{i}$ must be even for $n-r<i<s$.*

We now give a little history. It is common to let $r *_F s$ denote the smallest n for which a sums-of-squares formula of type $[r,s,n]$ exists. Many papers have studied lower bounds on $r *_F s$, but for a long time such results were known only for fields of characteristic 0: one reduces to a geometric problem over $\mathbb{R}$, and then topological methods are used to obtain the bounds (see [L] for a summary). In this paper we begin the process of extending such results to characteristic p, replacing the topological methods by those of motivic homotopy theory.

The most classical result along these lines is Theorem 1.2 for the particular case $F = \mathbb{R}$, which leads to lower bounds for $r *_{\mathbb{R}} s$. It seems to have been proven in three places, namely [B], [Ho], and [St]; but in modern times the given condition on binomial coefficients is usually called the 'Hopf condition'. The paper [S1] gives some history, and explains how K. Y. Lam and T. Y. Lam deduced the condition for arbitrary fields of characteristic 0. Problem C of [L, p. 188] explicitly asked whether the same condition holds over fields of characteristic $p > 2$. Work on this question had previously been done by Adem [A1], [A2] and Yuzvinsky [Y] for special values of r, s, and n. In [SS] a weaker version of the condition was proved for arbitrary fields and arbitrary values of r, s, and n.

Stiefel's proof of the condition for $F = \mathbb{R}$ used Stiefel-Whitney classes; Behrend's (which worked over any formally real field) used some basic intersection theory; and Hopf deduced it using singular cohomology. Our proof of the general theorem uses a variation of Hopf's method and *motivic* cohomology. It can be regarded as purely algebraic—at least, as 'algebraic' as things like group cohomology and algebraic K-theory. These days it is perhaps not so clear that there exists a point where topology ends and algebra begins.

We now explain Hopf's proof, and our generalization, in more detail. Given a sums-of-squares formula of type $[r, s, n]$, one has in particular a bilinear map $\phi\colon F^r \times F^s \to F^n$ given by $(x_1, \dots, x_r; y_1, \dots y_s) \mapsto (z_1, \dots, z_n)$. If we let q be the quadratic form on F^k given by $q(w_1, \dots, w_k) = w_1^2 + \dots + w_k^2$, then we have $q(\phi(x, y)) = q(x)q(y)$. When $F = \mathbb{R}$ one has that $q(w) = 0$ only when $w = 0$, and so ϕ restricts to a map $(\mathbb{R}^r - 0) \times (\mathbb{R}^s - 0) \to (\mathbb{R}^n - 0)$. The bilinearity of ϕ tells us, in particular, that we can quotient by scalar-multiplication to get $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}$.

On mod 2 cohomology this gives $\mathbb{Z}/2[x]/x^n \to \mathbb{Z}/2[a]/a^r \otimes \mathbb{Z}/2[b]/b^s$, and the bilinearity of ϕ shows that $x \mapsto a + b$. Since $x^n = 0$ and we have a ring map, it follows that $(a + b)^n = 0$ in the target ring. The Hopf condition falls out immediately.

This proof used, in a seemingly crucial way, the fact that over $\mathbb{R}$ a sum of squares is 0 only when all the numbers were zero to begin with. This of course does not work over fields of characteristic p (or over $\mathbb{C}$, for that matter). Our bilinear form gives us a map of schemes $\phi\colon \mathbb{A}^r \times \mathbb{A}^s \to \mathbb{A}^n$, but we cannot say that it restricts to $(\mathbb{A}^r - 0) \times (\mathbb{A}^s - 0) \to (\mathbb{A}^n - 0)$ as we did above.

To remedy the situation, let Q_k denote the projective quadric in $\mathbb{P}^{k+1}$ defined by the equation $w_1^2 + \dots + w_{k+2}^2 = 0$. The bilinear map ϕ induces

$$(\mathbb{P}^{r-1} - \mathrm{Q}_{r-2}) \times (\mathbb{P}^{s-1} - \mathrm{Q}_{s-2}) \to (\mathbb{P}^{n-1} - \mathrm{Q}_{n-2}).$$

In effect, we have removed all possible numbers whose sum-of-squares would give us zero. Let DQ_k denote the *deleted quadric* $\mathbb{P}^k - \mathrm{Q}_{k-1}$ (our convention is that the subscript on a scheme always denotes its dimension). We will compute the mod 2 motivic cohomology of DQ_k (Theorem 2.3), find that it is close to being a truncated polynomial algebra, and repeat Hopf's argument in this new context. As an amusing exercise (cf. [Ln, 6.3]) one can show that over the field $\mathbb{C}$ the space DQ_k—with the complex topology—has the same homotopy type as $\mathbb{R}P^k$; so our argument is in some sense 'the same' as Hopf's in this case.

The idea of using deleted quadrics to deduce the Hopf condition first appeared in [SS]. In that paper the Chow groups of the deleted quadrics were computed, but these are only enough to deduce a weaker version of the Hopf condition (one that is approximately half as powerful). This is explained further in Remark 2.7. On the other hand, we should point out that the full power of motivic cohomology is not completely necessary in this paper: one can also derive the Hopf condition using étale cohomology, by the same arguments (see Remark 2.8). Since in this case computing étale cohomology involves exactly the same steps as computing motivic cohomology, we have gone ahead and computed the stronger invariant.

1.3. *Organization.* Section 2 shows how to deduce the Hopf condition from a few easily stated facts about motivic cohomology. Section 3 outlines in more detail the basic properties of motivic cohomology needed in the rest of the paper. This list is somewhat extensive, but our hope is that it will be accessible to readers not yet acquainted with the motivic theory—most of the properties are analogs of familiar things about singular cohomology. Finally, Section 4 carries out the necessary calculations. We also include an appendix on the Chow groups of quadrics, as several facts about these play a large role in the paper.

2. The basic argument

Because of the nature of the computations that we will make, we use slightly different definitions for the varieties Q_n and DQ_n than those in Section 1. These definitions will remain in effect for the entire paper. Unfortunately, the usefulness of these choices will not become clear until Section 4.

From now on the field F is always assumed not to have characteristic 2.

Definition 2.1. When $n = 2k$, let Q_n be the *projective quadric* in $\mathbb{P}^{n+1}$ defined by the equation $a_1b_1 + a_2b_2 + \cdots + a_{k+1}b_{k+1} = 0$. When $n = 2k+1$, let Q_n be the *projective quadric* in $\mathbb{P}^{n+1}$ defined by the equation $a_1b_1 + a_2b_2 + \cdots + a_{k+1}b_{k+1} + c^2 = 0$. In either case, let DQ_{n+1} be $\mathbb{P}^{n+1} - \mathrm{Q}_n$.

Note that Q_0 is isomorphic to $\operatorname{Spec} F \amalg \operatorname{Spec} F$, and $\mathrm{Q}_1 \cong \mathbb{P}^1$. One possible isomorphism $\mathbb{P}^1 \to \mathrm{Q}_1$ sends $[x, y]$ to $[-x^2, y^2, xy]$.

Occasionally we will need to equip DQ_{n+1} with a basepoint, in which case we will always choose $[1, 1, 0, 0, \ldots, 0]$ (although the choice turns out not to matter).

LEMMA 2.2. *Suppose that the ground field F has a square root of -1 (call it i). Then Q_n is isomorphic to the projective quadric in $\mathbb{P}^{n+1}$ defined by the equation $w_1^2 + \cdots + w_{n+2}^2 = 0$.*

Proof. When $n = 2k$, use the change of coordinates $a_j = w_{2j-1} + iw_{2j}$, $b_j = w_{2j-1} - iw_{2j}$. When $n = 2k+1$, use the same formulas as above for $1 \leq j \leq k+1$ and also let $c = w_{n+2}$. □

We regard $\mathbb{P}^{2k} \hookrightarrow \mathbb{P}^{2k+1}$ as the subscheme defined by $a_{k+1} = b_{k+1}$, and we regard $\mathbb{P}^{2k-1} \hookrightarrow \mathbb{P}^{2k}$ as the subscheme defined by $c = 0$. These choices have the advantage that they give us inclusions $\mathrm{Q}_{n-2} \hookrightarrow \mathrm{Q}_{n-1}$ and $\mathrm{DQ}_{n-1} \hookrightarrow \mathrm{DQ}_n$.

The following theorem states the computation of the motivic cohomology ring $H^{*,*}(\mathrm{DQ}_n; \mathbb{Z}/2)$. In order to understand the statement, the reader needs to know just a few basic facts about motivic cohomology; a more complete account of these facts appears in Section 3. First, $H^{*,*}(-; \mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes, taking its values in bi-graded commutative rings of characteristic 2. If we set $\mathbb{M}_2 = H^{*,*}(\operatorname{Spec} F; \mathbb{Z}/2)$, the map induced by $X \to \operatorname{Spec} F$ makes $H^{*,*}(X; \mathbb{Z}/2)$ into an $\mathbb{M}_2$-algebra. It is known that $\mathbb{M}_2^{0,0} \cong \mathbb{Z}/2$, $\mathbb{M}_2^{0,1} \cong \mathbb{Z}/2$, and the generator $\tau \in \mathbb{M}_2^{0,1}$ is not nilpotent.

THEOREM 2.3. *Assume that every element of F is a square and that* $\mathrm{char}(F) \neq 2$.

(a) *If $n = 2k+1$ then $H^{*,*}(\mathrm{DQ}_n; \mathbb{Z}/2) \cong \mathbb{M}_2[a, b]/(a^2 = \tau b, b^{k+1})$, where a has degree $(1, 1)$ and b has degree $(2, 1)$.*

(b) *If $n = 2k$ then $H^{*,*}(\mathrm{DQ}_n; \mathbb{Z}/2) \cong \mathbb{M}_2[a, b]/(a^2 = \tau b, b^{k+1}, ab^k)$ where a and b are as in part* (a).

(c) *The map $H^{*,*}(\mathrm{DQ}_{n+1}; \mathbb{Z}/2) \to H^{*,*}(\mathrm{DQ}_n; \mathbb{Z}/2)$ sends a to a, and b to b.*

In fact, b is the unique nonzero class in $H^{2,1}$, and a is the unique nonzero class in $H^{1,1}$ that becomes zero when restricted to the basepoint $\operatorname{Spec} F \to \mathrm{DQ}_n$. These facts are needed below in the proof of Proposition 2.5. See the comments before Proposition 4.6 for more details.

Note that if τ were equal to 1 then the above rings would be truncated polynomial algebras (in analogy with the singular cohomology of $\mathbb{R}P^n$).

A more general version of this theorem, without any assumptions on F, appears as Theorem 4.9. The proof is slightly involved, and so will be deferred until Section 4. However, let us at least record how the above statements follow from the more general version:

Proof. If every element of F is a square, then $\mathbb{M}_2^{1,1} = 0$ (see Section 3.2). Therefore, in Theorem 4.9 both ρ and ε are zero. This gives us the formulas in part (a) and (b). Part (c) is Proposition 4.6. □

For us, the most important consequence of the theorem is the following:

Corollary 2.4. *In* $H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2)$ *we have* $a^{n+1}=0$ *and* $a^i\neq 0$ *for* $i\leq n$.

Proof. The claims are immediate from the calculation since all the powers of τ are nonzero. □

Proof of Theorem 1.2. Suppose we have a sums-of-squares formula of type $[r,s,n]$ over F. This remains true if we extend F, and so we may as well assume that every element of F is a square. Therefore, Theorem 2.3 applies.

As explained in Section 1, the sums-of-squares formula gives a map $p\colon \mathrm{DQ}_{r-1}\times\mathrm{DQ}_{s-1}\to\mathrm{DQ}_{n-1}$ (this uses Lemma 2.2) and we will consider the induced map on motivic cohomology. There is a Künneth formula for computing motivic cohomology of products of certain 'cellular' varieties (see Proposition 3.9), and the deleted quadrics belong to this class by Proposition 4.2. In order to apply Proposition 3.9, we also have to observe that $H^{*,*}(\mathrm{DQ}_{r-1};\mathbb{Z}/2)$ is free over $\mathbb{M}_2$, which is apparent from Theorem 2.3.

Therefore p^* is a map

$$H^{*,*}(\mathrm{DQ}_{n-1};\mathbb{Z}/2)\to H^{*,*}(\mathrm{DQ}_{r-1};\mathbb{Z}/2)\otimes_{\mathbb{M}_2}H^{*,*}(\mathrm{DQ}_{s-1};\mathbb{Z}/2).$$

We will use the letters a and b to denote the generators of $H^{*,*}(\mathrm{DQ}_{n-1};\mathbb{Z}/2)$, a_1 and b_1 for the generators of $H^{*,*}(\mathrm{DQ}_{r-1};\mathbb{Z}/2)$, and a_2 and b_2 for the generators of $H^{*,*}(\mathrm{DQ}_{s-1};\mathbb{Z}/2)$.

We show in the following proposition that $p^*(a)=a_1+a_2$. Since the above corollary says that $a^n=0$, it will follow that $(a_1+a_2)^n=0$. Using the corollary again, this can only happen if $\binom{n}{i}$ is even for $n-r<i<s$. □

Proposition 2.5. *Suppose that F is a field of characteristic not* 2 *in which every element is a square. If p^*, a, a_1, and a_2 are as in the above proof, then $p^*(a)=a_1+a_2$.*

Before we can give the proof, we need to state a few more properties of motivic cohomology. Once again, more details are given in Section 3. First, $\mathbb{M}_2^{p,q}$ is nonzero only in the range $q\geq 0$. Second, when every element of F is a square one has $\mathbb{M}_2^{1,1}=0$. Finally, motivic cohomology is $\mathbb{A}^1$-homotopy invariant in the following sense. Let i_0 and i_1 denote the inclusions $\{0\}\hookrightarrow\mathbb{A}^1$ and $\{1\}\hookrightarrow\mathbb{A}^1$, respectively. If $H\colon X\times\mathbb{A}^1\to Y$ is a map of smooth schemes, then the composites $H(\mathrm{Id}\times i_0)$ and $H(\mathrm{Id}\times i_1)$ induce the same map $H^{*,*}(Y;\mathbb{Z}/2)\to H^{*,*}(X;\mathbb{Z}/2)$. Such a map H is called an $\mathbb{A}^1$-*homotopy* from $H(\mathrm{Id}\times i_0)$ to $H(\mathrm{Id}\times i_1)$.

Proof. Because $p^*(a)$ has degree $(1,1)$, it must be of the form $\varepsilon_1a_1+\varepsilon_2a_2+m\cdot 1$, where m belongs to $\mathbb{M}_2^{1,1}$ and ε_1 and ε_2 belong to $\mathbb{M}_2^{0,0}\cong\mathbb{Z}/2$. Since

$\mathbb{M}_2^{1,1} = 0$ under our assumptions on F, we can ignore m. To show that $\varepsilon_1 = 1$, in light of Theorem 2.3(c) it would suffice to verify that the map

$$\mathrm{DQ}_1 \times \{*\} \to \mathrm{DQ}_{r-1} \times \mathrm{DQ}_{s-1} \to \mathrm{DQ}_{n-1}$$

is $\mathbb{A}^1$-homotopic to the standard inclusion $\mathrm{DQ}_1 \hookrightarrow \mathrm{DQ}_{n-1}$. (A similar argument will show that $\varepsilon_2 = 1$.) Actually we will not quite do this, but instead verify that the composition

$$j\colon \mathrm{DQ}_1 \times \{*\} \to \mathrm{DQ}_{r-1} \times \mathrm{DQ}_{s-1} \to \mathrm{DQ}_{n-1} \hookrightarrow \mathrm{DQ}_{n+1}$$

is $\mathbb{A}^1$-homotopic to the standard inclusion. By Theorem 2.3(c) again, this is enough.

For the rest of this section we will use the coordinates $w_1, \dots, w_{n+2}$ on $\mathbb{P}^{n+1}$ given in Lemma 2.2. Recall that ϕ is our bilinear map $F^r \times F^s \to F^n$. Let $e_1, \dots, e_k$ be the standard basis for F^k, and let $\phi(e_1, e_1) = (u_1, \dots, u_n)$ and $\phi(e_2, e_1) = (v_1, \dots, v_n)$. Then the map $j\colon \mathrm{DQ}_1 \to \mathrm{DQ}_{n+1}$ has the form

$$[a, b] \mapsto [u_1 a + v_1 b, u_2 a + v_2 b, \dots, u_n a + v_n b, 0, 0],$$

and the sums-of-squares formula satisfied by ϕ tells us that

$$u_1^2 + \cdots + u_n^2 = 1, \quad v_1^2 + \cdots + v_n^2 = 1, \quad \text{and} \quad u_1 v_1 + \cdots + u_n v_n = 0.$$

Note that the standard inclusion $\mathrm{DQ}_1 \hookrightarrow \mathrm{DQ}_{n+1}$ has the same description but where $(u_1, \dots, u_n) = (1, 0, \dots, 0)$ and $(v_1, \dots, v_n) = (0, 1, 0, \dots, 0)$. The following lemma gives the desired $\mathbb{A}^1$-homotopy, since both the map j and the standard inclusion are homotopic to the map $[a, b] \mapsto [0, 0, \dots, 0, a, b]$. □

For the following statement, recall that we are still using the coordinates on $\mathbb{P}^{n+1}$ given by Lemma 2.2.

Lemma 2.6. *Suppose that F contains a square root of -1. Let u and v be vectors in F^n such that $\Sigma_j u_j^2 = 1 = \Sigma_j v_j^2$ and $\Sigma_j u_j v_j = 0$. Then the map $f : \mathrm{DQ}_1 \to \mathrm{DQ}_{n+1}$ given by*

$$[a, b] \mapsto [u_1 a + v_1 b, u_2 a + v_2 b, \dots, u_n a + v_n b, 0, 0]$$

is $\mathbb{A}^1$-homotopic to the map $[a, b] \mapsto [0, 0, \dots, 0, a, b]$.

Proof. Let i be a square root of -1. First define a homotopy $\mathrm{DQ}_1 \times \mathbb{A}^1 \to \mathrm{DQ}_{n+1}$ by the formula

$$([a, b], t) \mapsto [u_1 a + v_1 b, u_2 a + v_2 b, \dots, u_n a + v_n b, ta - tib, tia + tb].$$

This shows that f is homotopic to g, where g is the map

$$[a, b] \mapsto [u_1 a + v_1 b, u_2 a + v_2 b, \dots, u_n a + v_n b, a - ib, ia + b].$$

Now define another homotopy $\mathrm{DQ}_1 \times \mathbb{A}^1 \to \mathbb{P}^{n+1}$ by the formula

$$([a, b], t) \mapsto [tu_1 a + tv_1 b, tu_2 a + tv_2 b, \dots, tu_n a + tv_n b, a - tib, tia + b].$$

The assumptions on the u's and v's imply that the sum of the squares in the image is exactly equal to a^2+b^2, which is nonzero because $[a,b]$ lies in DQ_1. So this is actually a homotopy $\mathrm{DQ}_1 \times \mathbb{A}^1 \to \mathrm{DQ}_{n+1}$, showing that g is homotopic to the desired map. □

Remark 2.7. In [SS] a weaker version of the Hopf condition was obtained by computing the Chow ring $\mathrm{CH}^*(\mathrm{DQ}_n)$, which essentially corresponds to the subring of $H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2)$ generated by b (see Property (A) in Section 3). This amounts to seeing about half of what motivic cohomology sees.

Remark 2.8. When F has a square root of -1, a theorem of [Lv] says that the étale cohomology ring $H^*_{\mathrm{et}}(\mathrm{DQ}_n;\mu_2^{\otimes *})$ is isomorphic to

$$H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2)[\tau^{-1}] \cong H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2) \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}]$$

(see Property (I) below). Since $H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2)$ is free over $\mathbb{M}_2$, this localization is particularly simple: it is precisely a truncated polynomial algebra $\mathbb{M}_2[\tau^{-1}][a]/a^{n+1}$. So the Hopf condition could have been proven using étale cohomology.

Remark 2.9. When every element of F is a square, it follows from the proof of the Milnor conjecture [V2] that $\mathbb{M}_2 \cong \mathbb{Z}/2[\tau]$. We never needed this, but it is useful to keep in mind.

3. Review of motivic cohomology

The theory now called motivic cohomology was first developed in two main places, namely [Bl1] and [VSF] (together with many associated papers). The paper [V3] proved that the two approaches give isomorphic theories. Below we recall the basic properties of motivic cohomology needed in the paper. For various reasons it is difficult to give simple references to [VSF] so most of our citations will be to [SV, Sec. 3] and the lecture notes [MVW].

3.1. *Basic properties.* For every field F, motivic cohomology is a contravariant functor $H^{*,*}(-)$ from the category of smooth schemes of finite type over F to the category of bi-graded commutative rings. Commutativity means that if $a \in H^{p,q}(X)$ and $b \in H^{s,t}(X)$ then $ab = (-1)^{ps}ba$. For the basic construction we refer the reader to [SV, Sec. 3] or [MVW, Sec. 3]. The list of properties below is far from complete, and in some cases we only give crude versions of more interesting properties—but this is all we will need in the present paper.

The scheme $\operatorname{Spec} F$ will often be denoted by "pt", and we denote $H^{*,*}(pt)$ by $\mathbb{M}$. The ring $\mathbb{M}$ can be very complicated (and is, in general, unknown). The motivic cohomology of a scheme is naturally a graded-commutative algebra over $\mathbb{M}$.

Property A. The graded subring $\oplus_n H^{2n,n}(X)$ is naturally isomorphic to the Chow ring $\mathrm{CH}^*(X)$ [Bl1, p. 268], [MVW, p. 4; Lect. 17].

In particular, $\mathbb{M}^{0,0} = \mathbb{Z}$. In general, $H^{*,*}(X)$ is isomorphic to the higher Chow groups of X [V3, Cor. 1.2].

Property B. For a closed inclusion $j\colon Z \hookrightarrow X$ of smooth schemes of codimension c, there is a long exact sequence of the form

$$\cdots \to H^{*-2c,*-c}(Z) \xrightarrow{j_!} H^{*,*}(X) \to H^{*,*}(X-Z) \to H^{*-2c+1,*-c}(Z) \to \cdots .$$

The map $j_!$ is called the 'Gysin map' or the 'pushforward', and it is a map of $\mathbb{M}$-modules. The long exact sequence is called the *Gysin*, *localization*, or *purity* sequence [Bl1, Sec. 3], [Bl2].

Property C. Let i_0 and i_1 denote the inclusions $\{0\} \hookrightarrow \mathbb{A}^1$ and $\{1\} \hookrightarrow \mathbb{A}^1$, respectively. If $H\colon X \times \mathbb{A}^1 \to Y$ is a map of smooth schemes, then the composites $H(\mathrm{Id}\times i_0)$ and $H(\mathrm{Id}\times i_1)$ induce the same map $H^{*,*}(Y) \to H^{*,*}(X)$. Such a map H is called an $\mathbb{A}^1$-homotopy from $H(\mathrm{Id} \times i_0)$ to $H(\mathrm{Id} \times i_1)$ [Bl1, Sec. 2], [SV, Prop. 4.2].

Property D. $H^{*,*}(\mathbb{P}^n) = \mathbb{M}[t]/(t^{n+1})$, where t has degree $(2,1)$ [SV, Prop. 4.4].

Property E. If $E \to B$ is an algebraic fiber bundle (i.e., a map which is locally a product in the Zariksi topology) whose fiber is an affine space $\mathbb{A}^n$, then $H^{*,*}(B) \to H^{*,*}(E)$ is an isomorphism.

Property (E) is easy to prove by induction on the size of a trivializing cover, and by use of the Mayer-Vietoris sequence [SV, Prop. 4.1] together with Property (C).

Property F. $\mathbb{M}^{p,q} = 0$ if $q < 0$, if $p > q \geq 0$, or if $q = 0$ and $p < 0$ [MVW, p. 4; Th. 3.5].

Property G. $\mathbb{M}^{1,1} = F^*$ and $\mathbb{M}^{0,1} = 0$ [Bl1, Th. 6.1], [MVW, p. 4,(2)].

3.2. *Finite coefficients.* For every $n \in \mathbb{Z}$ there is also a theory $H^{*,*}(-;\mathbb{Z}/n)$ which is related to $H^{*,*}(-)$ by a natural long exact sequence of the form

$$(3.3)\quad \cdots \to H^{*,*}(X) \xrightarrow{\times n} H^{*,*}(X) \to H^{*,*}(X;\mathbb{Z}/n) \to H^{*+1,*}(X) \xrightarrow{\times n} \cdots .$$

For the definition see [MVW, Def. 3.4]. The theory satisfies the analogs of Properties (B) through (F) above.

Let $\mathbb{M}_2$ denote $H^{*,*}(pt;\mathbb{Z}/2)$. Since $\mathbb{M}$ may contain 2-torsion, $\mathbb{M}_2$ is not necessarily the same as $\mathbb{M}/(2)$—rather, there is a long exact sequence of the form

$$\cdots \to \mathbb{M}^{p,q} \xrightarrow{\times 2} \mathbb{M}^{p,q} \to \mathbb{M}_2^{p,q} \to \mathbb{M}^{p+1,q} \to \cdots .$$

This sequence, together with Property (F) and the fact that $\mathbb{M}^{0,0} = \mathbb{Z}$, tells us that $\mathbb{M}_2^{0,0} = \mathbb{Z}/2$. Note that $H^{*,*}(X;\mathbb{Z}/2)$ is naturally a commutative algebra over $\mathbb{M}_2$.

Since $\mathbb{M}^{1,1} = F^*$ and $\mathbb{M}^{0,1} = \mathbb{M}^{2,1} = 0$, we get the exact sequence

$$0 \to \mathbb{M}_2^{0,1} \to F^* \xrightarrow{\times 2} F^* \to \mathbb{M}_2^{1,1} \to 0 \tag{3.4}$$

where the map $F^* \to F^*$ sends x to x^2. The usual notation is to let $\tau \in \mathbb{M}_2^{0,1}$ denote the class which maps to -1, and to let $\rho \in \mathbb{M}_2^{1,1}$ denote the image of -1. If F has a square root of -1 then $\rho = 0$. Moreover, if every element of F is a square then $\mathbb{M}_2^{1,1} = 0$.

3.5. *The Bockstein.* The Bockstein map $\beta\colon H^{*,*}(-;\mathbb{Z}/2) \to H^{*+1,*}(-;\mathbb{Z}/2)$ is defined in the usual manner from the maps in the sequence (3.3). A direct consequence of the definition (as in topology) is that $\beta^2 = 0$. Note that $\beta(\tau) = \rho$.

Property H. For all $a, b \in H^{*,*}(X;\mathbb{Z}/2)$, $\beta(ab) = \beta(a)b + a\beta(b)$ [Lv, Lem. 6.1].

3.6. *Relation with étale cohomology.* There is a natural map of bi-graded rings $\eta\colon H^{*,*}(X;\mathbb{Z}/n) \to H^*_{\mathrm{et}}(X;\mu_n^{\otimes *})$ (cf. [MVW, Th. 10.2], for example). In the case $n = 2$, the element τ maps to the class of -1 in $H^0_{\mathrm{et}}(pt;\mu_2) \cong \{1,-1\}$, and multiplication by this class is an isomorphism on étale cohomology. Note in particular that this implies that the powers of τ are all nonzero in $H^{0,*}(pt;\mathbb{Z}/2)$.

Property I. The induced map $H^{*,*}(X;\mathbb{Z}/2)[\tau^{-1}] \to H^*_{\mathrm{et}}(X;\mu_2^{\otimes *})$ is an isomorphism for any smooth scheme X, provided that F has a square root of -1 [Lv].

The construction of the map η from [MVW] makes it clear that the Bockstein on $H^{*,*}(-;\mathbb{Z}/2)$ (which can be regarded as induced by the extension $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$) is compatible with the Bockstein on étale cohomology induced by $0 \to \mu_2 \to \mu_4 \to \mu_2 \to 0$. If the field contains a square root of -1 then we can identify μ_4 with $\mathbb{Z}/4$, and of course μ_2 with $\mathbb{Z}/2$. These observations will be used in the proof of Theorem 4.9.

3.7. *Reduced cohomology.* Given any basepoint of a scheme X (i.e., a map $pt \to X$), the kernel of the induced map $H^{*,*}(X) \to H^{*,*}(pt)$ is the reduced cohomology of X and is denoted by $\tilde{H}^{*,*}(X)$. A similar definition applies with $\mathbb{Z}/n$-coefficients. The above map has a splitting (induced by $X \to pt$), and thus $H^{*,*}(X) \cong \mathbb{M} \oplus \tilde{H}^{*,*}(X)$ as $\mathbb{M}$-modules. Similarly, $H^{*,*}(X;\mathbb{Z}/2) \cong \mathbb{M}_2 \oplus \tilde{H}^{*,*}(X;\mathbb{Z}/2)$.

3.8. *A Künneth theorem.* Let $\mathcal{C}$ denote the smallest class of smooth schemes satisfying the following properties:

(1) $\mathcal{C}$ contains the affine spaces $\mathbb{A}^k$.

(2) If $Z \hookrightarrow X$ is a closed inclusion of smooth schemes and $\mathcal{C}$ contains two of X, Z, and $X - Z$, then it also contains the third.

(3) If $E \to B$ is an algebraic fiber bundle whose fiber is an affine space, then $E \in \mathcal{C}$ if and only if $B \in \mathcal{C}$.

The following result is a modest generalization of [J, Th. 4.5], and can be proven using the same techniques. A complete proof, for a more general class of schemes than $\mathcal{C}$, is given in [DI, Th. 8.12].

PROPOSITION 3.9. *Suppose X and Y are smooth schemes, with at least one of them belonging to $\mathcal{C}$. If either $H^{*,*}(X)$ or $H^{*,*}(Y)$ is free as an $\mathbb{M}$-module, then there is a Künneth isomorphism of bi-graded rings*

$$H^{*,*}(X) \otimes_{\mathbb{M}} H^{*,*}(Y) \cong H^{*,*}(X \times Y).$$

Similarly, if either $H^{*,*}(X;\mathbb{Z}/n)$ or $H^{*,*}(Y;\mathbb{Z}/n)$ is free as an $H^{*,*}(pt;\mathbb{Z}/n)$-module, then there is a Künneth isomorphism of bi-graded rings

$$H^{*,*}(X;\mathbb{Z}/n) \otimes_{H^{*,*}(pt;\mathbb{Z}/n)} H^{*,*}(Y;\mathbb{Z}/n) \cong H^{*,*}(X \times Y;\mathbb{Z}/n).$$

4. Computations

In this section F is an arbitrary ground field not of characteristic 2. We will study the quadrics Q_n and DQ_n. Note that in this generality Lemma 2.2 does not apply; therefore, Q_n and DQ_n cannot necessarily be redefined in terms of sums of squares. We assume $\mathrm{char}(F) \neq 2$ so that Q_n is smooth for all n, not just even n.

PROPOSITION 4.1. *If n is odd, $H^{*,*}(\mathrm{Q}_n)$ is a free module over $\mathbb{M}$ with generators in degrees $(0,0),(2,1),(4,2),\ldots,(2n,n)$. If n is even, $H^{*,*}(\mathrm{Q}_n)$ is a free $\mathbb{M}$-module with generators in degrees $(0,0),(2,1),\ldots,(2n,n)$ plus an extra generator in degree $(n,\frac{n}{2})$.*

Proof. The proof is by induction. The result for Q_0 is obvious, and the result for $\mathrm{Q}_1 \cong \mathbb{P}^1$ is Property (D).

Except for the base cases in the previous paragraph, the argument for the odd and even cases is identical. We give details only for the even case, and let $n = 2k$. Let Z be the $(n-1)$-dimensional subscheme defined by $a_1 = 0$, and let $U = \mathrm{Q}_n - Z$. Note that Z is singular (it is the projective cone on Q_{n-2}), and that $U \cong \mathbb{A}^n$. Let $Q' = \mathrm{Q}_n - \{[0,1,0,0,\ldots,0]\}$, and let $Z' = Z - \{[0,1,0,0,\ldots,0]\}$. Then $Z' \hookrightarrow Q'$ is a smooth pair, with complement $\mathbb{A}^n$. So the localization sequence for $Z' \hookrightarrow Q'$ gives an isomorphism $\tilde{H}^{*,*}(Q') \cong H^{*-2,*-1}(Z')$. The projection map $Z' \to \mathrm{Q}_{n-2}$ which forgets the first two homogeneous coordinates is a fiber bundle with fiber $\mathbb{A}^1$; hence $H^{*,*}(Z') \cong H^{*,*}(\mathrm{Q}_{n-2})$ by Property (E).

Taking the computations of the previous paragraph together, we conclude that $H^{*,*}(Q') \cong \mathbb{M} \oplus H^{*-2,*-1}(\mathrm{Q}_{n-2})$. By induction, this is free over $\mathbb{M}$ with one generator in each degree $(0,0),(2,1),\ldots,(2n-2,n-1)$ plus an extra generator in degree $(n,\frac{n}{2})$.

Finally, we consider the localization sequence for $\{[0,1,0,\ldots,0]\} \hookrightarrow \mathrm{Q}_n$. This has the form

$$\cdots \leftarrow H^{*-2n+1,*-n}(pt) \xleftarrow{\delta} H^{*,*}(Q') \leftarrow H^{*,*}(\mathrm{Q}_n) \leftarrow H^{*-2n,*-n}(pt) \leftarrow \cdots .$$

The generators for $H^{*,*}(Q')$ (as an $\mathbb{M}$-module) must map to zero under δ for dimension reasons. It follows that

$$0 \leftarrow H^{*,*}(Q') \leftarrow H^{*,*}(\mathrm{Q}_n) \leftarrow H^{*-2n,*-n}(pt) \leftarrow 0$$

a short exact sequence of $\mathbb{M}$-modules, in which the outer terms are known to be free. So the middle term is a direct sum of the outer terms. The right term provides a generator of degree $(2n,n)$, and the left term provides the rest of the generators. □

The above proof also shows the following:

Proposition 4.2. *The schemes* Q_n *and* DQ_n *belong to the class* $\mathcal{C}$ *from Section* 3.8.

Proof. If one knows by induction that Q_{n-2} belongs to $\mathcal{C}$ then so do Z', Q', and Q_n, in that order.

The fact that projective spaces belong to the class $\mathcal{C}$ is trivial: one uses the standard algebraic cell decomposition (cf. [F, 1.9.1]). Then since Q_{n-1} and $\mathbb{P}^n$ are both in $\mathcal{C}$, so is DQ_n. □

By Proposition 4.1, in order to understand the ring structure on $H^{*,*}(\mathrm{Q}_n)$ it suffices just to understand the subring $H^{2*,*}(\mathrm{Q}_n) \cong \mathrm{CH}^*(\mathrm{Q}_n)$, because the $\mathbb{M}$-algebra generators lie in degrees $(2*,*)$. The computation of this Chow ring is well-known; the additive computation can be found in [Sw, 13.3], for instance, and the ring structure is stated in [KM]. For the reader's convenience, and because we need several of the auxiliary facts, we give a complete account in Appendix A. These ideas lead to the following result, whose proof is essentially the content of Theorem A.4 and Theorem A.10.

PROPOSITION 4.3.

(a) *If* $n = 2k+1$, *then as a ring* $H^{*,*}(\mathrm{Q}_n) = \mathbb{M}[x,y]/(x^{k+1} - 2y, y^2)$ *where* x *has degree* $(2,1)$ *and* y *has degree* $(2k+2, k+1)$.

(b) *If* $n = 2k$ *and* k *is odd, then* $H^{*,*}(\mathrm{Q}_n) = \mathbb{M}[x,y]/(x^{k+1} - 2xy, y^2)$ *where* x *has degree* $(2,1)$ *and* y *has degree* $(2k, k)$.

(c) *If* $n = 2k$ *and* k *is even, then* $H^{*,*}(\mathrm{Q}_n) = \mathbb{M}[x,y]/(x^{k+1} - 2xy, y^2 - x^k y)$ *where* x *has degree* $(2,1)$ *and* y *has degree* $(2k, k)$.

We will now consider the motivic cohomology of the deleted quadrics DQ_n. The idea is to use the localization sequence

(4.4)

$$\cdots \leftarrow H^{*-1,*-1}(\mathrm{Q}_{n-1}) \leftarrow H^{*,*}(\mathrm{DQ}_n) \xleftarrow{i^*} H^{*,*}(\mathbb{P}^n) \xleftarrow{j_!} H^{*-2,*-1}(\mathrm{Q}_{n-1}) \leftarrow \cdots .$$

By Proposition 4.1 the cohomology of Q_{n-1} has generators as an $\mathbb{M}$-module in degrees $(2*, *)$, so we can completely determine the $\mathbb{M}$-module map $j_!$ just by understanding the pushforward map $\mathrm{CH}^{*-1}(\mathrm{Q}_{n-1}) \to \mathrm{CH}^*(\mathbb{P}^n)$ of Chow groups. For the quadrics, this is discussed in detail in the appendix: all maps are either the identity or multiplication by 2. However, a problem now occurs. Because the ground ring $\mathbb{M}$ might have 2-torsion, the kernel and cokernel of $j_!$ will not necessarily be free over $\mathbb{M}$—so we run into complicated extension problems. As a result, we have not been able to compute the integral motivic cohomology of DQ_n. The problem goes away if we work with $\mathbb{Z}/2$ coefficients.

PROPOSITION 4.5. *If* F *is a field with* $\mathrm{char}(F) \neq 2$, *then* $H^{*,*}(\mathrm{DQ}_n; \mathbb{Z}/2)$ *is a free* $\mathbb{M}_2$*-module with one generator in degree* $(i, \lceil \frac{i}{2} \rceil)$ *for each* $0 \leq i \leq n$, *where* $\lceil \frac{i}{2} \rceil$ *is the smallest integer that is at least* $\frac{i}{2}$.

Proof. The argument from Proposition 4.1 shows that $H^{*,*}(\mathrm{Q}_{n-1}; \mathbb{Z}/2)$ is free over $\mathbb{M}_2$ on the same set of generators as before, and the map of subrings $H^{2*,*}(\mathrm{Q}_{n-1}) \to H^{2*,*}(\mathrm{Q}_{n-1}; \mathbb{Z}/2)$ is just quotienting by the ideal (2).

By Lemma A.6, we know that the Gysin map

$$j_! \colon H^{2i,i}(\mathrm{Q}_{n-1}) \to H^{2i+2,i+1}(\mathbb{P}^n)$$

is multiplication by 2 for $0 \leq i < \frac{n-1}{2}$, and is an isomorphism for $\frac{n-1}{2} < i \leq n-1$. If n is odd, then it is the fold map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ for $i = \frac{n-1}{2}$.

The goal is to use the $\mathbb{Z}/2$-analog of (4.4), so we first have to understand the Gysin map $j_!$ with $\mathbb{Z}/2$-coefficients. Since $H^{2*,*}(\mathrm{Q}_{n-1}; \mathbb{Z}/2)$ and $H^{2*,*}(\mathbb{P}^n; \mathbb{Z}/2)$ are both obtained from integral cohomology simply by quotienting by the ideal (2), it follows that the Gysin map with $\mathbb{Z}/2$-coefficients is an isomorphism, zero, or the fold map in all degrees $(2*, *)$. Since the generators (as $\mathbb{M}_2$-modules) live in these degrees, we find that the kernel and cokernel of $j_!\colon H^{*,*}(\mathrm{Q}_{n-1};\mathbb{Z}/2) \to H^{*,*}(\mathbb{P}^n;\mathbb{Z}/2)$ are both free over $\mathbb{M}_2$.

If $n = 2k$, then the generators for $\operatorname{coker} j_!$ are in degrees $(0,0)$, $(2,1)$, $(4,2),\dots,(2k,k)$, and the generators for $\ker j_!$ are in degrees $(0,0),(2,1),\dots,$ $(2k-2,k-1)$. If $n = 2k+1$, then the generators are the same, except that $\ker j_!$ has another generator in degree $(2k,k)$.

From the $\mathbb{Z}/2$-analog of (4.4), we have the short exact sequence

$$0 \leftarrow \ker j_! \leftarrow H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2) \leftarrow \operatorname{coker} j_! \leftarrow 0.$$

It follows that the middle group is also free over $\mathbb{M}_2$. Be aware that the left map shifts degrees by $(-1,-1)$. □

We know $\mathbb{M}_2^{0,0} \cong \mathbb{Z}/2$. From Property (F) it follows that $\mathbb{M}_2^{p,q} = 0$ if $q < 0$, if $q = 0$ and $p < 0$, or if $p > q \geq 0$. So the above calculation shows that $H^{1,1}(\mathrm{DQ}_n;\mathbb{Z}/2) \cong \mathbb{M}_2^{1,1} \oplus \mathbb{M}_2^{0,0}$, where the first summand comes from the motivic cohomology of $\operatorname{Spec} F$. Hence, there is a unique nonzero element $a \in \tilde{H}^{1,1}(\mathrm{DQ}_n;\mathbb{Z}/2)$. When $n > 1$ the calculation gives $H^{2,1}(\mathrm{DQ}_n;\mathbb{Z}/2) \cong \mathbb{Z}/2$, and we let b denote the unique nonzero element. For $n = 1$ we have $\mathrm{DQ}_1 \cong \mathbb{A}^1 - 0$, and it is known that $H^{2,1}(\mathbb{A}^1 - 0;\mathbb{Z}/2) = 0$ (see, for instance, [V1, Lem. 6.8]). In this case we define $b = 0$ by convention.

Proposition 4.6. *The map $H^{*,*}(\mathrm{DQ}_{n+1};\mathbb{Z}/2) \to H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2)$ induced by the inclusion takes a to a and b to b.*

Proof. In light of the definitions of a and b in the previous paragraph, we just need to show that $H^{i,1}(\mathrm{DQ}_{n+1};\mathbb{Z}/2) \to H^{i,1}(\mathrm{DQ}_n;\mathbb{Z}/2)$ is surjective for $i = 1$ or $i = 2$. Consider the diagram

$$\begin{tikzcd}
H^{i+1,1}(\mathbb{P}^{n+1};\mathbb{Z}/2) \arrow[d] & H^{i-1,0}(\mathrm{Q}_n;\mathbb{Z}/2) \arrow[l] \arrow[d] & H^{i,1}(\mathrm{DQ}_{n+1};\mathbb{Z}/2) \arrow[l] \arrow[d] & H^{i,1}(\mathbb{P}^{n+1};\mathbb{Z}/2) \arrow[l] \arrow[d] \\
H^{i+1,1}(\mathbb{P}^{n};\mathbb{Z}/2) & H^{i-1,0}(\mathrm{Q}_{n-1};\mathbb{Z}/2) \arrow[l] & H^{i,1}(\mathrm{DQ}_{n};\mathbb{Z}/2) \arrow[l] & H^{i,1}(\mathbb{P}^{n};\mathbb{Z}/2) \arrow[l]
\end{tikzcd}$$

in which the rows are localization sequences. The left and right vertical maps are isomorphisms. Finally, the cohomology groups of the quadrics are both isomorphic to $\mathbb{M}_2^{i-1,0}$, and the map between them is the identity. It follows from a diagram chase that the desired map is surjective. □

Lemma 4.7. *$\beta(a) = b$ in $H^{*,*}(\mathrm{DQ}_n;\mathbb{Z}/2)$, where β is the Bockstein.*

Proof. We look at the long exact sequence

$$\cdots \to H^{1,1}(\mathrm{DQ}_n) \xrightarrow{\times 2} H^{1,1}(\mathrm{DQ}_n) \to H^{1,1}(\mathrm{DQ}_n;\mathbb{Z}/2) \xrightarrow{\delta} H^{2,1}(\mathrm{DQ}_n) \to \cdots.$$

The localization sequence (4.4) for integral cohomology, together with the identification of $j_!$ in Lemma A.6, show that $\mathrm{DQ}_n \hookrightarrow \mathbb{P}^n$ induces an isomorphism on $H^{1,1}(-;\mathbb{Z})$. It follows that if a were the mod 2 reduction of an integral class, it would also be the image of a class in $H^{1,1}(\mathbb{P}^n;\mathbb{Z}/2)$. But $* \hookrightarrow \mathbb{P}^n$ induces an isomorphism on $H^{1,1}(-;\mathbb{Z}/2)$, whereas the class a in $H^{1,1}(\mathrm{DQ}_n;\mathbb{Z}/2)$ restricts to zero on the basepoint. We conclude that a cannot be the mod 2 reduction of an integral class, and therefore $\delta(a)$ is nonzero.

The sequence (4.4) (again with our knowledge of $j_!$) also shows that $H^{2,1}(\mathrm{DQ}_n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$, with the generator in the image of the map $H^{2,1}(\mathbb{P}^n;\mathbb{Z}) \to H^{2,1}(\mathrm{DQ}_n;\mathbb{Z})$. It follows that $\delta(a)$ is the unique nonzero element of $H^{2,1}(\mathrm{DQ}_n;\mathbb{Z})$, and the mod 2 reduction of $\delta(a)$ is b. □

We need one more lemma before stating the final result.

LEMMA 4.8. *$H^{2k+1,k+1}(\mathrm{DQ}_{2k+2};\mathbb{Z}/2) \to H^{2k+1,k+1}(\mathrm{DQ}_{2k+1};\mathbb{Z}/2)$ is injective.*

Proof. Consider the diagram

$$\begin{tikzcd}
H^{2k,k}(\mathrm{Q}_{2k+1};\mathbb{Z}/2) \arrow[d] & H^{2k+1,k+1}(\mathrm{DQ}_{2k+2};\mathbb{Z}/2) \arrow[l] \arrow[d] & H^{2k+1,k+1}(\mathbb{P}^{2k+2};\mathbb{Z}/2) \arrow[l] \arrow[d] & \arrow[l] \\
H^{2k,k}(\mathrm{Q}_{2k};\mathbb{Z}/2) & H^{2k+1,k+1}(\mathrm{DQ}_{2k+1};\mathbb{Z}/2) \arrow[l] & H^{2k+1,k+1}(\mathbb{P}^{2k+1};\mathbb{Z}/2) \arrow[l] & \arrow[l, "f"']
\end{tikzcd}$$

in which the rows are portions of localization sequences. We first claim that the map labelled f (in the lower right corner) is zero. Note that the domain of f is $H^{2k-1,k}(\mathrm{Q}_{2k};\mathbb{Z}/2) \cong \mathbb{M}_2^{1,1}x$ where x is the generator of $H^{2k-2,k-1}(\mathrm{Q}_{2k};\mathbb{Z}/2) \cong \mathrm{CH}^{k-1}(\mathrm{Q}_{2k}) \otimes \mathbb{Z}/2$. The codomain is isomorphic to $\mathbb{M}_2^{1,1} \cdot y$, where y is the generator of $H^{2k,k}(\mathbb{P}^{2k+1};\mathbb{Z}/2) \cong \mathrm{CH}^k(\mathbb{P}^{2k+1}) \otimes \mathbb{Z}/2$. These two facts follow from Proposition 4.1 and Properties (D) and (F). The map f is a Gysin map, and is therefore a map of $\mathbb{M}_2$-modules. But the Gysin map takes x to $2y$ by Lemma A.6, and so after reducing mod 2 the image of x is zero.

Now note that the right-most vertical map is an isomorphism. A diagram chase would give us the desired result, if we knew that the left vertical map was injective. But this map equals the mod 2 reduction of the map $\mathrm{CH}^k(\mathrm{Q}_{2k+1}) \to \mathrm{CH}^k(\mathrm{Q}_{2k})$. We look at the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] & \mathrm{CH}^k(\mathrm{Q}_{2k+1}) \arrow[r] & \mathrm{CH}^k(\mathrm{Q}_{2k}) \arrow[r, equal] & \mathbb{Z} \oplus \mathbb{Z} \\
\mathbb{Z} \arrow[r, equal] & \mathrm{CH}^k(\mathbb{P}^{2k+2}) \arrow[u, "\cong"] \arrow[r, "\cong"] & \mathrm{CH}^k(\mathbb{P}^{2k+1}) \arrow[u] \arrow[r, equal] & \mathbb{Z}\,.
\end{tikzcd}$$

Proposition A.3 shows that the left vertical map is an isomorphism. Lemma A.9 identifies the right vertical map as the diagonal, and from that information the result follows at once. □

THEOREM 4.9. *Let F be a field with* $\mathrm{char}(F) \neq 2$.

(a) *If* $n = 2k+1$ *then* $H^{*,*}(\mathrm{DQ}_n; \mathbb{Z}/2) \cong \mathbb{M}_2[a, b]/(a^2 = \rho a + \tau b, b^{k+1})$ *where* a *has degree* $(1, 1)$ *and* b *has degree* $(2, 1)$.

(b) *If* $n = 2k$, *there exists an element* ε *in* $\mathbb{M}_2^{1,1}$ *such that* $H^{*,*}(\mathrm{DQ}_n; \mathbb{Z}/2) \cong \mathbb{M}_2[a, b]/(a^2 = \rho a + \tau b, b^{k+1}, ab^k = \varepsilon b^k)$ *where* a *and* b *are as in* (a).

Remark 4.10. We have not been able to identify the class ε in any nontrivial case. This is not important for proving the Hopf condition, but it would be satisfying to resolve the issue of whether ε is equal to 0, or ρ, or some other element.

Proof. For convenience we will drop subscripts and superscripts: $Q = \mathrm{Q}_{n-1}$, $\mathbb{P} = \mathbb{P}^n$, and $\mathrm{DQ} = \mathrm{DQ}_n$. We know $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2)$ additively by Proposition 4.5, so that we just need to determine the ring structure.

Note that the map $H^{2i,i}(\mathbb{P}) \to H^{2i,i}(\mathrm{DQ})$ is surjective because it is the map $\mathrm{CH}^i(\mathbb{P}) \to \mathrm{CH}^i(\mathrm{DQ})$. Therefore, the nonzero element t of $H^{2,1}(\mathbb{P}; \mathbb{Z}/2)$ goes to the nonzero element b of $H^{2,1}(\mathrm{DQ}; \mathbb{Z}/2)$. Then t^i maps to b^i, and surjectivity implies that b^i must be the unique nonzero element in $H^{2i,i}(\mathrm{DQ}; \mathbb{Z}/2)$ for $1 \leq i \leq \frac{n}{2}$.

Lemma 4.7 showed that $\beta(a) = b$. Since $\beta^2 = 0$ one has $\beta(b) = 0$, so that Property (H) implies that $\beta(ab^i) = b^{i+1}$. In particular ab^i is nonzero for $0 \leq i \leq \frac{n}{2} - 1$.

Now $\tilde{H}^{1,1}(\mathrm{DQ}; \mathbb{Z}/2) \cong (\mathbb{M}_2^{0,0})a$ and $H^{2i-1,i}(\mathrm{DQ}; \mathbb{Z}/2) \cong \mathbb{M}_2^{0,0} \oplus \mathbb{M}_2^{1,1} b^{i-1}$ for $1 \leq i \leq \frac{n+1}{2}$, where the first factor arises from the generator in degree $(2i-1, i)$. Property (H) and the fact that $\mathbb{M}_2^{2,1} = 0$ implies that $\beta(x) = 0$ for any $x \in \mathbb{M}_2^{1,1} b^{i-1}$. So we cannot have $ab^i \in \mathbb{M}_2^{1,1} b^{i-1}$.

Based on our knowledge of $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2)$ as an $\mathbb{M}_2$-module, we can now conclude that when $n = 2k$ the classes $1, b, b^2, \ldots, b^k$ and $a, ab, ab^2, \ldots, ab^{k-1}$ are a free basis for $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2)$ over $\mathbb{M}_2$.

The argument is slightly harder when $n = 2k + 1$, because we must show that ab^k is nonzero (even though its Bockstein is zero). However, we already know that ab^k is nonzero in $H^{*,*}(\mathrm{DQ}_{n+1}; \mathbb{Z}/2)$. The map

$$H^{2k+1,k+1}(\mathrm{DQ}_{n+1}; \mathbb{Z}/2) \to H^{2k+1,k+1}(\mathrm{DQ}_n; \mathbb{Z}/2)$$

is an injection by Lemma 4.8 and takes ab^k to ab^k by Proposition 4.6. It follows that $1, b, \ldots, b^k, a, ab, \ldots, ab^k$ is a free basis for $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2)$ when $n = 2k + 1$.

We next identify a^2. This part of the argument exactly parallels [V1, pp. 20-21]. The class $a^2 \in \tilde{H}^{2,2}(\mathrm{DQ}; \mathbb{Z}/2)$ must be a linear combination over $\mathbb{M}_2$ of the elements a and b: $a^2 = Aa + Bb$ where $A \in \mathbb{M}_2^{1,1}$ and $B \in \mathbb{M}_2^{0,1} \cong \mathbb{Z}/2$. To identify A it is sufficient to look at the image of a^2 under $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2) \to H^{*,*}(\mathrm{DQ}_1; \mathbb{Z}/2)$, since $Aa + Bb$ goes to Aa under this map by Proposition 4.6 and the fact that $b = 0$ in $H^{*,*}(\mathrm{DQ}_1; \mathbb{Z}/2)$. Note that DQ_1 is isomorphic to

$\mathbb{A}^1 - 0$, and one knows that $H^{*,*}(\mathbb{A}^1 - 0; \mathbb{Z}/2) \cong \mathbb{M}_2[a]/(a^2 = \rho a)$ by [V1, Lem. 6.8]. So $A = \rho$.

To identify B, let K be the field consisting of F with a square root of -1 adjoined (unless F already has a square root of -1, in which case $K = F$). Let DQ_K be the base change of DQ along the map $\operatorname{Spec} K \to \operatorname{Spec} F$. Under the induced map $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2) \to H^{*,*}(\mathrm{DQ}_K; \mathbb{Z}/2)$, ρ maps to zero; so $\rho a + Bb$ maps to Bb. Hence it suffices to assume that F contains a square root of -1 and show that $a^2 = \tau b$.

Under the map $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2) \to H^*_{\mathrm{et}}(\mathrm{DQ}; \mu_2^{\otimes *})$ the element τ becomes invertible (cf. Property (I)), and so we can write $a = \tau a'$ (in $H^*_{\mathrm{et}}(\mathrm{DQ}; \mu_2^{\otimes *})$), for some $a' \in H^1_{\mathrm{et}}(pt; \mu_2^0)$. This group is sheaf cohomology with coefficients in the constant sheaf $\mathbb{Z}/2$; if β_{et} is the Bockstein on étale cohomology induced by $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ one has that $\beta_{\mathrm{et}}(a') = (a')^2$ by a standard property of the Bockstein on sheaf cohomology (the proof is the same as the one in topology). Our remarks in Section 3.6 show that the Bocksteins in motivic and étale cohomology are compatible, because F has a square root of -1. So we now compute that

$$a^2 = \tau^2 (a')^2 = \tau^2 \beta_{\mathrm{et}}(a') = \tau \cdot \beta(\tau a') = \tau \cdot \beta(a) = \tau \cdot b \tag{4.11}$$

in $H^*_{\mathrm{et}}(\mathrm{DQ}; \mu_2^{\otimes *})$. Note that the third equality uses the analog of Property (H) for étale cohomology, together with the fact that $\beta(\tau) = \rho = 0$ (by our assumption on F).

As a consequence of (4.11), we have in particular that a^2 is nonzero in $H^{*,*}(\mathrm{DQ}; \mathbb{Z}/2)[\tau^{-1}]$. But $a^2 = Bb$, so B must be nonzero. From the sequence (3.4) we recall that $\mathbb{M}_2^{0,1} = \{0, \tau\}$, and so $B = \tau$. We have therefore shown that $a^2 = \rho a + \tau b \in H^{2,2}(\mathrm{DQ}; \mathbb{Z}/2)$.

This finishes part (a) of the theorem. For part (b) we just observe that $ab^k \in H^{2k+1,k+1}(\mathrm{DQ}; \mathbb{Z}/2)$, and $H^{2k+1,k+1}(\mathrm{DQ}; \mathbb{Z}/2) \cong \mathbb{M}_2^{1,1} b^k$. So for some $\varepsilon \in \mathbb{M}_2^{1,1}$ we have $\varepsilon b^k = ab^k$. This finishes part (b). □

Remark 4.12. When n is odd, the cohomology of DQ_n is the same as the cohomology of the scheme $(\mathbb{A}^n - 0)/\pm 1$, which was essentially computed by Voevodsky in [V1, Th. 6.10]. With some effort it can be proven that these two schemes are $\mathbb{A}^1$-homotopy equivalent.

Appendix A. Chow groups of quadrics

This appendix contains a calculation of the Chow rings of the quadrics Q_n, as well as various pushforward and pullback maps. This is classical, but the details are useful and we do not have a suitable reference. We assume a basic familiarity with the Chow ring; see [F] or [H, App. A].

Let $\mathrm{CH}_i(X)$ be the Chow group of dimension i cycles on X. If $Z \hookrightarrow X$ is a closed subscheme there is an exact sequence

$$\mathrm{CH}_i(Z) \to \mathrm{CH}_i(X) \to \mathrm{CH}_i(X - Z) \to 0$$

where the first map is pushforward and the second map is restriction.

If $X \subseteq \mathbb{P}^n$ is a closed subscheme, we let $\not\Sigma X \subseteq \mathbb{P}^{n+1}$ denote the projective cone on X. Let $\not\Sigma\colon \mathrm{CH}_i(X) \to \mathrm{CH}_{i+1}(\not\Sigma X)$ be the map sending a cycle to the projective cone on the cycle, and recall that this is an isomorphism for $i \geq 0$. Also note that $\mathrm{CH}_0(\not\Sigma X) = \mathbb{Z}$ no matter what X is. Finally, recall that $\mathrm{CH}_i(\mathbb{A}^n) = 0$ if $i \neq n$, whereas $\mathrm{CH}_n(\mathbb{A}^n) = \mathbb{Z}$.

When X is nonsingular one defines $\mathrm{CH}^i(X) = \mathrm{CH}_{\dim X - i}(X)$.

The following discussion is modeled on [Sw, 13.3].

A.1. *The odd-dimensional case.* Consider the quadric $\mathrm{Q}_{2k+1} \hookrightarrow \mathbb{P}^{2k+2}$ defined by $a_1 b_1 + \cdots + a_{k+1} b_{k+1} + c^2 = 0$. We let j be the inclusion.

Lemma A.2. *For all* $0 \leq i \leq 2k+1$, *the Chow group* $\mathrm{CH}_i(\mathrm{Q}_{2k+1})$ *is isomorphic to* $\mathbb{Z}$. *The pushforward map* $j_* : \mathrm{CH}_i(\mathrm{Q}_{2k+1}) \to \mathrm{CH}_i(\mathbb{P}^{2k+2})$ *is an isomorphism if* $0 \leq i \leq k$, *and is multiplication by* 2 (*as a map* $\mathbb{Z} \to \mathbb{Z}$) *if* $k+1 \leq i \leq 2k+1$.

Proof. The first claim follows immediately from Proposition 4.1 and Property (A).

The proof of the second statement is by induction. The base case Q_1 is isomorphic to $\mathbb{P}^1$, and Q_1 is imbedded in $\mathbb{P}^2$ as a degree two hypersurface. So j_* is an isomorphism for $i = 0$ and is multiplication by 2 for $i = 1$.

If Z is the closed subscheme defined by $a_1 = 0$, we know $\mathrm{Q}_{2k+1} - Z \cong \mathbb{A}^{2k+1}$ and $Z \cong \not\Sigma \mathrm{Q}_{2k-1}$. The resulting localization sequence gives us a diagram

$$\begin{tikzcd}
\mathrm{CH}_i(\not\Sigma \mathrm{Q}_{2k-1}) \arrow[r] \arrow[d] & \mathrm{CH}_i(\mathrm{Q}_{2k+1}) \arrow[r] \arrow[d, "j_*"] & \mathrm{CH}_i(\mathbb{A}^{2k+1}) \\
\mathrm{CH}_i(\not\Sigma \mathbb{P}^{2k}) \arrow[r] & \mathrm{CH}_i(\mathbb{P}^{2k+2}) &
\end{tikzcd}$$

in which the top row is exact. Since $\not\Sigma \mathbb{P}^{2k}$ is isomorphic to $\mathbb{P}^{2k+1}$, the bottom horizontal arrow is an isomorphism for all $0 \leq i \leq 2k+1$, and both groups in the bottom row are isomorphic to $\mathbb{Z}$.

For $0 \leq i \leq 2k$, the first two groups in the top row are also isomorphic to $\mathbb{Z}$. For $0 \leq i \leq k$, the left vertical arrow is known by induction to be an isomorphism. The only possibility is that the map j_* is an isomorphism in this range.

Now for $k+1 \leq i \leq 2k$, the left vertical arrow is known by induction to be multiplication by 2. Since the upper left horizontal arrow is a surjection, the only possibility is that the map j_* is multiplication by 2.

Finally, for the case $i = 2k+1$ note that Q_{2k+1} is a degree 2 hypersurface in $\mathbb{P}^{2k+2}$. Thus, the fundamental class $[\mathrm{Q}_{2k+1}]$ maps to twice the generator of $\mathrm{CH}_{2k+1}(\mathbb{P}^{2k+2})$. $\square$

By analyzing the above proof, one can give explicit generators for $\mathrm{CH}_i(\mathrm{Q}_{2k+1})$. If $0 \leq i \leq k$, the generator is the class of the cycle determined by setting all coordinates equal to zero except for $b_1, \dots, b_{i+1}$. Note that this cycle is isomorphic to $\mathbb{P}^i$. On the other hand, if $k+1 \leq i \leq 2k+1$, then the generator is the class of the cycle determined by setting $a_1, \dots, a_{2k+1-i}$ equal to zero. Note that this cycle is the iterated projective cone on $\mathrm{Q}_{2i-2k-1}$, and also the intersection of Q_{2k+1} with a copy of $\mathbb{P}^{i+1}$.

We next want to compute the ring structure on $\mathrm{CH}^*(\mathrm{Q}_{2k+1})$ as well as the pullback map $j^* : \mathrm{CH}_i(\mathbb{P}^{2k+2}) \to \mathrm{CH}_{i-1}(\mathrm{Q}_{2k+1})$. It is easier to do the latter first.

Proposition A.3. *The map* $j^*\colon \mathrm{CH}_i(\mathbb{P}^{2k+2}) \to \mathrm{CH}_{i-1}(\mathrm{Q}_{2k+1})$ *is an isomorphism if* $k+2 \leq i \leq 2k+2$ *and is multiplication by* 2 *if* $1 \leq i \leq k+1$.

Proof. The projection formula $j_*(a \cdot j^* b) = (j_* a) \cdot b$ gives us

$$j_*(j^*[\mathbb{P}^i]) = j_*([\mathrm{Q}_{2k+1}] \cdot j^*[\mathbb{P}^i]) = j_*([\mathrm{Q}_{2k+1}]) \cdot [\mathbb{P}^i] = 2[\mathbb{P}^{2k+1}] \cdot [\mathbb{P}^i] = 2[\mathbb{P}^{i-1}].$$

In other words the composition $j_* j^* : \mathrm{CH}_i(\mathbb{P}^{2k+2}) \to \mathrm{CH}_{i-1}(\mathbb{P}^{2k+2})$ is multiplication by 2. When $1 \leq i \leq k+1$, the map $j_*\colon \mathrm{CH}_{i-1}(\mathrm{Q}_{2k+1}) \to \mathrm{CH}_{i-1}(\mathbb{P}^{2k+2})$ is an isomorphism, so j^* must be multiplication by 2. When $k+2 \leq i \leq 2k+2$, the map j_* is multiplication by 2, so j^* must be an isomorphism. $\square$

It is now easy to deduce the ring structure on $\mathrm{CH}^*(\mathrm{Q}_{2k+1})$, using the map from $\mathrm{CH}^*(\mathbb{P}^{2k+2})$. Note that when $k = 0$ we are looking at $\mathrm{Q}_1 \cong \mathbb{P}^1$, and so $\mathrm{CH}^*(\mathrm{Q}_1)$ is isomorphic to $\mathbb{Z}[y]/y^2$, where y has degree 1.

Theorem A.4. *If* $k \geq 0$, *then* $\mathrm{CH}^*(\mathrm{Q}_{2k+1}) \cong \mathbb{Z}[x, y]/(x^{k+1} - 2y, y^2)$, *where* x *has degree* 1 *and* y *has degree* $k+1$.

Proof. The map $j^*\colon \mathrm{CH}^i(\mathbb{P}^{2k+2}) \to \mathrm{CH}^i(\mathrm{Q}_{2k+1})$ (which now preserves the grading because we are grading by codimension) is an isomorphism if $0 \leq i \leq k$ and is multiplication by 2 if $k+1 \leq i \leq 2k+1$. This follows immediately from the previous proposition simply by regrading.

Let t be the generator of $\mathrm{CH}^1(\mathbb{P}^{2k+2})$, and let $x = j^*(t)$. Then $x^{k+1} = j^*(t^{k+1})$ is twice a generator of $\mathrm{CH}^{k+1}(\mathrm{Q}_{2k+1})$, and we let y be this generator. The desired isomorphism of rings follows immediately from our knowledge of the groups $\mathrm{CH}^*(\mathrm{Q}_{2k+1})$ and the description of j^* in the previous paragraph. $\square$

A.5. *The even-dimensional case.* This case is a little harder. The quadric Q_{2k} is defined by $a_1 b_1 + \cdots + a_{k+1} b_{k+1} = 0$. As before, let j be the inclusion

$Q_{2k} \hookrightarrow \mathbb{P}^{2k+1}$. Many of the results from the previous section carry over to this section with identical proofs.

The base case is Q_0, which is $pt \amalg pt$. Note that $j_* : CH_0(Q_0) \to CH_0(\mathbb{P}^1)$ is the fold map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$.

We already know from Proposition 4.1 that the Chow group $CH_i(Q_{2k})$ is isomorphic to $\mathbb{Z}$ for all $0 \leq i \leq 2k$, except that $CH_k(Q_{2k})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The same arguments as in the proof of Lemma A.2 allow us to conclude that the pushforward map $j_* : CH_i(Q_{2k}) \to CH_i(\mathbb{P}^{2k+1})$ is an isomorphism if $0 \leq i \leq k-1$, is multiplication by 2 if $k+1 \leq i \leq 2k$, and is the fold map if $i = k$. We summarize these facts (with cohomological grading, and for both the even and odd cases) in the following lemma—this result is critical for the computations in Section 4.

LEMMA A.6. *For any n, the map $j_*\colon CH^i(Q_{n-1}) \to CH^{i+1}(\mathbb{P}^n)$ is multiplication by 2 for $0 \leq i < \frac{n-1}{2}$, and is an isomorphism for $\frac{n-1}{2} < i \leq n-1$. If n is odd, then it is the fold map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ for $i = \frac{n-1}{2}$.*

Once again, one can give explicit generators for $CH_i(Q_{2k})$. For $i \neq k$, the description of these generators is the same as in the odd case. For $i = k$, one generator is determined by $b_1 = b_2 = \cdots = b_{k+1} = 0$, and the other generator is determined by $a_1 = b_2 = \cdots = b_{k+1} = 0$. We let α and β represent these two codimension k cycles.

LEMMA A.7. *Let α' be the cycle determined by $a_1 = a_2 = \cdots = a_{k+1} = 0$, and let β' be the cycle determined by $b_1 = a_2 = \cdots = a_{k+1} = 0$. If k is odd, then $\alpha = \alpha'$ and $\beta = \beta'$ in $CH_k(Q_{2k})$. If k is even, then $\alpha = \beta'$ and $\beta = \alpha'$ in $CH_k(Q_{2k})$.*

Proof. The result is a consequence of Theorems II and III in [HP, §XIII.4]. Alternatively, one can easily write down explicit homotopies. For instance, if k is odd let $H\colon \mathbb{P}^k \times \mathbb{A}^1 \to Q_{2k}$ be given by

$$[a_1, a_2, \ldots, a_{k+1}], t \mapsto [(1-t)a_1, ta_2, (1-t)a_2, -ta_1, (1-t)a_3, ta_4, \ldots, -ta_k]$$

(where the last four coordinates are $(1-t)a_k, ta_{k+1}, (1-t)a_{k+1}, -ta_k$). Let Z denote the image of the closed inclusion $\mathbb{P}^k \times \mathbb{A}^1 \hookrightarrow Q_{2k} \times \mathbb{A}^1$ given by $(x,t) \mapsto (H(x,t), t)$. Then Z gives a rational equivalence between α and α' by intersecting Z with $Q_{2k} \times \{0\}$ and $Q_{2k} \times \{1\}$, similarly to [F, Ex. 2.6.6]. The same kind of homotopy allows one to deduce the other rational equivalences as well. □

LEMMA A.8. *Let $[*]$ be the fundamental class of a point in $CH^{2k}(Q_{2k})$. If k is odd, then $\alpha \cdot \alpha = 0 = \beta \cdot \beta$ and $\alpha \cdot \beta = [*]$ in the Chow ring $CH^*(Q_{2k})$. If k is even, then $\alpha \cdot \alpha = [*] = \beta \cdot \beta$ and $\alpha \cdot \beta = 0$.*

Proof. When k is odd, $\alpha \cdot \alpha = \alpha \cdot \alpha'$. However, α and α' do not intersect, so $\alpha \cdot \alpha' = 0$. Similarly, $\beta \cdot \beta = 0$. Now α and β' intersect transversely at a point, so $\alpha \cdot \beta = \alpha \cdot \beta' = [*]$. Similar arguments apply to the even case. □

As in Proposition A.3, the map $j^*\colon \mathrm{CH}_i(\mathbb{P}^{2k+1}) \to \mathrm{CH}_{i-1}(\mathrm{Q}_{2k})$ is an isomorphism if $k+2 \le i \le 2k+1$ and is multiplication by 2 if $1 \le i \le k$. After regrading by codimension, this says $j^*\colon \mathrm{CH}^i(\mathbb{P}^{2k+1}) \to \mathrm{CH}^i(\mathrm{Q}_{2k})$ is an isomorphism for $0 \le i < k$ and multiplication by 2 for $k < i \le 2k$. The same argument with the projection formula also shows that when $i = k$, j^* takes the generator to $u\alpha + (2-u)\beta$ for some $u \in \mathbb{Z}$.

Lemma A.9. *The map* $j^*\colon \mathrm{CH}^k(\mathbb{P}^{2k+1}) \to \mathrm{CH}^k(\mathrm{Q}_{2k})$ *sends the generator* t^k *to* $\alpha + \beta$.

Proof. We already know that $j^*(t^{2k}) = 2[*]$, where $[*]$ is the fundamental class of a point in Q_{2k} and is also the generator of $\mathrm{CH}^{2k}(\mathrm{Q}_{2k})$. Therefore,

$$\begin{aligned} 2[*] = j^*(t^{2k}) = (j^*(t^k))^2 &= (u\alpha + (2-u)\beta)^2 \\ &= u^2\alpha^2 + 2u(2-u)\alpha\beta + (2-u)^2\beta^2. \end{aligned}$$

If k is odd, Lemma A.8 lets us rewrite this equation as $2[*] = 2u(2-u)[*]$, so that $u = 1$. If k is even, Lemma A.8 gives $2[*] = (u^2 + (2-u)^2)[*]$, so that again $u = 1$. □

Theorem A.10. *If* k *is odd, then there is an isomorphism of rings* $\mathrm{CH}^*(\mathrm{Q}_{2k}) \cong \mathbb{Z}[x,y]/(x^{k+1} - 2xy, y^2)$, *where* x *has degree* 1 *and* y *has degree* k. *If* k *is even, then* $\mathrm{CH}^*(\mathrm{Q}_{2k}) \cong \mathbb{Z}[x,y]/(x^{k+1} - 2xy, y^2 - x^k y)$, *where* x *has degree* 1 *and* y *has degree* k.

Proof. Let t be the generator $[\mathbb{P}^{2k}]$ of $\mathrm{CH}^1(\mathbb{P}^{2k+1})$, and let $x = j^*(t)$. Then $j^*(t^i) = x^i$. As we know that j^* takes generators to generators for $0 \le i \le k-1$, it follows that x^i is a generator for $\mathrm{CH}^i(\mathrm{Q}_{2k})$ in these dimensions.

Now $x^k = j^*(t^k) = \alpha + \beta$ by the previous lemma. If we let y equal α, then x^k and y are two generators for $\mathrm{CH}^k(\mathrm{Q}_{2k})$. Note that Lemma A.8 implies that $x^k y = [*]$ since $\alpha(\alpha+\beta) = [*]$ in both the even and odd cases.

Next we can compute that

$$j_*(x^i \cdot y) = j_*(j^*(t^i) \cdot y) = t^i \cdot j_* y = t^i \cdot t^k = t^{k+i}$$

for $1 \le t \le k$. Since j_* is an isomorphism in codimension $k+i$, it follows that $x^i y$ is a generator in $\mathrm{CH}^{k+i}(\mathrm{Q}_{2k})$.

Now $x^{k+1} = j^*(t^{k+1}) = j^*(j_*(xy)) = 2xy$. Also, for dimension reasons $x^{k+1}y = 0$. Finally, Lemma A.8 shows that $y^2 = 0$ if k is odd and $y^2 = [*] = x^k y$ if k is even.

Thus, we have shown that the additive generators for $\mathrm{CH}^*(\mathrm{Q}_{2k})$ are 1, x, $x^2, \ldots, x^k$, y, $xy, \ldots, x^k y$, where the elements are listed in order of increasing degree. Moreover, we have constructed a ring map from the desired ring to $\mathrm{CH}^*(\mathrm{Q}_{2k})$ which is an additive isomorphism. □

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