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Pseudodifferential operators on manifolds with a Lie structure at infinity

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Abstract

We define and study an algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ of pseudodifferential operators canonically associated to a noncompact, Riemannian manifold M_0 whose geometry at infinity is described by a Lie algebra of vector fields $\mathcal{V}$ on a compactification M of M_0 to a compact manifold with corners. We show that the basic properties of the usual algebra of pseudodifferential operators on a compact manifold extend to $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$. We also consider the algebra $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ of differential operators on M_0 generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$, and show that $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is a microlocalization of $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$. Our construction solves a problem posed by Melrose in 1990. Finally, we introduce and study semi-classical and "suspended" versions of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$.

Contents

Introduction

Let (M_0, g_0) be a complete, noncompact Riemannian manifold. It is a fundamental problem to study the geometric operators on M_0. As in the compact case, pseudodifferential operators provide a powerful tool for that purpose, provided that the geometry at infinity is taken into account. One needs, however, to restrict to suitable classes of noncompact manifolds.

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Let M be a compact manifold with corners such that $M_0 = M \smallsetminus \partial M$, and assume that the geometry at infinity of M_0 is described by a Lie algebra of vector fields $\mathcal{V} \subset \Gamma(M; TM)$; that is, M_0 is a Riemannian manifold with a Lie structure at infinity, Definition 1.3. In [27], Melrose has formulated a far reaching program to study the analytic properties of geometric differential operators on M_0. An important ingredient in Melrose's program is to define a suitable pseudodifferential calculus $\Psi^\infty_{\mathcal{V}}(M_0)$ on M_0 adapted in a certain sense to $(M, \mathcal{V})$. This pseudodifferential calculus was called a "microlocalization of $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$" in [27], where $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ is the algebra of differential operators on M_0 generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$. (See §2.)

Melrose and his collaborators have constructed the algebras $\Psi^\infty_{\mathcal{V}}(M_0)$ in many special cases, see for instance [9], [21], [22], [23], [26], [28], [30], [47], and especially [29]. One of the main reasons for considering the compactification M is that the geometric operators on manifolds with a Lie structure at infinity identify with degenerate differential operators on M. This type of differential operator appears naturally, for example, also in the study of boundary value problems on manifolds with singularities. Numerous important results in this direction were obtained also by Schulze and his collaborators, who typically worked in the framework of the Boutet de Monvel algebras. See [39], [40] and the references therein. Other important cases in which this program was completed can be found in [15], [16], [17], [35], [37]. An earlier important motivation for the construction of these algebras was the method of layer potentials for boundary value problems and questions in analysis on locally symmetric spaces. See for example [4], [5], [6], [8], [18], [19], [24], [32].

An outline of the construction of the algebras $\Psi^\infty_{\mathcal{V}}(M_0)$ was given by Melrose in [27], provided certain compact manifolds with corners (blow-ups of M^2 and M^3) can be constructed. In the present paper, we modify the blow-up construction using Lie groupoids, thus completing the construction of the algebras $\Psi^\infty_{\mathcal{V}}(M_0)$. Our method relies on recent progress achieved in [2], [7], [35].

The explicit construction of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ microlocalizing $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ in the sense of [27] is, roughly, as follows. First, $\mathcal{V}$ defines an extension of TM_0 to a vector bundle $A \to M$ ($M_0 = M \smallsetminus \partial M$). Let $V_r := \{d(x,y) < r\} \subset M_0^2$ and $(A)_r = \{v \in A, \|v\| < r\}$. Let $r > 0$ be less than the injectivity radius of M_0 and $V_r \ni (x,y) \mapsto (x, \tau(x,y)) \in (A)_r$ be a local inverse of the Riemannian exponential map $TM_0 \ni v \mapsto \exp_x(-v) \in M_0 \times M_0$. Let χ be a smooth function on A with support in $(A)_r$ and $\chi = 1$ on $(A)_{r/2}$. For any $a \in S^m_{1,0}(A^*)$, we define

$$(1) \quad \big[a_i(D)u\big](x) = (2\pi)^{-n} \int_{M_0} \left(\int_{T^*_x M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x,y)) a(x,\eta) u(y)\, d\eta \right) dy.$$

The algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is then defined as the linear span of the operators $a_\chi(D)$ and $b_\chi(D)\exp(X_1)\dots\exp(X_k)$, $a \in S^{\infty}(A^*)$, $b \in S^{-\infty}(A^*)$, and $X_j \in \mathcal{V}$, and where $\exp(X_j) : \mathcal{C}^{\infty}_{\mathrm{c}}(M_0) \to \mathcal{C}^{\infty}_{\mathrm{c}}(M_0)$ is defined as the action on functions associated to the flow of the vector field X_j.

The operators $b_\chi(D)\exp(X_1)\dots\exp(X_k)$ are needed to make our space closed under composition. The introduction of these operators is in fact a crucial ingredient in our approach to Melrose's program. The results of [7], [35] are used to show that $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is closed under composition, which is the most difficult step in the proof.

A closely related situation is encountered when one considers a product of a manifold with a Lie structure at infinity M_0 by a Lie group G and operators G invariant on $M_0 \times G$. We obtain in this way an algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0; G)$ of G–invariant pseudodifferential operators on $M_0 \times G$ with similar properties. The algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0; G)$ arises in the study of the analytic properties of differential geometric operators on some higher dimensional manifolds with a Lie structure at infinity. When $G = \mathbb{R}^q$, this algebra is slightly smaller than one of Melrose's suspended algebras and plays the same role, namely, it appears as a quotient of an algebra of the form $\Psi^{\infty}_{1,0,\mathcal{V}'}(M_0')$, for a suitable manifold M_0'. The quotient map $\Psi^{\infty}_{1,0,\mathcal{V}'}(M_0') \to \Psi^{\infty}_{1,0,\mathcal{V}}(M_0; G)$ is a generalization of Melrose's indicial map. A convenient approach to indicial maps is provided by groupoids [17].

We also introduce a semi-classical variant of the algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$, denoted $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0[[h]])$, consisting of semi-classical families of operators in $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$. For all these algebras we establish the usual mapping properties between appropriate Sobolev spaces.

The article is organized as follows. In Section 1 we recall the definition of manifolds with a Lie structure at infinity and some of their basic properties, including a discussion of compatible Riemannian metrics. In Section 2 we define the spaces $\Psi^{m}_{1,0,\mathcal{V}}(M_0)$ and the principal symbol maps. Section 3 contains the proof of the crucial fact that $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is closed under composition, and therefore it is an algebra. We do this by showing that $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is the homomorphic image of $\Psi^{\infty}_{1,0}(\mathcal{G})$, where $\mathcal{G}$ is any d-connected Lie groupoid integrating A (d–connected means that the fibers of the domain map d are connected). In Section 4 we establish several other properties of the algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ that are similar and analogous to the properties of the algebra of pseudodifferential operators on a compact manifold. In Section 5 we define the algebras $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0[[h]])$ and $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0; G)$, which are generalizations of the algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$. The first of these two algebras consists of the semi-classical (or adiabatic) families of operators in $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$. The second algebra is a subalgebra of the algebra of G–invariant, properly supported pseudodifferential operators on $M_0 \times G$, where G is a Lie group.

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1. Manifolds with a Lie structure at infinity

For the convenience of the reader, let us recall the definition of a Riemannian manifold with a Lie structure at infinity and some of its basic properties.

1.1. *Preliminaries.* In the sequel, by a *manifold* we shall always understand a C^∞-manifold *possibly with corners*, whereas a *smooth manifold* is a C^∞-manifold *without corners* (and without boundary). By definition, every point p in a manifold with corners M has a coordinate neighborhood diffeomorphic to $[0,\infty)^k \times \mathbb{R}^{n-k}$ such that the transition functions are smooth up to the boundary. If p is mapped by this diffeomorphism to $(0,\dots,0,x_{k+1},\dots,x_n)$, we shall say that p is a point of *boundary depth* k and write $\operatorname{depth}(p) = k$. The closure of a connected component of points of boundary depth k is called a *face of codimension* k. Faces of codimension 1 are also-called *hyperfaces.* For simplicity, we always assume that each hyperface H of a manifold with corners M is an embedded submanifold and has a defining function, that is, that there exists a smooth function $x_H \geq 0$ on M such that

$$H = \{x_H = 0\} \;\text{ and }\; dx_H \neq 0 \;\text{ on }\; H.$$

For the basic facts on the analysis of manifolds with corners we refer to the forthcoming book [25]. We shall denote by ∂M the union of all nontrivial faces of M and by M_0 the interior of M, i.e., $M_0 := M \smallsetminus \partial M$. Recall that a map $f : M \to N$ is a *submersion of manifolds with corners* if df is surjective at any point and $df_p(v)$ is an inward pointing vector if, and only if, v is an inward pointing vector. In particular, the sets $f^{-1}(q)$ are smooth manifolds (no boundary or corners).

To fix notation, we shall denote the sections of a vector bundle $V \to X$ by $\Gamma(X,V)$, unless X is understood, in which case we shall write simply $\Gamma(V)$. A Lie subalgebra $\mathcal{V} \subseteq \Gamma(M,TM)$ of the Lie algebra of all smooth vector fields on M is said to be *a structural Lie algebra of vector fields* provided it is a finitely generated, projective $\mathcal{C}^\infty(M)$-module and each $V \in \mathcal{V}$ is tangent to all hyperfaces of M.

Definition 1.1. A *Lie structure at infinity* on a smooth manifold M_0 is a pair $(M,\mathcal{V})$, where M is a compact manifold, possibly with corners, and

$\mathcal{V} \subset \Gamma(M, TM)$ is a structural Lie algebra of vector fields on M with the following properties:

(a) M_0 is diffeomorphic to the interior $M \smallsetminus \partial M$ of M.

(b) For any vector field X on M_0 and any $p \in M_0$, there are a neighborhood V of p in M_0 and a vector field $Y \in \mathcal{V}$, such that $Y = X$ on V.

A manifold with a Lie structure at infinity will also be called a *Lie manifold*.

Here are some examples.

Examples 1.2. (a) Take $\mathcal{V}_b$ to be the set of all vector fields tangent to all faces of a manifold with corners M. Then $(M, \mathcal{V}_b)$ is a manifold with a Lie structure at infinity.

(b) Take $\mathcal{V}_0$ to be the set of all vector fields vanishing on all faces of a manifold with corners M. Then $(M, \mathcal{V}_0)$ is a Lie manifold. If ∂M is a smooth manifold (i.e., if M is a manifold with boundary), then $\mathcal{V}_0 = r\Gamma(M; TM)$, where r is the distance to the boundary.

(c) As another example consider a manifold with smooth boundary and consider the vector fields $\mathcal{V}_{\mathrm{sc}} = r\mathcal{V}_b$, where r and $\mathcal{V}_b$ are as in the previous examples.

These three examples are, respectively, the "b-calculus", the "0-calculus," and the "scattering calculus" from [29]. These examples are typical and will be referred to again below. Some interesting and highly nontrivial examples of Lie structures at infinity on $\mathbb{R}^n$ are obtained from the N-body problem [45] and from strictly pseudoconvex domains [31]. Further examples of Lie structures at infinity were discussed in [2].

If M_0 is compact without boundary, then it follows from the above definition that $M = M_0$ and $\mathcal{V} = \Gamma(M, TM)$, so that a Lie structure at infinity on M_0 gives no additional information on M_0. The interesting cases are thus the ones when M_0 is noncompact.

Elements in the enveloping algebra $\mathrm{Diff}^*_{\mathcal{V}}(M)$ of $\mathcal{V}$ are called *$\mathcal{V}$-differential operators on M*. The order of differential operators induces a filtration $\mathrm{Diff}^m_{\mathcal{V}}(M)$, $m \in \mathbb{N}_0$, on the algebra $\mathrm{Diff}^*_{\mathcal{V}}(M)$. Since $\mathrm{Diff}^*_{\mathcal{V}}(M)$ is a $\mathcal{C}^\infty(M)$-module, we can introduce $\mathcal{V}$-differential operators acting between sections of smooth vector bundles $E, F \to M$, $E, F \subset M \times \mathbb{C}^N$ by

$$\mathrm{Diff}^*_{\mathcal{V}}(M; E, F) := e_F M_N(\mathrm{Diff}^*_{\mathcal{V}}(M)) e_E\,, \tag{2}$$

where $e_E, e_F \in M_N(\mathcal{C}^\infty(M))$ are the projections onto E and, respectively, F. It follows that $\mathrm{Diff}^*_{\mathcal{V}}(M; E, E) =: \mathrm{Diff}^*_{\mathcal{V}}(M; E)$ is an algebra that is closed under adjoints.

Let $A \to M$ be a vector bundle and $\varrho : A \to TM$ a vector bundle map. We shall also denote by ϱ the induced map $\Gamma(M, A) \to \Gamma(M, TM)$ between the smooth sections of these bundles. Suppose a Lie algebra structure on $\Gamma(M, A)$ is given. Then the pair (A, ϱ) together with this Lie algebra structure on $\Gamma(A)$ is called a *Lie algebroid* if $\varrho([X, Y]) = [\varrho(X), \varrho(Y)]$ and $[X, fY] = f[X, Y] + (\varrho(X)f)Y$ for any smooth sections X and Y of A and any smooth function f on M. The map $\varrho : A \to TM$ is called the *anchor of* A. We have also denoted by ϱ the induced map $\Gamma(M, A) \to \Gamma(M, TM)$. We shall also write $Xf := \varrho(X)f$.

If $\mathcal{V}$ is a structural Lie algebra of vector fields, then $\mathcal{V}$ is projective, and hence the Serre-Swan theorem [13] shows that there exists a smooth vector bundle $A_{\mathcal{V}} \to M$ together with a natural map

$$\begin{array}{ccc} \varrho_{\mathcal{V}} : A_{\mathcal{V}} & \longrightarrow & TM \\ & \searrow \quad \swarrow & \\ & M & \end{array} \tag{3}$$

such that $\mathcal{V} = \varrho_{\mathcal{V}}(\Gamma(M, A_{\mathcal{V}}))$. The vector bundle $A_{\mathcal{V}}$ turns out to be a *Lie algebroid* over M.

We thus see that there exists an equivalence between structural Lie algebras of vector fields $\mathcal{V} = \Gamma(A_{\mathcal{V}})$ and Lie algebroids $\varrho : A \to TM$ such that the induced map $\Gamma(M, A) \to \Gamma(M, TM)$ is injective and has range in the Lie algebra $\mathcal{V}_b(M)$ of all vector fields that are tangent to all hyperfaces of M. Because A and $\mathcal{V}$ determine each other up to isomorphism, we sometimes specify a Lie structure at infinity on M_0 by the pair (M, A). The definition of a manifold with a Lie structure at infinity allows us to identify M_0 with $M \smallsetminus \partial M$ and $A|_{M_0}$ with TM_0.

We now turn our attention to Riemannian structures on M_0. Any metric on A induces a metric on $TM_0 = A|_{M_0}$. This suggests the following definition.

Definition 1.3. A manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$, $\mathcal{V} = \Gamma(M, A)$, and with metric g_0 on TM_0 obtained from the restriction of a metric g on A is called a *Riemannian manifold with a Lie structure at infinity.*

The geometry of a Riemannian manifold (M_0, g_0) with a Lie structure $(M, \mathcal{V})$ at infinity has been studied in [2]. For instance, (M_0, g_0) is necessarily of infinite volume and complete. Moreover, all the covariant derivatives of the Riemannian curvature tensor are bounded. Under additional mild assumptions, we also know that the injectivity radius is bounded from below by a positive constant, i.e., (M_0, g_0) is of bounded geometry. (A *manifold with bounded geometry* is a Riemannian manifold with positive injectivity radius and with bounded covariant derivatives of the curvature tensor; see [41] and references therein.) A useful property is that all geometric operators on M_0 that are associated to a metric on A are $\mathcal{V}$-differential operators (i.e., in $\mathrm{Diff}^m_{\mathcal{V}}(M)$ [2]).

On a Riemannian manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$, $\mathcal{V} = \Gamma(M, A)$, the exponential map $\exp_p : T_pM_0 \to M_0$ is well-defined for all $p \in M_0$ and extends to a differentiable map $\exp_p : A_p \to M$ depending smoothly on $p \in M$. A convenient way to introduce the exponential map is via the geodesic spray, as done in [2]. A related phenomenon is that any vector field $X \in \Gamma(A)$ is integrable, which is a consequence of the compactness of M. The resulting diffeomorphism of M_0 will be denoted ψ_X.

PROPOSITION 1.4. *Let F_0 be an open boundary face of M and $X \in \Gamma(M; A)$. Then the diffeomorphism ψ_X maps F_0 to itself.*

Proof. This follows right away from the assumption that all vector fields in $\mathcal{V}$ are tangent to all faces [2]. □

2. Kohn-Nirenberg quantization and pseudodifferential operators

Throughout this section M_0 will be a fixed manifold with Lie structure at infinity $(M, \mathcal{V})$ and $\mathcal{V} := \Gamma(A)$. We shall also fix a metric g on $A \to M$, which induces a metric g_0 on M_0. We are going to introduce a pseudodifferential calculus on M_0 that microlocalizes the algebra of $\mathcal{V}$-differential operators $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ on M given by the Lie structure at infinity.

2.1. *Riemann-Weyl fibration.* Fix a Riemannian metric g on the bundle A, and let $g_0 = g|_{M_0}$ be its restriction to the interior M_0 of M. We shall use this metric to trivialize all density bundles on M. Denote by $\pi : TM_0 \to M_0$ the natural projection. Define

$$\Phi : TM_0 \longrightarrow M_0 \times M_0, \quad \Phi(v) := (x, \exp_x(-v)),\ x = \pi(v). \tag{4}$$

Recall that for $v \in T_xM$ we have $\exp_x(v) = \gamma_v(1)$ where γ_v is the unique geodesic with $\gamma_v(0) = \pi(v) = x$ and $\gamma_v'(0) = v$. It is known that there is an open neighborhood U of the zero-section M_0 in TM_0 such that $\Phi|_U$ is a diffeomorphism onto an open neighborhood V of the diagonal $M_0 = \Delta_{M_0} \subseteq M_0 \times M_0$.

To fix notation, let E be a real vector space together with a metric or a vector bundle with a metric. We shall denote by $(E)_r$ the set of all vectors v of E with $|v| < r$.

We shall also assume from now on that r_0, the injectivity radius of (M_0, g_0), is positive. We know that this is true under some additional mild assumptions and we conjectured that the injectivity radius is always positive [2]. Thus, for each $0 < r \leq r_0$, the restriction $\Phi|_{(TM_0)_r}$ is a diffeomorphism onto an open neighborhood V_r of the diagonal Δ_{M_0}. It is for this reason that we need the positive injectivity radius assumption.

We continue, by slight abuse of notation, to write Φ for that restriction. Following Melrose, we shall call Φ the *Riemann-Weyl fibration.* The inverse of Φ is given by

$$M_0 \times M_0 \supseteq V_r \ni (x,y) \longmapsto (x, \tau(x,y)) \in (TM_0)_r \,,$$

where $-\tau(x,y) \in T_xM_0$ is the tangent vector at x to the shortest geodesic $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

2.2. *Symbols and conormal distributions.* Let $\pi : E \to M$ be a smooth vector bundle with orthogonal metric g. Let

$$\langle \xi \rangle := \sqrt{1 + g(\xi,\xi)}. \tag{5}$$

We shall denote by $S^m_{1,0}(E)$ the symbols of type $(1,0)$ in Hörmander's sense [12]. Recall that they are defined, in local coordinates, by the standard estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(\xi)| \leq C_{K,\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \quad \pi(\xi) \in K,$$

where K is a compact subset of M trivializing E (i.e., $\pi^{-1}(K) \simeq K \times \mathbb{R}^n$) and α and β are multi-indices. If $a \in S^m_{1,0}(E)$, then its image in $S^m_{1,0}(E)/S^{m-1}_{1,0}(E)$ is called the *principal symbol* of a and denoted $\sigma^{(m)}(a)$. A symbol a will be called *homogeneous of degree μ if* $a(x, \lambda\xi) = \lambda^\mu a(x,\xi)$ for $\lambda > 0$ and $|\xi|$ and $|\lambda\xi|$ are large. A symbol $a \in S^m_{1,0}(E)$ will be called *classical* if there exist symbols $a_k \in S^{m-k}_{1,0}(E)$, homogeneous of degree $m-k$, such that $a - \sum_{j=0}^{N-1} a_k \in S^{m-N}_{1,0}(E)$. Then we identify $\sigma^{(m)}(a)$ with a_0. (See any book on pseudodifferential operators or the corresponding discussion in [3].)

We now specialize to the case $E = A^*$, where $A \to M$ is the vector bundle such that $\mathcal{V} = \Gamma(M, A)$. Recall that we have fixed a metric g on A. Let $\pi : A \to M$ and $\overline{\pi} : A^* \to M$ be the canonical projections. Then the inverse of the Fourier transform $\mathcal{F}^{-1}_{\text{fiber}}$, along the fibers of A^* gives a map

$$\mathcal{F}^{-1}_{\text{fiber}} : S^m_{1,0}(A^*) \longrightarrow \mathcal{C}^{-\infty}(A) := \mathcal{C}^\infty_c(A)' \,, \quad \langle \mathcal{F}^{-1}_{\text{fiber}} a, \varphi \rangle := \langle a, \mathcal{F}^{-1}_{\text{fiber}} \varphi \rangle, \tag{6}$$

where $a \in S^m_{1,0}(A^*)$, φ is a smooth, compactly supported function, and

$$\mathcal{F}^{-1}_{\text{fiber}}(\varphi)(\xi) := (2\pi)^{-n} \int_{\overline{\pi}(\zeta) = \pi(\xi)} e^{i\langle \xi, \zeta \rangle} \varphi(\zeta)\, d\zeta. \tag{7}$$

Then $I^m(A, M)$ is defined as the image of $S^m_{1,0}(A^*)$ through the above map. We shall call this space the space of distributions on A *conormal* to M. The spaces $I^m(TM_0, M_0)$ and $I^m(M_0^2, \Delta_{M_0}) = I^m(M_0^2, M_0)$ are defined similarly. In fact, these definitions are special cases of the following more general definition. Let $X \subset Y$ be an embedded submanifold of a manifold with corners Y. On a small neighborhood V of X in Y we define a structure of a vector bundle over X, such that X is the zero section of V, as a bundle V is isomorphic to the normal bundle of X in Y. Then we define the space of *distributions on Y that are conormal of order m to X,* denoted $I^m(Y, X)$, to be the space of distributions on M that are smooth on $Y \smallsetminus X$ and, that are, in a tubular neighborhood $V \to X$ of X in Y, the inverse Fourier transforms of elements in $S^m(V^*)$ along the fibers of $V \to X$. For simplicity, we have ignored the density factor. For more details on conormal distributions we refer to [11], [12], [42] and the forthcoming book [25] (for manifolds with corners).

The main use of spaces of conormal distributions is in relation to pseudodifferential operators. For example, since we have

$$I^m(M_0^2, M_0) \subseteq \mathcal{C}^{-\infty}(M_0^2) := \mathcal{C}_c^\infty(M_0^2)',$$

we can associate to a distribution in $K \in I^m(M_0^2, M_0)$ a continuous linear map $T_K : \mathcal{C}_c^\infty(M_0) \to \mathcal{C}^{-\infty}(M_0) := \mathcal{C}_c^\infty(M_0)'$, by the Schwartz kernel theorem. Then a well known result of Hörmander [11], [12] states that T_K is a pseudodifferential operator on M_0 and that all pseudodifferential operators on M_0 are obtained in this way, for various values of m. This defines a map

$$T : I^m(M_0^2, M_0) \to \mathrm{Hom}(\mathcal{C}_c^\infty(M_0), \mathcal{C}^{-\infty}(M_0)). \tag{8}$$

Recall now that $(A)_r$ denotes the set of vectors of norm $< r$ of the vector bundle A. We agree to write $I^m_{(r)}(A, M)$ for all $k \in I^m(A, M)$ with $\operatorname{supp} k \subseteq (A)_r$. The space $I^m_{(r)}(TM_0, M_0)$ is defined in an analogous way. Then restriction defines a map

$$\mathcal{R} : I^m_{(r)}(A, M) \longrightarrow I^m_{(r)}(TM_0, M_0). \tag{9}$$

Recall that r_0 denotes the injectivity radius of M_0 and that we assume $r_0 > 0$. Similarly, the Riemann–Weyl fibration Φ of Equation (4) defines, for any $0 < r \leq r_0$, a map

$$\Phi_* : I^m_{(r)}(TM_0, M_0) \to I^m(M_0^2, M_0). \tag{10}$$

We shall also need various subspaces of conormal distributions, which we shall denote by including a subscript as follows:

- "cl" to designate the distributions that are "classical," in the sense that they correspond to classical pseudodifferential operators,
- "c" to denote distributions that have compact support,
- "pr" to indicate operators that are properly supported or distributions that give rise to such operators.

For instance, $I^m_c(Y, X)$ denotes the space of compactly supported conormal distributions, so that $I^m_{(r)}(A, M) = I^m_c((A)_r, M)$. Occasionally, we shall use the double subscripts "cl,pr" and "cl,c." Note that "c" implies "pr".

2.3. *Kohn-Nirenberg quantization.* For notational simplicity, we shall use the metric g_0 on M_0 (obtained from the metric on A) to trivialize the half-density bundle $\Omega^{1/2}(M_0)$. In particular, we identify $\mathcal{C}_c^\infty(M_0, \Omega^{1/2})$ with $\mathcal{C}_c^\infty(M_0)$. Let $0 < r \leq r_0$ be arbitrary. Each smooth function χ, with $\chi = 1$ close to $M \subseteq A$ and support contained in the set $(A)_r$, induces a map $q_{\Phi,\chi} : S^m_{1,0}(A^*) \longrightarrow I^m(M_0^2, M_0)$,

$$q_{\Phi,\chi}(a) := \Phi_* \left(\mathcal{R}\left(\chi \mathcal{F}^{-1}_{\mathrm{fiber}}(a)\right)\right). \tag{11}$$

Let $a_\chi(D)$ be the operator on M_0 with distribution kernel $q_{\Phi,\chi}(a)$, defined using the Schwartz kernel theorem, i.e., $a_\chi(D) := T \circ q_{\Phi,\chi}(a)$. Following Melrose, we call the map $q_{\Phi,\chi}$ the *Kohn-Nirenberg quantization map.* It will play an important role in what follows.

For further reference, let us make the formula for the induced operator $a_\chi(D) : \mathcal{C}_c^\infty(M_0) \to \mathcal{C}_c^\infty(M_0)$ more explicit. Neglecting the density factors in the formula, we obtain for $u \in \mathcal{C}_c^\infty(M_0)$,

$$a_\chi(D)u(x) = \int_{M_0} (2\pi)^{-n} \int_{T_x^* M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x,y)) a(x,\eta) u(y)\, d\eta\, dy\,. \tag{12}$$

Specializing to the case of Euclidean space $M_0 = \mathbb{R}^n$ with the standard metric we have $\tau(x, y) = x - y$, and hence

$$a_\chi(D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\eta} \chi(x, x-y) a(x,\eta) u(y)\, d\eta\, dy\,, \tag{13}$$

i.e., the well-known formula for the Kohn-Nirenberg-quantization on $\mathbb{R}^n$, if $\chi = 1$. The following lemma states that, up to regularizing operators, the above quantization formulas do not depend on χ.

LEMMA 2.1. *Let $0 < r \leq r_0$. If χ_1 and χ_2 are smooth functions with support $(A)_r$ and $\chi_j = 1$ in a neighborhood of $M \subseteq A$, then $(\chi_1 - \chi_2)\mathcal{F}^{-1}_{\mathrm{fiber}}(a)$ is a smooth function, and hence $a_{\chi_1}(D) - a_{\chi_2}(D)$ has a smooth Schwartz kernel. Moreover, the map $S^m_{1,0}(A^*) \to \mathcal{C}^\infty(A)$ that maps $a \in S^m_{1,0}(A^*)$ to $(\chi_1 - \chi_2)\mathcal{F}^{-1}_{\mathrm{fiber}}(a)$ is continuous, where the right-hand side is endowed with the topology of uniform C^∞-convergence on compact subsets.*

Proof. Since the singular supports of $\chi_1\mathcal{F}^{-1}_{\mathrm{fiber}}(a)$ and $\chi_2\mathcal{F}^{-1}_{\mathrm{fiber}}(a)$ are contained in the diagonal Δ_{M_0} and $\chi_1 - \chi_2$ vanishes there, we have that $(\chi_1 - \chi_2)\mathcal{F}^{-1}_{\mathrm{fiber}}(a)$ is a smooth function.

To prove the continuity of the map $S^m_{1,0}(A^*) \ni a \mapsto (\chi_1 - \chi_2)\mathcal{F}^{-1}_{\mathrm{fiber}}(a) \in \mathcal{C}^\infty(A)$, it is enough, using a partition of unity, to assume that $A \to M$ is a trivial bundle. Then our result follows from the standard estimates for oscillatory integrals (i.e., by formally writing $|v|^2 \int e^{i\langle v,\xi\rangle} a(\xi) d\xi = -\int (\Delta_\xi e^{i\langle v,\xi\rangle}) a(\xi) d\xi$ and then integrating by parts; see [12], [33], [43], [44] for example). $\square$

We now verify that the quantization map $q_{\Phi,\chi}$, Equation (11), gives rise to pseudodifferential operators.

LEMMA 2.2. *Let $r \leq r_0$ be arbitrary. For each $a \in S^m_{1,0}(A^*)$ and each $\chi \in \mathcal{C}^\infty_c((A)_r)$ with $\chi = 1$ close to $M \subseteq A$, the distribution $q_{\Phi,\chi}(a)$ is the Schwartz-kernel of a pseudodifferential operator $a_\chi(D)$ on M_0, which is properly supported if $r < \infty$ and has principal symbol $\sigma^{(\mu)}(a) \in S^m_{1,0}(E)/S^{m-1}_{1,0}(E)$. If $a \in S^\mu_{cl}(A^*)$, then $a_\chi(D)$ is a classical pseudodifferential operator.*

Proof. Denote also by $\chi : I^m(TM_0, M_0) \to I^m_{(r)}(TM_0, M_0)$ the "multiplication by χ" map. Then

$$a_\chi(D) = T \circ \Phi_* \circ \mathcal{R} \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}(a) := T_{\Phi_*(\mathcal{R}(\chi \mathcal{F}^{-1}_{\text{fiber}}(a)))} = T \circ q_{\Phi,\chi}(a) \tag{14}$$

where T is defined as in Equation (8). Hence $a_\chi(D)$ is a pseudodifferential operator by Hörmander's result mentioned above [11], [12] (stating that the distribution conormal to the diagonal is exactly the Schwartz kernel of pseudodifferential operators. Since $\chi\mathcal{R}(a)$ is properly supported, so will be the operator $a_\chi(D)$).

For the statement about the principal symbol, we use the principal symbol map for conormal distributions [11], [12], and the fact that the restriction of the anchor $A \to TM$ to the interior $A|_{M_0}$ is the identity. (This also follows from Equation (13) below.) This proves our lemma. □

Let us denote by $\Psi^m(M_0)$ the space of pseudodifferential operators of order $\leq m$ on M_0 (no support condition). We then have the following simple corollary.

COROLLARY 2.3. *The map $\sigma_{tot} : S^m_{1,0}(A^*) \to \Psi^m(M_0)/\Psi^{-\infty}(M_0)$,*

$$\sigma_{tot}(a) := a_\chi(D) + \Psi^{-\infty}(M_0)$$

is independent of the choice of the function $\chi \in \mathcal{C}^\infty_c((A)_r)$ used to define $a_\chi(D)$ in Lemma 2.2.

Proof. This follows right away from Lemma 2.2. □

Let us remark that our pseudodifferential calculus depends on more than just the metric.

Remark 2.4. Non-isomorphic Lie structures at infinity can lead to the same metric on M_0. An example is provided by $\mathbb{R}^n$ with the standard metric, which can be obtained either from the radial compactification of $\mathbb{R}^n$ with the scattering calculus, or from $[-1, 1]^n$ with the b-calculus. See Examples 1.2 and the paragraph following it. The pseudodifferential calculi obtained from these Lie algebra structures at infinity will be, however, different.

The above remark readily shows that not all pseudodifferential operators in $\Psi^m(M_0)$ are of the form $a_\chi(D)$ for some symbol $a \in S^m_{1,0}(A^*)$, not even if we assume that they are properly supported, because they do not have the correct behavior at infinity. Moreover, the space $T \circ q_{\Phi,\chi}(S^\infty_{1,0}(A^*))$ of all pseudodifferential operators of the form $a_\chi(D)$ with $a \in S^\infty_{1,0}(A^*)$ is not closed under composition. In order to obtain a suitable space of pseudodifferential operators that is closed under composition, we are going to include more (but not all) operators of order $-\infty$ in our calculus.

Recall that we have fixed a manifold M_0, a Lie structure at infinity (M, A) on M_0, and a metric g on A with injectivity radius $r_0 > 0$. Also, recall that any $X \in \Gamma(A) \subset \mathcal{V}_b$ generates a global flow $\Psi_X : \mathbb{R} \times M \to M$. Evaluation at $t = 1$ yields a diffeomorphism $\Psi_X(1, \cdot) : M \to M$, whose action on functions is denoted

$$\psi_X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M). \tag{15}$$

We continue to assume that the injectivity radius r_0 of our fixed manifold with a Lie structure at infinity $(M, \mathcal{V})$ is strictly positive.

Definition 2.5. Fix $0 < r < r_0$ and $\chi \in \mathcal{C}^\infty_c((A)_r)$ such that $\chi = 1$ in a neighborhood of $M \subseteq A$. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0)$ of *pseudodifferential operators generated by the Lie structure at infinity* (M, A) is the linear space of operators $\mathcal{C}^\infty_c(M_0) \to \mathcal{C}^\infty_c(M_0)$ generated by $a_\chi(D)$, $a \in S^m_{1,0}(A^*)$, and $b_\chi(D)\psi_{X_1} \dots \psi_{X_k}$, $b \in S^{-\infty}(A^*)$ and $X_j \in \Gamma(A)$, $\forall j$.

Similarly, the space $\Psi^m_{cl,\mathcal{V}}(M_0)$ of *classical pseudodifferential operators generated by the Lie structure at infinity* (M, A) is obtained by using classical symbols a in the construction above.

It is implicit in the above definition that the spaces $\Psi^{-\infty}_{1,0,\mathcal{V}}(M_0)$ and $\Psi^{-\infty}_{cl,\mathcal{V}}(M_0)$ are the same. They will typically be denoted by $\Psi^{-\infty}_{\mathcal{V}}(M_0)$. As usual, we shall denote

$$\Psi^\infty_{1,0,\mathcal{V}}(M_0) := \cup_{m\in\mathbb{Z}} \Psi^m_{1,0,\mathcal{V}}(M_0) \quad \text{and} \quad \Psi^\infty_{cl,\mathcal{V}}(M_0) := \cup_{m\in\mathbb{Z}} \Psi^m_{cl,\mathcal{V}}(M_0).$$

At first sight, the above definition depends on the choice of the metric g on A. However, we shall soon prove that this is not the case.

As for the usual algebras of pseudodifferential operators, we have the following basic property of the principal symbol.

PROPOSITION 2.6. *The principal symbol establishes isomorphisms*

$$\sigma^{(m)} : \Psi^m_{1,0,\mathcal{V}}(M_0)/\Psi^{m-1}_{1,0,\mathcal{V}}(M_0) \to S^m_{1,0}(A^*)/S^{m-1}_{1,0}(A^*) \tag{16}$$

and

$$\sigma^{(m)} : \Psi^m_{cl,\mathcal{V}}(M_0)/\Psi^{m-1}_{cl,\mathcal{V}}(M_0) \to S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*). \tag{17}$$

Proof. This follows from the classical case of the spaces $\Psi^m(M_0)$ by Lemma 2.2. $\square$

3. The product

We continue to denote by $(M, \mathcal{V})$, $\mathcal{V} = \Gamma(A)$, a fixed manifold with a Lie structure at infinity and with positive injectivity radius. In this section we want to show that the space $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is an algebra (i.e., it is closed under multiplication) by showing that it is the homomorphic image of the algebra $\Psi^{\infty}_{1,0}\mathcal{G})$ of pseudodifferential operators on any d-connected groupoid $\mathcal{G}$ integrating A (Theorem 3.2).

First we need to fix the terminology and to recall some definitions and constructions involving groupoids.

3.1. *Groupoids.* Here is first an abstract definition that will be made more clear below. Recall that a small category is a category whose morphisms form a set. A *groupoid* is a small category all of whose morphisms are invertible. Let $\mathcal{G}$ denote the set of morphisms and M denote the set of objects of a given groupoid. Then each $g \in \mathcal{G}$ will have a domain $d(g) \in M$ and a range $r(g) \in M$ such that the product $g_1 g_2$ is defined precisely when $d(g_1) = r(g_2)$. Moreover, it follows that the multiplication (or composition) is associative and every element in $\mathcal{G}$ has an inverse. We shall identify the set of objects M with their identity morphisms via a map $\iota : M \to \mathcal{G}$. One can think then of a groupoid as being a group, except that the multiplication is only partially defined. By abuse of notation, we shall use the same notation for the groupoid and its set of morphisms ($\mathcal{G}$ in this case). An intuitive way of thinking of a groupoid with morphisms $\mathcal{G}$ and objects M is to think of the elements of $\mathcal{G}$ as being arrows between the points of M. The points of M will be called *units*, by identifying an object with its identity morphism. There will be structural maps $d, r : \mathcal{G} \to M$, *domain and range*, $\mu : \{(g, h), d(g) = r(h)\} \to \mathcal{G}$, *multiplication*, $\mathcal{G} \ni g \to g^{-1} \in \mathcal{G}$, *inverse*, and $\iota : M \to \mathcal{G}$ satisfying the usual identities satisfied by the composition of functions.

A *Lie groupoid* is a groupoid $\mathcal{G}$ such that the space of arrows $\mathcal{G}$ and the space of units M are manifolds with corners, all its structural maps (i.e., multiplication, inverse, domain, range, ι) are differentiable, the domain and range maps (i.e., d and r) are submersions. By the definition of a submersion of manifolds with corners, the submanifolds $\mathcal{G}_x := d^{-1}(x)$ and $\mathcal{G}^x := r^{-1}(x)$ are smooth (so they have no corners or boundary), for any $x \in M$. Also, it follows that that M is an embedded submanifold of $\mathcal{G}$.

The *d–vertical tangent space to* $\mathcal{G}$, denoted $T_{\mathrm{vert}}\mathcal{G}$, is the union of the tangent spaces to the fibers of $d : \mathcal{G} \to M$; that is,

$$T_{\mathrm{vert}}\mathcal{G} := \cup_{x \in M} T\mathcal{G}_x = \ker d_*, \tag{18}$$

the union being a disjoint union, with topology induced from the inclusion $T_{\mathrm{vert}}\mathcal{G} \subset T\mathcal{G}$. The *Lie algebroid of* $\mathcal{G}$, denoted $A(\mathcal{G})$ is defined to be the restriction of the d–vertical tangent space to the set of units M, that is,

$A(\mathcal{G}) = \cup_{x\in M} T_x\mathcal{G}_x$, a vector bundle over M. The space of sections of $A(\mathcal{G})$ identifies canonically with the space of sections of the d-vertical tangent bundle (= d-vertical vector fields) that are *right invariant* with respect to the action of $\mathcal{G}$. It also implies a canonical isomorphism between the vertical tangent bundle and the pull-back of $A(\mathcal{G})$ via the range map $r : \mathcal{G} \to M$:

$$r^*A(\mathcal{G}) \simeq T_{\mathrm{vert}}\mathcal{G}. \tag{19}$$

The structure of Lie algebroid on $A(\mathcal{G})$ is induced by the Lie brackets on the spaces $\Gamma(T\mathcal{G}_x)$, $\mathcal{G}_x := d^{-1}(x)$. This is possible since the Lie bracket of two right invariant vector fields is again right invariant. The anchor map in this case is given by the differential of r, $r_* : A(\mathcal{G}) \to TM$.

Let $\mathcal{G}$ be a Lie groupoid with units M, then there is associated to it a pseudodifferential calculus (or algebra of pseudodifferential operators) $\Psi^\infty_{1,0}(\mathcal{G})$, whose operators of order m form a linear space denoted $\Psi^m_{1,0}(\mathcal{G})$, $m \in \mathbb{R}$, such that $\Psi^m_{1,0}(\mathcal{G})\Psi^{m'}_{1,0}(\mathcal{G}) \subset \Psi^{m+m'}_{1,0}(\mathcal{G})$. This calculus is defined as follows: $\Psi^m_{1,0}(\mathcal{G})$ consists of smooth families of pseudodifferential operators (P_x), $P_x \in \Psi^m_{1,0}(\mathcal{G}_x)$, $x \in M$, that are right invariant with respect to multiplication by elements of $\mathcal{G}$ and are "uniformly supported." To define what uniformly supported means, let us observe that the right invariance of the operators P_x implies that their distribution kernels K_{P_x} descend to a distribution $k_P \in I^m(\mathcal{G}, M)$. Then the family $P = (P_x)$ is called *uniformly supported* if, by definition, k_P has compact support. If P is uniformly supported, then each P_x is properly supported. The right invariance condition means, for $P = (P_x) \in \Psi^\infty_{1,0}(\mathcal{G})$, that right multiplication $\mathcal{G}_x \ni g' \mapsto g'g \in \mathcal{G}_y$ maps P_y to P_x, whenever $d(g) = y$ and $r(g) = x$. By definition, the evaluation map

$$\Psi^\infty_{1,0}(\mathcal{G}) \ni P = (P_x) \mapsto e_z(P) := P_z \in \Psi^\infty_{1,0}(\mathcal{G}_z) \tag{20}$$

is an algebra morphism for any $z \in M$. If we require that the operators P_x be classical of order $\mu \in \mathbb{C}$, we obtain spaces $\Psi^\mu_{cl}(\mathcal{G})$ having similar properties. These spaces were considered in [35].

All results and constructions above remain true for classical pseudodifferential operators. This gives the algebra $\Psi^\infty_{cl}(\mathcal{G})$ consisting of families $P = (P_x)$ of *classical* pseudodifferential operators satisfying all the previous conditions.

Assume that the interior M_0 of M is an invariant subset. Recall that the so-called *vector representation* $\pi_M : \Psi^\infty_{1,0}(\mathcal{G}) \to \mathrm{End}(\mathcal{C}^\infty_{\mathrm{c}}(M_0))$ associates to a pseudodifferential operator P on $\mathcal{G}$ a pseudodifferential operator $\pi_M(P) : \mathcal{C}^\infty_{\mathrm{c}}(M_0) \to \mathcal{C}^\infty_{\mathrm{c}}(M_0)$ [17]. This representation π_M is defined as follows. If $\varphi \in \mathcal{C}^\infty_{\mathrm{c}}(M_0)$, then $\varphi \circ r$ is a smooth function on $\mathcal{G}$, and we can let the family (P_x) act along each $\mathcal{G}_x$ to obtain the function $P(\varphi\circ r)$ on $\mathcal{G}$ defined by $P(\varphi \circ r)|_{\mathcal{G}_x} = P_x(\varphi \circ r|_{\mathcal{G}_x})$. The fact that P_x is a smooth family guarantees that $P(\varphi \circ r)$ is also smooth. Using then the fact that r is a submersion, so that locally it is a product map, we obtain that $P(\varphi \circ r) = \varphi_0 \circ r$, for some function $\varphi_0 \in \mathcal{C}_c^\infty(M_0)$. We shall then let

$$\pi_M(P)\varphi = \varphi_0. \tag{21}$$

The fact that P is uniformly supported guarantees that φ_0 will also have compact support in M_0. A more explicit description of π_M in the case of Lie manifolds will be obtained in the proof of Theorem 3.2, more precisely, Equation (27).

A Lie groupoid $\mathcal{G}$ with units M is said to *integrate* A if $A(\mathcal{G}) \simeq A$ as vector bundles over M. Recall that the groupoid $\mathcal{G}$ is called *d–connected* if $\mathcal{G}_x := d^{-1}(x)$ is a connected set, for any $x \in M$. If there exists a Lie groupoid $\mathcal{G}$ integrating A, then there exists also a d–connected Lie groupoid with this property. (Just take for each x the connected component of x in $\mathcal{G}_x$.)

Our plan to show that $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is an algebra, is then to prove that it is the image under π_M of $\Psi^\infty_{1,0}(\mathcal{G})$, for a Lie groupoid $\mathcal{G}$ integrating A, $\Gamma(M, A) = \mathcal{V}$. In fact, any d-connected Lie groupoid will satisfy this, by Theorem 3.2. This requires the following deep result due to Crainic and Fernandes [7] stating that the Lie algebroids associated to Lie manifolds are integrable.

THEOREM 3.1 (Cranic–Fernandes). *Any Lie algebroid arising from a Lie structure at infinity is actually the Lie algebroid of a Lie groupoid* (*i.e.*, *it is* integrable).

This theorem should be thought of as an analog of Lie's third theorem stating that every finite dimensional Lie algebra is the Lie algebra of a Lie group. However, the analog of Lie's theorem for Lie algebroids does not hold: there are Lie algebroids which are not Lie algebroids to a Lie groupoid [20].

A somewhat weaker form of the above theorem, which is however enough for the proof of Melrose's conjecture, was obtained [34].

We are now ready to state and prove the main result of this section. We refer to [17] or [35] for the concepts and results on groupoids and algebras of pseudodifferential operators on groupoids not explained below or before the statement of this theorem.

THEOREM 3.2. *Let M_0 be a manifold with a Lie structure at infinity, $(M, \mathcal{V})$, $A = A_\mathcal{V}$, as above. Also, let $\mathcal{G}$ be a d-connected groupoid with units M and with $A(\mathcal{G}) \simeq A$. Then $\Psi^m_{1,0,\mathcal{V}}(M_0) = \pi_M(\Psi^m_{1,0}(\mathcal{G}))$ and $\Psi^m_{cl,\mathcal{V}}(M_0) = \pi_M(\Psi^m_{cl}(\mathcal{G}))$.*

Proof. We shall consider only the first equality. The case of classical operators can be treated in exactly the same way.

Here is first, briefly, the idea of the proof. Let $P = (P_x) \in \Psi^m_{1,0}(\mathcal{G})$. Then the Schwartz kernels of the operators P_x form a smooth family of conormal distributions in $I^m(\mathcal{G}_x^2, \mathcal{G}_x)$ that descends, by right invariance, to a distribution

$k_P \in I^m_c(\mathcal{G}, M)$ (i.e., to a compactly supported distribution on $\mathcal{G}$, conormal to M) called the *convolution kernel of* P. The map $P \mapsto k_P$ is an isomorphism [35] with inverse

$$T : I^m_c(\mathcal{G}, M) \to \Psi^m_{1,0}(\mathcal{G}). \tag{22}$$

Fix a metric on $A \to M$. The resulting exponential map (reviewed below) then gives rise, for $r > 0$ small enough, to an open embedding

$$\alpha : (A)_r \to \mathcal{G}, \tag{23}$$

which is a diffeomorphism onto its image. This diffeomorphism then gives rise to an embedding

$$\alpha_* : I^m_{(r)}(A, M) := I^m_c((A)_r, M) \to I^m(\mathcal{G}, M) \tag{24}$$

such that for each χ as above

$$\pi_M\big(\alpha_*(\chi\mathcal{F}^{-1}_{\text{fiber}}(a))\big) = a_\chi(D) \in \Psi^m(M_0). \tag{25}$$

This will allow us to show that $\pi_M(\Psi^m_{1,0}(\mathcal{G}))$ contains the linear span of all operators P of the form $P = a_\chi(D)$, $a \in S^m_{1,0}(A^*)$, $m \in \mathbb{Z}$ fixed. This reduces the problem to verifying that

$$\pi_M(\Psi^{-\infty}(\mathcal{G})) = \Psi^{-\infty}_{\mathcal{V}}(M_0). \tag{26}$$

Using a partition of unity, this in turn will be reduced to Equation (25). Now let us provide the complete details.

Let $\mathcal{G}^x_x := d^{-1}(x) \cap r^{-1}(x)$, which is a group for any $x \in M_0$, by the axioms of a groupoid. Then $\mathcal{G}^x_x \simeq \mathcal{G}^y_y$ whenever there exists $g \in \mathcal{G}$ with $d(g) = x$ and $r(g) = y$ (conjugate by g). We can assume, without loss of generality, that M is connected. Let $\Gamma := \mathcal{G}^x_x$, for some fixed $x \in M_0$. Our above informal description of the proof can be conveniently formalized and visualized using the following diagram whose morphisms are as defined below:

$$\begin{tikzcd}
S^m_{1,0}(A) \arrow[r, "\mathcal{F}^{-1}_{\text{fiber}}"] & I^m(A, M) \arrow[r, "\chi"] & I^m_{(r)}(A, M) \arrow[r, "\mathcal{R}"] \arrow[d, "\alpha_*"'] & I^m_{(r)}(TM_0, M_0) \arrow[d, "\Phi_*"] \\
& \Psi^m_{1,0}(\mathcal{G}) \arrow[r, "\cong"] \arrow[d, "e_x"'] & I^m_c(\mathcal{G}, M) \arrow[r, "l_*"] \arrow[d, "\mu_1^*"'] & I^m(M_0^2, M_0) \arrow[d, equal] \\
& \Psi^m_{pr}(\mathcal{G}_x)^\Gamma \arrow[r, "\cong"] & I^m_{pr}(\mathcal{G}_x^2, \mathcal{G}_x)^\Gamma \arrow[r, "\tilde{r}_*"] \arrow[d, "\cong"'] & I^m(M_0^2, M_0) \arrow[d, "\cong"] \\
& & \Psi^m_{pr}(\mathcal{G}_x)^\Gamma \arrow[r, "r_*"] & \Psi^m_{1,0}(M_0).
\end{tikzcd}$$

We now define the morphisms appearing in the above diagram in such a way that it will turn out to be a commutative diagram. The bottom three rectangles will trivially turn out to be commutative. Recall that the index "pr" means "properly supported."

Next, recall that the maps $\mathcal{F}_{\text{fiber}}^{-1}$ (the fiberwise inverse Fourier transform), χ (the multiplication by the cut-off function χ), $\mathcal{R}$ (the restriction map), Φ_* (induced by the inverse of the exponential map), and e_x (evaluation of the family (P_y) at $y = x$) have already been defined.

We let $\mu_1(g', g) = g'g^{-1}$, and we let μ_1^* be the map induced at the level of kernels by μ_1 by pull-back (which is seen to be defined in this case because μ_1 is a submersion and its range is transverse to M).

The four isomorphisms not named are the "T isomorphisms" and their inverses defined in various places earlier (identifying spaces of conormal distributions with spaces of pseudodifferential operators). More precisely, the top isomorphism is from [35] and all the other isomorphisms are the canonical identifications between pseudodifferential operators and distributions on product spaces that are conormal to the diagonal (via the Schwartz kernels). In fact, the top isomorphism T is completely determined by the requirement that the left-most square (containing e_x) be commutative.

It is a slightly more difficult task to define r_*. We shall have to make use minimally of groupoid theory. Let $y \in M$ be arbitrary for a moment. Since the Lie algebra of $\mathcal{G}_y^y$ is isomorphic to the kernel of the anchor map $\varrho : A(\mathcal{G})_y \to T_yM$, we see that $\mathcal{G}_y^y$ is a discrete group if, and only if, $y \in M_0$. Then

$$r_* : T_y\mathcal{G}_y = A(\mathcal{G})_y \to T_yM_0$$

is an isomorphism, if and only if, $y \in M_0$.

Let $x \in M_0$ be our fixed point. Then $r : \mathcal{G}_x \to M_0$ is a surjective local diffeomorphism. Also $\Gamma := \mathcal{G}_x^x$ acts freely on $\mathcal{G}_x$ and $\mathcal{G}_x/\Gamma = M_0$. Hence $r : \mathcal{G}_x \to M_0$ is a covering map with group Γ, and $\mathcal{C}^\infty(\mathcal{G}_x)^\Gamma = \mathcal{C}^\infty(M_0)$. Let $P = (P_y) \in \Psi^m_{1,0}(\mathcal{G})$. Since P_x is Γ-invariant and properly supported, the map $P_x : \mathcal{C}^\infty(\mathcal{G}_x) \to \mathcal{C}^\infty(\mathcal{G}_x)$, descends to a map $\mathcal{C}^\infty(M_0) \to \mathcal{C}^\infty(M_0)$, which is by definition $r_*(P)$. More precisely, if φ is a smooth function on M_0, then $\varphi \circ r$, is a Γ-invariant function on $\mathcal{G}_x$. Hence $P(\varphi \circ r)$ is defined (because P is properly supported) and is also Γ-invariant. Thus there exists a function $\varphi_0 \in \mathcal{C}^\infty(M_0)$ such that $P(\varphi \circ r) = \varphi_0 \circ r$. The operator $r_*(P)$ is then given by $r_*(P)\varphi := \varphi_0$. This definition of r_* provides us with the following simpler definition of the vector representation π_M:

$$\pi_M(P) = r_*(e_x(P)). \tag{27}$$

We also obtain that

$$\pi_M \circ T = r_* \circ e_x \circ T = r_* \circ T \circ \mu_1^*, \tag{28}$$

by the commutativity of the left-most rectangle.

The commutativity of the bottom rectangle completely determines the morphism $\tilde{r}_*$. However, we shall also need an explicit description of this map which can be obtained as follows. Recall that $\mathcal{G}_x$ is a covering of M_0 with group $\Gamma := \mathcal{G}_x^x$. Hence we can identify $I^m_{pr}(\mathcal{G}_x^2, \mathcal{G}_x)^\Gamma$ with $I^m_{pr}((\mathcal{G}_x^2)^\Gamma, \mathcal{G}_x^\Gamma)$. The map $\tau : (\mathcal{G}_x^2)/\Gamma \to M_0^2$ is also a covering map. This allows us to identify a distribution with small support in $(\mathcal{G}_x^2)/\Gamma$ with a distribution with support in a small subset of M_0^2. These identifications then extend by summation along the fibers of $\tau : (\mathcal{G}_x^2)/\Gamma \to M_0^2$ to define a distribution $\tau_*(u) \in \mathcal{D}'(M_0^2)$, for any distribution u on $(\mathcal{G}_x^2)/\Gamma$ whose support intersects only finitely many components of $\tau^{-1}(U)$, for any connected locally trivializing open set $U \subset M_0$. The morphism $\tilde{r}_*$ identifies then with τ_*. Also, observe for later use that

$$\tau(g', g) = (r(g'), r(g)) = (r(g'g^{-1}), d(g'g^{-1})) = (r(\mu_1(g', g)), d(\mu_1(g', g))). \tag{29}$$

Next, we must set $l_* := \tilde{r}_* \circ \mu_1^*$, by the commutativity requirement. For this morphism we have a similar, but simpler, description of $l_*(u)$. Namely, $l_*(u)$ is obtained by first restricting a distribution u to $d^{-1}(M_0) = r^{-1}(M_0)$ and then by applying to this restriction the push-forward map defined by $(d, r) : d^{-1}(M_0) \to M_0^2$ (that is, we sum over open sets in $\mathcal{G}_x$ covering sets in the base M_0^2). Equation (29) guarantees that this alternative description of l_* satisfies $l_* := \tilde{r}_* \circ \mu_1^*$.

To define α_*, recall that we have fixed a metric on A. This metric then lifts via $r : \mathcal{G} \to M$ to $T_{\mathrm{vert}}\mathcal{G} \simeq r^*A(\mathcal{G})$, by Equations (18) and (19). The induced metrics on the fibers of $\mathcal{G}_y$, $y \in M$, give rise, using the (geodesic) exponential map, to maps

$$A_y \simeq A(\mathcal{G})_y = T_y\mathcal{G}_y \to \mathcal{G}_y.$$

These maps give rise to an application $(A)_r \to \mathcal{G}$, which, by the inverse mapping theorem, is seen to be a diffeomorphism onto its image. It, moreover, sends the zero section of A to the units of $\mathcal{G}$. Then α_* is the resulting map at the level of conormal distributions. (Note that $\mathcal{G}_y$ is complete.)

We have now completed the definition of all morphisms in our diagram. To prove that our diagram is commutative, it remains to prove that

$$l_* \circ \alpha_* = \Phi_* \circ \mathcal{R}.$$

This however follows from the above description of the map l_*, since (d, r) is injective on $\alpha((A)_r)$ and $r : \mathcal{G}_x \to M_0$ is an isometric covering, thus preserving the exponential maps.

The commutativity of the above diagram finally shows that

$$\begin{aligned} a_\chi(D) := T \circ q_{\Phi,\chi}(a) &= T \circ \Phi_* \circ \mathcal{R} \circ \chi \circ \mathcal{F}^{-1}_{\mathrm{fiber}}(a) \\ &= \pi_M \circ T \circ \alpha_* \circ \chi \circ \mathcal{F}^{-1}_{\mathrm{fiber}}(a) = \pi_M(Q), \end{aligned} \tag{30}$$

where $Q = T \circ \alpha_* \circ \chi \circ \mathcal{F}_{\mathrm{fiber}}^{-1}(a)$ and $a \in S^m_{1,0}(A^*)$. Thus every operator of the form $a_\chi(D)$ is in the range of π_M.

We notice for the rest of our argument that the definition of the vector representation π_M can be extended by the same formula to arbitrary right invariant families of operators $P = (P_x)$, $P_x : \mathcal{C}^\infty(\mathcal{G}_x) \to \mathcal{C}^\infty(\mathcal{G}_x)$, such that the induced operator $P : \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}) \to \cup \mathcal{C}^\infty(\mathcal{G}_x)$ has range in $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G})$. We shall use this in the following case. Let $X \in \mathcal{V}$. Then X defines by integration a diffeomorphism of M, see Equation (15). Let $\tilde{X}$ be its lift to a d-vertical vector field on $\mathcal{G}$ (i.e., on each $\mathcal{G}_x$ we obtain a vector field, and this family of vector fields is right invariant). A result from [14, Appendix] (see also [34]) then shows that $\tilde{X}$ can be integrated to a global flow. Let us denote by $\tilde{\psi}_X$ the *family* of diffeomorphisms of each $\mathcal{G}_x$ obtained in this way, as well as their action on functions. It follows then from the definition that

$$\pi_M(\tilde{\psi}_X) = \psi_X. \tag{31}$$

The Equations (30) and (31) then give

$$\pi_M(Q\tilde{\psi}_{X_1} \dots \tilde{\psi}_{X_n}) = a_\chi(D)\psi_{X_1} \dots \psi_{X_n} \in \Psi^{-\infty}_{1,0,\mathcal{V}}(M_0), \tag{32}$$

for any $a \in S^{-\infty}(A^*)$ and $Q = T \circ \alpha_* \circ \chi \circ \mathcal{F}_{\mathrm{fiber}}^{-1}(a)$. Also $Q\tilde{\psi}_{X_1} \dots \tilde{\psi}_{X_n} \in \Psi^{-\infty}(\mathcal{G})$, since the product of a regularizing operator with the operator induced by a diffeomorphism is regularizing. We have thus proved that $\pi_M(\Psi^m_{1,0}(\mathcal{G})) \supset \Psi^m_{1,0,\mathcal{V}}(M_0)$. Let us now prove the opposite inclusion, that is that $\pi_M(\Psi^m_{1,0}(\mathcal{G})) \subset \Psi^m_{1,0,\mathcal{V}}(M_0)$.

Let $Q \in \Psi^m_{1,0}(\mathcal{G})$ be arbitrary and let $b = T^{-1}(Q)$. Let χ_0 be a smooth function on $\mathcal{G}$ that is equal to 1 in a neighborhood of M in $\mathcal{G}$ and with support in $\alpha((A)_r)$ and such that $\chi = 1$ on the support of $\chi_0 \circ \alpha$. Then $b_0 := \chi_0 b$ is in the range of $\alpha_* \circ \chi \circ \mathcal{F}_{\mathrm{fiber}}^{-1}$, because any distribution $u \in I^m_{(r)}(A, M)$ is in the range of $\mathcal{F}_{\mathrm{fiber}}^{-1}$, if $r < \infty$. Then the difference $b - b_0$ is smooth. Because $\mathcal{G}$ is d-connected, we can use a construction similar to the one used to define b_0 and a partition of unity argument to obtain that

$$T(b - b_0) = \sum_{j=1}^{l} T(b_j)\tilde{\psi}_{X_{j1}} \dots \tilde{\psi}_{X_{jn}} \tag{33}$$

for some distributions $b_j \in \chi I^{-\infty}_{(r)}(A, M)$ and vector fields $X_{jk} \in \mathcal{V}$. Let a_j be such that $b_j = \alpha_* \circ \chi \circ \mathcal{F}_{\mathrm{fiber}}^{-1}(a_j)$, for $a_0 \in S^m_{1,0}(A^*)$ and $a_j \in S^{-\infty}_{1,0}(A^*)$, if $j > 0$. Then Equations (30) and (32) show that

$$\pi_M(Q) = a_0(D) + \sum_{j=1}^{l} a_j(D)\psi_{X_{j1}} \dots \psi_{X_{jn}} \in \Psi^m_{1,0,\mathcal{V}}(M_0). \tag{34}$$

We have thus proved that $\pi_M(\Psi^m_{1,0}(\mathcal{G})) = \Psi^m_{1,0,\mathcal{V}}(M_0)$, as desired. This completes our proof. $\square$

Since the map π_M respects adjoints: $\pi_M(P^*) = \pi_M(P)^*$, [17], we obtain the following corollary.

Corollary 3.3. *The algebras* $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ *and* $\Psi^{\infty}_{cl,\mathcal{V}}(M_0)$ *are closed under taking adjoints.*

We end this section with three remarks.

Remark 3.4. Equation (33) is easily understood in the case of groups, when it amounts to the possibility of covering any given compact set by finitely many translations of a given open neighborhood of the identity. The argument in general is the same as the argument used to define the basic coordinate neighborhoods on $\mathcal{G}$ in [34]. The basic coordinate neighborhoods on $\mathcal{G}$ were used in that paper to define the smooth structure on the groupoid $\mathcal{G}$.

Remark 3.5. We suspect that any proof of the fact that $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is closed under multiplication is equivalent to the integrability of A. In fact, Melrose has implicitly given some evidence for this in [27] for particular $(M, \mathcal{V})$, by showing that the kernels of the pseudodifferential operators on M_0 that he constructed naturally live on a modified product space $M^2_{\mathcal{V}}$. In his case $M^2_{\mathcal{V}}$ was a blow-up of the product $M \times M$, and hence was a *larger* compactification of the product $M_0 \times M_0$. The kernels of his operators naturally extended to conormal distributions on this larger product $M^2_{\mathcal{V}}$. The product and adjoint were defined in terms of suitable maps between $M^2_{\mathcal{V}}$ and some fibered product spaces $M^3_{\mathcal{V}}$, which are suitable blow-ups of M^3 and hence larger compactifications of M_0^3. This in principle leads to a solution of the problem of microlocalizing $\mathcal{V}$ that we stated in the introduction whenever one can define the spaces $M^2_{\mathcal{V}}$ and $M^3_{\mathcal{V}}$. Let us also mention here that Melrose's approach usually leads to algebras that are slightly larger than ours.

Remark 3.6. Let $\mathcal{G}$ be a Lie groupoid such that the map π_M is an isomorphism and let $N \subset M$ be a face of M; then we obtain a generalized indicial map

$$R_N : \Psi^{\infty}_{1,0,\mathcal{V}}(M_0) \simeq \Psi^{\infty}_{1,0}(\mathcal{G}) \to \Psi^{\infty}_{1,0}(\mathcal{G}_N).$$

In applications, the algebras $\Psi^{\infty}_{1,0}(\mathcal{G}_N)$ often turn out to be isomorphic to the algebras $\Psi^{\infty}_{1,0,\mathcal{V}_1}(N_0; G)$ studied in the last section of this paper. In fact, this is the motivation for introducing the algebras $\Psi^{\infty}_{1,0,\mathcal{V}_1}(N_0; G)$.

4. Properties of $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$

Theorem 3.2 has several consequences similar to the results in [21], [28], [29], [30], [38], [40].

4.1. *Basic properties.* We obtain that the algebras $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ and $\Psi^{\infty}_{cl,\mathcal{V}}(M_0)$ are independent of the choices made to define them and, thus, depend only on the Lie structure at infinity $(M, \mathcal{V})$.

COROLLARY 4.1. *The spaces $\Psi^{m}_{1,0,\mathcal{V}}(M_0)$ and $\Psi^{m}_{cl,\mathcal{V}}(M_0)$ are independent of the choice of the metric on A and the function χ used to define it, but depend, in general, on the Lie structure at infinity (M, A) on M_0.*

Proof. The space $\Psi^{m}_{1,0}(\mathcal{G})$ does not depend on the metric on A or on the function χ and neither does the vector representation π_M. Then, by using Theorem 3.2, we see that the proof is the same for classical operators. □

An important consequence is that $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ and

$$\Psi^{\infty}_{cl,\mathcal{V}}(M_0) = \cup_{m\in\mathbb{Z}} \Psi^{m}_{cl,\mathcal{V}}(M_0)$$

are filtered algebras, as it is the case of the usual algebra of pseudodifferential operators on a compact manifold.

PROPOSITION 4.2. *By the above notation,*

$$\Psi^{m}_{1,0,\mathcal{V}}(M_0)\Psi^{m'}_{1,0,\mathcal{V}}(M_0) \subseteq \Psi^{m+m'}_{1,0,\mathcal{V}}(M_0) \quad \textit{and}$$
$$\Psi^{m}_{cl,\mathcal{V}}(M_0)\Psi^{m'}_{cl,\mathcal{V}}(M_0) \subseteq \Psi^{m+m'}_{cl,\mathcal{V}}(M_0)\,,$$

for all $m, m' \in \mathbb{C} \cup \{-\infty\}$.

Proof. Use Theorem 3.2 and the fact that π_M preserves the product. □

Part (i) of the following result is an analog of a standard result about the b-calculus [28], whereas the second part shows the independence of diffeomorphisms of the algebras $\Psi^{\infty}_{cl,\mathcal{V}}(M_0)$, in the framework of manifolds with a Lie structure at infinity. Recall that if $X \in \Gamma(A)$, we have denoted by $\psi_X := \Psi_X(1,\cdot) : M \to M$ the diffeomorphism defined by integrating X (and specializing at $t = 1$).

PROPOSITION 4.3. (i) *Let x be a defining function of some hyperface of M. Then*

$$x^s\Psi^{m}_{1,0,\mathcal{V}}(M_0)x^{-s} = \Psi^{m}_{1,0,\mathcal{V}}(M_0) \quad \textit{and} \quad x^s\Psi^{m}_{cl,\mathcal{V}}(M_0)x^{-s} = \Psi^{m}_{cl,\mathcal{V}}(M_0)$$

for any $s \in \mathbb{C}$.

(ii) *Similarly,*

$$\psi_X\Psi^{m}_{1,0,\mathcal{V}}(M_0)\psi_X^{-1} = \Psi^{m}_{1,0,\mathcal{V}}(M_0) \quad \textit{and} \quad \psi_X\Psi^{m}_{cl,\mathcal{V}}(M_0)\psi_X^{-1} = \Psi^{m}_{cl,\mathcal{V}}(M_0),$$

for any $X \in \Gamma(A)$.

Proof. We have that $x^s \Psi^m_{cl}(\mathcal{G}) x^{-s} = \Psi^m_{cl}(\mathcal{G})$, for any $s \in \mathbb{C}$, by [17]. A similar result for type $(1,0)$ operators is proved in the same way as in [17]. This proves (a) because $\pi_M(x^s P x^{-s}) = x^s \pi_M(P) x^{-s}$.

Similarly, using the notations of Theorem 3.2, we have $\tilde{\psi}_X \Psi^m_{cl}(\mathcal{G}) \tilde{\psi}_X^{-1} = \Psi^m_{cl}(\mathcal{G})$, for any $X \in \Gamma(A) = \mathcal{V}$. By the diffeomorphism invariance of the space of pseudodifferential operators, $\tilde{\psi}_X P \tilde{\psi}_X^{-1}$ defines a right invariant family of pseudodifferential operators on $\mathcal{G}$ for any such right invariant family $P = (P_x)$, as in the proof of Theorem 3.2. To check that the family $P_1 := \psi_X P \psi_X^{-1}$ has a compactly supported convolution kernel, denote by

$$(\mathcal{G})_a = \{g, \operatorname{dist}(g, d(g)) \leq a\}.$$

Then observe that $\operatorname{supp}(\tilde{\psi}_X P \tilde{\psi}_X^{-1}) \subset \mathcal{G}_{d+2\|X\|}$ whenever $\operatorname{supp}(P) \subset (\mathcal{G})_d$. Then use Equation (31) to conclude the result.

The proof for type $(1,0)$ operators is the same. □

Let us notice that the same proof gives (ii) above for any diffeomorphism of M_0 that extends to an automorphism of (M, A). Recall that an automorphism of the Lie algebroid $\pi : A \to M$ is a morphism of vector bundles (φ, ψ), $\varphi : M \to M$, $\psi : A \to A$, such that φ and ψ are diffeomorphisms, $\pi \circ \psi = \varphi \circ \pi$, and we have the following compatibility with the anchor map ϱ:

$$\varrho \circ \psi = \varphi_* \circ \varrho.$$

4.2. *Mapping properties.* Let $H^s(M_0)$ be the domain of $(1+\Delta)^{s/2}$, where Δ is the (positive) Laplace operator on M_0 defined by the metric, if $s \geq 0$. The space $H^{-s}(M_0)$, $s \geq 0$, is defined by duality, the duality form being the pairing of distributions with test functions.

COROLLARY 4.4. *Each operator $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$, $P : \mathcal{C}^\infty_c(M_0) \to \mathcal{C}^\infty(M_0)$, extends to continuous linear operators $P : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ and $P : H^s(M_0) \to H^{s-m}(M_0)$. The space $H^m(M_0)$, $m \geq 0$, identifies with the domain of P with the graph topology and $H^{-m}(M_0) = PL^2(M_0) + L^2(M_0)$, for any elliptic $P \in \Psi^m_{1,0}(M_0)$.*

Proof. The first part is a direct consequence of the definition since any $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ is properly supported. The last part follows from the results of [1] and [3].

We now sketch the proof for the benefit of the reader. It follows from the explicit form of the kernels of operators $T \in \Psi^{-n-1}_{1,0,\mathcal{V}}(M_0)$, $n = \dim(M_0)$, that such a T is bounded on $L^2(M_0)$. Using the symbolic properties of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$, namely Proposition 2.6 and Proposition 4.2, it then follows that any $T \in \Psi^0_{1,0,\mathcal{V}}(M_0)$ is bounded on $L^2(M_0)$ (the details are the same as in [17] or [3]). Using again the symbolic properties of $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$, we prove as in [3] that the domain of the closure of P and $PL^2(M_0) + L^2(M_0)$ is independent of P elliptic of order m. Let us denote by H_m the domain of the closure of P and $H_{-m} = PL^2(M_0) + L^2(M_0)$. Then it is proved in [3] that $T : H_r \to H_{r-m}$ is bounded, for any T of order m. In [1] it is proved using partitions of unity that $T : H^r(M_0) \to H^{r-m}(M_0)$ is bounded for any T of order m. This shows that $H_r = H^r(M_0)$ for any $r \in \mathbb{R}$. □

4.3. *Quantization.* We have the following quantization properties of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$.

For any $X \in \Gamma(A)$, denote by $a_X : A^* \to \mathbb{C}$ the function defined by $a_X(\xi) = \xi(X)$. Then there exists a unique Poisson structure on A^* such that $\{a_X, a_Y\} = a_{[X,Y]}$. It is related to the Poisson structure $\{\,\cdot\,,\,\cdot\,\}^{T^*M}$ on T^*M via the formula

$$\{f_1 \circ \varrho^*, f_2 \circ \varrho^*\}^{T^*M} = \{f_1, f_2\} \circ \varrho^*,$$

where $\varrho^* : T^*M \to A^*$ denotes the dual to the anchor map ϱ. In particular, $\{\,\cdot\,,\,\cdot\,\}$ and $\{\,\cdot\,,\,\cdot\,\}^{T^*M}$ coincide on M_0.

Proposition 4.5. *For any $P \in \Psi^m_{1,0}(M_0)$ and any $Q \in \Psi^{m'}_{1,0}(M_0)$, where $\{\,\cdot\,,\,\cdot\,\}$ is the usual Poisson bracket on A^*,*

$$\sigma^{(m+m'-1)}([P,Q]) = \{\sigma^{(m)}(P), \sigma^{(m')}(Q)\}.$$

Proof. The Poisson structure on T^*M_0 is induced from the Poisson structure on A^*. In turn, the Poisson structure on T^*M_0 determines the Poisson structure on A^*, because T^*M_0 is dense in A^*. The desired result then follows from the similar result that is known for pseudodifferential operators on M_0 and the Poisson bracket on T^*M_0. □

We conclude with the following result, which is independent of the previous considerations, but sheds some light on them. The invariant differential operators on $\mathcal{G}$ are generated by d–vertical invariant vector fields on $\mathcal{G}$, that is by $\Gamma(A(\mathcal{G}))$. We have by definition that $\pi_M = \varrho : \Gamma(M; A(\mathcal{G})) \to \Gamma(M; TM)$, and hence π_M maps the algebra of invariant differential operators onto $\mathcal{G}$ to $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$. In particular, the proof of Theorem 3.2 (more precisely Equation (30)) can be used to prove the following result, which we will however prove also without making appeal to Theorem 3.2.

Proposition 4.6. *Let $X \in \Gamma(A)$ and denote by $a_X(\xi) = \xi(X)$ the associated linear function on A^*. Then $a_{X,\chi} \in S^1(A^*)$ and $a_X(D) = -iX$. Moreover,*

$$\{a_\chi(D),\, a = \text{ polynomial in each fiber}\} = \mathrm{Diff}^*_{\mathcal{V}}(M_0).$$

Proof. We continue to use a fixed metric on A to trivialize any density bundle. Let $u = \mathcal{F}^{-1}_{\text{fiber}}(a)$, where $a \in S^m_{cl}(A^*)$ is polynomial in each fiber.

By the Fourier inversion formula (and integration by parts), u is supported on M, which is the same thing as saying that u is a distribution of the form $\langle u, f\rangle = \int_M P_0 f(x) d\mathrm{vol}(x)$, with P_0 a differential operator acting along the fibers of $A \to M$ and $f \in \mathcal{C}_{\mathrm{c}}^{\infty}(A)$. It then follows from the definition of $a_\chi(D)$, from the formula above for $u = \mathcal{F}_{\mathrm{fiber}}^{-1}(a)$, and from the fact that $\chi = 1$ in a neighborhood of the support of u that

$$a_\chi(D) f(x) = [P_0 f(\exp_x(-v))]|_{v=0}, \quad v \in T_x M_0. \tag{35}$$

Let $X_1, X_2, \dots, X_m \in \Gamma(A)$ and

$$a = a_{X_1} a_{X_2} \dots a_{X_m} \in S^m(A^*). \tag{36}$$

Then the differential operator P_0 above is given by the formula

$$P_0 f(x) = \int_{A_x^*} a(\xi) \mathcal{F}^{-1} f(\xi),$$

with the inverse Fourier transform $\mathcal{F}^{-1}$ being defined along the fiber A_x. Hence

$$P_0 = i^m X_1 X_2 \dots X_m,$$

with each X_j being identified with the family of constant-coefficient differential operators along the fibers of $A \to M$ that acts along A_x as the derivation in the direction of $X_j(x)$.

For any $X \in A$, we shall denote by ψ_{tX} the one-parameter subgroup of diffeomorphisms of M generated by X. (Note that ψ_{tX} is defined for any t because M is compact and X is tangent to all faces of M.) We thus obtain an action of ψ_{tX} on functions by $[\psi_{tX}(f)](x) = f(\exp(tX)x)$. Then the differential operator P_0, associated to a as in Equation (36), is given by

$$P_0(f \circ \exp)|_M = i^m \big[\partial_1 \partial_2 \dots \partial_m \psi_{t_1 X_1 + t_2 X_2 + \dots + t_m X_m} f\big]|_{t_1 = \dots = t_m = 0}. \tag{37}$$

Then Equations (35) and (37) give

$$a_\chi(D) f = i^m \big[\partial_1 \dots \partial_m \exp(-t_1 X_1 - \dots - t_m X_m) f\big]|_{t_1 = \dots = t_m = 0}, \tag{38}$$

In particular, $a_X(D) = -iX$, for any $X \in \Gamma(A)$.

This proves that

$$a_\chi(D) \in \mathrm{Diff}_{\mathcal{V}}^*(M_0), \tag{39}$$

by the Campbell-Hausdorff formula [10], [36], which states that $a_\chi(D)$ is generated by $X_1, X_2, \dots, X_n$ (and their Lie brackets), and hence that it is generated by $\mathcal{V}$, which was assumed to be a Lie algebra.

Let us prove now that any differential operator $P \in \mathrm{Diff}_{\mathcal{V}}^*(M_0)$ is of the form $a_\chi(D)$, for some polynomial symbol a on A^*. This is true if P has degree zero. Indeed, assume P is the multiplication by $f \in \mathcal{C}^\infty(M)$. Lift f to an order zero symbol on A^*, by letting this extension be constant in each fiber. Then $P = f(D)$. We shall prove our statement by induction on the degree

m of P. By linearity, we can reduce to the case $P = i^{-m} X_1 \dots X_m$, where $X_1, \dots, X_m \in \Gamma(A)$. Let $a = a_{X_1} \dots a_{X_m}$. Then

$$\sigma_m(a_\chi(D))(\xi) = a(\xi) = X_1(\xi) \dots X_m(\xi) = \sigma_m(P),$$

and hence $Q := a_\chi(D) - i^{-m} X_1 \dots X_m \in \mathrm{Diff}_{\mathcal{V}}^{m-1}(M_0)$. By the induction hypothesis, $Q = b_\chi(D)$ for some polynomial symbol of order at most $m-1$ on A^*. This completes the proof. □

From this we obtain the following corollary.

Corollary 4.7. *Let* $\mathrm{Diff}(M_0)$ *be the algebra of all differential operators on* M_0. *Then*

$$\Psi_{1,0,\mathcal{V}}^{\infty}(M_0) \cap \mathrm{Diff}(M_0) = \mathrm{Diff}_{\mathcal{V}}^{*}(M_0).$$

Proof. We know from the above proposition that

$$\Psi_{1,0,\mathcal{V}}^{\infty}(M_0) \cap \mathrm{Diff}(M_0) \supset \mathrm{Diff}_{\mathcal{V}}^{*}(M_0).$$

Conversely, assume $P \in \Psi_{1,0,\mathcal{V}}^{m}(M_0) \cap \mathrm{Diff}(M_0)$. We shall prove by induction on m that $P \in \mathrm{Diff}_{\mathcal{V}}^{m}(M_0)$. If $m = 0$ then P is the multiplication with a smooth function f on M_0. But then $f = \sigma^{(0)}(P) \in S^0(A^*)$ is constant along the fibers of $A^* \to M$, and hence $f \in \mathcal{C}^\infty(M) \subset \mathrm{Diff}_{\mathcal{V}}^{*}(M_0)$.

Assume now that the statement is proved for P of order $< m$. We shall prove it also for P of order m. Then $a := \sigma^{(m)}(P)$ is a polynomial symbol in $S^m(A^*)$. Thus $a_\chi(D) \in \mathrm{Diff}_{\mathcal{V}}^{m}(M_0)$, by Proposition 4.6. But then $\sigma^{(m)}(P - a_\chi(D)) = 0$, by Lemma 2.2, and hence $P - a_\chi(D) \in \Psi_{1,0,\mathcal{V}}^{m-1}(M_0) \cap \mathrm{Diff}(M_0)$. By the induction hypothesis $P - a_\chi(D) \in \mathrm{Diff}_{\mathcal{V}}^{m-1}(M_0)$. This completes the proof. □

5. Group actions and semi-classical limits

One of the most convenient features of manifolds with a Lie structure at infinity is that questions about analysis of these manifolds often reduce to questions about analysis of simpler manifolds. These simpler manifolds are manifolds of the same dimension but endowed with certain nontrivial group actions. Harmonic analysis techniques then allow us to ultimately reduce our questions to analysis of lower dimensional manifolds with a Lie structure at infinity. In this section, we discuss the algebras $\Psi_{1,0,\mathcal{V}}^{\infty}(M_0, G)$ that generalize the algebras $\Psi_{1,0,\mathcal{V}}^{\infty}(M_0)$ when group actions are considered. These algebras are necessary for the reductions mentioned above and are typically the range of (generalized) indicial maps. Then we discuss a semi-classical version of the algebra $\Psi_{1,0,\mathcal{V}}^{\infty}(M_0)$.

5.1. *Group actions.* We shall consider the following setting. Let M_0 be a manifold with a Lie structure at infinity (M, A), and $\mathcal{V} = \Gamma(A)$, as above. Also, let G be a Lie group with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$. We shall denote by $\mathfrak{g}_M$ the bundle $M \times \mathfrak{g} \to M$. Then

$$\mathcal{V}_G := \mathcal{V} \oplus \mathcal{C}^\infty(M, \mathfrak{g}) \simeq \Gamma(A \oplus \mathfrak{g}_M) \tag{40}$$

has the structure of a Lie algebra with respect to the bracket $[\,\cdot\,,\,\cdot\,]$ which is defined such that on $\mathcal{C}^\infty(M, \mathfrak{g})$ it coincides with the pointwise bracket, on $\mathcal{V}$ it coincides with the original bracket, and, for any $X \in \mathcal{V}$, $f \in \mathcal{C}^\infty(M)$, and $Y \in \mathfrak{g}$, we have

$$[X, f \otimes Y] := X(f) \otimes Y\,.$$

(Here $f \otimes Y$ denotes the function $\xi : M \to \mathfrak{g}$ defined by $\xi(m) = f(m)Y \in \mathfrak{g}$.)

The main goal of this subsection is to indicate how the results of Section (2) extend to $\mathcal{V}_G$, after we replace A with $A \oplus \mathfrak{g}_M$, M_0 with $M_0 \times G$, and M with $M \times G$. The resulting constructions and definitions will yield objects on $M \times G$ that are invariant with respect to the action of G on itself by *right* translations.

We now proceed by analogy with the construction of the operators $a_\chi(D)$ in Subsection 2.3. First, we identify a section of $\mathcal{V}_G := \mathcal{V} \oplus \mathcal{C}^\infty(M, \mathfrak{g}) \simeq \Gamma(A \oplus \mathfrak{g}_M)$ with a right G-invariant vector field on $M_0 \times G$. At the level of vector bundles, this corresponds to the map

$$p : T(M_0 \times G) = TM_0 \times TG \to TM_0 \times \mathfrak{g}, \tag{41}$$

where the map $TG \to \mathfrak{g}$ is defined by means of the trivialization of TG by *right* invariant vector fields. Let $p_1 : M \times G \to M$ be the projection onto the first component and $p_1^* A$ be the lift of A to $M \times G$ via p_1.

The map p defined in Equation (41) can then be used to define the lift

$$p^*(u) \in I^m(p_1^* A \oplus TG, M \times G), \tag{42}$$

for any distribution $u \in I^m(A \oplus \mathfrak{g}_M, M)$. In particular, $p^*(u)$ will be a right G-invariant distribution. Then we define $\mathcal{R}$ to be the restriction of distributions from $p_1^* A \oplus TG$ to distributions on $TM_0 \times TG = T(M_0 \times G)$.

We endow $M_0 \times G$ with the metric obtained from a metric on A and a right invariant metric on G. This allows us to define the exponential map, thus obtaining, as in Section 2, a differentiable map

$$\Phi : (TM_0 \times TG)_r = (T(M_0 \times G))_r \to (M_0 \times G)^2 \tag{43}$$

that is a diffeomorphism onto an open neighborhood of the diagonal, provided that $r < r_0$, where r_0 is the injectivity radius of $M_0 \times G$. We shall denote as before by

$$\Phi_* : I_c^m((TM_0 \times TG)_r, M_0 \times G) \to I_c^m((M_0 \times G)^2, M_0 \times G)$$

the induced map on conormal distributions.

The inverse Fourier transform will give a map

$$\mathcal{F}^{-1}_{\text{fiber}} : S^m_{1,0}(A^* \oplus \mathfrak{g}^*_M) \longrightarrow I^m(A \oplus \mathfrak{g}_M, M), \tag{44}$$

defined by the same formula as before (Equation (6)). Finally, we shall also need a smooth function χ on $A \oplus \mathfrak{g}_M$ that is equal to 1 in a neighborhood of the zero section and has support inside $(A \oplus \mathfrak{g}_M)_r$.

We can then define the quantization map in the G-equivariant case by

(45)

$$q_{\Phi,\chi,G} := \Phi_* \circ \mathcal{R} \circ p^* \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}} : S^m_{1,0}(A^* \oplus \mathfrak{g}^*_M) \longrightarrow \mathcal{I}^m((M_0 \times G)^2, M_0 \times G).$$

The main difference from the definition in Equation (11) is that we included the map p^*, which is the lift of distributions in $I^m(A \oplus \mathfrak{g}_M, M)$ to G-invariant distributions in $I^m(p_1^* A \oplus TG, M_0 \otimes G)$; see Equation (42). Then

$$a_\chi(D) = T \circ q_{\Phi,\chi,G}, \tag{46}$$

as before.

With this definition of the quantization map, all the results of the previous sections remain valid, with the appropriate modifications. In particular, we obtain the following definition of the algebra of G-equivariant pseudodifferential operators associated to (A, M, G).

Definition 5.1. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0, G)$ of *G-equivariant pseudodifferential operators generated by the Lie structure at infinity* (M, A) is the linear space of operators $\mathcal{C}^\infty_c(M_0 \times G) \to \mathcal{C}^\infty_c(M_0 \times G)$ generated by $a_\chi(D)$, $a \in S^m_{1,0}(A^* \oplus \mathfrak{g}^*_M)$, and $b_\chi(D)\psi_{X_1} \dots \psi_{X_k}$, $b \in S^{-\infty}(A^* \oplus \mathfrak{g}^*_M)$ and $X_j \in \Gamma(A \oplus \mathfrak{g}_M)$.

The space $\Psi^m_{cl,\mathcal{V}}(M_0, G)$ of *classical G-equivariant pseudodifferential operators generated by the Lie structure at infinity* (M, A) is defined similarly, but using classical symbols a.

With this definition, all the results on the algebras $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ extend right away to the spaces $\Psi^m_{1,0,\mathcal{V}}(M_0, G)$ and $\Psi^m_{cl,\mathcal{V}}(M_0, G)$. In particular, these spaces are algebras for $m = 0$, are independent of the choice of the metric on A used to define them, and have the usual symbolic properties of the algebras of pseudodifferential operators.

The only thing that may need more explanation is what we replace π_M with in the G-equivariant case, because there we no longer use the vector representation. Let $\mathcal{G}$ be a groupoid integrating A, $\Gamma(A) = \mathcal{V}$. Then $\mathcal{G} \times G$ integrates $A \oplus \mathfrak{g}_M$. If $P = (P_x) \in \Psi^m_{1,0}(\mathcal{G} \times G)$, then we consider $\pi_0(P)$ to be the operator induced by P_x on $(\mathcal{G}_x/\mathcal{G}^x_x) \times G$, $x \in M_0$, the later space being a quotient of $(\mathcal{G} \times G)_x$. We shall then use π_0 instead of π_M in the G-equivariant case. (By the proof of Theorem 3.2, $\pi_0 = \pi_M$, if G is reduced to a point.)

5.2. *Indicial maps.* The main reason for considering the algebras $\Psi^m_{1,0,\mathcal{V}}(M_0, G)$ and their classical counterparts is the following. Let $(M, \mathcal{V})$, $\mathcal{V} = \Gamma(M, A)$, be a manifold with a Lie structure at infinity. Let $N_0 \subset M$ be a submanifold such that $T_x N_0 = \varrho(A_x)$ for any $x \in N_0$. Moreover, assume that N_0 is completely contained in an open face $F \subset M$ such that $N := \overline{N_0}$ is a submanifold with corners of F and $N_0 = N \smallsetminus \partial N$. Then the restriction $A|_{N_0}$ is such that the Lie bracket on $\mathcal{V} = \Gamma(A)$ descends to a Lie bracket on $\Gamma(A|_{N_0})$. (This is due to the fact that the space I of functions vanishing on N is invariant for derivations in $\mathcal{V}$. Then $I\mathcal{V}$ is an ideal of $\mathcal{V}$, and hence $\mathcal{V}/I\mathcal{V} \simeq \Gamma(A|_N)$ is naturally a Lie algebra.)

Assume now that there exists a Lie group G and a vector bundle $A_1 \to N$ such that $A|_N \simeq A_1 \oplus \mathfrak{g}_N$ and $\mathcal{V}_1 := \mathcal{V}|_N \simeq \Gamma(A_1)$. Then $\mathcal{V}_1$ is a Lie algebra and (N_0, N, A_1) is also a manifold with a Lie structure at infinity. In many cases (certainly for many of the most interesting examples) one obtains for any Lie group H a natural morphism

$$R_N : \Psi^m_{1,0,\mathcal{V}}(M_0; H) \to \Psi^m_{1,0,\mathcal{V}_1}(N_0; G \times H). \tag{47}$$

For example, the generalizations of the morphisms considered in [17] are of the form (47). However, we do not know exactly what the conditions are under which the morphism R_N above is defined.

Let $\mathfrak{h} = \operatorname{Lie} H$ and $\mathfrak{h}_N = M \times \operatorname{Lie} H$. Then, at the level of kernels the morphism defined by Equation (47) corresponds to the restriction maps

$$r_N : I^m(A^* \oplus \mathfrak{h}_N^*, M) \to I^*(A^*|_N \oplus \mathfrak{h}_N^*, N) \simeq I^*(A_1^* \oplus \mathfrak{g}_N \oplus \mathfrak{h}_N^*, N)$$

in the sense that $R_N(a_\chi(D)) = (r_N(a))_\chi(D)$.

5.3. *Semi-classical limits.* We now define the algebra $\Psi^m_{1,0,\mathcal{V}}(M_0[[h]])$, an element of which will be, roughly speaking, a semi-classical family of operators (T_t), $T_t \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ $t \in (0,1]$. See [46] for some applications of semi-classical analysis.

Definition 5.2. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0[[h]])$ of *pseudodifferential operators generated by the Lie structure at infinity* (M, A) is the linear space of families of operators $T_t : \mathcal{C}^\infty_c(M_0 \times G) \to \mathcal{C}^\infty_c(M_0 \times G)$, $t \in (0,1]$, generated by

$$a_\chi(t, tD)\,, \quad a \in S^m_{1,0}([0,1] \times A^* \oplus \mathfrak{g}_M^*),$$

and

$$b_\chi(t, tD)\psi_{tX_1(t)} \dots \psi_{tX_k(t)}, \quad b \in S^{-\infty}([0,1] \times A^* \oplus \mathfrak{g}_M^*),$$
$$X_j \in \Gamma([0,1] \times A \oplus \mathfrak{g}_M).$$

The space $\Psi^m_{cl,\mathcal{V}}(M_0[[h]])$ of *semi-classical families of pseudodifferential operators generated by the Lie structure at infinity* (M, A) is defined similarly, but using classical symbols a.

Thus we consider families of operators (T_t), $T_t \in \Psi^m_{1,0,\mathcal{V}}(M_0)$, defined in terms of data a, b, X_k, that extend smoothly to $t = 0$, with the interesting additional feature that the cotangent variable is rescaled as $t \to 0$.

Again, all the results on the algebras $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ extend right away to the spaces $\Psi^m_{1,0,\mathcal{V}}(M_0[[h]])$ and $\Psi^m_{cl,\mathcal{V}}(M_0[[h]])$, except maybe Proposition 4.6 and its Corollary 4.7, that need to be properly reformulated.

Another variant of the above constructions is to consider families of manifolds with a Lie structure at infinity. The necessary changes are obvious though, and we will not discuss them here.

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