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The Calderón problem with partial data

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Abstract

In this paper we improve an earlier result by Bukhgeim and Uhlmann [1], by showing that in dimension $n \geq 3$, the knowledge of the Cauchy data for the Schrödinger equation measured on possibly very small subsets of the boundary determines uniquely the potential. We follow the general strategy of [1] but use a richer set of solutions to the Dirichlet problem. This implies a similar result for the problem of Electrical Impedance Tomography which consists in determining the conductivity of a body by making voltage and current measurements at the boundary.

1. Introduction

The Electrical Impedance Tomography (EIT) inverse problem consists in determining the electrical conductivity of a body by making voltage and current measurements at the boundary of the body. Substantial progress has been made on this problem since Calderón's pioneer contribution [3], and is also known as Calderón's problem, in the case where the measurements are made on the whole boundary. This problem can be reduced to studying the Dirichlet-to-Neumann (DN) map associated to the Schrödinger equation. A key ingredient in several of the results is the construction of complex geometrical optics for the Schrödinger equation (see [14] for a survey). Approximate complex geometrical optics solutions for the Schrödinger equation concentrated near planes are constructed in [6] and concentrated near spheres in [8].

Much less is known if the DN map is only measured on part of the boundary. The only previous result that we are aware of, without assuming any *a priori* condition on the potential besides being bounded, is in [1]. It is shown there that if we measure the DN map restricted to, roughly speaking, slightly more than half of the boundary then one can determine uniquely the potential. The proof relies on a Carleman estimate with an exponential weight with a linear phase. The Carleman estimate can also be used to construct complex geometrical optics solutions for the Schrödinger equation. We are able in this paper to improve significantly on this result. We show that measuring the DN map on an arbitrary open subset of the boundary we can determine uniquely the potential. We do this by proving a more general Carleman estimate (Proposition 3.2) with exponential nonlinear weights. This Carleman estimate allows also to construct a much wider class of complex geometrical optics than previously known (§4). We now state more precisely the main results.

In the following, we let $\Omega \subset\subset \mathbf{R}^n$, be an open connected set with C^∞ boundary. For the main results, we will also assume that $n \geq 3$. If $q \in L^\infty(\Omega)$, then we consider the operator $-\Delta + q : L^2(\Omega) \to L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ as a bounded perturbation of minus the usual Dirichlet Laplacian. $-\Delta + q$ then has a discrete spectrum, and we assume

$$0 \text{ is not an eigenvalue of } -\Delta + q : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega). \tag{1.1}$$

Under this assumption, we have a well-defined Dirichlet to Neumann map

$$\mathcal{N}_q : H^{\frac{1}{2}}(\partial\Omega) \ni v \mapsto \partial_\nu u_{|\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega), \tag{1.2}$$

where ν denotes the exterior unit normal and u is the unique solution in

$$H_\Delta(\Omega) := \{u \in H^1(\Omega);\ \Delta u \in L^2(\Omega)\} \tag{1.3}$$

of the problem

$$(-\Delta + q)u = 0 \text{ in } \Omega,\ u_{|\partial\Omega} = v. \tag{1.4}$$

See [1] for more details, here we have slightly modified the choice of the Sobolev indices.

Let $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$, where $\mathrm{ch}\,(\Omega)$ denotes the convex hull of Ω. Define the front and the back faces of $\partial\Omega$ by

$$F(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu(x) \leq 0\},\ B(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu(x) > 0\}. \tag{1.5}$$

The main result of this work is the following:

THEOREM 1.1. *With* Ω, x_0, $F(x_0)$, $B(x_0)$ *defined as above, let* $q_1, q_2 \in L^\infty(\Omega)$ *be two potentials satisfying* (1.1) *and assume that there exist open neighborhoods* $\widetilde{F}$, $\widetilde{B} \subset \partial\Omega$ *of* $F(x_0)$ *and* $B(x_0) \cup \{x \in \partial\Omega; (x - x_0) \cdot \nu = 0\}$ *respectively, such that*

$$\mathcal{N}_{q_1} u = \mathcal{N}_{q_2} u \text{ in } \widetilde{F}, \text{ for all } u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathcal{E}'(\widetilde{B}). \tag{1.6}$$

Then $q_1 = q_2$.

Notice that by Green's formula $\mathcal{N}_q^* = \mathcal{N}_{\overline{q}}$. It follows that $\widetilde{F}$ and $\widetilde{B}$ can be permuted in (1.6) and we get the same conclusion.

If $\widetilde{B} = \partial\Omega$ then we obtain the following result.

THEOREM 1.2. *With* Ω, x_0, $F(x_0)$, $B(x_0)$ *defined as above, let* $q_1, q_2 \in L^\infty(\Omega)$ *be two potentials satisfying* (1.1) *and assume that there exists a neighborhood* $\widetilde{F} \subset \partial\Omega$ *of* $F(x_0)$, *such that*

$$\mathcal{N}_{q_1} u = \mathcal{N}_{q_2} u \text{ in } \widetilde{F}, \text{ for all } u \in H^{\frac{1}{2}}(\partial\Omega). \tag{1.7}$$

Then $q_1 = q_2$.

We have the following easy corollary,

COROLLARY 1.3. *With* Ω *as above, let* $x_1 \in \partial\Omega$ *be a point such that the tangent plane* H *of* $\partial\Omega$ *at* x_1 *satisfies* $\partial\Omega \cap H = \{x_1\}$. *Assume in addition, that* Ω *is strongly starshaped with respect to* x_1. *Let* $q_1, q_2 \in L^\infty(\Omega)$ *and assume that there exists a neighborhood* $\widetilde{F} \subset \partial\Omega$ *of* x_1, *such that* (1.7) *holds. Then* $q_1 = q_2$.

Here we say that Ω is strongly star shaped with respect to x_1 if every line through x_1 which is not contained in the tangent plane H cuts the boundary $\partial\Omega$ at precisely two distinct points, x_1 and x_2, and the intersection at x_2 is transversal.

Theorem 1.1 has an immediate consequence for the Calderón problem.

Let $\gamma \in C^2(\overline{\Omega})$ be a strictly positive function on $\overline{\Omega}$. Given a voltage potential f on the boundary, the equation for the potential in the interior, under the assumption of no sinks or sources of current in Ω, is

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega, \quad u_{|\partial\Omega} = f.$$

The Dirichlet-to-Neumann map is defined in this case as follows:

$$\mathcal{N}_\gamma(f) = (\gamma \partial_\nu u)_{|\partial\Omega}.$$

It extends to a bounded map

$$\mathcal{N}_\gamma : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega).$$

As a direct consequence of Theorem 1.1 we have

COROLLARY 1.4. *Let* $\gamma_i \in C^2(\overline{\Omega})$, $i = 1, 2$, *be strictly positive. Assume that* $\gamma_1 = \gamma_2$ *on* $\partial\Omega$ *and*

$$\mathcal{N}_{\gamma_1} u = \mathcal{N}_{\gamma_2} u \text{ in } \widetilde{F}, \text{ for all } u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathcal{E}'(\widetilde{B}).$$

Then $\gamma_1 = \gamma_2$.

Here $\widetilde{F}$ and $\widetilde{B}$ are as in Theorem 1.1. It is well known (see for instance [14]) that one can relate $\mathcal{N}_\gamma$ and $\mathcal{N}_q$ in the case that $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ with $\gamma > 0$ by the formula

$$\mathcal{N}_q(f) = (\gamma^{-\frac{1}{2}})_{|\partial\Omega} \mathcal{N}_\gamma(\gamma^{-\frac{1}{2}} f) + \frac{1}{2}\left(\gamma^{-1}\partial_\nu \gamma\right)_{|\partial\Omega} f. \tag{1.8}$$

The Kohn-Vogelius result [9] implies that $\gamma_1 = \gamma_2$ and $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ on $\widetilde{F} \cap \widetilde{B}$. Then using (1.8) and Theorem 1.1 we immediately get Corollary 1.4.

A brief outline of the paper is as follows. In Section 2 we review the construction of weights that can be used in proving Carleman estimates. In Section 3 we derive the Carleman estimate (Proposition 3.2) that we shall use in the construction of complex geometrical optics solutions for the Schrödinger equation. In Sections 4, 5 we use the Carleman estimate for solutions of the inhomogeneous Schrödinger equation vanishing on the boundary. This leads to show that, under the conditions of Theorems 1.1 and 1.2, the difference of the potentials is orthogonal in L^2 to a family of oscillating functions which are real-analytic. For simplicity we first prove Theorem 1.2. In Section 6 we end the proof of Theorem 1.2 by choosing this family appropriately and using the wave front set version of Holmgren's uniqueness theorem. Finally in Section 7 we prove the more general result Theorem 1.1.

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2. Remarks about Carleman weights in the variable coefficient case

In this section we review the construction of weights that can be used in proving Carleman estimates. The discussion is a little more general than what will actually be needed, but much of the section can be skipped at the first reading and we will indicate where.

Let $\widetilde{\Omega} \subset \mathbf{R}^n$, $n \geq 2$ be an open set, and let $G(x) = (g^{ij}(x))$ a positive definite real symmetric $n \times n$-matrix, depending smoothly on $x \in \widetilde{\Omega}$. Put

$$p(x, \xi) = \langle G(x)\xi | \xi \rangle. \tag{2.1}$$

Let $\varphi \in C^\infty(\widetilde{\Omega}; \mathbf{R})$ with $\varphi'(x) \neq 0$ everywhere, and consider

$$p(x, \xi + i\varphi'_x(x)) = a(x, \xi) + ib(x, \xi), \tag{2.2}$$

so that with the usual automatic summation convention:

$$a(x, \xi) = g^{ij}(x)\xi_i \xi_j - g^{ij}(x)\varphi'_{x_i}\varphi'_{x_j}, \tag{2.3}$$

$$b(x, \xi) = 2\langle G(x)\varphi'(x) | \xi \rangle = 2g^{\mu\nu}\varphi'_{x_\mu}\xi_\nu. \tag{2.4}$$

Readers, who are not interested in routine calculations, may go directly to the conclusion of this section.

A direct computation gives the Hamilton field $H_a = a'_\xi \cdot \partial_x - a'_x \cdot \partial_\xi$ of a:

$$H_a = 2g^{ij}(x)\xi_j\partial_{x_i} - \partial_{x_\nu}(g^{ij})\xi_i\xi_j\partial_{\xi_\nu} + \partial_{x_\nu}(g^{ij})\varphi'_{x_i}\varphi'_{x_j}\partial_{\xi_\nu} + 2g^{ij}\varphi''_{x_i,x_\nu}\varphi'_{x_j}\partial_{\xi_\nu}, \tag{2.5}$$

and

$$\begin{aligned}\frac{1}{2}H_a b =& 2g^{ij}\xi_j g^{\mu\nu}\varphi''_{x_i,x_\mu}\xi_\nu + 2g^{ij}\xi_j\partial_{x_i}(g^{\mu\nu})\varphi'_{x_\mu}\xi_\nu - \partial_{x_\nu}(g^{ij})\xi_i\xi_j g^{\mu\nu}\varphi'_{x_\mu} \\ &+\partial_{x_\nu}(g^{ij})\varphi'_{x_i}\varphi'_{x_j}g^{\mu\nu}\varphi'_{x_\mu} + 2g^{ij}\varphi''_{x_i,x_\nu}\varphi'_{x_j}g^{\mu\nu}\varphi_{x_\mu} \\ =& 2\langle\varphi''_{xx}|G\xi\otimes G\xi\rangle + 2\langle\varphi''_{xx}|G\varphi'_x\otimes G\varphi'_x\rangle + 2\langle\partial_x G|G\xi\otimes\varphi'_x\otimes\xi\rangle \\ &-\langle\partial_x G|G\varphi'_x\otimes\xi\otimes\xi\rangle + \langle\partial_x G|G\varphi'_x\otimes\varphi'_x\otimes\varphi'_x\rangle.\end{aligned} \tag{2.6}$$

Here we use the straight forward scalar products between tensors of the same size (2 or 3) and consider that the first index in the 3 tensor $\partial_x G$ is the one corresponding to the differentiations ∂_{x_j}. We also notice that φ'_x, ξ are naturally cotangent vectors, while $G\varphi'_x, G\xi$ are tangent vectors. We want this Poisson bracket to be ≥ 0 or even $\equiv 0$ on the set $a = b = 0$, i.e. on the set given by

$$\langle G|\xi\otimes\xi - \varphi'_x\otimes\varphi'_x\rangle = 0,\ \langle G|\varphi'_x\otimes\xi\rangle = 0. \tag{2.7}$$

Observation 1. If φ is a distance function in the sense that $\langle G|\varphi'_x\otimes\varphi'_x\rangle \equiv 1$, then if we differentiate in the direction $G\varphi'_x$, we get

$$0 = (G\varphi'_x\cdot\partial_x)\langle G|\varphi'_x\otimes\varphi'_x\rangle = \langle\partial_x G|G\varphi'_x\otimes\varphi'_x\otimes\varphi'_x\rangle + 2\langle\varphi''_{xx}|G\varphi'_x\otimes G\varphi'_x\rangle.$$

From this we see that two terms in the final expression in (2.6) cancel and we get

$$\frac{1}{2}H_a b = 2\langle\varphi''_{xx}|G\xi\otimes G\xi\rangle + 2\langle\partial_x G|G\xi\otimes\varphi'_x\otimes\xi\rangle - \langle\partial_x G|G\varphi'_x\otimes\xi\otimes\xi\rangle. \tag{2.8}$$

Observation 2. If we replace $\varphi(x)$ by $\psi(x) = f(\varphi(x))$, then

$$\begin{aligned}\psi'_x &= f'(\varphi(x))\varphi'_x \\ \psi''_{xx} &= f''(\varphi(x))\varphi'_x\otimes\varphi'_x + f'(\varphi(x))\varphi''_{xx}.\end{aligned}$$

If ξ satisfies (2.7), then it is natural to replace ξ by $\eta = f'(\varphi)\xi$, in order to preserve this condition (for the new symbol) and we see that all terms in the final member of (2.6), when restricted to $a = b = 0$, become multiplied by $f'(\varphi)^3$ except the second one which becomes replaced by

$$f'(\varphi)^3 2\langle\varphi''_{xx}|G\varphi'_x\otimes G\varphi'_x\rangle + 2f''(\varphi(x))f'(\varphi(x))^2\langle G|\varphi'_x\otimes\varphi'_x\rangle^2.$$

(For the first term in (2.6) we also use that $\langle\varphi_x'\otimes\varphi_x'|G\xi\otimes G\xi\rangle = \langle\varphi_x'|G\xi\rangle^2 = 0$.) Thus we get after the two substitutions $\varphi\mapsto\psi = f(\varphi(x))$, $\xi\mapsto\eta = f'(\varphi(x))\xi$:

$$\begin{aligned}(2.9)\quad \frac{1}{2}H_a b(x,\eta) = {}& 2f''(\varphi(x))f'(\varphi(x))^2\|\varphi_x'\|_g^4 + f'(\varphi)^3\big(2\langle\varphi_{xx}''|G\xi\otimes G\xi\rangle \\ &+2\langle\varphi_{xx}''|G\varphi_x'\otimes G\varphi_x'\rangle + 2\langle\partial_x G|G\xi\otimes\varphi_x'\otimes\xi\rangle \\ &-\langle\partial_x G|G\varphi_x'\otimes\xi\otimes\xi\rangle + \langle\partial_x G|G\varphi_x'\otimes\varphi_x'\otimes\varphi_x'\rangle\big),\end{aligned}$$

with $\eta = f'(\varphi)\xi$, ξ satisfying (2.7), so that η satisfies the same condition (with φ replaced by ψ):

$$(2.10)\qquad \langle G|\eta\otimes\eta - \psi_x'\otimes\psi_x'\rangle = \langle G|\psi_x'\otimes\eta\rangle = 0.$$

Moreover $\|\varphi_x'\|_g^2 = \langle G|\varphi_x'\otimes\varphi_x'\rangle$ by definition.

Conclusion. To get $H_a b\geq 0$ whenever (2.7) is satisfied, it suffices to start with a function φ with nonvanishing gradient, and then replace φ by $f(\varphi)$ with $f'>0$ and f''/f' sufficiently large. This kind of convexification ideas are very old and used recently in a related context by Lebeau-Robbiano [10], Burq [2]. For later use, we needed to spell out the calculations quite explicitly.

3. Carleman estimate

We use from now on semiclassical notation (see for instance [4]).

Let $P_0 = -h^2\Delta = \sum(hD_{x_j})^2$, with $D_{x_j} = \frac{1}{i}\partial_{x_j}$. Let φ, $\widetilde{\Omega}$ be as in the beginning of Section 2. Then

$$(3.1)\qquad e^{\varphi/h}\circ P_0\circ e^{-\varphi/h} = \sum_{j=1}^{n}(hD_{x_j} + i\partial_{x_j}\varphi)^2 = A + iB,$$

where A, B are the formally selfadjoint operators:

$$(3.2)\qquad A = (hD)^2 - (\varphi_x')^2,\ B = \sum(\partial_{x_j}\varphi\circ hD_{x_j} + hD_{x_j}\circ\partial_{x_j}\varphi)$$

having the Weyl symbols (for the semi-classical quantization)

$$(3.3)\qquad a = \xi^2 - (\varphi_x')^2,\ b = 2\varphi_x'\cdot\xi.$$

We assume that φ has non vanishing gradient and is a limiting Carleman weight in the sense that

$$(3.4)\qquad \{a,b\}(x,\xi) = 0,\ \text{when } a(x,\xi) = b(x,\xi) = 0.$$

Here $\{a,b\} = a_\xi'\cdot b_x' - a_x'\cdot b_\xi'$ is the Poisson bracket (as in (2.6)):

$$(3.5)\qquad \{a,b\} = 4\langle\varphi_{xx}''(x)|\xi\otimes\xi + \varphi_x'\otimes\varphi_x'\rangle.$$

On the x-dependent hypersurface in ξ-space, given by $b(x,\xi) = 0$, we know that the quadratic polynomial $\{a,b\}(x,\xi)$ vanishes when $\xi^2 = (\varphi'_x)^2$. It follows that

$$\{a,b\}(x,\xi) = c(x)(\xi^2 - (\varphi'_x)^2), \text{ for } b(x,\xi) = 0, \tag{3.6}$$

where $c(x) \in C^\infty(\widetilde{\Omega}; \mathbf{R})$. Then consider

$$\{a,b\}(x,\xi) - c(x)(\xi^2 - (\varphi'_x)^2),$$

which is a quadratic polynomial in ξ, vanishing when $\varphi'_x(x) \cdot \xi = 0$ It follows that this is of the form $\ell(x,\xi)b(x,\xi)$ where $\ell(x,\xi)$ is affine in ξ with smooth coefficients, and we end up with

$$\{a,b\} = c(x)a(x,\xi) + \ell(x,\xi)b(x,\xi). \tag{3.7}$$

But $\{a,b\}$ contains no linear terms in ξ, so we know that $\ell(x,\xi)$ is linear in ξ.

The commutator $[A,B]$ can be computed directly: and we get

$$\begin{aligned}[A,B] = \frac{h}{i}\Big(\sum_{j,k} \big[(hD_{x_j} \circ \varphi''_{x_jx_k} + \varphi''_{x_jx_k} \circ hD_{x_j})hD_{x_k} \\ + hD_{x_k}(hD_{x_j} \circ \varphi''_{x_jx_k} + \varphi''_{x_jx_k} \circ hD_{x_j})\big] \\ + 4\langle \varphi''_{xx}, \varphi'_x(x) \otimes \varphi'_x(x)\rangle\Big).\end{aligned}$$

The Weyl symbol of $[A,B]$ as a semi-classical operator is

$$\frac{h}{i}\{a,b\} + h^3 p_0(x),$$

Combining this with (3.7), we get with a new p_0:

$$i[A,B] = h\left(\frac{1}{2}(c(x) \circ A + A \circ c) + \frac{1}{2}(LB + BL) + h^2 p_0(x)\right), \tag{3.8}$$

where L denotes the Weyl quantization of ℓ.

We next derive the Carleman estimate for $u \in C_0^\infty(\Omega)$, $\Omega \subset\subset \widetilde{\Omega}$: Start from $P_0 u = v$ and let $\widetilde{u} = e^{\varphi/h}u$, $\widetilde{v} = e^{\varphi/h}v$, so that

$$(A + iB)\widetilde{u} = \widetilde{v}. \tag{3.9}$$

Using the formal selfadjointness of A, B, we get

$$\|\widetilde{v}\|^2 = ((A - iB)(A + iB)\widetilde{u}|\widetilde{u}) = \|A\widetilde{u}\|^2 + \|B\widetilde{u}\|^2 + (i[A,B]\widetilde{u}|\widetilde{u}). \tag{3.10}$$

Using (3.8), we get for $u \in C_0^\infty(\Omega)$:

$$\begin{aligned}\|\widetilde{v}\|^2 &\geq \|A\widetilde{u}\|^2 + \|B\widetilde{u}\|^2 - \mathcal{O}(h)(\|A\widetilde{u}\|\|\widetilde{u}\| + \|L\widetilde{u}\|\|B\widetilde{u}\|) - \mathcal{O}(h^3)\|\widetilde{u}\|^2 \\ &\geq \frac{2}{3}\|A\widetilde{u}\|^2 + \frac{1}{2}\|B\widetilde{u}\|^2 - \mathcal{O}(h^2)(\|\widetilde{u}\|^2 + \|L\widetilde{u}\|^2). \\ &\geq \frac{1}{2}(\|A\widetilde{u}\|^2 + \|B\widetilde{u}\|^2) - \mathcal{O}(h^2)\|\widetilde{u}\|^2,\end{aligned} \tag{3.11}$$

where in the last step we used the *a priori* estimate

$$\|h\nabla\widetilde{u}\|^2 \leq \mathcal{O}(1)(\|A\widetilde{u}\|^2 + \|\widetilde{u}\|^2),$$

which follows from the classical ellipticity of A.

Now we could try to use that B is associated to a nonvanishing gradient field (and hence without any closed or even trapped trajectories in $\widetilde{\Omega}$), to obtain the Poincaré estimate:

$$h\|\widetilde{u}\| \leq \mathcal{O}(1)\|B\widetilde{u}\|. \tag{3.12}$$

We see that (3.12) is not quite good enough to absorb the last term in (3.11). In order to remedy for this, we make a slight modification of φ by introducing

$$\varphi_\varepsilon = f\circ\varphi, \text{ with } f = f_\varepsilon \tag{3.13}$$

to be chosen below, and write $a_\varepsilon + ib_\varepsilon$ for the conjugated symbol. We saw in Section 2 and especially in (2.9) that the Poisson bracket $\{a_\varepsilon, b_\varepsilon\}$, becomes with φ equal to the original weight:

$$\{a_\varepsilon, b_\varepsilon\}(x, f'(\varphi)\eta) = f'(\varphi)^3\left(\{a,b\}(x,\eta) + \frac{4f''(\varphi)}{f'(\varphi)}\|\varphi_x'\|^4\right), \quad \text{when } a(x,\eta) = b(x,\eta) = 0. \tag{3.14}$$

The substitution $\xi \to f'(\varphi)\eta$ is motivated be the fact that if $a(x,\eta) = b(x,\eta) = 0$, then $a_\varepsilon(x, f'(\varphi)\eta) = b_\varepsilon(x, f'(\varphi)\eta) = 0$. Now let

$$f_\varepsilon(\lambda) = \lambda + \varepsilon\lambda^2/2, \tag{3.15}$$

with $0 \leq \varepsilon \ll 1$, so that

$$\frac{4f''(\varphi)}{f'(\varphi)} = \frac{4\varepsilon}{1+\varepsilon\varphi} = 4\varepsilon + \mathcal{O}(\varepsilon^2).$$

In view of (3.14), (3.4), we get

$$\{a_\varepsilon, b_\varepsilon\}(x,\xi) = 4f_\varepsilon''(\varphi)(f_\varepsilon'(\varphi))^2\|\varphi'\|^4 \approx 4\varepsilon\|\varphi_x'\|^4, \tag{3.16}$$

when $a_\varepsilon(x,\xi) = b_\varepsilon(x,\xi) = 0$, so instead of (3.7), we get

$$\{a_\varepsilon, b_\varepsilon\} = 4f_\varepsilon''(\varphi)(f_\varepsilon'(\varphi))^2\|\varphi_x'\|^4 + c_\varepsilon(x)a_\varepsilon(x,\xi) + \ell_\varepsilon(x,\xi)b_\varepsilon(x,\xi), \tag{3.17}$$

with $\ell_\varepsilon(x,\xi)$ linear in ξ.

Instead of (3.11), we get with $\widehat{u} = e^{\varphi_\varepsilon/h}u$, $\widehat{v} = e^{\varphi_\varepsilon/h}v$ when $P_0u = v$:

$$\|\widehat{v}\|^2 \geq h(4\varepsilon + \mathcal{O}(\varepsilon^2))\int \|\varphi_x'\|^4|\widehat{u}(x)|^2dx + \frac{1}{2}\|A_\varepsilon\widehat{u}\|^2 + \frac{1}{2}\|B_\varepsilon\widehat{u}\|^2 - \mathcal{O}(h^2)\|\widehat{u}\|^2, \tag{3.18}$$

while the analogue of (3.12) remains uniformly valid when ε is small:

$$h\|\widehat{u}\| \leq \mathcal{O}(1)\|B_\varepsilon \widehat{u}\|, \tag{3.19}$$

even though we will not use this estimate.

Choose $h \ll \varepsilon \ll 1$, so that (3.18) gives

$$\|\widehat{v}\|^2 \geq \varepsilon h\|\widehat{u}\|^2 + \frac{1}{2}\|A_\varepsilon \widehat{u}\|^2 + \frac{1}{2}\|B_\varepsilon \widehat{u}\|^2. \tag{3.20}$$

We want to transform this into an estimate for $\widetilde{u}, \widetilde{v}$. From the special form of A_ε, we see that

$$\|hD\widehat{u}\|^2 \leq (A_\varepsilon \widehat{u}|\widehat{u}) + \mathcal{O}(1)\|\widehat{u}\|^2,$$

leading to

$$\|hD\widehat{u}\|^2 \leq \frac{1}{2}\|A_\varepsilon \widehat{u}\|^2 + \mathcal{O}(1)\|\widehat{u}\|^2.$$

Combining this with (3.20), we get

$$\|\widehat{v}\|^2 \geq \frac{\varepsilon h}{C_0}(\|\widehat{u}\|^2 + \|hD\widehat{u}\|^2) + \left(\frac{1}{2} - \mathcal{O}(\varepsilon h)\right)\|A_\varepsilon \widehat{u}\|^2 + \frac{1}{2}\|B_\varepsilon \widehat{u}\|^2. \tag{3.21}$$

Write $\varphi_\varepsilon = \varphi + \varepsilon g$, where $g = g_\varepsilon$ is $\mathcal{O}(1)$ with all its derivatives. We have

$$\widehat{u} = e^{\varepsilon g/h}\widetilde{u},\ \widehat{v} = e^{\varepsilon g/h}\widetilde{v},$$

so

$$hD\widehat{u} = e^{\varepsilon g/h}(hD\widetilde{u} + \frac{\varepsilon}{i}g'\widetilde{u}) = e^{\frac{\varepsilon}{h}g}(hD\widetilde{u} + \mathcal{O}(\varepsilon)\widetilde{u}),$$

and

$$\begin{aligned}\|\widehat{u}\|^2 + \|hD\widehat{u}\|^2 &\geq \|e^{\varepsilon g/h}\widetilde{u}\|^2 + \|e^{\varepsilon g/h}hD\widetilde{u}\|^2 \\ &\quad -C\varepsilon\|e^{\varepsilon g/h}\widetilde{u}\|\|e^{\varepsilon g/h}hD\widetilde{u}\| - C\varepsilon^2\|e^{\varepsilon g/h}\widetilde{u}\|^2 \\ &\geq (1 - C\varepsilon)(\|e^{\varepsilon g/h}\widetilde{u}\|^2 + \|e^{\varepsilon g/h}hD\widetilde{u}\|^2),\end{aligned}$$

so from (3.21) we obtain after increasing C_0 by a factor $(1 + \mathcal{O}(\varepsilon))$:

$$\|e^{\varepsilon g/h}\widetilde{v}\|^2 \geq \frac{\varepsilon h}{C_0}(\|e^{\varepsilon g/h}\widetilde{u}\|^2 + \|e^{\varepsilon g/h}hD\widetilde{u}\|^2). \tag{3.22}$$

If we take $\varepsilon = Ch$ with $C \gg 1$ but fixed, then $\varepsilon g/h$ is uniformly bounded in Ω and we get the Carleman estimate

$$h^2(\|\widetilde{u}\|^2 + \|hD\widetilde{u}\|^2) \leq C_1\|\widetilde{v}\|^2. \tag{3.23}$$

This clearly extends to solutions of the equation

$$(-h^2\Delta + h^2 q)u = v, \tag{3.24}$$

if $q \in L^\infty$ is fixed, since we can start by applying (3.23) with $\widetilde{v}$ replaced by $\widetilde{v} - h^2 q\widetilde{u}$. Summing up the discussion so far, we have

PROPOSITION 3.1. *Let P_0, $\widetilde{\Omega}$, φ be as in the beginning of this section and assume that φ is a limiting Carleman weight in the sense that* (3.4) *holds. Let $\Omega \subset\subset \widetilde{\Omega}$ be open and let $q \in L^\infty(\Omega)$. Then if $u \in C_0^\infty(\Omega)$, we have*

$$h(\|e^{\varphi/h}u\| + \|hDe^{\varphi/h}u\|) \leq C\|e^{\varphi/h}(-h^2\Delta + h^2q)u\|, \tag{3.25}$$

where C depends on Ω, and $h > 0$ is small enough so that $Ch\|q\|_{L^\infty(\Omega)} \leq 1/2$.

We next establish a Carleman estimate when $P_0u = v$, $u \in C^\infty(\Omega)$, $u_{|\partial\Omega} = 0$ and $\Omega \subset\subset \widetilde{\Omega}$ is a domain with C^∞ boundary. As before, we let $\widehat{u} = e^{\varphi/h}u$, $\widehat{v} = e^{\varphi/h}v$, with $\varphi = \varphi_\varepsilon$, $0 \leq \varepsilon \ll 1$. With $A = A_\varepsilon$, $B = B_\varepsilon$, we have

$$(A + iB)\widehat{u} = \widehat{v}, \tag{3.26}$$

and

$$\begin{aligned} \|\widehat{v}\|^2 &= ((A + iB)\widehat{u}|(A + iB)\widehat{u}) \\ &= \|A\widehat{u}\|^2 + \|B\widehat{u}\|^2 + i((B\widehat{u}|A\widehat{u}) - (A\widehat{u}|B\widehat{u})), \end{aligned} \tag{3.27}$$

Using that B is a first order differential operator and that

$$\widehat{u}\big|_{\partial\Omega} = 0,$$

we see that

$$(A\widehat{u}|B\widehat{u}) = (BA\widehat{u}|\widehat{u}). \tag{3.28}$$

Similarly, we have

$$(B\widehat{u}|(\varphi_x')^2\widehat{u}) = ((\varphi_x')^2 B\widehat{u}|\widehat{u}). \tag{3.29}$$

Finally, we use Green's formula, with ν denoting the exterior unit normal, to transform

$$(B\widehat{u}| - h^2\Delta\widehat{u})_\Omega = -h^2(B\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega} + (-h^2\Delta B\widehat{u}|\widehat{u})_\Omega,$$

where we also used that $\widehat{u}\big|_{\partial\Omega} = 0$.

On $\partial\Omega$, we have

$$B = 2(\varphi_x' \cdot \nu)\frac{h}{i}\partial_\nu + B',$$

where B' acts along the boundary, so using again the Dirichlet condition, we get

$$(B\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega} = \frac{2h}{i}((\varphi_x' \cdot \nu)\partial_\nu\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega}.$$

Putting together the calculations and using (3.2) for A, we get

$$\|\widehat{v}\|^2 = \|A\widehat{u}\|^2 + \|B\widehat{u}\|^2 + i([A, B]\widehat{u}|\widehat{u}) - 2h^3((\varphi_x' \cdot \nu)\partial_\nu\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega}. \tag{3.30}$$

Let

$$\partial\Omega_\pm = \{x \in \partial\Omega;\ \pm\varphi_x' \cdot \nu \geq 0\}. \tag{3.31}$$

Notice that $\partial\Omega_\pm$ are independent of ε. We rewrite (3.30) as

$$\begin{aligned}(3.32)\quad & -2h^3((\varphi'_x\cdot\nu)\partial_\nu\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega_-}+i([A,B]\widehat{u}|\widehat{u})+\|A\widehat{u}\|^2+\|B\widehat{u}\|^2\\ &\qquad =\|\widehat{v}\|^2+2h^3((\varphi'_x\cdot\nu)\partial_\nu\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega_+}.\end{aligned}$$

This is analogous to (3.10) and the extra boundary terms can be added in the discussion leading from (3.18) to (3.21) and we get instead of (3.21):

$$\begin{aligned}(3.33)\quad & -2h^3((\varphi'_x\cdot\nu)\partial_\nu\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega_-}+\tfrac{\varepsilon h}{C_0}(\|\widehat{u}\|^2+\|hD\widehat{u}\|^2)\\ &+(\tfrac{1}{2}-\mathcal{O}(\varepsilon h))\|A_\varepsilon\widehat{u}\|^2+\tfrac{1}{2}\|B_\varepsilon\widehat{u}\|^2\\ &\le\|\widehat{v}\|^2+2h^3((\varphi'_x\cdot\nu)\partial_\nu\widehat{u}|\partial_\nu\widehat{u})_{\partial\Omega_+},\end{aligned}$$

with $\varphi=\varphi_\varepsilon$, provided $\varepsilon\gg h$. Fixing $\varepsilon=Ch$ for $C\gg 1$, we get with $\varphi=\varphi_{\varepsilon_0}$ for some $C_0>0$:

$$\begin{aligned}(3.34)\quad & -\frac{h^3}{C_0}((\varphi'_x\cdot\nu)\partial_\nu\widetilde{u}|\partial_\nu\widetilde{u})_{\partial\Omega_-}+\frac{h^2}{C_0}(\|\widetilde{u}\|^2+\|hD\widetilde{u}\|^2)\\ &\qquad\le\|\widetilde{v}\|^2+C_0h^3((\varphi'_x\cdot\nu)\partial_\nu\widetilde{u}|\partial_\nu\widetilde{u})_{\partial\Omega_+}.\end{aligned}$$

Here we recall that $-h^2\Delta u=v$, $\widetilde{u}=e^{\varphi/h}u$, $\widetilde{v}=e^{\varphi/h}v$, $\varphi=\varphi_{\varepsilon_0}$, $u_{|\partial\Omega}=0$.

If $q\in L^\infty$, we get for $h^2(-\Delta+q)u=v$, $u_{|\partial\Omega}=0$, by applying (3.34) with $\widetilde{v}$ replaced by $\widetilde{v}-h^2q\widetilde{u}$:

$$\begin{aligned}(3.35)\quad & -\frac{h^3}{C_0}((\varphi'_x\cdot\nu)\partial_\nu\widetilde{u}|\partial_\nu\widetilde{u})_{\partial\Omega_-}+\frac{h^2}{C_0}(\|\widetilde{u}\|^2+\|hD\widetilde{u}\|^2)\\ &\qquad\le\|\widetilde{v}\|^2+C_0h^3((\varphi'_x\cdot\nu)\partial_\nu\widetilde{u}|\partial_\nu\widetilde{u})_{\partial\Omega_+}.\end{aligned}$$

Here $\widetilde{u},\widetilde{v}$ are defined as before.

Summing up, we have

Proposition 3.2. *Let $\widetilde{\Omega},\varphi$ be as in Proposition* 3.1. *Let $\Omega\subset\subset\widetilde{\Omega}$ be an open set with C^∞ boundary and let $q\in L^\infty(\Omega)$. Let ν denote the exterior unit normal to $\partial\Omega$ and define $\partial\Omega_\pm$ as in* (3.31). *Then there exists a constant $C_0>0$, such that for every $u\in C^\infty(\overline{\Omega})$ with $u_{|\partial\Omega}=0$, we have for $0<h\ll 1$:*

$$\begin{aligned}(3.36)\quad & -\frac{h^3}{C_0}((\varphi'_x\cdot\nu)e^{\varphi/h}\partial_\nu u|e^{\varphi/h}\partial_\nu u)_{\partial\Omega_-}+\frac{h^2}{C_0}(\|e^{\varphi/h}u\|^2+\|e^{\varphi/h}h\nabla u\|^2)\\ &\le\|e^{\varphi/h}(-h^2\Delta+h^2q)u\|^2+C_0h^3((\varphi'_x\cdot\nu)e^{\varphi/h}\partial_\nu u|e^{\varphi/h}\partial_\nu u)_{\partial\Omega_+},\end{aligned}$$

Remark. If φ is a limiting Carleman weight, then so is $-\varphi$. With $\widetilde{u}=e^{-\varphi/h}u$, $\widetilde{v}=e^{-\varphi/h}v$, we still have (3.35), provided we permute $\partial\Omega_-$ and $\partial\Omega_+$ and change the signs in front of the boundary terms, so that they remain positive.

4. Construction of complex geometrical optics solutions

Let $H^s(\mathbf{R}^n)$ denote the semi-classical Sobolev space of order s, equipped with the norm $\|\langle hD\rangle^s u\|$. We define $H^s(\Omega)$, $H_0^s(\Omega)$ in the usual way, when $\Omega \subset\subset \mathbf{R}^n$ has smooth boundary. (3.23) can be written

$$h\|u\|_{H^1} \leq C\|e^{\varphi/h}P_0 e^{-\varphi/h}u\|,\ u \in C_0^\infty(\Omega), \tag{4.1}$$

when $P_0 = -h^2\Delta$. Here we let $\Omega \subset \widetilde{\Omega}$ be as in Section 3. Recall that $P_{0,\varphi} = e^{\varphi/h}P_0 e^{-\varphi/h}$ has the semiclassical Weyl symbol $\xi^2 - {\varphi_x'}^2 + 2i\varphi_x' \cdot \xi = a + ib$, which is elliptic in the region $|\xi| \geq 2|\varphi'(x)|$. It is therefore clear that (4.1) can be extended to:

$$h\|u\|_{H^{-s+1}} \leq C_{s,\Omega}\|e^{\varphi/h}P_0 e^{-\varphi/h}u\|_{H^{-s}},\ u \in C_0^\infty(\Omega), \tag{4.2}$$

for every fixed $s \in \mathbf{R}$. With $q \in L^\infty(\widetilde{\Omega})$, we put

$$P = -h^2(\Delta - q),\ P_\varphi = e^{\varphi/h}Pe^{-\varphi/h} = P_{0,\varphi} + h^2 q.$$

If $0 \leq s \leq 1$, we have

$$\|qu\|_{H^{-s}} \leq \|qu\| \leq \|q\|_{L^\infty}\|u\| \leq \|q\|_{L^\infty}\|u\|_{H^{-s+1}},$$

and for $h > 0$ small enough, we get from (4.2):

$$h\|u\|_{H^{-s+1}} \leq C_{s,\Omega}\|e^{\varphi/h}Pe^{-\varphi/h}u\|_{H^{-s}}. \tag{4.3}$$

The Hahn-Banach theorem now implies in the usual way:

Proposition 4.1. *Let $0 \leq s \leq 1$. Then for $h \geq 0$ small enough, for every $v \in H^{s-1}(\Omega)$, there exists $u \in H^s(\Omega)$ such that*

$$e^{-\varphi/h}Pe^{\varphi/h}u = v,\ h\|u\|_{H^s} \leq C\|v\|_{H^{s-1}}. \tag{4.4}$$

This result remains valid, when q is complex valued. In that case we replace P in (4.3) by $\overline{P} = -h^2\Delta + \overline{q}$.

We next construct certain WKB-solutions to the homogeneous equation. Recall that a, b are in involution on the joint zero set $J:\ a = b = 0$ in view of (3.7). At the points of J we also see that the Hamilton fields

$$H_a = 2(\xi \cdot \partial_x + \langle \varphi_{xx}''\varphi_x' | \partial_\xi\rangle),\ H_b = 2(\varphi_x' \cdot \partial_x - \langle \varphi_{xx}''\xi | \partial_\xi\rangle) \tag{4.5}$$

are linearly independent and even have linearly independent x-space projections. We conclude that J is an involutive manifold such that each bicharacteristic leaf (of dimension 2) has a base space projection which is also a nice submanifold of dimension 2. It follows that we have plenty of smooth local solutions to the Hamilton-Jacobi problem

$$a(x, \psi'(x)) = b(x, \psi'(x)) = 0. \tag{4.6}$$

Indeed, if $(x_0, \xi_0) \in J$, and we let $H \subset \Omega$ be a submanifold of codimension 2 passing through x_0 transversally to the projection of the bicharacteristic leaf through (x_0, ξ_0), then we have a unique local solution of (4.6), with $\psi_{|H} = \widetilde{\psi}$, if $\widetilde{\psi}$ is a smooth real-valued function on H such that $\widetilde{\psi}'(x_0)$ is equal to the projection of (x_0, ξ_0) in $T^*_{x_0}(H)$.

Since we need some explicit control of the size of the domain of definition of ψ, we now give a more down-to-earth construction. (4.6) can be written more explicitly as

$$\psi'(x)^2 - \varphi'(x)^2 = 0,\ \varphi'(x) \cdot \psi'(x) = 0. \tag{4.7}$$

First restrict the attention to the hypersurface $G = \varphi^{-1}(C_0)$ for some fixed constant C_0, and let g denote the restriction of ψ to G. Then we get the necessary condition that

$$g'(x)^2 = \varphi'(x)^2, \tag{4.8}$$

where $g'(x)^2$ is the square of the norm of the differential for the metric dual to e_0, the induced Euclidean metric. Now (4.8) is a standard eikonal equation on G and we can find solutions of the form $g(x) = \operatorname{dist}(x, \Gamma)$, where Γ is either a point or a hypersurface in G and dist denotes the distance on G with respect to the metric $\varphi'(x)^2 e_0(dx)$. Of course, we will have to be careful, since such distance functions in general will develop singularities, and in the following we restrict G if necessary, so that the function g is smooth. With g solving (4.8), we define ψ to be the extension of g which is constant along the integral curves of the field $\varphi'(x) \cdot \partial_x$:

$$\varphi'(x) \cdot \partial_x \psi(x) = 0,\ \psi_{|G} = g. \tag{4.9}$$

Then the second equation in (4.7) holds by construction, and the first equation is fulfilled at the points of G. In order to verify that equation also away from G, we consider,

$$\varphi'(x) \cdot \partial_x ({\psi'}^2 - {\varphi'}^2) = 2(\langle \psi'' \varphi' | \psi' \rangle - \langle \varphi'' \varphi' | \varphi' \rangle). \tag{4.10}$$

Taking the gradient of the second equation in (4.7), we get $\varphi'' \psi' + \psi'' \varphi' = 0$, and hence

$$\varphi'(x) \cdot \partial_x ({\psi'}^2 - {\varphi'}^2) = -2(\langle \varphi'' \psi' | \psi' \rangle + \langle \varphi'' \varphi' | \varphi' \rangle) = -\frac{1}{2}\{a, b\}(x, \psi') \tag{4.11}$$

$$= -\frac{1}{2} c(x)({\psi'}^2 - {\varphi'}^2) - \ell(x, \psi') \varphi' \cdot \psi'$$

$$= -\frac{1}{2} c(x)({\psi'}^2 - {\varphi'}^2). \tag{4.12}$$

Thus

$$\left(\varphi'(x)\cdot\partial_x + \frac{c(x)}{2}\right)(\psi'^2-\varphi'^2)=0,\ (\psi'^2-\varphi'^2)_{|G}=0, \tag{4.13}$$

and we conclude that $\psi'^2-\varphi'(x)^2=0$.

Summing up the discussion so far, we have seen that if φ is a limiting Carleman weight, and the open set Ω is a union of integral segments of $\varphi'(x)\cdot\partial_x$ all crossing the smooth hypersurface $G\subset\varphi^{-1}(C_0)$, then if g is smooth solution to the eikonal equation (4.8) *on G and we define ψ to be the solution of* (4.9), *we get a solution of* (4.6).

(4.6) implies that

$$p(x,i\varphi'(x)+\psi'(x))=0, \tag{4.14}$$

which is the eikonal equation for the construction of WKB-solutions of the form $u(x;h)=a(x;h)e^{\frac{1}{h}(-\varphi+i\psi)}$ of $P_0u\approx 0$. If we try a smooth and independent of h, we get

$$\begin{aligned} e^{-\frac{1}{h}(-\varphi+i\psi)}P_0e^{\frac{1}{h}(-\varphi+i\psi)}a &= e^{-\frac{i}{h}\psi}P_{0,\varphi}e^{\frac{i}{h}\psi}a\\ &= \big(((hD+\psi'_x)^2-{\varphi'_x}^2)+i(\varphi'(x)(hD+\psi')\\ &\quad +(hD+\psi')\varphi'))\big)a\\ &= (hL-h^2\Delta)a, \end{aligned} \tag{4.15}$$

where L is the transport operator given by

$$L=\psi'D+D\psi'+i(\varphi'D+D\varphi'). \tag{4.16}$$

Along the projection of each bicharacteristic leaf this is an elliptic operator of Cauchy-Riemann type and if we assume that the leaves are open and simply connected, then (see [5]) there exists a nonvanishing smooth function $a\in C^\infty$ such that

$$La=0. \tag{4.17}$$

Recall that $q\in L^\infty(\widetilde{\Omega})$. Assume that a in (4.17) is well-defined in a neighborhood of $\overline{\Omega}$. Then from (4.15), we see that with $P=P_0+h^2q$:

$$Pe^{\frac{1}{h}(-\varphi+i\psi)}a=e^{-\varphi/h}h^2d, \tag{4.18}$$

with $d=\mathcal{O}(1)$ in L^∞ and hence in L^2. Now apply Proposition 4.1 with φ replaced by $-\varphi$, to find $r\in H^1(\Omega)$ with $h\|r\|_{H^1}\le Ch^2$, such that

$$e^{\varphi/h}Pe^{-\varphi/h}e^{i\psi/h}r=-h^2d,$$

i.e.

$$P(e^{\frac{1}{h}(-\varphi+i\psi)}(a+r))=0. \tag{4.19}$$

5. More use of the Carleman estimate

In Section 3 we derived a Carleman estimate for $e^{\varphi/h}u$ when $h^2(-\Delta+q)u = v$ when φ is a smooth limiting Carleman weight with nonvanishing gradient. In order to stick close to the paper [1], we write the corresponding estimate for $e^{-\varphi/h}u$, when $(-\Delta+q)u = v$, $u_{|\partial\Omega} = 0$:

$$\begin{aligned}(5.1)\quad & \frac{h^3}{C_0}((\varphi'_x\cdot\nu)e^{-\varphi/h}\partial_\nu u|e^{-\varphi/h}\partial_\nu u)_{\partial\Omega_+} + \frac{h^2}{C_0}(\|e^{-\varphi/h}u\|^2 + \|e^{-\varphi/h}h\nabla u\|^2) \\ & \qquad \le h^4\|e^{-\varphi/h}v\|^2 - C_0h^3((\varphi'_x\cdot\nu)e^{-\varphi/h}\partial_\nu u|e^{-\varphi/h}\partial_\nu u)_{\partial\Omega_-},\end{aligned}$$

where ν is the exterior unit normal and $\Omega_\pm = \{x\in\partial\Omega;\ \pm\nu\cdot\varphi' > 0\}$.

Let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials. Let

$$(5.2)\quad u_2 = e^{\frac{1}{h}(\varphi+i\psi_2)}(a_2 + r_2(x;h)),\ \text{with } (\Delta - q_2)u_2 = 0,\ \|r_2\|_{H^1} = \mathcal{O}(h).$$

Here ψ_2 is chosen as in Section 4 so that $(\varphi')^2 = (\psi_2')^2 = \varphi'\cdot\psi_2' = 0$ and so that the integral leaves of the commuting vector fields $\varphi'\cdot\partial_x, \psi'\cdot\partial_x$ are simply connected in Ω. a_2 is smooth in a neighborhood of $\overline{\Omega}$ and everywhere nonvanishing.

Let $\mathcal{N}_q$ be the Dirichlet to Neumann map for the potential q and let

$$\begin{aligned}\partial\Omega_{-,\varepsilon_0} &= \{x\in\partial\Omega;\ \nu(x)\cdot\varphi'_x(x) < \varepsilon_0\},\\ \partial\Omega_{+,\varepsilon_0} &= \{x\in\partial\Omega;\ \nu(x)\cdot\varphi'_x(x) \ge \varepsilon_0\},\end{aligned}$$

for some fixed $\varepsilon_0 > 0$, so that $\partial\Omega_{+,\varepsilon_0} \subset \partial\Omega_+$, $\partial\Omega_- \subset \partial\Omega_{-,\varepsilon_0}$. Here $\nu(x)$ denotes the unit outer normal to $\partial\Omega$.

Assume

$$(5.3)\qquad \mathcal{N}_{q_1}(f) = \mathcal{N}_{q_2}(f),\ \text{in } \partial\Omega_{-,\varepsilon_0},\ \text{for all } f\in H^{\frac12}(\partial\Omega).$$

Let $u_1 \in H^1(\Omega)$ solve

$$(5.4)\qquad (\Delta - q_1)u_1 = 0,\ u_{1|\partial\Omega} = u_{2|\partial\Omega}.$$

Then by the assumption (5.3), we have

$$(5.5)\qquad \partial_\nu u_1 = \partial_\nu u_2 \text{ in } \partial\Omega_{-,\varepsilon_0}.$$

Put $u = u_1 - u_2$, $q = q_2 - q_1$, so that

$$(5.6)\qquad \operatorname{supp}(\partial_\nu u_{|\partial\Omega}) \subset \partial\Omega_{+,\varepsilon_0},$$

and

$$(5.7)\qquad (\Delta - q_1)u = (\Delta - q_1)u_2 = qu_2,\ u_{|\partial\Omega} = 0.$$

For $v\in H^1(\Omega)$ with $\Delta v\in L^2(\Omega)$, we get using (5.6),(5.7) and Green's formula:

$$(5.8)\quad \int_\Omega qu_2\overline{v}dx = \int_\Omega(\Delta - q_1)u\overline{v}dx = \int_\Omega u\overline{(\Delta - \overline{q_1})v}dx + \int_{\partial\Omega_{+,\varepsilon_0}}(\partial_\nu u)\overline{v}\,S(dx).$$

As in Section 4 we can construct

$$v = e^{-\frac{1}{h}(\varphi + i\psi_1)}(a_1 + r_1), \tag{5.9}$$

with ψ_1 satisfying $\varphi' \cdot \psi_1' = 0$, $(\varphi')^2 = (\psi_1')^2$, with $a_1(x)$ nonvanishing and smooth, and with $\|r_1\|_{H^1(\Omega)} = \mathcal{O}(h)$, so that

$$(\Delta - \overline{q}_1)v = 0. \tag{5.10}$$

Then (5.8) becomes

$$\int_\Omega q e^{\frac{i}{h}(\psi_1+\psi_2)}(a_2 + r_2)\overline{(a_1 + r_1)}dx = \int_{\partial\Omega_{+,\varepsilon_0}} (\partial_\nu u) e^{-\frac{1}{h}(\varphi - i\psi_1)}\overline{(a_1 + r_1)}\, S(dx). \tag{5.11}$$

We shall work with ψ_1, ψ_2, φ slightly h-dependent in such a way that

$$\frac{1}{h}(\psi_1 + \psi_2) \to f,\ \ h \to 0. \tag{5.12}$$

Recall that

$$\|r_j\|_{H^1} = \mathcal{O}(h). \tag{5.13}$$

Then using that $q \in L^\infty$, we see that the left-hand side of (5.11) converges to

$$\int_\Omega a_2 \overline{a}_1 q(x) e^{if(x)} dx. \tag{5.14}$$

For the right-hand side of (5.11), we have, using (5.1), for $(\Delta - q_1)$ and (5.7):

$$\begin{aligned}
&|\int_{\partial\Omega_{+,\varepsilon_0}} (\partial_\nu u) e^{-\frac{1}{h}(\varphi - i\psi_1)}(\overline{a_1 + r_1})\, S(dx)|^2 \\
&\leq \|a_1 + r_1\|^2_{\partial\Omega_{+,\varepsilon_0}} \int_{\partial\Omega_{+,\varepsilon_0}} (e^{-\varphi/h}|\partial_\nu u|)^2 S(dx) \\
&\leq \|a_1 + r_1\|^2_{\partial\Omega_{+,\varepsilon_0}} \frac{1}{\varepsilon_0} \int_{\partial\Omega_{+,\varepsilon_0}} (\varphi' \cdot \nu)(e^{-\varphi/h}|\partial_\nu u|)^2 S(dx) \\
&\leq \frac{1}{\varepsilon_0}\|a_1 + r_1\|^2_{\partial\Omega_{+,\varepsilon_0}} (C_0 h \|e^{-\varphi/h} q u_2\|^2 - C_0^2 \int_{\partial\Omega_-} (\varphi' \cdot \nu)(e^{-\varphi/h}|\partial_\nu u|)^2 S(dx)).
\end{aligned} \tag{5.15}$$

Here $\partial_\nu u = 0$ on $\partial\Omega_-$, and using also (5.2), we get

$$|\int_{\partial\Omega_{+,\varepsilon_0}} (\partial_\nu u) e^{-\frac{1}{h}(\varphi - i\psi_1)}(\overline{a_1 + r_1}) S(dx)|^2 \leq \frac{C_0 h}{\varepsilon_0}\|a_1 + r_1\|^2_{\partial\Omega_{+,\varepsilon_0}} \|q(a_2 + r_2)\|^2. \tag{5.16}$$

Here $\|q(a_2+r_2)\|^2 = \mathcal{O}(1)$, by (5.13). Since $r_1 = \mathcal{O}(h)$ in the semiclassical H^1-norm, we have $r_1 = \mathcal{O}(1)$ in the standard $(h = 1)$ H^1-norm. Hence

$$r_{1|_{\partial\Omega}} = \mathcal{O}(1) \text{ in } L^2. \tag{5.17}$$

Consequently, the right-hand side of (5.11) tends to 0, when $h \to 0$, and letting $h \to 0$ there, we get

$$\int_{\Omega} q(x)a_2(x)\overline{a}_1(x)e^{if(x)}dx = 0, \tag{5.18}$$

for all f that can be attained as limits in (5.12).

Finally, we remark that if φ is real-analytic, then in the above constructions, we may arrange so that ψ_j and a_j have the same property.

6. End of the proof of Theorem 1.2

From now on, we assume that the dimension n is ≥ 3. We choose $\varphi(x) = \ln|x - x_0|$ for x_0 varying in a small open set separated from $\overline{\Omega}$ by some fixed affine hyperplane H. Notice that φ is a limiting Carleman weight in the sense of (3.4). We need a sufficiently rich family of functions f in (5.18) and recall that these functions are the ones that appear in (5.12) with ψ_j analytic near $\overline{\Omega}$ and satisfying $(\psi_j')^2 = (\varphi')^2$, $\psi_j' \cdot \varphi' = 0$. Changing the sign of ψ_2 we can also view f as a limit $\frac{1}{h}(\psi_1 - \psi_2)$ for suitable such h-dependent functions ψ_j. More precisely, we can take an analytic family $\psi(x, \alpha)$ depending on the additional parameters $\alpha = (\alpha_1, \dots, \alpha_k)$, with $\psi(\cdot, \alpha)$ satisfying

$$(\psi_x')^2 = (\varphi_x')^2,\ \psi_x' \cdot \varphi_x' = 0, \tag{6.1}$$

and then take

$$f(x) = \langle \psi_\alpha'(x, \alpha), \nu(\alpha)\rangle, \tag{6.2}$$

where $\nu(\alpha)$ is a tangent vector in the α-variables.

We first discuss the choice of ψ. Since φ_x' is radial, with respect to x_0, the second condition in (6.1) means that $\psi(x)$ is positively homogeneous of degree 0 with respect to $x - x_0$. A necessary and sufficient condition for ψ (at least if we work in some cone with vertex at x_0) is then that

$$(\psi_x')^2 = (\varphi_x')^2, \tag{6.3}$$

on a suitable open subset $x_0 + r_0 W$ of $x_0 + r_0 S^{n-1}$, for some fixed $r_0 > 0$. The necessity is obvious and the sufficiency follows easily by extending ψ to be a positively homogeneous function of degree 0 in the variables $x - x_0$.

Here is an explicit choice of a suitable open set in (6.3): Let $r_0 > 0$ be large enough so that $\overline{\Omega} \subset B(x_0, r_0)$. Let $x_0 + r_0 W \subset \partial B(x_0, r_0)$ be defined by

$$x_0 + r_0 W = \partial B(x_0, r_0) \cap H_+, \tag{6.4}$$

where H_+ is the open half-space delimited by the affine hyper-plane H, for which $x_0 \notin H_+$ (so that $\overline{\Omega} \subset H_+$). Then $\overline{\Omega}$ is contained in the open cone $x_0 + \mathbf{R}_+ W$, so if we choose ψ on $x_0 + r_0 W$ as in (6.3) and extend by homogeneity, we know that ψ will be smooth near $\overline{\Omega}$.

Let $y_0 \in \partial B(0,1) \setminus \overline{W}$ be such that the antipodal point $-y_0$ also is outside $\overline{W}$ and define

$$\psi(x,y) = d_{S^{n-1}}(x,y). \tag{6.5}$$

Then $\psi \in C^\infty(W \times \mathrm{neigh}\,(y_0))$ and the function $\psi((x-x_0)/|x-x_0|, y) \in C^\infty(\overline{\Omega} \times \mathrm{neigh}\,(y_0))$ will satisfy (6.1). Since the domain of definition does not contain antipodal points, we remark that

$$\psi''_{x,y} \text{ is of rank } n-2 \text{ and } \mathcal{R}(\psi''_{x,y}) = (\psi'_x)^\perp, \mathcal{N}(\psi''_{x,y}) = (\psi'_y). \tag{6.6}$$

This follows from basic properties of the geodesic flow (and remains true more generally for $\psi(x,y) = d(x,y)$ on a Riemannian manifold as long as x, y are not conjugate points.)

For $x \in W \subset S^{n-1}$, $(y,\nu) \in TS^{n-1}$, $y \in \mathrm{neigh}\,(y_0)$, we put

$$\widetilde{f}(x; y, \nu) = \psi'_y(x,y) \cdot \nu. \tag{6.7}$$

Then

$$\widetilde{f}'_x(x; y, \nu) = \psi''_{x,y}(x,y)(\nu). \tag{6.8}$$

In view of (6.6), we see that this vanishes precisely when $\nu \parallel \psi'_y(x,y)$, i.e. when ν is parallel to the (arrival) direction of the minimal geodesic from x to y. Restricting ν to nonvanishing directions which are close to be parallel to the plane H, we can assure that

$$\widetilde{f}'_x(x; y, \nu) \neq 0. \tag{6.9}$$

LEMMA 6.1. *$\widetilde{f}''_{x,(y,\nu)}$ has maximal rank $n-1$.*

Proof. We already know that $\widetilde{f}''_{x,\nu} = \psi''_{x,y}$ is of rank $n-2$ and that the image of this matrix is equal to $(\psi'_x)^\perp$. Consequently, we consider

$$g(y) = \psi'_x(x,y_0) \cdot \psi''_{x,y}(x,y)(\nu) = \langle \psi''_{x,y}(x,y) | \psi'_x(x,y_0) \otimes \nu \rangle$$

as a function of $y \in \mathrm{neigh}\,(y_0)$. The function vanishes for $y = y_0$ and can also be written

$$\begin{aligned} &\langle \psi''_{x,y}(x,y) | (\psi'_x(x,y_0) - \psi'_x(x,y)) \otimes \nu \rangle \\ &\qquad = \langle \psi''_{x,y}(x,y)(\nu) | \psi''_{x,y}(x,y)(y_0 - y) \rangle + \mathcal{O}((y_0-y)^2). \end{aligned}$$

From this expression, we see that the y-differential is nonvanishing and hence the range of $\widetilde{f}''_{x,(y,\nu)}$ contains vectors that are not orthogonal to $\psi'_x(x,y)$. $\square$

Now consider

$$\Psi(x; y, \widetilde{x}) = \psi\Big(\frac{x - \widetilde{x}}{|x - \widetilde{x}|}, y\Big) \in C^\infty(\overline{\Omega} \times \mathrm{neigh}\,(y_0, S^{n-1}) \times \mathrm{neigh}\,(x_0, \mathbf{R}^n)). \tag{6.10}$$

Ψ is analytic, real and satisfies (6.1) with $\varphi(x) = \Phi(x, \widetilde{x}) = \ln|x - \widetilde{x}|$. We can take $\alpha = y$ and (6.2) becomes

$$f(x) = f(x;\theta) = \langle \Psi'_y | \nu \rangle,\ \ \theta = (y, \widetilde{x}, \nu), \tag{6.11}$$

with $(y, \nu) \in TS^{n-1}$. Lemma 6.1 shows that $f''_{x,(y,\nu)}$ has rank $n-1$ and indeed the image of this matrix is the tangent space of $\partial B(\widetilde{x}, |x - \widetilde{x}|)$ at x. Since f'_x is a nonvanishing element of $T_x(\partial B(\widetilde{x}, |x - \widetilde{x}|))$, we can vary $\widetilde{x}$ infinitesimally to see that $f''_{x,\widetilde{x}}(\widetilde{\mu}) \notin T_x(\partial B(\widetilde{x}, |x - \widetilde{x}|))$ for a suitable $\widetilde{\mu} \in \mathbf{R}^n$. It is then clear that

$$f''_{x,\theta} = f''_{x,(y,\widetilde{x},\nu)} \text{ has maximal rank } n, \tag{6.12}$$

and hence that the map

$$\operatorname{neigh}(\overline{\Omega}) \ni x \mapsto f'_\theta(x, \theta) \in \mathbf{R}^{3n-2} \tag{6.13}$$

has injective differential.

Lemma 6.2. *The map* (6.13) *is injective.*

Proof. Let $x_1, x_2 \in \operatorname{neigh}(\overline{\Omega})$ be two points with

$$f'_\theta(x_1, \theta) = f'_\theta(x_2, \theta), \tag{6.14}$$

for some $\theta = (y, \widetilde{x}, \nu)$. Taking the ν-component of this relation, we get

$$\psi'_y(\widetilde{x}_1, y) = \psi'_y(\widetilde{x}_2, y),\ \ \widetilde{x}_j = \frac{x_j - \widetilde{x}}{|x_j - \widetilde{x}|}. \tag{6.15}$$

This means that $\widetilde{x}_1$, $\widetilde{x}_2$, y belong to the same geodesic γ and this geodesic is minimal (i.e. distance minimizing) on some segment that contains these three points in its interior. If $\widetilde{x}_1 \neq \widetilde{x}_2$, we may assume that $d(\widetilde{x}_2, y) < d(\widetilde{x}_1, y)$. For $y \in \operatorname{neigh}(y_0, S^{n-1})$, we have

$$d(\widetilde{x}_1, \widetilde{x}_2) + d(\widetilde{x}_2, y) - d(\widetilde{x}_1, y) =: g(y),\ \ g(y) \sim d(y, \gamma)^2.$$

It follows that

$$f(\widetilde{x}_2; y, \widetilde{x}, \nu) - f(\widetilde{x}_1; y, \widetilde{x}, \nu) = g'(y) \cdot \nu,$$

and using that ν is not parallel to $\dot{\gamma}$ at y_0, we see that this function has a nonvanishing y-gradient at $y = y_0$, in contradiction with (6.14). Thus, $\widetilde{x}_1 = \widetilde{x}_2$, or in other words, x_1 and x_2 belong to the same half-ray through $\widetilde{x}$.

Taking the $\widetilde{x}$-component of (6.14), we get

$$\nabla_{\widetilde{x}} \langle \psi'_y(\frac{x - \widetilde{x}}{|x - \widetilde{x}|}, y), \nu \rangle_{|x=x_1} = \nabla_{\widetilde{x}} \langle \psi'_y(\frac{x - \widetilde{x}}{|x - \widetilde{x}|}, y), \nu \rangle_{|x=x_2}.$$

These quantities are clearly nonvanishing and if $x_1 \neq x_2$, they differ by a factor $\neq 1$, since $\nabla_{\widetilde{x}}(\frac{x-\widetilde{x}}{|x-\widetilde{x}|})$ is homogeneous of degree -1 in $x - \widetilde{x}$. Thus $x_1 = x_2$. $\square$

Now apply (5.18) with $f(x) = f(x, \theta)$:

$$\int e^{if(x,\theta)} a_2 \overline{a}_1 q(x) dx = 0, \tag{6.16}$$

where a_2, a_1 are analytic nonvanishing functions of $x, y, \widetilde{x}$ in a neighborhood of $\overline{\Omega} \times \{y_0\} \times \{x_0\}$. Since $f(x, \theta) = f(x; y, \widetilde{x}, \nu)$ depends linearly on ν, we can replace ν by $\lambda\nu$ and get

$$\int e^{i\lambda f(x,\theta)} a_2 \overline{a}_1 q(x) dx = 0, \ \lambda \geq 1. \tag{6.17}$$

Now represent θ by some analytic real coordinates $\theta_1, \theta_2, \ldots, \theta_N$ near some fixed given point $\theta_0 = (y_0, x_0, \nu_0)$. If $x, z \in \overline{\Omega}$, $w \in \mathrm{neigh}\,(\theta_0)$, we consider the function

$$\theta \mapsto -f(z, \theta) + f(x, \theta) + \frac{i}{2}(\theta - w)^2. \tag{6.18}$$

For $x = z$, we have the unique nondegenerate critical point $\theta = w$, while for $x \neq z$ there is no real critical point in view of Lemma 6.2. For $x \approx z$ we have a unique complex critical point which is close to w, and we introduce the corresponding critical value

$$\psi(z, x, w) = \mathrm{v.c.}_\theta \left(-f(z, \theta) + f(x, \theta) + \frac{i}{2}(\theta - w)^2 \right). \tag{6.19}$$

From (6.13) and standard estimates on critical values in connection with the complex stationary phase method ([11, 13]), we deduce that

$$\mathrm{Im}\, \psi(z, x, w) \sim (z - x)^2, \ z, x \in \overline{\Omega}, \ z \approx x. \tag{6.20}$$

Moreover, when $x = z$, we have

$$\psi'_z(z, z, w) = -f'_z(z, w), \ \psi'_x(z, z, w) = f'_z(z, w), \ \psi(z, z, w) = 0. \tag{6.21}$$

We now multiply (6.17) by $\chi(\theta - w) e^{i\lambda \frac{i}{2}(\theta - w)^2 - i\lambda f(z,\theta)}$, and integrate with respect to θ, to get

$$\int e^{i\lambda\psi(z,x,w)} a(z, x, w; \lambda) \chi(z - x) q(x) dx = \mathcal{O}(e^{-\frac{\lambda}{C}}). \tag{6.22}$$

Here χ denote (different) standard cutoffs to a neighborhood of 0, and a is an elliptic classical analytic symbol of order 0.

Now restrict w to an n-dimensional manifold Σ which passes through θ_0, and write $(z, -f'_z(z, \theta)) = (\alpha_x, \alpha_\xi) = \alpha$. Then we rewrite (6.22) as

$$\int e^{i\lambda\psi(\alpha,x)} a(\alpha, x; h) \chi(\alpha_x - x) q(x) dx = \mathcal{O}(e^{-\frac{\lambda}{C}}), \tag{6.23}$$

implying that

$$(z, -f'_z(z, \theta_0)) \notin \mathrm{WF}_a(q), \tag{6.24}$$

since we can apply the standard FBI-approach ([13]). Notice that (6.20), (6.21) give:

$$\psi(\alpha,x)=(\alpha_x-x)\cdot\alpha_\xi+\mathcal{O}((\alpha_x-x)^2),\ \operatorname{Im}\psi(\alpha,x)\sim(\alpha_x-x)^2, \tag{6.25}$$

and we can choose Σ so that the map neigh $(z_0)\times\Sigma\ni(z,\theta)\mapsto(z,-f'_z(z,\theta))$ is local diffeomorphism near any given fixed point $z_0\in\overline{\Omega}$.

End of the proof of Theorem 1.2. Fix θ_0 as above, so that $0\neq -f'_z(z,\theta_0)\notin \mathrm{WF}_a(q)$ for all z in some neighborhood of $\overline{\Omega}$. (Notice that q now denotes the extension by 0 of the originally defined function on Ω.) Let z_0 be a point in supp (q), where $f(\cdot,\theta_0)_{|\mathrm{supp}\,(q)}$ is minimal. Then $-f'_z(z_0,\theta_0)$ belongs to the exterior conormal cone of supp (q) at z_0 and we get a contradiction between (6.24) and the fact that all such exterior conormal directions have to belong to $WF_a(q)$. (This is the wavefront version of Holmgren's uniqueness theorem, due to Hörmander ([7]) and Sato-Kawai-Kashiwara (remark by Kashiwara in [12]).) □

7. Complex geometrical optics solutions with Dirichlet data on part of the boundary

In this section we prove Theorem 1.1.

We first use the Carleman estimate (3.36) and the Hahn-Banach theorem to construct CGO solutions for the conjugate operator $P_\varphi^*=(e^{\frac{\varphi}{h}}Pe^{-\frac{\varphi}{h}})^*$ where $*$ denotes the adjoint. Notice that P_φ^* has the same form as P_φ except that q is replaced by $\bar q$ and φ by $-\varphi$.

PROPOSITION 7.1. *Let* φ *be as in* (3.36). *Let* $v\in H^{-1}(\Omega)$,

$$v_-\in L^2(\partial\Omega_-;(-\varphi'\cdot\nu)S(dx)).$$

Then $\exists\, u\in H^0(\Omega)$ *such that*

$$P_\varphi^* u=v,\quad u\big|_{\partial\Omega_-}=v_-.$$

Moreover

$$\|u\|_{H^0}+\sqrt{h}\|(\varphi'\cdot\nu)^{-\frac12}u\|_{\partial\Omega_+}\le C\left(\frac1h\|v\|_{H^{-1}}+\sqrt{h}\|(-\varphi'\cdot\nu)^{-\frac12}v_-\|_{\partial\Omega_-}\right). \tag{7.1}$$

Proof. We use the Carleman estimate (3.36). Let v as in the proposition. For $w\in(H_0^1\cap H^2)(\Omega)$ we have

$$|(w|v)_\Omega+(h\partial_\nu w|v_-)_{\partial\Omega_-}|\le\|w\|_{H^1}\|v\|_{H^{-1}}+\left((-\varphi'\cdot\nu)^{\frac12}h\partial_\nu w|(-\varphi'\cdot\nu)^{-\frac12}v_-\right)_{\partial\Omega_-}.$$

Therefore

$$
\begin{aligned}
&|(w,v)_\Omega + (h\partial_\nu w|v_-)_{\partial\Omega_-}| \\
&\le C\left(\frac{1}{h}\|v\|_{H^{-1}} h\|w\|_{H^1} + \frac{1}{\sqrt{h}}\|(-\varphi'\cdot\nu)^{-\frac12}v_-\|_{\partial\Omega_-}\sqrt{h}\|(-\varphi'\cdot\nu)^{\frac12}h\partial_\nu w\|_{\partial\Omega_-}\right).
\end{aligned}
$$

Now by using (3.36) we get

$$
\begin{aligned}
&|(w,v)_\Omega + (h\partial_\nu w|v_-)_{\partial\Omega_-}| \\
&\le C\left(\frac{1}{h}\|v\|_{H^{-1}} + \frac{1}{\sqrt{h}}\|(-\varphi'\cdot\nu)^{-\frac12}v_-\|_{\partial\Omega_-})(\|P_\varphi w\| + \sqrt{h}\|(\varphi'\cdot\nu)^{\frac12}h\partial_\nu w\|_{\partial\Omega_+}\right).
\end{aligned}
$$

By the Hahn-Banach theorem, $\exists\, u \in H^0(\Omega)$, $u_+ \in L^2(\partial\Omega_+, (\varphi'\cdot\nu)^{-\frac12}dS)$, u_+ on $\partial\Omega_+$ such that

(7.2)
$$
(w,v)_\Omega + (h\partial_\nu w|v_-)_{\partial\Omega_-} = (P_\varphi w|u) + (h\partial_\nu w|u_+)_{\partial\Omega_+}, \quad \forall\, w \in (H_0^1\cap H^2)(\Omega)
$$

with

(7.3)
$$
\|u\|_{H^0} + \frac{1}{\sqrt{h}}\|(\varphi'\cdot\nu)^{-\frac12}u_+\|_{\partial\Omega_+} \le C\left(\frac{1}{h}\|v\|_{H^{-1}} + \frac{1}{\sqrt{h}}\|(-\varphi'\cdot\nu)^{-\frac12}v_-\|_{\partial\Omega_-}\right).
$$

Since $P_\varphi = -h^2\Delta +$ a first order operator, and $w\big|_{\partial\Omega} = 0$ we have $(P_\varphi w|u) = (w|P_\varphi^* u) - h^2(\partial_\nu w|u)_{\partial\Omega}$.

Using this in (7.2) we obtain

$$
0 = (w|v - P_\varphi^* u) + h((\partial_\nu w|1_{\partial\Omega_-}v_-)_{\partial\Omega} - (\partial_\nu w|1_{\partial\Omega_+}u_+)_{\partial\Omega} + (\partial_\nu w|hu)_{\partial\Omega})
$$

where $1_{\partial\Omega_\pm}$ denotes the indicator function of $\partial\Omega_\pm$.

By varying w in $(H_0^1\cap H^2)(\Omega)$ we get

$$
P_\varphi^* u = v, \quad hu\big|_{\partial\Omega} = -1_{\partial\Omega_-}v_- + 1_{\partial\Omega_+}u_+.
$$

which implies the proposition after replacing v_- above by $-hv_-$. □

Let $W_- \subset \partial\Omega_-$ be an arbitrary strict open subset of $\partial\Omega_-$. We next want to modify the choice of u_2 in (5.2) so that $u_2\big|_{W_-} = 0$.

Proposition 7.2. *Let a_2, φ, ψ_2 be as in* (5.2). *Then we can construct a solution of*

(7.4)
$$
P\widetilde{u}_2 = 0, \quad \widetilde{u}_2\big|_{\overline{W}_-} = 0
$$

of the form

(7.5)
$$
\widetilde{u}_2 = e^{\frac1h(\varphi + i\psi_2)}(a_2 + \widetilde{r}_2) + u_r
$$

where $u_r = e^{i\frac{l}{h}}b(x;h)$ *with* b *a symbol of order zero in* h *and*

$$\operatorname{Im} l(x) = -\varphi(x) + k(x) \tag{7.6}$$

where $k(x) \sim \operatorname{dist}(x, \partial\Omega_-)$ *in a neighborhood of* $\partial\Omega_-$ *and* b *has its support in that neighborhood. Moreover,* $\|\widetilde{r}_2\|_{H^0} = \mathcal{O}(h)$, $\widetilde{r}_2\big|_{\partial\Omega_-} = 0$, $\|(\varphi'\cdot\nu)^{-\frac{1}{2}}\widetilde{r}_2\|_{\partial\Omega_+} = \mathcal{O}(h^{\frac{1}{2}})$.

Proof. We start by constructing a WKB solution u in Ω of

$$-h^2\Delta u = 0, u\big|_{\partial\Omega_-} = e^{\frac{1}{h}(\varphi+i\psi_2)}(\chi a_2)\big|_{\partial\Omega_-} \tag{7.7}$$

where $\chi \in C_0^\infty(\partial\Omega_-)$, $\chi = 1$ on $\bar{W}_-$.

We try $u = e^{\frac{i}{h}l(x)}b(x;h)$. The eikonal equation for l is

$$\begin{aligned} (l')^2 &= 0 \text{ to infinite order at } \partial\Omega \\ l\big|_{\partial\Omega_-} &= \psi_2 - i\varphi. \end{aligned} \tag{7.8}$$

Of course $g := \psi_2 - i\varphi$ is a solution but we look for the second solution, corresponding to having u equal to a “reflected wave”. We decompose on $\partial\Omega_-$

$$g' = g'_t + g'_\nu$$

where t denotes the tangential part and ν the normal part.

Then in order to satisfy the eikonal equation we need

$$0 = (g'_t)^2 + (g'_\nu)^2.$$

Therefore we can solve (7.8) to ∞-order at $\partial\Omega_-$ with l satisfying

$$l\big|_{\partial\Omega_-} = g\big|_{\partial\Omega_-}, \partial_\nu l\big|_{\partial\Omega_-} = -\partial_\nu g\big|_{\partial\Omega_-}.$$

By the definition of $\partial\Omega_-$ we have

$$\partial_\nu \operatorname{Im} g = -\partial_\nu\varphi > 0 \text{ on } \partial\Omega_-.$$

Since ν is the unit exterior normal we have that (7.6) is satisfied.

Solving also the transport equation to ∞-order, at the boundary we get a symbol b of order 0 with support arbitrarily close to suppχ, such that

$$\begin{cases} -h^2\Delta(e^{\frac{il}{h}}b(x;h)) &= e^{\frac{il}{h}}\mathcal{O}((\operatorname{dist}(x,\partial\Omega))^\infty + h^\infty) \\ e^{\frac{il}{h}}\Big|_{\partial\Omega} &= e^{\frac{ig}{h}}\chi a_2\Big|_{\partial\Omega}. \end{cases}$$

Our new WKB input to u_2 will be

$$(e^{\frac{ig}{h}}a_2 - e^{i\frac{l}{h}}b).$$

Instead of (4.18) we get

$$P(e^{i\frac{g}{h}}a_2 - e^{\frac{il}{h}}b) = e^{\frac{\varphi}{h}}h^2 d \tag{7.9}$$

where $d = \mathcal{O}(1)$ in $L^2(\Omega)$.

Using Proposition 7.1 we can solve

$$e^{-\frac{\varphi}{h}}Pe^{\frac{\varphi}{h}}(e^{i\frac{\psi_2}{h}}\widetilde{r}_2) = -h^2 d$$
$$\widetilde{r}_2|_{\partial\Omega_-} = 0$$

with

$$\|\widetilde{r}_2\|_{H^0} + \sqrt{h}\|(\varphi'\cdot\nu)^{-\frac{1}{2}}\widetilde{r}_2\|_{\partial\Omega_+} \le \frac{C}{h}\|h^2 d\|_{H^{-1}} = \mathcal{O}(h).$$

Thus

$$\|\widetilde{r}_2\| = \mathcal{O}(h), \quad \|(\varphi'\cdot\nu)^{-\frac{1}{2}}\widetilde{r}_2\|_{\partial\Omega_+} = \mathcal{O}(\sqrt{h}). \tag{7.10}$$

Now we take

$$u_2 = e^{\frac{1}{h}(\varphi+i\psi_2)}(a_2+\widetilde{r}_2) - e^{\frac{il}{h}}b. \tag{7.11}$$

Clearly $Pu_2 = 0$, $u_2\big|_{\partial\Omega} = 0$ in $\overline{W}_-$. □

Proof of Theorem 1.1. Let $u_2 = \widetilde{u}_2$ be as in Prop 7.2. Let $u_1 \in H_\Delta(\Omega)$ (see [1]) solve (5.4)

$$(\Delta - q_1)u_1 = 0, \quad u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega} \qquad (\mu_1 \in H_\Delta(\Omega) \text{ since } \mu_2 \text{ does}).$$

By construction we have that $\operatorname{supp} u_i\big|_{\partial\Omega}\cap\overline{W_-} = \emptyset$, $i = 1, 2$. As in Section 5, let $u = u_1 - u_2$, $q = q_1 - q_2$. Then (5.6) and (5.7) are valid and in fact $u \in H^2(\Omega)$ so that the Green's formula (5.8) is also valid. Now choose v as in (5.9), (5.10). Then instead of (5.11) we get

$$\int_\Omega qe^{\frac{i}{h}(\psi_1+\psi_2)}\overline{(a_1+r_1)}(a_2+\widetilde{r}_2)dx - \int_\Omega qe^{\frac{il}{h}-\frac{\varphi}{h}+i\frac{\psi_1}{h}}b(\overline{a_1+r_1})dx = \int_{\partial\Omega_{+,\varepsilon_0}}(\partial_\nu u)e^{-\frac{1}{h}(\varphi-i\psi_1)}\overline{(a_1+r_1)}dS. \tag{7.12}$$

The second term of the LHS is what is different from (5.11). Because of (7.6) this term goes to 0 as h goes to zero, since

$$|e^{\frac{il}{h}-\frac{\varphi}{h}+\frac{i\psi_1}{h}}| = e^{-\frac{k(x)}{h}},$$

and q, b, a_1, are bounded and $\|r_1\|_{H^0} \to 0$, $h \to 0$. Therefore we get, instead of (5.16),

$$\left|\int_{\partial\Omega_{+,\varepsilon_0}}(\partial_\nu u)e^{-\frac{1}{h}(\varphi-i\psi_1)}(\overline{a_1+r_1})S(dx)\right|^2 \le \frac{Coh}{\varepsilon}\|a_1+r_1\|^2_{\partial\Omega_{+,\varepsilon_0}}\|e^{-\frac{\varphi}{h}}qu_2\|^2.$$

The previous estimates imply that

$$\|e^{-\frac{\varphi}{h}}qu_2\|, \ \|a_1+r_1\|_{\partial\Omega_{+,\varepsilon_0}} = \mathcal{O}(1).$$

Consequently the RHS of (7.12) tends to 0 as $h \to 0$ and we get (5.18) as before, namely

$$\int_\Omega q(x)a_2(x)\overline{a_1(x)}e^{if(x)}dx = 0. \tag{7.13}$$

Now the arguments of Section 6 imply that $q = 0$ finishing the proof of Theorem 1.1. $\square$

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