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Monopoles and lens space surgeries

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Abstract

Monopole Floer homology is used to prove that real projective three-space cannot be obtained from Dehn surgery on a nontrivial knot in the three-sphere. To obtain this result, we use a surgery long exact sequence for monopole Floer homology, together with a nonvanishing theorem, which shows that monopole Floer homology detects the unknot. In addition, we apply these techniques to give information about knots which admit lens space surgeries, and to exhibit families of three-manifolds which do not admit taut foliations.

1. Introduction

Let K be a knot in S^3. Given a rational number r, let $S^3_r(K)$ denote the oriented three-manifold obtained from the knot complement by Dehn filling with slope r. The main purpose of this paper is to prove the following conjecture of Gordon (see [18], [19]):

Theorem 1.1. *Let U denote the unknot in S^3, and let K be any knot. If there is an orientation-preserving diffeomorphism $S^3_r(K) \cong S^3_r(U)$ for some rational number r, then $K = U$.*

To amplify the meaning of this result, we recall that $S^3_r(U)$ is the manifold $S^1 \times S^2$ in the case $r = 0$ and is a lens space for all nonzero r. More specifically, with our conventions, if $r = p/q$ in lowest terms, with $p > 0$, then $S^3_r(U) = L(p, q)$ as oriented manifolds. The manifold $S^3_{p/q}(K)$ in general has first homology group $\mathbb{Z}/p\mathbb{Z}$, independent of K. Because the lens space $L(2, q)$

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is $\mathbb{RP}^3$ for all odd q, the theorem implies (for example) that $\mathbb{RP}^3$ cannot be obtained by Dehn filling on a nontrivial knot.

Various cases of the Theorem 1.1 were previously known. The case $r = 0$ is the "Property R" conjecture, proved by Gabai [15], and the case where r is nonintegral follows from the cyclic surgery theorem of Culler, Gordon, Luecke, and Shalen [7]. The case where $r = \pm 1$ is a theorem of Gordon and Luecke; see [20] and [21]. Thus, the advance here is the case where r is an integer with $|r| > 1$, though our techniques apply for any nonzero rational r. In particular, we obtain an independent proof for the case of the Gordon-Luecke theorem. (Gabai's result is an ingredient of our argument.)

The proof of Theorem 1.1 uses the Seiberg-Witten monopole equations, and the monopole Floer homology package developed in [23]. Specifically, we use two properties of these invariants. The first key property, which follows from the techniques developed in [25], is a nonvanishing theorem for the Floer groups of a three-manifold admitting a taut foliation. When combined with the results of [14], [15], this nonvanishing theorem shows that Floer homology can be used to distinguish $S^1 \times S^2$ from $S^3_0(K)$ for nontrivial K. The second property that plays a central role in the proof is a surgery long exact sequence, or exact triangle. Surgery long exact sequences of a related type were introduced by Floer in the context of instanton Floer homology; see [5] and [12]. The form of the surgery long exact sequence which is used in the topological applications at hand is a natural analogue of a corresponding result in the Heegaard Floer homology of [35] and [34]. In fact, the strategy of the proof presented here follows closely the proof given in [33].

Given these two key properties, the proof of Theorem 1.1 has the following outline. For integral p, we shall say that a knot K is *p-standard* if $S^3_p(K)$ cannot be distinguished from $S^3_p(U)$ by its Floer homology groups. (A more precise definition is given in Section 3; see also Section 6.) We can rephrase the nonvanishing theorem mentioned above as the statement that, if K is 0-standard, then K is unknotted. A surgery long exact sequence, involving the Floer homology groups of $S^3_{p-1}(K)$, $S^3_p(K)$ and S^3, shows that if K is p-standard for $p > 0$, then K is also $(p-1)$ standard. By induction, it follows that if K is p-standard for some $p > 0$, then $K = U$. This gives the theorem for positive integers p. When $r > 0$ is nonintegral, we prove (again by using the surgery long exact sequence) that if $S^3_r(K)$ is orientation-preservingly diffeomorphic to $S^3_r(U)$, then K is also p-standard, where p is the smallest integer greater than r. This proves Thoerem 1.1 for all positive r. The case of negative r can be deduced by changing orientations and replacing K by its mirror-image.

As explained in Section 8, the techniques described here for establishing Theorem 1.1 can be readily adapted to other questions about knots admitting lens space surgeries. For example, if K denotes the $(2,5)$ torus knot, then it is easy to see that $S^3_9(K) \cong L(9,7)$, and $S^3_{11}(K) \cong L(11,4)$. Indeed, a result described in Section 8 shows that any lens space which is realized as integral surgery on a knot in S^3 with Seifert genus two is diffeomorphic to one of these two lens spaces. Similar lists are given when $g = 3$, 4, and 5. Combining these methods with a result of Goda and Teragaito, we show that the unknot and the trefoil are the only knots which admits a lens space surgery with $p = 5$. In another direction, we give obstructions to a knot admitting Seifert fibered surgeries, in terms of its genus and the degree of its Alexander polynomial.

Finally, in Section 9, we give some applications of these methods to the study of taut (coorientable) foliations, giving several families of three-manifolds which admit no taut foliation. One infinite family of hyperbolic examples is provided by the $(-2, 3, 2n + 1)$ pretzel knots for $n \geq 3$: it is shown that all Dehn fillings with sufficiently large surgery slope r admit no taut foliation. The first examples of hyperbolic three-manifolds with this property were constructed by Roberts, Shareshian, and Stein in [39]; see also [6]. In another direction, we show that if L is a nonsplit alternating link, then the double-cover of S^3 branched along L admits no taut foliation. Additional examples include certain plumbings of spheres and certain surgeries on the Borromean rings, as described in this section.

Outline. The remaining sections of this paper are as follows. In Section 2, we give a summary of the formal properties of the Floer homology groups developed in [23]. We do this in the simplest setting, where the coefficients are $\mathbb{Z}/2$. In this context we give precise statements of the nonvanishing theorem and surgery exact sequence. With $\mathbb{Z}/2$ coefficients, the nonvanishing theorem is applicable only to knots with Seifert genus $g > 1$. In Section 3, we use the nonvanishing theorem and the surgery sequence to prove Theorem 1.1 for all integer p, under the additional assumption that the genus is not 1. (This is enough to cover all cases of the theorem that do not follow from earlier known results, because a result of Goda and Teragaito [17] rules out genus-1 counterexamples to the theorem.)

Section 4 describes some details of the definition of the Floer groups, and the following two sections give the proof of the surgery long-exact sequence (Theorem 2.4) and the nonvanishing theorem. In these three sections, we also introduce more general (local) coefficients, allowing us to state the nonvanishing theorem in a form applicable to the case of Seifert genus 1. The surgery sequence with local coefficients is stated as Theorem 5.12. In Section 6, we discuss a refinement of the nonvanishing theorem using local coefficients. At this stage we have the machinery to prove Theorem 1.1 for integral r and any K, without restriction on genus. In Section 7, we explain how repeated applications of the long exact sequence can be used to reduce the case of nonintegral surgery slopes to the case where the surgery slopes are integral, so providing a proof of Theorem 1.1 in the nonintegral case that is independent of the cyclic surgery theorem of [7].

In Section 8, we describe several further applications of the same techniques to other questions involving lens-space surgeries. Finally, we give some applications of these techniques to studying taut foliations on three-manifolds in Section 9.

Remark on orientations. Our conventions about orientations and lens spaces have the following consequences. If a 2-handle is attached to the 4-ball along an attaching curve K in S^3, and if the attaching map is chosen so that the resulting 4-manifold has intersection form (p), then the oriented boundary of the 4-manifold is $S^3_p(K)$. For positive p, the lens space $L(p,1)$ coincides with $S^3_p(U)$ as an oriented 3-manifold. This is not consistent with the convention that $L(p,1)$ is the quotient of S^3 (the oriented boundary of the unit ball in $\mathbb{C}^2$) by the cyclic group of order p lying in the center of $U(2)$.

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2. Monopole Floer homology

2.1. *The Floer homology functors.* We summarize the basic properties of the Floer groups constructed in [23]. In this section we will treat only monopole Floer homology with coefficients in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Our three-manifolds will always be smooth, oriented, compact, connected and without boundary unless otherwise stated. To each such three-manifold Y, we associate three vector spaces over $\mathbb{F}$,

$$\widecheck{HM}_\bullet(Y), \quad \widehat{HM}_\bullet(Y), \quad \overline{HM}_\bullet(Y).$$

These are the monopole Floer homology groups, read "HM-to", "HM-from", and "HM-bar" respectively. They come equipped with linear maps i_*, j_* and p_* which form a long exact sequence

$$\cdots \xrightarrow{i_*} \widecheck{HM}_\bullet(Y) \xrightarrow{j_*} \widehat{HM}_\bullet(Y) \xrightarrow{p_*} \overline{HM}_\bullet(Y) \xrightarrow{i_*} \widecheck{HM}_\bullet(Y) \xrightarrow{j_*} \cdots . \tag{1}$$

A *cobordism* from Y_0 to Y_1 is an oriented, connected 4-manifold W equipped with an orientation-preserving diffeomorphism from ∂W to the disjoint union of $-Y_0$ and Y_1. We write $W : Y_0 \to Y_1$. We can form a category, in which the objects are three-manifolds, and the morphisms are diffeomorphism classes of cobordisms. The three versions of monopole Floer homology are functors from this category to the category of vector spaces. That is, to each $W : Y_0 \to Y_1$, there are associated maps

$$\begin{aligned} \widecheck{HM}(W) &: \widecheck{HM}_\bullet(Y_0) \to \widecheck{HM}_\bullet(Y_1) \\ \widehat{HM}(W) &: \widehat{HM}_\bullet(Y_0) \to \widehat{HM}_\bullet(Y_1) \\ \overline{HM}(W) &: \overline{HM}_\bullet(Y_0) \to \overline{HM}_\bullet(Y_1). \end{aligned}$$

The maps i_*, j_* and p_* provide natural transformations of these functors. In addition to their vector space structure, the Floer groups come equipped with a distinguished endomorphism, making them modules over the polynomial ring $\mathbb{F}[U]$. This module structure is respected by the maps arising from cobordisms, as well as by the three natural transformations.

These Floer homology groups are set up so as to be gauge-theory cousins of the Heegaard homology groups $HF^+(Y)$, $HF^-(Y)$ and $HF^\infty(Y)$ defined in [35]. Indeed, if $b_1(Y) = 0$, then the monopole Floer groups are conjecturally isomorphic to (certain completions of) their Heegaard counterparts.

2.2. *The nonvanishing theorem.* A *taut foliation* $\mathcal{F}$ of an oriented 3-manifold Y is a C^0 foliation of Y with smooth, oriented 2-dimensional leaves, such that there exists a closed 2-form ω on Y whose restriction to each leaf is everywhere positive. (Note that all foliations which are taut in this sense are automatically coorientable. There is a slightly weaker notion of tautness in the literature which applies even in the non-coorientable case, i.e. that there is a transverse curve which meets all the leaves. Of course, when $H^1(Y;\mathbb{Z}/2\mathbb{Z}) = 0$, all foliations are coorientable, and hence these two notions coincide.) We write $e(\mathcal{F})$ for the Euler class of the 2-plane field tangent to the leaves, an element of $H^2(Y;\mathbb{Z})$. The proof of the following theorem is based on the techniques of [25] and makes use of the results of [9].

THEOREM 2.1. *Suppose Y admits a smooth taut foliation $\mathcal{F}$ and is not $S^1 \times S^2$. If either* (a) $b_1(Y) = 0$, *or* (b) $b_1(Y) = 1$ *and $e(\mathcal{F})$ is nontorsion, then the image of $j_* : \widecheck{HM}_\bullet(Y) \to \widehat{HM}_\bullet(Y)$ is nonzero.*

The restriction to the two cases (a) and (b) in the statement of this theorem arises from our use of Floer homology with coefficients $\mathbb{F}$. The smoothness condition can also be relaxed somewhat. These issues are discussed in Section 6 below, where we give a more general nonvanishing result, Theorem 6.1, using Floer homology with local coefficients.

Note that j_* for $S^2 \times S^1$ is trivial in view of the following:

PROPOSITION 2.2. *If Y is a three-manifold which admits a metric of positive scalar curvature, then the image of j_* is zero.*

According to Gabai's theorem from [15], if K is a nontrivial knot, then $S^3_0(K)$ admits a taut foliation $\mathcal{F}$, and is not $S^1 \times S^2$. Furthermore, if the Seifert genus of K is greater than 1, then $\mathcal{F}$ is smooth and $e(\mathcal{F})$ is nontorsion. As a consequence, we have:

COROLLARY 2.3. *The image of* $j_* : \widetilde{HM}_\bullet(S^3_0(K)) \to \widehat{HM}_\bullet(S^3_0(K))$ *is nonzero if the Seifert genus of* K *is* 2 *or more, and is zero if* K *is the unknot.* □

2.3. *The surgery exact sequence.* Let M be an oriented 3-manifold with torus boundary. Let γ_1, γ_2, γ_3 be three oriented simple closed curves on ∂M with algebraic intersection numbers

$$(\gamma_1 \cdot \gamma_2) = (\gamma_2 \cdot \gamma_3) = (\gamma_3 \cdot \gamma_1) = -1.$$

Define γ_n for all n so that $\gamma_n = \gamma_{n+3}$. Let Y_n be the closed 3-manifold obtained by filling along γ_n: that is, we attach $S^1 \times D^2$ to M so that the curve $\{1\} \times \partial D^2$ is attached to γ_n. There is a standard cobordism W_n from Y_n to Y_{n+1}. The cobordism is obtained from $[0,1] \times Y_n$ by attaching a 2-handle to $\{1\} \times Y_n$, with framing γ_{n+1}. Note that these orientation conventions are set up so that $W_{n+1} \cup_{Y_{n+1}} W_n$ always contains a sphere with self-intersection number -1.

THEOREM 2.4. *There is an exact sequence*

$$\cdots \longrightarrow \widetilde{HM}_\bullet(Y_{n-1}) \xrightarrow{F_{n-1}} \widetilde{HM}_\bullet(Y_n) \xrightarrow{F_n} \widetilde{HM}_\bullet(Y_{n+1}) \longrightarrow \cdots,$$

in which the maps F_n *are given by the cobordisms* W_n. *The same holds for* $\widehat{HM}_\bullet$ *and* $\overline{HM}_\bullet$.

The proof of the theorem is given in Section 5.

2.4. *Gradings and completions.* The Floer groups are *graded* vector spaces, but there are two caveats: the grading is not by $\mathbb{Z}$, and a completion is involved. We explain these two points.

Let J be a set with an action of $\mathbb{Z}$, not necessarily transitive. We write $j \mapsto j + n$ for the action of $n \in \mathbb{Z}$ on J. A vector space V is *graded by* J if it is presented as a direct sum of subspaces V_j indexed by J. A homomorphism $h : V \to V'$ between vector spaces graded by J has degree n if $h(V_j) \subset V'_{j+n}$ for all j.

If Y is an oriented 3-manifold, we write $J(Y)$ for the set of homotopy-classes of oriented 2-plane fields (or equivalently nowhere-zero vector fields) ξ on Y. To define an action of $\mathbb{Z}$, we specify that $[\xi] + n$ denotes the homotopy class $[\tilde{\xi}]$ obtained from $[\xi]$ as follows. Let $B^3 \subset Y$ be a standard ball, and let $\rho : (B^3, \partial B^3) \to (SO(3), 1)$ be a map of degree $-2n$, regarded as an automorphism of the trivialized tangent bundle of the ball. Outside the ball B^3, we take $\tilde{\xi} = \xi$.

Inside the ball, we define

$$\tilde{\xi}(y) = \rho(y)\xi(y).$$

The structure of $J(Y)$ for a general three-manifold is as follows (see [25], for example). A 2-plane field determines a Spin^c structure on Y, so we can first write

$$J(Y) = \bigcup_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} J(Y,\mathfrak{s}),$$

where the sum is over all isomorphism classes of Spin^c structures. The action of $\mathbb{Z}$ on each $J(Y,\mathfrak{s})$ is transitive, and the stabilizer is the subgroup of $2\mathbb{Z}$ given by the image of the map

$$x \mapsto \langle c_1(\mathfrak{s}), x\rangle \tag{2}$$

from $H_2(Y;\mathbb{Z})$ to $\mathbb{Z}$. In particular, if $c_1(\mathfrak{s})$ is torsion, then $J(Y,\mathfrak{s})$ is an affine copy of $\mathbb{Z}$.

For each $j \in J(Y)$, there are subgroups

$$\begin{gathered}\widecheck{HM}_j(Y) \subset \widecheck{HM}_\bullet(Y)\\ \widehat{HM}_j(Y) \subset \widehat{HM}_\bullet(Y)\\ \overline{HM}_j(Y) \subset \overline{HM}_\bullet(Y),\end{gathered}$$

and there are internal direct sums which we denote by $\widecheck{HM}_*$, $\widehat{HM}_*$ and $\overline{HM}_*$:

$$\begin{aligned}\widecheck{HM}_*(Y) &= \bigoplus_j \widecheck{HM}_j(Y) \subset \widecheck{HM}_\bullet(Y)\\ \widehat{HM}_*(Y) &= \bigoplus_j \widehat{HM}_j(Y) \subset \widehat{HM}_\bullet(Y)\\ \overline{HM}_*(Y) &= \bigoplus_j \overline{HM}_j(Y) \subset \overline{HM}_\bullet(Y).\end{aligned}$$

The $\bullet$ versions are obtained from the $*$ versions as follows. For each $\mathfrak{s}$ with $c_1(\mathfrak{s})$ torsion, pick an arbitrary $j_0(\mathfrak{s})$ in $J(Y,\mathfrak{s})$. Define a decreasing sequence of subspaces $\widehat{HM}[n] \subset \widehat{HM}_*(Y)$ by

$$\widehat{HM}[n] = \bigoplus_{\mathfrak{s}} \bigoplus_{m\geq n} \widehat{HM}_{j_0(\mathfrak{s})-m}(Y),$$

where the sum is over torsion Spin^c structures. Make the same definition for the other two variants. The groups $\widecheck{HM}_\bullet(Y)$, $\widehat{HM}_\bullet(Y)$ and $\overline{HM}_\bullet(Y)$ are the completions of the direct sums $\widecheck{HM}_*(Y)$ etc. with respect to these decreasing filtrations. However, in the case of $\widecheck{HM}$, the subspace $\widecheck{HM}[n]$ is eventually zero for large n, so the completion has no effect. From the decomposition of $J(Y)$

into orbits, we have direct sum decompositions

$$\begin{aligned}\widecheck{HM}_\bullet(Y) &= \bigoplus_{\mathfrak{s}} \widecheck{HM}_\bullet(Y,\mathfrak{s})\\ \widehat{HM}_\bullet(Y) &= \bigoplus_{\mathfrak{s}} \widehat{HM}_\bullet(Y,\mathfrak{s})\\ \overline{HM}_\bullet(Y) &= \bigoplus_{\mathfrak{s}} \overline{HM}_\bullet(Y,\mathfrak{s}).\end{aligned}$$

Each of these decompositions has only finitely many nonzero terms.

The maps i_*, j_* and p_* are defined on the $*$ versions and have degree 0, 0 and -1 respectively, while the endomorphism U has degree -2. The maps induced by cobordisms do not have a degree and do not always preserve the $*$ subspace: they are continuous homomorphisms between complete filtered vector spaces.

To amplify the last point above, consider a cobordism $W : Y_0 \to Y_1$. The homomorphisms $\widecheck{HM}(W)$ etc. can be written as sums

$$\widecheck{HM}(W) = \sum_{\mathfrak{s}} \widecheck{HM}(W,\mathfrak{s}),$$

where the sum is over $\mathrm{Spin}^c(W)$: for each $\mathfrak{s} \in \mathrm{Spin}^c(W)$, we have

$$\widecheck{HM}(W,\mathfrak{s}) : \widecheck{HM}_\bullet(Y_0,\mathfrak{s}_0) \to \widecheck{HM}_\bullet(Y_1,\mathfrak{s}_1),$$

where $\mathfrak{s}_0$ and $\mathfrak{s}_1$ are the resulting Spin^c structures on the boundary components. The above sum is not necessarily finite, but it is convergent. The individual terms $\widecheck{HM}(W,\mathfrak{s})$ have a well-defined degree, in that for each $j_0 \in J(Y_0,\mathfrak{s}_0)$ there is a unique $j_1 \in J(Y_1,\mathfrak{s}_1)$ such that

$$\widecheck{HM}(W,\mathfrak{s}) : \widecheck{HM}_{j_0}(Y_0,\mathfrak{s}_0) \to \widecheck{HM}_{j_1}(Y_1,\mathfrak{s}_1).$$

The same remarks apply to $\widehat{HM}$ and $\overline{HM}$. The element j_1 can be characterized as follows. Let ξ_0 be an oriented 2-plane field in the class j_0, and let I be an almost complex structure on W such that: (i) the planes ξ_0 are invariant under $I|_{Y_0}$ and have the complex orientation; and (ii) the Spin^c structure associated to I is $\mathfrak{s}$. Let ξ_1 be the unique oriented 2-plane field on Y_1 that is invariant under I. Then $j_1 = [\xi_1]$. For future reference, we introduce the notation

$$j_0 \overset{\mathfrak{s}}{\sim} j_1$$

to denote the relation described by this construction.

2.4.1. *Remark.* Because of the completion involved in the definition of the Floer groups, the $\mathbb{F}[U]$-module structure of the groups $\widehat{HM}_*(Y,\mathfrak{s})$ (and its companions) gives rise to an $\mathbb{F}[[U]]$-module structure on $\widehat{HM}_\bullet(Y,\mathfrak{s})$, whenever $c_1(\mathfrak{s})$ is torsion. In the nontorsion case, the action of U on $\widehat{HM}_*(Y,\mathfrak{s})$ is actually nilpotent, so again the action extends. In this way, each of $\widecheck{HM}_\bullet(Y)$,

$\widehat{HM}_\bullet(Y)$ and $\overline{HM}_\bullet(Y)$ become modules over $\mathbb{F}[[U]]$, with continuous module multiplication.

2.5. *Canonical* mod 2 *gradings.* The Floer groups have a canonical grading mod 2. For a cobordism $W : Y_0 \to Y_1$, let us define

$$\iota(W) = \frac{1}{2}\big(\chi(W) + \sigma(W) - b_1(Y_1) + b_1(Y_0)\big),$$

where χ denotes the Euler number, σ the signature, and b_1 the first Betti number with real coefficients. Then we have the following proposition.

PROPOSITION 2.5. *There is one and only one way to decompose the grading set $J(Y)$ for all Y into even and odd parts in such a way that the following two conditions hold*:

(1) *The gradings $j \in J(S^3)$ for which $\widecheck{HM}_j(S^3)$ is nonzero are even.*

(2) *If $W : Y_0 \to Y_1$ is a cobordism and $j_0 \overset{\mathfrak{s}}{\sim} j_1$ for some* Spin^c *structure $\mathfrak{s}$ on W, then j_0 and j_1 have the same parity if and only if $\iota(W)$ is even.*

This result gives provides a canonical decomposition

$$\widecheck{HM}_\bullet(Y) = \widecheck{HM}_{\text{even}}(Y) \oplus \widecheck{HM}_{\text{odd}}(Y),$$

with a similar decomposition for the other two flavors. With respect to these mod 2 gradings, the maps i_* and j_* in the long exact sequence have even degree, while p_* has odd degree. The maps resulting from a cobordism W have even degree if and only if $\iota(W)$ is even.

2.6. *Computation from reducible solutions.* While the groups $\widecheck{HM}_\bullet(Y)$ and $\widehat{HM}_\bullet(Y)$ are subtle invariants of Y, the group $\overline{HM}_\bullet(Y)$ by contrast can be calculated knowing only the cohomology ring of Y. This is because the definition of $\overline{HM}_\bullet(Y)$ involves only the *reducible* solutions of the Seiberg-Witten monopole equations (those where the spinor is zero). We discuss here the case that Y is a rational homology sphere.

When $b_1(Y) = 0$, the number of different Spin^c structures on Y is equal to the order of $H_1(Y;\mathbb{Z})$, and $J(Y)$ is the union of the same number of copies of $\mathbb{Z}$. The contribution to $\overline{HM}_\bullet(Y)$ from each Spin^c structure is the same:

PROPOSITION 2.6. *Let Y be a rational homology sphere and $\mathfrak{t}$ a* Spin^c *structure on Y. Then*

$$\overline{HM}_\bullet(Y, \mathfrak{t}) \cong \mathbb{F}[U^{-1}, U]]$$

as topological $\mathbb{F}[[U]]$-modules, where the right-hand side denotes the ring of formal Laurent series in U that are finite in the negative direction.

The maps $\overline{HM}_\bullet(W)$ arising from cobordisms between rational homology spheres are also standard, as the next proposition states.

PROPOSITION 2.7. *Suppose* $W : Y_0 \to Y_1$ *is a cobordism between rational homology spheres, with* $b_1(W) = 0$, *and suppose that the intersection form on* W *is negative definite. Let* $\mathfrak{s}$ *be a* Spin^c *structure on* W, *and suppose* $j_0 \overset{\mathfrak{s}}{\sim} j_1$. *Then*

$$\overline{HM}(W, \mathfrak{s}) : \overline{HM}_{j_0}(Y_0) \to \overline{HM}_{j_1}(Y_1)$$

is an isomorphism. On the other hand, if the intersection form on W *is not negative definite, then* $\overline{HM}(W, \mathfrak{s})$ *is zero, for all* $\mathfrak{s}$.

Proposition 2.7 follows from the fact that for each Spin^c structure over W, there is a unique reducible solution.

The last part of the proposition above holds in a more general form. Let W be a cobordism between 3-manifolds that are not necessarily rational homology spheres, and let $b^+(W)$ denote the dimension of a maximal positive-definite subspace for the quadratic form on the image of $H^2(W, \partial W; \mathbb{R})$ in $H^2(W; \mathbb{R})$.

PROPOSITION 2.8. *If the cobordism* $W : Y_0 \to Y_1$ *has* $b^+(W) > 0$, *then the map* $\overline{HM}(W)$ *is zero.*

2.7. *Gradings and rational homology spheres.* We return to rational homology spheres, and cobordisms between them. If W is such a cobordism, then $H^2(W, \partial W; \mathbb{Q})$ is isomorphic to $H^2(W; \mathbb{Q})$, and there is therefore a quadratic form

$$Q : H^2(W; \mathbb{Q}) \to \mathbb{Q}$$

given by $Q(e) = (\bar{e} \smile \bar{e})[W, \partial W]$, where $\bar{e} \in H^2(W, \partial W; \mathbb{Q})$ is a class whose restriction to W is e. We will simply write e^2 for $Q(e)$.

LEMMA 2.9. *Let* $W, W' : Y_0 \to Y_1$ *be two cobordisms between a pair of rational homology spheres* Y_0 *and* Y_1. *Let* j_0 *and* j_1 *be classes of oriented* 2*-plane fields on the* 3*-manifolds and suppose that*

$$j_0 \overset{\mathfrak{s}}{\sim} j_1$$
$$j_0 \overset{\mathfrak{s}'}{\sim} j_1$$

for Spin^c *structure* $\mathfrak{s}$ *and* $\mathfrak{s}'$ *on the two cobordisms. Then*

$$c_1^2(\mathfrak{s}) - 2\chi(W) - 3\sigma(W) = c_1^2(\mathfrak{s}') - 2\chi(W') - 3\sigma(W'),$$

where χ *and* σ *denote the Euler number and signature.*

Proof. Every 3-manifold equipped with a 2-plane field ξ is the boundary of some almost-complex manifold (X, I) in such a way that ξ is invariant under I;

so bearing in mind the definition of the relation $\overset{\mathfrak{s}}{\sim}$, and using the additivity of all the terms involved, we can reduce the lemma to a statement about closed almost-complex manifolds. The result is thus a consequence of the fact that

$$c_1^2(\mathfrak{s})[X] - 2\chi(X) - 3\sigma(X) = 0$$

for the canonical Spin^c structure on a closed, almost-complex manifold X. $\square$

Essentially the same point leads to the definition of the following $\mathbb{Q}$-valued function on $J(Y)$, and the proof that it is well-defined:

Definition 2.10. For a three-manifold Y with $b_1(Y) = 0$ and $j \in J(Y)$ represented by an oriented 2-plane field ξ, we define $h(j) \in \mathbb{Q}$ by the formula

$$4h(j) = c_1^2(X, I) - 2\chi(X) - 3\sigma(X) + 2,$$

where X is a manifold whose oriented boundary is Y, and I is an almost-complex structure such that the 2-plane field ξ is I-invariant and has the complex orientation. The quantity $c_1^2(X, I)$ is to be interpreted again using the natural isomorphism $H^2(X, \partial X; \mathbb{Q}) \cong H^2(X; \mathbb{Q})$.

The map $h : J(Y) \to \mathbb{Q}$ satisfies $h(j + 1) = h(j) + 1$.

Now let $\mathfrak{s}$ be a Spin^c structure on a rational homology sphere Y, and consider the exact sequence

$$0 \to \mathrm{im}(p_*) \hookrightarrow \overline{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{i_*} \mathrm{im}(i_*) \to 0, \tag{3}$$

where $p_* : \widehat{HM}_\bullet(Y, \mathfrak{s}) \to \overline{HM}_\bullet(Y, \mathfrak{s})$. The image of p_* is a closed, nonzero, proper $\mathbb{F}[U^{-1}, U]]$-submodule of $\overline{HM}_\bullet(Y, \mathfrak{s})$; and the latter is isomorphic to $\mathbb{F}[U^{-1}, U]]$ by Proposition 2.6. The only such submodules of $\mathbb{F}[U^{-1}, U]]$ are the submodules $U^r\mathbb{F}[[U]]$ for $r \in \mathbb{Z}$. It follows that the short exact sequence above is isomorphic to the short exact sequence

$$0 \to \mathbb{F}[[U]] \to \mathbb{F}[U^{-1}, U]] \to \mathbb{F}[U^{-1}, U]]/\mathbb{F}[[U]] \to 0.$$

This observation leads to a $\mathbb{Q}$-valued invariant of Spin^c structures on rational homology spheres, after Frøyshov [13]:

Definition 2.11. Let Y be an oriented rational homology sphere and $\mathfrak{s}$ a Spin^c structure. We define (by either of two equivalent formulae)

$$\begin{aligned} Fr(Y, \mathfrak{s}) &= \min\{\, h(k) \mid i_* : \overline{HM}_k(Y, \mathfrak{s}) \to \widetilde{HM}_k(Y, \mathfrak{s}) \text{ is nonzero}\,\}, \\ &= \max\{\, h(k) + 2 \mid p_* : \widehat{HM}_{k+1}(Y, \mathfrak{s}) \to \overline{HM}_k(Y, \mathfrak{s}) \text{ is nonzero}\,\}. \end{aligned}$$

When j_* is zero, sequence (3) determines everything, and we have:

Corollary 2.12. *Let Y be a rational homology sphere for which the map j_* is zero. Then for each* Spin^c *structure $\mathfrak{s}$, the short exact sequence*

$$0 \to \widehat{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{p_*} \overline{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{i_*} \widetilde{HM}_\bullet(Y, \mathfrak{s}) \to 0$$

is isomorphic as a sequence of topological $\mathbb{F}[[U]]$*-modules to the sequence*

$$0 \to \mathbb{F}[[U]] \to \mathbb{F}[U^{-1}, U]] \to \mathbb{F}[U^{-1}, U]]/\mathbb{F}[[U]] \to 0.$$

Furthermore, if $j_{\min}$ *denotes the lowest degree in which* $\widetilde{HM}_{j_{\min}}(Y, \mathfrak{s})$ *is nonzero, then* $h(j_{\min}) = Fr(Y, \mathfrak{s})$.

2.8. *The conjugation action.* Let Y be a three-manifold, equipped with a spin bundle W. The bundle $\overline{W}$ which is induced from W with the conjugate complex structure naturally inherits a Clifford action from the one on W. This correspondence induces an involution on the set of Spin^c structures on Y, denoted $\mathfrak{s} \mapsto \overline{\mathfrak{s}}$.

Indeed, this conjugation action descends to an action on the Floer homology groups:

Proposition 2.13. *Conjugation induces a well-defined involution on* $\widetilde{HM}_\bullet(Y)$, *sending* $\widetilde{HM}(Y, \mathfrak{s}) \mapsto \widetilde{HM}(Y, \overline{\mathfrak{s}})$. *Indeed, conjugation induces involutions on the other two theories as well, which are compatible with the maps* i_*, j_*, *and* p_*.

3. Proof of Theorem 1.1 in the simplest cases

In this section, we prove Theorem 1.1 for the case that the surgery coefficient is an integer and the Seifert genus of K is not 1.

3.1. *The Floer groups of lens spaces.* We begin by describing the Floer groups of the 3-sphere. There is only one Spin^c structure on S^3, and j_* is zero because there is a metric of positive scalar curvature. Corollary 2.12 is therefore applicable. It remains only to say what $j_{\min}$ is, or equivalently what the Frøyshov invariant is.

Orient S^3 as the boundary of the unit ball in $\mathbb{R}^4$ and let $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$ be the subgroups of $\mathrm{SO}(4)$ that act trivially on the anti-self-dual and self-dual 2-forms respectively. Let ξ_+ and ξ_- be 2-plane fields invariant under $\mathrm{SU}(2)_-$ and $\mathrm{SU}(2)_+$ respectively. Our orientation conventions are set up so that $[\xi_-] = [\xi_+] + 1$.

Proposition 3.1. *The least* $j \in J(S^3)$ *for which* $\widetilde{HM}_j(S^3)$ *is nonzero is* $j = [\xi_-]$. *The largest* $j \in J(S^3)$ *for which* $\widehat{HM}_j(S^3)$ *is nonzero is* $[\xi_+] = [\xi_-] - 1$. *The Frøyshov invariant of* S^3 *is therefore given by*:

$$Fr(S^3) = h([\xi_-]) = 0.$$

We next describe the Floer groups for the lens space $L(p, 1)$, realized as $S^3_p(U)$ for an integer $p > 0$. The short description is provided by Corollary 2.12, because j_* is zero. To give a longer answer, we must describe the 2-plane field in which the generator of $\widecheck{HM}$ lies, for each Spin^c structure. Equivalently, we must give the Frøyshov invariants.

We first pin down the grading set $J(Y)$ for $Y = S^3_p(K)$ and $p > 0$. For a general knot K, we have a cobordism

$$W(p) : S^3_p(K) \to S^3,$$

obtained by the addition of a single 2-handle. The manifold $W(p)$ has $H_2(W(p)) = \mathbb{Z}$, and a generator has self-intersection number $-p$. A choice of orientation for a Seifert surface for K picks out a generator $h = h_{W(p)}$. For each integer n, there is a unique Spin^c structure $\mathfrak{s}_{n,p}$ on $W(p)$ with

$$\langle c_1(\mathfrak{s}_{n,p}), h\rangle = 2n - p. \tag{4}$$

We denote the Spin^c structure on $S^3_p(K)$ which arises from $\mathfrak{s}_{n,p}$ by $\mathfrak{t}_{n,p}$; it depends only on n mod p. Define $j_{n,p}$ to be the unique element of $J(S^3_p(K), \mathfrak{t}_{n,p})$ satisfying

$$j_{n,p} \overset{\mathfrak{s}_{n,p}}{\sim} [\xi_+],$$

where ξ_+ is the 2-plane field on S^3 described above. Like $\mathfrak{t}_{n,p}$, the class $j_{n,p}$ depends on our choice of orientation for the Seifert surface. Our convention implies that $j_{0,1} = [\xi_+]$ on $S^3_1(U) = S^3$. If $n \equiv n'$ mod p, then $j_{n,p}$ and $j_{n',p}$ belong to the same Spin^c structure, so they differ by an element of $\mathbb{Z}$ acting on $J(Y)$. The next lemma calculates that element of $\mathbb{Z}$.

Lemma 3.2.

$$j_{n,p} - j_{n',p} = \frac{(2n-p)^2 - (2n'-p)^2}{4p}.$$

Proof. We can equivalently calculate $h(j_{n,p}) - h(j_{n',p})$. We can compare $h(j_{n,p})$ to $h([\xi_+])$ using the cobordism $W(p)$, which tells us

$$4h(j_{n,p}) = 4h([\xi_+]) - c_1^2(\mathfrak{s}_{n,p}) + 2\chi(W(p)) + 3\sigma(W(p)),$$

and hence

$$\begin{aligned} 4h(j_{n,p}) &= -4 + \frac{(2n-p)^2}{p} + 2 - 3 \\ &= \frac{(2n-p)^2}{p} - 5. \end{aligned}$$

The result follows. □

Now we can state the generalization of Proposition 3.1.

Proposition 3.3. *Let n be in the range $0 \le n \le p$. The least $j \in J(Y, \mathfrak{t}_{n,p})$ for which $\widecheck{HM}_j(S^3_p(U), \mathfrak{t}_{n,p})$ is nonzero is $j_{n,p} + 1$. The largest $j \in$*

$J(S^3_p(U),\mathfrak{t}_n)$ for which $\widehat{HM}_j(S^3_p(U),\mathfrak{t}_{n,p})$ is nonzero is $j_{n,p}$. Equivalently, the Frøyshov invariant of $(S^3_p(U),\mathfrak{t}_{n,p})$ is given by:

$$\begin{aligned} Fr(S^3_p(U),\mathfrak{t}_{n,p}) &= h(j_{n,p})+1 \\ &= \frac{(2n-p)^2}{4p}-\frac{1}{4}. \end{aligned} \tag{5}$$

The meaning of this last result may be clarified by the following remarks. By Proposition 2.7, we have an isomorphism

$$\overline{HM}(W(p),\mathfrak{s}_{n,p}) : \overline{HM}_{j_{n,p}}(S^3_p(U),\mathfrak{t}_{n,p}) \to \overline{HM}_{[\xi+]}(S^3);$$

and because j_* is zero for lens spaces, the map

$$p_* : \widehat{HM}_{j_{n,p}+1}(S^3_p(U),\mathfrak{t}_{n,p}) \to \overline{HM}_{j_{n,p}}(S^3_p(U),\mathfrak{t}_{n,p})$$

is an isomorphism. Proposition 3.3 is therefore equivalent to the following corollary:

Corollary 3.4. *The map*

$$\widehat{HM}(W(p),\mathfrak{s}_{n,p}) : \widehat{HM}_\bullet(S^3_p(U),\mathfrak{t}_{n,p}) \to \widehat{HM}_\bullet(S^3)$$

is an isomorphism, whenever $0 \le n \le p$.

A proof directly from the definitions is sketched in Section 4.14. See also Proposition 7.5, which yields a more general result by a more formal argument.

We can now be precise about what it means for $S^3_p(K)$ to resemble $S^3_p(U)$ in its Floer homology.

Definition 3.5. For an integer $p > 0$, we say that K is p-standard if

(1) the map $j_* : \widetilde{HM}_\bullet(S^3_p(K)) \to \widehat{HM}_\bullet(S^3_p(K))$ is zero; and

(2) for $0 \le n \le p$, the Frøyshov invariant of the Spin^c structure $\mathfrak{t}_{n,p}$ on $S^3_p(K)$ is given by the same formula (5) as in the case of the unknot.

For $p = 0$, for the sake of expediency, we say that K is weakly 0-standard if the map j_* is zero for $S^3_0(K)$.

Observe that $\mathfrak{t}_{n,p}$ depended on an orientation Seifert surface for the knot K. Letting $\mathfrak{t}^+_{n,p}$ and $\mathfrak{t}^-_{n,p}$ be the two possible choices using the two orientations of the Seifert surface, it is easy to see that $\mathfrak{t}^+_{n,p}$ is the conjugate of $\mathfrak{t}^-_{n,p}$. In fact, since the Frøyshov invariant is invariant under conjugation, it follows that our notation of p-standard is independent of the choice of orientation.

If $p > 0$ and j_* is zero, the second condition in the definition is equivalent to the assertion that $\widehat{HM}(W(p),\mathfrak{s}_{n,p})$ is an isomorphism for n in the same range:

Corollary 3.6. *If K is p-standard and $p > 0$, then*

$$\widehat{HM}(W(p), \mathfrak{s}_{n,p}) : \widehat{HM}_\bullet(S^3_p(K), \mathfrak{t}_{n,p}) \to \widehat{HM}_\bullet(S^3)$$

is an isomorphism for $0 \le n \le p$. Conversely, if j_* is zero for $S^3_p(K)$ and the above map is an isomorphism for $0 \le n \le p$, then K is p-standard.

The next lemma tells us that a counterexample to Theorem 1.1 would be a p-standard knot.

Lemma 3.7. *If $S^3_p(K)$ and $S^3_p(U)$ are orientation-preserving diffeomorphic for some integer $p > 0$, then K is p-standard.*

Proof. Fix an integer n, and let $\psi : S^3_p(K) \to S^3_p(U)$ be a diffeomorphism. To avoid ambiguity, let us write $\mathfrak{t}^K_{n,p}$ and $\mathfrak{t}^U_{n,p}$ for the Spin^c structures on these two 3-manifolds, obtained as above. Because j_* is zero for $S^3_p(K)$ and $\overline{HM}(W(p), \mathfrak{s}_{n,p})$ is an isomorphism, the map

$$\widehat{HM}(W(p), \mathfrak{s}_{n,p}) : \widehat{HM}_\bullet(S^3_p(K), \mathfrak{t}^K_{n,p}) \to \widehat{HM}_\bullet(S^3)$$

is injective. Making a comparison with Corollary 3.6, we see that

$$Fr(S^3_p(K), \mathfrak{t}^K_{n,p}) \le Fr(S^3_p(U), \mathfrak{t}^U_{n,p})$$

for $0 \le n \le p$. So

$$\sum_{n=0}^{p-1} Fr(S^3_p(K), \mathfrak{t}^K_{n,p}) \le \sum_{n=0}^{p-1} Fr(S^3_p(U), \mathfrak{t}^U_{n,p}).$$

On the other hand, as n runs from 0 to $p-1$, we run through all Spin^c structures once each; and because the manifolds are diffeomorphic, we must have equality of the sums. The Frøyshov invariants must therefore agree term by term, and K is therefore p-standard. □

3.2. *Exploiting the surgery sequence.* When the surgery coefficient is an integer and the genus is not 1, Theorem 1.1 is now a consequence of the following proposition and Corollary 2.3, whose statement we can rephrase as saying that a weakly 0-standard knot has genus 1 or is unknotted.

Proposition 3.8. *If K is p-standard for some integer $p \ge 1$, then K is weakly 0-standard.*

Proof. Suppose that K is p-standard, so that in particular, j_* is zero for $S^3_p(K)$. We apply Theorem 2.4 to the following sequence of cobordisms

$$\cdots \longrightarrow S^3_{p-1}(K) \xrightarrow{W_0} S^3_p(K) \xrightarrow{W_1} S^3 \xrightarrow{W_2} S^3_{p-1}(K) \longrightarrow \cdots$$

to obtain a commutative diagram with exact rows and columns,

$$\begin{tikzcd}
{} & {} \arrow[d, "j_*"] & {} \arrow[d, "0"] & {} \arrow[d, "0"] & {} \\
{} \arrow[r] & \widehat{HM}_\bullet(S^3_{p-1}(K)) \arrow[r, "\widehat{HM}(W_0)"] \arrow[d, "p_*"] & \widehat{HM}_\bullet(S^3_p(K)) \arrow[r, "\widehat{HM}(W_1)"] \arrow[d, "p_*"] & \widehat{HM}_\bullet(S^3) \arrow[r] \arrow[d, "p_*"] & {} \\
{} \arrow[r] & \overline{HM}_\bullet(S^3_{p-1}(K)) \arrow[r, "\overline{HM}(W_0)"] \arrow[d, "i_*"] & \overline{HM}_\bullet(S^3_p(K)) \arrow[r, "\overline{HM}(W_1)"] \arrow[d, "i_*"] & \overline{HM}_\bullet(S^3) \arrow[r] \arrow[d, "i_*"] & {} \\
{} \arrow[r] & \widecheck{HM}_\bullet(S^3_{p-1}(K)) \arrow[r, "\widecheck{HM}(W_0)"] \arrow[d, "j_*"] & \widecheck{HM}_\bullet(S^3_p(K)) \arrow[r, "\widecheck{HM}(W_1)"] \arrow[d, "0"] & \widecheck{HM}_\bullet(S^3) \arrow[r] \arrow[d, "0"] & {} \\
{} & {} & {} & {} & .
\end{tikzcd}$$

In the case $K = U$, the cobordism W_1 is diffeomorphic (preserving orientation) to $N \setminus B^4$, where N is a tubular neighborhood of a 2-sphere with self-intersection number $-p$; and W_2 has a similar description, containing a sphere with self-intersection $(p-1)$. In general, the cobordism W_1 is the manifold we called $W(p)$ above.

Lemma 3.9. *The maps*

$$\begin{aligned}
\overline{HM}(W_1) &: \overline{HM}_\bullet(S^3_p(K)) \to \overline{HM}_\bullet(S^3) \\
\widehat{HM}(W_1) &: \widehat{HM}_\bullet(S^3_p(K)) \to \widehat{HM}_\bullet(S^3)
\end{aligned}$$

are zero if $p = 1$ and are surjective if $p \geq 2$.

Proof. We write

$$\widehat{HM}_\bullet(S^3_p(K)) = \bigoplus_{n=0}^{p-1} \widehat{HM}_\bullet(S^3_p(K), \mathfrak{t}_{n,p}).$$

If n is in the range $0 \leq n \leq p-1$, the map $\widehat{HM}_\bullet(W_1, \mathfrak{s}_{n,p})$ is an isomorphism by Corollary 3.6, which gives identifications

$$\begin{tikzcd}
\widehat{HM}_\bullet(S^3_p(K), \mathfrak{t}_{n,p}) \arrow[r, "\widehat{HM}(W_1, \mathfrak{s}_{n,p})"] \arrow[d] & \widehat{HM}_\bullet(S^3) \arrow[d] \\
\mathbb{F}[[U]] \arrow[r, "1"] & \mathbb{F}[[U]].
\end{tikzcd}$$

For $n' \equiv n \bmod p$, under the same identifications, $\widehat{HM}(W_1, \mathfrak{s}_{n',p})$ becomes multiplication by U^r, where

$$r = (j_{n',p} - j_{n,p})/2.$$

This difference was calculated in Lemma 3.2. Taking the sum over all $\mathfrak{s}_{n',p}$, we see that

$$\sum_{n'\equiv n\ (p)} \widehat{HM}(W_1, \mathfrak{s}_{n',p}) : \widehat{HM}_\bullet(S^3_p(K), \mathfrak{t}_{n,p}) \to \widehat{HM}_\bullet(S^3)$$

is isomorphic (as a map of vector spaces) to the map $\mathbb{F}[[U]] \to \mathbb{F}[[U]]$ given by multiplication by the series

$$\sum_{n'\equiv n\ (p)} U^{((2n'-p)^2-(2n-p)^2)/8p} \in \mathbb{F}[[U]].$$

When $n = 0$, this series is 0 as the terms cancel in pairs. For all other n in the range $1 \le n \le p-1$, the series has leading coefficient 1 (the contribution from $n' = n$) and is therefore invertible. Taking the sum over all residue classes, we obtain the result for $\widehat{HM}$. The case of $\overline{HM}$ is similar, but does not depend on Corollary 3.6. ☐

We can now prove Proposition 3.8 by induction on p. Suppose first that $p \ge 2$ and let K be p-standard. The lemma above tells us that $\widehat{HM}(W_1)$ is surjective, and from the exactness of the rows it follows that $\widehat{HM}(W_0)$ is injective. Commutativity of the diagram shows that $\overline{HM}(W_0) \circ p_*$ is injective, where $p_* : \widehat{HM}_\bullet(S^3_{p-1}(K)) \to \overline{HM}_\bullet(S^3_{p-1}(K))$. It follows that

$$j_* : \widetilde{HM}_\bullet(S^3_{p-1}(K)) \to \widehat{HM}_\bullet(S^3_{p-1}(K))$$

is zero, by exactness of the columns. To show that K is $(p-1)$-standard, we must examine its Frøyshov invariants.

Fix n in the range $0 \le n \le p-2$, and let

$$e \in \overline{HM}_{j_{n,p-1}}(S^3_{p-1}(K), \mathfrak{t}_{n,p-1})$$

be the generator. To show that the Frøyshov invariants of $S^3_{p-1}(K)$ are standard is to show that

$$e \in \operatorname{image}\bigl(p_* : \widehat{HM}_\bullet(S^3_{p-1}(K)) \to \overline{HM}_\bullet(S^3_{p-1}(K))\bigr).$$

From the diagram, this is equivalent to showing

$$\overline{HM}(W_0)(e) \in \operatorname{image}\bigl(p_* : \widehat{HM}_\bullet(S^3_p(K)) \to \overline{HM}_\bullet(S^3_p(K))\bigr).$$

Suppose on the contrary that $\overline{HM}(W_0)(e)$ does not belong to the image of p_*. This means that there is a Spin^c structure $\mathfrak{u}$ on W_0 such that

$$j_{n,p-1} \overset{\mathfrak{u}}{\sim} j_{m,p} + x$$

for some integer $x > 0$, and m in the range $0 \le m \le p-1$. There is a unique Spin^c structure $\mathfrak{w}$ on the composite cobordism

$$X = W_1 \circ W_0 : S^3_{p-1}(K) \to S^3$$

whose restriction to W_0 is $\mathfrak{u}$ and whose restriction to W_1 is $\mathfrak{s}_{m,p}$. We have

$$j_{n,p-1} \overset{\mathfrak{w}}{\sim} [\xi_+] + x.$$

On the other hand, the composite cobordism X is diffeomorphic to the cobordism $W(p-1)\#\overline{\mathbb{CP}}^2$ (a fact that we shall return to in Section 5), and we can therefore write (in a self-evident notation)

$$\mathfrak{w} = \mathfrak{s}_{n',p-1}\#\mathfrak{s}$$

for some Spin^c structure $\mathfrak{s}$ on $\overline{\mathbb{CP}}^2$, and some n' equivalent to n mod p. From Lemma 2.9, we see that

$$\begin{aligned} c_1^2(\mathfrak{w}) - 2\chi(X) - 3\sigma(X) & \\ &= 4x + c_1^2(\mathfrak{s}_{n,p-1}) - 2\chi(W(p-1)) - 3\sigma(W(p-1)) \end{aligned}$$

or in other words

$$\frac{(2n-p+1)^2 - (2n'-p+1)^2}{p-1} + c_1^2(\mathfrak{s}) + 1 = 4x.$$

But n is in the range $0 \le n \le p-1$ and $c_1^2(\mathfrak{s})$ has the form $-(2k+1)^2$ for some integer k, so the left-hand side is not greater than 0. This contradicts the assumption that x is positive, and completes the argument for the case $p \ge 2$.

In the case $p = 1$, the maps $\overline{HM}(W_1)$, $\widehat{HM}(W_1)$ and $\widetilde{HM}(W_1)$ are all zero. A diagram chase again shows that j_* is zero for $S_0^3(K)$, so K is weakly 0-standard. $\square$

4. Construction of monopole Floer homology

4.1. *The configuration space and its blow-up.* Let Y be an oriented 3-manifold, equipped with a Riemannian metric. Let $\mathcal{B}(Y)$ denote the space of isomorphism classes of triples $(\mathfrak{s}, A, \Phi)$, where $\mathfrak{s}$ is a Spin^c structure, A is a Spin^c connection of Sobolev class $L^2_{k-1/2}$ in the associated spin bundle $S \to Y$, and Φ is an $L^2_{k-1/2}$ section of S. Here $k - 1/2$ is any suitably large Sobolev exponent, and we choose a half-integer because there is a continuous restriction map $L^2_k(X) \to L^2_{k-1/2}(Y)$ when X has boundary Y. The space $\mathcal{B}(Y)$ has one component for each isomorphism class of Spin^c structure, so we can write

$$\mathcal{B}(Y) = \bigcup_{\mathfrak{s}} \mathcal{B}(Y, \mathfrak{s}).$$

We call an element of $\mathcal{B}(Y)$ *reducible* if Φ is zero and irreducible otherwise. If we choose a particular Spin^c structure from each isomorphism class, we can construct a space

$$\mathcal{C}(Y) = \bigcup_{\mathfrak{s}} \mathcal{C}(Y, \mathfrak{s}),$$

where $\mathcal{C}(Y,\mathfrak{s})$ is the space of all pairs (A,Φ), a Spin^c connection and section for the chosen S. Then we can regard $\mathcal{B}(Y)$ as the quotient of $\mathcal{C}(Y)$ by the gauge group $\mathcal{G}(Y)$ of all maps $u: Y\to S^1$ of class $L^2_{k+1/2}$.

The space $\mathcal{B}(Y)$ is a Banach manifold except at the locus of reducibles; the reducible locus $\mathcal{B}^{\mathrm{red}}(\mathrm{Y})$ is itself a Banach manifold, and the map

$$\begin{aligned}\mathcal{B}(Y)&\to\mathcal{B}^{\mathrm{red}}(Y)\\ [\mathfrak{s},A,\Phi]&\mapsto[\mathfrak{s},A,0]\end{aligned}$$

has fibers $L^2_{k-1/2}(S)/S^1$, which is a cone on a complex projective space. We can resolve the singularity along the reducibles by forming a real, oriented blow-up,

$$\pi:\mathcal{B}^\sigma(Y)\to\mathcal{B}(Y).$$

We define $\mathcal{B}^\sigma(Y)$ to be the space of isomorphism classes of quadruples $(\mathfrak{s},A,s,\phi)$, where ϕ is an element of $L^2_{k-1/2}(S)$ with unit L^2 norm and $s\geq 0$. The map π is

$$\pi:[\mathfrak{s},A,s,\phi]\mapsto[\mathfrak{s},A,s\phi].$$

This blow-up is a Banach manifold with boundary: the boundary consists of points with $s=0$ (we call these *reducible*), and the restriction of π to the boundary is a map

$$\pi:\partial\mathcal{B}^\sigma(Y)\to\mathcal{B}^{\mathrm{red}}(Y)$$

with fibers the projective spaces associated to the vector spaces $L^2_{k-1/2}(S)$.

4.2. *The Chern-Simons-Dirac functional.* After choosing a preferred connection A_0 in a spinor bundle S for each isomorphism class of Spin^c structure, we can define the Chern-Simons-Dirac functional $\mathcal{L}$ on $\mathcal{C}(Y)$ by

$$\mathcal{L}(A,\Phi)=-\frac{1}{8}\int_Y(A^{\mathrm{t}}-A_0^{\mathrm{t}})\wedge(F_{A^{\mathrm{t}}}+F_{A_0^{\mathrm{t}}})+\frac{1}{2}\int_Y\langle D_A\Phi,\Phi\rangle\,d\mathrm{vol}.$$

Here A^{t} is the associated connection in the line bundle $\Lambda^2 S$. The formal gradient of $\mathcal{L}$ with respect to the L^2 metric $\|\Phi\|^2+\frac{1}{4}\|A^{\mathrm{t}}-A_0^{\mathrm{t}}\|^2$ is a "vector field" $\tilde{\mathcal{V}}$ on $\mathcal{C}(Y)$ that is invariant under the gauge group and orthogonal to its orbits. We use quotation marks, because $\tilde{\mathcal{V}}$ is a section of the $L^2_{k-3/2}$ completion of the tangent bundle. Away from the reducible locus, $\tilde{\mathcal{V}}$ descends to give a vector field (in the same sense) $\mathcal{V}$ on $\mathcal{B}(Y)$. Pulling back by π, we obtain a vector field $\mathcal{V}^\sigma$ on the interior of the manifold-with-boundary $\mathcal{B}^\sigma(Y)$. This vector field extends smoothly to the boundary, to give a section

$$\mathcal{V}^\sigma:\mathcal{B}^\sigma(Y)\to\mathcal{T}_{k-3/2}(Y),$$

where $\mathcal{T}_{k-3/2}(Y)$ is the $L^2_{k-3/2}$ completion of $T\mathcal{B}^\sigma(Y)$. This vector field is tangent to the boundary at $\partial\mathcal{B}^\sigma(Y)$. The Floer groups $\widecheck{HM}(Y)$, $\widehat{HM}(Y)$ and $\overline{HM}(Y)$ will be defined using the Morse theory of the vector field $\mathcal{V}^\sigma$ on $\mathcal{B}^\sigma(Y)$.

4.2.1. *Example.* Suppose that $b_1(Y)$ is zero. For each Spin^c structure $\mathfrak{s}$, there is (up to isomorphism) a unique connection A in the associated spin bundle with $F_{A^t} = 0$, and there is a corresponding zero of the vector field $\mathcal{V}$ at the point $\alpha = [\mathfrak{s}, A_0, 0]$ in $\mathcal{B}^{\mathrm{red}}(Y)$. The vector field $\mathcal{V}^\sigma$ has a zero at the point $[\mathfrak{s}, A_0, 0, \phi]$ in $\partial\mathcal{B}^\sigma(Y)$ precisely when ϕ is a unit eigenvector of the Dirac operator D_A. If the spectrum of D_A is simple (i.e. no repeated eigenvalues), then the set of zeros of $\mathcal{V}^\sigma$ in the projective space $\pi^{-1}(\alpha)$ is a discrete set, with one point for each eigenvalue.

4.3. *Four-manifolds.* Let X be a compact oriented Riemannian 4-manifold (possibly with boundary), and write $\mathcal{B}(X)$ for the space of isomorphism classes of triples $(\mathfrak{s}, A, \Phi)$, where $\mathfrak{s}$ is a Spin^c structure, A is a Spin^c connection of class L^2_k and Φ is an L^2_k section of the associated half-spin bundle S^+. As in the 3-dimensional case, we can form a blow-up $\mathcal{B}^\sigma(X)$ as the space of isomorphism classes of quadruples $(\mathfrak{s}, A, s, \phi)$, where $s \geq 0$ and $\|\phi\|_{L^2(X)} = 1$. If Y is a boundary component of X, then there is a partially-defined restriction map

$$r : \mathcal{B}^\sigma(X) \dashrightarrow \mathcal{B}^\sigma(Y)$$

whose domain of definition is the set of configurations $[\mathfrak{s}, A, s, \phi]$ on X with $\phi|_Y$ nonzero. The map r is given by

$$[\mathfrak{s}, A, s, \phi] \mapsto [\mathfrak{s}|_Y, A|_Y, s/c, c\phi|_Y],$$

where $1/c$ is the L^2 norm of $\phi|_Y$. (We have identified the spin bundle S on Y with the restriction of S^+.) When X is cylinder $I \times Y$, with I a compact interval, we have a similar restriction map

$$r_t : \mathcal{B}^\sigma(I \times Y) \dashrightarrow \mathcal{B}^\sigma(Y)$$

for each $t \in I$.

If X is noncompact, and in particular if $X = \mathbb{R} \times Y$, then our definition of the blow-up needs to be modified, because the L^2 norm of ϕ need not be finite. Instead, we define $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$ as the space of isomorphism classes of quadruples $[\mathfrak{s}, A, \psi, \mathbb{R}^+\phi]$, where A is a Spin^c connection of class $L^2_{l,\mathrm{loc}}$, the set $\mathbb{R}^+\phi$ is the closed ray generated by a nonzero spinor ϕ in $L^2_{k,\mathrm{loc}}(X; S^+)$, and ψ belongs to the ray. (We write $\mathbb{R}^+$ for the *nonnegative* reals.) This is the usual way to define the blow up of a vector space at 0, without the use of a norm. The configuration is *reducible* if ψ is zero.

4.4. *The four-dimensional equations.* When X is compact, the Seiberg-Witten monopole equations for a configuration $\gamma = [\mathfrak{s}, A, s, \phi]$ in $\mathcal{B}^\sigma(X)$ are the equations

$$\begin{aligned} \frac{1}{2}\rho(F^+_{A^t}) - s^2(\phi\phi^*)_0 &= 0 \\ D^+_A\phi &= 0, \end{aligned} \tag{6}$$

where $\rho : \Lambda^+(X) \to i\mathfrak{su}(S^+)$ is Clifford multiplication and $(\phi\phi^*)_0$ denotes the traceless part of this hermitian endomorphism of S^+. When X is noncompact, we can write down essentially the same equations using the "norm-free" definition of the blow-up, $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$. In either case, we write these equations as

$$\mathcal{F}(\gamma) = 0.$$

In the compact case, we write

$$M(X) \subset \mathcal{B}^\sigma(X)$$

for the set of solutions. We draw attention to the noncompact case by writing $M_{\mathrm{loc}}(X) \subset \mathcal{B}^\sigma_{\mathrm{loc}}(X)$.

Take X to be the cylinder $\mathbb{R} \times Y$, and suppose that $\gamma = [\mathfrak{s}, A, \psi, \mathbb{R}^+\phi]$ is an element of $M_{\mathrm{loc}}(\mathbb{R} \times Y)$. A unique continuation result implies that the restriction of ϕ to each slice $\{t\} \times Y$ is nonzero; so there is a well-defined restriction

$$\check{\gamma}(t) = r_t(\gamma) \in \mathcal{B}^\sigma(Y)$$

for all t. We have the following relation between the equations $\mathcal{F}(\gamma) = 0$ and the vector field $\mathcal{V}^\sigma$:

LEMMA 4.1. *If γ is in $M_{\mathrm{loc}}(\mathbb{R} \times Y)$, then the corresponding path $\check{\gamma}$ is a smooth path in the Banach manifold-with-boundary $\mathcal{B}^\sigma(Y)$ satisfying*

$$\frac{d}{dt}\check{\gamma}(t) = -\mathcal{V}^\sigma.$$

Every smooth path $\check{\gamma}$ satisfying the above condition arises from some element of $M_{\mathrm{loc}}(\mathbb{R} \times Y)$ in this way.

We should note at this point that our sign convention is such that the 4-dimensional Dirac operator D_A^+ on the cylinder $\mathbb{R} \times Y$, for a connection A pulled back from Y, is equivalent to the equation

$$\frac{d}{dt}\phi + D_A\phi = 0$$

for a time-dependent section of the spin bundle $S \to Y$.

Next we define the moduli spaces that we will use to construct the Floer groups.

Definition 4.2. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two zeros of the vector field $\mathcal{V}^\sigma$ in the blow-up $\mathcal{B}^\sigma(Y)$. We write $M(\mathfrak{a}, \mathfrak{b})$ for the set of solutions $\gamma \in M_{\mathrm{loc}}(\mathbb{R} \times Y)$ such that the corresponding path $\check{\gamma}(t)$ is asymptotic to $\mathfrak{a}$ as $t \to -\infty$ and to $\mathfrak{b}$ as $t \to +\infty$.

Let $W : Y_0 \to Y_1$ be an oriented cobordism, and suppose the metric on W is cylindrical in collars of the two boundary components. Let W^* be the cylindrical-end manifold obtained by attaching cylinders $\mathbb{R}^- \times Y_0$ and $\mathbb{R}^+ \times Y_1$. From a solution γ in $M_{\mathrm{loc}}(W^*)$, we obtain paths $\check{\gamma}_0 : \mathbb{R}^- \to \mathcal{B}^\sigma(Y_0)$ and $\check{\gamma}_1 : \mathbb{R}^+ \to \mathcal{B}^\sigma(Y_1)$. The following moduli spaces will be used to construct the maps on the Floer groups arising from the cobordism W:

Definition 4.3. Let $\mathfrak{a}$ and $\mathfrak{b}$ be zeros of the vector field $\mathcal{V}^\sigma$ in $\mathcal{B}^\sigma(Y_0)$ and $\mathcal{B}^\sigma(Y_1)$ respectively. We write $M(\mathfrak{a}, W^*, \mathfrak{b})$ for the set of solutions $\gamma \in M_{\mathrm{loc}}(W^*)$ such that the corresponding paths $\check{\gamma}_0(t)$ and $\check{\gamma}_1(t)$ are asymptotic to $\mathfrak{a}$ and $\mathfrak{b}$ as $t \to -\infty$ and $t \to +\infty$ respectively.

4.4.1. *Example.* In example 4.2.1, suppose the spectrum is simple, let $\mathfrak{a}_\lambda \in \partial\mathcal{B}^\sigma(Y)$ be the critical point corresponding to the eigenvalue λ, and let ϕ_λ be a corresponding eigenvector of D_{A_0}. Then the reducible locus $M^{\mathrm{red}}(\mathfrak{a}_\lambda, \mathfrak{a}_\mu)$ in the moduli space $M(\mathfrak{a}_\lambda, \mathfrak{a}_\mu)$ is the quotient by $\mathbb{C}^*$ of the set of solutions ϕ to the Dirac equation

$$\frac{d}{dt}\phi + D_{A_0}\phi = 0$$

on the cylinder, with asymptotics

$$\phi \sim \begin{cases} C_0 e^{-\lambda t}\phi_\lambda, & \text{as } t \to -\infty, \\ C_1 e^{-\mu t}\phi_\mu, & \text{as } t \to +\infty, \end{cases}$$

for some nonzero constants C_0, C_1.

4.5. *Transversality and perturbations.* Let $\mathfrak{a} \in \mathcal{B}^\sigma(Y)$ be a zero of $\mathcal{V}^\sigma$. The derivative of the vector field at this point is a Fredholm operator on Sobolev completions of the tangent space,

$$\mathcal{D}_\mathfrak{a}\mathcal{V}^\sigma : \mathcal{T}_{k-1/2}(Y)_\mathfrak{a} \to \mathcal{T}_{k-3/2}(Y)_\mathfrak{a}.$$

Because of the blow-up, this operator is not symmetric (for any simple choice of inner product on the tangent space); but its spectrum is real and discrete. We say that $\mathfrak{a}$ is *nondegenerate* as a zero of $\mathcal{V}^\sigma$ if 0 is not in the spectrum. If $\mathfrak{a}$ is a nondegenerate zero, then it is isolated, and we can decompose the tangent space as

$$\mathcal{T}_{k-1/2} = \mathcal{K}^+_\mathfrak{a} \oplus \mathcal{K}^-_\mathfrak{a},$$

where $\mathcal{K}^+_\mathfrak{a}$ and $\mathcal{K}^-_\mathfrak{a}$ are the closures of the sum of the generalized eigenvectors belonging to positive (respectively, negative) eigenvalues. The *stable manifold* of $\mathfrak{a}$ is the set

$$\mathcal{S}_\mathfrak{a} = \{\, r_0(\gamma) \mid \gamma \in M_{\mathrm{loc}}(\mathbb{R}^+ \times Y),\ \lim_{t \to +\infty} r_t(\gamma) = \mathfrak{a} \,\}.$$

The unstable manifold $\mathcal{U}_\mathfrak{a}$ is defined similarly. If $\mathfrak{a}$ is nondegenerate, these are locally closed Banach submanifolds of $\mathcal{B}^\sigma(Y)$ (possibly with boundary), and their tangent spaces at $\mathfrak{a}$ are the spaces $\mathcal{K}^+_{\mathfrak{a}}$ and $\mathcal{K}^-_{\mathfrak{a}}$ respectively. Via the map $\gamma \mapsto \gamma_0$, we can identify $M(\mathfrak{a}, \mathfrak{b})$ with the intersection

$$M(\mathfrak{a}, \mathfrak{b}) = \mathcal{S}_{\mathfrak{b}} \cap \mathcal{U}_{\mathfrak{a}}.$$

In general, there is no reason to expect that the zeros are all nondegenerate. (In particular, if $b_1(Y)$ is nonzero then the reducible critical points are never isolated.) To achieve nondegeneracy we perturb the equations, replacing the Chern-Simons-Dirac functional $\mathcal{L}$ by $\mathcal{L} + f$, where f belongs to a suitable class $\mathcal{P}(Y)$ of gauge-invariant functions on $\mathcal{C}(Y)$. We write $\tilde{\mathfrak{q}}$ for the gradient of f on $\mathcal{C}(Y)$, and $\mathfrak{q}^\sigma$ for the resulting vector field on the blow-up. Instead of the flow equation of Lemma 4.1, we now look (formally) at the equation

$$\frac{d}{dt}\check{\gamma}(t) = -\mathcal{V}^\sigma - \mathfrak{q}^\sigma.$$

Solutions of this perturbed flow equation correspond to solutions $\gamma \in \mathcal{B}^\sigma(\mathbb{R}\times Y)$ of an equation $\mathcal{F}_{\mathfrak{q}}(\gamma) = 0$ on the 4-dimensional cylinder. We do not define the class of perturbations $\mathcal{P}(Y)$ here (see [23]).

The first important fact is that we can choose a perturbation f from the class $\mathcal{P}(Y)$ so that all the zeros of $\mathcal{V}^\sigma + \mathfrak{q}^\sigma$ are nondegenerate. From this point on we suppose that such a perturbation is chosen. We continue to write $M(\mathfrak{a}, \mathfrak{b})$ for the moduli spaces, $\mathcal{S}_{\mathfrak{a}}$ and $\mathcal{U}_{\mathfrak{a}}$ for the stable and unstable manifolds, and so on, without mention of the perturbation. The irreducible zeros will be a finite set; but as in Example 4.2.1, the number of reducible critical points will be infinite. In general, there is one reducible critical point $\mathfrak{a}_\lambda$ in the blow-up for each pair (α, λ), where $\alpha = [\mathfrak{s}, A, 0]$ is a zero of the restriction of $\mathcal{V} + \mathfrak{q}$ to $\mathcal{B}^{\mathrm{red}}(Y)$, and λ is an eigenvalue of a perturbed Dirac operator $D_{A,\mathfrak{q}}$. The point $\mathfrak{a}_\lambda$ is given by $[\mathfrak{s}, A, 0, \phi_\lambda]$, where ϕ_λ is a corresponding eigenvector, just as in the example.

Definition 4.4. We say that a reducible critical point $\mathfrak{a} \in \partial\mathcal{B}^\sigma(Y)$ is boundary-stable if the normal vector to the boundary at $\mathfrak{a}$ belongs to $\mathcal{K}^+_{\mathfrak{a}}$. We say $\mathfrak{a}$ is boundary-unstable if the normal vector belongs to $\mathcal{K}^-_{\mathfrak{a}}$.

In our description above, the critical point $\mathfrak{a}_\lambda$ is boundary-stable if $\lambda > 0$ and boundary-unstable if $\lambda < 0$. If $\mathfrak{a}$ is boundary-stable, then $\mathcal{S}_{\mathfrak{a}}$ is a manifold-with-boundary, and $\partial\mathcal{S}_{\mathfrak{a}}$ is the reducible locus $\mathcal{S}^{\mathrm{red}}_{\mathfrak{a}}$. The unstable manifold $\mathcal{U}_{\mathfrak{a}}$ is then contained in $\partial\mathcal{B}^\sigma(Y)$. If $\mathfrak{a}$ is boundary-unstable, then $\mathcal{U}_{\mathfrak{a}}$ is a manifold-with-boundary, while $\mathcal{S}_{\mathfrak{a}}$ is contained in the $\partial\mathcal{B}^\sigma(Y)$.

The Morse-Smale condition for the flow of the vector field $\mathcal{V}^\sigma + \mathfrak{q}^\sigma$ would ask that the intersection $\mathcal{S}_{\mathfrak{b}} \cap \mathcal{U}_{\mathfrak{a}}$ is a transverse intersection of Banach submanifolds in $\mathcal{B}^\sigma(Y)$, for every pair of critical points. We cannot demand this condition, because if $\mathfrak{a}$ is boundary-stable and $\mathfrak{b}$ is boundary-unstable, then $\mathcal{U}_{\mathfrak{a}}$ and $\mathcal{S}_{\mathfrak{b}}$ are both contained in $\partial\mathcal{B}^\sigma(Y)$. In this special case, the best we can ask is that the intersection be transverse in the boundary.

Definition 4.5. We say that the moduli space $M(\mathfrak{a},\mathfrak{b})$ is boundary-obstructed if $\mathfrak{a}$ and $\mathfrak{b}$ are both reducible, $\mathfrak{a}$ is boundary-stable and $\mathfrak{b}$ is boundary-unstable.

Definition 4.6. We say that a moduli space $M(\mathfrak{a},\mathfrak{b})$ is regular if the intersection $\mathcal{S}_{\mathfrak{a}} \cap \mathcal{U}_{\mathfrak{b}}$ is transverse, either as an intersection in the Banach manifold-with-boundary $\mathcal{B}^\sigma(Y)$ or (in the boundary-obstructed case) as an intersection in $\partial\mathcal{B}^\sigma(Y)$. We say the perturbation is regular if:

(1) all the zeros of $\mathcal{V}^\sigma + \mathfrak{q}^\sigma$ are nondegenerate;

(2) all the moduli spaces are regular; and

(3) there are no reducible critical points in the components $\mathcal{B}^\sigma(Y,\mathfrak{s})$ belonging to Spin^c structures $\mathfrak{s}$ with $c_1(\mathfrak{s})$ nontorsion.

The class $\mathcal{P}(Y)$ is large enough to contain regular perturbations, and we suppose henceforth that we have chosen a perturbation of this sort. The moduli spaces $M(\mathfrak{a},\mathfrak{b})$ will be either manifolds or manifolds-with-boundary, and the latter occurs only if $\mathfrak{a}$ is boundary-unstable and $\mathfrak{b}$ is boundary-stable. We write $M^{\mathrm{red}}(\mathfrak{a},\mathfrak{b})$ for the reducible configurations in the moduli space

4.5.1. *Remark.* The moduli space $M(\mathfrak{a},\mathfrak{b})$ cannot contain any irreducible elements if $\mathfrak{a}$ is boundary-stable or if $\mathfrak{b}$ is boundary-unstable.

We can decompose $M(\mathfrak{a},\mathfrak{b})$ according to the relative homotopy classes of the paths $\check{\gamma}(t)$: we write

$$M(\mathfrak{a},\mathfrak{b}) = \bigcup_z M_z(\mathfrak{a},\mathfrak{b}),$$

where the union is over all relative homotopy classes z of paths from $\mathfrak{a}$ to $\mathfrak{b}$. For any points $\mathfrak{a}$ and $\mathfrak{b}$ and any relative homotopy class z, we can define an integer $\mathrm{gr}_z(\mathfrak{a},\mathfrak{b})$ (as the index of a suitable Fredholm operator), so that

$$\dim M_z(\mathfrak{a},\mathfrak{b}) = \begin{cases} \mathrm{gr}_z(\mathfrak{a},\mathfrak{b}) + 1, & \text{in the boundary-obstructed case,} \\ \mathrm{gr}_z(\mathfrak{a},\mathfrak{b}), & \text{otherwise,} \end{cases}$$

whenever the moduli space is nonempty. The quantity $\mathrm{gr}_z(\mathfrak{a},\mathfrak{b})$ is additive along composite paths. We refer to $\mathrm{gr}_z(\mathfrak{a},\mathfrak{b})$ as the *formal dimension* of the moduli space $M_z(\mathfrak{a},\mathfrak{b})$.

Let $W : Y_0 \to Y_1$ be a cobordism, and suppose $\mathfrak{q}_0$ and $\mathfrak{q}_1$ are regular perturbations for the two 3-manifolds. Form the Riemannian manifold W^* by attaching cylindrical ends as before. We perturb the equations $\mathcal{F}(\gamma) = 0$ on the compact manifold W by a perturbation $\mathfrak{p}$ that is supported in cylindrical collar-neighborhoods of the boundary components. The perturbation $\mathfrak{p}$ near the boundary component Y_i is defined by a t-dependent element of $\mathcal{P}(Y_i)$, equal to $\mathfrak{q}_0$ in a smaller neighborhood of the boundary. We continue to denote the solution set of the perturbed equations $\mathcal{F}_{\mathfrak{p}}(\gamma) = 0$ by $M(W) \subset \mathcal{B}^\sigma(W)$. This is a Banach manifold with boundary, and there is a restriction map

$$r_{0,1} : M(W) \to \mathcal{B}^\sigma(Y_0) \times \mathcal{B}^\sigma(Y_1).$$

The cylindrical-end moduli space $M(\mathfrak{a}, W^*, \mathfrak{b})$ can be regarded as the inverse image of $\mathcal{U}_{\mathfrak{a}} \times \mathcal{S}_{\mathfrak{b}}$ under $r_{0,1}$:

$$\begin{tikzcd} M(\mathfrak{a}, W^*, \mathfrak{b}) \arrow[r] \arrow[d] & \mathcal{U}_{\mathfrak{a}} \times \mathcal{S}_{\mathfrak{b}} \arrow[d] \\ M(W) \arrow[r, "r_{0,1}"] & \mathcal{B}^\sigma(Y_0) \times \mathcal{B}^\sigma(Y_1). \end{tikzcd}$$

The moduli space $M(\mathfrak{a}, W^*, \mathfrak{b})$ is *boundary-obstructed* if $\mathfrak{a}$ is boundary-stable and $\mathfrak{b}$ is boundary-unstable.

Definition 4.7. If $M(\mathfrak{a}, W^*, \mathfrak{b})$ is not boundary-obstructed, we say that the moduli space is regular if $r_{0,1}$ is transverse to $\mathcal{U}_{\mathfrak{a}} \times \mathcal{S}_{\mathfrak{b}}$. In the boundary-obstructed case, $M(\mathfrak{a}, W^*, \mathfrak{b})$ consists entirely of reducibles, and we say that it is regular if the restriction

$$r^{\mathrm{red}}_{0,1} : M^{\mathrm{red}}(W) \to \partial\mathcal{B}^\sigma(Y_0) \times \partial\mathcal{B}^\sigma(Y_1)$$

is transverse to $\mathcal{U}_{\mathfrak{a}} \times \mathcal{S}_{\mathfrak{b}}$.

One can always choose the perturbation $\mathfrak{p}$ on W so that the moduli spaces $M(\mathfrak{a}, W^*, \mathfrak{b})$ are all regular. Each moduli space has a decomposition

$$M(\mathfrak{a}, W^*, \mathfrak{b}) = \bigcup_z M_z(\mathfrak{a}, W^*, \mathfrak{b})$$

indexed by the connected components z of the fiber $r^{-1}_{0,1}(\mathfrak{a}, \mathfrak{b})$ of the map

$$r_{0,1} : \mathcal{B}^\sigma(W) \dashrightarrow \mathcal{B}^\sigma(Y_0) \times \mathcal{B}^\sigma(Y_1).$$

The set of these components is a principal homogeneous space for the group $H^2(W, Y_0 \cup Y_1; \mathbb{Z})$. We can define an integer $\mathrm{gr}_z(\mathfrak{a}, W, \mathfrak{b})$ which is additive for composite cobordisms, such that the dimension of the nonempty moduli spaces is given by:

$$\dim M_z(\mathfrak{a}, W^*, \mathfrak{b}) = \begin{cases} \mathrm{gr}_z(\mathfrak{a}, W, \mathfrak{b}) + 1, & \text{in the boundary-obstructed case,} \\ \mathrm{gr}_z(\mathfrak{a}, W, \mathfrak{b}), & \text{otherwise.} \end{cases}$$

4.6. *Compactness.* We suppose now that a regular perturbation $\mathfrak{q}$ is fixed. The moduli space $M(\mathfrak{a}, \mathfrak{b})$ has an action of $\mathbb{R}$, by translations of the cylinder $\mathbb{R} \times Y$. We write $\breve{M}(\mathfrak{a}, \mathfrak{b})$ for the quotient by $\mathbb{R}$ of the nonconstant solutions:

$$\breve{M}(\mathfrak{a}, \mathfrak{b}) = \{\, \gamma \in M(\mathfrak{a}, \mathfrak{b}) \mid r_t(\gamma) \text{ is nonconstant} \,\}/\mathbb{R}.$$

We refer to elements of $\breve{M}(\mathfrak{a},\mathfrak{b})$ as unparametrized trajectories. The space $\breve{M}_z(\mathfrak{a},\mathfrak{b})$ has a compactification: the space of broken (unparametrized) trajectories $\breve{M}_z^+(\mathfrak{a},\mathfrak{b})$. This space is the union of all products

$$\breve{M}_{z_1}(\mathfrak{a}_0,\mathfrak{a}_1)\times\cdots\times\breve{M}_{z_l}(\mathfrak{a}_{l-1},\mathfrak{a}_l), \tag{7}$$

where $\mathfrak{a}_0=\mathfrak{a}$, $\mathfrak{a}_l=\mathfrak{b}$ and the composite of the paths z_i is z.

Because of the presence of boundary-obstructed trajectories, the enumeration of the strata that contribute to the compactification is more complicated than it would be for a Morse-Smale flow. For example:

LEMMA 4.8. *If $\breve{M}_z(\mathfrak{a},\mathfrak{b})$ is zero-dimensional, then it is compact. If $\breve{M}_z(\mathfrak{a},\mathfrak{b})$ is one-dimensional and contains irreducible trajectories, then the nonempty products* (7) *that contribute to the compactification $\breve{M}_z^+(\mathfrak{a},\mathfrak{b})$ are of two types*:

(1) *products $\breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times\breve{M}_{z_2}(\mathfrak{a}_1,\mathfrak{b})$ with two factors*;

(2) *products $\breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times\breve{M}_{z_2}(\mathfrak{a}_1,\mathfrak{a}_2)\times\breve{M}_{z_3}(\mathfrak{a}_2,\mathfrak{b})$ with three factors, of which the middle one is boundary-obstructed.*

The situation for the reducible parts of the moduli spaces is simpler. If $\breve{M}_z^{\mathrm{red}}(\mathfrak{a},\mathfrak{b})$ has dimension one, then its compactification involves only broken trajectories with two components,

$$\breve{M}_{z_1}^{\mathrm{red}}(\mathfrak{a},\mathfrak{a}_1)\times\breve{M}_{z_2}^{\mathrm{red}}(\mathfrak{a}_1,\mathfrak{b}).$$

In [23], gluing theorems are proved that describe the structure of the compactification $\breve{M}_z^+(\mathfrak{a},\mathfrak{b})$ near a stratum of the type (7). In the case of a 1-dimensional moduli space containing irreducibles (as in the lemma above), the compactification is a C^0 manifold with boundary in a neighborhood of the strata of the first type. At a point belonging to a stratum of the second type (with three factors), the structure of the compactification is more complicated: a neighborhood of such a point can be embedded in the positive quadrant $\mathbb{R}^+\times\mathbb{R}^+$ as the zero set of a continuous function that is strictly positive on the positive x-axis, strictly negative on the positive y-axis, and zero at the origin. We refer to this structure (more general than a 1-manifold with boundary) as a codimension-1 δ-structure. Despite the extra complication, spaces with this structure share with compact 1-manifolds the fact that the number of boundary points is even:

LEMMA 4.9. *Let $N=N_1\cup N_0$ be a compact space, containing an open subset N_1 that is a smooth* 1*-manifold and a closed complement N_0 that is a finite set. Suppose N has a codimension-*1 *δ-structure in the neighborhood of each point of N_0. Then $|N_0|$ is even.*

4.6.1. *Remark.* In the case that $\mathfrak{a}$ is boundary-unstable and $\mathfrak{b}$ is boundary-stable, the space $\breve{M}_z(\mathfrak{a},\mathfrak{b})$ is already a manifold-with-boundary before compactification: the boundary is $\breve{M}_z^{\mathrm{red}}(\mathfrak{a},\mathfrak{b})$.

The moduli spaces $M_z(\mathfrak{a},W^*,\mathfrak{b})$ can be compactified in a similar way. For example, if $M_z(\mathfrak{a},W^*,\mathfrak{b})$ contains irreducibles and is one-dimensional, then it has a compactification obtained by adding strata that are products of either two or three factors. Those involving two factors have one of the two possible shapes

$$\begin{gathered}\breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times M_{z_2}(\mathfrak{a}_1,W^*,\mathfrak{b})\\ M_{z_1}(\mathfrak{a},W^*,\mathfrak{b}_1)\times \breve{M}_{z_2}(\mathfrak{b}_1,\mathfrak{b})\end{gathered}\tag{8}$$

where the $\mathfrak{a}$'s belong to $\mathcal{B}^\sigma(Y_0)$ and the $\mathfrak{b}$'s belong to $\mathcal{B}^\sigma(Y_1)$. Those involving three factors have one of the three possible shapes

$$\begin{gathered}\breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times \breve{M}_{z_2}(\mathfrak{a}_1,\mathfrak{a}_2)\times M_{z_3}(\mathfrak{a}_2,W^*,\mathfrak{b})\\ \breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times M_{z_2}(\mathfrak{a}_1,W^*,\mathfrak{b}_1)\times \breve{M}_{z_3}(\mathfrak{b}_1,\mathfrak{b})\\ M_{z_1}(\mathfrak{a},W^*,\mathfrak{b}_1)\times \breve{M}_{z_2}(\mathfrak{b}_1,\mathfrak{b}_2)\times \breve{M}_{z_3}(\mathfrak{b}_2,\mathfrak{b}).\end{gathered}\tag{9}$$

In the case of three factors, the middle factor is boundary-obstructed. All these strata are finite sets, and the compactification $M^+(\mathfrak{a},W^*,\mathfrak{b})$ has a codimension-1 δ-structure at each point.

4.7. *Three Morse complexes.* Let $\mathfrak{C}^s$, $\mathfrak{C}^u$ and $\mathfrak{C}^o$ denote the set of critical points (zeros of $\mathcal{V}^\sigma+\mathfrak{q}^\sigma$ in $\mathcal{B}^\sigma(Y)$) that are boundary-stable, boundary-unstable, and irreducible respectively. Let

$$C^s(Y),\quad C^u(Y),\quad C^o(Y)$$

denote vector spaces over $\mathbb{F}$, with bases $e_{\mathfrak{a}}$ indexed by the elements $\mathfrak{a}$ of these three sets. For every pair of critical points $\mathfrak{a}$, $\mathfrak{b}$, we define

$$n_z(\mathfrak{a},\mathfrak{b})=\begin{cases}|\breve{M}_z(\mathfrak{a},\mathfrak{b})| \bmod 2, & \text{if } \dim \breve{M}_z(\mathfrak{a},\mathfrak{b})=0,\\ 0, & \text{otherwise.}\end{cases}$$

From these, we construct linear maps

$$\begin{aligned}\partial^o_o &: C^o(Y)\to C^o(Y)\\ \partial^o_s &: C^o(Y)\to C^s(Y)\\ \partial^u_o &: C^u(Y)\to C^o(Y)\\ \partial^u_s &: C^u(Y)\to C^s(Y)\end{aligned}$$

by the formulae

$$\partial^o_o e_{\mathfrak{a}}=\sum_{\mathfrak{b}\in\mathfrak{C}^o}\sum_z n_z(\mathfrak{a},\mathfrak{b})e_{\mathfrak{b}},\quad (\mathfrak{a}\in\mathfrak{C}^o),$$

and so on. The four maps correspond to the four cases in which a space of trajectories can contain irreducibles; see Remark 4.5.1 above.

Along with the $n_z(\mathfrak{a}, \mathfrak{b})$, we define quantities $\bar{n}_z(\mathfrak{a}_\mathfrak{b})$ using the reducible parts of the moduli spaces:

$$\bar{n}_z(\mathfrak{a}, \mathfrak{b}) = \begin{cases} |\breve{M}_z^{\mathrm{red}}(\mathfrak{a}, \mathfrak{b})| \bmod 2, & \text{if } \dim \breve{M}_z^{\mathrm{red}}(\mathfrak{a}, \mathfrak{b}) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

These are used similarly as the matrix entries of linear maps

$$\begin{aligned} \bar{\partial}^s_s &: C^s(Y) \to C^s(Y) \\ \bar{\partial}^s_u &: C^s(Y) \to C^u(Y) \\ \bar{\partial}^u_s &: C^u(Y) \to C^s(Y) \\ \bar{\partial}^u_u &: C^u(Y) \to C^u(Y). \end{aligned}$$

Note that the maps $\bar{\partial}^u_s$ and ∂^u_s are different. The former counts reducible elements in zero-dimensional moduli spaces $\breve{M}_z^{\mathrm{red}}(\mathfrak{a}, \mathfrak{b})$, where the corresponding irreducible moduli space $\breve{M}(\mathfrak{a}, \mathfrak{b})$ will be 1-dimensional.

LEMMA 4.10. *We have the following identities*:

$$\begin{aligned} \partial^o_o \partial^o_o + \partial^u_o \bar{\partial}^s_u \partial^o_s &= 0 \\ \partial^o_s \partial^o_o + \bar{\partial}^s_s \partial^o_s + \partial^u_s \bar{\partial}^s_u \partial^o_s &= 0 \\ \partial^o_o \partial^u_o + \partial^u_o \bar{\partial}^u_u + \partial^u_o \bar{\partial}^s_u \partial^u_s &= 0 \\ \bar{\partial}^u_s + \partial^o_s \partial^u_o + \bar{\partial}^s_s \partial^u_s + \partial^u_s \bar{\partial}^u_u + \partial^u_s \bar{\partial}^s_u \partial^u_s &= 0. \end{aligned}$$

Proof. All four identities are proved by enumerating the end-points of all the 1-dimensional moduli spaces $\breve{M}^+_z(\mathfrak{a}, \mathfrak{b})$ that contain irreducibles, using Lemma 4.8 and Lemma 4.9. In the last identity of the four, the extra term $\bar{\partial}^u_s$ is accounted for by Remark 4.6.1. □

Using the reducible parts of the moduli spaces, we obtain the simpler result:

LEMMA 4.11. *We have the following identities*:

$$\begin{aligned} \bar{\partial}^s_s \bar{\partial}^s_s + \bar{\partial}^u_s \bar{\partial}^s_u &= 0 \\ \bar{\partial}^s_s \bar{\partial}^u_s + \bar{\partial}^u_s \bar{\partial}^u_u &= 0 \\ \bar{\partial}^u_u \bar{\partial}^s_u + \bar{\partial}^s_u \bar{\partial}^s_s &= 0 \\ \bar{\partial}^u_u \bar{\partial}^u_u + \bar{\partial}^s_u \bar{\partial}^u_s &= 0. \end{aligned}$$ □

Definition 4.12. We construct three vector spaces with differentials, $(\check{C}(Y), \check{\partial})$, $(\hat{C}(Y), \hat{\partial})$ and $(\bar{C}(Y), \bar{\partial})$, by setting

$$\begin{aligned} \check{C}(Y) &= C^o(Y) \oplus C^s(Y) \\ \hat{C}(Y) &= C^o(Y) \oplus C^u(Y) \\ \bar{C}(Y) &= C^s(Y) \oplus C^u(Y), \end{aligned}$$

and defining

$$\check{\partial} = \begin{bmatrix} \partial^o_o & \partial^u_o \bar{\partial}^s_u \\ \partial^o_s & \bar{\partial}^s_s + \partial^u_s \bar{\partial}^s_u \end{bmatrix}, \hat{\partial} = \begin{bmatrix} \partial^o_o & \partial^u_o \\ \bar{\partial}^s_u \partial^o_s & \bar{\partial}^u_u + \bar{\partial}^s_u \partial^u_s \end{bmatrix}, \bar{\partial} = \begin{bmatrix} \bar{\partial}^s_s & \bar{\partial}^u_s \\ \bar{\partial}^s_u & \bar{\partial}^u_u \end{bmatrix}.$$

The proof that the differentials $\check{\partial}$, $\hat{\partial}$ and $\bar{\partial}$ each have square zero follows by elementary manipulation of the identities in the previous two lemmas. We define the Floer homology groups

$$\widecheck{HM}_*(Y), \quad \widehat{HM}_*(Y), \quad \overline{HM}_*(Y)$$

as the homology of the three complexes above. Each of these is a sum of subspaces contributed by the connected components $\mathcal{B}^\sigma(Y, \mathfrak{s})$ of $\mathcal{B}^\sigma(Y)$, so that

$$\widecheck{HM}_*(Y) = \bigoplus_{\mathfrak{s}} \widecheck{HM}_*(Y, \mathfrak{s})$$

for example. After choosing a base-point, we can grade the complex $\check{C}(Y, \mathfrak{s})$ by $\mathbb{Z}/d\mathbb{Z}$, where $d\mathbb{Z}$ is the subgroup of $\mathbb{Z}$ arising as the image of the map

$$z \mapsto \mathrm{gr}_z(\mathfrak{a}, \mathfrak{a})$$

from $\pi_1(\mathcal{B}^\sigma(Y, \mathfrak{s}), \mathfrak{a})$ to $\mathbb{Z}$. This image is contained in $2\mathbb{Z}$ and coincides with the image of the map (2). The $\bullet$ versions of the Floer groups are obtained by completion, as explained in Section 2.4.

To motivate the formalism a little, it may be helpful to say that the construction of these complexes can also be carried out (with less technical difficulty) in the case that we replace $\mathcal{B}^\sigma(Y)$ by a finite-dimensional manifold with boundary, $(B, \partial B)$. In the finite-dimensional case, the complexes compute respectively the ordinary homology groups,

$$H_*(B; \mathbb{F}), \quad H_*(B, \partial B; \mathbb{F}), \quad H_*(\partial B; \mathbb{F}).$$

The long exact sequence (1) is analogous to the long exact sequence of a pair $(B, \partial B)$. The maps i_*, j_* and p_* arise from maps i, j, p on the chain complexes of Definition 4.12, given by the matrices

$$i = \begin{bmatrix} 0 & \partial^u_o \\ 1 & \partial^u_s \end{bmatrix}, \qquad j = \begin{bmatrix} 1 & 0 \\ 0 & \bar{\partial}^s_u \end{bmatrix}, \qquad p = \begin{bmatrix} \partial^o_s & \partial^u_s \\ 0 & 1 \end{bmatrix}.$$

The exactness of the sequence is a formal consequence of the identities.

Up to canonical isomorphism, the Floer groups are independent of the choice of metric and perturbation that are involved in their construction. As in Floer's original argument [11], this independence follows from the more general construction of maps from cobordisms, and their properties.

4.8. *Maps from cobordisms.* Let $W : Y_0 \to Y_1$ be a cobordism equipped with a Riemannian metric and a regular perturbation $\mathfrak{p}$ so that the moduli spaces $M_z(\mathfrak{a}, W^*, \mathfrak{b})$ are regular. For each pair of critical points $\mathfrak{a}$, $\mathfrak{b}$ belonging to Y_0 and Y_1 respectively, let

$$m_z(\mathfrak{a}, W, \mathfrak{b}) = \begin{cases} |M_z(\mathfrak{a}, W^*, \mathfrak{b})| \bmod 2, & \text{if } \dim M_z(\mathfrak{a}, W^*, \mathfrak{b}) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\bar{m}_z(\mathfrak{a}, W, \mathfrak{b})$ for reducible critical points similarly, using $M_z^{\mathrm{red}}(\mathfrak{a}, W^*, \mathfrak{b})$. These provide the matrix entries of eight linear maps

$$\begin{aligned} m_o^o &: C_\bullet^o(Y_0) \to C_\bullet^o(Y_1) \\ m_s^o &: C_\bullet^o(Y_0) \to C_\bullet^s(Y_1) \\ m_o^u &: C_\bullet^u(Y_0) \to C_\bullet^o(Y_1) \\ m_s^u &: C_\bullet^u(Y_0) \to C_\bullet^s(Y_1) \end{aligned}$$

and

$$\begin{aligned} \bar{m}_s^s &: C_\bullet^s(Y_0) \to C_\bullet^s(Y_1) \\ \bar{m}_u^s &: C_\bullet^s(Y_0) \to C_\bullet^u(Y_1) \\ \bar{m}_s^u &: C_\bullet^u(Y_0) \to C_\bullet^s(Y_1) \\ \bar{m}_u^u &: C_\bullet^u(Y_0) \to C_\bullet^u(Y_1), \end{aligned}$$

with definitions parallel to the those of the maps ∂_o^o etc. above. For example,

$$\bar{m}_s^s(e_\mathfrak{a}) = \sum_{\mathfrak{b} \in \mathfrak{C}^s(Y_1)} \sum_z \bar{m}_z(\mathfrak{a}, W, \mathfrak{b}) e_\mathfrak{b}, \quad (\mathfrak{a} \in \mathfrak{C}^s(Y_0)).$$

The bullets denote completion, which is necessary because, for a given $\mathfrak{a}$, there are infinitely many $\mathfrak{b}$ for which $\bar{m}_{\mathfrak{b},z}^{\mathfrak{a}}$ may be nonzero for some z. (For a given $\mathfrak{a}$ and $\mathfrak{b}$, only finitely many z can contribute.) Again, m_s^u and $\bar{m}_s^u$ are different maps. By enumerating the boundary points of 1-dimensional moduli spaces $M^+(\mathfrak{a}, W^*, \mathfrak{b})$ and appealing to Lemma 4.9, we obtain identities involving these operators. For example, by considering such moduli spaces for which the endpoints $\mathfrak{a}$ and $\mathfrak{b}$ are both irreducible, we obtain the identity

$$m_o^o \partial_o^o + \partial_o^o m_o^o + \partial_o^u \bar{\partial}_u^s m_s^o + \partial_o^u \bar{m}_u^s \partial_s^o + m_o^u \bar{\partial}_u^s \partial_s^o = 0. \tag{10}$$

The five terms in this identity enumerate the boundary points of each of the five types described in (8) and (9).

We combine these linear maps to define maps

$$\begin{aligned} \check{m}(W) &: \check{C}_\bullet(Y_0) \to \check{C}_\bullet(Y_1), \\ \hat{m}(W) &: \hat{C}_\bullet(Y_0) \to \hat{C}_\bullet(Y_1), \\ \bar{m}(W) &: \bar{C}_\bullet(Y_0) \to \bar{C}_\bullet(Y_1), \end{aligned}$$

by the formulae

$$\check{m}(W) = \begin{bmatrix} m_o^o & m_o^u \bar{\partial}_u^s + \partial_o^u \bar{m}_u^s \\ m_s^o & \bar{m}_s^s + m_s^u \bar{\partial}_u^s + \partial_s^u \bar{m}_u^s \end{bmatrix},$$

$$\hat{m}(W) = \begin{bmatrix} m_o^o & m_o^u \\ \bar{m}_u^s \partial_s^o + \bar{\partial}_u^s m_s^o & \bar{m}_u^u + \bar{m}_u^s \partial_s^u + \bar{\partial}_u^s m_s^u \end{bmatrix},$$

and

$$\bar{m}(W) = \begin{bmatrix} \bar{m}^s_s & \bar{m}^u_s \\ \bar{m}^s_u & \bar{m}^u_u \end{bmatrix}.$$

Identities such as (10) supply the proof of:

PROPOSITION 4.13. *The maps $\check{m}(W)$, $\hat{m}(W)$ and $\bar{m}(W)$ are chain maps, and they commute up to homotopy with i, j and p.*

We define $\widecheck{HM}(W)$, $\widehat{HM}(W)$ and $\overline{HM}(W)$ to be the maps on the Floer homology groups arising from the chain maps $\check{m}(W)$, $\hat{m}(W)$ and $\bar{m}(W)$.

4.9. *Families of metrics.* The chain maps $\check{m}(W)$ depend on a choice of Riemannian metric g and perturbation $\mathfrak{p}$ on W. Let P be a smooth manifold, perhaps with boundary, and let g_p and $\mathfrak{p}_p$ be a smooth family of metrics and perturbations on W, for $p \in P$. We suppose that there are collar neighborhoods of the boundary components Y_0 and Y_1 on which all the g_p are equal to the same fixed cylindrical metrics and on which all the $\mathfrak{p}_p$ agree with the given regular perturbations $\mathfrak{q}_0$ and $\mathfrak{q}_1$. We can form a *parametrized* moduli space over P, as the union

$$\begin{aligned} M(\mathfrak{a}, W^*, \mathfrak{b})_P &= \bigcup_p \{p\} \times M(\mathfrak{a}, W^*, \mathfrak{b})_p \\ &\subset P \times \mathcal{B}^\sigma_{\mathrm{loc}}(W^*). \end{aligned}$$

Regularity for such moduli spaces is defined as the transversality of the map

$$r_{0,1} : M(W)_P \to \mathcal{B}^\sigma(Y_0) \times \mathcal{B}^\sigma(Y_1)$$

to the submanifold $\mathcal{U}_{\mathfrak{a}} \times \mathcal{S}_{\mathfrak{b}}$, with the usual adaptation in the boundary-obstructed cases. If P has boundary Q, then we take regularity of $M(\mathfrak{a}, W^*, \mathfrak{b})_P$ to include also the condition that $M(\mathfrak{a}, W^*, \mathfrak{b})_Q$ is regular. Given any family of metrics g_p for $p \in P$, and any family of perturbations $\mathfrak{p}_q$ for $q \in Q$ such that the moduli spaces $M(\mathfrak{a}, W^*, \mathfrak{b})_Q$ are regular, we can choose an extension of the family $\mathfrak{p}_q$ to all of P in such a way that all the moduli spaces $M(\mathfrak{a}, W^*, \mathfrak{b})_P$ are regular also.

Now suppose that P is compact, with boundary Q. For each $\mathfrak{a}$ and $\mathfrak{b}$, define

$$m_z(\mathfrak{a}, W, \mathfrak{b})_P = \begin{cases} |M^P_z(\mathfrak{a}, W^*, \mathfrak{b})| \bmod 2, & \text{if } \dim M^P_z(\mathfrak{a}, W^*, \mathfrak{b}) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Use the moduli spaces of reducible solutions to define $\bar{m}_z(\mathfrak{a}, W, \mathfrak{b})_P$ similarly. From these, we construct linear maps

$$\check{m}(W)_P : \check{C}_\bullet(Y_0) \to \check{C}_\bullet(Y_1)$$

with companion maps $\hat{m}(W)_P$ and $\bar{m}(W)_P$. If the boundary Q is empty, then these are chain maps, just as in the case that P is a point: the proof is by enumeration of the boundary points in the compactifications of 1-dimensional moduli spaces $M_z(\mathfrak{a}, W^*, \mathfrak{b})_P$. (The compactification is constructed as the parametrized union of the moduli spaces $M_z^+(\mathfrak{a}, W^*, \mathfrak{b})_p$ over P.)

If Q is nonempty, then the boundary of $M_z(\mathfrak{a}, W^*, \mathfrak{b})_P$ has an additional contribution, namely the moduli space $M_z(\mathfrak{a}, W^*, \mathfrak{b})_Q$. Identities such as (10) therefore have an additional term: we have, for example (one of eight similar identities),

(11)
$$(m^o_o)_P\partial^o_o + \partial^o_o(m^o_o)_P + \partial^u_o\bar{\partial}^s_u(m^o_s)_P + \partial^u_o(\bar{m}^s_u)_P\partial^o_s + (m^u_o)_P\bar{\partial}^s_u\partial^o_s = (m^o_o)_Q.$$

The maps $\check{m}(W)_P$ etc. are no longer chain maps: instead, we have

$$\begin{aligned}\check{\partial}\check{m}(W)_P + \check{m}(W)_P\check{\partial} &= \check{m}(W)_Q\\ \hat{\partial}\hat{m}(W)_P + \hat{m}(W)_P\hat{\partial} &= \hat{m}(W)_Q\\ \bar{\partial}\bar{m}(W)_P + \bar{m}(W)_P\bar{\partial} &= \bar{m}(W)_Q.\end{aligned}$$

Thus $\check{m}(W)_Q$ is chain-homotopic to zero, and $\check{m}(W)_P$ provides the chain-homotopy. If we take P to be the interval $[0,1]$ and Q to be the boundary $\{0,1\}$, we obtain:

COROLLARY 4.14. *The chain maps* $\check{m}(W)_0$ *and* $\check{m}(W)_1$ *from* $\check{C}_\bullet(Y_0)$ *to* $\check{C}_\bullet(Y_1)$, *corresponding to two different choices of metric and regular perturbation on the interior of* W, *are chain homotopic, and therefore induce the same map on Floer homology. The same holds for the other two flavors.*

4.10. *Composing cobordisms.* Let $W : Y_0 \to Y_2$ be a composite cobordism

$$W : Y_0 \xrightarrow{W_0} Y_1 \xrightarrow{W_1} Y_2.$$

Equip W with a metric which is cylindrical near the two boundary components as well as in a neighborhood of $Y_1 \subset W$, and let $\mathfrak{p}$ be a perturbation that agrees with the regular perturbations $\mathfrak{q}_i$ near Y_i for $i = 0, 1, 2$. For each $T \geq 0$, let $W(T) \cong W$ be the Riemannian manifold obtained by cutting along Y_1 and inserting a cylinder $[-T, T] \times Y_1$ with the product metric. We can form the parametrized union

$$\bigcup_{T\geq 0} M_z(\mathfrak{a}, W(T)^*, \mathfrak{b}). \tag{12}$$

Since the manifolds $W(T)$ are all copies of W with varying metric, this can be seen as an example of a parametrized moduli space $M_z(\mathfrak{a}, W^*, \mathfrak{b})_P$ of the sort we have been considering, with $P \cong [0, \infty)$. We add a fiber over $T = \infty$ by setting

$$W(\infty)^* = W_0^* \amalg W_1^*,$$

a disjoint union of the two cylindrical end manifolds. We define

$$M_z(\mathfrak{a}, W(\infty)^*, \mathfrak{b})$$

to be the union of products

$$\bigcup_{\mathfrak{c}} \bigcup_{z_0, z_1} M_{z_0}(\mathfrak{a}, W_0^*, \mathfrak{c}) \times M_{z_1}(\mathfrak{c}, W_1^*, \mathfrak{b}). \tag{13}$$

The union is over all pairs of classes z_0, z_1 with composite z and all $\mathfrak{c} \in \mathfrak{C}(Y_1)$. Putting in this extra fiber, we have a family

$$M_z(\mathfrak{a}, W^*, \mathfrak{b})_{\bar{P}} = \bigcup_{T \in [0,\infty]} \{T\} \times M_z(\mathfrak{a}, W(T)^*, \mathfrak{b}),$$

parametrized by the space $\bar{P} \cong [0, \infty]$. The moduli space just defined is a noncompact manifold with boundary. The boundary consists of the union of the two fibers over $T = 0$ and $T = \infty$, together with the reducible locus $M_z^{\mathrm{red}}(\mathfrak{a}, W^*, \mathfrak{b})_{\bar{P}}$ in the case that the moduli space contains both reducibles and irreducibles. It is contained in a compact space

$$M_z^+(\mathfrak{a}, W^*, \mathfrak{b})_{\bar{P}} = \bigcup_{T \in [0,\infty]} \{T\} \times M_z^+(\mathfrak{a}, W(T)^*, \mathfrak{b}),$$

where for $T = \infty$ a typical element of $M_z^+(\mathfrak{a}, W(T)^*, \mathfrak{b})$ is a quintuple

$$(\gamma_0, \gamma_{01}, \gamma_1, \gamma_{12}, \gamma_2),$$

where γ_i is a broken trajectory for Y_i (possibly with zero components) and γ_{01}, γ_{12} belong to the moduli spaces of W_0 and W_1.

LEMMA 4.15. *If $M_z(\mathfrak{a}, W^*, \mathfrak{b})_P$ is zero-dimensional, then it is compact. If $M_z(\mathfrak{a}, W^*, \mathfrak{b})_P$ is one-dimensional and contains irreducible trajectories, then the compactification $M_z^+(\mathfrak{a}, W^*, \mathfrak{b})_{\bar{P}}$ is a* 1*-dimensional space with a codimension-*1 *δ-structure at all boundary points. The boundary points are of the following types*:

(1) *the fiber over $T = 0$, namely the space $M_z(\mathfrak{a}, W^*, \mathfrak{b})$ for $W(0) = W$;*

(2) *the fiber over $T = \infty$, namely the union of products* (13);

(3) *products of two factors, of one of the forms*

$$\breve{M}_{z_1}(\mathfrak{a}, \mathfrak{a}_1) \times M_{z_2}(\mathfrak{a}_1, W^*, \mathfrak{b})_P$$
$$M_{z_1}(\mathfrak{a}, W^*, \mathfrak{b}_1)_P \times \breve{M}_{z_2}(\mathfrak{b}_1, \mathfrak{b})$$

(*cf.* (8) *above*);

(4) *products of three factors, of one of the forms*

$$\breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times\breve{M}_{z_2}(\mathfrak{a}_1,\mathfrak{a}_2)\times M_{z_3}(\mathfrak{a}_2,W^*,\mathfrak{b})_P$$
$$\breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times M_{z_2}(\mathfrak{a}_1,W^*,\mathfrak{b}_1)_P\times\breve{M}_{z_3}(\mathfrak{b}_1,\mathfrak{b})$$
$$M_{z_1}(\mathfrak{a},W^*,\mathfrak{b}_1)_P\times\breve{M}_{z_2}(\mathfrak{b}_1,\mathfrak{b}_2)\times\breve{M}_{z_3}(\mathfrak{b}_2,\mathfrak{b})$$

(*cf.* (9) *above*);

(5) *parts of the fiber* $M_z^+(\mathfrak{a},W(\infty)^*,\mathfrak{b})$ *over* $T=\infty$ *of one the forms*

$$\breve{M}_{z_1}(\mathfrak{a},\mathfrak{a}_1)\times M_{z_2}(\mathfrak{a}_1,W_0^*,\mathfrak{c})\times M_{z_2}(\mathfrak{c},W_1^*,\mathfrak{b})$$
$$M_{z_1}(\mathfrak{a},W_0^*,\mathfrak{c}_1)\times\breve{M}_{z_2}(\mathfrak{c}_1,\mathfrak{c})\times M_{z_3}(\mathfrak{c},W_1^*,\mathfrak{b})$$
$$M_{z_1}(\mathfrak{a},W_0^*,\mathfrak{c})\times M_{z_2}(\mathfrak{c},W_1^*,\mathfrak{b}_1)\times\breve{M}_{z_3}(\mathfrak{b}_1,\mathfrak{b}),$$

where the middle factor is boundary-obstructed in each case;

(6) *the reducible locus* $M_z^{\mathrm{red}}(\mathfrak{a},W^*,\mathfrak{b})_P$ *in the case that the moduli space contains both irreducibles and reducibles* (*which requires* $\mathfrak{a}$ *to be boundary-unstable and* $\mathfrak{b}$ *to be boundary-stable*).

Following a familiar pattern, we now count the elements in the zero-dimensional moduli spaces, to obtain elements of $\mathbb{F}$:

$$m_z(\mathfrak{a},W,\mathfrak{b})_P=\begin{cases}|M_z(\mathfrak{a},W^*,\mathfrak{b})_P| \bmod 2, & \text{if } \dim M_z(\mathfrak{a},W^*,\mathfrak{b})_P=0,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\bar{m}_z(\mathfrak{a},W,\mathfrak{b})_P=\begin{cases}|M_z^{\mathrm{red}}(\mathfrak{a},W^*,\mathfrak{b})_P| \bmod 2, & \text{if } \dim M_z^{\mathrm{red}}(\mathfrak{a},W^*,\mathfrak{b})_P=0,\\ 0, & \text{otherwise.}\end{cases}$$

These become the matrix entries of maps

$$\begin{aligned}K_o^o&: C_\bullet^o(Y_0)\to C_\bullet^o(Y_2)\\ K_s^o&: C_\bullet^o(Y_0)\to C_\bullet^s(Y_2)\\ K_o^u&: C_\bullet^u(Y_0)\to C_\bullet^o(Y_2)\\ K_s^u&: C_\bullet^u(Y_0)\to C_\bullet^s(Y_2)\end{aligned}$$

and

$$\begin{aligned}\bar{K}_s^s&: C_\bullet^s(Y_0)\to C_\bullet^s(Y_2)\\ \bar{K}_u^s&: C_\bullet^s(Y_0)\to C_\bullet^u(Y_2)\\ \bar{K}_s^u&: C_\bullet^u(Y_0)\to C_\bullet^s(Y_2)\\ \bar{K}_u^u&: C_\bullet^u(Y_0)\to C_\bullet^u(Y_2),\end{aligned}$$

just as we defined m_o^o and its companions. From Lemma 4.15 and Lemma 4.9 we obtain identities involving these operators, as usual. For example, as an operator $C^o(Y_0)\to C^o(Y_2)$, we have

$$\begin{aligned}m_o^o(W)+m_o^o(W_1)m_o^o(W_0)+K_o^o\partial_o^o+\partial_o^oK_o^o+K_o^u\bar{\partial}_u^s\partial_s^o+\partial_o^u\bar{K}_u^s\partial_s^o+\partial_o^u\bar{\partial}_u^sK_s^o\\ +m_o^u(W_1)\bar{m}_u^s(W_0)\partial_s^o+m_o^u(W_1)\bar{\partial}_u^sm_s^o(W_0)+\partial_o^u\bar{m}_u^s(W_1)m_s^o(W_0)=0.\end{aligned}$$

The ten terms in this identity correspond to the ten possibilities listed in the first five cases of the lemma above. (The final case of the lemma does not apply.)

We combine the pieces K_o^o etc. to define a map

$$\check{K} : \check{C}_\bullet(Y_0) \to \check{C}_\bullet(Y_2)$$

by the matrix

$$\check{K} = \begin{bmatrix} K_o^o & K_o^u \bar{\partial}_u^s + m_o^u(W_1)\bar{m}_u^s(W_0) + \partial_o^u \bar{K}_u^s \\ K_s^o & \bar{K}_s^s + K_s^u \bar{\partial}_u^s + m_s^u(W_1)\bar{m}_u^s(W_0) + \partial_s^u \bar{K}_u^s \end{bmatrix}.$$

PROPOSITION 4.16. *We have the equality*

$$\check{\partial}\check{K} + \check{K}\check{\partial} = \check{m}(W_1)\check{m}(W_0) + \check{m}(W)$$

as maps $\check{C}_\bullet(Y_0) \to \check{C}_\bullet(Y_2)$. *At the level of homology therefore, we have*

$$\widecheck{HM}(W_1) \circ \widecheck{HM}(W_0) = \widecheck{HM}(W).$$

There are similar identities for the other two flavors of Floer homology.

Proof. The chain-homotopy identity is a formal consequence of the ten-term identity above, together with its seven companions and the corresponding identities for the ∂ and m operators. □

4.11. *The module structure.* We describe now a way to define the $\mathbb{F}[U]$-modules structure on Floer homology. A different and more general approach is taken in [23], but the result is the same, and the present version of the definition (based on [8]) is simpler to describe. Let $W : Y_0 \to Y_1$ be a cobordism, and let $w_1, \dots, w_p \in W$ be chosen points. Let $B_1, \dots, B_p$ be standard ball neighborhoods of these points. The space $\mathcal{B}^\sigma(B_q)$ is a Hilbert manifold with boundary; and because it arises as a free quotient by the gauge group $\mathcal{G}$ of L^2_{k+1} maps $u : B_q \to S^1$, there is a natural line bundle L_q on $\mathcal{B}^\sigma(B_q)$ associated to the homomorphism $u \mapsto u(w_q)$ from $\mathcal{G}$ to S^1.

Because of unique continuation, there is a well-defined restriction map

$$r_q : M(W) \to \mathcal{B}^\sigma(B_q),$$

and hence also

$$r_q : M(\mathfrak{a}, W^*, \mathfrak{b}) \to \mathcal{B}^\sigma(B_q),$$

for all $\mathfrak{a}$ and $\mathfrak{b}$. Let s_q be a smooth section of L_q, and let $V_q \subset \mathcal{B}^\sigma(B_q)$ be its zero set. Omitting the restriction maps that are implied by our notation, we now consider the moduli spaces

$$M_z(\mathfrak{a}, W^*, \{w_1, \dots, w_p\}, \mathfrak{b}) \subset M_z(\mathfrak{a}, W^*, \mathfrak{b})$$

defined as the intersection

$$M_z(\mathfrak{a}, W^*, \mathfrak{b}) \cap V_1 \cap \cdots \cap V_p.$$

We can choose the sections s_q so that, for all $\mathfrak{a}$ and $\mathfrak{b}$, their pull-backs of $s_1, \ldots, s_q$ to $M(\mathfrak{a}, W^*, \mathfrak{b})$ have transverse zero sets. The above intersection is then a smooth manifold.

We repeat verbatim the construction of the chain maps $\check{m}(W)$, $\hat{m}(W)$ and $\bar{m}(W)$ from Section 4.8, but replacing $M_z(\mathfrak{a}, W^*, \mathfrak{b})$ by the lower-dimensional moduli space $M_z(\mathfrak{a}, W^*, \{w_1, \ldots, w_p\}, \mathfrak{b})$ throughout. In this way, we construct maps that we temporarily denote by

$$\widecheck{HM}(W, \{w_1, \ldots, w_p\}) : \widecheck{HM}_\bullet(Y_0) \to \widecheck{HM}_\bullet(Y_1),$$

with similar maps for the other two flavors. As a special case, we define a map

$$U : \widecheck{HM}_\bullet(Y) \to \widecheck{HM}_\bullet(Y)$$

by taking $p = 1$ and taking W to be the cylinder $[0, 1] \times Y$. The proof of the composition law for composite cobordisms adapts to prove that U^p is equal to the map arising from the cylindrical cobordism with p base-points; and more generally,

$$\widecheck{HM}(W, \{w_1, \ldots, w_p\}) = U^p \widecheck{HM}(W),$$

(a formula which then makes the notation $\widecheck{HM}(W, \{w_1, \ldots, w_p\})$ obsolete).

4.12. *Local coefficients.* There is a variant of Floer homology, using local coefficients. We continue to work over the field $\mathbb{F} = \mathbb{Z}/2$, and we consider a local system of $\mathbb{F}$-vector spaces, Γ on $\mathcal{B}^\sigma(Y)$. This means that for each points $\mathfrak{a}$ in $\mathcal{B}^\sigma(Y)$ we have a vector space $\Gamma_\mathfrak{a}$ over $\mathbb{F}$, and for each relative homotopy-class of paths z from $\mathfrak{a}$ to $\mathfrak{b}$ we have an isomorphism

$$\Gamma(z) : \Gamma_\mathfrak{a} \to \Gamma_\mathfrak{b}.$$

These should satisfy the composition law for composite paths $\Gamma(z_1 \circ z_2) = \Gamma(z_2) \circ \Gamma(z_1)$. Given such a local system, and given as usual a Riemannian metric and regular perturbation for Y, we introduce vector spaces $C^o(Y; \Gamma)$, $C^s(Y; \Gamma)$ and $C^u(Y; \Gamma)$, defining them as

$$\bigoplus_\mathfrak{a} \Gamma_\mathfrak{a},$$

where the sum is over all critical points in $\mathcal{B}^\sigma(Y)$ that are irreducible, boundary-stable or boundary-unstable respectively. We define a map $\partial^o_o : C^o(Y; \Gamma) \to C^o(Y; \Gamma)$ by the formula

$$\partial^o_o(e) = \sum_{\mathfrak{b} \in \mathfrak{C}^o} \sum_z n_z(\mathfrak{a}, \mathfrak{b}) \Gamma(z)(e), \quad (e \in \Gamma_\mathfrak{a}),$$

where $n_z(\mathfrak{a}, \mathfrak{b})$ is defined as before. This map, along with its companions ∂^o_s etc., are then used to define the differential

$$\check{\partial} : \check{C}(Y; \Gamma) \to \check{C}(Y; \Gamma)$$

for the complex $\check{C}(Y; \Gamma) = C^o(Y; \Gamma) \oplus C^s(Y; \Gamma)$. Proceeding as before, we construct the Floer group $\widecheck{HM}_\bullet(Y; \Gamma)$, and also its companions $\widehat{HM}_\bullet(Y; \Gamma)$ and $\overline{HM}_\bullet(Y; \Gamma)$.

Let $W : Y_0 \to Y_1$ now be a cobordism, and suppose local systems Γ_i are given on Y_i for $i = 0, 1$. The restriction maps

$$r_i : \mathcal{B}^\sigma(W) \dashrightarrow \mathcal{B}^\sigma(Y_i), \quad (i = 1, 2)$$

are only partially defined, but the pull-backs $r_i^*(\Gamma_i)$ provide well-defined local systems on $\mathcal{B}^\sigma(W)$. This is because a local system Γ on $\mathcal{B}^\sigma(Y)$ is, in a canonical way, the pull-back of a local system on $\mathcal{B}(Y)$, and the restriction maps to $\mathcal{B}(Y_i)$ are everywhere defined.

Definition 4.17. A W-morphism from the local system Γ_0 on $\mathcal{B}^\sigma(Y_0)$ to the local system Γ_1 on $\mathcal{B}^\sigma(Y_1)$ is an isomorphism of local systems,

$$\Gamma_W : r_0^*(\Gamma_0) \to r_1^*(\Gamma_1).$$

Given $\mathfrak{a}$ in $\mathcal{B}^\sigma(Y_0)$ and $\mathfrak{b}$ in $\mathcal{B}^\sigma(Y_1)$, and given a choice z of a connected component in $r_{0,1}^{-1}(\mathfrak{a}, \mathfrak{b})$, a W-morphism provides us with an isomorphism

$$\Gamma_W(z) : \Gamma_\mathfrak{a} \to \Gamma_\mathfrak{b},$$

which behaves as expected with respect to composition on either side with paths in $\mathcal{B}^\sigma(Y_i)$. We can use Γ_W to define maps

$$m^o_o : C^o(Y_0; \Gamma_0) \to C^o(Y_1; \Gamma_1),$$

and so on, just as in Section 4.8 above: for example, we define

$$m^o_o(e) = \sum_{\mathfrak{b} \in \mathfrak{C}^o(Y_1)} \sum_z m_z(\mathfrak{a}, W, \mathfrak{b})\, \Gamma_W(z)(e), \quad (e \in \Gamma_{0,\mathfrak{a}}).$$

The result is a map

$$\widecheck{HM}(W; \Gamma_W) : \widecheck{HM}_\bullet(Y_0; \Gamma_0) \to \widecheck{HM}_\bullet(Y_1; \Gamma_1),$$

with companion maps on $\widehat{HM}$ and $\overline{HM}$. The proof of independence of the choice of metric and perturbation on W, and the proof of the composition law (with the obvious notion of composition of W-morphisms), carry over with straightforward modifications.

4.12.1. *Support of local systems.* Let Y be a three-manifold, and fix an open subset $M \subset Y$. There is a partially-defined restriction map $\rho_M : \mathcal{B}^\sigma(Y) \dashrightarrow \mathcal{B}^\sigma(M)$. A local system Γ over $\mathcal{B}^\sigma(M)$ induces a local system $\rho_M^*(\Gamma)$ over

$\mathcal{B}^\sigma(Y)$ by pull-back. (The fact that the ρ_M is only partially defined is again of no consequence, as above.)

Definition 4.18. A local system over $\mathcal{B}^\sigma(Y)$ which is obtained as the pull-back of one over $\mathcal{B}^\sigma(M)$ is said to be supported on M.

Similarly, suppose we have a cobordism $W\colon Y_0 \longrightarrow Y_1$, equipped with an open set $B \subset W$, and let $M_0 = B \cap Y_0$ and $M_1 = B \cap Y_1$. Let Γ_0 and Γ_1 be local systems on the M_i. Again, we have partially-defined restriction maps

$$\rho_0\colon \mathcal{B}^\sigma(B) \dashrightarrow \mathcal{B}^\sigma(M_i), \quad (i = 1, 2)$$

which induce well-defined local systems $\rho_i^*(\Gamma_i)$ over $\mathcal{B}^\sigma(B)$. A B-morphism of local systems is an isomorphism

$$\Gamma_B\colon \rho_0^*(\Gamma_0) \longrightarrow \rho_1^*(\Gamma_1)$$

of local systems over $\mathcal{B}^\sigma(B)$. Using the restriction map

$$\rho\colon \mathcal{B}^\sigma(W) \dashrightarrow \mathcal{B}^\sigma(B),$$

we can pull back a B-morphism Γ_B of local systems to obtain a W-morphism

$$\rho_B^*(\Gamma_B)\colon r_0^*(\rho_0^*(\Gamma_0)) \longrightarrow r_1^*(\rho_1^*(\Gamma_1)).$$

Such a W-morphism is said to be *supported on B*.

4.12.2. *Example.* Let η be a C^∞ singular 1-cycle in Y with real coefficients. Given a relative homotopy class of paths z from $\mathfrak{a}$ to $\mathfrak{b}$ in $\mathcal{B}^\sigma(Y)$, let us choose a representative path $\tilde{z}$, and let $[A_{\tilde{z}}, s, \phi_{\tilde{z}}]$ be the corresponding element of $\mathcal{B}^\sigma([0,1] \times Y)$. Define

$$f_\eta(z) = (i/2\pi) \int_{[0,1]\times\eta} F_{A^{\mathrm{t}}_{\tilde{z}}}.$$

This depends only on η and z.

Let $\mathbb{K}$ be an integral domain of characteristic 2, and let

$$\mu : \mathbb{R} \to \mathbb{K}^\times$$

be a homomorphism from the additive group $\mathbb{R}$ to the multiplicative group of units in $\mathbb{K}$. We can construct a local system Γ_η on $\mathcal{B}^\sigma(Y)$ by declaring that $\Gamma_{\eta,\mathfrak{a}}$ is $\mathbb{K}$ for all $\mathfrak{a}$, and that

$$\Gamma_\eta(z) : \Gamma_{\eta,\mathfrak{a}} \to \Gamma_{\eta,\mathfrak{b}}$$

is multiplication by the unit $\mu(f_\eta(z))$ in $\mathbb{K}^\times$. For definiteness, we henceforth take $\mathbb{K}$ to be the field of fractions of the group ring $\mathbb{F}[\mathbb{R}]$, and μ to be the natural inclusion

$$\mu : \mathbb{R} \to \mathbb{F}[\mathbb{R}]^\times \subset \mathbb{K}^\times.$$

Now let $W : Y_0 \to Y_1$ be a cobordism, let η_0, η_1 be 1-cycles as above, and let Γ_{η_i} be the corresponding local systems. Suppose we are given a C^∞ singular 2-chain ν in W, with

$$\partial\nu = \eta_1 - \eta_0.$$

Given a component z in $r_{0,1}^{-1}(\mathfrak{a}, \mathfrak{b})$, we choose a representative $[A_z, s, \phi_z]$ in $\mathcal{B}^\sigma(W)$, and we extend our notation above by setting

$$f_\nu(z) = (i/2\pi) \int_\nu F_{A_z}.$$

We can define a W-morphism $\Gamma_{W,\nu} : \Gamma_{\eta_0} \to \Gamma_{\eta_1}$ by specifying that the isomorphism

$$\Gamma_{W,\nu}(z) : \Gamma_{\eta_0,\mathfrak{a}} \to \Gamma_{\eta_1,\mathfrak{b}}$$

is given by multiplication by $\mu(f_\nu(z))$. There are corresponding maps

$$\begin{aligned}
\widecheck{HM}(W; \Gamma_{W,\nu}) &: \widecheck{HM}_\bullet(Y_0; \Gamma_{\eta_0}) \to \widecheck{HM}_\bullet(Y_1; \Gamma_{\eta_1})\\
\widehat{HM}(W; \Gamma_{W,\nu}) &: \widehat{HM}_\bullet(Y_0; \Gamma_{\eta_0}) \to \widehat{HM}_\bullet(Y_1; \Gamma_{\eta_1})\\
\overline{HM}(W; \Gamma_{W,\nu}) &: \overline{HM}_\bullet(Y_0; \Gamma_{\eta_0}) \to \overline{HM}_\bullet(Y_1; \Gamma_{\eta_1}).
\end{aligned}$$

We can consider these constructions as defining functors on an extension of our cobordism category. We have a category whose objects are pairs (Y, η), where Y is a 3-manifold (compact, connected and oriented as usual) and η is a C^∞ singular 1-cycle with real coefficients. The morphisms are diffeomorphism classes of pairs (W, ν), where ν is a 2-cycle and $\partial\nu = \eta_1 - \eta_0$.

From the definitions, it follows that if $\tilde{\nu} = \nu + \partial\theta$ for some C^∞ 3-chain θ, then $\Gamma_{W,\nu}$ and $\Gamma_{W,\tilde{\nu}}$ are equal. As a consequence, there are isomorphisms (for example)

$$\widecheck{HM}_\bullet(Y; \Gamma_\eta) \cong \widecheck{HM}_\bullet(Y; \Gamma_{\eta'})$$

whenever $[\eta] = [\eta']$ in $H_1(Y; \mathbb{R})$. However, to specify a particular isomorphism, one must express $\eta - \eta'$ as a boundary. Indeed, suppose that $\partial\nu_1 = \eta - \eta' = \partial\nu_2$, then the two isomorphisms differ by the automorphism which on $\widecheck{HM}_\bullet(Y, \mathfrak{t}; \Gamma_\eta)$ is given by multiplication by $\mu(\langle c_1(\mathfrak{t}), [\nu - \nu'] \rangle)$. In particular, when Y is a rational homology three-sphere and η is a cycle, then there is a canonical identification

$$\widecheck{HM}(Y; \Gamma_\eta) \cong \widecheck{HM}(Y) \otimes \mathbb{K}.$$

If the 1-cycle η is contained in $M \subset Y$, then the local coefficient system Γ_η is supported on M, in the sense of the definition above. Moreover, suppose $W : Y_0 \to Y_1$ is a cobordism, $B \subset W$ is an open set with $M_i = B \cap Y_i$, and η_i are 1-cycles in M_i. Let ν be a 2-cycle with $\partial\nu = \eta_1 - \eta_0$. Then the W-morphism $\Gamma_{W,\nu} : \Gamma_{\eta_0} \to \Gamma_{\eta_1}$ is supported on B if ν is contained in B.

We conclude this section by noting the following result. (The only particular property of our choice of $\mathbb{K}$ and μ that is used here is the fact that $1-\mu(t)$ is a unit, for all nonzero t.)

LEMMA 4.19. *If* $[\eta]$ *is nonzero in* $H_1(Y;\mathbb{R})$, *then* $\overline{HM}_\bullet(Y;\Gamma_\eta) = 0$ *and hence* $j_* : \widecheck{HM}_\bullet(Y;\Gamma_\eta) \to \widehat{HM}_\bullet(Y;\Gamma_\eta)$ *is an isomorphism.*

Proof. If $c_1(\mathfrak{s})$ is nontorsion, there are no reducible critical points, so $\overline{HM}_\bullet(Y;\Gamma_\eta)$ has contributions only from those $\mathfrak{s}$ with torsion first Chern class. For each such $\mathfrak{s}$, the space of reducibles in $\mathcal{B}(Y,\mathfrak{s})$ has the homotopy type of the torus $H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$, for it deformation-retracts onto the torus T of Spin^c connections A in S with $F_{A^t} = 0$. The lemma is a consequence of the fact that the ordinary homology group $H_*(T;\Gamma_\eta)$ with local coefficients is zero. Details are given in [23]. $\square$

4.13. *Duality and pairings.* Along with the chain complex $(\check{C}_*(Y),\check{\partial})$ and its companions, we have the corresponding cochain complexes,

$$(\check{C}^*(Y),\check{\partial}^*), \quad (\hat{C}^*(Y),\hat{\partial}^*), \quad (\bar{C}^*(Y),\bar{\partial}^*),$$

and the monopole Floer cohomology groups $\widecheck{HM}^*(Y)$, $\widehat{HM}^*(Y)$, $\overline{HM}^*(Y)$. These are modules over $\mathbb{F}[U]$, with U now acting with degree 2. To form the $\bullet$ versions $\widecheck{HM}^\bullet(Y)$, we should now complete in the direction of increasing degree, so that they again become modules over $\mathbb{F}[[U]]$. We can do the same with local coefficients, and we have nondegenerate pairings of $\mathbb{K}$-vector spaces

$$\widecheck{HM}^j(Y;\Gamma_{-\eta}) \times \widecheck{HM}_j(Y;\Gamma_\eta) \to \mathbb{K}, \tag{14}$$

for any C^∞ real 1-cycle η.

Let $-Y$ denote the oriented manifold obtained from Y by reversing the orientation. The spaces $\mathcal{B}^\sigma(Y)$ and $\mathcal{B}^\sigma(-Y)$ can be canonically identified, though the change of orientation changes the sign of the functional $\mathcal{L}$, and so changes the vector field $\mathcal{V}^\sigma$ to $-\mathcal{V}^\sigma$. If $\mathfrak{q}$ is a regular perturbation for Y, we can select $-\mathfrak{q}$ as regular perturbation for $-Y$. The notion of boundary-stable and boundary-unstable are interchanged when the vector field changes sign, so we have identifications

$$\begin{aligned}\mathfrak{C}^o(Y) &= \mathfrak{C}^o(-Y)\\ \mathfrak{C}^s(Y) &= \mathfrak{C}^u(-Y)\\ \mathfrak{C}^u(Y) &= \mathfrak{C}^s(-Y).\end{aligned}$$

The moduli space $M(\mathfrak{a},\mathfrak{b})$ for the cylinder $\mathbb{R}\times Y$ is the same as the moduli space $M(\mathfrak{b},\mathfrak{a})$ for the cylinder $\mathbb{R}\times(-Y)$. So the operator ∂^o_s on $-Y$, for example, becomes $(\partial^u_o)^*$ for Y. In this way, the boundary map $\hat{\partial}$ for $-Y$ becomes the operator $\check{\partial}^*$ for Y, and so on. Thus we obtain the following proposition, which we state also for local coefficients.

Proposition 4.20. *There are canonical isomorphisms*

$$\begin{aligned}\mathrm{D}&:\widecheck{HM}_j(-Y)\to\widehat{HM}^j(Y)\\ \mathrm{D}&:\widehat{HM}_j(-Y)\to\widecheck{HM}^j(Y)\\ \mathrm{D}&:\overline{HM}_j(-Y)\to\overline{HM}^j(Y).\end{aligned}$$

If η is a real 1-cycle in Y, and Γ_η the corresponding local system with fiber $\mathbb{K}$, then there are also isomorphisms

$$\begin{aligned}\mathrm{D}&:\widecheck{HM}_j(-Y;\Gamma_\eta)\to\widehat{HM}^j(Y;\Gamma_\eta)\\ \mathrm{D}&:\widehat{HM}_j(-Y;\Gamma_\eta)\to\widecheck{HM}^j(Y;\Gamma_\eta)\\ \mathrm{D}&:\overline{HM}_j(-Y;\Gamma_\eta)\to\overline{HM}^j(Y;\Gamma_\eta).\end{aligned}$$

Here j belongs to the grading set $J(Y)$, which we can identify canonically with $J(-Y)$, because the notion of an oriented 2-plane field on Y makes no reference to the orientation of the manifold.

Note that the canonical identification $J(Y)=J(-Y)$ does not respect the action of $\mathbb{Z}$: there is a sign change, so $j+n$ becomes $j-n$ if the orientation of Y is reversed.

If we combine the isomorphisms D with the pairing (14), we obtain a nondegenerate pairing of vector spaces over $\mathbb{F}$,

$$\langle -\,,\,-\rangle_{\mathrm{D}}:\widehat{HM}_*(-Y)\times\widecheck{HM}_*(Y)\to\mathbb{F}.$$

With local coefficients, there is a nondegenerate pairing of vector spaces over $\mathbb{K}$:

$$\langle -\,,\,-\rangle_{\mathrm{D}}:\widehat{HM}_*(-Y;\Gamma_{-\eta})\times\widecheck{HM}_*(Y;\Gamma_\eta)\to\mathbb{K}.$$

4.14. *Calculations for lens spaces.* Let $S^3_p(U)$ be the lens space obtained by integer surgery on the unknot, for some $p>0$, and let $W(p):S^3_p(U)\to S^3$ be the surgery cobordism, as described in Section 3.1. Corollary 3.4 states that the map

$$\widehat{HM}(W(p),\mathfrak{s}_{n,p}):\widehat{HM}_\bullet(S^3_p(U),\mathfrak{t}_{n,p})\to\widehat{HM}_\bullet(S^3) \tag{15}$$

is an isomorphism if $0\le n\le p$, an assertion which is equivalent to the calculation of the Frøyshov invariant of $(S^3_p(U),\mathfrak{t}_{n,p})$ as given in Proposition 3.1. We shall now provide a proof of this result.

Recall that we have injective maps

$$\begin{aligned}p_*:\widehat{HM}_\bullet(S^3_p(U),\mathfrak{t}_{n,p})&\hookrightarrow\overline{HM}_\bullet(S^3_p(U),\mathfrak{t}_{n,p})\\ p_*:\widehat{HM}_\bullet(S^3)&\hookrightarrow\overline{HM}_\bullet(S^3)\end{aligned}$$

because these 3-manifolds have positive scalar curvature for their standard metrics, cf. Proposition 2.2. The complex $\bar{C}_*(S^3_p(U))$ can be described using the material of Example 4.2.1. In $\mathcal{B}(S^3_p(U), \mathfrak{t}_{n,p})$, we have a unique critical point $\alpha = [0, A_n, 0]$, with A^{t}_n flat. After a choice of small perturbation, we can assume that the perturbation $D_{A_n,\mathfrak{q}}$ of the Dirac operator D_{A_n} has simple spectrum; and there is then one nondegenerate critical point $\mathfrak{a}_\lambda$ in the blow-up $\mathcal{B}^\sigma(S^3_p(U), \mathfrak{t}_{n,p})$ for each eigenvalue λ of $D_{A_n,\mathfrak{q}}$. The differentials in the Floer complexes are all zero, and we see that

$$\begin{aligned}\overline{HM}_*(S^3_p(U), \mathfrak{t}_{n,p}) &= \bar{C}_*(S^3_p(U), \mathfrak{t}_{n,p})\\ &= \bigoplus_{\lambda} \mathbb{F}\, e_{\mathfrak{a}_\lambda}.\end{aligned}$$

The image $p_*\widehat{HM}_*(S^3_p(U), \mathfrak{t}_{n,p})$ is the subspace

$$\hat{C}_*(S^3_p(U), \mathfrak{t}_{n,p}) = \bigoplus_{\lambda<0} \mathbb{F}\, e_{\mathfrak{a}_\lambda},$$

which is generated as an $\mathbb{F}[U]$-module by the element $e_{\mathfrak{a}_{\lambda_{-1}}}$, where λ_{-1} is the first negative eigenvalue. Let $\mathfrak{b}_{\mu_{-1}}$ similarly denote the critical point in $\mathcal{B}^\sigma(S^3)$ corresponding to the generator of $p_*\widehat{HM}_*(S^3)$; so μ_{-1} is the first negative eigenvalue of the perturbed Dirac operator on S^3.

The assertion that the map (15) is surjective is equivalent to the assertion that

$$\widehat{HM}(W(p), \mathfrak{s}_{n,p})(e_{\mathfrak{a}_{\lambda_{-1}}}) = e_{\mathfrak{b}_{\mu_{-1}}}.$$

The moduli space $M^{\mathrm{red}}_{\mathfrak{s}_{n,p}}(\mathfrak{a}_{\lambda_{-1}}, W(p), \mathfrak{b}_{\mu_{-1}})$ can be identified with the space of equivalence classes of pairs $[A, \phi]$, where

- A is a Spin^c connection for $\mathfrak{s}_{n,p}$ on $W(p)^*$, satisfying $F^+_{A^{\mathrm{t}}} = 0$ and such that A^{t} is asymptotic to a flat connection on both ends; and
- ϕ is a section of S^+ on $W(p)^*$ satisfying a small-perturbation of the Dirac equation $D^+_{A,\mathfrak{p}}\phi = 0$ and having asymptotics

$$\begin{aligned}|\phi| &= O(e^{-\lambda_{-1}t}), t \to -\infty,\\ |\phi| &= O(e^{-\mu_{-1}t}), t \to +\infty,\end{aligned}$$

  on the two ends of $W(p)^*$.

(The equivalence relation is generated by the action of gauge treansformations and scaling ϕ by nonzero complex scalars.)

The connection A satisfying the first condition is unique up to gauge transformation. Given A, the spinors Φ satisfying the second condition form an open subset of a projective space of complex dimension $2d$, where d is the L^2 index of the Dirac operator D^+_A on $W(p)^*$. The moduli space is a point if $d = 0$. So the surjectivity of (15) is eventually equivalent to the next lemma.

LEMMA 4.21. *Let A be a* Spin^c *connection for a* Spin^c *structure* $\mathfrak{s}$ *on* $W(p)^*$, *and suppose that that* F_{A^t} *is anti-self-dual and is asymptotically zero on both ends. Then the index of the Fredholm operator*

$$D_A^+ : L^2_{1,A}(W(p)^*, S^+)) \to L^2(W(p)^*, S^-)$$

is zero if the pairing of $c_1(\mathfrak{s})$ *with a generator* h *of* $H_2(W(p); \mathbb{Z})$ *satisfies*

$$-p \leq \langle c_1(\mathfrak{s}), h \rangle \leq p.$$

Proof. If we change the orientation of $W(p)$, make a conformal change, and add two points, we obtain a Kähler orbifold $\bar{W}$, the weighted projective space obtained as the quotient of $\mathbb{C}^3 \setminus \{0\}$ by the action of $\mathbb{C}^*$ with weights $(1, 1, p)$. Because of its conformal invariance, we can equivalently study the index of the Dirac operator on $\bar{W}$. Let the Kähler form ω be normalized so as to have integral 1 on h, and let L_k be the orbifold line bundle on $\bar{W}$ with curvature $c_1(L_k) = k[\omega]$. Let $\mathfrak{s}_0$ be the canonical Spin^c structure (which has $c_1(\mathfrak{s}_0) = -(p+2)[\omega]$) and let $\mathfrak{s}_k$ be the Spin^c structure on $\bar{W}$ obtained by tensoring $\mathfrak{s}_0$ with L_k. Let A_k be a Spin^c connection for $\mathfrak{s}_k$ compatible with the holomorphic structure.

The kernel of $D^+_{A_k}$ is isomorphic to the orbifold Dolbeault cohomology group

$$H^{0,0}(\bar{W}; L_k) \oplus H^{0,2}(\bar{W}; L_k).$$

If $k < 0$, then $H^{0,0}$ vanishes because L_k then has negative degree. If $k > -p-2$, then $H^{0,2}$ vanishes, by Serre duality. So the kernel of the Dirac operator is zero for $-p-2 < k < 0$; or equivalently $-p-2 < c_1(\mathfrak{s}_k) < p+2$. The cokernel is $H^{0,1}(\bar{W}; L_k)$, which vanishes for all k. □

5. Proof of the surgery long exact sequence

We return to the notation of Theorem 2.4. Before starting the proof in earnest, we explain the simple argument which shows that the composites $\widecheck{HM}(W_{n+1}) \circ \widecheck{HM}(W_n)$ are zero. We use the composition law to equate this map to $\widecheck{HM}(X_n)$, where X_n is the composite cobordism,

$$X_n = W_n \cup_{Y_{n+1}} W_{n+1}$$

from Y_n to Y_{n+2}. Recall that each cobordism W_n arises from the addition of a single 2-handle. The core of the 2-handle in W_{n+1} attaches to the cocore of the 2-handle in W_n to form an embedded 2-sphere,

$$E_n \subset X_n.$$

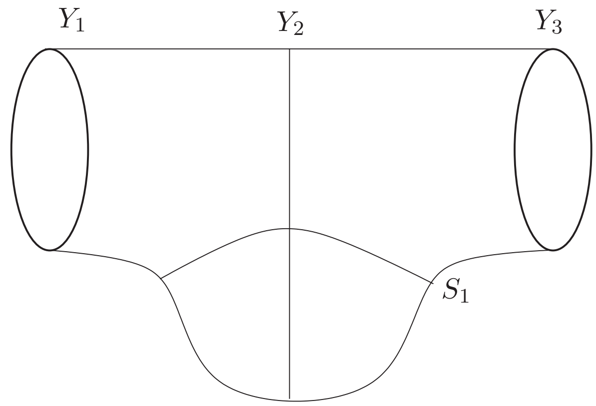

Figure 1: *Breaking up the composite.* This indicates two ways of breaking up the composite cobordism X_1. Y_2 separates X_1 into W_1 and W_2, while S_1 separates it into B_1 and Z_1.

This sphere has self-intersection number -1, and the boundary S_n of a tubular neighborhood of E_n is an embedded 3-sphere, giving an alternative decomposition

$$X_n = B_n \#_{S_n} Z_n,$$

where here B_n is another cobordism from Y_n to Y_{n+2} (obtained by a two-handle addition), punctured at point, and Z_n is the tubular neighborhood. (See Figure 1 for a schematic sketch of X_1.)

There is a diffeomorphism $\tau : X_n \to X_n$ which is the identity on B_n with

$$\tau_*[E_n] = -[E_n]$$

in $H_2(X_n; \mathbb{Z})$. If $\mathfrak{s}$ is a Spin^c structure on X_n, then $\langle c_1(\mathfrak{s}), [E_n] \rangle$ is odd; so τ acts on Spin$^c(X_n)$ without fixed points. We can write

$$\widetilde{HM}(X_n) = \sum_{\mathfrak{s}} \widetilde{HM}(X_n, \mathfrak{s}),$$

and by diffeomorphism-invariance, the contributions from $\mathfrak{s}$ and $\tau^*(\mathfrak{s})$ are equal. Since $\mathbb{F}$ has characteristic 2, the sum vanishes.

The proof that the sequence is exact is considerably harder than the proof that the composites are zero. We use the following straightforward result from homological algebra.

Lemma 5.1. *Let $\{C_n\}_{n \in \mathbb{Z}/3\mathbb{Z}}$ be a collection of chain complexes over $\mathbb{Z}/2\mathbb{Z}$ and let*

$$\{f_n \colon C_n \longrightarrow C_{n+1}\}_{n \in \mathbb{Z}/3\mathbb{Z}}$$

be a collection of chain maps with the following two properties:

(1) *the composite* $f_{n+1} \circ f_n : C_n \longrightarrow C_{n+2}$ *is chain-homotopic to zero, by a chain homotopy* H_n:

$$\partial \circ H_n + H_n \circ \partial = f_{n+1} \circ f_n;$$

(2) *the sum*

$$\psi_n = f_{n+2} \circ H_n + H_{n+1} \circ f_n \colon C_n \longrightarrow C_n$$

(*which is a chain map*) *induces isomorphisms on homology*, $(\psi_n)_* : H_*(C_n) \to H_*(C_n)$.

Then the sequence

$$\cdots \longrightarrow H_*(Y_{n-1}) \xrightarrow{(f_{n-1})_*} H_*(Y_n) \xrightarrow{(f_n)_*} H_*(Y_{n+1}) \longrightarrow \cdots$$

is exact.

We wish to apply the lemma to the chain maps $\check{m}(W_n)$; and while we know that the composites $\check{m}(W_{n+1})\check{m}(W_n)$ induce the zero map on Floer homology, we need an explicit chain-homotopy in order to apply the lemma. That is our goal in the next subsection.

5.1. *The first chain homotopy.* We now construct the required null-homotopy of $\check{m}(W_{n+1})\check{m}(W_n)$. Take $n = 1$, and equip X_1 with a metric g which is product-like near both the separating hypersurfaces Y_2 and S_1. We can arrange that the metric on S_1 is obtained by taking the round metric on S^3 and flattening it near a Clifford torus $Y_2 \cap S_1$ (all that is necessary here is that the metric is product-like normal to the torus $Y_2 \cap S_1$ and can be connected to a round metric through metrics with positive scalar curvature). From this, we construct a family of metrics $Q(S_1, Y_2)$ parametrized by $T \in \mathbb{R}$ as follows. When the parameter T for the family is negative, we insert a cylinder $[T, -T] \times S_1$ normal to S_1, and when it is positive, we insert a cylinder $[-T, T] \times Y_2$ normal to Y_2.

There is a corresponding parametrized moduli space

$$M_z(\mathfrak{a}, X_1^*, \mathfrak{b})_Q.$$

As in Section 4.10, we can complete the family of Riemannian manifolds: at $T = -\infty$ we obtain the disjoint union

$$X_1(-\infty)^* = B_1^* \amalg Z_1^*,$$

and at $T = +\infty$ we obtain

$$X_1(+\infty)^* = W_1^* \amalg W_2^*.$$

The manifold B_1^* has three cylindrical ends. There is now a moduli space

$$M_z(\mathfrak{a}, X_1^*, \mathfrak{b})_{\bar{Q}} \to \bar{Q}(S_1, Y_2)$$

over $\bar{Q}(S_1, Y_2) = [-\infty, \infty]$ and its compactification $M_z^+(\mathfrak{a}, X_1^*, \mathfrak{b})_{\bar{Q}}$, involving broken trajectories.

Define quantities $m_z(\mathfrak{a}, X_1, \mathfrak{b})_Q \in \mathbb{F}$ by counting elements in zero-dimensional moduli spaces $M_z(\mathfrak{a}, X_1^*, \mathfrak{b})_Q$ in the now familiar way, and define $\bar{m}_z(\mathfrak{a}, X_1, \mathfrak{b})_Q$ similarly, using $M_z^{\mathrm{red}}(\mathfrak{a}, X_1^*, \mathfrak{b})_Q$. We use these as the matrix entries of the linear map

$$H_o^o : C^o(Y_1) \to C^o(Y_3) \tag{16}$$

and its seven companions H_s^o, H_o^u, H_s^u, $\bar{H}_s^s$, $\bar{H}_u^s$, $\bar{H}_s^u$ and $\bar{H}_u^u$; and from these we construct a map $\check{H}_1$ by the same formula that defined the chain-homotopy $\check{K}$ in Section 4.10:

$$\check{H}_1 = \begin{bmatrix} H_o^o & H_o^u \bar{\partial}_u^s + m_o^u(W_2)\bar{m}_u^s(W_1) + \partial_o^u \bar{H}_u^s \\ H_s^o & \bar{H}_s^s + H_s^u \bar{\partial}_u^s + m_s^u(W_2)\bar{m}_u^s(W_1) + \partial_s^u \bar{H}_u^s \end{bmatrix}.$$

PROPOSITION 5.2. *If the chosen perturbation on $S_1 \cong S^3$ is sufficiently small, then we have*

$$\check{\partial} \circ \check{H}_1 + \check{H}_1 \circ \check{\partial} = \check{m}(W_2) \circ \check{m}(W_1)$$

as chain maps from $\check{C}_\bullet(Y_1)$ to $\check{C}_\bullet(Y_3)$.

Proof. The formula closely resembles the formula involving $\check{K}$ from Proposition 4.16. The chain homotopy $\check{K}$ was defined using the family of metrics parametrized by the positive half, $[0, \infty]$, of the family $\bar{Q}$. The fiber over $T = 0$ contributed the extra term $\check{m}(W)$ in the previous formula.

To prove the present proposition, we proceed as before, obtaining identities involving H_o^o and its companions by examining 1-dimensional moduli spaces $M_z^+(\mathfrak{a}, X_1^*, \mathfrak{b})_{\bar{Q}}$ and counting their boundary points. The new phenomena occur in examining the fiber of $T = -\infty$.

A typical element of $M_z^+(\mathfrak{a}, X_1^*, \mathfrak{b})_{\bar{Q}}$ in the fiber over $T = -\infty$ is a quintuple

$$(\breve{\gamma}_{Y_1}, \breve{\gamma}_{S_1}, \breve{\gamma}_{Y_3}, \gamma_{B_1}, \gamma_{Z_1}),$$

where the first three are broken trajectories on the corresponding cylinders (each of these may be empty) and γ_{B_1} and γ_{Z_1} are solutions on the corresponding cylindrical-end manifolds.

To understand which of these decompositions occur, we must understand the Floer complex for the three-sphere S_1. Since the S_1 has positive scalar curvature and is simply-connected, there is a unique (reducible) critical point in $\mathcal{B}(S_1)$. After a small perturbation, the critical points in $\mathcal{B}^\sigma(S_1)$ still lie over a single reducible configuration. We label these critical points in $\mathcal{B}^\sigma(S_1)$ as $\mathfrak{a}_{\lambda_i}$, where λ_i are the eigenvalues of a self-adjoint Fredholm operator obtained as a small perturbation of the Dirac operator on S^3. (See Example 4.2.1.) We assume that the λ_i are strictly increasing as i runs through $\mathbb{Z}$ and that λ_0 is the first positive eigenvalue:

$$\cdots \lambda_{-2} < \lambda_{-1} < 0 < \lambda_0 < \lambda_1 < \cdots . \tag{17}$$

It is a consequence of this description that $\breve{\gamma}_{S_1}$ *a priori* live in even-dimensional moduli spaces. In fact, by counting dimensions, we see that the trajectories $\breve{\gamma}_{S_1}$ in this fiber are empty.

We claim that the possibilities for γ_{Z_1} come in pairs. Specifically, regard Z_1 as a manifold with boundary S_1 (so that $-S_1$ is a boundary component of B_1). For each critical point, we have moduli spaces $M_z(Z_1, \mathfrak{a}_{\lambda_i})$. The choice of z is equivalent in this instance to a choice of Spin^c structure $\mathfrak{s}$ on Z_1, which in turn is determined by its first Chern class. We write z_k for the component corresponding to the Spin^c structure $\mathfrak{s}$ with $\langle c_1(\mathfrak{s}), [E_1]\rangle = 2k-1$.

LEMMA 5.3. *The following hold for a sufficiently small perturbation on S_1.*

(1) *The moduli spaces $M_z(Z_1, \mathfrak{a}_{\lambda_i})$ contain no irreducibles. They are empty for $i \geq 0$.*

(2) *For $i < 0$, the moduli space $M_{z_k}(Z_1, \mathfrak{a}_{\lambda_i})$ consists of a single point when it has formal dimension equal to zero.*

(3) *The formal dimensions of $M_{z_k}(Z_1, \mathfrak{a}_{\lambda_i})$ and $M_{z_{1-k}}(Z_1, \mathfrak{a}_{\lambda_i})$ are the same.*

Proof. The formal dimensions of all the moduli spaces $M_{z_k}(Z_1, \mathfrak{a}_{\lambda_0})$ are the same as the moduli spaces for the corresponding Spin^c structures over $\overline{\mathbb{CP}}^2$. It follows that the formal dimension of $M_{z_k}(Z_1, \mathfrak{a}_{\lambda_i})$ is $-k(k-1) - 2i - 1$ if $i \geq 0$, and $-k(k-1) - 2i - 2$ if $i < 0$. Thus, the formal dimensions of all the moduli spaces $M_z(Z_1, \mathfrak{a}_{\lambda_i})$ with $i \geq 0$ are negative, and these moduli spaces are therefore empty. If $i < 0$, then $\mathfrak{a}_{\lambda_i}$ is boundary-unstable, so the corresponding moduli space contains no irreducibles. That the zero-dimensional moduli spaces are points is the same phenomenon that underlies Proposition 2.7. The last statement is a consequence of the diffeomorphism $\tau : Z_1 \to Z_1$ which is the identity on the cylindrical end and sends $[E_1]$ to $-[E_1]$. □

From the lemma, it follows that the number of end-points of a 1-dimensional moduli space $M_z^+(\mathfrak{a}, X_1^*, \mathfrak{b})_{\bar{Q}}$ which lie over $T = -\infty$ is even. The identities which we obtain from these moduli spaces are therefore the same as the identities for K_o^o etc. in Section 4.10, but without the term from $T = 0$. For example, we have

$$\begin{aligned} m_o^o(W_1)m_o^o(W_0) + H_o^o\partial_o^o + \partial_o^o H_o^o + H_o^u\bar{\partial}_u^s\partial_s^o + \partial_o^u\bar{H}_u^s\partial_s^o + \partial_o^u\bar{\partial}_u^s H_s^o \\ + m_o^u(W_1)\bar{m}_u^s(W_0)\partial_s^o + m_o^u(W_1)\bar{\partial}_u^s m_s^o(W_0) + \partial_o^u\bar{m}_u^s(W_1)m_s^o(W_0) = 0. \end{aligned}$$

The chain identity in the proposition follows from this identity and its companions. □

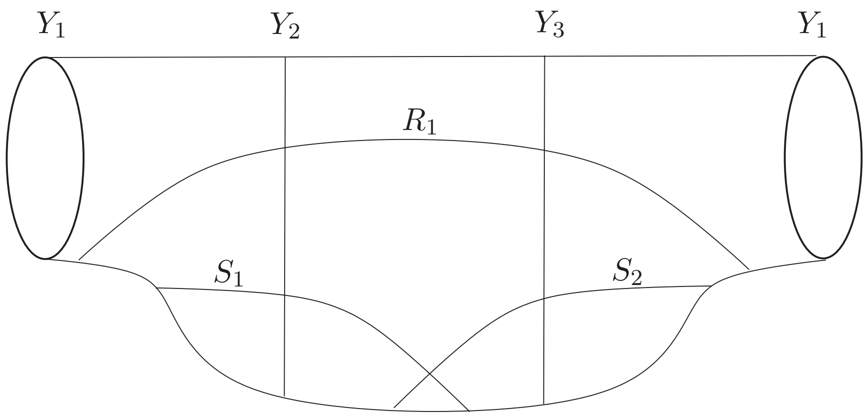

Figure 2: *Breaking up the triple-composite.* This indicates the five hypersurfaces which separate V_1.

5.2. *The second chain homotopy.* Proposition 5.2 gives us the first chain homotopy required by Lemma 5.1. Our next goal is to to construct the second homotopy required by the lemma. This will be constructed by counting points in moduli spaces associated to a two-parameter family of metrics on a four-manifold.

Specifically, consider the four-manifold V_1 obtained as

$$V_1 = W_1 \cup_{Y_2} W_2 \cup_{Y_3} W_3.$$

This four-manifold contains the 2-spheres E_1 and E_2, and the 3-spheres S_1 and S_2 which bound their tubular neighborhoods. The spheres E_1 and E_2 intersect transversely in a single point, with intersection number 1. The 3-spheres S_1 and S_2 intersect transversely in a 2-torus. Let N_1 be a regular neighborhood of $E_1 \cup E_2$ containing the 3-spheres S_1 and S_2. The boundary of N_1 is a separating hypersurface R_1 in V_1, diffeomorphic to $S^1 \times S^2$. The manifold N_1 is diffeomorphic to the complement of the neighborhood of a standard circle in $\overline{\mathbb{CP}}^2$, and gives a decomposition

$$V_1 = U_1 \cup_{R_1} N_1.$$

The manifold U_1 is obtained topologically by removing a neighborhood of the curve $\{0\} \times K$ from the cylindrical cobordism $[-1,1] \times Y_1$, where K is the core of the solid torus $S^1 \times D^2$ that was used in the Dehn filling to create Y_1.

In all, we have five separating hypersurfaces Y_2, R_1, Y_3, S_2, and S_1, as pictured in Figure 2. These are arranged cyclically so that any one intersects only its two neighbors. For any two of these surfaces, say S and S', which do *not* intersect, we can construct a 2-parameter family of metrics $P(S, S')$ parametrized by $\mathbb{R}^+ \times \mathbb{R}^+$, by inserting cylinders $[-T_S, T_S] \times S$ and $[-T_{S'}, T_{S'}] \times S'$. In the usual way, we complete this family to obtain a family of Riemannian

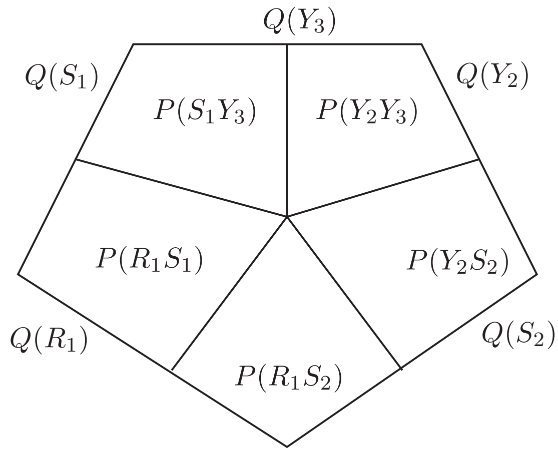

Figure 3: *The two-parameter family of metrics.* This is a schematic illustration of the two-parameter family of metrics $\bar{P}$, parameterized by a pentagon. The five regions parameterize the five two-parameter families of metrics where the metrics are varied normal to two of the five three-manifolds. Any two two-parameter families meet along an edge which parameterizes metrics where only one of the five three-manifolds is pulled out. The five edges on the boundary parameterize metrics where one of the five three-manifolds is stretched completely out.

manifolds over the "square"

$$\bar{P}(S, S') \cong [0, \infty] \times [0, \infty].$$

There are five such families of metrics, corresponding to the five pairs of disjoint separating surfaces. The squares fit together along five common edges, corresponding to families of metrics where just one of the lengths T_S is nonzero. In this way we set up a two-parameter family of metrics $\bar{P} = \bar{P}(R_1, Y_2, Y_3, S_1, S_2)$, as the union of five squares $\bar{P}(S, S')$, as shown in Figure 3. For each of the five hypersurfaces S, there are two edges of $\bar{P}$ where $T_S = \infty$. We denote the union of these two edges by $\bar{Q}_S$. Thus,

$$\partial \bar{P} = \bar{Q}_{S_2} \cup \bar{Q}_{Y_2} \cup \bar{Q}_{Y_3} \cup \bar{Q}_{S_2} \cup \bar{Q}_{R_1}.$$

By a small adjustment, we can arrange throughout the family that the metrics on R_1, S_1, and S_2 are standard.

For each pair of critical points $\mathfrak{a}$, $\mathfrak{b}$ in $\mathfrak{C}(Y_1)$, we now have a (parametrized) moduli space $M_z(\mathfrak{a}, V_1^*, \mathfrak{b})_P$ and its compactification $M_z^+(\mathfrak{a}, V_1^*, \mathfrak{b})_{\bar{P}}$. If this moduli space is zero-dimensional, then it is compact. As usual, we define quantities

$$m_z(\mathfrak{a}, V_1, \mathfrak{b})_{\bar{P}}, \quad \bar{m}_z(\mathfrak{a}, V_1, \mathfrak{b})_{\bar{P}}$$

by counting points (mod 2) in zero-dimensional moduli spaces $M_z(\mathfrak{a}, V_1^*, \mathfrak{b})_P$ and $M_z^{\mathrm{red}}(\mathfrak{a}, V_1^*, \mathfrak{b})_P$ respectively. These are the matrix entries of maps such as

$$G_o^o : C_\bullet^o(Y_1) \to C_\bullet^o(Y_1)$$

and its seven companions G_s^o, G_o^u, G_s^u, $\bar{G}_s^s$, $\bar{G}_u^s$, $\bar{G}_s^u$ and $\bar{G}_u^u$.

Now suppose that $M_z(\mathfrak{a}, V_1^*, \mathfrak{b})_P$ has dimension 1. In the first instance, let us suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are in $\mathfrak{C}^o(Y_1)$. As in the earlier settings, we will obtain an identity

$$A_o^o = 0$$

for an operator $A_o^o : C_\bullet^o(Y_1) \to C_\bullet^o(Y_1)$ by enumerating mod 2 the endpoints of the compactification $M_z^+(\mathfrak{a}, V_1^*, \mathfrak{b})_{\bar{P}}$, and summing over all $\mathfrak{a}$, $\mathfrak{b}$ and z. First, there are the endpoints which lie over the interior of $P \subset \bar{P}$. These arise from strata with either two factors,

$$\breve{M}_{z_1}(\mathfrak{a}, \mathfrak{a}_1) \times M_{z_2}(\mathfrak{a}_1, V_1^*, \mathfrak{b})_P$$
$$M_{z_1}(\mathfrak{a}, V_1^*, \mathfrak{b}_1)_P \times \breve{M}_{z_2}(\mathfrak{b}_1, \mathfrak{b}),$$

or three:

$$\breve{M}_{z_1}(\mathfrak{a}, \mathfrak{a}_1) \times \breve{M}_{z_2}(\mathfrak{a}_1, \mathfrak{a}_2) \times M_{z_3}(\mathfrak{a}_2, V_1^*, \mathfrak{b})_P$$
$$\breve{M}_{z_1}(\mathfrak{a}, \mathfrak{a}_1) \times M_{z_2}(\mathfrak{a}_1, V_1^*, \mathfrak{b}_1)_P \times \breve{M}_{z_3}(\mathfrak{b}_1, \mathfrak{b})$$
$$M_{z_1}(\mathfrak{a}, V_1^*, \mathfrak{b}_1)_P \times \breve{M}_{z_2}(\mathfrak{b}_1, \mathfrak{b}_2) \times \breve{M}_{z_3}(\mathfrak{b}_2, \mathfrak{b}),$$

just as in Lemma 4.15. In the case of three factors, the middle one is boundary-obstructed. Together, these terms contribute

$$G_o^o \partial_o^o + \partial_o^o G_o^o + \partial_o^u \bar{\partial}_u^s G_s^o + \partial_o^u \bar{G}_u^s \partial_s^o + G_o^u \bar{\partial}_u^s \partial_s^o \tag{18a}$$

to the operator A_o^o. The remaining terms of A_o^o come from boundary points in the moduli space that lie over one of the five parts $\bar{Q}_S$ of the boundary $\partial\bar{P}$. In the case that $S = S_1$ or S_2, the contribution from $\bar{Q}_S$ is zero. This is because when T_{S_1} or T_{S_2} becomes infinite, the manifold V_1 splits off either Z_1^* or Z_2^*, so we can apply Lemma 5.3 to see that the total number of endpoints over $\bar{Q}_{S_1}$ and $\bar{Q}_{S_2}$ is even. (This is the same mechanism involved in the proof of Proposition 5.2.)

Next we analyze the endpoints which lie over $\bar{Q}_{Y_3}$. When $T_{Y_3} = \infty$, the manifold V_1^* decomposes as a disjoint union $X_1^* \cup W_3^*$, where X_1^* is the composite cobordism above. In the family parametrized by $\bar{Q}_{Y_3}$, the metric on W_3^* is constant, while the family of metrics on the component X_1^* is the same family $\bar{Q}$ that appeared as $\bar{Q}(S_1, Y_2)$ in Section 5.1. Endpoints lying over the interior part of the edge, $Q_{Y_3} \subset \bar{Q}_{Y_3}$ may belong to strata with two factors, which have the form

$$M_{z_1}(\mathfrak{a}, X_1^*, \mathfrak{a}_1)_{\bar{Q}} \times M_{z_2}(\mathfrak{a}_1, W_3^*, \mathfrak{b});$$

or they may belong to strata with three factors, one of which is boundary obstructed, as in case 5 of Lemma 4.15. Altogether, these terms contribute four terms to A^o_o,

$$\text{(18b)}\quad m^o_o(W_3)H^o_o(X_1) + m^u_o(W_3)\bar{H}^s_u(X_1)\partial^o_s \\ + m^u_o(W_3)\bar{\partial}^s_u H^o_s(X_1) + \partial^u_o \bar{m}^s_u(W_3)H^o_s(X_1)$$

where the operators $H^*_* = H^*_*(X_1)$ are those defined at (16). The contributions from endpoints lying over Q_{Y_2} are similar: we obtain

$$\text{(18c)}\quad H^o_o(X_2)m^o_o(W_1) + H^u_o(X_2)\bar{m}^s_u(W_1)\partial^o_s \\ + H^u_o(X_2)\bar{\partial}^s_u m^o_s(W_1) + \partial^u_o \bar{H}^s_u(X_2)m^o_s(W_1).$$

There is also one possible type of endpoint that occurs at the vertex of $\bar{P}$ where $\bar{Q}_{Y_3}$ and $\bar{Q}_{Y_2}$ meet: these lie in a moduli space

$$M_{z_1}(\mathfrak{a}, W^*_1, \mathfrak{a}_1) \times M_{z_2}(\mathfrak{a}_1, W^*_2, \mathfrak{a}_2) \times M_{z_3}(\mathfrak{a}_2, W^*_3, \mathfrak{b}),$$

where the middle factor is boundary-obstructed. These contribute a term

$$\text{(18d)}\qquad m^u_o(W_3)\bar{m}^s_u(W_2)m^o_s(W_1)$$

to A^o_o.

When $T_{R_1} = \infty$, we have a decomposition of V^*_1 into two pieces

$$N^*_1 \cup U^*_1,$$

where U_1 and N_1 are as above. The manifold U^*_1 has three ends. We regard U_1 as a cobordism from $R_1 \amalg Y_1$ to Y_1, and N_1 as a manifold with oriented boundary R_1. The 1-parameter family of metrics $\bar{Q}_{R_1}$ is constant on U_1, and we have moduli spaces

$$M_z(N_1, \mathfrak{a}')_{\bar{Q}} \quad \text{and} \quad M_z(\mathfrak{a}', \mathfrak{a}, U^*_1, \mathfrak{b})$$

or $\mathfrak{a}' \in \mathfrak{C}(R_1)$ and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{C}(Y_1)$. Here $\bar{Q} \cong [-\infty, \infty]$ is the family of metrics $\bar{Q}(S_1, S_2)$ on U_1 which stretches along S_1 when T is negative and S_2 when T is positive. On N_1 we can count points in zero-dimensional moduli spaces $M_z(N_1, \mathfrak{a}')_{\bar{Q}}$, and so define elements

$$\begin{aligned} n_s &\in C^s_\bullet(R_1), \\ n_o &\in C^o_\bullet(R_1) \\ \bar{n}_s &\in C^s_\bullet(R_1) \\ \bar{n}_u &\in C^u_\bullet(R_1). \end{aligned}$$

(The last two count points in zero-dimensional moduli spaces $M^{\text{red}}_z(N_1, \mathfrak{a}')$.) The situation simplifies slightly, on account of the following lemma.

LEMMA 5.4. *If the perturbation on R_1 is sufficiently small, then there are no irreducible critical points (so n_o is zero), and no irreducible trajectories on $\mathbb{R} \times R_1$. The perturbation can be chosen so that the one-dimensional reducible trajectories come in pairs, so $\bar{\partial}^s_s$, $\bar{\partial}^s_u$, $\bar{\partial}^u_s$ and $\bar{\partial}^u_u$ are all zero. The invariant $\bar{n}_s$ is zero also.*

Proof. We postpone the proof to Section 5.3 below, where we also calculate $\bar{n}_u$. □

The zero-dimensional moduli spaces $M_z(\mathfrak{a}', \mathfrak{a}, U_1^*, \mathfrak{b})$ provide the matrix entries of maps

$$m^{uo}_o : C^u_\bullet(R_1) \otimes C^o_\bullet(Y_1) \to C^o_\bullet(Y_1)$$

as well as m^{uu}_o, m^{uo}_s and m^{uu}_s, while the reducible parts of these moduli spaces define $\bar{m}^{ss}_s$, $\bar{m}^{ss}_u$, $\bar{m}^{su}_s$, $\bar{m}^{su}_u$, $\bar{m}^{us}_s$, $\bar{m}^{us}_u$, $\bar{m}^{uu}_s$ and $\bar{m}^{uu}_u$. Of these eight maps defined by zero-dimensional moduli spaces of reducible solutions, the maps $\bar{m}^{ss}_s$, $\bar{m}^{su}_u$ and $\bar{m}^{us}_u$ arise from boundary-obstructed moduli spaces. The moduli spaces $M_z(\mathfrak{a}', \mathfrak{a}, U_1^*, \mathfrak{b})$ contributing to $\bar{m}^{ss}_u$ are *doubly* boundary obstructed (or boundary obstructed with corank 2, in the notation of [23]): these zero-dimensional moduli spaces have formal dimension $\mathrm{gr}_z(\mathfrak{a}', \mathfrak{a}, U_1, \mathfrak{b}) = -2$.

We can now enumerate the end-points belonging to $\bar{Q}_{R_1}$ in the 1-dimensional moduli spaces $M^+_z(\mathfrak{a}, V_1^*, \mathfrak{b})_{\bar{P}}$ that contribute to A^o_o. First there are points belonging to strata with two factors, of the form

$$M_{z_1}(N_1^*, \mathfrak{a}')_{\bar{Q}} \times M_{z_2}(\mathfrak{a}', \mathfrak{a}, U_1^*, \mathfrak{b}),$$

where $\mathfrak{a}'$ is necessarily boundary-unstable (so the solution on N_1^* is reducible). Next we should look for points belonging to strata with three factors, one of which is boundary-obstructed; but when $\mathfrak{a}$ and $\mathfrak{b}$ are irreducible, there are no such contributions. Finally, there are points belonging to strata with four factors, one of which is doubly boundary-obstructed. These have the form

$$M_{z_1}(N_1^*, \mathfrak{a}')_{\bar{Q}} \times M_{z_2}(\mathfrak{a}, \mathfrak{a}_1) \times M_{z_3}(\mathfrak{a}', \mathfrak{a}_1, U_1^*, \mathfrak{b}_1) \times M_{z_4}(\mathfrak{b}_1, \mathfrak{b}),$$

where $\mathfrak{a}'$ is boundary-stable, $\mathfrak{a}_1$ is boundary-stable, and $\mathfrak{b}_1$ is boundary-unstable. From these we obtain the final two terms in A^o_o:

$$m^{uo}_o(\bar{n}_u \otimes \cdot) + \partial^u_o \bar{m}^{ss}_u (n_s \otimes \partial^o_s(\cdot)). \tag{18e}$$

The identity $A^o_o = 0$ has sixteen terms, from (18a)–(18e):

$$\begin{aligned} G^o_o\partial^o_o &+ \partial^o_o G^o_o + \partial^u_o \bar{\partial}^s_u G^o_s + \partial^u_o \bar{G}^s_u \partial^o_s + G^u_o \bar{\partial}^s_u \partial^o_s \\ &+ m^o_o(W_3) H^o_o(X_1) + m^u_o(W_3) \bar{H}^s_u(X_1) \partial^o_s \\ &+ m^u_o(W_3) \bar{\partial}^s_u H^o_s(X_1) + \partial^u_o \bar{m}^s_u(W_3) H^o_s(X_1) \\ &+ H^o_o(X_2) m^o_o(W_1) + H^u_o(X_2) \bar{m}^s_u(W_1) \partial^o_s \\ &+ H^u_o(X_2) \bar{\partial}^s_u m^o_s(W_1) + \partial^u_o \bar{H}^s_u(X_2) m^o_s(W_1) \\ &+ m^u_o(W_3) \bar{m}^s_u(W_2) m^o_s(W_1) + m^{uo}_o(\bar{n}_u \otimes \cdot) + \partial^u_o \bar{m}^{ss}_u(n_s \otimes \partial^o_s(\cdot)) = 0. \end{aligned}$$

There are three similar identities, $A^o_s = 0$, $A^u_o = 0$ and $A^u_s = 0$, coming from the three other types of 1-dimensional moduli spaces that contain irreducibles. In full, these are

$$\begin{aligned}
A^o_s =& \bar\partial^s_s G^o_s + \bar G^s_s \partial^o_s + \partial^o_s G^o_o + G^o_s \partial^o_o + \partial^u_s \bar\partial^s_u G^o_s + \partial^u_s \bar G^s_u \partial^o_s + G^u_s \bar\partial^s_u \partial^o_s \\
&+ \bar m^{ss}_s (n_s \otimes \partial^o_s \cdot) + \partial^u_s \bar m^{ss}_u (n_s \otimes \partial^o_s \cdot) \\
&+ m^{uo}_s (\overline{n}_u \otimes \cdot) \\
&+ \bar H^s_s (X_2) m^o_s (W_1) + H^o_s (X_2) m^o_o (W_1) \\
&+ \partial^u_s \bar H^s_u (X_2) m^o_s (W_1) + H^u_s (X_2) \bar\partial^s_u m^o_s (W_1) + H^u_s (X_2) \bar m^s_u (W_1) \partial^o_s \\
&+ \bar m^s_s (W_3) H^o_s (X_1) + m^o_s (W_3) H^o_o (X_1) \\
&+ \partial^u_s \bar m^s_u (W_3) H^o_s (X_1) + m^u_s (W_3) \bar\partial^s_u H^o_s (X_1) + m^u_s (W_3) \bar H^s_u (X_1) \partial^o_s \\
&+ m^u_s (W_3) \bar m^s_u (W_2) m^o_s (W_1),
\end{aligned}$$

$$\begin{aligned}
A^u_o =& \partial^o_o G^u_o + G^o_o \partial^u_o + \partial^u_o \bar G^u_u + G^u_o \bar\partial^u_u + \partial^u_o \bar\partial^s_u G^u_s + \partial^u_o \bar G^s_u \partial^u_s + G^u_o \bar\partial^s_u \partial^u_s \\
&+ \partial^u_o \bar m^{ss}_u (n_s \otimes \partial^u_s \cdot) + \partial^u_o \bar m^{su}_u (n_s \otimes \cdot) \\
&+ m^{uu}_o (\overline{n}_u \otimes \cdot) \\
&+ H^o_o (X_2) m^u_o (W_1) + H^u_o (X_2) \bar m^u_u (W_1) \\
&+ \partial^u_o \bar H^s_u (X_2) m^u_s (W_1) + H^u_o (X_2) \bar\partial^s_u m^u_s (W_1) + H^u_o (X_2) \bar m^s_u (W_1) \partial^u_s \\
&+ m^o_o (W_3) H^u_o (X_1) + m^u_o (W_3) \bar H^u_u (X_1) \\
&+ \partial^u_o \bar m^s_u (W_3) H^u_s (X_1) + m^u_o (W_3) \bar\partial^s_u H^u_s (X_1) + m^u_o (W_3) \bar H^s_u (X_1) \partial^u_s \\
&+ m^u_o (W_3) \bar m^s_u (W_2) m^o_s (W_1),
\end{aligned}$$

$$\begin{aligned}
A^u_s =& \bar G^u_s + \bar\partial^s_s G^u_s + \bar G^s_s \partial^u_s + \partial^u_s \bar G^u_u + G^u_s \bar\partial^u_u + \partial^u_s \bar\partial^s_u G^u_s + \partial^u_s \bar G^s_u \partial^u_s + G^u_s \bar\partial^s_u \partial^u_s \\
&+ \partial^o_s G^u_o + G^o_s \partial^u_o \\
&+ \partial^u_s \bar m^{ss}_u (n_s \otimes \partial^u_s \cdot) + \partial^u_s \bar m^{su}_u (n_s \otimes \cdot) + \bar m^{su}_s (n_s \otimes \cdot) + \bar m^{ss}_s (n_s \otimes \partial^u_s \cdot) \\
&+ m^{uu}_s (\overline{n}_u \otimes \cdot) \\
&+ \bar H^s_s (X_2) m^u_s (W_1) + H^u_s (X_2) \bar m^u_u (W_1) \\
&+ \partial^u_s \bar H^s_u (X_2) m^u_s (W_1) + H^u_s (X_2) \bar\partial^s_u m^u_s (W_1) \\
&+ H^u_s (X_2) \bar m^s_u (W_1) \partial^u_s + H^o_s (X_2) m^u_o (W_1) \\
&+ \bar m^s_s (W_3) H^u_s (X_1) + m^u_s (W_3) \bar H^u_u (X_1) \\
&+ \partial^u_s \bar m^s_u (W_3) H^u_s (X_1) + m^u_s (W_3) \bar\partial^s_u H^u_s (X_1) \\
&+ m^u_s (W_3) \bar H^s_u (X_1) \partial^u_s + m^o_s (W_3) H^u_o (X_1) \\
&+ m^u_s (W_3) \bar m^s_u (W_2) m^u_s (W_1).
\end{aligned}$$

There are four simpler identities involving only the reducible moduli spaces: these are the vanishing of expressions $\bar A^s_s$, $\bar A^s_u$, $\bar A^u_s$ and $\bar A^u_u$, where for example

$$\begin{aligned}
\bar A^s_s =& \bar G^s_s \bar\partial^s_s + \bar G^u_s \bar\partial^s_u + \bar\partial^s_s \bar G^s_s + \bar\partial^u_s \bar G^s_u + \bar m^s_s (W_3) \bar H^s_s (X_1) + \bar m^u_s (W_3) \bar H^s_u (X_1) \\
&+ \bar H^s_s (X_2) \bar m^s_s (W_1) + \bar H^u_s (X_2) \bar m^s_u (W_1) + \bar m^{us}_s (\bar n_u \otimes \cdot).
\end{aligned}$$

We define an operator

$$\check{L} : \check{C}_\bullet(Y_1) \to \check{C}_\bullet(Y_1)$$

by combining some of the contributions from $\bar{Q}_{R_1}$: we write

$$\check{L} = \begin{bmatrix} L^o_o & L^u_o \bar{\partial}^s_u + \partial^u_o \bar{L}^s_u \\ L^o_s & \bar{L}^s_s + L^u_s \bar{\partial}^s_u + \partial^u_s \bar{L}^s_u \end{bmatrix}, \tag{19}$$

where

$$L^o_o = m^{uo}_o(\bar{n}_u \otimes \cdot),$$

and so on. (The term $\bar{n}_u$ appears in the definition of all of these.) In words, when $\mathrm{gr}_z(\mathfrak{a}, V_1, \mathfrak{b}) = -1$ the $e_{\mathfrak{b}}$ component of $\check{L}(e_{\mathfrak{a}})$ counts points in the zero-dimensional strata of $\breve{M}^+_z(\mathfrak{a}, \mathfrak{b})$ which are broken along a critical point in $\mathfrak{C}^u(R_1)$.

We define $\check{G} : \check{C}_\bullet(Y_1) \to \check{C}_\bullet(Y_1)$ by the formula

$$\check{G} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where

$$\begin{aligned} a &= G^o_o \\ b &= \partial^u_o \bar{G}^s_u + G^u_o \bar{\partial}^s_u + m^u_o \bar{H}^s_u + H^u_o \bar{m}^s_u + \partial^u_o(\bar{m}^{ss}_u(n_s \otimes \cdot)) \\ c &= G^o_s \\ d &= \bar{G}^s_s + \partial^u_s \bar{G}^s_u + G^u_s \bar{\partial}^s_u + m^u_s \bar{H}^s_u + H^u_s \bar{m}^s_u + \partial^u_s \bar{m}^{ss}_u(n_s \otimes \cdot) + \bar{m}^{ss}_s(n_s \otimes \cdot). \end{aligned}$$

Here, we have written $m^u_o \bar{H}^s_u$ for example as an abbreviation for $m^u_o(W_3)\bar{H}^s_u(X_1)$, because no ambiguities arise in the formulae. In words, if $\mathrm{gr}(\mathfrak{a}, V^*_1, \mathfrak{b}) = -2$, the $e_{\mathfrak{b}}$ component of $\check{G}(e_{\mathfrak{a}})$ counts points in the zero-dimensional strata of $\breve{M}^+_z(\mathfrak{a}, \mathfrak{b})$.

Proposition 5.5. *We have the identity*

$$\check{\partial} \circ \check{G} + \check{G} \circ \check{\partial} = \check{m}_3 \circ \check{H}_1 + \check{H}_2 \circ \check{m}_1 + \check{L},$$

where $\check{m}_3 = \check{m}(W_3)$ and $\check{H}_1$, $\check{H}_2$ are the operators from Proposition 5.2, *using X_1 and X_2 respectively.*

Proof. In addition to the identities $A^*_* = 0$ and $\overline{A}^*_* = 0$, there are the identities arising from pieces of the cobordism. For example, we can consider the three-ended manifold U^*_1. On this, once again, we can enumerate ends of the one-dimensional moduli spaces $M_z(\mathfrak{a}', \mathfrak{a}, U^*_1, \mathfrak{b})$. These give relations which are formally similar to the relations coming from a two-ended cobordism from Y_1 to Y_2 (since differentials for the Floer homology of R_1 are trivial, cf. Lemma 5.4). Looking at ends of irreducible moduli spaces, we get four relations of the type B_o^{uo}, B_s^{uo}, B_o^{uu} and B_s^{uu} For example, the relation of the form $B_s^{uu}=0$, can be written out as:

$$
\begin{aligned}
B_s^{uu} &= \bar{m}_s^{uu}(\cdot\otimes\cdot)+\bar{m}_s^{us}(\cdot\otimes\partial_s^u(\cdot))+m_s^{uu}(\cdot\otimes\bar{\partial}_u^u(\cdot)) \\
&\quad+\bar{\partial}_s^s m_s^{uu}(\cdot\otimes\cdot)+\partial_s^u\bar{m}_u^{uu}(\cdot\otimes\cdot).
\end{aligned}
\tag{20}
$$

In addition to these, we have eight relations coming from looking at ends of reducible moduli spaces $\overline{B}_*^{**}=0$ (where here each $*$ can be either symbol u or s). There are the relations coming from ends of moduli spaces for the X_1 and X_2 (with its two-parameter families of metrics), the ends of the moduli spaces for W_i, and finally, the ends of moduli spaces for $\mathbb{R}\times Y_i$. Putting all these together, we get the proposition. □

We can put the above outlined proof of Proposition 5.5 into a more conceptual framework. Throughout the following discussion, we fix two critical points $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathrm{gr}(\mathfrak{a},V_1^*,\mathfrak{b})=-1$ (where both $\mathfrak{a}$ and $\mathfrak{b}$ are in $\mathfrak{C}^s\cup\mathfrak{C}^o$, if we are considering the case of $\widecheck{HM}$, for example). We count the ends of those one-dimensional strata in $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$. Clearly, these ends count points in the zero-dimensional strata in $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$. We claim that the total sum of these zero-dimensional strata counts the $e_{\mathfrak{b}}$ component of the image of $e_{\mathfrak{a}}$ under the map

$$
\check{\partial}\circ\check{G}+\check{G}\circ\check{\partial}+\check{m}_3\circ\check{H}_1+\check{H}_2\circ\check{m}_1+\check{L},
\tag{21}
$$

which must therefore be zero.

The verification can be broken into the following three steps. Recall that the strata of $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$ consist of fibered products over various critical points of moduli spaces. We call these critical points with multiplicity (if the same critical point appears more than once) *break points* for the stratum. We say that a stratum in $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$ has a *good break* if at least one of its break points lies in $(\mathfrak{C}^s\cup\mathfrak{C}^o)(Y_i)$ (with $i\in\{1,2,3\}$) or in $\mathfrak{C}^u(R_1)$. One must verify first that the nonempty, zero-dimensional strata in $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$ which have a good break have, in fact, a unique (i.e. with multiplicity one) good break. This follows from a straightforward dimension count, after listing all possible good breaks. Indeed, in view of the definitions of the maps $\check{G}$, $\check{H}$, $\check{m}$, and $\check{L}$, the above dimension counts show that the strata for which the good break occurs along Y_1 are counted in $\check{\partial}\circ\check{G}+\check{G}\circ\check{\partial}$, those where it occurs along Y_3 are counted in $\check{m}_3\circ\check{H}_1$, those where it occurs along Y_2 are counted in $\check{H}_2\circ\check{m}_1$, and those where it occurs along R_1 are counted in $\check{L}$. Second, one verifies that any of the zero-dimensional strata with one good break appear uniquely as boundaries of one-dimensional moduli spaces in $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$. Finally, one verifies that any of the zero-dimensional strata in $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$ which have no good break, and hence are not accounted for in Equation (21), are counted exactly twice: they appear in the boundaries of two distinct one-dimensional strata in $M_z^+(\mathfrak{a},V_1^*,\mathfrak{b})_{\overline{P}}$.

5.3. *Calculation.* The plan of the proof is to deduce Theorem 2.4 from Lemma 5.1. We have already constructed the chain homotopies referred to in the first part of the lemma: these are the chain homotopies $\check{H}_n$. To verify the hypothesis in the second part of the lemma, it is enough to verify that $\check{L}$ induces isomorphisms in homology, because of Proposition 5.5. That is the content of the next proposition.

PROPOSITION 5.6. *The map $\check{L} : \check{C}_\bullet(Y_1) \to \check{C}_\bullet(Y_1)$ induces isomorphisms in homology. Indeed, the resulting map on $\widecheck{HM}_\bullet(Y_1)$ is multiplication by the power series*

$$\sum_{k\geq 0} U^{k(k+1)/2},$$

which has leading coefficient 1.

We begin by examining the Floer complexes of the manifold $R_1 = S^1 \times S^2$, equipped with a standard metric and small regular perturbation $\mathfrak{q}$ from the class $\mathcal{P}(R_1)$. With no perturbation, the critical points (A, Φ) of the Chern-Simons-Dirac functional all belong to the Spin^c structure $\mathfrak{s}_0$ with $c_1(\mathfrak{s}_0) = 0$. They are simply the reducible solutions $(A, 0)$ with A^{t} flat. In $\mathcal{B}(R_1)$, they form a circle. For any flat connection A, the corresponding Dirac operator D_A on R_1 has no kernel, so there is no spectral flow between any two points in the circle. We can choose our small perturbation to restrict to a standard Morse function on this circle, with one maximum α^1 and one minimum α^0. We also need to arrange that the corresponding perturbed Dirac operators at these two points have simple eigenvalues, and we choose the perturbation small enough so as not to introduce any spectral flow on the paths joining α^1 to α^0. In the blow-up $\mathcal{B}^\sigma(R_1)$, each of these critical points gives rise to a collection of critical points $\mathfrak{a}_i^0$ and $\mathfrak{a}_i^1$, corresponding to the eigenvalues λ_i of the perturbed Dirac operator. We again assume these eigenvalues are labeled in increasing order, with λ_0 the first positive eigenvalue, as in (17). The points $\mathfrak{a}_i^0$ and $\mathfrak{a}_i^1$ are boundary-stable when $i \geq 0$ and boundary-unstable when $i < 0$.

The trajectories on $\mathbb{R} \times R_1$ are all reducible, because R_1 has positive scalar curvature. Their images in $\mathcal{B}(R_1)$ are therefore either constant paths at α^0 or α^1, or one of the two trajectories γ, γ' from α^1 to α^0 on the circle. For each i, there are two trajectories γ_i and γ_i' lying over γ and γ' respectively in a moduli space $M_z(\mathfrak{a}_i^1, \mathfrak{a}_i^0)$. These are the only trajectories belonging to 1-dimensional moduli spaces, so all boundary maps are zero in the Floer complexes. Thus $\widecheck{HM}_\bullet(R_1) \cong \check{C}_\bullet(R_1)$, which has generators

$$e_i^0, e_i^1, \qquad (i \geq 0)$$

corresponding to the critical points $\mathfrak{a}_i^0$ and $\mathfrak{a}_i^1$, while $\hat{C}_\bullet(R_1)$ has generators

$$e_i^0, e_i^1, \qquad (i < 0).$$

We identify $J(Y,\mathfrak{s}_0)$ with $\mathbb{Z}$ in such a way that e^0_0 belongs to $\check{C}_0(R_1)$. Then e^μ_i is in grading $\mu+2i$ for i positive and in grading $\mu+2i+1$ for i negative.

The manifold N_1 has boundary R_1, and its homology is generated by the classes $[E_1]$ and $[E_2]$ of the two spheres. A Spin^c structure $\mathfrak{t}$ on N_1 whose restriction to R_1 is $\mathfrak{s}_0$ is uniquely determined by the evaluation of $c_1(\mathfrak{t})$ on $[E_1]$. For each $k\in\mathbb{Z}$, we write $\mathfrak{t}_k$ for the Spin^c structure whose first Chern class evaluates to $2k+1$ on $[E_1]$. The Spin^c structures $\mathfrak{t}_k$ and $\mathfrak{t}_{-1-k}$ are complex conjugates. We write

$$M_k(N_1^*,\mathfrak{a}')_{\bar{Q}}$$

for the union of the moduli spaces belonging to components z which give rise the Spin^c structure $\mathfrak{t}_k$. The family $\bar{Q}$ is the same 1-parameter family $\bar{Q}(S_1,S_2)$ that appeared above. The following lemma and its two corollaries are straightforward.

LEMMA 5.7. *The dimension of the moduli space $M_k(N_1^*,\mathfrak{a}^\mu_i)_{\bar{Q}}$ is given by*

$$\dim M_k(N_1^*,\mathfrak{a}^\mu_i)_{\bar{Q}}=\begin{cases}-\mu-k(k+1)-2i, & i\geq 0,\\ -\mu-k(k+1)-2i-1, & i<0.\end{cases}\qquad\square$$

COROLLARY 5.8. *The only nonempty moduli spaces $M_k(N_1^*,\mathfrak{a}')_{\overline{Q}}$ with $\mathfrak{a}'$ boundary-stable occur when $k=0$ or -1 and $\mathfrak{a}'=\mathfrak{a}^0_0$, in which case the moduli space is zero-dimensional. The moduli spaces $M^{\mathrm{red}}_k(N_1^*,\mathfrak{a}')_{\bar{Q}}$ are empty for all boundary-stable $\mathfrak{a}'$.* $\square$

COROLLARY 5.9. *The zero-dimensional moduli spaces $M_k(N_1^*,\mathfrak{a}')_{\bar{Q}}$, with $\mathfrak{a}'$ boundary-unstable, are the moduli spaces*

$$M_k(N_1^*,\mathfrak{a}^1_{i_k})_{\bar{Q}},\qquad i_k=-1-k(k+1)/2.\tag{22}$$ $\square$

At this point, we have verified all parts of Lemma 5.4, and in addition we can now express $\bar{n}_u\in C^u_\bullet(R_1)$ as

$$\bar{n}_u=\sum_{k\in\mathbb{Z}}a_k\,e^1_{-1-k(k+1)/2}$$

where a_k counts points in the moduli space (22) (which consists entirely or reducibles, because $\mathfrak{a}^1_{i_k}$ is boundary-unstable).

LEMMA 5.10. *For all $k\in\mathbb{Z}$, the sum a_k+a_{-1-k} is* 1 mod 2.

Proof. Let g be any Riemannian metric on N_1^* that is standard on the end. Fix a Spin^c structure $\mathfrak{t}_k$ on N_1. Because N_1^* has no first homology and no self-dual, square-integrable harmonic 2-forms, there is a unique Spin^c connection

$$A=A(k,g)$$

in the associated spin bundle $S^+ \to N_1$ with L^2 curvature satisfying the abelian anti-self-duality equation $F^+_{A^{\mathrm{t}}} = 0$. On the cylindrical end, A^{t} is asymptotically flat, so A defines a point

$$\theta_k(g) \in \mathcal{S},$$

where $\mathcal{S} \subset \mathcal{B}(R_1)$ is the circle of flat Spin^c connections. This depends only on k and g.

Fix a Spin structure on R_1 whose associated Spin^c structure is $\mathfrak{s}_0$. This fixes an isomorphism between $\mathfrak{s}_0$ and its complex conjugate. Complex conjugation now gives an involution on the circle, $\sigma : \mathcal{S} \to \mathcal{S}$, with two fixed points s_+ and s_-. The isomorphism between $\mathfrak{s}_0$ and its conjugate extends to an isomorphism $\bar{\mathfrak{t}}_k \to \mathfrak{t}_{-1-k}$, and we therefore have

$$\theta_{-1-k}(g) = \sigma\theta_k(g). \tag{23}$$

Consider now the family of metrics $\bar{Q} = \bar{Q}(S_1, S_2)$ on N_1. As T goes to $-\infty$, the manifold N_1^* decomposes into two pieces, one of which has cylindrical ends $\mathbb{R}^- \times S_1$ and $\mathbb{R}^+ \times R_1$. This piece, call it T_1^*, carries no L^2 harmonic 2-forms (it is a punctured $S^2 \times D^2$ with cylindrical ends), so the map θ_k extends to continuously to $T = -\infty$ and $\theta_k(-\infty)$ is one of s_+ or s_-. The same applies to $T = +\infty$, so we obtain a map

$$\theta_k : \bar{Q} \to \mathcal{S}$$

with

$$\theta_k(\pm\infty) \in \{s_+, s_-\}.$$

We also have $\theta_k(+\infty) = \theta_{-1-k}(+\infty)$, and the same with $T = -\infty$. Thus the two maps

$$\theta_k, \theta_{-1-k} : [-\infty, \infty] \to \mathcal{S}$$

together define a mod 2 1-cycle in $\mathcal{S}$. The statement of the lemma follows from the assertion that this 1-cycle has nonzero degree mod 2.

Specifically, let $\Theta_k \colon S^1 \longrightarrow \mathcal{S}$ be the cycle obtained by joining θ_k and θ_{-1-k}. We show that for generic $x \in \mathcal{S}$, the space $\Theta_k^{-1}(x)$ is cobordant to

$$\left(M_k^{\mathrm{ab}}(N_1^*, \mathfrak{a}^1)_{\bar{Q}} \bigcup M_{-1-k}^{\mathrm{ab}}(N_1^*, \mathfrak{a}^1)_{\bar{Q}}\right) \times M^{\mathrm{ab}}(\mathfrak{a}^1, ([0,1] \times R_1)^*, \mathcal{S}) \times_{\mathcal{S}} \{x\} \times S. \tag{24}$$

Here, $M_k^{\mathrm{ab}}(N_1^*, \mathfrak{a}^1)_{\bar{Q}}$ denotes the moduli space of solutions to the perturbed abelian anti-self-duality equiations. Similarly, $M^{\mathrm{ab}}(\mathfrak{a}^1, ([0,1] \times R_1)^*, \mathcal{S})$ denotes the moduli space of solutions to the perturbed abelian anti-self-duality equations, where we use the perturbation at the $t = \infty$ and no perturbation at the $t = -\infty$ end. In particular, this moduli space admits a map by taking the limit as $t \mapsto \infty$ to $\mathcal{S}$. The claimed compact cobordism is induced by taking a one-parameter family of perturbations indexed by $T \in \mathbb{R}^+$ on N_1^* which are supported on ever-longer pieces of the attached cylinder. The fiber over $T = 0$ of this cobordism is $\{x\} \times_{\mathcal{S}} \bigl(M_k^{\mathrm{ab}}(N_1^*, \mathcal{S}) \cup M_{-1-k}^{\mathrm{ab}}(N_1^*, \mathcal{S})\bigr)$, whose number of points coincides with the stated degree, while the fiber over $T = \infty$ is the space described in (24). In particular, if the stated degree is odd, then so is the number of points in $M_k^{\mathrm{ab}}(N_1^*, \mathfrak{a}^1) \cup M_{-1-k}(N_1^*, \mathfrak{a}^1)$. But $M_k^{\mathrm{ab}}(N_1^*, \mathfrak{a}^1)$ is identified with $M_k(N_1^*, \mathfrak{a}^1_{i_k})$.

It remains now to show that the degree of Θ_k is nonzero modulo two. This in turn is equivalent to saying that

$$\theta_k(\infty) \neq \theta_k(-\infty),$$

because of the relationship (23). So we must prove that $\theta_k : \bar{Q} \to \mathcal{S}$ is a path joining s_- to s_+.

To get a concrete model for θ_k, choose a standard closed curve δ representing the generator of $H_1(R_1)$, and let $\Sigma \subset N_1^*$ be a topological open disk with a cylindrical end $\mathbb{R}^+ \times \delta$. To pin it down, we make Σ disjoint from $Z_1 \subset N_1$ and have geometric intersection 1 with $E_2 \subset Z_2$. If we write $A = A(k, g)$ again for the anti-self-dual connection, then

$$\theta_k(g) = \exp \frac{1}{2} \int_\Sigma F_{A^{\mathrm{t}}}$$

is a model for the map θ_k as a map from the circle. (The factor of $1/2$ is there because of the relationship between A and A^{t}.) When $T = -\infty$, the surface Σ is contained in the piece T_1^* on which the connection A^{t} has become flat. So with this model, $\theta_k(-\infty) = 1$. When $T = +\infty$, the surface Σ decomposes into two pieces: one is a cylinder contained in T_2^*, which contributes nothing to the integral; and the other is a disk Δ with cylindrical end contained in $Z_2^* \cong \overline{\mathbb{CP}}^2 \setminus B^4$. The curvature $F_{A^{\mathrm{t}}}$ has exponential decay on the end of Z_2^* and its integral on Δ is equal to its integral on any compact surface $\Delta' \subset Z_2^*$ having the same intersection with E_2. Since Δ has intersection 1 with E_2 and $c_1(\mathfrak{t}_k)$ evaluates to $-(2k+1)$ on E_2, we have

$$\int_\Delta F_{A^{\mathrm{t}}} = (2\pi/i)(2k+1).$$

So $\theta_k(\infty) = -1$, and we have the result. □

For each integer $i < 0$, we can define a map

$$\check{L}[i] : \check{C}_\bullet(Y_1) \to \check{C}_\bullet(Y_1)$$

by repeating the definition of $\check{L}$ above, but replacing $\bar{n}_u$ in the formulae by the basis vector e_i^1. Because there are no differentials on R_1, the map $\check{L}[i]$ is a chain map, and from the formula for $\bar{n}_u$, we have

$$\begin{aligned}\check{L} &= \sum_{k\in\mathbb{Z}} a_k\, \check{L}[-1-k(k+1)/2] \\ &= \sum_{k\geq 0} \check{L}[-1-k(k+1)/2],\end{aligned}$$

where in the second line we have used Lemma 5.10. Proposition 5.6 now follows from:

LEMMA 5.11. *The map $\check{L}[i] : \check{C}_\bullet(Y_1) \to \check{C}_\bullet(Y_1)$ for $i<0$ gives rise to the map $\widecheck{HM}_\bullet(Y_1) \to \widecheck{HM}_\bullet(Y_1)$ given by multiplication by U^{-i-1}.*

Proof. We use the fact that the manifold U_1 (whose moduli spaces define $\check{L}$) can be realised as the complement in $[-1,1]\times Y_1$ of the tubular neighborhood of a curve K: thus

$$[-1,1]\times Y_1 = U_1 \cup_{R_1} N(K),$$

where $N(K) \cong S^1 \times B^3$ and R_1 is the oriented boundary of $N(K)$. Referring to the definition of the action of U^p from Section 4.11, we choose p basepoints $w_1,\dots w_p$ in the interior of $N(K)$, and use these together with the cylindrical cobordism $[-1,1]\times Y_1$ to define a chain map

$$\check{m}([-1,1]\times Y_1, \{w_1,\dots,w_p\}) : \check{C}_\bullet(Y_1) \to \check{C}_\bullet(Y_1) \tag{25}$$

which induces the map U^p on $\widecheck{HM}_\bullet(Y_1)$.

We can use any Riemannian metric on the cylinder in the construction of this chain map. We choose a metric in which $N(K)$ has positive scalar curvature, the metric on R_1 is standard, and R_1 has a product neighborhood. We then consider the family of metrics parametrized by $Q=[0,\infty)$ obtained by inserting a cylinder $[-T,T]\times R_1$. By an argument similar to our previous analysis, we obtain in this way a chain-homotopy between the chain map (25) and the chain map

$$\sum_{j<0} b_j\, \check{L}[j],$$

where

$$\sum_{j<0} b_j e^1_j = \bar{n}_u(N(K))$$

is the element of $\hat{C}_\bullet(R_1) \cong \widehat{HM}_\bullet(R_1)$ obtained by counting points in moduli spaces on $N(K)^*$ with p base-points. That is,

$$b_j = |M(N(K), \mathfrak{a}^1_j) \cap V_1 \cap \cdots \cap V_p| \bmod 2,$$

or zero if the intersection is not zero-dimensional. An examination of dimensions shows that the only contributions occur when $j=-1-p$. The moduli consists of reducibles, so the calculation of b_j is straightforward: we have $b_j = 1$ when $j = -1 - p$. Thus the sum above has just one term, and the map U^p is equal to the map arising from the chain map $\check{L}[-1-p]$. □

With the verification of Proposition 5.6, the proof of Theorem 2.4 is complete for the case of $\widecheck{HM}_\bullet$. The other two case have similar proofs. In the case of $\overline{HM}_\bullet$, the formulae are considerably simpler. The exactness in the case of $\widehat{HM}_\bullet$ could also be deduced from the other two cases, by chasing the square diagram in which the columns are the i, j, p exact sequences and the rows are the surgery cobordism sequences.

5.4. *Local coefficients.* We describe here a refinement of the long exact sequence, with local coefficients. As in Section 2.3, we consider a 3-manifold M with torus boundary, and let γ_0, γ_1, γ_2 be three oriented simple closed curves on ∂M with algebraic intersection

$$(\gamma_0 \cdot \gamma_1) = (\gamma_1 \cdot \gamma_2) = (\gamma_2 \cdot \gamma_0) = -1.$$

We again write $W_n \colon Y_n \longrightarrow Y_{n+1}$ for the 2-handle cobordisms.

The interior M^o can be viewed as an open subset of Y_i for all i. Fix local coefficient systems Γ_i over Y_i which are supported in $M^o \subset Y_i$, in the sense of Definition 4.18. Moreover, the set $[0,1] \times M^o$ can be viewed as an open subset of W_n (where here $\{0\} \times M^o \subset Y_n$ and $\{1\} \times M^o \subset Y_{n+1}$). Let

$$\Gamma_{W_n} \colon \Gamma_n \longrightarrow \Gamma_{n+1}$$

be a W_n-morphism of the local system which is supported in $[0,1] \times M^o$.

THEOREM 5.12. *Let Γ_n be local systems on the Y_n, and let $\Gamma_{W_n} : \Gamma_n \to \Gamma_{n+1}$ be morphisms of local systems. Suppose these satisfy the support condition just described. Then, the induced maps with local coefficients $\check{F}_n = \widecheck{HM}(W_n; \Gamma_{W_n})$ fit into a long exact sequence of the form*

$$\cdots \longrightarrow \widecheck{HM}_\bullet(Y_{n-1}; \Gamma_{n-1}) \xrightarrow{\check{F}_{n-1}} \widecheck{HM}_\bullet(Y_n; \Gamma_n) \xrightarrow{\check{F}_n} \widecheck{HM}_\bullet(Y_{n+1}; \Gamma_{n+1}) \longrightarrow \cdots.$$

There are also corresponding long exact sequences for the other two variants of Floer homology.

Proof. To prove exactness, we once again appeal to Lemma 5.1. Indeed, the homotopies $\check{H}'_n \colon \check{C}(Y_n; \Gamma_n) \longrightarrow \check{C}(Y_{n+2}; \Gamma_{n+2})$ are constructed as before, only now the entries contain also the X_n-morphisms gotten by composing the morphisms Γ_{W_n} and $\Gamma_{W_{n+1}}$ (we add the primes to distinguish the homotopies here from the ones appearing in the discussion in Subsection 5.1). Observe that this composite morphism of local systems is supported in the complement of Z_n (so as in Lemma 5.4, the contributions from Z_i still drop out in pairs). In fact, the proof of Proposition 5.5 carries over, as well, since the triple-composite morphism of local systems is supported in a complement of the N_i.

In the same way, the map $\check{L}'$ is seen to be multiplication gotten by multiplying the chain map induced by the triple-composite $\Gamma_{W_{n+2}} \circ \Gamma_{W_{n+1}} \circ \Gamma_{W_n}$ (which is supported in the complement of N_n) by the power series $\check{L}$. In particular, the map $\check{L}'$ is a quasi-isomorphism, too. $\square$

Specializing to the local system determined by cycles (Example 4.12.2), we get the following:

COROLLARY 5.13. *Let Y_0, Y_1, Y_2 be as above, with the additional property that $H_1(Y_0;\mathbb{Z}) \cong \mathbb{Z}$ and $H_1(Y_1;\mathbb{Z}) = H_1(Y_2;\mathbb{Z}) = 0$. Fix a cycle η in M^o which generates the image of $H_1(M;\mathbb{Z})$ in $H_1(M;\mathbb{R})$. In this case, we have a long exact sequence*

$$\cdots \longrightarrow \widecheck{HM}_\bullet(Y_{-1}) \otimes \mathbb{K} \xrightarrow{\check{F}_{-1}} \widecheck{HM}_\bullet(Y_0;\Gamma_{\eta_0}) \xrightarrow{\check{F}_0} \widecheck{HM}_\bullet(Y_1) \otimes \mathbb{K} \xrightarrow{\check{F}_1} \cdots,$$

in which the map $\check{F}_1$ can be expressed in terms of the usual maps induced by cobordisms, by the following formula:

$$\check{F}_1 = \sum_{\mathfrak{s} \in \mathrm{Spin}^c(W_1)} \mu(\langle c_1(\mathfrak{s}), [h]\rangle) \cdot \widecheck{HM}(W_1, \mathfrak{s}),$$

where $[h] \in H_2(W_1;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.

Proof. We apply Theorem 5.12 in the following setting. We let Γ_n be the local system on Y_n induced by the chain $\eta \subset M^o \subset Y_n$. Indeed, in the cobordisms $W_n \colon Y_n \to Y_{n+1}$, we choose two-chains ν_n which are products $\nu_n = [0,1] \times \eta \subset [0,1] \times M^o \subset W_n$. The chain ν_n induces a W_n-morphism of the local system Γ_{W_n,ν_n} which is supported in $[0,1] \times M^o$.

Recall that the isomorphism class of $\widecheck{HM}(Y;\Gamma_\eta)$ depends only on the homology class of η. In fact, since both Y_1 and Y_2 are homology three-spheres, the cycle is null-homologous, so it follows at once that for $i = 1, 2$,

$$\widecheck{HM}(Y_i;\Gamma_{\eta_i}) \cong \widecheck{HM}(Y_i) \otimes \mathbb{K}. \tag{26}$$

We argue that the chain $\nu_1 \subset W_1$ represents a generator of $H_2(W_1, \partial W_1;\mathbb{Z}) \cong \mathbb{Z}$. To see why, recall that inside Y_1, γ_0 can be viewed as a knot, with a Seifert surface Σ. Pushing the interior of the Seifert surface into $[0,1] \times Y_1$, which we then cap off inside the added two-handle, we obtain a generator $[\widehat{\Sigma}]$ for $H_2(W_1;\mathbb{Z})$. On the other hand, our chain ν_1 is gotten by $[0,1] \times \eta$, and η links the knot γ_0 once. Thus, it follows that the oriented intersection number of $\widehat{\Sigma}$ with ν_1 is ± 1, and hence $[\nu_1]$ is also a generator of $H_2(W_1, \partial W_1)$. Now, for each Spin^c structure on W_1, we see that composing with the identification from Equation (26), we see that the map $\widecheck{HM}(W_1;\Gamma_{W_1,\nu}, \mathfrak{s})$ is identified with $\mu(\langle c_1(\mathfrak{s}), [\nu]]\rangle) \cdot \widecheck{HM}(W_1, \mathfrak{s})$, where here $[\nu] \in H_2(W_1;\mathbb{Z})$ is the unique homology class corresponding to ν_1. The result now follows. $\square$

6. Proof of the nonvanishing theorem

6.1. *Statement of the sharper result.* We now turn to the proof of Theorem 2.1. There is a sharper version of this nonvanishing theorem, which involves the Floer groups with local coefficients.

THEOREM 6.1. *Suppose Y admits a taut foliation $\mathcal{F}$ and is not $S^1 \times S^2$. Suppose either that $\mathcal{F}$ is smooth, or that $\mathcal{F}$ has holonomy and is smooth in the complement of the closed leaves. Let η be a C^∞ singular 1-cycle in Y whose homology class $[\eta]$ satisfies*

$$\mathrm{P.D.}[\eta] = [\omega] + t\, e(\mathcal{F}) \in H^2(Y;\mathbb{R})$$

where ω is closed 2-form which is positive on the leaves of $\mathcal{F}$ and $t \in \mathbb{R}$. Then the image of the map

$$j_* : \widetilde{HM}_k(Y;\Gamma_\eta) \to \widehat{HM}_k(Y;\Gamma_\eta)$$

is nonzero, where $k \in J(Y)$ is the homotopy class of the 2-plane given by the tangent planes to $\mathcal{F}$.

As an application, we consider the case of the manifold $Y = S^3_0(K)$, where $K \neq U$. By Gabai's theorem [15] again, this manifold has a taut foliation $\mathcal{F}$. In the case of a genus-1 knot, Gabai's foliation is not guaranteed to be smooth; but it does have holonomy and is smooth away from genus-1 closed leaves, so the theorem applies. If ω is closed and positive on the leaves, then the cohomology class $[\omega]$ will be nonzero, because Gabai's foliation has a compact leaf. We therefore have the following corollary. Unlike Corollary 2.3, this result applies also to genus one knots:

COROLLARY 6.2. *Suppose $K \neq U$, let g be the Seifert genus of K, and let $\mathfrak{s}$ be a Spin^c structure on $S^3_0(K)$ for which $c_1(\mathfrak{s})$ is $2g-2$ times a generator for $H^2(S^3_0(K);\mathbb{Z})$. Then there is a $k \in J(Y,\mathfrak{s})$ such that the image of*

$$j_* : \widetilde{HM}_k(S^3_0(K);\Gamma_\eta) \to \widehat{HM}_k(S^3_0(K);\Gamma_\eta)$$

is nonzero whenever the homology class $[\eta]$ is nonzero. By contrast, j_* is zero for $S^3_0(U)$, for all η. □

The proof of the theorem is based on the results of [25]. Let X be a compact oriented 4-manifold with oriented boundary Y. We assume X is connected, but may allow Y to be disconnected. Let ξ be an oriented contact structure on Y, compatible with the orientation of Y. If α is a 1-form on Y whose kernel is the field of 2-planes ξ, then the orientation condition can be expressed as the condition that $\alpha \wedge d\alpha > 0$. Let $\mathfrak{s}$ be the Spin^c structure on Y determined by ξ, and let $\mathfrak{s}_X$ be any extension of $\mathfrak{s}$ to X. Note that the space of isomorphism classes of such extensions, denoted by $\mathrm{Spin}^c(X,\xi)$, is an affine space for the group $H^2(X, Y; \mathbb{Z})$. Generalizing the monopole invariants of closed 4-manifolds, the paper [25] defines an invariant $\mathfrak{m}(X, \xi, \mathfrak{s}_X)$ associated to such data. Neglecting orientations, we can take it to be an element of $\mathbb{F} = \mathbb{Z}/2$. We review some of the properties of this invariant.

The 2-plane field ξ picks out not just a Spin^c structure $\mathfrak{s}$ on Y, but also a preferred nowhere-vanishing section Φ_0 of the spin bundle $S \to Y$. When $\mathfrak{s}_X$ is given, we can interpret Φ_0 as a section of $S^+_X|_Y$, and there is a relative second Chern class (or Euler class) which we use as the definition of $\mathrm{gr}(X, \xi, \mathfrak{s}_X)$:

$$\mathrm{gr}(X, \xi, \mathfrak{s}_X) = \langle c_2(S^+_X, \Phi_0), [X, \partial X] \rangle \in \mathbb{Z}.$$

The condition $\mathrm{gr}(X, \xi, \mathfrak{s}_X) = 0$ is equivalent to the existence of an almost complex structure on X for which the 2-planes ξ are complex and such that the associated Spin^c structure is $\mathfrak{s}_X$.

THEOREM 6.3 ([25]). (1) *The invariant* $\mathfrak{m}(X, \xi, \mathfrak{s}_X)$ *is nonzero only if* $\mathrm{gr}(X, \xi, \mathfrak{s}_X) = 0$, *and vanishes for all but finitely many* $\mathfrak{s}_X \in \mathrm{Spin}^c(X, \xi)$.

(2) *Suppose* X *carries a symplectic form* ω *that is positive on the oriented 2-plane field* ξ. *Let* $\mathfrak{s}_\omega \in \mathrm{Spin}^c(X, \xi)$ *be the canonical* Spin^c *structure which* ω *determines on* X. *Then*

$$\mathfrak{m}(X, \xi, \mathfrak{s}_\omega) = 1.$$

(3) *Let* ω *and* $\mathfrak{s}_\omega$ *be as in the previous item, let* $e \in H^2(X, \partial X; \mathbb{Z})$, *and let* $\mathfrak{s}_\omega + e \in \mathrm{Spin}^c(X, \omega)$ *denote the* Spin^c *structure with spin bundle* $S = S_\omega \otimes L$, *where* L *is the line bundle, trivialized on* ∂X, *with relative first Chern class* $c_1(L) = e$. *Then, if* $\mathfrak{m}(X, \xi, \mathfrak{s}_\omega + e) \neq 0$, *it follows that*

$$\langle e \smile \omega, [X, \partial X] \rangle \geq 0,$$

with equality only if $e = 0$. □

We combine the individual invariants $\mathfrak{m}(X, \xi, \mathfrak{s}_X)$ into a generating function. Recall that $\mathbb{K}$ is the field of fractions of $\mathbb{F}[\mathbb{R}]$, and that $\mu : \mathbb{R} \to \mathbb{K}^\times$ is the canonical homomorphism. Given a reference Spin^c structure $\mathfrak{s}_0$ on X extending $\mathfrak{s}$, we define a function

$$\mathfrak{m}^*(X, \xi, \mathfrak{s}_0) : H_2(X, Y; \mathbb{R}) \to \mathbb{K}$$

by the formula

$$\mathfrak{m}^*(X, \xi, \mathfrak{s}_0)(h) = \sum_e \mathfrak{m}(X, \xi, \mathfrak{s}_0 + e)\, \mu\langle 2e, h\rangle.$$

As a corollary of Theorem 6.3 we have:

COROLLARY 6.4. *If X carries a symplectic form ω positive on ξ then*

$$\mathfrak{m}^*(X, \xi, \mathfrak{s}_\omega)(\text{P.D.}[\omega]) \neq 0.$$

Further, if the intersection form on $H^2(X, \partial X)$ is trivial then

$$\mathfrak{m}^*(X, \xi, \mathfrak{s}_\omega)(\text{P.D.}([\omega] + t\, c_1(\mathfrak{s}_\omega))) \neq 0$$

for all $t \in \mathbb{R}$.

Proof. The first statement is an immediate consequence of Theorem 6.3. For the second statement, we note that the condition $\text{gr}(X, \xi, \mathfrak{s}_\omega + e) = 0$ is equivalent to

$$e \smile (e + c_1(\mathfrak{s}_\omega)) = 0,$$

or simply to $e \smile c_1(\mathfrak{s}_\omega) = 0$ when the cup product on the relative cohomology is zero; so

$$\mathfrak{m}^*(X, \xi, \mathfrak{s}_\omega)(\text{P.D.}([\omega] + t\, c_1(\mathfrak{s}_\omega))) = \mathfrak{m}^*(X, \xi, \mathfrak{s}_\omega)(\text{P.D.}[\omega]). \qquad \square$$

6.2. *Construction of the invariants of* (X, ξ). We review the construction of the invariant $\mathfrak{m}(X, \xi, \mathfrak{s}_X)$ from [25]. Let α again be the 1-form defining the 2-plane field ξ on Y, let ω_0 be the symplectic form $d(t^2\alpha/2)$ on $[1, \infty) \times Y$, and let g_0 be a compatible metric. Attaching $[1, \infty) \times Y$ to X, we obtain a complete Riemannian manifold X^+ with an expanding conical end. On the end, there is a canonical Spin^c connection A_0 and spinor Φ_0 of unit length. We choose a reference Spin^c structure $\mathfrak{s}_0$ on X, extending the Spin^c structure on the end, and extend A_0 and Φ_0 arbitrarily. For a pair (A, Φ) consisting of a Spin^c connection in $\mathfrak{s}_X$ and a section of S_X^+, we consider the equations on X^+:

$$\begin{aligned} \frac{1}{2}\rho(F^+_{A^t}) - (\Phi\Phi^*)_0 &= \frac{1}{2}\rho(F^+_{A_0^t}) - (\Phi_0\Phi_0^*)_0 + \varepsilon, \\ D_A^+\Phi &= 0, \end{aligned} \tag{27}$$

where ε is a perturbation term: an exponentially decaying section of $i\mathfrak{su}(S_X^+)$. There is a moduli space $M(X^+, \mathfrak{s}_X)$ consisting of all gauge-equivalence classes of solutions (A, Φ) on X^+ which are asymptotically equal to (A_0, Φ_0), in that

$$\begin{aligned} A - A_0 &\in L^2_k \\ \Phi - \Phi_0 &\in L^2_{k, A_0}. \end{aligned} \tag{28}$$

For generic ε, this moduli space is a smooth manifold, and

$$\dim M(X^+, \mathfrak{s}_X) = \text{gr}(X, \xi, \mathfrak{s}_X)$$

if the moduli space is nonempty. Note that the asymptotic conditions mean that Φ is nonzero, so there are no reducible solutions in the moduli space. The moduli space is compact, and $\mathfrak{m}(X, \xi, \mathfrak{s}_X)$ is defined as the number of points in the moduli space, mod 2, or as zero if the dimension is positive.

6.3. *Floer chains from contact structures.* Let Z now denote the half-infinite cylinder $\mathbb{R}^+ \times (-Y)$, which has oriented boundary $\{0\} \times Y$. Given a contact structure ξ defined by a 1-form α as above, we again form the symplectic cone $[1, \infty) \times Y$, with oriented boundary $\{1\} \times (-Y)$, and attach this to Z. The result is a complete Riemannian manifold Z^+ with one cylindrical end and one expanding conical end; the latter carries the symplectic form ω_0.

On Z^+, we write down monopole equations which resemble the equations (27) on the conical end and resemble the perturbed equations $\mathcal{F}_{\mathfrak{p}} = 0$ on the cylindrical end. A convenient way to make this construction is as a fiber product. To do this, we choose a regular perturbation $\mathfrak{q}$ for the equations on Y, and let $\mathfrak{p}$ be a t-dependent perturbation on the cylinder Z that is equal to $\mathfrak{q}$ on the end and is zero near the boundary. For each critical point $\mathfrak{a}$ in $\mathcal{B}^\sigma(-Y)$, there is a moduli space

$$M(\mathbb{R}^+ \times (-Y), \mathfrak{a}) \subset \mathcal{B}^\sigma(\mathbb{R}^+ \times (-Y)).$$

This is a Banach manifold with a restriction map

$$r_0 : M(\mathbb{R}^+ \times (-Y), \mathfrak{a}) \to \mathcal{B}^\sigma(\{0\} \times (-Y)) = \mathcal{B}^\sigma(Y).$$

There is also the moduli space M_1 of solutions to the equations (27) on the conical manifold $[1, \infty) \times Y$, with asymptotic conditions (28). We choose the term ε so that the right-hand side of the first equation vanishes near the boundary $\{1\} \times Y$. This is another Banach manifold; and because the solutions are irreducible, there is a restriction map to blown-up configuration space of the boundary, because of a unique continuation argument:

$$r_1 : M_1 \to \mathcal{B}^\sigma(\{1\} \times Y) = \mathcal{B}^\sigma(Y).$$

Definition 6.5. We define the moduli space $M(Z^+, \mathfrak{a})$ as the fiber product of the maps r_0 and r_1.

Although the fiber product makes a convenient definition, we can also regard $M(Z^+, \mathfrak{a})$ as a subspace of $\mathcal{B}^\sigma_{\mathrm{loc}}(Z^+)$. Given γ in $M(Z^+, \mathfrak{a})$, we can define a path $\check{\gamma}$ in $\mathcal{B}^\sigma(Y)$ by restricting γ to the slices $\{t\} \times Y$, first in the cylindrical end then in the conical end. If γ, γ' are two solutions, then the corresponding paths $\check{\gamma}$ and $\check{\gamma}'$ both have limit point $\mathfrak{a}$ on the cylindrical end and have the same asymptotics on the conical end; so there is a well-defined difference element in $\pi_1(\mathcal{B}^\sigma(Y), \mathfrak{a})$. In this way, we partition $M(Z^+, \mathfrak{a})$ into components of different topological type:

$$M(Z^+, \mathfrak{a}) = \bigcup_z M_z(Z^+, \mathfrak{a}).$$

Once again, we can count points in zero-dimensional moduli spaces, to define:

$$m_z(Z^+,\mathfrak{a}) = \begin{cases} |M_z(Z^+,\mathfrak{a})| \bmod 2, & \text{if } \dim M_z(Z^+,\mathfrak{a}) = 0,\\ 0, & \text{otherwise.}\end{cases}$$

The compactness results of [25] tell us that $m_z(Z^+,\mathfrak{a})$ is zero for all but finitely many $\mathfrak{a}$ and z. Because $M_z(Z^+,\mathfrak{a})$ consists only of irreducibles, $\mathfrak{a}$ must be either irreducible or boundary-stable on $-Y$ if the moduli space is to be nonempty. (Note that the notion of "boundary-stable" for a critical point $\mathfrak{a}$ depends on the orientation: $\mathfrak{C}^s(-Y)$ is the same as $\mathfrak{C}^u(Y)$.) Taking the irreducible and boundary-stable elements in turn, we define an element of the complex $\check{C}_*(-Y) = C^o(-Y) \oplus C^s(-Y)$, by

$$\check{\psi} = (\psi^o, \psi^s), \tag{29}$$

where

$$\psi^o = \sum_{\mathfrak{a}\in\mathfrak{C}^o} \sum_z m_z(Z^+,\mathfrak{a})\, e_{\mathfrak{a}}$$

and

$$\psi^s = \sum_{\mathfrak{a}\in\mathfrak{C}^s} \sum_z m_z(Z^+,\mathfrak{a})\, e_{\mathfrak{a}}.$$

We have:

LEMMA 6.6. *The element $\check{\psi}$ in $\check{C}_*(-Y)$ is closed*; *that is*, $\check{\partial}\check{\psi} = 0$.

Proof. As usual, this is proved by counting the boundary points of 1-dimensional moduli spaces $M_z(Z^+,\mathfrak{b})$, augmented by the observation that there are no reducible solutions in these moduli spaces. Specifically, counting boundary points in the case that $\mathfrak{b}$ is irreducible gives the identity

$$\partial^o_o\psi^o + \partial^u_o\bar{\partial}^s_u\psi^s = 0,$$

while the case that $\mathfrak{b}$ is boundary-stable provides the identity

$$\partial^o_s\psi^o + \bar{\partial}^s_s\psi^s + \partial^u_s\bar{\partial}^s_u\psi^s = 0.$$

Together these tell us that $\check{\partial}\check{\psi} = 0$. □

Next we extend the construction of $\check{\psi}$ to the Floer complex with local coefficients. Let η be a C^∞ real 1-cycle in Y, and let η^+ be the corresponding noncompact 2-chain in Z^+. Like Z^+, the 2-chain η^+ has one cylindrical end and one expanding conical end; it is oriented so that the cylindrical end coincides with $-(\mathbb{R}^+ \times \eta)$. Extend the connection A_0 from the conical end to all of Z^+ in such a way that it is translation-invariant and in temporal gauge on the cylindrical end. Let γ belong to $M_z(Z^+, \mathfrak{a})$, and let A be the corresponding Spin^c connection on Z^+. The integral

$$f(z) = (i/2\pi) \int_{\eta^+} (F_{A^{\mathfrak{t}}} - F_{A_0^{\mathfrak{t}}})$$

is finite, and depends on γ through only its homotopy class z. Let $\Gamma_{-\eta}$ be the local system on $\mathcal{B}^\sigma(-Y)$ defined in Example 4.12.2, and denote the generator 1 in $\Gamma_{\mathfrak{a}} \cong \mathbb{K}$ by $e_{\mathfrak{a}}$. Then we can define an element

$$\check{\psi}_\eta = (\psi^o_\eta, \psi^s_\eta) \in \check{C}(-Y; \Gamma_{-\eta})$$

by the formulae

$$\psi^o_\eta = \sum_{\mathfrak{a}\in\mathfrak{C}^o} \sum_z m_z(Z^+, \mathfrak{a}) \mu(f(z))\, e_{\mathfrak{a}}$$
$$\psi^s_\eta = \sum_{\mathfrak{a}\in\mathfrak{C}^s} \sum_z m_z(Z^+, \mathfrak{a}) \mu(f(z))\, e_{\mathfrak{a}}.$$

As in the previous case, we have:

LEMMA 6.7. *The element $\check{\psi}_\eta$ in $\check{C}_*(-Y; \Gamma_{-\eta})$ is closed.* □

6.4. *Proof of Theorem* 6.1. We recall Eliashberg and Thurston's construction [9], whereby a taut foliation on Y leads to a symplectic form ω_W on the cylinder

$$W = [-1, 1] \times Y$$

and contact structures $\xi_\pm$ on the boundary components. First suppose that the foliation $\mathcal{F}$ is smooth. Let α be a 1-form defining the tangents to the foliation, and let ω be a closed 2-form positive on the leaves. Set

$$\omega_W = d(t\alpha) + \omega$$

on W. This form is symplectic. According to [9, Theorem 2.4.1], there exist smooth contact structures ξ_+ and ξ_-, compatible with the orientations of Y and $-Y$ respectively, which are C^0 close to the tangent plane field of the foliation. We regard these as contact structures on the two boundary components $\{1\} \times Y$ and $\{-1\} \times Y$ of W; the C^0-close condition means that ω_W will be positive on these 2-plane fields. Now suppose that $\mathcal{F}$ is only C^0, but still has smooth leaves, has nontrivial holonomy, and is smooth in the complement of the closed leaves. The smooth contact structures ξ_+ and ξ_- still exist, according to [9, Prop. 2.9.4]. However, we cannot define ω_W by the formula above, because the 1-form α which defines $\mathcal{F}$ is not differentiable. Instead, let β be a smooth 1-form which is a C^0 approximation to α, close enough that $\beta \wedge \omega$ is still a positive 3-form on Y, and consider the 2-form

$$\omega_W = \varepsilon d(t\beta) + \omega,$$

where ε is a positive real parameter. The square of this 2-form is the 4-form

$$2\varepsilon dt \wedge \beta \wedge \omega + 2\varepsilon^2 t dt \wedge \beta \wedge d\beta.$$

The first term is a positive 4-form on the cylinder. The second term is not necessarily positive. However, because of the ε^2 in the second term, the sum of the two terms is positive if ε is small enough. Similarly, ω_W is positive on the 2-planes of ξ_+ and ξ_- once ε is small enough, even though $d(t\beta)$ has indeterminate sign.

If we regard W as a manifold with a contact structure $\xi = (\xi_-, \xi_+)$ on its boundary $\{-1, 1\} \times Y$, then we can construct the invariants

$$\mathfrak{m}(W, \xi, \mathfrak{s}_W)$$

for Spin^c structures $\mathfrak{s}_W$ extending the standard Spin^c structure determined by the 2-plane field on the boundary, as above. On the other hand, the contact structure ξ_+ provides a cycle

$$\check{\psi}(\xi_+) = \big(\psi^o(\xi^+), \psi^s(\xi^+)\big) \in \check{C}_*(-Y),$$

and from ξ_- we similarly obtain a cycle

$$\check{\psi}(\xi_-) = \big(\psi^o(\xi^-), \psi^s(\xi^-)\big) \in \check{C}_*(Y).$$

Let

$$\check{\Psi}(\xi_+) \in \widetilde{HM}_*(-Y) \check{\Psi}(\xi_-) \in \widetilde{HM}_*(Y)$$

be the homology classes of these.

PROPOSITION 6.8. *We have the pairing formula*

$$\sum_{\mathfrak{s}_W} \mathfrak{m}(W, \xi, \mathfrak{s}_W) = \big\langle j_* \check{\Psi}(\xi_+), \check{\Psi}(\xi_-) \big\rangle_{\mathrm{D}},$$

where $\langle - , - \rangle_{\mathrm{D}}$ *is the duality pairing*

$$\widehat{HM}_*(-Y) \otimes \widetilde{HM}_*(Y) \to \mathbb{F},$$

from Section 4.13.

Proof. Let $Z^+(\xi_+)$ be the manifold with one cylindrical end and one conical end, obtained by applying the construction of the previous subsection to ξ_+ on Y, and let $Z^+(\xi_-)$ be constructed similarly using ξ_-. As above, we have counting-invariants

$$m_z(Z^+(\xi_+), \mathfrak{a}) \in \mathbb{F}, (\mathfrak{a} \in \mathfrak{C}^o(-Y) \cup \mathfrak{C}^s(-Y)),$$
$$m_z(Z^+(\xi_-), \mathfrak{a}) \in \mathbb{F}, (\mathfrak{a} \in \mathfrak{C}^o(Y) \cup \mathfrak{C}^s(Y)).$$

In the case of the first of these two, we can regard $\mathfrak{a}$ as a critical point in $\mathcal{B}^\sigma(Y)$ via the identification

$$\mathfrak{C}^o(-Y) \cup \mathfrak{C}^s(-Y)) = \mathfrak{C}^o(Y) \cup \mathfrak{C}^u(Y)).$$

If we unravel the pairing on the right-hand side of the formula in the proposition, we find it is equal to

$$\sum_{\mathfrak{a}\in\mathfrak{C}^o(Y)} m_{z_1}(Z^+(\xi_-),\mathfrak{a})m_{z_2}(Z^+(\xi_+),\mathfrak{a}) + \sum_{\mathfrak{a}\in\mathfrak{C}^s(Y)}\sum_{\mathfrak{b}\in\mathfrak{C}^u(Y)} m_{z_1}(Z^+(\xi_-),\mathfrak{a})n_{z_2}(\mathfrak{a},\mathfrak{b})m_{z_3}(Z^+(\xi_+),\mathfrak{b}) \tag{30}$$

where $n_z(\mathfrak{a},\mathfrak{b})$ counts unparametrized boundary-obstructed trajectories as in Section 4.7. On the other hand, we can consider the 1-parameter family of metrics on W parametrized by $Q=[0,\infty)$ in which the length of the cylinder is increased. There is a corresponding parametrized moduli space

$$M(W^+,\mathfrak{s}_W)_Q$$

associated to the manifold W^+ with two conical ends. The map to Q is proper, and there is a compactification $M(W^+,\mathfrak{s}_W)_{\bar{Q}}$ over $\bar{Q}=[0,\infty]$, where at ∞ the manifold W^+ becomes the disjoint union of $Z^+(\xi_-)$ and $Z^+(\xi_+)$. If we look at the union of all 1-dimensional moduli spaces $M(W^+,\mathfrak{s}_W)_{\bar{Q}}$ and count the endpoints of these, then the contributions from endpoints lying over $0\in\bar{Q}$ is equal to the left-hand side in the proposition, while the contribution from the endpoints lying over ∞ is the sum (30). $\square$

We now reformulate this proposition for local coefficients. Because ξ_- and ξ_+ are both C^0-close to $\mathcal{F}$, there is a canonical choice of reference Spin^c structure $\mathfrak{s}_0$ on W, and we can write an arbitrary $\mathfrak{s}_W$ as

$$\mathfrak{s}_W=\mathfrak{s}_0+e,$$

for some $e\in H^2(W,\partial W;\mathbb{Z})$. Let η be a 1-cycle in Y, and set

$$h_\eta=[-1,1]\times[\eta]$$

regarded as a class in $H_2(W,\partial W;\mathbb{R})$. Let

$$\mathfrak{m}^*(W,\xi,\mathfrak{s}_0):H_2(W,\partial W;\mathbb{R})\to\mathbb{K}$$

be as in Section 6.1. The construction of $\check{\psi}_\eta$ leads to classes

$$\check{\Psi}_\eta(\xi_+)\in\widetilde{HM}_*(-Y;\Gamma_{-\eta})\check{\Psi}_\eta(\xi_-)\in\widetilde{HM}_*(Y;\Gamma_\eta).$$

Just as in the case of coefficients $\mathbb{F}$, we have:

Proposition 6.9. *We have the pairing formula*

$$\mathfrak{m}^*(W,\xi,\mathfrak{s}_0)(h_\eta)=\left\langle j_*\check{\Psi}_\eta(\xi_+),\check{\Psi}_\eta(\xi_-)\right\rangle_{\mathrm{D}},$$

where $\langle\,-\,,\,-\,\rangle_{\mathrm{D}}$ *is the duality pairing*

$$\widehat{HM}_*(-Y;\Gamma_{-\eta})\otimes\widetilde{HM}_*(Y;\Gamma_\eta)\to\mathbb{F},$$

from Section 4.13, *and* j_* *is the map* $\widetilde{HM}_*(-Y;\Gamma_{-\eta})\to\widehat{HM}_*(-Y;\Gamma_{-\eta})$. $\square$

Now we conclude the proof of Theorem 6.1. Suppose the class $[\eta]$ in $H_1(Y;\mathbb{R})$ satisfies

$$\mathrm{P.D.}[\eta] = [\omega] + t\,c_1(\mathcal{F}),$$

so that the class h_η in $H_2(W, \partial W; \mathbb{R})$ satisfies

$$\mathrm{P.D.}[h_\eta] = [\omega_W] + t\,c_1(\mathfrak{s}_{\omega_W}).$$

The intersection form on $H^2(W, \partial W)$ is trivial, so Corollary 6.4 tells us that $\mathfrak{m}^*(W, \xi, \mathfrak{s}_0)(h_\eta)$ is nonzero. (Note that $\mathfrak{s}_0$ and $\mathfrak{s}_{\omega_W}$ are the same.) From the proposition above, it follows that $j_*\check{\Psi}_\eta(\xi^+)$ is nonzero; and in particular the image of j_* is nontrivial in $\widehat{HM}_*(-Y;\Gamma_{-\eta})$. The hypotheses of the theorem are symmetrical with respect to orientation, so j_* has nonzero image also in $\widehat{HM}_*(Y;\Gamma_\eta)$. ∎

6.5. *Application to knots of genus one.* Using the nonvanishing theorem with local coefficients, we can now complete the proof of Theorem 1.1 for knots of genus 1, in the case that the surgery coefficient is an integer. The arguments of Section 3 continue to show that if K is p-standard then K is $(p-1)$-standard, for integers $p \geq 2$. We need therefore only prove the following result, by a method applicable to genus 1.

PROPOSITION 6.10. *If K is p-standard for $p = 1$, then K is the unknot.*

Proof. Consider the long exact sequence sequence with local coefficients, in the form given in Corollary 5.13, applied to 3-manifolds S^3, $S^3_1(K)$ and $S^3_0(K)$:

$$\cdots \longrightarrow \widetilde{HM}_\bullet(S^3) \otimes \mathbb{K} \xrightarrow{\check{F}_{-1}} \widetilde{HM}_\bullet(S^3_0(K);\Gamma_{\eta_0}) \xrightarrow{\check{F}_0} \widetilde{HM}_\bullet(S^3_1(K)) \otimes \mathbb{K} \xrightarrow{\check{F}_1} \cdots .$$

If K is 1-standard, then the map $\check{F}_1$ is entirely determined by the map induced on $\overline{HM}$, which in turn is determined using Proposition 2.7. Consequently, the map $\check{F}_1$ is given by multiplication by

$$\sum_{n\geq 0} U^{n(n+1)/2} \cdot (\mu(2n+1) + \mu(-2n-1)),$$

thought of as a map from $\mathbb{K}[U^{-1}, U]/\mathbb{K}[U]$ ($\cong \widetilde{HM}(S^3_1(K)) \otimes \mathbb{K} \cong \widetilde{HM}(S^3) \otimes \mathbb{K}$) to itself. Since the coefficient of U^0 is a nonzero element of $\mathbb{K}$, this map is an isomorphism. By the long exact sequence, we can conclude that $\widetilde{HM}(S^3_0(K);\Gamma_{\eta_0}) = 0$. It follows now from Corollary 6.2 that K is the unknot. ∎

7. The case of nonintegral r

We now turn to the proof of Theorem 1.1 in the case where r is nonintegral. Some of the results proved along the way apply in more general settings, and will be used later.

Our strategy here is to show that if there is an orientation-preserving diffeomorphism $S^3_r(K) \cong S^3_r(U)$ for $r > 0$, then K is p-standard where p is the smallest integer greater than r. In this way, we reduce to the case of integral r, which is proved earlier in the paper. Of course, the case where $r < 0$ once again follows from the case where $r > 0$, by reflecting the knot.

LEMMA 7.1. *Let $W\colon Y_1 \to Y_2$ be a cobordism which contains a sphere with self-intersection number zero $S \subset W$ which represents a nontrivial homology class in $H_2(W, \partial W; \mathbb{Z})$. Then, for each* Spin^c *structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$, the induced map on Floer homologies are trivial.*

Proof. Since $S^1 \times S^2$ admits a metric of positive scalar curvature, it follows that if $\widecheck{HM}(W, \mathfrak{s})$ is nontrivial, then $\langle c_1(\mathfrak{s}), [S]\rangle = 0$.

We must now prove that even if $\langle c_1(\mathfrak{s}), [S]\rangle = 0$, then the induced map $\widecheck{HM}(W, \mathfrak{s})$ is trivial. To see this, we pass to the blow-up $W^\# = W \# \overline{\mathbb{CP}}^2$. Fix a two-sphere E in $W^\#$ supported in $\overline{\mathbb{CP}}^2$ with square -1. Given $\mathfrak{s} \in \mathrm{Spin}^c(W)$, there is a Spin^c structure $\widehat{\mathfrak{s}} \in \mathrm{Spin}^c(W^\#)$ which extends $\mathfrak{s}$ and with the additional property that $\langle c_1(\widehat{\mathfrak{s}}), [E]\rangle = +1$. By the blow-up formula, $\widecheck{HM}(W, \mathfrak{s}) = \widecheck{HM}(W^\#, \widehat{\mathfrak{s}}) = \widecheck{HM}(W^\#, \widehat{\mathfrak{s}} + PD(E))$. Since $[S] - [E]$ can also be represented by a sphere with self-intersection number -1, $\widecheck{HM}(W^\#, \widehat{\mathfrak{s}} + PD(E)) = \widecheck{HM}(W^\#, \widehat{\mathfrak{s}} + PD[S])$. By repeatedly using the same argument, we have that $\widecheck{HM}(W, \mathfrak{s}) = \widecheck{HM}(W^\#, \widehat{\mathfrak{s}} + kPD[S])$ for all $k \in \mathbb{Z}$. However, our hypotheses on S ensure that $\{\widehat{\mathfrak{s}} + kPD[S]\}_{k\in\mathbb{Z}}$ is an infinite collection of Spin^c structures; but for any fixed $\eta \in \widecheck{HM}(Y_1)$, there can be only finitely many Spin^c structures for which the map $\widecheck{HM}(W^\#, \mathfrak{t})(\eta)$ is nontrivial. It now follows that $\widecheck{HM}(W, \mathfrak{s}) = 0$. Since $b^+(W) > 0$, it follows from Proposition 2.8 that the map $\overline{HM}(W, \mathfrak{s}) = 0$ as well, and an easy diagram chase now also shows that $\widehat{HM}(W, \mathfrak{s}) = 0$. ∎

PROPOSITION 7.2. *Let M be a three-manifold with torus boundary, and choose oriented curves γ_1 and γ_2 with $\gamma_1 \cdot \gamma_2 = -1$. Fix also a cycle $\eta \subset M$. Letting γ_3 be a curve representing $[\gamma_1] + [\gamma_2]$ and γ_4 be a curve representing $[\gamma_1] - [\gamma_2]$. Consider the surgery long exact sequences*

$$\cdots \longrightarrow \widecheck{HM}_\bullet(Y_1;\Gamma_\eta) \xrightarrow{\check{F}_1} \widecheck{HM}_\bullet(Y_2;\Gamma_\eta) \xrightarrow{\check{F}_2} \widecheck{HM}_\bullet(Y_3;\Gamma_\eta) \longrightarrow \cdots$$
$$\cdots \longrightarrow \widecheck{HM}_\bullet(Y_2;\Gamma_\eta) \xrightarrow{\check{G}_2} \widecheck{HM}_\bullet(Y_1;\Gamma_\eta) \xrightarrow{\check{G}_1} \widecheck{HM}_\bullet(Y_4;\Gamma_\eta) \xrightarrow{\check{G}_4} \cdots$$

where here Y_i is obtained from M by filling γ_i and $\check{F}_i$ and $\check{G}_i$ are maps induced by the cobordisms equipped with the product cycles $[0,1] \times \eta$, thought of as supported in the complement of the two-handle additions, as in Theorem 5.12. *Then we have that $\check{F}_1 \circ \check{G}_2 = 0 = \check{G}_2 \circ \check{F}_1$. For composites of the maps belonging to the long exact sequences for the other two Floer homologies $\overline{HM}$ and $\widehat{HM}$, we have an analogous vanishing results* (*e.g.* $\widehat{F}_1 \circ \widehat{G}_2 = 0$).

Proof. Let $A_1 \colon Y_1 \longrightarrow Y_2$ denote the cobordism inducing the map $\check{F}_1$, and and $B_2 \colon Y_2 \longrightarrow Y_1$ denote the cobordism inducing the map $\check{G}_2$, so that the union $B_2 \cup_{Y_2} A_1 \colon Y_1 \longrightarrow Y_1$ is a cobordism (i.e. $\check{F}_1 = \widecheck{HM}(A_1; \Gamma_{A_1,\nu})$ and $\check{G}_2 = \widecheck{HM}(B_2; \Gamma_{B_2,\nu})$, where here the chains ν are induced by the product cycles $[0,1] \times \eta$, thought of as supported the complement of two-handle additions).

Inside the composite cobordism $W = B_2 \cup_{Y_2} A_1$, one can find a sphere with self-intersection number S equal to zero, which represents a nontrivial homology class in $H_2(W, \partial W; \mathbb{Z})$. Specifically, suppose that A_1 is built from Y_1 by attaching a two-handle along K_1 (with some framing) and B_2 is obtained by then attaching a two-handle along K_2 (with some other framing), then the two-sphere S corresponds to K_2, and it is homologically nontrivial since the homology class corresponding to K_1 intersects it once (cf. Figure 4).

It follows from the composition law for cobordisms, together with Lemma 7.1 that

$$0 = \widecheck{HM}(W; \Gamma_{W,\nu}) = \widecheck{HM}(B_2; \Gamma_{B_2,\nu}) \circ \widecheck{HM}(A_1; \Gamma_{A_1,\nu}) = \check{G}_2 \circ \check{F}_1.$$

The composite $\check{F}_1 \circ \check{G}_2$ vanishes in the same way. □

Proposition 7.3. *Let K be a knot in S^3, and fix a cycle $\eta \in S^3 - K$ whose homology class generates $H_1(S^3 - K; \mathbb{R})$. Let $r_0, r_1 \in \mathbb{Q} \cup \{\infty\}$ with r_0, r_1 nonnegative. Suppose moreover that if we write r_0 and r_1 as fractions in their lowest terms $r_i = p_i/q_i$ (where here all p_i, q_i are nonnegative integers), then $p_0 q_1 - p_1 q_0 = 1$. Then, we have a short exact sequence of the form*:

$$0 \longrightarrow \widecheck{HM}_\bullet(S^3_{r_1}(K); \Gamma_\eta) \longrightarrow \widecheck{HM}_\bullet(S^3_{r_2}(K); \Gamma_\eta) \longrightarrow \widecheck{HM}_\bullet(S^3_{r_0}(K); \Gamma_\eta) \longrightarrow 0,$$

where here $r_2 = (p_0 + p_1)/(q_0 + q_1)$.

Proof. We prove the result by induction on $q_2 = q_0 + q_1$.

In the case where $q_0 + q_1 = 1$, it follows that $q_0 = 0$ and $q_1 = p_0 = 1$. Now, Theorem 5.12 gives us an exact sequence

$$\begin{aligned} &\longrightarrow \widecheck{HM}_\bullet(S^3_{p_1+1}(K); \Gamma_\eta) \\ &\qquad \longrightarrow \widecheck{HM}_\bullet(S^3; \Gamma_\eta) \xrightarrow{\widecheck{HM}(W; \Gamma_{W,\nu})} \widecheck{HM}_\bullet(S^3_{p_1}(K); \Gamma_\eta) \longrightarrow \cdots . \end{aligned}$$

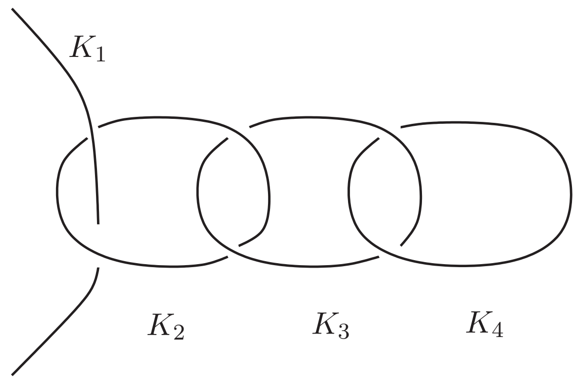

Figure 4: *Handle decomposition for Proposition* 7.2. Consider the link pictured above, where here K_1 is thought of as any initial framed knot in the three-manifold Y_1, and K_2, K_3, and K_4 are unknots with the property that K_i links K_{i-1} and K_{i+1} geometrically once (for $i = 2, 3$). The three cobordisms $A_1\colon Y_1 \to Y_2$, $A_2\colon Y_2 \to Y_3$, and $A_3\colon Y_3 \to Y_1$ which induce maps fitting into the long exact sequence are given as follows. A_1 is specified by the framed knot K_1, A_2 is specified by the framed knot K_2 (thought of as a knot in Y_2) with framing -1, while A_3 is specified by the framed knot K_3 with framing -1. Three cobordisms $B_2\colon Y_2 \to Y_1$, $B_1\colon Y_1 \to Y_4$, $B_4\colon Y_4 \to Y_2$ are specified as follows. B_2 is specified by K_2 (thought of as a knot inside Y_2) with framing 0, while B_1 is specified by K_3 with framing -1, and B_4 is specified by K_4 with framing -1. In particular, the cobordism $B_2 \cup_{Y_2} A_1\colon Y_1 \to Y_1$ is specified by the link $K_1 \cup K_2$, where here K_2 is given framing 0. In this composite cobordism, K_2 corresponds to a homologically nontrivial sphere with self-intersection number zero.

We claim that the map $\widetilde{HM}(W; \Gamma_\nu) \equiv 0$. This follows from commutativity of the diagram,

$$\begin{tikzcd}
\overline{HM}_\bullet(S^3; \Gamma_\eta) \arrow[r, "\overline{HM}_\bullet(W;\Gamma_{W,\eta})"] \arrow[d, "i_*"'] & \overline{HM}_\bullet(S^3_{p_1}(K); \Gamma_\eta) \arrow[d, "i'_*"'] \\
\widetilde{HM}_\bullet(S^3; \Gamma_\eta)) \arrow[r, "\widetilde{HM}_\bullet(W;\Gamma_{W,\eta})"] & \widetilde{HM}_\bullet(S^3_{p_1}(K); \Gamma_\eta)
\end{tikzcd}$$

bearing in mind that $i_*\colon \overline{HM}_\bullet(S^3; \Gamma_\eta) \longrightarrow \widetilde{HM}_\bullet(S^3; \Gamma_\eta)$ is surjective, together with the fact that $\overline{HM}_\bullet(W; \Gamma_{W,\eta}) \equiv 0$, which is analyzed in two cases. In the case where $p_1 > 0$, $\overline{HM}(W) \equiv 0$ since $b_2^+(W) = 1$, in view of Proposition 2.8; while in the case where $p_1 = 0$, it follows from the fact that $\overline{HM}_\bullet(S^3_0(K); \Gamma_\eta) = 0$, in view of Lemma 4.19. (Note that the present case of Lemma 4.19 follows at once from Proposition 2.7, together with the surgery long exact sequence on the level of $\overline{HM}$ with local coefficients.)

For the inductive step, Theorem 5.12 gives a long exact sequence

$$\longrightarrow \widecheck{HM}_\bullet(S^3_{r_2};\Gamma_\eta) \longrightarrow \widecheck{HM}_\bullet(S^3_{r_0};\Gamma_\eta) \xrightarrow{F} \widecheck{HM}_\bullet(S^3_{r_1};\Gamma_\eta) \longrightarrow \cdots .$$

Let $r_3 = (p_0 - p_1)/(q_0 - q_1)$. If $q_0 > q_1$, then by induction on the denominator, we have a short exact sequence

$$0 \longrightarrow \widecheck{HM}_\bullet(S^3_{r_1};\Gamma_\eta) \xrightarrow{G} \widecheck{HM}_\bullet(S^3_{r_0};\Gamma_\eta) \longrightarrow \widecheck{HM}_\bullet(S^3_{r_3};\Gamma_\eta) \longrightarrow 0.$$

By Proposition 7.2, it follows that $G \circ F = 0$. Since G is injective, it follows that F is the trivial map.

In the case where $q_0 < q_1$, by induction on the denominator, we have

$$0 \longrightarrow \widecheck{HM}_\bullet(S^3_{r_3}(K);\Gamma_\eta) \longrightarrow \widecheck{HM}_\bullet(S^3_{r_1}(K);\Gamma_\eta) \xrightarrow{G} \widecheck{HM}_\bullet(S^3_{r_0}(K);\Gamma_\eta) \longrightarrow 0.$$

Now, since $F \circ G = 0$ and G is surjective, it follows that $F \equiv 0$.

In the final case, where $q_0 = q_1$, it follows that $q_0 = q_1 = 1$ and that $p_0 = p_1 + 1$. Exactness now follows from the short exact sequence

$$0 \longrightarrow \widecheck{HM}_\bullet(S^3_{p_1}(K);\Gamma_\eta) \longrightarrow \widecheck{HM}_\bullet(S^3_{p_1+1}(K);\Gamma_\eta) \longrightarrow \widecheck{HM}_\bullet(S^3;\Gamma_\eta) \longrightarrow 0$$

which was established earlier. □

PROPOSITION 7.4. *Let K be a knot in S^3 and suppose that*

$$j\colon \widecheck{HM}_\bullet(S^3_r(K)) \longrightarrow \widehat{HM}_\bullet(S^3_r(K))$$

is trivial for some nonintegral, rational $r > 0$. Let p be the smallest integer greater than or equal to r, then

$$j\colon \widecheck{HM}_\bullet(S^3_p(K)) \longrightarrow \widehat{HM}_\bullet(S^3_p(K))$$

is trivial, as well.

Proof. Note that for $r > 0$, if η is any real cycle in the rational homology sphere $S^3_r(K)$, then the map

$$j_\eta\colon \widecheck{HM}_\bullet(S^3_r(K);\Gamma_\eta) \longrightarrow \widehat{HM}_\bullet(S^3_r(K);\Gamma_\eta)$$

is nontrivial if and only if the corresponding map

$$j\colon \widecheck{HM}_\bullet(S^3_r(K)) \longrightarrow \widehat{HM}_\bullet(S^3_r(K))$$

is; in fact since η is null-homologous, we have identifications

$$\widecheck{HM}_\bullet(S^3_r(K);\Gamma_\eta) \cong \widecheck{HM}_\bullet(S^3_r(K)) \otimes \mathbb{K}$$

and

$$\widehat{HM}_\bullet(S^3_r(K);\Gamma_\eta) \cong \widehat{HM}_\bullet(S^3_r(K)) \otimes \mathbb{K},$$

under which the map $j \otimes \mathrm{Id}_{\mathbb{K}}$ is identified with j_η.

Write $r = p/q$ in its lowest terms. Since p and q are relatively prime, we can find a pair of integers a and b with the property that $aq - bp = \pm 1$. Since $p > 0$ and $q > 1$, it follows that a and b must have the same sign, or $a = 0$. Without loss of generality, we can assume that a and b are both nonnegative. By simultaneously subtracting multiples of p off from a and multiples of q off from b, we can arrange for $0 \le a < p$ and $0 < b < q$. If $aq - bp = +1$, let $r_0 = a/b$, $r_1 = (p-a)/(q-b)$ and $r_2 = r = p/q$, while if $aq - bp = -1$, we let $r_0 = (p-a)/(q-b)$ and $r_1 = a/b$. In both cases, the short exact sequence from Proposition 7.3 holds, giving us the following diagram, where the rows and columns are exact

$$\begin{array}{ccccc}
\overline{HM}_\bullet(S^3_r) & \xrightarrow{\overline{F}} & \overline{HM}_\bullet(S^3_{r_0}) & \longrightarrow & 0 \\
{\scriptstyle i^r_*}\big\downarrow & & {\scriptstyle i^{r_0}_*}\big\downarrow & & \\
\widecheck{HM}_\bullet(S^3_r) & \xrightarrow{\check{F}} & \widecheck{HM}_\bullet(S^3_{r_0}) & \longrightarrow & 0 \\
\big\downarrow & & & & \\
0. & & & &
\end{array}$$

It follows at once that $i^{r_0}_*$ is surjective as well. Note that the denominator of r_0 is smaller than that of r, and there are no integers between r and r_0; hence by induction on this denominator, the result follows from the long exact sequence which connects p_*, i_*, and j_*. □

Let $K \subset S^3$ be a knot and $r > 0$ be a rational number. We can construct a map

$$\sigma_r \colon \mathrm{Spin}^c(S^3_r(K)) \longrightarrow \mathrm{Spin}^c(S^3_r(U))$$

as follows. Consider the Hirzebruch-Jung continued fractions expansion of r

$$r = [a_1, .., a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}, \tag{31}$$

where $a_1 \ge 1$ and $a_i \ge 2$ for $i > 1$. Consider the four-manifold whose Kirby calculus picture is given $K = K_1$ followed by a chain of unknots $K_2, \dots, K_n$, where K_i links K_{i-1} and K_{i+1} once; and the framing of K_i is $-a_i$. After deleting a ball, this gives a cobordism

$$W_r(K) \colon S^3_r(K) \longrightarrow S^3.$$

It is straightforward to see that any Spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(K))$ can be extended to a Spin^c structure over W. Let $\mathfrak{s}$ denote such an extension. Next, let $W_r(U)$ denote the corresponding cobordism for the unknot

$$W_r(U) \colon L(p, q) \longrightarrow S^3.$$

By the construction of these cobordisms, there is a distinguished identification

$$\tau\colon H^2(W_r(U);\mathbb{Z}) \xrightarrow{\cong} H^2(W_r(K);\mathbb{Z}).$$

Let, $\mathfrak{s}' \in \mathrm{Spin}^c(W_r(U))$ denote the Spin^c structure structure with $\tau(c_1(\mathfrak{s}')) = c_1(\mathfrak{s})$. It is straightforward to see that the correspondence which sends $\mathfrak{t}$ to the restriction of $\mathfrak{s}'$ to $S^3_r(U) \subset \partial W_r(U)$ induces a well-defined map σ_r as stated in the beginning of the paragraph.

Proposition 7.5. *Let K be a knot in S^3. Then, for all $\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(K))$, we have that*

$$\max_{\{\mathfrak{s}\in\mathrm{Spin}^c(W_r(U))\big|\mathfrak{s}|_{S^3_r(U)}=\sigma_r(\mathfrak{t})\}} c_1(\mathfrak{s})^2 + \mathrm{rk} H_2(W_r(U)) \leq -4Fr(\mathfrak{t}),$$

with equality when $K = U$.

Proof. Throughout this proof, we let $\alpha \in \overline{HM}_{\xi_-}(S^3)$ be the nonzero class supported in the summand corresponding to the two-plane field $\xi_- \in J(S^3)$ (recall that $h(\xi_-) = 0$). Note that $i_*(\alpha) \in \widecheck{HM}_\bullet(S^3)$ is nontrivial. Note that $W_r(K)\colon S^3_r(K) \longrightarrow S^3$ is a cobordism between rational homology three-spheres with $b_2^+(W_r(K)) = 0$. Since for each $\mathfrak{s} \in \mathrm{Spin}^c(W_r(K))$, the induced map $\overline{HM}(W_r(K),\mathfrak{s})$ is an isomorphism (cf. Proposition 2.7), we get a diagram

$$\begin{tikzcd}
\overline{HM}_*(S^3_r(K),\mathfrak{t}) \arrow[r, "\overline{HM}(W_r(K)\text{,}\mathfrak{s})", "\cong"'] \arrow[d, "i_*"'] & \overline{HM}_*(S^3) \arrow[d, "i_*"] \\
\widecheck{HM}_*(S^3_r(K),\mathfrak{t}) \arrow[r, "\widecheck{HM}(W_r(K)\text{,}\mathfrak{s})"] & \widecheck{HM}_*(S^3),
\end{tikzcd}$$

where here $\mathfrak{t} = \mathfrak{s}|_{S^3_r(K)}$. It follows that the map $\widecheck{HM}(W_r(K),\mathfrak{s}) \circ i_*$ is surjective, and hence there is a two-plane-field $j \in J(\mathfrak{t})$ and an element $\beta \in \overline{HM}_j(S^3_r(K))$ with the property that $\widecheck{HM}(W_r(K),\mathfrak{s})(i_*(\beta)) = i_*(\alpha)$.

Now, by the dimension formula

$$-4h(j) = c_1(\mathfrak{s})^2 - 2\chi(W_r(K)) - 3\sigma(W_r(K)) = c_1(\mathfrak{s})^2 + \mathrm{rk} H_2(W_r(K)).$$

But since $i_*(\beta) \in \widecheck{HM}_j(S^3_r(K))$ is a nonzero-homogeneous element in the image of i_*, it follows that $h(j) \geq Fr(\mathfrak{t})$. Putting these together, we have shown that

$$c_1(\mathfrak{s})^2 + \mathrm{rk} H_2(W_r(K)) \leq -4Fr(\mathfrak{t}), \tag{32}$$

for any $\mathfrak{s}$ which extends $\mathfrak{t}$ over $W_r(K)$. Note that the left-hand-side of this equation depends only on the homological properties of $W_r(K)$, and hence can be replaced by $W_r(U)$ as in the statement of the proposition.

It remains to show that for given $\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(U))$, there is a Spin^c structure for which equality holds in Equation (32).

To this end, we claim that there is a cobordism

$$V_r(U)\colon S^3 \longrightarrow S^3_r(U),$$

with the property that $X = W_r(U) \cup_{S^3_r(U)} V_r(U)\colon S^3 \longrightarrow S^3$ is a negative-definite four-manifold (indeed, it is obtained from the cylinder $[0,1] \times S^3$ by a sequence of blow-ups). Moreover, each Spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(U))$ can be extended to a Spin^c structure $\mathfrak{u} \in \mathrm{Spin}^c(X)$ with the property that $c_1(\mathfrak{u})^2 + \mathrm{rk} H_2(X) = 0$ (i.e. so that its square is maximal). Concretely, $V_r(U)$ is constructed from a plumbing of spheres with multiplicities $\{b_1, \dots, b_m\}$ chosen so that

$$1 = [a_1, \dots, a_n, 1, b_1, \dots, b_m].$$

The property of Spin^c structures with minimal square can be proved by induction on the size of the expansion.

Given this fact, note that the composite

$$i'_* \circ \overline{HM}(V_r(U), \mathfrak{u}|_{V_r(U)})\colon \overline{HM}_\bullet(S^3) \longrightarrow \widetilde{HM}_\bullet(S^3_r(U))$$

is once again surjective for all $\mathfrak{u} \in \mathrm{Spin}^c(X)$ (as it is the composite of an isomorphism with a surjection). It follows that if there were no extension of $\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(U))$ to $W_r(U)$ for which equality holds in Equation (32), then the composite map $\widetilde{HM}(X, \mathfrak{u}) = \widetilde{HM}(W_r(U), \mathfrak{u}|_{W_r(U)}) \circ \widetilde{HM}(V_r(U), \mathfrak{u}_{V_r(U)})$ would have kernel for any $\mathfrak{u} \in \mathrm{Spin}^c(U)$ with $\mathfrak{u}|_{S^3_r(U)} = \mathfrak{t}$. But for any choice of $\mathfrak{u} \in \mathrm{Spin}^c(X)$ with $c_1(\mathfrak{u}|_{W_r(U)})^2 + \mathrm{rk} H_2(W_r(U); \mathbb{Z}) = 0$, this map is an isomorphism. □

The following result reduces Theorem 1.1 in the case where r is nonintegral to the integral case:

THEOREM 7.6. *Suppose that there is an orientation-preserving diffeomorphism $S^3_r(K) \cong S^3_r(U)$ for some nonintegral $r > 0$. Then K is p-standard, where p is the smallest integer greater than r.*

Proof. Since $S^3_r(K)$ is orientation-preserving diffeomorphic to $S^3_r(U)$, it follows that

$$\sum_{\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(K))} Fr(S^3_r(K), \mathfrak{t}) = \sum_{\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(U))} Fr(S^3_r(U), \mathfrak{t}).$$

Since σ_r is a bijection, it follows from this equation together with Proposition 7.5 that in fact $Fr(S^3_r(K), \mathfrak{t}) = Fr(S^3_r(U), \sigma_r(\mathfrak{t}))$ for all $\mathfrak{t} \in \mathrm{Spin}^c(S^3_r(K))$. Since j is trivial on $\widetilde{HM}_\bullet(S^3_r(U)) = \widetilde{HM}_\bullet(S^3_r(K))$, it follows from Proposition 7.4 that j is trivial on $\widetilde{HM}_\bullet(S^3_p(K))$ as well.

We claim that $Fr(S^3_p(K), \mathfrak{t}) = Fr(S^3_p(U), \sigma_p(\mathfrak{t}))$ for all $\mathfrak{t} \in \mathrm{Spin}^c(S^3_p(U))$. To see this, note that by construction, we can decompose

$$W_r(K) = V \cup_{S^3_p(K)} W_p(K),$$

where $V\colon S^3_r(K) \longrightarrow S^3_p(K)$ is obtained from the $n-1$ two-handle additions (specified by $K_2, \dots, K_n$). We claim that any $\mathfrak{u} \in \mathrm{Spin}^c(S^3_p(K))$ admits an extension $\mathfrak{s}$ over all of $W_r(K)$, so that the induced map

$$\widetilde{HM}(W_r(K), \mathfrak{s})\colon \widetilde{HM}_\bullet(S^3_r(K), \mathfrak{s}|W_r(K)) \longrightarrow \widetilde{HM}_\bullet(S^3)$$

is an isomorphism. Now, corresponding to the decomposition, we can write

$$\widetilde{HM}(W_r(K), \mathfrak{s}) = \widetilde{HM}(W_p(K), \mathfrak{s}|_{W_p(K)}) \circ \widetilde{HM}(V, \mathfrak{s}|_V).$$

Since $b_2^+(V) = 0$, $\overline{HM}(V, \mathfrak{s}|_V)$ induces an isomorphism (cf. Proposition 2.7), and j_* is trivial for both $S^3_r(K)$ and $S^3_p(K)$, it follows easily that $\widetilde{HM}(V, \mathfrak{s}|_V)$ is surjective. It follows that

$$\widetilde{HM}(W_p, \mathfrak{s}|_{W_p(K)})\colon \widetilde{HM}_\bullet(S^3_p(K), \mathfrak{t}) \longrightarrow \widetilde{HM}_\bullet(S^3)$$

is an isomorphism, and hence that $Fr(S^3_p(K), \mathfrak{t}) = Fr(S^3_p(U), \sigma_p(\mathfrak{t}))$. From this, it follows readily that K is p-standard. □

8. Further applications for lens space surgeries

In this section, we use the surgery long exact sequence, along with some earlier results, to study the more general problem of lens space surgeries.

Recall the following result of Meng and Taubes [27] (reformulated in the context of monopole Floer homology):

THEOREM 8.1. *Let K be a knot in S^3, and write its symmetrized Alexander polynomial as*

$$\Delta_K(T) = a_0 + \sum_i a_i(T^i + T^{-i}).$$

We fix a generator $h \in H_2(S^3_0(K); \mathbb{Z})$, and let $\widetilde{HM}_\bullet(S^3_0(K), i)$ denote the Floer homology of $S^3_0(K)$ with local coefficients determined by any cycle ν which generates $H_1(S^3_0(K); \mathbb{R})$, evaluated in the Spin^c *stucture $\mathfrak{s}$ with $\langle c_1(\mathfrak{s}), [h]\rangle = 2i$. Then*

$$\chi_{\mathbb{K}}(\widetilde{HM}(S^3_0(K), i)) = -\sum_{j=1}^{\infty} j a_{|i|+j}.$$

(*Where here the left-hand-side is the Euler characteristic over the field $\mathbb{K}$ of fractions of the group-ring $\mathbb{F}[\mathbb{R}]$, and the sign is determined by the canonical mod two grading on Floer homology described in subsection* 2.5.)

Combining this with the results of this paper, we obtain the following necessary criterion for a $S^3_p(K)$ to be a lens space:

THEOREM 8.2. *Let K be a knot in S^3 with the property that some integer surgery on K gives a lens space, then the Seifert genus of K coincides with the degree of the symmetrized Alexander polynomial of K.*

Proof. If the genus of K is bigger than the degree of the Alexander polynomial, then the according to the Meng-Taubes theorem, we know that $\chi(\widetilde{HM}(S^3_0(K), g-1)) = 0$, while by the nonvanishing result, Corollary 6.2, $\widetilde{HM}(S^3_0(K), g-1) \neq 0$, and hence $\widetilde{HM}_{\text{odd}}(S^3_0(K)) \neq 0$. Thus, by Theorem 5.12, we have the exact sequence

$$\longrightarrow \widetilde{HM}_{\text{odd}}(S^3;\Gamma_\eta) \longrightarrow \widetilde{HM}_{\text{odd}}(S^3_0(K);\Gamma_\eta) \longrightarrow \widetilde{HM}_{\text{odd}}(S^3_1(K);\Gamma_\eta) \longrightarrow .$$

Since $\widetilde{HM}_{\text{odd}}(S^3) = 0$, it follows that $\widetilde{HM}_{\text{odd}}(S^3_1(K)) \neq 0$ (whether or not we use the local coefficient system Γ_η).

Indeed, if $\widetilde{HM}_{\text{odd}}(S^3_p(K)) \neq 0$, it follows easily that $\widetilde{HM}_{\text{odd}}(S^3_{p+1}(K)) \neq 0$, since in this case, the sequence of Theorem 2.4 takes the form

$$\longrightarrow \widetilde{HM}_\bullet(S^3) \longrightarrow \widetilde{HM}_\bullet(S^3_p(K)) \longrightarrow \widetilde{HM}_\bullet(S^3_{p+1}(K)) \longrightarrow,$$

where here F_p and F_{p+1} preserve the absolute $\mathbb{Z}/2\mathbb{Z}$ grading (cf. Section 2.5). In particular, this makes it impossible for $S^3_n(K)$ to be a lens space (for integral $n > 0$). $\square$

In [31], it is shown that if K is any knot with the property that $S^3_p(K) = L(p, q)$, then the Alexander polynomial of K is uniquely determined up to a finite indeterminacy. Indeed, this algorithm is concrete: its input is the Milnor torsion for $L(p, q)$ and $L(p, 1)$, and the finite indeterminacy depends on the homology class of the induced knot in $L(p, q)$.

Consider the rational numbers $d(-L(p, q), i)$ associated to a lens space $L(p, q)$ (recall that we have fixed here the orientation convention that $L(p, q) = S^3_{p/q}(U)$), and an element $i \in \mathbb{Z}/p\mathbb{Z}$, determined by the recursive formula

$$\begin{aligned} d(-L(1,1),0) &= 0 \\ d(-L(p,q),i) &= \left(\frac{pq-(2i+1-p-q)^2}{4pq}\right) - d(-L(q,r),j), \end{aligned}$$

where r and j are the reductions modulo q of p and i respectively. (Note that these numbers turn out to agree with the Frøyshov invariants of the lens space $-L(p, q)$, under a particular identification $\text{Spin}^c(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$.)

The following can be found in Corollary 7.5 of [31]:

THEOREM 8.3. *The lens space $L(p, q)$ is obtained as surgery on a knot $K \subset S^3$ only if there is a one-to-one correspondence*

$$\sigma\colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \text{Spin}^c(L(p, q))$$

with the following symmetries:

- $\sigma(-[i]) = \overline{\sigma([i])}$
- *there is an isomorphism $\phi\colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ with the property that*
$$\sigma([i]) - \sigma([j]) = \phi([i-j]),$$

with the following properties. For $i \in \mathbb{Z}$, let $[i]$ denote its reduction modulo p, and define

$$t_i = \begin{cases} -d(L(p,q),\sigma[i]) + d(L(p,1),[i]) & \text{if } 2|i| \leq p \\ 0 & \text{otherwise}, \end{cases}$$

then the Laurent polynomial

$$L_\sigma(T) = 1 + \sum_i \left(\frac{t_{i-1}}{2} - t_i + \frac{t_{i+1}}{2} \right) T^i = \sum_i a_i \cdot T^i$$

has integral coefficients, and all the $t_i \leq 0$. Indeed, if $S^3_p(K) \cong L(p,q)$, then its Alexander polynomial has the form $L_\sigma(T)$ for some choice of σ as above.

By combining Theorem 8.2, results of [31], and work of Goda and Teragaito [17], we obtain the following:

Corollary 8.4. *If K is a knot with the property that for some $p \in \mathbb{Z}$, $S^3_p(K)$ is a lens space and $|p| < 9$, then K is either the unknot or the trefoil.*

Proof. In view of Theorem 8.3, it is now an experiment in numerology to see that if $S^3_p(K)$ is a lens space with $|p| < 9$, then Alexander polynomial of K is either trivial or $T - 1 + T^{-1}$ (see the list at the end of Section 10 of [31]). In view of Theorem 8.2, it follows that the genus of K is zero or one. Combining this with a theorem of Goda and Teragaito [17], according to which the only genus one knot which admits lens space surgeries is the trefoil, the corollary is complete. □

As another application, we obtain the following bound on the Seifert genus g of K in terms of the order of the lens space.

Corollary 8.5. *Let K be a knot in S^3 with the property that for some integer p, $S^3_p(K)$ is a lens space, then $2g - 1 \leq p$.*

Proof. The bound $2d-1 \leq p$ where d is the degree of the Alexander polynomial of K follows immediately from the algorithm described in Theorem 8.3; the rest follows from Theorem 8.2. □

The bound on the Seifert genus stated above is still fairly coarse, and can usually be improved for fixed p and q using Theorem 8.2, combined with the algorithm for determining the Alexander polynomial of K given in Theorem 8.3.

It is interesting to compare Corollary 8.5 with a conjecture of Goda and Teragaito for hyperbolic knots which admit lens space surgeries. They conjecture that for such a knot, the order p of the fundamental group is related with the Seifert genus g by the inequalities $2g + 8 \leq p \leq 4g - 1$. Indeed they prove (Theorem 1.1 of [17]) that if K is a hyperbolic knot in S^3, and if $S^3_p(K)$ is a lens space, then $|p| \leq 12g - 7$. They restrict to the hyperbolic case, since the case of nonhyperbolic knots yielding lens space surgeries is completely understood, cf. [4], [42], [43]. The only such knots with lens space surgeries are torus knots, and the $(2, 2pq \pm 1)$-cable of a (p, q) torus knot (in which case the resulting lens space is $L(4pq \pm 1, 4q^2)$).

The condition that $t_i \leq 0$ from Theorem 8.3 has an improvement, using the Floer homology for knots (see [30] and [36]). Specifically, in Corollary 1.3 of [29], it is shown that if K is a knot on which some integral surgery is a lens space, then all the nonzero coefficients of its Alexander polynomial are ± 1, and they alternate in sign.

Combining all this information, we can give stronger constraints on the lens spaces which can be obtained by surgeries on knots with a fixed Seifert genus. As an illustration, we have the following:

COROLLARY 8.6. *The only lens spaces which can be obtained by positive integer surgery on a knot in S^3 with Seifert genus 2 are orientation-preserving diffeomorphic to $L(9, 7)$ and $L(11, 4)$.*

Proof. According to Theorem 8.2, combined with Corollary 1.3 of [29] (which states that the nonzero coefficients of the Alexander polynomial all have absolute value one and alternate in sign), we see that if K has genus two and some integral surgery on it gives a lens space, then Δ_K is either $T^{-2} - 1 + T^2$ or $T^{-2} - T^{-1} + 1 - T + T^2$. In the first case, K is neither a torus knot nor the $(2, 2pq \pm 1)$-cable of a (p, q) torus knot, and hence it must be hyperbolic (cf. [4], [42], [43]). According to Goda and Teragaito's bound, $|p| \leq 17$. But this is now ruled out by Corollary 7.5 of [31]. In the second case, we can rule out the possibility that K is hyperbolic and $p \neq 9, 11$ in the same manner. In these remaining cases, the algorithm of Theorem 8.3 forces $S^3_p(K)$ to be one of the two listed possibilities. □

The above procedure is purely algorithmic, and can be repeated for higher genera. For instance, if positive integral surgery on a genus three knot gives a lens space, then that lens space is contained in the list

$$L(11, 9), \qquad L(13, 10), \qquad L(13, 9), \qquad L(15, 4)$$

(all of which are realized by torus knots); in the genus four case, the list is

$$L(14, 11), \qquad L(16, 9), \qquad L(17, 13), \qquad L(19, 5)$$

(again, all of these are realized by torus knots). In the genus five case, the list reads

$$L(18, 13), \qquad L(19, 11), \qquad L(21, 16), \qquad L(23, 6),$$

where now the last two examples are realized by a torus knot, and the first two are realized by the $(-2, 3, 7)$ pretzel knot (cf. [10]).

In a different direction, we can combine Theorem 8.2 with properties of the Heegaard Floer homology for knots (see [30] or [36]) to obtain the following result on the four-ball genera of knots admitting lens space surgeries, compare also [24]:

COROLLARY 8.7. *Let K be a knot in S^3 with the property that $S^3_p(K) = L(p,q)$. Then, the Seifert genus, the four-ball genus, and the degree of the Alexander polynomial all coincide.*

Proof. According to [29] (cf. Corollary 1.6 in that reference), the four-ball genus of K is bounded below by the degree of the Alexander polynomial of K. The equality of the three quantities follows from the fact that the four-ball genus is less than or equal to the Seifert genus of K, together with Theorem 8.2. □

8.1. *Seifert fibered surgeries.* We give an application of the long exact sequence to the question of when a knot in S^3 admits a Seifert fibered surgery. To state the strongest form, it is useful to pin down orientations.

Let Y be a Seifert fibered space with $b_1(Y) = 0$ or 1. Such a manifold can be realized as the boundary of a four-manifold $W(\Gamma)$ obtained by plumbing two-spheres according to a weighted tree Γ. Here, the weights are thought of as a map m from the set of vertices of Γ to $\mathbb{Z}$.

Definition 8.8. Let Y be an oriented Seifert fibered three-manifold with $b_1(Y) = 0$ or 1. We say that Y has a positive Seifert fibered orientation if it can be presented as the oriented boundary of a plumbing of spheres $W(\Gamma)$ along a weighted tree Γ so that with $b^-(W(\Gamma)) = 0$. If Y does not have a positive Seifert fibered orientation, then $-Y$ does, and we say that Y is *negatively oriented.*

Moreover, either orientation on any lens space is a positive Seifert orientation; similarly, either orientation on a Seifert fibered space with $b_1(Y) = 1$ is a positive Seifert orientation. Finally, if Y is the quotient of a circle bundle $\pi\colon N \longrightarrow \Sigma$ over a Riemann surface by a finite group of orientation-preserving automorphisms G, and if N is oriented as a circle bundle with positive degree, then the induced orientation on Y is a positive Seifert orientation.

The basic property of the monopole Floer homology of Seifert fibered spaces we will use is the following result, which follows quickly from [28]. Or, alternatively, using Theorem 2.4, one can adapt the proof of the corresponding result for Heegaard Floer homology (cf. Corollary 1.4 of [32]), to the context of Seiberg-Witten monopoles.

THEOREM 8.9. *If Y is a positively oriented Seifert fibered rational homology three-sphere, then $\widecheck{HM}_\bullet(Y)$ is supported entirely in even degrees.*

Sometimes, we will consider Seifert fibered spaces with first Betti number equal to one. One could adapt techniques of [28] to this situation, as well, but it is quicker now to appeal to the surgery long exact sequence. The relevant fact in this case is the following:

COROLLARY 8.10. *If Y_0 is a Seifert fibered space with $b_1(Y_0) = 1$, and η is a generator of $H_1(Y_0;\mathbb{R})$, then $\widetilde{HM}_\bullet(Y_0;\Gamma_\eta)$ is supported entirely in odd degrees.*

Proof. Express Y_0 as the boundary of a plumbing W of spheres with $b^-(W) = 0$, and let Y_1 denote the new Seifert fibered space obtained by increasing the multiplicity of the central node by one, and let Y_2 denote the plumbing of spheres obtained by deleting the central node. (The latter space, Y_2 is a connected sum of lens spaces.) We have that the Floer homology groups Y_0, Y_1, and Y_2 fit into a long exact sequence as in Theorem 5.12. In fact, since Y_2 can be given a positive scalar curvature metric, $i_*\colon \overline{HM}(Y_2;\Gamma_\eta) \longrightarrow \widetilde{HM}_\bullet(Y_2;\Gamma_\eta)$ is surjective, and hence, since $\overline{HM}(Y_0;\Gamma_\eta) = 0$ (cf. Lemma 4.19), it follows that the map from $\widetilde{HM}_\bullet(Y_2;\Gamma_\eta) \longrightarrow \widetilde{HM}_\bullet(Y_0;\Gamma_\eta)$ is trivial. Thus, we get the short exact sequence

$$0 \longrightarrow \widetilde{HM}_\bullet(Y_0;\Gamma_\eta) \xrightarrow{\widetilde{HM}(W_0)} \widetilde{HM}_\bullet(Y_1;\Gamma_\eta) \xrightarrow{\widetilde{HM}(W_1)} \widetilde{HM}_\bullet(Y_2;\Gamma_\eta) \longrightarrow 0.$$

Since $\widetilde{HM}_\bullet(Y_1;\Gamma_\eta)$ is supported in even degrees and the map $\widetilde{HM}(W_0)$ reverses the canonical mod two grading (cf. Proposition 2.5), the result follows. □

Another application of the surgery long exact sequence gives the following:

PROPOSITION 8.11. *Let K be a knot in S^3. Then, for all $r = p/q > 0$, we have that*

$$\mathrm{rk}\widetilde{HM}_{\mathrm{odd}}(S^3_{p/q};\Gamma_\nu) = q \cdot \mathrm{rk}\widetilde{HM}_{\mathrm{even}}(S^3_0(K);\Gamma_\nu).$$

Proof. This follows immediately from Proposition 7.3. □

We obtain the following direct generalization of Theorem 8.2:

THEOREM 8.12. *Let K be a knot whose Alexander polynomial $\Delta_K(T)$ has degree strictly less than its Seifert genus. Then, there is no rational number $r \geq 0$ with the property that $S^3_r(K)$ is a positively oriented Seifert fibered space.*

Proof. In view of Theorem 8.1, the condition on K ensures that $\widetilde{HM}_{\mathrm{even}}(S^3_0(K);\Gamma_\eta) \neq 0$ (and also that $\widetilde{HM}_{\mathrm{odd}}(S^3_0(K);\Gamma_\eta) \neq 0$, but we do not use this here). The case where $r = 0$ now is ruled out by the latter fact, together with Corollary 8.10. For the case where $r > 0$, $\widetilde{HM}_{\mathrm{odd}}(S^3_r(K);\Gamma_\eta) \neq 0$ in view of Proposition 8.11, and hence Theorem 8.9 shows that it is never a positively oriented Seifert fibered space. □

It is a more subtle problem to detect whether $S^3_r(K)$ is a negative Seifert fibered space for $r \geq 0$. We include the following:

THEOREM 8.13. *If K is a knot whose Seifert genus g is strictly greater than the degree of its Alexander polynomial, and also $g > 1$, then $S^3_{1/n}(K)$ is not a Seifert fibered space for any integer n.*

Proof. As usual, by reflecting K if necessary, it suffices to consider the case where $n > 0$. $S^3_{1/n}(K)$ is not a positively oriented Seifert fibered space, according to Theorem 8.12. Thus, we are left with the case where $n > 0$ and Y is a negatively oriented Seifert fibered space.

We claim that if Y is a negatively oriented Seifert fibered space, then the cokernel of $i_\bullet \colon \overline{HM}_\bullet(Y) \longrightarrow \widetilde{HM}_\bullet(Y)$ is supported entirely in odd degrees. This follows easily from duality.

Now, in view of Corollary 2.3, our hypothesis on the knot K ensures that $\widetilde{HM}_{\mathrm{odd}}(S^3_0(K)) \neq 0$. Indeed, letting ξ be any nontrivial element of $\widetilde{HM}_{\mathrm{odd}}(S^3_0(K), g-1) \subset \widetilde{HM}_{\mathrm{odd}}(S^3_0(K))$, its conjugate $\overline{\xi}$ lies in the summand $\widetilde{HM}_{\mathrm{odd}}(S^3_0(K), -g+1)$, and hence it is linearly independent of ξ. Consider the surgery exact sequence, following Proposition 7.3

$$0 \longrightarrow \widetilde{HM}_\bullet(Y_0(K)) \xrightarrow{F} \widetilde{HM}_\bullet(Y_{1/n}(K)) \longrightarrow \widetilde{HM}_\bullet(Y_{\frac{1}{n-1}}) \longrightarrow 0,$$

where here F reverses the mod 2 degree. Since F is injective, if $\eta = F(\xi)$, then $\overline{\eta}$ is linearly independent from η. It follows that η is not in the image of $i_* \colon \overline{HM}(Y_{1/n}(K)) \longrightarrow \widetilde{HM}_\bullet(Y_{1/n}(K))$, since conjugation acts trivially on $\overline{HM}(Y_{1/n}(K))$. In view of the previous paragraph, $S^3_{1/n}(K)$ is not negatively Seifert fibered for $n > 0$. □

9. Foliations

The exact sequence, together with Theorem 2.1 can be used to exhibit large classes of three-manifolds admitting no (coorientable) taut foliations. (Recall that all foliations we consider in this paper are coorientable, so we drop this modifier from the statements of our results.)

Definition 9.1. A *monopole L-space* is a rational homology three-sphere Y for which

$$j_* \colon \widetilde{HM}_\bullet(Y) \longrightarrow \widehat{HM}_\bullet(Y)$$

is trivial.

Examples include all lens spaces, and indeed all three-manifolds with positive scalar curvature. By Theorem 2.1, a monopole L-space admits no taut foliations.

Proposition 9.2. *Let M be a connected, oriented three-manifold with torus boundary, equipped with three oriented, simple closed curves γ_0, γ_1, and γ_2 as in Theorem 2.4. Suppose moreover that Y_0, Y_1, and Y_2 are rational homology three-spheres, with the property that*

$$|H_1(Y_2;\mathbb{Z})| = |H_1(Y_0;\mathbb{Z})| + |H_1(Y_1;\mathbb{Z})|.$$

Suppose also that Y_0 and Y_1 are monopole L-spaces. Then it follows that Y_2 is a monopole L-space, too.

Proof. It follows from the hypotheses that the cobordism $W_0\colon Y_0 \longrightarrow Y_1$ has $b_2^+(W_0) = 1$, and hence according to Section 2.6 the map $\overline{HM}(W_0)\colon \overline{HM}(Y_0) \longrightarrow \overline{HM}(Y_1)$ is trivial. Now, Theorem 2.4, and our hypothesis, gives the following diagram (where all rows and columns are exact):

$$\begin{tikzcd}
 & {} \arrow[d, "0"] & {} \arrow[d, "j_*"] & {} \arrow[d, "0"] & \\
{} \arrow[r] & \widehat{HM}_\bullet(Y_1) \arrow[r, "\widehat{HM}(W_1)"] \arrow[d, "p_*"] & \widehat{HM}_\bullet(Y_2) \arrow[r, "\widehat{HM}(W)"] \arrow[d, "p_*"] & \widehat{HM}_\bullet(Y_0) \arrow[r] \arrow[d, "p_*"] & {} \\
{} \arrow[r, "0"] & \overline{HM}_\bullet(Y_1) \arrow[r, "\overline{HM}(W_1)"] \arrow[d, "i_*"] & \overline{HM}_\bullet(Y_2) \arrow[r, "\overline{HM}(W)"] \arrow[d, "i_*"] & \overline{HM}_\bullet(Y_0) \arrow[r, "0"] \arrow[d, "i_*"] & {} \\
{} \arrow[r] & \widetilde{HM}_\bullet(Y_1) \arrow[r, "\widetilde{HM}(W_1)"] \arrow[d, "0"] & \widetilde{HM}_\bullet(Y_2) \arrow[r, "\widetilde{HM}(W)"] \arrow[d, "j_*"] & \widetilde{HM}_\bullet(Y_0) \arrow[r] \arrow[d, "0"] & {} \\
 & {} & {} & {} &
\end{tikzcd}.$$

Now, a diagram-chase shows that $j_* = 0$ as well. □

Using the above proposition, together with Theorem 2.1, we can find large classes of three-manifolds which admit no taut foliations. We list several here:

Corollary 9.3. *Let L be a nonsplit, alternating link in S^3, and let $\Sigma(L)$ denote the branched double-cover of S^3 along L. Then, $\Sigma(L)$ does not admit a taut foliation.*

Proof. For any link diagram for L, choose a crossing, and let L_0 and L_1 denote the two resolutions of L at the crossing, as pictured in Figure 5. It is easy to see that $\Sigma(L)$, $\Sigma(L_0)$, and $\Sigma(L_1)$ fit into an exact triangle as in Theorem 2.4. Furthermore, if we start with a connected, reduced alternating projection for L, then both L_0 and L_1 are connected, alternating projections and hence $\Sigma(L)$, $\Sigma(L_0)$ and $\Sigma(L_1)$ are rational homology spheres. Indeed,

$$|H_1(\Sigma(L))| = |H_1(\Sigma(L_0))| + |H_1(\Sigma(L_1)|$$

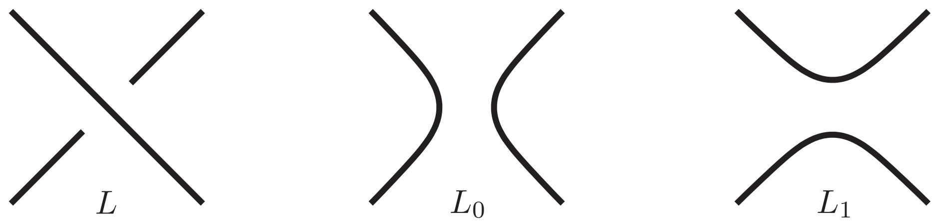

Figure 5: *Resolving link crossings.* Given a link with a crossing as labelled in L above, we have two resolutions L_0 and L_1, obtained by replacing the crossing by the two simplifications pictured above.

by classical knot theory (cf. [26]). The result now follows from Proposition 9.2, together with Theorem 2.1. (For more details, see the corresponding result in [33].) □

Definition 9.4. A *weighted graph* is a graph G equipped with an integer-valued function m on its vertices. The *degree* of a vertex v, written $d(v)$, is the number of edges which contain it.

As a special case of Corollary 9.3, we have the following:

COROLLARY 9.5. *Let G be a connected, weighted tree which satisfies the inequality $m(v) \geq d(v)$ at each vertex v, and for which the inequality is strict at at least one vertex. Let $Y(G)$ denote the three-manifold which is the boundary of the sphere-plumbing associated to the graph G. Then $Y(G,m)$ admits no taut foliations.*

Proof. We embed G as a planar graph, and attach one additional vertex v_0, which is connected to each of the vertices in v in G by $d(v)-m(v)$ edges. In this manner, we obtain a new graph G', which can be realized as the black graph for an alternating link (with connected projection). The branched double-cover of G' is easily seen to be $Y(G,m)$, and hence Corollary 9.3 applies. □

COROLLARY 9.6. *Let $K \subset S^3$ be a knot for which there is a positive rational number r with the property that $S^3_r(K)$ is a lens space or, more generally, a monopole L-space. Then, for all rational numbers $s \geq r$, $S^3_s(K)$ admits no taut foliation.*

Proof. By Proposition 7.4, if p denotes the smallest integer greater than r, then $S^3_p(K)$ is also a monopole L-space. The result now follows from repeated applications of Proposition 9.2. □

By a theorem of Thurston (cf. [40], [41]), if K is not a torus knot or a satellite knot, then $S^3_r(K)$ is hyperbolic for all but finitely many r. One can now use the above corollary to construct infinitely many hyperbolic three-manifolds with no taut foliation by, for example, starting with a hyperbolic knot which admits some lens space surgery. (A complete list of the known knots which admit lens space surgeries can be found in [1], see also [2], [16], [10].) The first examples of hyperbolic three-manifolds which admit no taut foliation were constructed in [39], see also [6]

Corollary 9.7. *Fix an odd integer $n \geq 7$, and let K be the $(-2,3,n)$ pretzel knot. For all $r \geq 2n+4$, then $S^3_r(K)$ admits no taut foliations.*

Proof. When $n = 7$, then $S^3_{18}(K)$ is a lens space. The result now follows from Corollary 9.6 (see also [22]). Indeed, for $n \geq 7$, it is well-known (cf. [3]), $S^3_{2n+4}(K)$ is a Seifert fibered space with Seifert invariants $(-2, 1/2, 1/4, (n-8)/(n-6))$. Repeated applications of Proposition 9.2 can be used to show that this Seifert fibered space is a monopole L-space, and hence again we can apply Corollary 9.6. □

These bounds can be sharpened. For example, when $n = 7$, $S^3_{17}(K)$ is a quotient of S^3 by a finite isometry group, and in particular, it has positive scalar curvature. Thus, for all $r \geq 17$, $S^3_r(K)$ admits no taut foliation. In fact, with some extra work, one can improve the bound in general to $r \geq n+2$. It is interesting to compare this with results of Roberts [37], [38], which constructs taut foliations on certain surgeries on fibered knots. For example, when K is the $(-2,3,7)$ pretzel knot, she shows that $S^3_r(K)$ admits a taut foliation for all $r < 1$.

In a similar vein, we have

Proposition 9.8. *For any three rational numbers $a, b, c \geq 1$, the three-manifold $M(a,b,c)$ obtained by performing a, b, and c surgery on the Borromean rings carries no taut foliations.*

Proof. First, we prove the case where a, b, and c are integers. In the basic case where $a = b = c$, the manifold $M(1,1,1)$ is the Poincaré homology sphere, which admits a metric of positive scalar curvature, and hence it is a monopole L-space. For the inductive step, we apply Proposition 9.2, to see that the fact that $L(b,1)\#L(c,1)$ and $M(a,b,c)$ are monopole L-spaces implies that so is $M(a+1,b,c)$. Repeated applications of the proposition also gives now the result for all rational numbers a, b, and c in the range. □

Note that this family includes the "Weeks manifold" $M(1, 5/2, 5)$, which is known by other methods not to admit any taut foliations; see [6].

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