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Weak mixing for interval exchange transformations and translation flows

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Abstract

We prove that a typical interval exchange transformation is either weakly mixing or it is an irrational rotation. We also conclude that a typical translation flow on a typical translation surface of genus $g \geq 2$ (with prescribed singularity types) is weakly mixing.

1. Introduction

Let $d \geq 2$ be a natural number and let π be an *irreducible* permutation of $\{1, \dots, d\}$; that is, $\pi\{1, \dots, k\} \neq \{1, \dots, k\}$, $1 \leq k < d$. Given $\lambda \in \mathbb{R}^d_+$, we define an *interval exchange transformation* (i.e.t.) $f := f(\lambda, \pi)$ in the usual way [CFS], [Ke]: we consider the interval

$$I := I(\lambda, \pi) = \left[0, \sum_{i=1}^{d} \lambda_i\right) \,, \tag{1.1}$$

break it into subintervals

$$I_i := I_i(\lambda, \pi) = \left[\sum_{j<i} \lambda_j, \sum_{j \leq i} \lambda_j\right), \quad 1 \leq i \leq d, \tag{1.2}$$

and rearrange the I_i according to π (in the sense that the i-th interval is mapped onto the $\pi(i)$-th interval). In other words, $f : I \to I$ is given by

$$x \mapsto x + \sum_{\pi(j)<\pi(i)} \lambda_j - \sum_{j<i} \lambda_j \,, \quad x \in I_i \,. \tag{1.3}$$

We are interested in the ergodic properties of i.e.t.'s. Obviously, they preserve the Lebesgue measure on I. Katok proved that i.e.t.'s and suspension flows over

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i.e.t.'s with roof function of bounded variation are never mixing [Ka], [CFS]. Then the fundamental work of Masur [M] and Veech [V2] established that almost every i.e.t. is uniquely ergodic (this means that, for every irreducible π and for Lebesgue almost every $\lambda \in \mathbb{R}^d_+$, $f(\lambda, \pi)$ is uniquely ergodic).

The question of whether the typical i.e.t. is weakly mixing is more delicate except if π is a *rotation* of $\{1, \dots, d\}$, that is, if π satisfies the following conditions: $\pi(i+1) \equiv \pi(i) + 1 \mod d$, for all $i \in \{1, \dots, d\}$. In that case $f(\lambda, \pi)$ is conjugate to a rotation of the circle, hence it is not weakly mixing, for every $\lambda \in \mathbb{R}^d_+$. After the work of Katok and Stepin [KS] (who proved weak mixing for almost all i.e.t.'s on 3 intervals), Veech [V4] established almost sure weak mixing for infinitely many irreducible permutations and asked whether the same property is true for any irreducible permutation which is not a rotation. In this paper, we give an affirmative answer to this question.

THEOREM A. *Let π be an irreducible permutation of $\{1, \dots, d\}$ which is not a rotation. For Lebesgue almost every $\lambda \in \mathbb{R}^d_+$, $f(\lambda, \pi)$ is weakly mixing.*

We should remark that *topological* weak mixing was established earlier (for almost every i.e.t. which is not a rotation) by Nogueira-Rudolph [NR].

We recall that a measure-preserving transformation f of a probability space (X, m) is said to be *weakly mixing* if for every pair of measurable sets A, $B \subset X$,

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} |m(f^{-k}A \cap B) - m(A)m(B)| = 0 . \tag{1.4}$$

It follows immediately from the definitions that every mixing transformation is weakly mixing and every weakly mixing transformation is ergodic. A classical theorem states that any *invertible* measure-preserving transformation f is weakly mixing if and only if it has *continuous spectrum*; that is, the only eigenvalue of f is 1 and the only eigenfunctions are constants [CFS], [P]. Thus it is possible to prove weak mixing by ruling out the existence of non-constant *measurable* eigenfunctions. This is in fact the standard approach which is also followed in this paper. Topological weak mixing is proved by ruling out the existence of non-constant *continuous* eigenfunctions. Analogous definitions and statements hold for flows.

1.1. *Translation flows.* Let M be a compact orientable *translation surface* of genus $g \geq 1$, that is, a surface with a finite or empty set Σ of conical singularities endowed with an atlas such that coordinate changes are given by translations in $\mathbb{R}^2$ [GJ1], [GJ2]. Equivalently, M is a compact surface endowed with a flat metric, with at most finitely many conical singularities and trivial holonomy. For a general flat surface the cone angles at the singularities are $2\pi(\kappa_1 + 1) \leq \cdots \leq 2\pi(\kappa_r + 1)$, where $\kappa_1, \dots, \kappa_r > -1$ are real numbers satisfying $\sum \kappa_i = 2g - 2$. If the surface has trivial holonomy, then $\kappa_i \in \mathbb{Z}_+$, for all $1 \leq i \leq r$, and there exists a parallel section of the unit tangent bundle T_1M, that is, a parallel vector field of unit length, well-defined on $M \setminus \Sigma$. A third, equivalent, point of view is to consider pairs (M, ω) of a compact Riemann surface M and a (non-zero) abelian differential ω. A flat metric on M (with $\Sigma := \{\omega = 0\}$) is given by $|\omega|$ and a parallel (horizontal) vector field v of unit length is determined by the condition $\omega(v) = 1$. The specification of the parameters $\kappa = (\kappa_1, \ldots, \kappa_r) \in \mathbb{Z}_+^r$ with $\sum \kappa_i = 2g - 2$ determines a finite dimensional *stratum* $\mathcal{H}(\kappa)$ of the moduli space of translation surfaces which is endowed with a natural complex structure and a Lebesgue measure class [V5], [Ko].

A *translation flow* F on a translation surface M is the flow generated by a parallel vector field of unit length on $M \setminus \Sigma$. The space of all translation flows on a given translation surface is naturally identified with the unit tangent space at any regular point; hence it is parametrized by the circle S^1. For all $\theta \in S^1$, the translation flow F_θ, generated by the vector field v_θ such that $e^{-i\theta}\omega(v_\theta) = 1$, coincides with the restriction of the geodesic flow of the flat metric $|\omega|$ to an invariant surface $M_\theta \subset T_1M$ (which is the graph of the vector field v_θ in the unit tangent bundle over $M \setminus \Sigma$).

We are interested in *typical* translation flows (with respect to the Haar measure on S^1) on *typical* translation surfaces (with respect to the Lebesgue measure class on a given stratum). In genus 1 there are no singularities and translation flows are linear flows on $\mathbb{T}^2$: they are typically uniquely ergodic but never weakly mixing. In genus $g \geq 2$, the unique ergodicity for a typical translation flow on the typical translation surface was proved by Masur [M] and Veech [V2]. This result was later strenghtened by Kerckhoff, Masur and Smillie [KMS] to include arbitrary translation surfaces.

As in the case of interval exchange transformations, the question of weak mixing of translation flows is more delicate than unique ergodicity, but it is widely expected that weak mixing holds typically in genus $g \geq 2$. We will show that it is indeed the case:

THEOREM B. *Let $\mathcal{H}(\kappa)$ be any stratum of the moduli space of translation surfaces of genus $g \geq 2$. For almost all translation surfaces $(M, \omega) \in \mathcal{H}(\kappa)$, the translation flow F_θ on (M, ω) is weakly mixing for almost all $\theta \in S^1$.*

Translation flows and i.e.t.'s are intimately related: the former can be viewed as suspension flows (of a particular type) over the latter. However, since the weak mixing property, unlike ergodicity, is not invariant under suspensions and time changes, the problems of weak mixing for translation flows and i.e.t.'s are independent of one another. We point out that (differently from the case of i.e.t.'s, where weak mixing had been proved for infinitely many combinatorics), there has been little progress on weak mixing for typical translation flows (in the measure-theoretic sense), except for *topological* weak mixing, proved in [L]. Gutkin and Katok [GK] proved weak mixing for a G_δ-dense set of translation flows on translation surfaces related to a class of rational polygonal billiards. We should point out that our results tell us nothing new about the dynamics of rational polygonal billiards (for the well-known reason that rational polygonal billiards yield zero measure subsets of the moduli space of all translation surfaces).

1.2. *Parameter exclusion.* To prove our results, we will perform a parameter exclusion to get rid of undesirable dynamics. With this in mind, instead of working in the direction of understanding the dynamics on the phase space (regularity of eigenfunctions[1], etc.), we will focus on analysis of the parameter space.

We analyze the parameter space of suspension flows over i.e.t.'s via a renormalization operator (i.e.t.'s correspond to the case of constant roof function). The renormalization operator acts non-linearly on i.e.t.'s and linearly on roof functions, so it has the structure of a cocycle (the Zorich cocycle) over the renormalization operator on the space of i.e.t.'s (the Rauzy-Zorich induction). One can work out a criterion for weak mixing (originally due to Veech [V4]) in terms of the dynamics of the renormalization operator.

An important ingredient in our analysis is the result of [F2] on the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle over the Teichmüller flow. This result is equivalent to the non-uniform hyperbolicity of the Zorich cocycle [Z3]. Actually we only need a weaker result, namely that the Kontsevich-Zorich cocycle, or equivalently the Zorich cocycle, has two positive Lyapunov exponents in the case of surfaces of genus at least 2.

In the case of translation flows, a "linear" parameter exclusion (on the roof function parameters) shows that "bad" roof functions form a small set (basically, each positive Lyapunov exponent of the Zorich cocycle gives one obstruction for the eigenvalue equation, which has only one free parameter). This argument is explained in Appendix A.

The situation for i.e.t.'s is much more complicated, since we have no freedom to change the roof function. We need to carry out a "non-linear" exclusion process, based on a statistical argument. This argument proves weak mixing at once for typical i.e.t.'s and typical translation flows. While for the linear exclusion it is enough to use the ergodicity of the renormalization operator on the space of i.e.t.'s, the statistical argument for the non-linear exclusion heavily uses its mixing properties.

[1] In this respect, we should remark that Yoccoz has pointed out to us the existence of "strange" eigenfunctions for certain values of the parameter.

1.3. *Outline.* We start this paper with basic background on cocycles over expanding maps. We then prove our key technical result, an abstract parameter exclusion scheme for "sufficiently random integral cocycles".

We then review known results on the renormalization dynamics for i.e.t.'s and show how the problem of weak mixing reduces to the abstract parameter exclusion theorem. The same argument also covers the case of translation flows.

In the appendix we present the linear exclusion argument, which is much simpler than the non-linear exclusion argument but is enough to deal with translation flows and yields an estimate on the Hausdorff dimension of the set of translation flows which are not weakly mixing.

2. Background

2.1. *Strongly expanding maps.* Let (Δ, μ) be a probability space. We say that a measurable transformation $T : \Delta \to \Delta$, which preserves the measure class of the measure μ, is *weakly expanding* if there exists a partition (modulo 0) $\{\Delta^{(l)}, l \in \mathbb{Z}\}$ of Δ into sets of positive μ-measure, such that, for all $l \in \mathbb{Z}$, T maps $\Delta^{(l)}$ onto Δ, $T^{(l)} := T|\Delta^{(l)}$ is invertible and $T^{(l)}_* \mu$ is equivalent to μ.

Let Ω be the set of all finite words with integer entries. The length (number of entries) of an element $\mathbf{l} \in \Omega$ will be denoted by $|\mathbf{l}|$. For any $\mathbf{l} \in \Omega$, $\mathbf{l} = (l_1, \ldots, l_n)$, we set $\Delta^{\mathbf{l}} := \{x \in \Delta, T^{k-1}(x) \in \Delta^{(l_k)}, 1 \leq k \leq n\}$ and $T^{\mathbf{l}} := T^n|\Delta^{\mathbf{l}}$. Then $\mu(\Delta^{\mathbf{l}}) > 0$.

Let $\mathcal{M} = \{\mu^{\mathbf{l}}, \mathbf{l} \in \Omega\}$, where

$$\mu^{\mathbf{l}} := \frac{1}{\mu(\Delta^{\mathbf{l}})} T^{\mathbf{l}}_* \mu. \tag{2.1}$$

We say that T is *strongly expanding* if there exists a constant $K > 0$ such that

$$K^{-1} \leq \frac{d\nu}{d\mu} \leq K, \quad \nu \in \mathcal{M}. \tag{2.2}$$

This has the following consequence. If $Y \subset \Delta$ is such that $\mu(Y) > 0$ then

$$K^{-2}\mu(Y) \leq \frac{T^{\mathbf{l}}_* \nu(Y)}{\mu(\Delta^{\mathbf{l}})} \leq K^2 \mu(Y), \quad \nu \in \mathcal{M}, \mathbf{l} \in \Omega. \tag{2.3}$$

2.2. *Projective transformations.* We let $\mathbb{P}^{p-1}_+ \subset \mathbb{P}^{p-1}$ be the projectivization of $\mathbb{R}^p_+$. We will call it the *standard simplex*. A *projective contraction* is a projective transformation taking the standard simplex into itself. Thus a projective contraction is the projectivization of some matrix $B \in \mathrm{GL}(\mathrm{p}, \mathbb{R})$ with non-negative entries. The image of the standard simplex by a projective contraction is called a *simplex*. We need the following simple but crucial fact.

LEMMA 2.1. *Let Δ be a simplex compactly contained in $\mathbb{P}_+^{p-1}$ and let $\{\Delta^{(l)}, l \in \mathbb{Z}\}$ be a partition of Δ (modulo sets of Lebesgue measure 0) into sets of positive Lebesgue measure. Let $T : \Delta \to \Delta$ be a measurable transformation such that, for all $l \in \mathbb{Z}$, T maps $\Delta^{(l)}$ onto Δ, $T^{(l)} := T|\Delta^{(l)}$ is invertible and its inverse is the restriction of a projective contraction. Then T preserves a probability measure μ which is absolutely continuous with respect to the Lebesgue measure on Δ and has a density which is continuous and positive on $\overline{\Delta}$. Moreover, T is strongly expanding with respect to μ.*

Proof. Let $d([x],[y])$ be the projective distance between $[x]$ and $[y]$:

$$d([x],[y]) = \sup_{1 \le i,j \le p} \left| \ln \frac{x_i y_j}{x_j y_i} \right| . \tag{2.4}$$

Let $\mathcal{N}$ be the class of absolutely continuous probability measures on Δ whose densities have logarithms which are p-Lipschitz with respect to the projective distance. Since Δ has finite projective diameter, it suffices to show that there exists $\mu \in \mathcal{N}$ invariant under T and such that $\mu^{\mathbf{l}} \in \mathcal{N}$ for all $\mathbf{l} \in \Omega$. Notice that $\mathcal{N}$ is compact in the weak* topology and convex.

Since $(T^{\mathbf{l}})^{-1}$ is the projectivization of some matrix $B^{\mathbf{l}} = (b^{\mathbf{l}}_{ij})$ in $\mathrm{GL(p,\mathbb{R})}$ with non-negative entries, we have

$$|\det D(T^{\mathbf{l}})^{-1}(x)| = \left[\frac{\|x\|}{\|B^{\mathbf{l}} \cdot x\|} \right]^p \det(B^{\mathbf{l}}), \tag{2.5}$$

so that

$$\frac{|\det D(T^{\mathbf{l}})^{-1}(y)|}{|\det D(T^{\mathbf{l}})^{-1}(x)|} = \left[\frac{\|B^{\mathbf{l}} \cdot x\|}{\|B^{\mathbf{l}} \cdot y\|} \frac{\|y\|}{\|x\|} \right]^p \le \sup_{1 \le i \le p} \left(\frac{x_i \|y\|}{y_i \|x\|} \right)^p \le e^{pd([x],[y])} . \tag{2.6}$$

Thus

$$\mathrm{Leb}^{\mathbf{l}} := \frac{1}{\mathrm{Leb}(\Delta^{\mathbf{l}})} T^{\mathbf{l}}_* \mathrm{Leb} \in \mathcal{N}, \tag{2.7}$$

and

$$\nu_n := \frac{1}{n} \sum_{k=0}^{n-1} T^k_* \mathrm{Leb} = \frac{1}{n} \sum_{\mathbf{l} \in \Omega, |\mathbf{l}| \le n} \mathrm{Leb}(\Delta^{\mathbf{l}}) \mathrm{Leb}^{\mathbf{l}} \in \mathcal{N}. \tag{2.8}$$

Let μ be any limit point of $\{\nu_n\}$ in the weak* topology. Then μ is invariant under T, belongs to $\mathcal{N}$ and, for any $\mathbf{l} \in \Omega$, $\mu^{\mathbf{l}}$ is a limit of

$$\nu^{\mathbf{l}}_n = \left(\sum_{\mathbf{l}^0 \in \Omega, |\mathbf{l}^0| \le n} \mathrm{Leb}(\Delta^{\mathbf{l}^0 \mathbf{l}}) \right)^{-1} \sum_{\mathbf{l}^0 \in \Omega, |\mathbf{l}^0| \le n} \mathrm{Leb}(\Delta^{\mathbf{l}^0 \mathbf{l}}) \mathrm{Leb}^{\mathbf{l}^0 \mathbf{l}} \in \mathcal{N}, \tag{2.9}$$

which implies that $\mu^{\mathbf{l}} \in \mathcal{N}$. ∎

2.3. *Cocycles.* A *cocycle* is a pair (T, A), where $T : \Delta \to \Delta$ and $A : \Delta \to \mathrm{GL}(\mathrm{p}, \mathbb{R})$, viewed as a linear skew-product $(x, w) \mapsto (T(x), A(x) \cdot w)$ on $\Delta \times \mathbb{R}^p$. Notice that $(T, A)^n = (T^n, A_n)$, where

$$A_n(x) = A(T^{n-1}(x)) \cdots A(x), \quad n \geq 0. \tag{2.10}$$

If (Δ, μ) is a probability space, μ is an invariant ergodic measure for T (in particular T is measurable) and

$$\int_\Delta \ln \|A(x)\| d\mu(x) < \infty, \tag{2.11}$$

we say that (T, A) is a *measurable cocycle.*

Let

$$E^s(x) := \{w \in \mathbb{R}^p, \lim \|A_n(x) \cdot w\| = 0\}, \tag{2.12}$$

$$E^{cs}(x) := \{w \in \mathbb{R}^p, \limsup \|A_n(x) \cdot w\|^{1/n} \leq 1\}. \tag{2.13}$$

Then $E^s(x) \subset E^{cs}(x)$ are subspaces of $\mathbb{R}^p$ (called the *stable* and *central stable* spaces respectively), and we have $A(x) \cdot E^{cs}(x) = E^{cs}(T(x))$, $A(x) \cdot E^s(x) = E^s(T(x))$. If (T, A) is a measurable cocycle, $\dim E^s$ and $\dim E^{cs}$ are constant almost everywhere.

If (T, A) is a measurable cocycle, the Oseledets Theorem [O], [KB] implies that $\lim \|A_n(x) \cdot w\|^{1/n}$ exists for almost every $x \in \Delta$ and for every $w \in \mathbb{R}^p$, and that there are p Lyapunov exponents $\theta_1 \geq \cdots \geq \theta_p$ characterized by

$$\begin{aligned} \#\{i, \theta_i = \theta\} = &\dim\{w \in \mathbb{R}^p, \lim \|A_n(x) \cdot w\|^{1/n} \leq e^\theta\} \\ &- \dim\{w \in \mathbb{R}^p, \lim \|A_n(x) \cdot w\|^{1/n} < e^\theta\}. \end{aligned} \tag{2.14}$$

Thus $\dim E^{cs}(x) = \#\{i, \theta_i \leq 0\}$.[2] Moreover, if $\lambda < \min\{\theta_i, \theta_i > 0\}$ then for almost every $x \in \Delta$, for every subspace $G_0 \subset \mathbb{R}^p$ transverse to $E^{cs}(x)$, there exists $C(x, G_0) > 0$ such that

$$\|A_n(x) \cdot w\| \geq C(x, G_0) e^{\lambda n} \|w\|, \quad \text{for all } w \in G_0(x). \tag{2.15}$$

Given $B \in \mathrm{GL}(\mathrm{p}, \mathbb{R})$, we define

$$\|B\|_0 = \max\{\|B\|, \|B^{-1}\|\}. \tag{2.16}$$

If the measurable cocycle (T, A) satisfies the stronger condition

$$\int_\Delta \ln \|A(x)\|_0 d\mu(x) < \infty, \tag{2.17}$$

we will call (T, A) a *uniform cocycle.*

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[2] It is also possible to show that $\dim E^s(x) = \#\{i, \theta_i < 0\}$.

LEMMA 2.2. *Let (T,A) be a uniform cocycle and let*

$$\omega(\kappa) = \sup_{\mu(U)\le\kappa} \sup_{N>0} \int_U \frac{1}{N} \ln \|A_N(x)\|_0 d\mu(x). \tag{2.18}$$

Then

$$\lim_{\kappa\to 0} \omega(\kappa) = 0 \ . \tag{2.19}$$

Proof. Let $\mathcal{B}_\kappa$ be the set of measures $\nu \le \mu$ with total mass at most κ. Notice that $T_*\mathcal{B}_\kappa \subset \mathcal{B}_\kappa$. Let

$$\omega_N(\kappa) = \sup_{\nu\in\mathcal{B}_\kappa} \int \frac{1}{N} \sum_{k=0}^{N-1} \ln \|A(T^k(x))\|_0 d\nu, \tag{2.20}$$

so that clearly

$$\omega(\kappa) \le \sup_{N>0} \omega_N(\kappa), \tag{2.21}$$

$$\omega_N(\kappa) = \sup_{\nu\in\mathcal{B}_\kappa} \int \frac{1}{N} \sum_{k=0}^{N-1} \ln \|A(x)\|_0 dT_*^k\nu \le \sup_{\nu\in\mathcal{B}_\kappa} \int \ln \|A(x)\|_0 d\nu. \tag{2.22}$$

Since $\ln \|A(x)\|_0$ is integrable,

$$\lim_{\kappa\to 0} \sup_{\nu\in\mathcal{B}_\kappa} \int \ln \|A(x)\|_0 d\nu = 0. \tag{2.23}$$

The result follows from (2.21), (2.22) and (2.23). □

We say that a cocycle (T,A) is *locally constant* if $T : \Delta \to \Delta$ is strongly expanding and $A|\Delta^{(l)}$ is a constant $A^{(l)}$, for all $l \in \mathbb{Z}$. In this case, for all $\mathbf{l} \in \Omega$, $\mathbf{l} = (l_1, \ldots, l_n)$, we let

$$A^{\mathbf{l}} := A^{(l_n)} \cdots A^{(l_1)}. \tag{2.24}$$

We say that a cocycle (T,A) is *integral* if $A(x) \in \mathrm{GL}(\mathrm{p}, \mathbb{Z})$, for almost all $x \in \Delta$. An integral cocycle can be regarded as a skew product on $\Delta \times \mathbb{R}^p/\mathbb{Z}^p$.

3. Exclusion of the weak-stable space

Let (T,A) be a cocycle. We define the *weak-stable space* at $x \in \Delta$ by

$$W^s(x) = \{w \in \mathbb{R}^p,\ \|A_n(x)\cdot w\|_{\mathbb{R}^p/\mathbb{Z}^p} \to 0\} \ , \tag{3.1}$$

where $\|\cdot\|_{\mathbb{R}^p/\mathbb{Z}^p}$ denotes the euclidean distance from the lattice $\mathbb{Z}^p \subset \mathbb{R}^p$. Now, it is immediate to see that, for almost all $x \in \Delta$, the space $W^s(x)$ is a union of translates of $E^s(x)$. If the cocycle is integral, $W^s(x)$ has a natural interpretation as the stable space at $(x, 0)$ of the zero section in $\Delta \times \mathbb{R}^p/\mathbb{Z}^p$. If the cocycle is bounded, that is, if the function $A : \Delta \to \mathrm{GL}(\mathrm{p}, \mathbb{R})$ is essentially bounded, then it is easy to see that $W^s(x) = \cup_{c \in \mathbb{Z}^p} E^s(x) + c$. In general $W^s(x)$ may be the union of uncountably many translates of $E^s(x)$.

Let $\Theta \subset \mathbb{P}^{p-1}$ be a compact set. We say that Θ is *adapted* to the cocycle (T, A) if $A^{(l)} \cdot \Theta \subset \Theta$ for all $l \in \mathbb{Z}$ and if, for almost every $x \in \Delta$,

$$\|A(x) \cdot w\| \geq \|w\|, \tag{3.2}$$

$$\|A_n(x) \cdot w\| \to \infty \tag{3.3}$$

whenever $w \in \mathbb{R}^p \setminus \{0\}$ projectivizes to an element of Θ.

Let $\mathcal{J} = \mathcal{J}(\Theta)$ be the set of lines in $\mathbb{R}^p$, parallel to some element of Θ and not passing through 0.

The main result in this section is the following.

THEOREM 3.1. *Let (T, A) be a locally constant integral uniform cocycle and let Θ be adapted to (T, A). Assume that for every line $J \in \mathcal{J} := \mathcal{J}(\Theta)$, $J \cap E^{cs}(x) = \emptyset$ for almost every $x \in \Delta$. Then if L is a line contained in $\mathbb{R}^p$ parallel to some element of Θ, $L \cap W^s(x) \subset \mathbb{Z}^p$ for almost every $x \in \Delta$.*

Remark 3.2. It is much easier to prove Theorem 3.1 if one assumes that $\int \|A\|^{1+\varepsilon} d\mu < \infty$ for some $\varepsilon > 0$, and certain parts of the proof become more transparent already under the condition $\int \|A\|^{\varepsilon} d\mu < \infty$. For the cocycles to which we will apply Theorem 3.1 in this paper, namely, uniformly hyperbolic inducings of the Zorich cocycle, it is well known that $\int \|A\| d\mu = \infty$, and it was recently shown in [AGY] that one can choose the cocycles so as to obtain $\int \|A\|^{1-\varepsilon} d\mu < \infty$.

The proof of Theorem 3.1 will take up the rest of this section.

For $J \in \mathcal{J}$, we let $\|J\|$ be the distance between J and 0.

LEMMA 3.3. *There exists $\varepsilon_0 > 0$ such that*

$$\lim_{n \to \infty} \sup_{J \in \mathcal{J}} \mu \left\{ x, \ln \frac{\|A_n(x) \cdot J\|}{\|J\|} < \varepsilon_0 n \right\} = 0 \,. \tag{3.4}$$

Proof. Let $C(x, J)$ be the largest real number such that

$$\|A_n(x) \cdot J\| \geq C(x, J) e^{\lambda n/2} \|J\|, \quad n \geq 0, \tag{3.5}$$

where $\lambda > 0$ is smaller than all positive Lyapunov exponents of (T, A). By the Oseledets Theorem [O], [KB], $C(x, J) \in [0, 1]$ is strictly positive for every $J \in \mathcal{J}$ and almost every $x \in \Delta$, and depends continuously on J for almost every x. Thus, for every $\delta > 0$ and $J \in \mathcal{J}$, there exists $C_\delta(J) > 0$ such that $\mu\{x, C(x,J) \leq C_\delta(J)\} < \delta$. By Fatou's Lemma, for any $C > 0$ the function $F(J) := \mu\{x, C(x,J) \leq C\}$ is upper semi-continuous; hence $\mu\{x, C(x,J') \leq C_\delta(J)\} < \delta$ for every J' in a neighborhood of J. By compactness, there exists $C_\delta > 0$ such that $\mu\{x, C(x,J) \leq C_\delta\} < \delta$ for every $J \in \mathcal{J}$ with $\|J\| = 1$, and hence for every $J \in \mathcal{J}$. The result now follows since $2\varepsilon_0 < \lambda$. □

For any $\delta < 1/10$, let $W^s_{\delta,n}(x)$ be the set of all $w \in B_\delta(0)$ such that $\|A_k(x) \cdot w\|_{\mathbb{R}^p/\mathbb{Z}^p} < \delta$ for all $k \leq n$, and let $W^s_\delta(x) = \cap W^s_{\delta,n}(x)$.

LEMMA 3.4. *There exists $\delta > 0$ such that for all $J \in \mathcal{J}$ and for almost every $x \in \Delta$, $J \cap W^s_\delta(x) = \emptyset$.*

Proof. For any $\delta < 1/10$, let $\phi_\delta(\mathbf{l}, J)$ be the number of connected components of the set $A^{\mathbf{l}}(J \cap B_\delta(0)) \cap B_\delta(\mathbb{Z}^p \setminus \{0\})$ and let $\phi_\delta(\mathbf{l}) := \sup_{J \in \mathcal{J}} \phi_\delta(\mathbf{l}, J)$. For any (fixed) $\mathbf{l} \in \Omega$ the function $\delta \mapsto \phi_\delta(\mathbf{l})$ is non-decreasing and there exists $\delta_{\mathbf{l}} > 0$ such that for $\delta < \delta_{\mathbf{l}}$ we have $\phi_\delta(\mathbf{l}) = 0$. We also have

$$\phi_\delta(\mathbf{l}) \leq \|A^{\mathbf{l}}\|_0 \,, \quad \mathbf{l} \in \Omega \,. \tag{3.6}$$

Given J with $\|J\| < \delta$ and $\mathbf{l} \in \Omega$, let $J_{\mathbf{l},1}, \ldots, J_{\mathbf{l},\phi_\delta(\mathbf{l},J)}$ be all the lines of the form $A^{\mathbf{l}} \cdot J - c$ where $A^{\mathbf{l}}(J \cap B_\delta(0)) \cap B_\delta(c) \neq \emptyset$ with $c \in \mathbb{Z}^p \setminus \{0\}$. Let $J_{\mathbf{l},0} = A^{\mathbf{l}} \cdot J$.

By definition we have

$$\|J_{\mathbf{l},k}\| < \delta, \quad k \geq 1 \,. \tag{3.7}$$

To obtain a lower bound we argue as follows. Let $w \in J_{\mathbf{l},k}$ satisfy $\|w\| = \|J_{\mathbf{l},k}\|$. Then $\|w - w'\| < \delta$ for some $w' \in A^{\mathbf{l}} \cdot (J \cap B_\delta(0)) - c$. Since J is parallel to some element of Θ, it is expanded by $A^{\mathbf{l}}$ (see (3.2)). It follows that $\|(A^{\mathbf{l}})^{-1} \cdot (w + c) - (A^{\mathbf{l}})^{-1} \cdot (w' + c)\| < \delta$, which implies $\|(A^{\mathbf{l}})^{-1} \cdot (w+c)\| < 2\delta$. Since $(A^{\mathbf{l}})^{-1} \cdot c \in \mathbb{Z}^p \setminus \{0\}$, we have

$$\|A^{\mathbf{l}}\|_0 \|w\| \geq \|(A^{\mathbf{l}})^{-1} \cdot c - (A^{\mathbf{l}})^{-1} \cdot (w + c)\| \geq 1 - 2\delta, \tag{3.8}$$

and finally we get

$$\|J_{\mathbf{l},k}\| \geq (1 - 2\delta)\|A^{\mathbf{l}}\|_0^{-1} \geq 2^{-1}\|A^{\mathbf{l}}\|_0^{-1} \,, \quad k \geq 1. \tag{3.9}$$

On the other hand, it is clear that

$$\|A^{\mathbf{l}}\|_0 \|J\| \geq \|J_{\mathbf{l},0}\| \geq \|A^{\mathbf{l}}\|_0^{-1} \|J\|. \tag{3.10}$$

Given measurable sets $X, Y \subset \Delta$ such that $\mu(Y) > 0$, we let

$$P_\nu(X|Y) = \frac{\nu(X \cap Y)}{\nu(Y)}, \quad \nu \in \mathcal{M}, \tag{3.11}$$

$$P(X|Y) = \sup_{\nu \in \mathcal{M}} P_\nu(X|Y). \tag{3.12}$$

For $N \in \mathbb{N} \setminus \{0\}$, let Ω^N be the set of all words of length N, and $\widehat{\Omega}^N$ be the set of all words of length a multiple of N.

For any $0 < \eta < 1/10$, select a finite set $Z \subset \Omega^N$ such that $\mu(\cup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}}) > 1 - \eta$. Since the cocycle is locally constant and uniform, there exists $0 < \eta_0 < 1/10$ such that, for all $\eta < \eta_0$,

$$\sum_{\mathbf{l} \in \Omega^N \setminus Z} \ln \|A^{\mathbf{l}}\|_0 \, \mu(\Delta^{\mathbf{l}}) < \frac{1}{10} \,. \tag{3.13}$$

CLAIM 3.5. *There exists $N_0 \in \mathbb{N} \setminus \{0\}$ such that, if $N > N_0$, then for every $J \in \mathcal{J}$ and every measurable set $Y \subset \Delta$ with $\mu(Y) > 0$,*

$$\inf_{\nu \in \mathcal{M}} \sum_{\mathbf{l}^1 \in Z} \ln \frac{\|J_{\mathbf{l}^1,0}\|}{\|J\|} P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right) \geq 2 \,. \tag{3.14}$$

Proof. By Lemma 3.3, for every $\kappa > 0$, there exists $N_0(\kappa) \in \mathbb{N}$ such that the following holds. For every $N > N_0(\kappa)$ and every $J \in \mathcal{J}$ there exists $Z' := Z'(N, J) \subset Z$ such that, for all $\mathbf{l} \in Z'$,

$$\ln \frac{\|A^{\mathbf{l}} \cdot J\|}{\|J\|} \geq \varepsilon_0 N, \tag{3.15}$$

$$\mu \left(\bigcup_{\mathbf{l} \in Z \setminus Z'} \Delta^{\mathbf{l}} \right) < \kappa. \tag{3.16}$$

We have

$$\begin{aligned}
(I) &:= \sum_{\mathbf{l}^1 \in Z'} \ln \frac{\|J_{\mathbf{l}^1,0}\|}{\|J\|} P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right) \\
&\geq \varepsilon_0 N P_\nu \left(\bigcup_{\mathbf{l} \in Z'} \Delta^{\mathbf{l}} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right) \\
&\geq \varepsilon_0 N \left[1 - K^4 P_\mu \left(\bigcup_{\mathbf{l} \in Z \setminus Z'} \Delta^{\mathbf{l}} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \right) \right] \\
&\geq \varepsilon_0 N \left(1 - K^4 \frac{\kappa}{1 - \eta} \right),
\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
(II) &:= \sum_{\mathbf{l}^1 \in Z \setminus Z'} \ln \frac{\|J_{\mathbf{l}^1,0}\|}{\|J\|} P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right) \\
&\geq - \sum_{\mathbf{l}^1 \in Z \setminus Z'} \ln \|A^{\mathbf{l}^1}\|_0 P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right) \\
&\geq - \sum_{\mathbf{l}^1 \in Z \setminus Z'} \ln \|A^{\mathbf{l}^1}\|_0 K^4 P_\mu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \right) \\
&\geq -K^4 \frac{1}{1-\eta} \int_{\cup_{\mathbf{l} \in Z \setminus Z'} \Delta^{\mathbf{l}}} \ln \|A^{\mathbf{l}}(x)\|_0 \, d\mu \\
&\geq -K^4 \frac{1}{1-\eta} \omega(\kappa) N
\end{aligned}
\tag{3.18}
$$

(where $\omega(\kappa)$ is as in Lemma 2.2), so that for any $\eta < 1/10$, for $\kappa > 0$ sufficiently small and for all $N > N_0(\kappa)$,

$$
\sum_{\mathbf{l}^1 \in Z} \ln \frac{\|J_{\mathbf{l}^1,0}\|}{\|J\|} P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right) \geq (I) + (II) \geq \frac{\varepsilon_0}{2} N. \tag{3.19}
$$

Hence the claim is proved since $N_0 \geq \max\{N_0(\kappa), 4\varepsilon_0^{-1}\}$. □

CLAIM 3.6. *Let $N > N_0$. There exists $\rho_0(Z) > 0$ such that, for every $0 < \rho < \rho_0(Z)$, every $J \in \mathcal{J}$ and every $Y \subset \Delta$ with $\mu(Y) > 0$,*

$$
\sup_{\nu \in \mathcal{M}} \sum_{\mathbf{l}^1 \in Z} \|J_{\mathbf{l}^1,0}\|^{-\rho} P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right) \leq (1-\rho) \|J\|^{-\rho}. \tag{3.20}
$$

Proof. Let

$$
\Phi(\nu, Y, \rho) := \sum_{\mathbf{l}^1 \in Z} \frac{\|J_{\mathbf{l}^1,0}\|^{-\rho}}{\|J\|^{-\rho}} P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right). \tag{3.21}
$$

Then $\Phi(\nu, Y, 0) = 1$ and

$$
\frac{d}{d\rho} \Phi(\nu, Y, \rho) = \sum_{\mathbf{l}^1 \in Z} - \ln \left(\frac{\|J_{\mathbf{l}^1,0}\|}{\|J\|} \right) \frac{\|J_{\mathbf{l}^1,0}\|^{-\rho}}{\|J\|^{-\rho}} P_\nu \left(\Delta^{\mathbf{l}^1} | \bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}} \cap T^{-N}(Y) \right),
$$

since Z is a finite set. By Claim 3.5 there exists $\rho_0(Z) > 0$ such that, for every $Y \subset \Delta$ with $\mu(Y) > 0$,

$$
\frac{d}{d\rho} \Phi(\nu, Y, \rho) \leq -1, \quad 0 \leq \rho \leq \rho_0(Z), \tag{3.22}
$$

which gives the result. □

At this point we fix $0 < \eta < \eta_0$, $N > N_0$, $Z \subset \Omega^N$, and $0 < \rho < \rho_0(Z)$ so that (3.13) and (3.20) hold and let $\delta < 1/10$ be so small that

$$\sum_{\mathbf{l} \in \Omega^N \setminus Z} \left[\rho \ln \|A^{\mathbf{l}}\|_0 + \ln \left(1 + \|A^{\mathbf{l}}\|_0 (2\delta)^\rho \right) \right] \mu(\Delta^{\mathbf{l}}) - \rho\mu(\bigcup_{\mathbf{l} \in Z} \Delta^{\mathbf{l}}) = \alpha < 0, \tag{3.23}$$

(this is possible by (3.13)) and

$$\phi_\delta(\mathbf{l}) = 0, \quad \mathbf{l} \in Z\,, \tag{3.24}$$

(this is possible since Z is finite).

Let $\Gamma_\delta^m(J) = \{x \in \Delta,\ J \cap W^s_{\delta, mN}(x) \neq \emptyset\}$. Our goal is to show that $\mu\left(\Gamma_\delta^m(J)\right) \to 0$ for every $J \in \mathcal{J}$. Let Ω be, as above, the set of all finite words with integer entries. Let Ω^N and $\widehat{\Omega}^N$ be, as above, the subset of all words of length N and the subset of all words of length a multiple of N, respectively. Let $\psi : \Omega^N \to \mathbb{Z}$ be such that $\psi(\mathbf{l}) = 0$ if $\mathbf{l} \in Z$ and $\psi(\mathbf{l}) \neq \psi(\mathbf{l}')$ whenever $\mathbf{l} \neq \mathbf{l}'$ and $\mathbf{l} \notin Z$. We let $\Psi : \widehat{\Omega}^N \to \Omega$ be given by $\Psi(\mathbf{l}^{(1)} \dots \mathbf{l}^{(m)}) = \psi(\mathbf{l}^{(1)}) \dots \psi(\mathbf{l}^{(m)})$, where the $\mathbf{l}^{(i)}$ are in Ω^N. We let $\widehat{\Delta}^{\mathbf{d}} = \cup_{\mathbf{l} \in \Psi^{-1}(\mathbf{d})} \Delta^{\mathbf{l}}$.

For $\mathbf{d} \in \Omega$, let $C(\mathbf{d}) \geq 0$ be the smallest real number such that

$$\sup_{\nu \in \mathcal{M}} P_\nu(\Gamma_\delta^m(J) | \widehat{\Delta}^{\mathbf{d}}) \leq C(\mathbf{d}) \|J\|^{-\rho}, \quad J \in \mathcal{J}. \tag{3.25}$$

It follows that $C(\mathbf{d}) \leq 1$ for all $\mathbf{d}$ (since $\Gamma_\delta^m(J) = \emptyset$, $\|J\| > \delta$).

CLAIM 3.7. *If* $\mathbf{d} = (d_1, \dots, d_m)$,

$$C(\mathbf{d}) \leq \prod_{d_i = 0} (1 - \rho) \prod_{d_i \neq 0, \psi(\mathbf{l}^i) = d_i} \|A^{\mathbf{l}^i}\|_0^\rho (1 + \|A^{\mathbf{l}^i}\|_0 (2\delta)^\rho). \tag{3.26}$$

Proof. Let $\mathbf{d} = (d_1, \dots, d_{m+1})$, $\mathbf{d}' = (d_2, \dots, d_{m+1})$. There are two possibilities: (1) If $d_1 = 0$, we have by (3.20) and (3.24)

$$P_\nu(\Gamma_\delta^{m+1}(J) | \widehat{\Delta}^{\mathbf{d}}) \leq \sum_{\mathbf{l}^1 \in Z} P(\Gamma_\delta^m(J_{\mathbf{l}^1, 0}) | \widehat{\Delta}^{\mathbf{d}'}) P_\nu(\Delta^{\mathbf{l}^1} | \widehat{\Delta}^{\mathbf{d}}) \leq (1 - \rho) C(\mathbf{d}') \|J\|^{-\rho}\,.$$

(2) If $d_1 \neq 0$, let $\mathbf{l}^1$ be given by $\psi(\mathbf{l}^1) = d_1$. Then either $\|J\| > \delta$ (and $P(\Gamma_\delta^{m+1}(J) | \widehat{\Delta}^{\mathbf{d}}) = 0$) or, by (3.6), (3.9) and (3.10),

$$\begin{aligned} P(\Gamma_\delta^{m+1}(J) | \widehat{\Delta}^{\mathbf{d}}) &\leq \sum_{k=0}^{\phi_\delta(\mathbf{l}^1)} P(\Gamma_\delta^m(J_{\mathbf{l}^1, k}) | \widehat{\Delta}^{\mathbf{d}'}) \\ &\leq C(\mathbf{d}') (\|J_{\mathbf{l}^1, 0}\|^{-\rho} + \phi_\delta(\mathbf{l}^1) \sup_{k \geq 1} \|J_{\mathbf{l}^1, k}\|^{-\rho}) \\ &\leq C(\mathbf{d}') \|J\|^{-\rho} \left(\|A^{\mathbf{l}^1}\|_0^\rho + \frac{2^\rho \|A^{\mathbf{l}^1}\|_0^{1+\rho}}{\|J\|^{-\rho}} \right) \\ &\leq C(\mathbf{d}') \|J\|^{-\rho} \left(\|A^{\mathbf{l}^1}\|_0^\rho + (2\delta)^\rho \|A^{\mathbf{l}^1}\|_0^{1+\rho} \right)\,. \end{aligned}$$

The result follows. $\square$

Let

$$\gamma(x) := \begin{cases} -\rho\,, & x \in \cup_{\mathbf{l}\in Z}\Delta^{\mathbf{l}}\,, \\ \rho \ln \|A^{\mathbf{l}}\|_0 + \ln\left(1 + \|A^{\mathbf{l}}\|_0 (2\delta)^{\rho}\right), & x \in \cup_{\mathbf{l}\in \Omega^N \setminus Z}\Delta^{\mathbf{l}}\,. \end{cases} \tag{3.27}$$

We have chosen $\delta > 0$ so that (see (3.23))

$$\int_{\Delta} \gamma(x) d\mu(x) = \alpha < 0. \tag{3.28}$$

Let $C_m(x) = C(\mathbf{d})$ for $x \in \widehat{\Delta}^{\mathbf{d}}$, $|\mathbf{d}| = m$. Then by (3.26)

$$\ln C_m(x) \le \sum_{k=0}^{m-1} \gamma\left(T^{kN}(x)\right) \tag{3.29}$$

so that, by Birkhoff's ergodic theorem, $C_m(x) \to 0$ for almost every $x \in \Delta$. By dominated convergence (since $C_m(x) \le 1$),

$$\lim_{m\to\infty} \int_{\Delta} C_m(x) d\mu(x) = 0. \tag{3.30}$$

Notice that

$$\mu(\Gamma_\delta^m(J)) \le \sum_{\mathbf{d}\in\Omega, |\mathbf{d}|=m} \mu(\widehat{\Delta}^{\mathbf{d}}) P_\mu(\Gamma_\delta^m(J) | \widehat{\Delta}^{\mathbf{d}}) \le \int_{\Delta} C_m(x) \|J\|^{-\rho} d\mu(x), \tag{3.31}$$

so that $\lim \mu\left(\Gamma_\delta^m(J)\right) = 0$.

Proof of Theorem 3.1. Assume that there exists a positive measure set X such that for every $x \in X$, there exists $w(x) \in (L \cap W^s(x)) \backslash \mathbb{Z}^p$. Thus, for every $\delta > 0$ and for every $x \in X$, there exists $n_0(x) > 0$ such that for every $n \ge n_0(x)$, there exists $c_n(x) \in \mathbb{Z}^p \setminus \{0\}$ such that $A_n(x) \cdot w(x) - c_n(x) \in W_\delta^s\left(T^n(x)\right)$.

If $A_n(x) \cdot L - c_n(x)$ passes through 0 for all $n \ge n_0$, we get a contradiction as follows. Since $A_n(x)$ expands L (see (3.3)) we get

$$\|A_{n-n_0}\left(T^{n_0}(x)\right)^{-1}\left(A_n(x) \cdot w(x) - c_n(x)\right)\| \to 0 \;.$$

In addition,

$$\begin{aligned} &A_{n-n_0}\left(T^{n_0}(x)\right)^{-1}\left(A_n(x) \cdot w(x) - c_n(x)\right) \\ &\qquad\qquad = A_{n_0}(x) \cdot w(x) - A_{n-n_0}\left(T^{n_0}(x)\right)^{-1} \cdot c_n(x)\;, \end{aligned} \tag{3.32}$$

so that $A_{n_0}(x) \cdot w(x) = c_{n_0}(x)$, a contradiction.

Thus for every $x \in X$ there exists $n(x) \ge n_0(x)$ such that $A_{n(x)} \cdot L - c_n(x)$ does not pass through 0; that is, $A_{n(x)} \cdot L - c_n(x) \in \mathcal{J}$. By restricting to a subset of X of positive measure, we may assume that $n(x)$, $A_{n(x)}(x)$ and $c_{n(x)}(x)$ do not depend on $x \in X$. Then $A_{n(x)}(x) \cdot L - c_{n(x)}(x) \in \mathcal{J}$ intersects $W_\delta^s(x')$ for all $x' \in T^{n(x)}(X)$ and $\mu\left(T^{n(x)}(X)\right) > 0$. This contradicts Lemma 3.4. $\square$

4. Renormalization schemes

Let $d \geq 2$ be a natural number and let $\mathfrak{S}_d$ be the space of *irreducible* permutations on $\{1, \dots, d\}$; that is, $\pi \in \mathfrak{S}_d$ if and only if $\pi\{1, \dots, k\} \neq \{1, \dots, k\}$ for $1 \leq k < d$. An i.e.t. $f := f(\lambda, \pi)$ on d intervals is specified by a pair $(\lambda, \pi) \in \mathbb{R}_+^d \times \mathfrak{S}_d$ as described in the introduction.

4.1. *Rauzy induction.* We recall the definition of the induction procedure first introduced by Rauzy in [R] (see also Veech [V1]). Let (λ, π) be such that $\lambda_d \neq \lambda_{\pi^{-1}(d)}$. Then the first return map under $f(\lambda, \pi)$ to the interval

$$\left[0, \sum_{i=1}^{d} \lambda_i - \min\{\lambda_{\pi^{-1}(d)}, \lambda_d\}\right) \tag{4.1}$$

can again be seen as an i.e.t. $f(\lambda', \pi')$ on d intervals as follows:

(1) If $\lambda_d < \lambda_{\pi^{-1}(d)}$, let

$$\lambda'_i = \begin{cases} \lambda_i, & 1 \leq i < \pi^{-1}(d), \\ \lambda_{\pi^{-1}(d)} - \lambda_d, & i = \pi^{-1}(d), \\ \lambda_d, & i = \pi^{-1}(d) + 1, \\ \lambda_{i-1}, & \pi^{-1}(d) + 1 < i \leq d, \end{cases} \tag{4.2}$$

$$\pi'(i) = \begin{cases} \pi(i), & 1 \leq i \leq \pi^{-1}(d), \\ \pi(d), & i = \pi^{-1}(d) + 1, \\ \pi(i-1), & \pi^{-1}(d) + 1 < i \leq d. \end{cases} \tag{4.3}$$

(2) If $\lambda_d > \lambda_{\pi^{-1}(d)}$, let

$$\lambda'_i = \begin{cases} \lambda_i, & 1 \leq i < d, \\ \lambda_d - \lambda_{\pi^{-1}(d)}, & i = d, \end{cases} \tag{4.4}$$

$$\pi'(i) = \begin{cases} \pi(i), & 1 \leq \pi(i) \leq \pi(d), \\ \pi(i) + 1, & \pi(d) < \pi(i) < d, \\ \pi(d) + 1, & \pi(i) = d. \end{cases} \tag{4.5}$$

In the first case, we will say that (λ', π') is obtained from (λ, π) by an elementary operation of type 1, and in the second case by an elementary operation of type 2. In both cases, π' is still an irreducible permutation.

Let $\mathcal{Q}_R : \mathbb{R}_+^d \times \mathfrak{S}_d \to \mathbb{R}_+^d \times \mathfrak{S}_d$ be the map defined by $\mathcal{Q}_R(\lambda, \pi) = (\lambda', \pi')$. Notice that $\mathcal{Q}_R$ is defined almost everywhere (in the complement of finitely many hyperplanes).

The *Rauzy class* of a permutation $\pi \in \mathfrak{S}_d$ is the set $\mathfrak{R}(\pi)$ of all $\tilde{\pi}$ that can be obtained from π by a finite number of elementary operations. It is a basic fact that the Rauzy classes partition $\mathfrak{S}_d$.

Let $\mathbb{P}_+^{d-1} \subset \mathbb{P}^{d-1}$ be the projectivization of $\mathbb{R}_+^d$. Since $\mathcal{Q}_R$ commutes with dilations

$$\mathcal{Q}_R(\alpha\lambda, \pi) = (\alpha\lambda', \pi')\,, \quad \alpha \in \mathbb{R} \setminus \{0\}\,, \tag{4.6}$$

$\mathcal{Q}_R$ projectivizes to a map $\mathcal{R}_R : \mathbb{P}_+^{d-1} \times \mathfrak{S}_d \to \mathbb{P}_+^{d-1} \times \mathfrak{S}_d$.

THEOREM 4.1 (Masur, [M], Veech, [V2]). *Let $\mathfrak{R} \subset \mathfrak{S}_d$ be a Rauzy class. Then $\mathcal{R}_R|\mathbb{P}_+^{d-1} \times \mathfrak{R}$ admits an ergodic conservative infinite absolutely continuous invariant measure, unique in its measure class up to a scalar multiple. Its density is a positive rational function.*

4.2. *Zorich induction.* Zorich [Z1] modified the Rauzy induction as follows. Given (λ, π), let $n := n(\lambda, \pi)$ be such that $Q_R^{n+1}(\lambda, \pi)$ is defined and, for $1 \leq i \leq n$, $\mathcal{Q}_R^i(\lambda, \pi)$ is obtained from $\mathcal{Q}_R^{i-1}(\lambda, \pi)$ by elementary operations of the same type, while $\mathcal{Q}_R^{n+1}(\lambda, \pi)$ is obtained from $\mathcal{Q}_R^n(\lambda, \pi)$ by an elementary operation of the other type. Then he sets

$$\mathcal{Q}_Z(\lambda, \pi) = \mathcal{Q}_R^{n(\lambda,\pi)}(\lambda, \pi). \tag{4.7}$$

Now, $\mathcal{Q}_Z : \mathbb{R}_+^d \times \mathfrak{S}_d \to \mathbb{R}_+^d \times \mathfrak{S}_d$ is defined almost everywhere (in the complement of countably many hyperplanes). We can again consider the projectivization of $\mathcal{Q}_Z$, denoted by $\mathcal{R}_Z$.

THEOREM 4.2 (Zorich, [Z1]). *Let $\mathfrak{R} \subset \mathfrak{S}_d$ be a Rauzy class. Then $\mathcal{R}_Z|\mathbb{P}_+^{d-1} \times \mathfrak{R}$ admits a unique ergodic absolutely continuous probability measure $\mu_{\mathfrak{R}}$. Its density is positive and analytic.*

4.3. *Cocycles.* Let (λ', π') be obtained from (λ, π) by the Rauzy or the Zorich induction. Let $f := f(\lambda, \pi)$. For any $x \in I' := I(\lambda', \pi')$ and $j \in \{1, \ldots, d\}$, let $r_j(x)$ be the number of intersections of the orbit $\{x, f(x), \ldots, f^k(x), \ldots\}$ with the interval $I_j := I_j(\lambda, \pi)$ before the first return time $r(x)$ of x to $I' := I(\lambda', \pi')$; that is, $r_j(x) := \#\{0 \leq k < r(x), f^k(x) \in I_j\}$. In particular, we have $r(x) = \sum_j r_j(x)$. Notice that $r_j(x)$ is constant on each $I'_i := I_i(\lambda', \pi')$ and for all i, $j \in \{1, \ldots, d\}$, let $r_{ij} := r_j(x)$ for $x \in I'_i$. Let $B := B(\lambda, \pi)$ be the linear operator on $\mathbb{R}^d$ given by the $d \times d$ matrix (r_{ij}). The function $B : \mathbb{R}_+^d \times \mathfrak{R} \to \mathrm{GL}(\mathrm{d}, \mathbb{R})$ yields a cocycle over the Rauzy induction and a related one over the Zorich induction, called respectively the *Rauzy cocycle* and the *Zorich cocycle* (denoted respectively by B^R and B^Z). We see immmediately that $B^R, B^Z \in \mathrm{GL}(\mathrm{d}, \mathbb{Z})$, and

$$B^Z(\lambda, \pi) = B^R\left(\mathcal{Q}_R^{n(\lambda,\pi)-1}(\lambda, \pi)\right) \cdots B^R(\lambda, \pi). \tag{4.8}$$

Notice that $\mathcal{Q}(\lambda, \pi) = (\lambda', \pi')$ implies $\lambda = B^*\lambda'$ (B^* denotes the adjoint of B). Thus

$$\langle \lambda, w \rangle = 0 \quad \text{if and only if} \quad \langle \lambda', B \cdot w \rangle = 0. \tag{4.9}$$

Obviously we can projectivize the cocycles B^R and B^Z.

THEOREM 4.3 (Zorich, [Z1]). *Let $\mathfrak{R} \subset \mathfrak{S}_d$ be a Rauzy class. Then*

$$\int_{\mathbb{P}_+^{d-1}\times\mathfrak{R}} \ln \|B^Z\|_0 \, d\mu_{\mathfrak{R}} < \infty \,. \tag{4.10}$$

4.4. *An invariant subspace.* Given a permutation $\pi \in \mathfrak{S}_d$, let σ be the permutation on $\{0, \dots, d\}$ defined by

$$\sigma(i) := \begin{cases} \pi^{-1}(1) - 1, & i = 0, \\ d, & i = \pi^{-1}(d), \\ \pi^{-1}(\pi(i)+1) - 1, & i \neq 0,\, \pi^{-1}(d)\,. \end{cases} \tag{4.11}$$

For every $j \in \{0, \dots, d\}$, let $S(j)$ be the orbit of j under σ. This defines a partition $\Sigma(\pi) := \{S(j),\, 0 \le j \le d\}$ of the set $\{0, \dots, d\}$. For every $S \in \Sigma(\pi)$, let $b^S \in \mathbb{R}^d$ be the vector defined by

$$b_i^S := \chi_S(i-1) - \chi_S(i), \quad 1 \le i \le d, \tag{4.12}$$

where χ_S denotes the characteristic function of S. Let $H(\pi)$ be the annihilator of the subspace generated by the set $\Upsilon(\pi) := \{b^S, S \in \Sigma(\pi)\}$. A basic fact from [V4] is that if $\mathcal{Q}(\lambda, \pi) := (\lambda', \pi')$ then

$$B(\lambda, \pi)^* \cdot \Upsilon(\pi') = \Upsilon(\pi)\,, \tag{4.13}$$

which implies

$$B(\lambda, \pi) \cdot H(\pi) = H(\pi')\,. \tag{4.14}$$

It follows that the dimension of $H(\pi)$ depends only on the Rauzy class of $\pi \in \mathfrak{S}_d$. Let $N(\pi)$ be the cardinality of the set $\Sigma(\pi)$. Veech showed in [V2] that the dimension of $H(\pi)$ is equal to $d - N(\pi) + 1$ and that the latter is in fact a non-zero even number equal to $2g$, where $g := g(\pi)$ is the genus of the Riemann surface obtained by the "zippered rectangles" construction. The space of zippered rectangles $\Omega(\pi)$ associated to a permutation $\pi \in \mathfrak{S}_d$ is the space of all triples (λ, h, a) where $\lambda \in \mathbb{R}_+^d$, h belongs to a closed convex cone with non-empty interior $H^+(\pi) \subset H(\pi)$ (specified by finitely many linear inequalities) and a belongs to a closed parallelepiped $Z(h, \pi) \subset \overline{\mathbb{R}_+^d}$ of dimension $N(\pi) - 1$. Given $\pi \in \mathfrak{S}_d$ and $(\lambda, h, a) \in \Omega(\pi)$, with h in the $H(\pi)$ interior of $H^+(\pi)$, it is possible to construct a closed translation surface $M := M(\lambda, h, a, \pi)$ of genus $g(\pi) = (d - N(\pi) + 1)/2$ by performing appropriate gluing operations on the union of the flat rectangles $R_i(\lambda, h) \subset \mathbb{C}$ having bases $I_i(\lambda, \pi)$ and heights h_i for $i \in \{1, \dots, d\}$. The gluing maps are translations specified by the permutation $\pi \in \mathfrak{S}_d$ and by the gluing 'heights' $a := (a_1, \dots, a_d) \in Z(h, \pi)$. The set $\Sigma \subset M$ of the singularities of M is in one-to-one correspondence with the set $\Sigma(\pi)$. In fact for any $S \in \Sigma(\pi)$, the surface M has exactly one conical singularity of total angle $2\pi\nu(S)$, where $\nu(S)$ is the cardinality of $S \cap \{1, \ldots, d-1\}$ [V2]. There is a natural local identification of the relative cohomology $H^1(M, \Sigma; \mathbb{R})$ with the space $\mathbb{R}_+^d$ of i.e.t.'s with fixed permutation $\pi \in \mathfrak{S}_d$. Under this identification the generators of $\Upsilon(\pi)$ correspond to integer elements of $H^1(M, \Sigma; \mathbb{R})$ and the quotient space of the space generated by $\Upsilon(\pi)$ coincides with the absolute cohomology $H^1(M, \mathbb{R})$. By the definition of $H(\pi)$ it follows that $H(\pi)$ is identified with the absolute homology $H_1(M, \mathbb{R})$ and that $H(\pi) \cap \mathbb{Z}^d$ is identified with $H_1(M, \mathbb{Z}) \subset H_1(M, \mathbb{R})$ (see [Z1, §9]), hence $H(\pi) \cap \mathbb{Z}^d$ is a co-compact lattice in $H(\pi)$ and in particular,

$$\operatorname{dist}\Big(H(\pi), \mathbb{Z}^d \setminus H(\pi)\Big) > 0 \ . \tag{4.15}$$

4.5. *Lyapunov exponents.* Let $\mathfrak{R} \subset \mathfrak{S}_d$ be a Rauzy class. We can consider the restrictions $B^R(\lambda, \pi)|H(\pi)$ and $B^Z(\lambda, \pi)|H(\pi)$, $([\lambda], \pi) \in \mathbb{P}_+^{d-1} \times \mathfrak{R}$, as integral cocycles over $\mathcal{R}_Z|\mathbb{P}_+^{d-1} \times \mathfrak{R}$. We will call these cocycles the Rauzy and Zorich cocycles respectively. The Zorich cocycle is uniform (with respect to the measure $\mu_{\mathfrak{R}}$) by Theorems 4.2 and 4.3.[3]

Let $\theta_1(\mathfrak{R}) \geq \cdots \geq \theta_{2g}(\mathfrak{R})$ be the Lyapunov exponents of the Zorich cocycle on $\mathbb{P}_+^{d-1} \times \mathfrak{R}$. In [Z1], Zorich showed that $\theta_i(\mathfrak{R}) = -\theta_{2g+1-i}(\mathbb{R})$ for all $i \in \{1, \ldots, 2g\}$ and that $\theta_1(\mathfrak{R}) > \theta_2(\mathfrak{R})$ (he derived the latter result from the non-uniform hyperbolicity of the Teichmüller flow proved earlier by Veech in [V3]). He also conjectured that $\theta_1(\mathfrak{R}) > \cdots > \theta_{2g}(\mathfrak{R})$. Part of this conjecture was proved by the second author in [F2].[4]

Theorem 4.4 (Forni, [F2]). *For any Rauzy class $\mathfrak{R} \subset \mathfrak{S}_d$ the Zorich cocycle on $\mathbb{P}_+^{d-1} \times \mathfrak{R}$ is non-uniformly hyperbolic. Thus*

$$\begin{aligned} \theta_1(\mathfrak{R}) > &\theta_2(\mathfrak{R}) \geq \cdots \geq \theta_g(\mathfrak{R}) > 0 \\ &> \theta_{g+1}(\mathfrak{R}) \geq \cdots \geq \theta_{2g-1}(\mathfrak{R}) > \theta_{2g}(\mathfrak{R}) \ . \end{aligned} \tag{4.16}$$

Actually [F2] proved the non-uniform hyperbolicity of a related cocycle (the Kontsevich-Zorich cocycle), which is a continuous time version of the Zorich cocycle. The relation between the two cocycles can be outlined as follows (see [V2], [V3], [V5], [Z3]). In [V2] Veech introduced a zippered-rectangles "moduli space" $\mathcal{M}(\mathfrak{R})$ as a quotient of the space $\Omega(\mathfrak{R})$ of all zippered rectangles associated to permutations in a given Rauzy class $\mathfrak{R}$. He also introduced a

[3] Strictly speaking, to fit into the setting of §2.3 we should fix an appropriate measurable trivialization of the bundle with fiber $H(\pi)$ at each $(\lambda, \pi) \in \mathbb{P}_+^{d-1} \times \mathfrak{R}(\pi)$ by selecting, for each $\tilde{\pi} \in \mathfrak{R}(\pi)$, an isomorphism $H(\tilde{\pi}) \to \mathbb{R}^{2g}$ that takes $H(\tilde{\pi}) \cap \mathbb{Z}^d$ to $\mathbb{Z}^{2g}$.

[4] The proof of the full conjecture was recently announced [AV].

zippered-rectangles flow on $\Omega(\mathfrak{R})$ which projects to a flow on the moduli space $\mathcal{M}(\mathfrak{R})$. By construction, the Rauzy induction is a factor of the return map of the zippered-rectangles flow to a cross-section $Y(\mathfrak{R}) \subset \mathcal{M}(\mathfrak{R})$. In fact, such a return map is a "natural extension" of the Rauzy induction. The Rauzy or Zorich cocycles are cocycles on the bundle with fiber $H(\pi)$ at $(\lambda, \pi) \in \mathbb{P}^{d-1} \times \mathfrak{R}$. We recall that the space $H(\pi)$ can be naturally identified with the real homology $H_1(M, \mathbb{R})$ of the surface $M := M(\lambda, h, a, \pi)$. There is a natural map from the zippered-rectangles "moduli space" $\mathcal{M}(\mathfrak{R})$ onto a connected component $\mathcal{C}$ of a stratum of the moduli space $\mathcal{H}_g$ of holomorphic (abelian) differentials on Riemann surfaces of genus g, and the zippered-rectangles flow on $\mathcal{M}(\mathfrak{R})$ projects onto the Teichmüller flow on $\mathcal{C} \subset \mathcal{H}_g$. The Kontsevich-Zorich cocycle, introduced in [Ko], is a cocycle over the Teichmüller flow on the *real cohomology bundle* over $\mathcal{H}_g$, that is, the bundle with fiber the real cohomology $H^1(M, \mathbb{R})$ at every point $[(M, \omega)] \in \mathcal{H}_g$. The Kontsevich-Zorich cocycle can be lifted to a cocycle over the zippered-rectangles flow. The return map of the lifted cocycle to the real cohomology bundle over the cross-section $Y(\mathfrak{R})$ projects onto a cocycle over the Rauzy induction, isomorphic (via Poincaré duality) to the Rauzy cocycle. It follows that the Lyapunov exponents of the Zorich cocycle on the Rauzy class $\mathfrak{R}$ are related to the exponents of the Kontsevich-Zorich cocycle on $\mathcal{C}$ [Ko], [F2],

$$
\begin{aligned}
\nu_1(\mathcal{C}) = 1 > & \nu_2(\mathcal{C}) \geq \cdots \geq \nu_g(\mathcal{C}) > 0 \\
& > \nu_{g+1}(\mathcal{C}) \geq \cdots \geq \nu_{2g-1}(\mathcal{C}) > \nu_{2g}(\mathcal{C}) = -1 \,,
\end{aligned}
\tag{4.17}
$$

by the formula $\nu_i(\mathcal{C}) = \theta_i(\mathfrak{R})/\theta_1(\mathfrak{R})$ for all $i \in \{1, \ldots, 2g\}$ (see [Z3, §4.5]). Thus the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle on every connected component of every stratum is equivalent to the hyperbolicity of the Zorich cocycle on every Rauzy class.

5. Exclusion of the central stable space for the Zorich cocycle

THEOREM 5.1. *Let $\pi \in \mathfrak{S}_d$ be a permutation with $g > 1$ and let $L \subset H(\pi)$ be a line not passing through 0. Let E^{cs} denote the central stable space of the Rauzy or Zorich cocycle. If $\dim E^{cs} < 2g-1$, then for almost every $[\lambda] \in \mathbb{P}_+^{d-1}$, $L \cap E^{cs}([\lambda], \pi) = \emptyset$.*

This theorem was essentially proved by Nogueira and Rudolph in [NR]. However, the result of [NR] is slightly different from what we need, although the modification is straightforward. We will therefore give a short sketch of the proof.

Proof. Following Nogueira-Rudolph [NR], we define $\pi \in \mathfrak{S}_d$ to be standard if $\pi(1) = d$ and $\pi(d) = 1$. They proved that every Rauzy class contains at least one standard permutation [NR], Lemma 3.2. Clearly it suffices to consider the case when π is standard.

Notice that

$$E^{cs}\left(\mathcal{R}_R([\lambda],\pi)\right) = B^R \cdot E^{cs}([\lambda],\pi) \tag{5.1}$$

for almost every $([\lambda],\pi)$. It is easy to see (using Perron-Frobenius together with (4.9)) that $E^{cs}([\lambda],\pi)$ is orthogonal to λ.

Define vectors $v^{(i)} \in \mathbb{R}^d$ by

$$v^{(i)}_j = \begin{cases} 1, & \pi(j) < \pi(i),\ j > i, \\ -1, & \pi(j) > \pi(i),\ j < i, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

It follows that $v^{(i)}$, $1 \le i \le d$, generate $H(\pi)$.

In [NR, §3], Nogueira and Rudolph showed that for $1 \le i \le d$ there exist $k_i \in \mathbb{N}$ and a component $D_i \subset \mathbb{P}^{d-1}_+ \times \{\pi\}$ of the domain of $\mathcal{R}^{k_i}_R$ such that $\mathcal{R}^{k_i}_R(D_i) = \mathbb{P}^{d-1}_+ \times \{\pi\}$. Defining $B_{(i)} = B^R\left(\mathcal{R}^{k_i-1}_R([\lambda],\pi)\right)\cdots B^R([\lambda],\pi)$, we have

$$B_{(i)} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} + z_i(v^{(i)} - v^{(d)}) - z_d\, v^{(d)}, \quad i \neq d, \tag{5.3}$$

$$B_{(d)} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} - z_d\, v^{(d)}. \tag{5.4}$$

We will now prove the desired statement by contradiction. If the conclusion of the theorem does not hold, a density point argument shows that there exists a set of positive measure of $[\lambda] \in \mathbb{P}^{d-1}_+$ and a line $L \subset H(\pi)$ parallel to an element of $\mathbb{P}^{d-1}_+$ such that

$$L \cap E^{cs}([\lambda],\pi) \neq \emptyset, \tag{5.5}$$

$$(B_{(i)} \cdot L) \cap E^{cs}([\lambda],\pi) \neq \emptyset\,, \quad 1 \le i \le d. \tag{5.6}$$

Write $L = \{h^{(1)} + t h^{(2)},\ t \in \mathbb{R}\}$ with $h^{(1)}$, $h^{(2)} \in H(\pi)$ linearly independent. Then from $L \cap E^{cs}([\lambda],\pi) \neq \emptyset$, we get

$$h^{(1)} - \frac{\langle \lambda, h^{(1)} \rangle}{\langle \lambda, h^{(2)} \rangle} h^{(2)} \in E^{cs}([\lambda],\pi)\,, \tag{5.7}$$

and similarly,

$$\begin{aligned}(h^{(1)} + h^{(1)}_i(v^{(i)} - v^{(d)}) - h^{(1)}_d v^{(d)}) - \frac{\langle \lambda, h^{(1)} + h^{(1)}_i(v^{(i)} - v^{(d)}) - h^{(1)}_d v^{(d)} \rangle}{\langle \lambda, h^{(2)} + h^{(2)}_i(v^{(i)} - v^{(d)}) - h^{(2)}_d v^{(d)} \rangle} \\ \times (h^{(2)} + h^{(2)}_i(v^{(i)} - v^{(d)}) - h^{(2)}_d v^{(d)}) \in E^{cs}([\lambda],\pi)\,,\end{aligned}$$

for $1 \le i < d$, and

$$(5.8)\qquad (h^{(1)} - h_d^{(1)} v^{(d)}) - \frac{\langle \lambda, h^{(1)} - h_d^{(1)} v^{(d)} \rangle}{\langle \lambda, h^{(2)} - h_d^{(2)} v^{(d)} \rangle} (h^{(2)} - h_d^{(2)} v^{(d)}) \in E^{cs}([\lambda], \pi)\,.$$

A computation then shows that

$$(5.9)\qquad v^{(1)}, \ldots, v^{(d)} \in E^{cs}([\lambda], \pi) + \{t h^{(2)},\, t \in \mathbb{R}\}\,,$$

for almost every such $[\lambda]$. Thus $E^{cs}([\lambda], \pi)$ has codimension at most 1 in $H(\pi)$, but this contradicts $\dim E^{cs} < 2g - 1 = \dim H(\pi) - 1$. □

6. Weak mixing for interval exchange tranformations

Weak mixing for the interval exchange transformation f is equivalent to the existence of no non-constant measurable solutions $\phi : I \to \mathbb{C}$ of the equation

$$(6.1)\qquad \phi\,(f(x)) = e^{2\pi i t} \phi(x),$$

for any $t \in \mathbb{R}$. This is equivalent to the following two conditions:

(1) f is ergodic;

(2) for any $t \in \mathbb{R} \setminus \mathbb{Z}$, there are no non-zero measurable solutions $\phi : I \to \mathbb{C}$ of the equation

$$(6.2)\qquad \phi\,(f(x)) = e^{2\pi i t} \phi(x)\ .$$

By [M], [V2], the first condition is not an obstruction to almost sure weak mixing: $f(\lambda, \pi)$ is ergodic for almost every $\lambda \in \mathbb{R}^d_+$. Our criterion to deal with the second condition is the following :

Theorem 6.1 (Veech, [V4, §7]). *For any Rauzy class $\mathfrak{R} \subset \mathfrak{S}_d$ there exists an open set $U_{\mathfrak{R}} \subset \mathbb{P}^{d-1}_+ \times \mathfrak{R}$ with the following property. Assume that the orbit of $([\lambda], \pi) \in \mathbb{P}^{d-1}_+ \times \mathfrak{R}$ under the Rauzy induction $\mathcal{R}_R$ visits $U_{\mathfrak{R}}$ infinitely many times. If there exists a non-constant measurable solution $\phi : I \to \mathbb{C}$ to the equation*

$$(6.3)\qquad \phi\,(f(x)) = e^{2\pi i t h_i} \phi(x)\,, \quad x \in I_i(\lambda, \pi)\,,$$

with $t \in \mathbb{R}$, $h \in \mathbb{R}^d$, then

$$(6.4)\qquad \lim_{\substack{n \to \infty \\ \mathcal{R}^n_R([\lambda], \pi) \in U_{\mathfrak{R}}}} \|B^R_n([\lambda], \pi) \cdot t h\|_{\mathbb{R}^d/\mathbb{Z}^d} = 0.$$

Notice that (6.3) reduces to (6.2) when $h = (1, \ldots, 1)$ and can thus be used to rule out eigenvalues for i.e.t.'s. The more general form (6.3) will be used in the case of translation flows.

We thank Jean-Christophe Yoccoz for pointing out to us that the above result is due to Veech (our original proof does not differ from Veech's). We will call it the *Veech criterion* for weak mixing. It has the following consequences:

Theorem 6.2 (Katok-Stepin, [KS]). *If $g = 1$ then either π is a rotation or $f(\lambda, \pi)$ is weakly mixing for almost every λ.*

(Of course Katok-Stepin's result predates the Veech criterion.)

Theorem 6.3 (Veech, [V4]). *Let $\pi \in \mathfrak{S}_d$. If $(1, \dots, 1) \notin H(\pi)$, then $f(\lambda, \pi)$ is weakly mixing for almost every $\lambda \in \mathbb{R}^d_+$.*

6.1. *Proof of Theorem* A (*Introduction*). By Theorems 6.2 and 6.3, it is enough to consider the case where $g > 1$ and $(1, \dots, 1) \in H(\pi)$. By the Veech criterion (Theorem 6.1), Theorem A is a consequence of the following:

Theorem 6.4. *Let $\mathfrak{R} \subset \mathfrak{S}_d$ be a Rauzy class with $g > 1$, let $\pi \in \mathfrak{R}$ and let $h \in H(\pi) \setminus \{0\}$. Let $U \subset \mathbb{P}^{d-1}_+ \times \mathfrak{R}$ be any open set. For almost every $[\lambda] \in \mathbb{P}^{d-1}_+$ the following holds: for every $t \in \mathbb{R}$, either $th \in \mathbb{Z}^d$ or*

$$\limsup_{\substack{n \to \infty \\ \mathcal{R}^n_R([\lambda],\pi) \in U}} \|B^R_n([\lambda], \pi) \cdot th\|_{\mathbb{R}^d/\mathbb{Z}^d} > 0. \tag{6.5}$$

Proof. We may assume that U intersects $\mathbb{P}^{d-1}_+ \times \{\pi\}$. For n sufficiently large there exists a connected component $\Delta \times \{\pi\} \subset \mathbb{P}^{d-1} \times \{\pi\}$ of the domain of $\mathcal{R}^n_Z$ which is compactly contained in U. Indeed, the connected component of the domain of $\mathcal{R}^n_R$ containing $([\lambda], \pi)$ shrinks to $([\lambda], \pi)$ as $n \to \infty$, for almost every $[\lambda] \in \mathbb{P}^{d-1}_+$ (this is exactly the criterion for unique ergodicity used in [V2]).

If the result does not hold, a density point argument implies that there exists $h \in H(\pi) \setminus \{0\}$ and a positive measure set of $[\lambda] \in \Delta$ such that

$$\lim_{\substack{n \to \infty \\ \mathcal{R}^n_R([\lambda],\pi) \in U}} \|B^R_n([\lambda], \pi) \cdot th\|_{\mathbb{R}^d/\mathbb{Z}^d} = 0, \quad \text{for some } t \in \mathbb{R} \text{ such that } th \notin \mathbb{Z}^d. \tag{6.6}$$

Let $T : \Delta \to \Delta$ be the map induced by $\mathcal{R}_Z$ on Δ. Then T is ergodic, and by Lemma 2.1, it is also strongly expanding. For almost every $\lambda \in \Delta$, let

$$A(\lambda) := B^Z(T^{r(\lambda)-1}(\lambda), \pi) \cdots B^Z(\lambda, \pi) | H(\pi)\,, \tag{6.7}$$

where $r(\lambda)$ is the return time of $\lambda \in \Delta$. Then the cocycle (T, A) is locally constant, integral and uniform, and $\Theta := \overline{\mathbb{P}^{d-1}_+}$ is adapted to (T, A). The central stable space of (T, A) coincides with the central stable space of $\big(\mathcal{R}_Z, B^Z | H(\pi)\big)$ almost everywhere. Using Theorem 5.1, we see that all the hypotheses of Theorem 3.1 are satisfied. Thus for almost every $[\lambda] \in \Delta$, the line

$L = \{th, t \in \mathbb{R}\}$ intersects the weak stable space in a subset of $H(\pi) \cap \mathbb{Z}^d$. This implies (together with (4.15)) that (6.6) fails for almost every $\lambda \in \Delta$, as required. □

7. Translation flows

7.1. *Special flows.* Any translation flow on a translation surface can be regarded, by considering its return map to a transverse interval, as a *special flow* (suspension flow) over some interval exchange transformation with a roof function constant on each sub-interval. For completeness we discuss weak mixing for general special flows over i.e.t.'s with sufficiently regular roof function. Thanks to recent results on the cohomological equation for i.e.t.'s [MMY], the general case can be reduced to the case of special flows with roof function constant on each sub-interval.

Let $F := F(\lambda, h, \pi)$ be the special flow over the i.e.t. $f := f(\lambda, \pi)$ with roof function specified by the vector $h \in \mathbb{R}^d_+$, that is, the roof function is constant, equal to h_i, on the sub-interval $I_i := I_i(\lambda, \pi)$, for all $i \in \{1, \ldots, d\}$. We remark that, by Veech's "zippered rectangles" construction (see §4), if F is a translation flow then necessarily $h \in H(\pi)$.

The phase space of F is the union of disjoint rectangles $D := \cup_i I_i \times [0, h_i)$, and the flow F is completely determined by the conditions $F_s(x, 0) = (x, s)$, $x \in I_i$, $s < h_i$, $F_{h_i}(x, 0) = (f(x), 0)$, for all $i \in \{1, \ldots, d\}$. Weak mixing for the flow F is equivalent to the existence of no non-constant measurable solutions $\phi : D \to \mathbb{C}$ of the equation

$$\phi(F_s(x)) = e^{2\pi its}\phi(x), \tag{7.1}$$

for any $t \in \mathbb{R}$; or, in terms of the i.e.t. f,

(1) f is ergodic;

(2) for any $t \neq 0$ there are no non-zero measurable solutions $\phi : I \to \mathbb{C}$ of equation (6.3).

THEOREM 7.1. *Let $\pi \in \mathfrak{S}_d$ with $g > 1$. For almost every $(\lambda, h) \in \mathbb{R}^d_+ \times (H(\pi) \cap \mathbb{R}^d_+)$, the special flow $F := F(\lambda, h, \pi)$ is weakly mixing.*

Proof. This is an immediate consequence of the Veech criterion and of Theorem 6.4. □

This theorem is all we need in the case of translation flows since it takes care of the case $h \in H(\pi)$. Let $H^\perp(\pi)$ be the orthogonal complement of $H(\pi)$ in $\mathbb{R}^d$. The case when $h \in \mathbb{R}^d_+$ has non-zero orthogonal projection on $H^\perp(\pi)$ is covered by the following:

LEMMA 7.2 (Veech, [V3]). *Assume that* $([\lambda], \pi) \in \mathbb{P}_+^d \times \mathfrak{S}_d$ *is such that* $\mathcal{Q}_R^n([\lambda], \pi)$ *is defined for all* $n > 0$. *If for some* $h \in \mathbb{R}^d$ *and* $t \in \mathbb{R}$,

$$\liminf_{n\to\infty} \|B_n^R([\lambda], \pi) \cdot th\|_{\mathbb{R}^d/\mathbb{Z}^d} = 0\,,$$

then the orthogonal projection of th *on* $H^\perp(\pi)$ *belongs to* $\mathbb{Z}^d$.

This lemma, together with the Veech criterion, can be used to establish typical weak mixing for special flows with some specific combinatorics (which must satisfy, in particular, $\dim H(\pi) \le d-1$). However, it does not help at all when $h \in H(\pi)$, which is the relevant case for translation flows.

THEOREM 7.3. *Let* $\pi \in \mathfrak{S}_d$ *with* $g > 1$ *and let* $h \in \mathbb{R}^d \setminus \{0\}$. *If* $U \subset \mathbb{P}_+^{d-1}$ *is any open set*, *then for almost every* $[\lambda] \in \mathbb{P}_+^{d-1}$ *and for every* $t \in \mathbb{R}$, *either* $th \in \mathbb{Z}^d$ *or*

$$\limsup_{\mathcal{R}_R^n([\lambda],\pi)\in U} \|B_n^R([\lambda], \pi) \cdot th\|_{\mathbb{R}^d/\mathbb{Z}^d} > 0. \tag{7.2}$$

Proof. If $h \in H^\perp(\pi)$, this is just a consequence of Lemma 7.2. So we assume that $h \notin H^\perp(\pi)$. Let w be the orthogonal projection of h on $H(\pi)$. By Theorem 6.4, there exists a full measure set of $[\lambda] \in \mathbb{P}_+^{d-1}$ such that if $tw \notin \mathbb{Z}^d$ then

$$\limsup_{\mathcal{R}_R^n([\lambda],\pi)\in U} \|B_n^R([\lambda], \pi) \cdot tw\|_{\mathbb{R}^d/\mathbb{Z}^d} > 0.$$

By Lemma 7.2, if (7.2) does not hold for some $t \in \mathbb{R}$, then $th = c + tw$ with $c \in \mathbb{Z}^d$. But this implies that

$$\|B_n^R([\lambda], \pi) \cdot tw\|_{\mathbb{R}^d/\mathbb{Z}^d} = \|B_n^R([\lambda], \pi) \cdot th\|_{\mathbb{R}^d/\mathbb{Z}^d},$$

and the result follows. □

THEOREM 7.4. *Let* $\pi \in \mathfrak{S}_d$ *with* $g > 1$. *For almost every* $(\lambda, h) \in \mathbb{R}_+^d \times \mathbb{R}_+^d$, *the special flow* $F := F(\lambda, h, \pi)$ *is weakly mixing.*

Proof. This follows immediately from Theorem 7.3 and the Veech criterion (Theorem 6.1) by Fubini's theorem. □

Following [MMY], we let $BV(\sqcup I_i)$ be the space of functions whose restrictions to each of the intervals I_i is a function of bounded variation, $BV_*(\sqcup I_i)$ be the hyperplane of $BV(\sqcup I_i)$ made of functions whose integral on the disjoint union $\sqcup I_i$ vanishes and $BV_*^1(\sqcup I_i)$ be the space of functions which are absolutely continuous on each I_i and whose first derivative belongs to $BV_*(\sqcup I_i)$.

THEOREM 7.5. *Let* $\pi \in \mathfrak{S}_d$ *with* $g > 1$. *For almost every* $\lambda \in \mathbb{R}_+^d$, *there exists a bounded surjective linear map* $\chi : BV_*^1(\sqcup I_i) \to \mathbb{R}^d$ *and a full measure* *set $\mathcal{F} \subset \mathbb{R}^d$ such that if $r \in BV_*^1(\sqcup I_i)$ is a strictly positive function with $\chi(r) \in \mathcal{F}$, then the special flow $F := F(\lambda, \pi; r)$ over the* i.e.t. $f := f(\lambda, \pi)$ *under the roof function r is weakly mixing.*

Proof. By the definition of a special flow over the map f and under the roof function r (see [CFS, Chap. 11]), the flow F has continuous spectrum if and only if

(1) f is ergodic,

(2) for any $t \neq 0$ there are no non-zero measurable solutions $\phi : I \to \mathbb{C}$ of the equation

$$\phi\,(f(x)) = e^{2\pi i t r(x)} \phi(x)\,, \quad x \in I\,. \tag{7.3}$$

By [MMY], under a full measure condition on $\lambda \in \mathbb{R}_+^d$ (a Roth-type condition), one can define a surjective bounded linear map $\chi : BV_*^1(\sqcup I_i) \to \mathbb{R}^d$ such that the cohomological equation

$$u\,(f(x)) - u(x) = r(x) - \chi_i(r)\,, \quad x \in I_i\,,$$

has a bounded measurable solution $u : I \to \mathbb{R}$ (the Roth-type condition also implies that f is uniquely ergodic). If for some $t \neq 0$ there exists a solution ϕ of equation (7.3), then the function $\psi : I \to \mathbb{C}$ given by

$$\psi(x) := e^{-2\pi i t u(x)} \phi(x)\,, \quad x \in I\,,$$

is a solution of the equation (6.3) for $h = \chi(r) \in \mathbb{R}^d$. The result then follows from Theorem 7.3 and from the Veech criterion (Theorem 6.1) by Fubini's theorem. □

7.2. *Proof of Theorem* B. Let $\mathcal{C}$ be a connected component of a stratum of the moduli space of holomorphic differentials of genus $g > 1$. By Veech's "zippered rectangles" construction, $\mathcal{C}$ can be locally parametrized by triples $(\lambda, h, a) \in \Omega(\mathfrak{R})$ where $\mathfrak{R}$ is the Rauzy class of some irreducible permutation π_0 with $g(\pi_0) = g$ (see [V2], [V3]). Moreover, this parametrization (which preserves the Lebesgue measure class) is such that the special flow $F := F(\lambda, h, \pi)$ is isomorphic to the vertical translation flow on the translation surface $M(\lambda, h, a, \pi)$. Thus Theorem B follows from Theorem 7.1 by Fubini's theorem.

Appendix. Linear exclusion

THEOREM A.1. *Let (T, A) be a measurable cocycle on $\Delta \times \mathbb{R}^p$. For almost every $x \in \Delta$, if $G \subset \mathbb{R}^p$ is any affine subspace parallel to a linear subspace $G_0 \subset \mathbb{R}^p$ transverse to the central stable space $E^{cs}(x)$, then the Hausdorff dimension of $G \cap W^s(x)$ is equal to* 0.

Proof. Let $x \in \Delta$. If $n \geq m \geq 0$ we let $S_{\delta,m,n}(x)$ be the set of $w \in G$ such that $A_k(x) \cdot w \in B_\delta(\mathbb{Z}^p)$, for all $m \leq k \leq n$. Thus

$$G \cap W^s(x) = \cap_{\delta>0} \cup_{m\geq 0} \cap_{n\geq m} S_{\delta,m,n}(x).$$

If $\delta < 1/2$, all connected components of $S_{\delta,m,n}(x)$ are convex open sets of diameter at most

$$2\delta\, C(x, G_0)^{-1} e^{-\lambda n},$$

where $C(x, G_0) > 0$ and $\lambda > 0$ are given by Oseledets Theorem as in (2.15). For $n \geq 0$, let $\rho_n(x)$ be the maximal number of connected components of $S_{\delta,m,n+1}(x)$ intersecting U, over all $m \leq n$ and all connected components U of $S_{\delta,m,n}(x)$. We have

$$\rho_n(x) \leq 1 + (3\delta \|A(T^n(x))\,\|)^p.$$

Let then

$$\beta_\delta(x) := \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left[1 + \left(3\delta \|A(T^k(x))\|\right)^p\right].$$

By Birkhoff's ergodic theorem, $\lim_{\delta\to 0} \beta_\delta(x) = 0$ for almost every $x \in \Delta$.

It follows that there exists a sequence $\varepsilon_\delta(x, n)$, with $\lim_{n\to\infty} \varepsilon_\delta(x, n) = 0$ for almost every $x \in \Delta$, such that each connected component U of $S_{\delta,m,m}(x)$ intersects at most

$$\prod_{k=m}^{n-1} \rho_k(x) \leq e^{\varepsilon_\delta(x,n)n}\, e^{\beta_\delta(x)n}$$

connected components of $S_{\delta,m,n}(x)$. Thus U intersects $\cap_{n\geq m} S_{\delta,m,n}(x)$ in a set of upper box dimension at most $\frac{\beta_\delta(x)}{\lambda}$. We conclude that $\cup_{m\geq 0} \cap_{n\geq m} S_{\delta,m,n}(x)$ has Hausdorff dimension at most $\frac{\beta_\delta(x)}{\lambda}$; hence $G \cap W^s(x)$ has Hausdorff dimension 0, for almost every $x \in \Delta$. ☐

Theorem A.1, together with Veech's criterion (Theorem 6.1) has the following consequence, which implies Theorem B.

THEOREM A.2. *Let $\pi \in \mathfrak{S}_d$. Then for almost every $\lambda \in \mathbb{R}^d_+$, the set of $h \in H(\pi) \cap \mathbb{R}^d_+$ such that the special flow $F(\lambda, h, \pi)$ is not weakly mixing has Hausdorff dimension at most $g(\pi) + 1$.*[5]

Proof. It is enough to show that, for almost every $x \in \Delta$, the weak stable space $W^s(x)$ of the cocycle (T, A), considered in the proof of Theorem 6.4, has Hausdorff dimension at most $g(\pi)$. In fact, in this case the set of $h \in H(\pi) \setminus \{0\}$ such that the line through h intersects $W^s(x)$ in some $w \neq 0$ has

[5] In order to get a weaker, measure zero, statement (when $g(\pi) > 1$), it is enough to assume as above that the Zorich cocycle has two positive Lyapunov exponents.

Hausdorff dimension at most $g(\pi)+1$, and the result then follows by Veech's criterion. Since $W^s(x) = (G_0 \cap W^s(x)) + E^s(x)$, where G_0 is any linear subspace transverse to the stable space $E^s(x)$, it is enough to show that $W^s(x) \cap G_0$ has Hausdorff dimension 0. This follows from the above Theorem A.1, since by the non-uniform hyperbolicity of (T, A) the central stable space $E^{cs}(x)$ and the stable space $E^s(x)$ coincide, for almost every $x \in \Delta$. $\square$

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References

[AGY] A. Avila, S. Gouëzel, and J.-C. Yoccoz, Exponential mixing for the Teichmüller flow, *Pub. Math. IHES* **104** (2006), 143–211.

[AV] A. Avila and M. Viana, Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture, to appear in *Acta Math.*

[CFS] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Grundlehren der Mathematisches Wissenschaften, **245**, Springer-Verlag, New York, 1982.

[F1] G. Forni, Solutions of the cohomological equation for area-preserving flows on higher genus surfaces, *Ann. of Math.* **146** (1997), 295–344.

[F2] ———, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, *Ann. of Math.* **155** (2002), 1–103.

[GJ1] E. Gutkin and C. Judge, The geometry and arithmetic of translation surfaces with applications to polygonal billiards, *Math. Res. Lett.* **3** (1996), 391–403.

[GJ2] ———, Affine mappings of translation surfaces: geometry and arithmetic, *Duke Math. J.* **103** (2000), 191–213.

[GK] E. Gutkin and A. B. Katok, Weakly mixing billiards, in *Holomorphic dynamics* (Mexico, 1986), *Lecture Notes in Math.* **1345**, 163–176, Springer-Verlag, New York, 1988.

[Ka] A. B. Katok, Interval exchange transformations and some special flows are not mixing, *Israel J. Math.* **35** (1980), 301–310.

[KB] A. B. Katok (with the collaboration of K. Burns), Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems, *Ergodic Theory Dynam. Systems* **14** (1994), 757–785.

[KS] A. B. Katok and A. M. Stepin, Approximations in ergodic theory, *Uspehi Mat. Nauk* **22** (1967), 81–106.

[Ke] M. S. Keane, Interval exchange transformations, Math. Z. **141** (1975), 25–31.

[KMS] S. Kerckhoff, H. Masur, and J. Smillie, Ergodicity of billiard flows and quadratic differentials, *Ann. of Math.* **124** (1986), 293–311.

[Ko] M. Kontsevich, Lyapunov exponents and Hodge theory, in *The Mathematical Beauty of Physics* (Saclay, 1996), *Adv. Ser. Math. Phys.* **24**, 318–332, World Sci. Publ., River Edge, NJ, 1997.

[L] I. Lucien, Melange faible topologique des flots sur les surfaces, *Ergodic Theory Dynam. Systems* **18** (1998), 963–984.

[MMY] S. Marmi, P. Moussa, and J.-C. Yoccoz, The cohomological equation for Roth type interval exchange maps, *Jour. Amer. Math. Soc.* **18** (2005), 823–872.

[M] H. Masur, Interval exchange transformations and measured foliations, *Ann. of Math.* **115** (1982), 169–200.

[NR] A. Nogueira and D. Rudolph, Topological weak-mixing of interval exchange maps, *Ergodic Theory Dynam. Systems* **17** (1997), 1183–1209.

[O] V. I. Oseledec, A multiplicative ergodic theorem. Characteristic Lyapunov exponents of dynamical systems, *Trans. Moscow Math. Soc.* **19** (1968), 179–210.

[P] K. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge, 1989.

[R] G. Rauzy, Échanges d'intervalles et transformations induites, *Acta Arith.* **34** (1979), 315–328.

[V1] W. A. Veech, Projective Swiss cheeses and uniquely ergodic interval exchange transformations, in *Ergodic Theory and Dynamical Systems*. I (College Park, MD, 1979–1980), *Progr. Math.* **10**, 113–193, Birkhaüser, Boston, Mass., 1981.

[V2] ———, Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math.* **115** (1982), 201–242.

[V3] ———, The Teichmüller geodesic flow, *Ann. of Math.* **124** (1986), 441–530.

[V4] ———, The metric theory of interval exchange transformations. I. Generic spectral properties, *Amer. J. Math.* **106** (1984), 1331–1359.

[V5] ———, Moduli spaces of quadratic differentials, *J. d'Analyse Math.* **55** (1990), 117–171.

[Z1] A. Zorich, Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents, *Ann. Inst. Fourier* (*Grenoble*) **46** (1996), 325–370.

[Z2] ———, Deviation for interval exchange transformations, *Ergodic Theory Dynam. Systems* **17** (1997), 1477–1499.

[Z3] ———, How do the leaves of a closed 1-form wind around a surface?, in *Pseudoperiodic Topology*, *Amer. Math. Soc. Transl.* **197**, 135–178, A.M.S., Providence, RI, 1999.

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