Weyl’s law for the cuspidal spectrum of $\text{SL}_n$

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Abstract

Let $\Gamma$ be a principal congruence subgroup of $\text{SL}_n(\mathbb{Z})$ and let $\sigma$ be an irreducible unitary representation of $\text{SO}(n)$. Let $N^\Gamma_{\text{cus}}(\lambda, \sigma)$ be the counting function of the eigenvalues of the Casimir operator acting in the space of cusp forms for $\Gamma$ which transform under $\text{SO}(n)$ according to $\sigma$. In this paper we prove that the counting function $N^\Gamma_{\text{cus}}(\lambda, \sigma)$ satisfies Weyl’s law. Especially, this implies that there exist infinitely many cusp forms for the full modular group $\text{SL}_n(\mathbb{Z})$.

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References

Let $G$ be a connected reductive algebraic group over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. An important problem in the theory of automorphic forms is the question of the existence and the construction of cusp forms for $\Gamma$. By Langlands’ theory of Eisenstein series [La], cusp forms are the building blocks of the spectral resolution of the regular representation of $G(\mathbb{R})$ in $L^2(\Gamma \backslash G(\mathbb{R}))$. Cusp forms are also fundamental in number theory. Despite their importance, very little is known about the existence of cusp forms in general. In this paper we will address the question of existence of cusp forms for the group $G = \text{SL}_n$. The main purpose of this paper is to prove that cusp forms exist in abundance for congruence subgroups of $\text{SL}_n(\mathbb{Z}), n \geq 2$. 
To formulate our main result we need to introduce some notation. For simplicity assume that $G$ is semisimple. Let $K_{\infty}$ be a maximal compact subgroup of $G(\mathbb{R})$ and let $X = G(\mathbb{R})/K_{\infty}$ be the associated Riemannian symmetric space. Let $Z(g_{\mathbb{C}})$ be the center of the universal enveloping algebra of the complexification of the Lie algebra $g$ of $G(\mathbb{R})$. Recall that a cusp form for $\Gamma$ in the sense of [La] is a smooth and $K_{\infty}$-finite function $\phi : \Gamma \backslash G(\mathbb{R}) \rightarrow \mathbb{C}$ which is a simultaneous eigenfunction of $Z(g_{\mathbb{C}})$ and which satisfies

$$\int_{\Gamma \backslash N_P(\mathbb{R}) \backslash N_{P}(\mathbb{R})} \phi(nx) \, dn = 0,$$

for all unipotent radicals $N_{P}$ of proper rational parabolic subgroups $P$ of $G$. We note that each cusp form $f \in C^\infty(\Gamma \backslash G(\mathbb{R}))$ is rapidly decreasing on $\Gamma \backslash G(\mathbb{R})$ and hence square integrable. Let $L^2(\Gamma \backslash G(\mathbb{R}))$ be the closure of the linear span of all cusp forms. Let $(\sigma, V_{\sigma})$ be an irreducible unitary representation of $K_{\infty}$. Set

$$L^2(\Gamma \backslash G(\mathbb{R}), \sigma) = (L^2(\Gamma \backslash G(\mathbb{R})) \otimes V_{\sigma})^{K_{\infty}}$$

and define $L^2_{\text{cusp}}(\Gamma \backslash G(\mathbb{R}), \sigma)$ similarly. Then $L^2_{\text{cusp}}(\Gamma \backslash G(\mathbb{R}), \sigma)$ is the space of cusp forms with fixed $K_{\infty}$-type $\sigma$. Let $\Omega_{G(\mathbb{R})} \in Z(g_{\mathbb{C}})$ be the Casimir element of $G(\mathbb{R})$. Then $-\Omega_{G(\mathbb{R})} \otimes \text{Id}$ induces a selfadjoint operator $\Delta_{\sigma}$ in the Hilbert space $L^2(\Gamma \backslash G(\mathbb{R}), \sigma)$ which is bounded from below. If $\Gamma$ is torsion free, $L^2(\Gamma \backslash G(\mathbb{R}), \sigma)$ is isomorphic to the space $L^2(\Gamma \backslash X, E_{\sigma})$ of square integrable sections of the locally homogeneous vector bundle $E_{\sigma}$ associated to $\sigma$, and $\Delta_{\sigma} = (\nabla^\sigma)^* \nabla^\sigma - \lambda_{\sigma} \text{Id}$, where $\nabla^\sigma$ is the canonical invariant connection and $\lambda_{\sigma}$ the Casimir eigenvalue of $\sigma$. This shows that $\Delta_{\sigma}$ is a second order elliptic differential operator. Especially, if $\sigma_0$ is the trivial representation, then $L^2(\Gamma \backslash G(\mathbb{R}), \sigma_0) \cong L^2(\Gamma \backslash X)$ and $\Delta_{\sigma_0}$ equals the Laplacian $\Delta$ of $X$.

The restriction of $\Delta_{\sigma}$ to the subspace $L^2_{\text{cusp}}(\Gamma \backslash G(\mathbb{R}), \sigma)$ has pure point spectrum consisting of eigenvalues $\lambda_0(\sigma) < \lambda_1(\sigma) < \cdots$ of finite multiplicity. We call it the cuspidal spectrum of $\Delta_{\sigma}$. A convenient way of counting the number of cusp forms for $\Gamma$ is to use their Casimir eigenvalues. For this purpose we introduce the counting function $N^\Gamma_{\text{cusp}}(\lambda, \sigma), \lambda \geq 0$, for the cuspidal spectrum of type $\sigma$ which is defined as follows. Let $\mathcal{E}(\lambda_i(\sigma))$ be the eigenspace corresponding to the eigenvalue $\lambda_i(\sigma)$. Then

$$N^\Gamma_{\text{cusp}}(\lambda, \sigma) = \sum_{\lambda_i(\sigma) \leq \lambda} \dim \mathcal{E}(\lambda_i(\sigma)).$$

For nonuniform lattices $\Gamma$ the selfadjoint operator $\Delta_{\sigma}$ has a large continuous spectrum so that almost all of the eigenvalues of $\Delta_{\sigma}$ will be embedded in the continuous spectrum. This makes it very difficult to study the cuspidal spectrum of $\Delta_{\sigma}$.

The first results concerning the growth of the cuspidal spectrum are due to Selberg [Se]. Let $H$ be the upper half-plane and let $\Delta$ be the hyperbolic
Laplacian of $H$. Let $N^\Gamma_{\text{cus}}(\lambda)$ be the counting function of the cuspidal spectrum of $\Delta$. In this case the cuspidal eigenfunctions of $\Delta$ are called \textit{Maass cusp forms}. Using the trace formula, Selberg [Se, p. 668] proved that for every congruence subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$, the counting function satisfies Weyl’s law, i.e.

\[
N^\Gamma_{\text{cus}}(\lambda) \sim \frac{\text{vol}(\Gamma \backslash H)}{4\pi} \lambda
\]

as $\lambda \to \infty$. In particular this implies that for congruence subgroups of $\text{SL}_2(\mathbb{Z})$ there exist as many Maass cusp forms as one can expect. On the other hand, it is conjectured by Phillips and Sarnak [PS] that for a nonuniform lattice $\Gamma$ of $\text{SL}_2(\mathbb{R})$ whose Teichmüller space $T$ is nontrivial and different from the Teichmüller space corresponding to the once-punctured torus, a generic lattice $\Gamma \in T$ has only finitely many Maass cusp forms. This indicates that the existence of cusp forms is very subtle and may be related to the arithmetic nature of $\Gamma$.

Let $d = \dim X$. It has been conjectured in [Sa] that for $\text{rank}(X) > 1$ and $\Gamma$ an irreducible lattice

\[
\limsup_{\lambda \to \infty} \frac{N^\Gamma_{\text{dis}}(\lambda)}{\lambda^{d/2}} = \frac{\text{vol}(\Gamma \backslash X)}{(4\pi)^{d/2} \Gamma(d/2 + 1)}
\]

where $\Gamma(s)$ denotes the gamma function. A lattice $\Gamma$ for which (0.2) holds is called by Sarnak \textit{essentially cuspidal}. An analogous conjecture was made in [Mu3, p. 180] for the counting function $N^\Gamma_{\text{dis}}(\lambda, \sigma)$ of the discrete spectrum of any Casimir operator $\Delta_\sigma$. This conjecture states that for any arithmetic subgroup $\Gamma$ and any $K_\infty$-type $\sigma$

\[
\limsup_{\lambda \to \infty} \frac{N^\Gamma_{\text{dis}}(\lambda, \sigma)}{\lambda^{d/2}} = \dim(\sigma) \frac{\text{vol}(\Gamma \backslash X)}{(4\pi)^{d/2} \Gamma(d/2 + 1)}
\]

Up to now these conjectures have been verified only in a few cases. In addition to Selberg’s result, Weyl’s law (0.2) has been proved in the following cases: For congruence subgroups of $G = \text{SO}(n,1)$ by Reznikov [Rez], for congruence subgroups of $G = R_{F/Q} \text{SL}_2$, where $F$ is a totally real number field, by Efrat [Ef, p. 6], and for $\text{SL}_3(\mathbb{Z})$ by St. Miller [Mil].

In this paper we will prove that each principal congruence subgroup $\Gamma$ of $\text{SL}_n(\mathbb{Z})$, $n \geq 2$, is essentially cuspidal, i.e. Weyl’s law holds for $\Gamma$. Actually we prove the corresponding result for all $K_\infty$-types $\sigma$. Our main result is the following theorem.

\textbf{Theorem 0.1.} For $n \geq 2$ let $X_n = \text{SL}_n(\mathbb{R})/\text{SO}(n)$. Let $d_n = \dim X_n$.

For every principal congruence subgroup $\Gamma$ of $\text{SL}_n(\mathbb{Z})$ and every irreducible unitary representation $\sigma$ of $\text{SO}(n)$ such that $\sigma|_{\mathbb{Z}_n} = \text{Id}$,

\[
N^\Gamma_{\text{cus}}(\lambda, \sigma) \sim \dim(\sigma) \frac{\text{vol}(\Gamma \backslash X_n)}{(4\pi)^{d_n/2} \Gamma(d_n/2 + 1)} \lambda^{d_n/2}
\]

as $\lambda \to \infty$. 


The method that we use is similar to Selberg’s method [Se]. In particular, it does not give any estimation of the remainder term. For \( n = 2 \) a much better estimation of the remainder term exists. Using the full strength of the trace formula, we can get a three-term asymptotic expansion of \( N^\lambda_{\text{cus}} \) with remainder term of order \( O(\sqrt{\lambda} / \log \lambda) \) [He, Th. 2.28], [Ve, Th. 7.3]. The method is based on the study of the Selberg zeta function. It is quite conceivable that the Arthur trace formula can be used to obtain a good estimation of the remainder term for arbitrary \( n \).

Next we reformulate Theorem 0.1 in the ad`elic language. Let \( G = \text{GL}_n \), regarded as an algebraic group over \( \mathbb{Q} \). Let \( \mathbb{A} \) be the ring of adeles of \( \mathbb{Q} \). Denote by \( A_G \) the split component of the center of \( G \) and let \( A_G(\mathbb{R})^0 \) be the component of 1 in \( A_G(\mathbb{R}) \). Let \( \xi_0 \) be the trivial character of \( A_G(\mathbb{R})^0 \) and denote by \( \Pi(G(\mathbb{A}), \xi_0) \) the set of equivalence classes of irreducible unitary representations of \( G(\mathbb{A}) \) whose central character is trivial on \( A_G(\mathbb{R})^0 \). Let \( L^2_{\text{cus}}(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})) \) be the subspace of cusp forms in \( L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})) \). Denote by \( \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0) \) the subspace of all \( \pi \) in \( \Pi(G(\mathbb{A}), \xi_0) \) which are equivalent to a subrepresentation of the regular representation in \( L^2_{\text{cus}}(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})) \). By [Sk] the multiplicity of any \( \pi \in \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0) \) in the space of cusp forms \( L^2_{\text{cus}}(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})) \) is one. Let \( A_f \) be the ring of finite adeles. Any irreducible unitary representation \( \pi \) of \( G(\mathbb{A}) \) can be written as \( \pi = \pi_{\infty} \otimes \pi_f \), where \( \pi_{\infty} \) and \( \pi_f \) are irreducible unitary representations of \( G(\mathbb{R}) \) and \( G(A_f) \), respectively. Let \( \mathcal{H}_{\pi, \infty} \) and \( \mathcal{H}_{\pi, f} \) denote the Hilbert space of the representation \( \pi_{\infty} \) and \( \pi_f \), respectively. Let \( K_f \) be an open compact subgroup of \( G(A_f) \). Denote by \( \mathcal{H}_{\pi, f}^{K_f} \) the subspace of \( K_f \)-invariant vectors in \( \mathcal{H}_{\pi, f} \). Let \( G(\mathbb{R})^1 \) be the subgroup of all \( g \in G(\mathbb{R}) \) with \( |\det(g)| = 1 \). Given \( \pi \in \Pi(G(\mathbb{A}), \xi_0) \), denote by \( \lambda_{\pi_K} \) the Casimir eigenvalue of the restriction of \( \pi_{\infty} \) to \( G(\mathbb{R})^1 \). For \( \lambda \geq 0 \) let \( \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0, \lambda) \) be the space of all \( \pi \in \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0) \) which satisfy \( |\lambda_{\pi_K}| \leq \lambda \). Set \( \varepsilon_{K_f} = 1 \), if \(-1 \in K_f \) and \( \varepsilon_{K_f} = 0 \) otherwise. Then we have

**Theorem 0.2.** Let \( G = \text{GL}_n \) and let \( d_n = \dim \text{SL}_n(\mathbb{R})/\text{SO}(n) \). Let \( K_f \) be an open compact subgroup of \( G(A_f) \) and let \( (\sigma, V_\sigma) \) be an irreducible unitary representation of \( O(n) \) such that \( \sigma(-1) = \text{Id} \) if \(-1 \in K_f \). Then

\[
\sum_{\pi \in \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0, \lambda)} \dim(\mathcal{H}_{\pi, f}^{K_f}) \dim(\mathcal{H}_{\pi_{\infty}} \otimes V_\sigma)^{O(n)} \sim \dim(\sigma) \frac{\text{vol}(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})/K_f)}{(4\pi)^{d_n/2}(d_n/2 + 1)} (1 + \varepsilon_{K_f}) \lambda^{d_n/2}
\]

as \( \lambda \to \infty \).

Here we have used that the multiplicity of any \( \pi \in \Pi(G(\mathbb{A}), \xi_0) \) in the space of cusp forms is one.
The asymptotic formula (0.5) may be regarded as the ad"elic version of Weyl's law for $GL_n$. A similar result holds if we replace $\xi_0$ by any unitary character of $A_G(\mathbb{R})^0$. If we specialize Theorem 0.2 to the congruence subgroup $K(N)$ which defines $\Gamma(N)$, we obtain Theorem 0.1.

Theorem 0.2 will be derived from the Arthur trace formula combined with the heat equation method. The heat equation method is a very convenient way to derive Weyl's law for the counting function of the eigenvalues of the Laplacian on a compact Riemannian manifold [Cha]. It is based on the study of the asymptotic behaviour of the trace of the heat operator. Our approach is similar. We will use the Arthur trace formula to compute the trace of the heat operator on the discrete spectrum and to determine its asymptotic behaviour as $t \to 0$.

We will now describe our method in more detail. Let $G(\mathbb{A})^1$ be the subgroup of all $g \in G(\mathbb{A})$ satisfying $|\det(g)| = 1$. Then $G(\mathbb{Q})$ is contained in $G(\mathbb{A})^1$ and the noninvariant trace formula of Arthur [A1] is an identity

$$
\sum_{\chi \in \mathcal{X}} J_{\chi}(f) = \sum_{\phi \in \mathcal{D}} J_{\phi}(f), \quad f \in C^\infty_c(G(\mathbb{A})^1),
$$

between distributions on $G(\mathbb{A})^1$. The left-hand side is the spectral side $J_{\text{spec}}(f)$ and the right-hand side the geometric side $J_{\text{geo}}(f)$ of the trace formula. The distributions $J_{\chi}$ are defined in terms of truncated Eisenstein series. They are parametrized by the set of cuspidal data $\mathcal{X}$. The distributions $J_{\phi}$ are parametrized by semisimple conjugacy in $G(\mathbb{Q})$ and are closely related to weighted orbital integrals on $G(\mathbb{A})^1$.

For simplicity we consider only the case of the trivial $K_\infty$-type. We choose a certain family of test functions $\tilde{\phi}_t^1 \in C^\infty_c(G(\mathbb{A})^1)$, depending on $t > 0$, which at the infinite place are given by the heat kernel $h_t \in C^\infty(G(\mathbb{R})^1)$ of the Laplacian on $X$, multiplied by a certain cutoff function $\varphi_t$, and which at the finite places are given by the normalized characteristic function of an open compact subgroup $K_f$ of $G(\mathbb{A})_f$. Then we evaluate the spectral and the geometric side at $\tilde{\phi}_t^1$ and study their asymptotic behaviour as $t \to 0$. Let $\Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)$ be the set of irreducible unitary representations of $G(\mathbb{A})$ which occur discretely in the regular representation of $G(\mathbb{A})$ in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A}))$. Given $\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)$, let $m(\pi)$ denote the multiplicity with which $\pi$ occurs in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A}))$. Let $\mathcal{H}_{\pi_{\infty}}^K$ be the space of $K_\infty$-invariant vectors in $\mathcal{H}_{\pi_{\infty}}$. Comparing the asymptotic behaviour of the two sides of the trace formula, we obtain

$$
\sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\pi) e^{t\lambda_\pi} \dim(\mathcal{H}_{\pi_f}^K) \dim(\mathcal{H}_{\pi_{\infty}}^K) 
$$

$$
\sim \frac{\text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1/K_f)}{(4\pi)^{d_n/2}} (1 + \varepsilon_{K_f}) t^{-d_n/2}
$$
as \( t \to 0 \), where the notation is as in Theorem 0.2. Applying Karamata's theorem [Fe, p. 446], we obtain Weyl's law for the discrete spectrum with respect to the trivial \( K_{\infty} \)-type. A nontrivial \( K_{\infty} \)-type can be treated in the same way. The discrete spectrum is the union of the cuspidal and the residual spectra. It follows from [MW] combined with Donnelly's estimation of the cuspidal spectrum [Do], that the order of growth of the counting function of the residual spectrum for \( GL_n \) is at most \( O(\lambda^{(d_n - 1)/2}) \) as \( \lambda \to \infty \). This implies (0.5).

To study the asymptotic behaviour of the geometric side, we use the fine \( \sigma \)-expansion [A10]

\[
J_{\text{geo}}(f) = \sum_{M \in \mathcal{L}} \sum_{\gamma \in (M \langle Q_S \rangle)_{M.S}} a^M(S, \gamma)J_M(f, \gamma),
\]  

which expresses the distribution \( J_{\text{geo}}(f) \) in terms of weighted orbital integrals \( J_M(\gamma, f) \). Here \( M \) runs over the set of Levi subgroups \( \mathcal{L} \) containing the Levi component \( M_0 \) of the standard minimal parabolic subgroup \( P_0 \), \( S \) is a finite set of places of \( \mathbb{Q} \), and \( (M \langle Q_S \rangle)_{M.S} \) is a certain set of equivalence classes in \( M \langle Q_S \rangle \). This reduces our problem to the investigation of weighted orbital integrals. The key result is that

\[
\lim_{t \to 0} t^{d_n/2} J_M(\phi^1_t, \gamma) = 0,
\]

unless \( M = G \) and \( \gamma = \pm 1 \). The contributions to (0.8) of the terms where \( M = G \) and \( \gamma = \pm 1 \) are easy to determine. Using the behaviour of the heat kernel \( h_t(\pm 1) \) as \( t \to 0 \), it follows that

\[
J_{\text{geo}}(\phi^1_t) \sim \frac{\text{vol}(\mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A})^1/K_f)}{(4\pi)^{d/2}} (1 + \varepsilon K_f) t^{-d_n/2}
\]

as \( t \to 0 \).

To deal with the spectral side, we use the results of [MS]. Let \( C^1(\mathbb{G}(\mathbb{A})^1) \) denote the space of integrable rapidly decreasing functions on \( \mathbb{G}(\mathbb{A})^1 \) (see [Mu2, §1.3] for its definition). By Theorem 0.1 of [MS], the spectral side is absolutely convergent for all \( f \in C^1(\mathbb{G}(\mathbb{A})^1) \). Furthermore, it can be written as a finite linear combination

\[
J_{\text{spec}}(f) = \sum_{M \in \mathcal{L}} \sum_{\mathcal{L}(M)} \sum_{P \in \mathcal{P}(M)} \sum_{s \in W^L(a_M)_{\text{reg}}} a_{M,s}J_{M,P}^L(f, s)
\]

of distributions \( J_{M,P}^L(f, s) \), where \( \mathcal{L}(M) \) is the set of Levi subgroups containing \( M \), \( \mathcal{P}(M) \) denotes the set of parabolic subgroups with Levi component \( M \) and \( W^L(a_M)_{\text{reg}} \) is a certain set of Weyl group elements. Given \( M \in \mathcal{L} \), the main ingredients of the distribution \( J_{M,P}^L(f, s) \) are generalized logarithmic derivatives of the intertwining operators

\[
M_{Q,P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad P, Q \in \mathcal{P}(M), \quad \lambda \in a^*_{M,\mathbb{C}},
\]
acting between the spaces of automorphic forms attached to \( P \) and \( Q \), respectively. First of all, Theorem 0.1 of [MS] allows us to replace \( \tilde{\phi}_1^t \) by a similar function \( \phi_1^t \in C^1(G(\mathbb{A})^1) \) which is given as the product of the heat kernel at the infinite place and the normalized characteristic function of \( K_f \). Consider the distribution where \( M = L = G \). Then \( s = 1 \) and

\[
J_G^G(\phi_1^t) = \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\pi)e^{\lambda_\pi} \dim(H^K_{\pi_f}) \dim(H^K_{\pi_\infty}).
\]

This is exactly the left-hand side of (0.7). Thus in order to prove (0.7) we need to show that for all proper Levi subgroups \( M \), all \( L \in \mathcal{L}(M) \), \( P \in \mathcal{P}(M) \) and \( s \in W_L(a_M)_{\text{reg}} \),

\[
J_M^P(\phi_1^t, s) = O(t^{-(d_n-1)/2})
\]

as \( t \to 0 \). This is the key result where we really need that our group is \( \text{GL}_n \). It relies on estimations of the logarithmic derivatives of intertwining operators for \( \lambda \in i a_M^* \). Given \( \pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0) \), let \( M_Q|_P(\pi, \lambda) \) be the restriction of the intertwining operator \( M_Q|_P(\lambda) \) to the subspace \( A_\pi^2(P) \) of automorphic forms of type \( \pi \). The intertwining operators can be normalized by certain meromorphic functions \( r_Q|_P(\pi, \lambda) \) [A7]. Thus

\[
M_Q|_P(\pi, \lambda) = r_Q|_P(\pi, \lambda)^{-1}N_Q|_P(\pi, \lambda),
\]

where \( N_Q|_P(\pi, \lambda) \) are the normalized intertwining operators. Using Arthur’s theory of \((G,M)\)-families [A5], our problem can be reduced to the estimation of derivatives of \( N_Q|_P(\pi, \lambda) \) and \( r_Q|_P(\pi, \lambda) \) on \( i a_M^* \). The derivatives of \( N_Q|_P(\pi, \lambda) \) can be estimated using Proposition 0.2 of [MS]. Let \( M = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_r} \). Then \( \pi = \otimes_i \pi_i \) with \( \pi_i \in \Pi_{\text{dis}}(\text{GL}_{n_i}(\mathbb{A})^1) \) and the normalizing factors \( r_Q|_P(\pi, \lambda) \) are given in terms of the Rankin-Selberg \( L \)-functions \( L(s, \pi_i \times \pi_j) \) and the corresponding \( \epsilon \)-factors \( \epsilon(s, \pi_i \times \pi_j) \). So our problem is finally reduced to the estimation of the logarithmic derivative of Rankin-Selberg \( L \)-functions on the line \( \text{Re}(s) = 1 \). Using the available knowledge of the analytic properties of Rankin-Selberg \( L \)-functions together with standard methods of analytic number theory, we can derive the necessary estimates.

In the proof of Theorems 0.1 and 0.2 we have used the following key results which at present are only known for \( \text{GL}_n \): 1) The nontrivial bounds of the Langlands parameters of local components of cuspidal automorphic representations [LRS] which are needed in [MS]; 2) The description of the residual spectrum given in [MW]; 3) The theory of the Rankin-Selberg \( L \)-functions [JPS].

The paper is organized as follows. In Section 2 we prove some estimations for the heat kernel on a symmetric space. In Section 3 we establish some estimates for the growth of the discrete spectrum in general. We are essentially using Donnelly’s result [Do] combined with the description of the
residual spectrum [MW]. The main purpose of Section 4 is to prove estimates for the growth of the number of poles of Rankin-Selberg $L$-functions in the critical strip. We use these results in Section 5 to establish the key estimates for the logarithmic derivatives of normalizing factors. In Section 6 we study the asymptotic behaviour of the spectral side $J_{\text{spec}}(\phi^1)$. Finally, in Section 7 we study the asymptotic behaviour of the geometric side, compare it to the asymptotic behaviour of the spectral side and prove the main results.

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1. Preliminaries

1.1. Fix a positive integer $n$ and let $G$ be the group $\text{GL}_n$ considered as an algebraic group over $\mathbb{Q}$. By a parabolic subgroup of $G$ we will always mean a parabolic subgroup which is defined over $\mathbb{Q}$. Let $P_0$ be the subgroup of upper triangular matrices of $G$. The Levi subgroup $M_0$ of $P_0$ is the group of diagonal matrices in $G$. A parabolic subgroup $P$ of $G$ is called standard, if $P \supset P_0$. By a Levi subgroup we will mean a subgroup of $G$ which contains $M_0$ and is the Levi component of a parabolic subgroup of $G$ defined over $\mathbb{Q}$. If $M \subset L$ are Levi subgroups, we denote the set of Levi subgroups of $L$ which contain $M$ by $L(M)$. Furthermore, let $\mathcal{F}(M)$ denote the set of parabolic subgroups of $L$ defined over $\mathbb{Q}$ which contain $M$, and let $\mathcal{P}(M)$ be the set of groups in $\mathcal{F}(M)$ for which $M$ is a Levi component. If $L = G$, we shall denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$. Write $\mathcal{L} = \mathcal{L}(M_0)$. Suppose that $P \in \mathcal{F}(M)$. Then

$$P = N_PM_P,$$

where $N_P$ is the unipotent radical of $P$ and $M_P$ is the unique Levi component of $P$ which contains $M$.

Let $M \in \mathcal{L}$ and denote by $A_M$ the split component of the center of $M$. Then $A_M$ is defined over $\mathbb{Q}$. Let $X(M)_{\mathbb{Q}}$ be the group of characters of $M$ defined over $\mathbb{Q}$ and set

$$a_M = \text{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R}).$$

Then $a_M$ is a real vector space whose dimension equals that of $A_M$. Its dual space is

$$a^*_M = X(M)_{\mathbb{Q}} \otimes \mathbb{R}.$$

Let $P$ and $Q$ be groups in $\mathcal{F}(M_0)$ with $P \subset Q$. Then there are a canonical surjection $a_P \to a_Q$ and a canonical injection $a^*_Q \hookrightarrow a^*_P$. The kernel of the first map will be denoted by $a^*_P/Q$. Then the dual vector space of $a^*_P$ is $a^*_P/a^*_Q$. 
Let $P \in \mathcal{F}(M_0)$. We shall denote the roots of $(P, A_P)$ by $\Sigma_P$, and the simple roots by $\Delta_P$. Note that for $GL_n$ all roots are reduced. They are elements in $X(A_P)_\mathbb{Q}$ and are canonically embedded in $a^*_P$.

For any $M \in \mathcal{L}$ there exists a partition $(n_1, \ldots, n_r)$ of $n$ such that

$$M = GL_{n_1} \times \cdots \times GL_{n_r}.$$  

Then $a^*_M$ can be canonically identified with $(\mathbb{R}^r)^*$ and the Weyl group $W(a_M)$ coincides with the group $S_r$ of permutations of the set $\{1, \ldots, r\}$.

1.2. Let $F$ be a local field of characteristic zero. If $\pi$ is an admissible representation of $GL_m(F)$, we shall denote by $\tilde{\pi}$ the contragredient representation to $\pi$. Let $\pi_i, i = 1, \ldots, r$, be irreducible admissible representations of the group $GL_{n_i}(F)$. Then $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ is an irreducible admissible representation of

$$M(F) = GL_{n_1}(F) \times \cdots \times GL_{n_r}(F).$$

For $s \in \mathbb{C}^r$ let $\pi_i[s_i]$ be the representation of $GL_{n_i}(F)$ which is defined by

$$\pi_i[s_i](g) = |\det(g)|^s \pi_i(g), \quad g \in GL_{n_i}(F).$$

Let

$$I^G_F(\pi, s) = \text{Ind}^{G(F)}_{P(F)}(\pi_1[s_1] \otimes \cdots \otimes \pi_r[s_r])$$

be the induced representation and denote by $\mathcal{H}_P(\pi)$ the Hilbert space of the representation $I^G_F(\pi, s)$. We refer to $s$ as the continuous parameter of $I^G_F(\pi, s)$. Sometimes we will write $I^G_F(\pi_1[s_1], \ldots, \pi_r[s_r])$ in place of $I^G_F(\pi, s)$.

1.3. Let $\mathcal{G}$ be a locally compact topological group. Then we denote by $\Pi(\mathcal{G})$ the set of equivalence classes of irreducible unitary representations of $\mathcal{G}$.

1.4. Let $M \in \mathcal{L}$. Denote by $A_M(\mathbb{R})^0$ the component of 1 of $A_M(\mathbb{R})$. Set

$$M(\mathbb{A})^1 = \bigcap_{\chi \in X(M)_0} \ker([\chi]).$$

This is a closed subgroup of $M(\mathbb{A})$, and $M(\mathbb{A})$ is the direct product of $M(\mathbb{A})^1$ and $A_M(\mathbb{R})^0$.

Given a unitary character $\xi$ of $A_M(\mathbb{R})^0$, denote by $L^2(M(\mathbb{Q}) \backslash M(\mathbb{A}), \xi)$ the space of all measurable functions $\phi$ on $M(\mathbb{Q}) \backslash M(\mathbb{A})$ such that

$$\phi(xm) = \xi(x)\phi(m), \quad x \in A_M(\mathbb{R})^0, \ m \in M(\mathbb{A}),$$

and $\phi$ is square integrable on $M(\mathbb{Q}) \backslash M(\mathbb{A})^1$. Let $L^2_{\text{dis}}(M(\mathbb{Q}) \backslash M(\mathbb{A}), \xi)$ denote the discrete subspace of $L^2(M(\mathbb{Q}) \backslash M(\mathbb{A}), \xi)$ and let $L^2_{\text{cus}}(M(\mathbb{Q}) \backslash M(\mathbb{A}), \xi)$ be the subspace of cusp forms in $L^2(M(\mathbb{Q}) \backslash M(\mathbb{A}), \xi)$. The orthogonal complement of $L^2_{\text{cus}}(M(\mathbb{Q}) \backslash M(\mathbb{A}), \xi)$ in the discrete subspace is the residual subspace $L^2_{\text{res}}(M(\mathbb{Q}) \backslash M(\mathbb{A}), \xi)$. Denote by $\Pi_{\text{dis}}(M(\mathbb{A}), \xi)$, $\Pi_{\text{cus}}(M(\mathbb{A}), \xi)$, and
\( \Pi_{\text{res}}(M(\mathbb{A}), \xi) \) the subspace of all \( \pi \in \Pi(M(\mathbb{A}), \xi) \) which are equivalent to a subrepresentation of the regular representation of \( M(\mathbb{A}) \) in \( L^2(M(\mathbb{Q}) \setminus M(\mathbb{A}), \xi) \), \( L^2_{\text{cus}}(M(\mathbb{Q}) \setminus M(\mathbb{A}), \xi) \), and \( L^2_{\text{res}}(M(\mathbb{Q}) \setminus M(\mathbb{A}), \xi) \), respectively.

Let \( \Pi_{\text{dis}}(M(\mathbb{A})^1) \) be the subspace of all \( \pi \in \Pi(M(\mathbb{A})^1) \) which are equivalent to a subrepresentation of the regular representation of \( M(\mathbb{A})^1 \) in

\[
L^2(M(\mathbb{Q}) \setminus M(\mathbb{A})^1).
\]

We denote by \( \Pi_{\text{res}}(M(\mathbb{A})^1) \) (resp. \( \Pi_{\text{res}}(M(\mathbb{A})^1) \)) the subspace of all \( \pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1) \) occurring in the cuspidal (resp. residual) subspace

\[
L^2_{\text{cus}}(M(\mathbb{Q}) \setminus M(\mathbb{A})^1) \text{ (resp. } L^2_{\text{res}}(M(\mathbb{Q}) \setminus M(\mathbb{A})^1) \text{)).}
\]

1.5. Let \( P \) be a parabolic subgroup of \( G \). We denote by \( \mathcal{A}^2(P) \) the space of square integrable automorphic forms on \( N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0 \setminus G(\mathbb{A}) \) (see [Mu2, §1.7]).

Given \( \pi \in \Pi_{\text{dis}}(M_P(\mathbb{A}), \xi_\mathbb{A}) \), let \( \mathcal{A}^2_{\pi}(P) \) be the subspace of \( \mathcal{A}^2(P) \) of automorphic forms of type \( \pi \) [A1, p. 925]. Let \( \pi \in \Pi(M_P(\mathbb{A})^1) \). We identify \( \pi \) with a representation of \( M_P(\mathbb{A}) \) which is trivial on \( A_P(\mathbb{R})^0 \). Hence we can define \( \mathcal{A}^2_{\pi}(P) \) for any \( \pi \in \Pi(M_P(\mathbb{A})^1) \). It is a space of square integrable functions on \( N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0 \setminus G(\mathbb{A}) \) such that for every \( x \in G(\mathbb{A}) \), the function

\[
\phi_x(m) = \phi(mx), \quad m \in M_P(\mathbb{A}),
\]

belongs to the \( \pi \)-isotypical subspace of the regular representation of \( M_P(\mathbb{A}) \) in the Hilbert space \( L^2(A_P(\mathbb{R})^0 M_P(\mathbb{Q}) \setminus M_P(\mathbb{A})) \).

2. Heat kernel estimates

In this section we shall prove some estimates for the heat kernel of the Bochner-Laplace operator acting on sections of a homogeneous vector bundle over a symmetric space. Let \( G \) be a connected, semisimple, algebraic group defined over \( \mathbb{Q} \). Let \( K_\infty \) be a maximal compact subgroup of \( G(\mathbb{R}) \) and let \( (\sigma, V_\sigma) \) be an irreducible unitary representation of \( K_\infty \) on a complex vector space \( V_\sigma \). Let \( \tilde{E}_\sigma = (G(\mathbb{R}) \times V_\sigma)/K_\infty \) be the associated homogeneous vector bundle over \( X = G(\mathbb{R})/K_\infty \). We equip \( \tilde{E}_\sigma \) with the \( G(\mathbb{R}) \)-invariant Hermitian fibre metric which is induced by the inner product in \( V_\sigma \). Let \( C^\infty(\tilde{E}_\sigma), \ C^\infty_c(\tilde{E}_\sigma) \) and \( L^2(\tilde{E}_\sigma) \) denote the space of smooth sections, the space of compactly supported smooth sections and the Hilbert space of square integrable sections of \( \tilde{E}_\sigma \), respectively. Then we have

\[
(2.1) \quad C^\infty(\tilde{E}_\sigma) = (C^\infty(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}, \quad L^2(\tilde{E}_\sigma) = (L^2(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}
\]

and similarly for \( C^\infty_c(\tilde{E}_\sigma) \). Let \( \Omega \in Z(g_V) \) be the Casimir element of \( G(\mathbb{R}) \) and let \( R \) be the right regular representation of \( G(\mathbb{R}) \) on \( C^\infty(G(\mathbb{R})) \). Let \( \Delta_\sigma \) be
the second order elliptic operator which is induced by \(-R(\Omega) \otimes \text{Id}\) in \(C^\infty(\tilde{E}_\sigma)\).

Let \(\tilde{\nabla}^\sigma\) be the canonical connection on \(\tilde{E}_\sigma\), and let \(\Omega_K\) be the Casimir element of \(K_\infty\). Let \(\lambda_\sigma = \sigma(\Omega_K)\) be the Casimir eigenvalue of \(\sigma\). Then with respect to the identification (2.1),

\[
(\tilde{\nabla}^\sigma)^* \tilde{\nabla}^\sigma = -R(\Omega) \otimes \text{Id} + \lambda_\sigma \text{Id}
\]

[2]

(\[\text{Mia, Prop. 1.1}\]) and therefore

\[
\tilde{\Delta}_\sigma = (\tilde{\nabla}^\sigma)^* \tilde{\nabla}^\sigma - \lambda_\sigma \text{Id}.
\]

Hence \(\tilde{\Delta}_\sigma : C^\infty_c(\tilde{E}_\sigma) \to L^2(\tilde{E}_\sigma)\) is essentially selfadjoint and bounded from below. We continue to denote its unique selfadjoint extension by \(\tilde{\Delta}_\sigma\). Let \(\exp(-t\tilde{\Delta}_\sigma)\) be the associated heat semigroup. The heat operator is a smoothing operator on \(L^2(\tilde{E}_\sigma)\) which commutes with the representation of \(G(\mathbb{R})\) on \(L^2(\tilde{E}_\sigma)\). Therefore, it is of the form

\[
e^{-t\tilde{\Delta}_\sigma}(\varphi) = \int_{G(\mathbb{R})} H_\sigma^\prime(g^{-1}g_1)(\varphi(g_1)) dg_1, \quad g \in G(\mathbb{R}),
\]

where \(\varphi \in (L^2(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}\) and \(H_\sigma^\prime : G(\mathbb{R}) \to \text{End}(V_\sigma)\) is in \(L^2 \cap C^\infty\) and satisfies the covariance property

\[
H_\sigma^\prime(g) = \sigma(k)H_\sigma^\prime(k^{-1}gk')\sigma(k')^{-1}, \quad g \in G(\mathbb{R}), k, k' \in K_\infty.
\]

In order to get estimates for \(H_\sigma^\prime\), we proceed as in [BM] and relate \(H_\sigma^\prime\) to the heat kernel of the Laplace operator of \(G(\mathbb{R})\) with respect to a left invariant metric on \(G(\mathbb{R})\). Let \(\mathfrak{g}\) and \(\mathfrak{k}\) denote the Lie algebras of \(G(\mathbb{R})\) and \(K_\infty\), respectively. Let \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) be the Cartan decomposition and let \(\theta\) be the corresponding Cartan involution. Let \(B(Y_1, Y_2)\) be the Killing form of \(\mathfrak{g}\). Set \(\langle Y_1, Y_2 \rangle = -B(Y_1, \theta Y_2)\), \(Y_1, Y_2 \in \mathfrak{g}\). By translation of \(\langle \cdot, \cdot \rangle\) we get a left invariant Riemannian metric on \(G(\mathbb{R})\). Let \(X_1, \cdots, X_p\) be an orthonormal basis for \(\mathfrak{p}\) with respect to \(B|_{\mathfrak{p} \times \mathfrak{p}}\) and let \(Y_1, \cdots, Y_k\) be an orthonormal basis for \(\mathfrak{k}\) with respect to \(-B|_{\mathfrak{k} \times \mathfrak{k}}\). Then we have

\[
\Omega = \sum_{i=1}^p X_i^2 - \sum_{i=1}^k Y_i^2 \quad \text{and} \quad \Omega_K = -\sum_{i=1}^k Y_i^2.
\]

Let

\[
P = -\Omega + 2\Omega_K = -\sum_{i=1}^p X_i^2 - \sum_{i=1}^k Y_i^2.
\]

Then \(R(P)\) is the Laplace operator \(\Delta_G\) on \(G(\mathbb{R})\) with respect to the left invariant metric defined above. The heat semigroup \(e^{-t\Delta_G}\) is represented by a smooth kernel \(p_t\), i.e.

\[
(e^{-t\Delta_G}f)(g) = \int_{G(\mathbb{R})} p_t(g^{-1}g')f(g')dg', \quad f \in L^2(G(\mathbb{R})), g \in G(\mathbb{R}),
\]
where $p_t \in C^\infty(G(\mathbb{R})) \cap L^2(G(\mathbb{R}))$. In fact, $p_t$ belongs to $L^1(G(\mathbb{R}))$ (see [N]) so that (2.7) can be written as

$$e^{-t\Delta_G} = R(p_t).$$

Let

$$Q = \int_{K_\infty} R(k) \otimes \sigma(k) \, dk$$

be the orthogonal projection of $L^2(G(\mathbb{R})) \otimes V_\sigma$ onto its $K_\infty$-invariant subspace $(L^2(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}$. By (2.6) we have

$$\bar{\Delta}_\sigma = -Q(R(\Omega) \otimes \text{Id})Q = Q(R(P) \otimes \text{Id})Q - 2Q(R(\Omega_K) \otimes \text{Id})Q = Q(\Delta_G \otimes \text{Id})Q - 2\lambda_\sigma \text{Id}_{L^2(\tilde{E}_\sigma)}.$$

Hence, we get

$$e^{-t\bar{\Delta}_\sigma} = Q(e^{-t\Delta_G} \otimes \text{Id})Q \cdot e^{t\lambda_\sigma}$$

which implies that

$$H_\sigma^t(g) = e^{t\lambda_\sigma} \int_{K_\infty} \int_{K_\infty} p_t(k^{-1}gk')\sigma(kk'^{-1}) \, dk \, dk'.$$

Let $C^1(G(\mathbb{R}))$ be Harish-Chandra’s space of integrable, rapidly decreasing functions on $G(\mathbb{R})$. Then (2.8) can be used to show that

$$H_\sigma^t \in (C^1(G(\mathbb{R})) \otimes \text{End}(V_\sigma))^{K_\infty \times K_\infty} \quad \text{[BM, Prop. 2.4].}$$

Now we turn to the estimation of the derivatives of $H_\sigma^t$. By (2.8), this problem can be reduced to the estimation of the derivatives of $p_t$. Let $\nabla$ denote the Levi-Civita connection and $\rho(g, g')$ the geodesic distance of $g, g' \in G(\mathbb{R})$ with respect to the left invariant metric. Then all covariant derivatives of the curvature tensor are bounded and the injectivity radius has a positive lower bound. Let $a = \dim G(\mathbb{R})$, $t \in \mathbb{N}_0$ and $T > 0$. Then it follows from Corollary 8 in [CLY] that there exist $C, c > 0$ such that

$$\|\nabla^l p_t(g)\| \leq Ct^{-(a+l)/2} \exp\left(-\frac{cp^2(g,1)}{t}\right)$$

for all $0 < t \leq T$ and $g \in G(\mathbb{R})$. By (2.8) and (2.10),

$$\|\nabla^l H^\sigma_t(g)\| \leq e^{2t\lambda_\sigma} \int_{K_\infty} \int_{K_\infty} \|\nabla^l p_t(k^{-1}gk')\|dkdk'$$

$$\leq C t^{-(a+l)/2} \int_{K_\infty} \int_{K_\infty} \exp\left(-\frac{cp^2(gk,k')}{t}\right)dkdk'.$$
for all $0 < t \leq T$. Choose the invariant Riemannian metric on $X$ which is defined by the restriction of the Killing form to $T_eX \cong \mathfrak{p}$. Then the canonical projection map $G(\mathbb{R}) \to X$ is a Riemannian submersion. Let $d(x, y)$ denote the geodesic distance on $X$. Then it follows that

$$\rho(g, e) \geq d(gK_{\infty}, K_{\infty}), \quad g \in G(\mathbb{R}).$$

Set $r(g) = d(gK_{\infty}, K_{\infty})$, $g \in G(\mathbb{R})$. Together with (2.11) we get the following result.

**Proposition 2.1.** Let $a = \dim G(\mathbb{R})$, $l \in \mathbb{N}_0$ and $T > 0$. There exist $C, c > 0$ such that

$$\|\nabla^l H_t^\sigma(g)\| \leq Ct^{-\frac{a+l}{2}} \exp\left(-\frac{cr^2(g)}{t}\right)$$

for all $0 < t \leq T$ and $g \in G(\mathbb{R})$.

We note that the exponent of $t$ on the right-hand side of (2.12) is not optimal. Using the method of Donnelly [Do2], this estimate can be improved for $l \leq 1$. Indeed by Theorem 3.1 of [Mu1],

**Proposition 2.2.** Let $n = \dim X$ and $T > 0$. There exist $C, c > 0$ such that

$$\|\nabla^l H_t^\sigma(g)\| \leq Ct^{-\frac{n}{2} - l} \exp\left(-\frac{cr^2(g)}{t}\right)$$

for all $0 < t \leq T$, $0 \leq l \leq 1$, and $g \in G(\mathbb{R})$.

We also need the asymptotic behaviour of the heat kernel on the diagonal. It is described by the following lemma.

**Lemma 2.3.** Let $n = \dim X$ and let $e \in G(\mathbb{R})$ be the identity element. Then

$$\text{tr} H_t^\sigma(e) = \frac{\dim(\sigma)}{(4\pi)^{n/2}} t^{-n/2} + O(t^{-(n-1)/2})$$

as $t \to 0$.

**Proof.** Note that for each $x \in X$, the injectivity radius at $x$ is infinite. Hence we can construct a parametrix for the fundamental solution of the heat equation for $\Delta_\sigma$ as in [Do2]. Let $\epsilon > 0$ and set

$$U_\epsilon = \{(x, y) \in X \times X \mid d(x, y) < \epsilon\}.$$ 

For any $l \in \mathbb{N}$ we define an approximate fundamental solution $P_l(x, y, t)$ on $U_\epsilon$ by the formula

$$P_l(x, y, t) = (4\pi t)^{-n/2} \exp \left(\frac{-d^2(x, y)}{4t}\right) \left(\sum_{i=0}^{l} \Phi_i(x, y)t^i\right),$$

where $\Phi_i(x, y)$ are suitable functions.
where the $\Phi_t(x, y)$ are smooth sections of $E_\sigma \otimes E_\sigma^*$ over $U_\varepsilon \times U_\varepsilon$ which are constructed recursively as in Theorem 2.26 of [BGV]. In particular, we have

$$\Phi_0(x, x) = \text{Id}_{V_\varepsilon}, \quad x \in X.$$ 

Let $\psi \in C^\infty(X \times X)$ be equal to 1 on $U_{\varepsilon/4}$ and 0 on $X \times X - U_{\varepsilon/2}$. Set

$$Q_t(x, y, t) = \psi(x, y) P_t(x, y, t).$$ 

If $l > n/2$, then the section $Q_t$ of $E_\sigma \otimes E_\sigma^*$ is a parametrix for the heat equation. Since $X$ is a Riemannian symmetric space, we get

$$H_t^\sigma(e) = \text{Id}_{V_\varepsilon} (4\pi t)^{-n/2} + O(t^{-(n-1)/2})$$

as $t \to 0$. This implies the lemma. \hfill \Box

3. Estimations of the discrete spectrum

In this section we shall establish a number of facts concerning the growth of the discrete spectrum. Let $M = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_r}$, $r \geq 1$, and let

$$M(\mathbb{R})_1 = M(\mathbb{R}) \cap M(\mathbb{A})_1.$$ 

Then $M(\mathbb{R}) = M(\mathbb{R})_1 \cdot A_M(\mathbb{R})^0$. Let $K_{M, \infty} \subset M(\mathbb{R})$ be the standard maximal compact subgroup. Then $K_{M, \infty}$ is contained in $M(\mathbb{R})_1$. Let

$$X_M = M(\mathbb{R})_1 / K_{M, \infty}$$

be the associated Riemannian symmetric space. Let $\Gamma_M \subset M(\mathbb{Q})$ be an arithmetic subgroup and let $(\tau, V_\tau)$ be an irreducible unitary representation of $K_{M, \infty}$ on $V_\tau$. Set

$$C^\infty(\Gamma_M \backslash M(\mathbb{R})_1, \tau) := (C^\infty(\Gamma_M \backslash M(\mathbb{R})_1) \otimes V_\tau)^{K_{M, \infty}}.$$ 

If $\Gamma_M$ is torsion free, then $\Gamma_M \backslash X_M$ is a Riemannian manifold and the homogeneous vector bundle $E_\tau$ over $X_M$, which is associated to $\tau$, can be pushed down to a vector bundle $E_\tau^\ast \to \Gamma_M \backslash X_M$. Then $C^\infty(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$ equals $C^\infty(\Gamma_M \backslash X_M, E_\tau^\ast)$, the space of smooth sections of $E_\tau^\ast$. Define $C_c^\infty(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$ and $L^2(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$ similarly. Let $\Omega_M(\mathbb{R})_1$ be the Casimir element of $M(\mathbb{R})_1$ and let $\Delta_\tau$ be the operator in $C^\infty(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$ which is induced by $-\Omega_M(\mathbb{R})_1 \otimes \text{Id}$. As unbounded operator in $L^2(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$ with domain $C_c^\infty(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$, $\Delta_\tau$ is essentially selfadjoint. Let $L^2_{\text{cus}}(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$ be the subspace of cusp forms of $L^2(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$. Then $L^2_{\text{cus}}(\Gamma_M \backslash M(\mathbb{R})_1, \tau)$ is an invariant subspace of $\Delta_\tau$, and $\Delta_\tau$ has pure point spectrum in this subspace consisting of eigenvalues $\lambda_0 < \lambda_1 < \cdots$ of finite multiplicity. Let $\mathcal{E}(\lambda_i)$ be the eigenspace of $\lambda_i$. Set

$$N_{\text{cus}}(\lambda, \tau) = \sum_{\lambda_i \leq \lambda} \dim \mathcal{E}(\lambda_i).$$
Let $d = \dim X_M$ and let

$$C_d = \frac{1}{(4\pi)^{d/2} \Gamma(\frac{d}{2} + 1)}$$

be Weyl's constant, where $\Gamma(s)$ denotes the gamma function. Then Donnelly [Do, Th. 9] has established the following basic estimation of the counting function of the cuspidal spectrum.

**Theorem 3.1.** For every $\tau \in \Pi(K_{M,\infty})$,

$$\limsup_{\lambda \to \infty} \frac{N_{\text{cusp}}(\lambda, \tau)}{\lambda^{d/2}} \leq C_d \dim(\tau) \text{vol}(\Gamma_M \backslash X_M).$$

Actually, Donnelly proved this theorem only for the case of a torsion free discrete group. However, it is easy to extend his result to the general case.

We shall now reformulate this theorem in the representation theoretic context. Let $\xi_0$ be the trivial character of $A_M(\mathbb{R})^0$ and let $\pi \in \Pi(M(\mathbb{A}), \xi_0)$. Let $m(\pi)$ be the multiplicity with which $\pi$ occurs in the regular representation of $M(\mathbb{R})$ in $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$. Then $\Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)$ consists of all $\pi \in \Pi(M(\mathbb{A}), \xi_0)$ with $m(\pi) > 0$. Write

$$\pi = \pi_{\infty} \otimes \pi_f,$$

where $\pi_{\infty} \in \Pi(M(\mathbb{R}))$ and $\pi_f \in \Pi(M(\mathbb{A}_f))$. Denote by $\mathcal{H}_{\pi_{\infty}}$ (resp. $\mathcal{H}_{\pi_f}$) the Hilbert space of the representation $\pi_{\infty}$ (resp. $\pi_f$). Let $K_{M,f}$ be an open compact subgroup of $M(\mathbb{A}_f)$ and let $\tau \in \Pi(K_{M,\infty})$. Denote by $\mathcal{H}_{\pi_{\infty}}(\tau)$ the $\tau$-isotypical subspace of $\mathcal{H}_{\pi_{\infty}}$ and let $\mathcal{H}_{\pi_f}^{K_{M,f}}$ be the subspace of $K_{M,f}$-invariant vectors in $\mathcal{H}_{\pi_f}$. Denote by $\lambda_{\pi}$ the Casimir eigenvalue of the restriction of $\pi_{\infty}$ to $M(\mathbb{R})^0$. Given $\lambda > 0$, let

$$\Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)_\lambda = \{ \pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0) \mid |\lambda_{\pi}| \leq \lambda \}.$$ 

Define $\Pi_{\text{cus}}(M(\mathbb{A}), \xi_0)_\lambda$ and $\Pi_{\text{res}}(M(\mathbb{A}), \xi_0)_\lambda$ similarly.

**Lemma 3.2.** Let $d = \dim X_M$. For every open compact subgroup $K_{M,f}$ of $M(\mathbb{A}_f)$ and every $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that

$$\sum_{\pi \in \Pi_{\text{cus}}(M(\mathbb{A}), \xi_0)_\lambda} m(\pi) \dim(\mathcal{H}_{\pi_f}^{K_{M,f}}) \dim(\mathcal{H}_{\pi_{\infty}}(\tau)) \leq C(1 + \lambda^{d/2})$$

for $\lambda \geq 0$.

**Proof.** Extending the notation of §1.4, we write $\Pi(M(\mathbb{R}), \xi_0)$ for the set of representations in $\Pi(M(\mathbb{R}))$ whose central character is trivial on $A_M(\mathbb{R})^0$. Given $\pi_{\infty} \in \Pi(M(\mathbb{R}), \xi_0)$, let $m(\pi_{\infty})$ be the multiplicity with which $\pi_{\infty}$ occurs discretely in the regular representation of $M(\mathbb{R})$ in
Let $\Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)$ be the subset of all $\pi_\infty \in \Pi(M(\mathbb{R}), \xi_0)$ which are equivalent to an irreducible subrepresentation of the regular representation of $M(\mathbb{R})$ in the Hilbert space $L^2(M(\mathbb{R}) \backslash M(\mathbb{A}))^{K_{M,f}}$. Given $\pi_\infty \in \Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)$, denote by $\lambda_{\pi_\infty}$ the Casimir eigenvalue of the restriction of $\pi_\infty$ to $M(\mathbb{R})^1$. For $\lambda \geq 0$, let

$$\Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)_\lambda = \{ \pi_\infty \in \Pi_{\text{cus}}(M(\mathbb{R}), \xi_0) \mid |\lambda_{\pi_\infty}| \leq \lambda \}.$$  

Then by (3.1), it suffices to show that for each $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that

$$\sum_{\pi_\infty \in \Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)_\lambda} m(\pi_\infty) \dim(\mathcal{H}_{\pi_\infty}(\tau)) \leq C(1 + \lambda^{d/2}).$$

To deal with this problem recall that there exist arithmetic subgroups $\Gamma_{M,i} \subset M(\mathbb{R})$, $i = 1, \ldots, l$, such that

$$M(\mathbb{Q}) \backslash M(\mathbb{A})/K_{M,f} \cong \bigsqcup_{i=1}^l (\Gamma_{M,i} \backslash M(\mathbb{R}))$$

(cf. [Mu1, §9]). Hence

(3.2) $$L^2(M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))^{K_{M,f}} \cong \bigoplus_{i=1}^l L^2(M(\mathbb{R})^0 \Gamma_{M,i} \backslash M(\mathbb{R}))$$

as $M(\mathbb{R})$-modules. For each $i$, $i = 1, \ldots, l$, and $\pi_\infty \in \Pi(M(\mathbb{R}))$ let $m_{\Gamma_{M,i}}(\pi_\infty)$ be the multiplicity with which $\pi_\infty$ occurs discretely in the regular representation of $M(\mathbb{R})$ in $L^2(M(\mathbb{R})^0 \Gamma_{M,i} \backslash M(\mathbb{R}))$. Then $m(\pi_\infty) = \sum_{i=1}^l m_{\Gamma_{M,i}}(\pi_\infty)$ and

$$\sum_{\pi_\infty \in \Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)_\lambda} m(\pi_\infty) \dim(\mathcal{H}_{\pi_\infty}(\tau))$$

$$= \sum_{i=1}^l \sum_{\pi_\infty \in \Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)_\lambda} m_{\Gamma_{M,i}}(\pi_\infty) \dim(\mathcal{H}_{\pi_\infty}(\tau)).$$

The interior sum can be interpreted as follows. Fix $i$ and set $\Gamma_M := \Gamma_{M,i}$. Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of $\Delta_\tau$ in the space of cusp forms $L^2_{\text{cus}}(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ and let $\mathcal{E}(\lambda_i)$ be the eigenspace of $\lambda_i$. By Frobenius reciprocity it follows that

$$\dim \mathcal{E}(\lambda_i) = \sum_{-\lambda_\infty = \lambda_i} m_{\Gamma_M}(\pi_\infty),$$
where the sum is over all $\pi_\infty \in \Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)$ such that the Casimir eigenvalue $\lambda_{\pi_\infty}$ equals $-\lambda_i$. Hence we obtain
\[
\sum_{\pi_\infty \in \Pi_{\text{cus}}(M(\mathbb{R}), \xi_0)_\lambda} \ m_{\Gamma, M}(\pi_\infty) \dim(\mathcal{H}_{\pi_\infty}(\tau)) = \lambda_{\text{cus}}^M(\lambda, \tau).
\]
Combined with Theorem 3.1 the desired estimation follows.

Next we consider the residual spectrum.

**Lemma 3.3.** Let $d = \dim X_M$. For every open compact subgroup $K_{M,f}$ of $M(\mathbb{A}_f)$ and every $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that
\[
\sum_{\pi \in \Pi_{\text{res}}(M(\mathbb{A}), \xi_0)} m(\pi) \dim(\mathcal{H}_{\pi_\infty}^{K_{M,f}}) \dim(\mathcal{H}_{\pi_\infty}(\tau)) \leq C(1 + \lambda^{(d-1)/2})
\]
for $\lambda \geq 0$.

**Proof.** We can assume that $M = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_r}$. Let $K_{M,f}$ be an open compact subgroup of $M(\mathbb{A}_f)$. There exist open compact subgroups $K_{i,f}$ of $\text{GL}_{n_i}(\mathbb{A}_f)$ such that $K_{1,f} \times \cdots \times K_{r,f} \subset K_{M,f}$. Thus we can replace $K_{M,f}$ by $K_{1,f} \times \cdots \times K_{r,f}$. Next observe that $K_{M,\infty} = O(n_1) \times \cdots \times O(n_r)$ and therefore, $\tau$ is given as $\tau = \tau_1 \otimes \cdots \otimes \tau_r$, where each $\tau_i$ is an irreducible unitary representation of $O(n_i)$. Finally note that every $\pi \in \Pi(M(\mathbb{A}), \xi_0)$ is of the form $\pi = \pi_1 \otimes \cdots \otimes \pi_r$. Hence we get $m(\pi) = \prod_{i=1}^r m(\pi_i)$ and
\[
\dim(\mathcal{H}_{\pi_\infty}^{K_{M,f}}) = \prod_{i=1}^r \dim(\mathcal{H}_{\pi_{i,\infty}}^{K_{i,f}}), \quad \dim(\mathcal{H}_{\pi_\infty}(\tau)) = \prod_{i=1}^r \dim(\mathcal{H}_{\pi_{i,\infty}}(\tau_i)).
\]
This implies immediately that it suffices to consider a single factor.

With the analogous notation the proof of the proposition is reduced to the following problem. For $m \in \mathbb{N}$ set $X_m = \text{SL}_m(\mathbb{R})/\text{SO}(m)$ and $d_m = \dim X_m$. Then we need to show that for every open compact subgroup $K_{m,f}$ of $\text{GL}_m(\mathbb{A}_f)$ and every $\tau \in \Pi(\text{O}(m))$ there exists $C > 0$ such that
\[
\sum_{\pi \in \Pi_{\text{res}}(\text{GL}_m(\mathbb{A}), \xi_0)_\lambda} m(\pi) \dim(\mathcal{H}_{\pi_{\infty}}^{K_{m,f}}) \dim(\mathcal{H}_{\pi_{\infty}}(\tau)) \leq C(1 + \lambda^{(d_m-1)/2})
\]
for $\lambda \geq 0$. To deal with this problem recall the description of the residual spectrum of $\text{GL}_m$ by Mœglin and Waldspurger [MW]. Let $\pi \in \Pi_{\text{res}}(\text{GL}_m(\mathbb{A}))$ and suppose that $\pi$ is trivial on $A_{\text{GL}_m}(\mathbb{R})^0$. There exist $k|m$, a standard parabolic subgroup $P$ of $\text{GL}_m$ of type $(l, \ldots, l)$, $l = m/k$, and a cuspidal automorphic representation $\rho$ of $\text{GL}_l$ which is trivial on $A_{\text{GL}_l}(\mathbb{R})^0$, such that $\pi$ is equivalent to the unique irreducible quotient $J(\rho)$ of the induced representation
\[
I_{P(\mathbb{A})}^{\text{GL}_m(\mathbb{A})}(\rho)((k-1)/2] \otimes \cdots \otimes \rho[(1-k)/2]).
\]
Here \( \rho[s] \) denotes the representation \( g \mapsto \rho(g) \) \( \det g \), \( s \in \mathbb{C} \). At the Archimedean place, the corresponding induced representation

\[
I_P^{GL_m}(\rho_{\infty}, k) := I_P^{GL_m(\mathbb{R})}(\rho_{\infty}([k - 1]/2) \otimes \cdots \otimes \rho_{\infty}([1 - k]/2))
\]

has also a unique irreducible quotient \( J(\rho_{\infty}) \). Comparing the definitions, we get \( J(\rho_{\infty}) = J(\rho_{\infty}) \). Hence the Casimir eigenvalue of \( \pi_{\infty} = J(\rho_{\infty}) \) equals the Casimir eigenvalue of \( J(\rho_{\infty}) \) which in turn coincides with the Casimir eigenvalue of \( \rho_{\infty} \). Then it follows that there exists \( C > 0 \) such that \( |\lambda_{\pi} - k\lambda_{\rho}| \leq C \) for all \( \pi \in \Pi_{res}(GL_m(\mathbb{A}), \xi_0) \). Using the main theorem of [MW, p. 606], we see that it suffices to fix \( l|m, l < m \), and to estimate

\[
\sum_{\rho \in \Pi_{cus}(GL_l(\mathbb{A}), \xi_0)} m(\rho) \dim(\mathcal{H}_{J(\rho), i}) \dim(\mathcal{H}_{J(\rho), \infty}(\tau)).
\]

First note that by [Sk], we have \( m(\rho) = 1 \) for all \( \rho \in \Pi_{cus}(GL_l(\mathbb{A}), \xi_0) \). So it remains to estimate the dimensions. We begin with the infinite place. Observe that \( \dim(\mathcal{H}_{J(\rho), \infty}(\tau)) = \dim(\tau)[J(\rho_{\infty})]_{O(m)} : \tau] \). Thus in order to estimate \( \dim(\mathcal{H}_{J(\rho), \infty}(\tau)) \) it suffices to estimate the multiplicity \( [J(\rho_{\infty})]_{O(m)} : \tau] \). Since \( J(\rho_{\infty}) \) is an irreducible quotient of \( I_P^{GL_m(\rho_{\infty}, k)} \), we have

\[
[J(\rho_{\infty})]_{O(m)} : \tau] \leq [I_P^{GL_m(\rho_{\infty}, k)}]_{O(m)} : \tau].
\]

Let \( K_{i, \infty} = O(l) \times \cdots \times O(l) \). Using Frobenius reciprocity as in [Kn, p. 208], we obtain

\[
[I_P^{GL_m(\rho_{\infty}, k)}]_{O(m)} : \tau] = \sum_{\omega \in \Pi(K_{i, \infty})} [(\rho_{\infty} \otimes \cdots \otimes \rho_{\infty})_{K_{i, \infty}} : \omega] \cdot [\tau|_{K_{i, \infty}} : \omega].
\]

Finally note that \( \omega = \omega_1 \otimes \cdots \otimes \omega_k \) with \( \omega_i \in \Pi(O(l)) \). Therefore we have

\[
[(\rho_{\infty} \otimes \cdots \otimes \rho_{\infty})_{K_{i, \infty}} : \omega] = \prod_{i=1}^k [\rho_{\infty}|_{O(l)} : \omega_i].
\]

At the finite places we proceed in an analogous way. This implies that there exist open compact subgroups \( K_{i,f} \) of \( GL_i(\mathbb{A}_f) \), \( i = 1, \ldots, k \) and \( \omega_1, \ldots, \omega_k \in \Pi(O(l)) \) such that (3.3) is bounded from above by a constant times

\[
\prod_{i=1}^k \left( \sum_{\rho \in \Pi_{cus}(GL_{i}(\mathbb{A}), \xi_0)} m(\rho) \dim(\mathcal{H}_{\rho, i}^{K_{i,f}}) \dim(\mathcal{H}_{\rho, \infty}(\omega_i)) \right).
\]

By Lemma 3.2 this term is bounded by a constant times \((1 + \lambda d_i/2)^k\), where \( d_i = l(l + 1)/2 - 1 \). Since \( m = k \cdot l \) and \( k > 1 \), we have

\[
d_l k = \frac{l(l + 1)k}{2} - k \leq \frac{m(m + 1)}{2} - 2 = d_m - 1.
\]

This proves the desired estimation in the case of \( M = GL_m \), and as explained above, this suffices to prove the lemma. \( \square \)
Combining Lemma 3.2 and Lemma 3.3, we obtain

**Proposition 3.4.** Let $d = \dim X_M$. For every open compact subgroup $K_{M,f}$ of $M(\mathbb{A}_f)$ and every $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)_{\lambda}} m(\pi) \dim(\mathcal{H}_{\pi,f}^{K_{M,f}}) \cdot \dim(\mathcal{H}_{\pi,\infty}(\tau)) \leq C(1 + \lambda^{d/2})$$

for $\lambda \geq 0$.

Next we restate Proposition 3.4 in terms of dimensions of spaces of automorphic forms. Let $P \in \mathcal{P}(M)$ and let $A^2(P)$ be the space of square integrable automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})\mathcal{A}_P(\mathbb{R})^0 \backslash G(\mathbb{A})$. Given $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)$, let $A^2_\pi(P)$ be the subspace of $A^2(P)$ of automorphic forms of type $\pi$ [A1, p. 925]. Let $K_{\infty}$ be the standard maximal compact subgroup of $G(\mathbb{R})$. Given an open compact subgroup $K_f$ of $G(\mathbb{A}_f)$ and $\sigma \in \Pi(K_{\infty})$, let $A_{\pi}(P)_{K_f}$ denote the subspace of $K_f$-invariant automorphic forms in $A^2_\pi(P)$ and let $A^2_\pi(P)_{K_f,\sigma}$ be the $\sigma$-isotypical subspace of $A^2_\pi(P)_{K_f}$.

**Proposition 3.5.** Let $d = \dim X_M$. For every open compact subgroup $K_f$ of $G(\mathbb{A}_f)$ and every $\sigma \in \Pi(K_{\infty})$ there exists $C > 0$ such that

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)_{\lambda}} \dim A^2_\pi(P)_{K_f,\sigma} \leq C(1 + \lambda^{d/2})$$

for $\lambda \geq 0$.

**Proof.** Let $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)$. Let $\mathcal{H}_P(\pi)$ be the Hilbert space of the induced representation $I_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi)$. There is a canonical isomorphism

$$j_P : \mathcal{H}_P(\pi) \otimes \text{Hom}_{M(\mathbb{A})}(\pi, I_{M(\mathbb{Q})A_P(\mathbb{R})^0}(\xi_0)) \rightarrow A^2_\pi(P),$$

which intertwines the induced representations. Let $\pi = \pi_{\infty} \otimes \pi_f$. Let $\mathcal{H}_P(\pi_{\infty})$ (resp. $\mathcal{H}_P(\pi_f)$) be the Hilbert space of the induced representation $I_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_{\infty})$ (resp. $I_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f)$). Denote by $\mathcal{H}_P(\pi_{\infty})_{\sigma}$ the $\sigma$-isotypical subspace of $\mathcal{H}_P(\pi_{\infty})$ and by $\mathcal{H}_P(\pi_f)_{K_f}$ the subspace of $K_f$-invariant vectors of $\mathcal{H}_P(\pi_f)$. Then it follows from (3.4) that

$$\dim A^2_\pi(P)_{K_f,\sigma} = m(\pi) \dim(\mathcal{H}_P(\pi_f)_{K_f}) \dim(\mathcal{H}_P(\pi_{\infty})_{\sigma}).$$

Using Frobenius reciprocity as in [Kn, p. 208] we get

$$[I_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_{\infty})_{|K_{\infty}} : \sigma] = \sum_{\tau \in \Pi(K_{M,\infty})} [\pi_{\infty}|_{K_{M,\infty}} : \tau] \cdot [\sigma|_{K_{M,\infty}} : \tau].$$

Hence we get

$$\dim(\mathcal{H}_P(\pi_{\infty})_{\sigma}) \leq \dim(\sigma) \sum_{\tau \in \Pi(K_{M,\infty})} \dim(\mathcal{H}_{\pi,\infty}(\tau)) [\sigma|_{K_{M,\infty}} : \tau].$$
Next we consider $\pi_f = \otimes_{p<\infty} \pi_p$. Replacing $K_f$ by a subgroup of finite index if necessary, we can assume that $K_f = \Pi_{p<\infty} K_p$. For any $p < \infty$, denote by $\mathcal{H}_P(\pi_p)$ the Hilbert space of the induced representation $I_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\pi_p)$. Let $\mathcal{H}_P(\pi_p)^{K_p}$ be the subspace of $K_p$-invariant vectors. Then $\dim \mathcal{H}_P(\pi_p)^{K_p} = 1$ for almost all $p$ and

$$\mathcal{H}_P(\pi_f)^{K_f} \cong \bigotimes_{p<\infty} \mathcal{H}_P(\pi_p)^{K_p}.$$  

Furthermore,

$$I_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\pi_p)^{K_p} = \left(I_{P(\mathbb{Z}_p)}^{G(\mathbb{Z}_p)}(\pi_p)^{K_p}\right)_{K_p \cap P(\mathbb{Z}_p)^{K_p}} \cong \bigoplus_{G(\mathbb{Z}_p)/K_p} \pi_p^{K_p \cap P(\mathbb{Z}_p)^{K_p}}.$$  

(3.7)

Let $K_{M,f} = K_f \cap M(\mathbb{A}_f)$. Using (3.5)–(3.7), it follows that in order to prove the proposition, it suffices to fix $\tau \in \Pi(K_{M,\infty})$ and to estimate

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}),\xi_0)} m(\pi) \dim(\mathcal{H}^{K_{M,f}}_{\pi_f}) \dim(\mathcal{H}_{\pi_\infty}(\tau)).$$

The proof is now completed by application of Proposition 3.4.

Finally we consider the analogous statement of Lemma 3.3 at the Archimedean place. For simplicity we consider only the case $M = G$. Let $K_{\infty}$ be the standard maximal compact subgroup of $G(\mathbb{R})$. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup and $\sigma \in \Pi(K_{\infty})$. Then the discrete subspace

$$L^2_{\text{dis}}(\Gamma \backslash G(\mathbb{R})^1, \sigma)$$

of $\Delta_{\sigma}$ decomposes as

$$L^2_{\text{dis}}(\Gamma \backslash G(\mathbb{R})^1, \sigma) = L^2_{\text{cus}}(\Gamma \backslash G(\mathbb{R})^1, \sigma) \oplus L^2_{\text{res}}(\Gamma \backslash G(\mathbb{R}), \sigma),$$

where $L^2_{\text{res}}(\Gamma \backslash G(\mathbb{R})^1, \sigma)$ is the subspace which corresponds to the residual spectrum of $\Delta_{\sigma}$. Let

$$L^2_{\text{res}}(\Gamma \backslash G(\mathbb{R})^1, \sigma) = \bigoplus_i \mathcal{E}_{\text{res}}(\lambda_i)$$

be the decomposition into eigenspaces of $\Delta_{\sigma}$. For $\lambda \geq 0$ set

$$N^T_{\text{res}}(\lambda, \sigma) = \sum_{\lambda_i \leq \lambda} \dim \mathcal{E}_{\text{res}}(\lambda_i).$$

**Proposition 3.6.** Let $d = G(\mathbb{R})^1/K_{\infty}$. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. For every $\sigma \in \Pi(K_{\infty})$ there exists $C > 0$ such that

$$N_{\text{res}}^T(\lambda, \sigma) \leq C(1 + \lambda^{(d-1)/2})$$

for $\lambda \geq 0$. 


Proof. First assume that $\Gamma \subset \text{SL}_n(\mathbb{Z})$. Let $\Gamma(N) \subset \Gamma$ be a congruence subgroup. Then

$$N_{\text{res}}^\Gamma(\lambda, \sigma) \leq N_{\text{res}}^{\Gamma(N)}(\lambda, \sigma).$$

Let

$$N = \prod_p p^{r_p}, \quad r_p \geq 0.$$ 

Set

$$K_p(N) = \{ k \in \text{GL}_n(\mathbb{Z}_p) \mid k \equiv 1 \mod p^{r_p} \mathbb{Z}_p \}$$

and

$$K(N) = \prod_{p<\infty} K_p(N).$$

Then $K(N)$ is an open compact subgroup of $G(\mathbb{A}_f)$ and

$$A_G(\mathbb{R})^0 G(\mathbb{Q}) \backslash G(\mathbb{A}) / K(N) \cong \bigcup_{(\mathbb{Z}/N\mathbb{Z})^*} (\Gamma(N) \backslash \text{SL}_n(\mathbb{R}))$$

(cf. [A9]). Hence

$$L^2_{\text{res}}(A_G(\mathbb{R})^0 G(\mathbb{Q}) \backslash G(\mathbb{A}))^{K(N)} \cong \bigoplus_{(\mathbb{Z}/N\mathbb{Z})^*} L^2_{\text{res}}(\Gamma(N) \backslash \text{SL}_n(\mathbb{R}))$$

as $\text{SL}_n(\mathbb{R})$-modules. Then

$$\sum_{\pi \in \Pi_{\text{res}}(G(\mathbb{A}), \xi_0)} m(\pi) \dim(\mathcal{H}_{\pi_\lambda}^{K(N)}) \dim(\mathcal{H}_{\pi_\infty}(\sigma)) = \varphi(N) N_{\text{res}}^{\Gamma(N)}(\lambda, \sigma),$$

where $\varphi(N) = \#[(\mathbb{Z}/N\mathbb{Z})^*]$. Put $M = G$ in Lemma 3.3. Then by Lemma 3.3 it follows that there exists $C > 0$ such that

$$N_{\text{res}}^{\Gamma(N)}(\lambda, \sigma) \leq C(1 + \lambda^{(d-1)/2}).$$

This proves the proposition for $\Gamma \subset \text{SL}_n(\mathbb{Z})$. Since an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ is commensurable with $G(\mathbb{Z})$, the general case can be easily reduced to this one. 

4. Rankin-Selberg $L$-functions

The main purpose of this section is to prove estimates for the number of zeros of Rankin-Selberg $L$-functions. We shall consider the Rankin-Selberg $L$-functions over an arbitrary number field, although in the present paper we shall use them only in the case of $\mathbb{Q}$. We begin with the description of the local $L$-factors.

Let $F$ be a local field of characteristic zero. Recall that any irreducible admissible representation of $\text{GL}_m(F)$ is given as a Langlands quotient: There
exist a standard parabolic subgroup \( P \) of type \((m_1, \ldots, m_r)\), discrete series representations \( \delta_i \) of \( \text{GL}_{m_i}(F) \) and complex numbers \( s_1, \ldots, s_r \) satisfying \( \text{Re}(s_1) \geq \text{Re}(s_2) \geq \cdots \geq \text{Re}(s_r) \) such that
\[
\pi = J_P^{\text{GL}_m}(\delta_1[s_1] \otimes \cdots \otimes \delta_r[s_r]),
\]
where the representation on the right is the unique irreducible quotient of the induced representation \( I_P^{\text{GL}_m}(\delta_1[s_1] \otimes \cdots \otimes \delta_r[s_r]) \) [MW, I.2]. Furthermore any irreducible generic representation \( \pi \) of \( \text{GL}_m(F) \) is equivalent to a fully induced representation \( I_{GL_m}(\delta_1[s_1] \otimes \cdots \otimes \delta_r[s_r]) \). If \( \pi \) is generic and unitary, it follows from the classification of the unitary dual of \( \text{GL}_m(F) \) that the parameters \( s_i \) satisfy
\[
|\text{Re}(s_i)| < 1/2, \quad i = 1, \ldots, r.
\]
Suppose that \( \pi \) is given as a Langlands quotient of the form (4.1). Then the \( L \)-function satisfies
\[
L(s, \pi) = \prod_j L(s + s_j, \delta_j)
\]
[J]. Furthermore, suppose that \( \pi_1 \) and \( \pi_2 \) are irreducible admissible representations of \( G_1 = \text{GL}_{m_1} (\mathbb{R}) \) and \( G_2 = \text{GL}_{m_2} (\mathbb{R}) \), respectively. Let
\[
\pi_i \cong J_{P_i}^{\text{GL}_{n_i}}(\tau_{i1}[s_{i1}], \ldots, \tau_{ir}[s_{ir}])
\]
be the Langlands parametrizations of \( \pi_i \), \( i = 1, 2 \). Then it follows from the multiplicativity of the local Rankin-Selberg \( L \)-factors [JPS, (9.4)], [Sh6] that
\[
L(s, \pi_1 \times \pi_2) = \prod_{i=1}^{r_1} \prod_{j=1}^{r_2} L(s + s_{i1} + s_{2j}, \tau_{1i} \times \tau_{2j}).
\]
This reduces the description of the local \( L \)-factors to the square-integrable case. Now we distinguish three cases according to the type of the field.

1. \( F \) non-Archimedean. Let \( \mathcal{O}_F \) denote the ring of integers of \( F \) and \( \mathfrak{P} \) the maximal ideal of \( \mathcal{O}_F \). Set \( q = \mathcal{O}_F/\mathfrak{P} \). The square-integrable case can be further reduced to the supercuspidal one. Finally for supercuspidal representations the \( L \)-factor is given by an elementary polynomial in \( q^{-s} \). For details see [JPS] (see also [MS]). If we put together all steps of the reduction, we get the following result. Let \( \pi_1 \) and \( \pi_2 \) be irreducible admissible representations of \( \text{GL}_{n_1}(F) \) and \( \text{GL}_{n_2}(F) \), respectively. Then there is a polynomial \( P_{\pi_1, \pi_2}(x) \) of degree at most \( n_1 \cdot n_2 \) with \( P_{\pi_1, \pi_2}(0) = 1 \) such that
\[
L(s, \pi_1 \times \pi_2) = P_{\pi_1, \pi_2}(q^{-s})^{-1}.
\]
In the special case where \( \pi_1 \) and \( \pi_2 \) are unitary and generic the \( L \)-factor has the following special form.
**Lemma 4.1.** Let $\pi_1$ and $\pi_2$ be irreducible unitary generic representations of $\text{GL}_{n_1}(F)$ and $\text{GL}_{n_2}(F)$, respectively. There exist complex numbers $a_i$, $i = 1, \ldots, n_1 \cdot n_2$, with $|a_i| < q$ such that

$$
L(s, \pi_1 \times \pi_2) = \prod_{i=1}^{n_1 \cdot n_2} (1 - a_i q^{-s})^{-1}.
$$

**Proof.** Let $\delta_1$ and $\delta_2$ be square-integrable representations of $\text{GL}_{d_1}(F)$ and $\text{GL}_{d_2}(F)$, respectively. As explained above there is a polynomial $P_{\delta_1, \delta_2}(x)$ of degree at most $d_1 \cdot d_2$ with $P_{\delta_1, \delta_2}(0) = 1$ such that

$$
L(s, \delta_1 \times \delta_2) = P_{\delta_1, \delta_2}(q^{-s})^{-1}.
$$

By (6) of [JPS, p. 445], $L(s, \delta_1 \times \delta_2)$ is holomorphic in the half-plane $\Re(s) > 0$. Hence $P_{\delta_1, \delta_2}(x)$ has no zeros in the unit disc. Thus there exist complex numbers $b_i$ with $|b_i| < 1$ such that

$$
L(s, \delta_1 \times \delta_2) = \prod_{i=1}^{d_1 \cdot d_2} (1 - b_i q^{-s})^{-1}.
$$

Now let $\pi_1$ and $\pi_2$ be unitary and generic. Then $L(s, \pi_1 \times \pi_2)$ can be written as a product of the form (4.4) and by (4.2) the parameters $s_{ij}$ satisfy $|\Re(s_{ij})| < 1/2$, $i = 1, 2$, $j = 1, \ldots, r_i$. With this and (4.6), the lemma follows. \hfill $\square$

If $F$ is Archimedean the $L$-factors are defined in terms of the $L$-factors attached to semisimple representations of the Weyl group $W_F$ by means of the Langlands correspondence [La1]. The structure of the $L$-factors are described, for example, in [MS, §3]. We briefly recall the result.

2. $F = \mathbb{R}$. First note that $\text{GL}_m(\mathbb{R})$ does not have square-integrable representations if $m \geq 3$. To describe the principal $L$-factors in the remaining cases $d = 1$ and $d = 2$, we define gamma factors by

$$
\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right), \quad \Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s).
$$

In the case $d = 1$, the unitary representations of $\text{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ are of the form $\psi_{\epsilon,t}(x) = \text{sign}^\epsilon(x)|x|^t$ with $\epsilon \in \{0, 1\}$ and $t \in i\mathbb{R}$. Then

$$
L(s, \psi_{\epsilon,t}) = \Gamma_\mathbb{R}(s + t + \epsilon).
$$

For $k \in \mathbb{Z}$ let $D_k$ be the $k$-th discrete series representation of $\text{GL}_2(\mathbb{R})$ with the same infinitesimal character as the $k$-dimensional representation. Then the unitary square-integrable representations of $\text{GL}_2(\mathbb{R})$ are unitary twists of $D_k$, $k \in \mathbb{Z}$, for which the $L$-factor is given by

$$
L(s, D_k) = \Gamma_\mathbb{C}(s + |k|/2).
$$
Let $\psi_\epsilon = \text{sign}^\epsilon$, $\epsilon \in \{0,1\}$. Then up to twists by unramified characters the following list describes the Rankin-Selberg $L$-factors in the square-integrable case:

$$L(s, D_{k_1} \times D_{k_2}) = \Gamma_C(s + |k_1 - k_2|/2) \cdot \Gamma_C(s + |k_1 + k_2|/2),$$

$$L(s, D_k \times \psi_\epsilon) = L(s, \psi_\epsilon \times D_k) = \Gamma_C(s + |k|/2),$$

$$L(s, \psi_{\epsilon_1} \times \psi_{\epsilon_2}) = \Gamma_R((s + \epsilon_{1,2})),$$

where $0 \leq \epsilon_{1,2} \leq 1$ with $\epsilon_{1,2} \equiv \epsilon_1 + \epsilon_2$ mod 2.

3. $F = \mathbb{C}$. There exist square-integrable representations of $GL_k(\mathbb{C})$ only if $k = 1$. For $r \in \mathbb{Z}$ let $\chi_r$ be the character of $GL_1(\mathbb{C}) = \mathbb{C}^\times$ which is given by $\chi(z) = (z/\overline{z})^r$, $z \in \mathbb{C}^\times$. Then

$$L(s, \chi_r) = \Gamma_C(s + |r|/2).$$

If $\chi_{r_1}$ and $\chi_{r_2}$ are two characters as above, then we have

$$L(s, \chi_{r_1} \times \chi_{r_2}) = \Gamma_C(s + |r_1 + r_2|/2).$$

Up to twists by unramified characters, these are all possibilities for the $L$-factors in the square-integrable case.

To summarize we obtain the following description of the local $L$-factors in the complex case. Let $\pi$ be an irreducible unitary representation of $GL_m(\mathbb{C})$. It is given by a Langlands quotient of the form

$$\pi = J_{B}^{GL_m}(\chi_1[s_1] \otimes \cdots \otimes \chi_m[s_m]),$$

where $B$ is the standard Borel subgroup of $GL_m$ and the $\chi_i$'s are characters of $GL_1(\mathbb{C}) = \mathbb{C}^\times$ which are defined by $\chi(z) = (z/\overline{z})^{r_i}$, $r_i \in \mathbb{Z}$, $i = 1, \ldots, m$. Then

$$L(s, \pi) = \prod_{i=1}^{m} \Gamma_C(s + s_i + |r_i|/2).$$

Let $\pi_1$ and $\pi_2$ be irreducible unitary representations of $GL_{m_1}(\mathbb{C})$ and $GL_{m_2}(\mathbb{C})$, respectively. Let $B_i \subset GL_{m_i}$ be the standard Borel subgroup. There exist characters $\chi_{ij}$ of $\mathbb{C}^\times$ of the form $\chi_{ij}(z) = (z/\overline{z})^{r_{ij}}$, $r_{ij} \in \mathbb{Z}$, and complex numbers $s_{ij}$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$, satisfying

$$\text{Re}(s_{i1}) \geq \cdots \geq \text{Re}(s_{im_1}), \quad |\text{Re}(s_{ij})| < 1/2,$$

such that

$$\pi_i = J_{B_i}^{GL_{m_i}}(\chi_{i1}[s_{i1}] \otimes \cdots \otimes \chi_{im_i}[s_{im_i}]), \quad i = 1, 2.$$

Then the Rankin-Selberg $L$-factor is given by

$$L(s, \pi_1 \times \pi_2) = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} \Gamma_C(s + s_{1i} + s_{2j} + |r_{1i} + r_{2j}|/2).$$
If $F = \mathbb{R}$, the $L$-factors have a similar form.

The description of the $L$-factors in the Archimedean case can be unified in the following way. By the duplication formula of the gamma function we have

$$\Gamma_C(s) = \Gamma_R(s)\Gamma_R(s + 1).$$

Let $F$ be Archimedean. Set $e_F = 1$, if $F = \mathbb{R}$, and $e_F = 2$, if $F = \mathbb{C}$. Let $\pi \in \Pi(\text{GL}_m(F))$. Then it follows from (4.13) and the definition of the $L$-factors, that there exist complex numbers $\mu_j(\pi)$, $j = 1, \ldots, me_F$, such that

$$L(s, \pi) = \prod_{j=1}^{me_F} \Gamma_R(s + \mu_j(\pi)).$$

The numbers $\mu_j(\pi)$ are determined by the Langlands parameters of $\pi$. Similarly, if $\pi_i \in \Pi(\text{GL}_m_i(F))$, $i = 1, 2$, it follows from the description of the Rankin-Selberg $L$-factors that there exist complex numbers $\mu_{j,k}(\pi_1 \times \pi_2)$ such that

$$L(s, \pi_1 \times \pi_2) = \prod_{j,k} \Gamma_R(s + \mu_{j,k}(\pi_1 \times \pi_2)).$$

**Lemma 4.2.** Let $F$ be Archimedean. There exists $C > 0$ such that

$$\sum_{j,k} |\mu_{j,k}(\pi_1 \times \pi_2)|^2 \leq C \left( \sum_i |\mu_i(\pi_1)|^2 + \sum_j |\mu_j(\pi_2)|^2 \right)$$

for all generic $\pi_i \in \Pi(\text{GL}_m_i(F))$, $i = 1, 2$.

**Proof.** First consider the case $F = \mathbb{C}$. Let $\pi_1$ and $\pi_2$ be irreducible unitary generic representations of $\text{GL}_{m_1}(\mathbb{C})$ and $\text{GL}_{m_2}(\mathbb{C})$, respectively. Write $\pi_i$ as the Langlands quotient of the form (4.11). Using (4.10) and (4.12) together with (4.13), it follows that it suffices to prove that there exist $C > 0$ such that

$$\sum_{j,k} |s_{1j} + s_{2k} + |r_{1j} + r_{2k}|/2|^2 \leq C \sum_{i,j} |s_{ij} + |r_{ij}|/2|^2$$

for all generic $\pi_i \in \Pi(\text{GL}_m_i(\mathbb{C}))$, $i = 1, 2$. This follows immediately, if we use the fact that the parameters $s_{ij}$ satisfy $|\text{Re}(s_{ij})| < 1/2$ and the $r_{ij}$’s are integers.

The proof in the case $F = \mathbb{R}$ is essentially the same. We only have to check the different possible cases for the $L$-factors as listed above. 

Next we consider the global Rankin-Selberg $L$-functions. Let $E$ be a number field and let $\mathbb{A}_E$ be the ring of adeles of $E$. Given $m \in \mathbb{N}$, let $\Pi_{\text{dis}}(\text{GL}_m(\mathbb{A}_E))$ and $\Pi_{\text{cus}}(\text{GL}_m(\mathbb{A}_E))$ be defined in the same way as in the case of $\mathbb{Q}$ (see §1.4). Recall that the Rankin-Selberg $L$-function attached to a pair of automorphic
representations $\pi_1$ of $\text{GL}_{m_1}(\mathbb{A}_E)$ and $\pi_2$ of $\text{GL}_{m_2}(\mathbb{A}_E)$ is defined by the Euler product

$$L(s, \pi_1 \times \pi_2) = \prod_v L(s, \pi_{1,v} \times \pi_{2,v}),$$

(4.16)

where $v$ runs over all places of $E$. The Euler product is known to converge in a certain half-plane $\text{Re}(s) > c$. Suppose that $\pi_1$ and $\pi_2$ are unitary cuspidal automorphic representations of $\text{GL}_{m_1}(\mathbb{A}_E)$ and $\text{GL}_{m_2}(\mathbb{A}_E)$, respectively. Then $L(s, \pi_1 \times \pi_2)$ has the following basic properties:

i) The Euler product $L(s, \pi_1 \times \pi_2)$ converges absolutely for all $s$ in the half-plane $\text{Re}(s) > 1$.

ii) $L(s, \pi_1 \times \pi_2)$ admits a meromorphic continuation to the entire complex plane with at most simple poles at 0 and 1.

iii) $L(s, \pi_1 \times \pi_2)$ is of order one and is bounded in vertical strips outside of the poles.

iv) It satisfies a functional equation of the form

$$L(s, \pi_1 \times \pi_2) = \epsilon(s, \pi_1 \times \pi_2) L(1 - s, \tilde{\pi}_1 \times \tilde{\pi}_2)$$

(4.17)

with

$$\epsilon(s, \pi_1 \times \pi_2) = W(\pi_1 \times \pi_2)(D_E^{m_1m_2}N(\pi_1 \times \pi_2))^{1/2-s},$$

(4.18)

where $D_E$ is the discriminant of $E$, $W(\pi_1 \times \pi_2)$ is a complex number of absolute value 1 and $N(\pi_1 \times \pi_2) \in \mathbb{N}$.

The absolute convergence of the Euler product (4.16) in the half-plane $\text{Re}(s) > 1$ was proved in [JS1]. The functional equation is established in [Sh1, Th. 4.1] combined with [Sh3, Prop. 3.1] and [Sh3, Th. 1]. See also [Sh5] for the general case. The location of the poles has been determined in the appendix of [MW]. Property iii) was proved in [RS, p. 280].

Now let $\pi_1 \in \Pi_{\text{dis}}(\text{GL}_{m_1}(\mathbb{A}_E))$ and $\pi_2 \in \Pi_{\text{dis}}(\text{GL}_{m_2}(\mathbb{A}_E))$. Using the description of the residual spectrum for $\text{GL}_n$ [MW], $L(s, \pi_1 \times \pi_2)$ can be expressed in terms of Rankin-Selberg $L$-functions attached to cuspidal automorphic representations. Indeed, by [MW] there exist $k_i \in \mathbb{N}$ with $k_i|m_i$, parabolic subgroups $P_i$ of $G_i = \text{GL}_{m_i}$ of type $(d_1, \ldots, d_1)$, $d_i = m_i/k_i$, and unitary cuspidal automorphic representations $\delta_i$ of $\text{GL}_{d_i}(\mathbb{A}_E)$ such that

$$\pi_i = J^G_{P_i}(\delta_i[(k_i - 1)/2] \otimes \cdots \otimes \delta_i[(1 - k_i)/2]),$$

(4.19)

where the right-hand side denotes the unique irreducible quotient of the induced representation $I^G_{P_i}(\delta_i[(k_i - 1)/2] \otimes \cdots \otimes \delta_i[(1 - k_i)/2])$. Set $k = k_1 + k_2 - 2$. 
Then it follows from (4.4) that
\begin{equation}
L(s, \pi_1 \times \pi_2) = \prod_{i=0}^{k_1-1} \prod_{j=0}^{k_2-1} L(s + k/2 - i - j, \delta_1 \times \delta_2).
\end{equation}

Using this equality and i)-iv) above, we deduce immediately the corresponding properties satisfied by \(L(s, \pi_1 \times \pi_2)\). Especially, \(L(s, \pi_1 \times \pi_2)\) satisfies a functional equation of the form (4.17) with an \(\epsilon\)-factor similar to (4.18).

We shall now investigate the logarithmic derivatives of the Rankin-Selberg \(L\)-functions. First we need to introduce some notation. Let \(\pi_i \in \Pi_{\text{dis}}(\text{GL}_{m_i}(A_E))\), \(i = 1, 2\). For each Archimedean place \(w\) of \(E\) let \(\mu_{j,k}(\pi_{1,w} \times \pi_{2,w})\), \(j = 1, \ldots, r_w\), \(k = 1, \ldots, h_w\), be the parameters attached to \((\pi_{1,w}, \pi_{2,w})\) by means of (4.15). Set
\begin{equation}
c(\pi_1 \times \pi_2) = \sum_{w|\infty} \sum_{j,k} |\mu_{j,k}(\pi_{1,w} \times \pi_{2,w})|.
\end{equation}
Let \(N(\pi_1 \times \pi_2)\) be the integer that is determined by the \(\epsilon\)-factor as in (4.18). Set
\begin{equation}
\nu(\pi_1 \times \pi_2) = D_E^{m_1,m_2} N(\pi_1 \times \pi_2) (2 + c(\pi_1 \times \pi_2)).
\end{equation}
We call \(\nu(\pi_1 \times \pi_2)\) the level of \((\pi_1, \pi_2)\). Given \(\pi \in \Pi_{\text{dis}}(\text{GL}_{m_2}(A_E))\), set
\[\pi_\infty = \otimes_{v|\infty} \pi_v, \quad \pi_f = \otimes_{v<\infty} \pi_v.\]

**Lemma 4.3.** For every \(\epsilon > 0\) there exists \(C > 0\) such that
\[\left| \frac{L'(s, \pi_1,f \times \pi_2,f)}{L(s, \pi_1,f \times \pi_2,f)} \right| \leq C\]
for all \(s\) in the half-plane \(\text{Re}(s) \geq 2 + \epsilon\) and all \(\pi_i \in \Pi_{\text{cusp}}(\text{GL}_{m_i}(A_E))\), \(i = 1, 2\).

**Proof.** Let \(\pi_i \in \Pi_{\text{cusp}}(\text{GL}_{m_i}(A_E))\), \(i = 1, 2\), and let \(v < \infty\). By [Sk], \(\pi_{1,v}\) and \(\pi_{2,v}\) are unitary generic representations. Hence by Lemma 4.1 there exist complex numbers \(a_i(v)\), \(i = 1, \ldots, m_1 \cdot m_2\), with
\begin{equation}
|a_i(v)| < N(v)
\end{equation}
such that
\[L(s, \pi_{1,v} \times \pi_{2,v}) = \prod_{i=1}^{m_1 \cdot m_2} (1 - a_i(v) N(v)^{-s})^{-1}.
\]
Suppose that \(\text{Re}(s) > 1\). By (4.23) we have \(|a_i(v)/N(v)^s| < 1\). Hence, taking the logarithmic derivative, we get
\[\frac{L'(s, \pi_{1,v} \times \pi_{2,v})}{L(s, \pi_{1,v} \times \pi_{2,v})} = -\sum_i \frac{a_i(v) \log N(v)}{N(v)^s(1 - a_i(v) N(v)^{-s})} \]
\[= - \log N(v) \sum_{k=1}^\infty \frac{a_i(v)^k}{N(v)^{sk}}.
\]
Suppose that \( \sigma = \Re(s) > 1 \). Then by (4.23) we get

\[
\left| \frac{L'(s, \pi_1, \pi_2)}{L(s, \pi_1, \pi_2)} \right| \leq m_1 m_2 \sum_{k=1}^{\infty} \frac{\log N(v)}{N(v)^{\sigma-1}k}.
\]

Let \( \zeta_E(s) \) be the Dedekind zeta function of \( E \). Let \( \epsilon > 0 \) and set \( \sigma = 2 + \epsilon \). Then for \( \Re(s) \geq \sigma \) we get

\[
\left| \frac{L'(s, \pi_1, \pi_2)}{L(s, \pi_1, \pi_2)} \right| \leq m_1 m_2 \left| \frac{\zeta_E'(\sigma - 1)}{\zeta_E(\sigma - 1)} \right|.
\]

**Lemma 4.4.** For every \( \epsilon > 0 \) there exists \( C > 0 \) such that

\[
\left| \frac{L'(s, \pi_1, \pi_2)}{L(s, \pi_1, \pi_2)} \right| \leq C \left( 1 + \log(|s| + c(\pi_1 \times \pi_2)) \right)
\]

for all \( s \) with \( \Re(s) \geq 1 + \epsilon \) and all \( \pi_i \in \Pi_{\text{cus}}(\text{GL}_{m_i}(\mathbb{A}_E)), i = 1, 2 \).

**Proof.** Let \( w|\infty \). By (4.15) we have

\[
L(s, \pi_1, \pi_2) = \prod_{j,k} \Gamma_R(s + \mu_{j,k}(\pi_1, \pi_2)).
\]

Since \( \pi_1, \pi_2 \) are unitary and generic, the complex numbers \( \mu_{j,k}(\pi_1, \pi_2) \) satisfy

\[
\Re(\mu_{j,k}(\pi_1, \pi_2)) > -1.
\]

Now recall that by Stirlings formula

\[
\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1})
\]

is valid as \( |s| \to \infty \), in the angle \( -\pi + \delta < \arg s < \pi - \delta \), for any fixed \( \delta > 0 \). Hence

\[
\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{2} \log \pi + \log s + O(|s|^{-1})
\]

holds in the same range of \( s \). Let \( \epsilon > 0 \). Using (4.24), (4.25) and (4.26), it follows that there exists \( C > 0 \) such that

\[
\left| \frac{L'(s, \pi_1, \pi_2)}{L(s, \pi_1, \pi_2)} \right| \leq C + \sum_{j,k} \log(|s| + |\mu_{j,k}(\pi_1, \pi_2)|).
\]

for all \( w|\infty \), all \( s \) with \( \Re(s) \geq 1 + \epsilon \) and all \( \pi_i \in \Pi_{\text{cus}}(\text{GL}_{m_i}(\mathbb{A}_E)), i = 1, 2 \).

This implies the lemma.

Let \( \pi_i \in \Pi_{\text{dis}}(\text{GL}_{m_i}(\mathbb{A}_E)), i = 1, 2 \), and \( T > 0 \) be given. Denote by \( N(T; \pi_1, \pi_2) \) the number of zeros of \( L(s, \pi_1, \pi_2) \), counted with multiplicity, which are contained in the disc of radius \( T \) centered at 0.
Proposition 4.5. There exists $C > 0$ such that

$$N(T; \pi_1, \pi_2) \leq CT \log(T + \nu(\pi_1 \times \pi_2))$$

for all $T \geq 1$ and all $\pi_i \in \Pi_{\text{dis}}(GL_m(A_E))$, $i = 1, 2$.

Proof. By (4.20) we can assume that $\pi_1$ and $\pi_2$ are unitary cuspidal automorphic representations. Set

$$\Lambda(s) = s^a(1-s)^a \left( D_E^{m_1 m_2} N(\pi_1 \times \pi_2) \right)^{s/2} L(s, \pi_1 \times \pi_2),$$

where $a$ denotes the order of the pole of $L(s, \pi_1 \times \pi_2)$ at $s = 1$. Note that $a$ can be at most 1. Since $\pi_i$ is unitary, we have $\tilde{\pi}_i = \pi_i$, $i = 1, 2$. Hence by (4.17), it follows that $\Lambda(s)$ satisfies the functional equation

$$\Lambda(s) = W(\pi_1 \times \pi_2) \left( D_E^{m_1 m_2} N(\pi_1 \times \pi_2) \right)^{1/2} \Lambda(1-\overline{s}).$$

Since $L(s, \pi_1 \times \pi_2)$ is of order one, $\Lambda(s)$ is an entire function of order one and hence, it admits a representation as a Weierstrass product of the form

$$\Lambda(s) = e^{A+B s} \prod_{\rho} (1-s/\rho)^e^{s/\rho},$$

where $A, B \in \mathbb{C}$ and the product runs over the set of zeros of $\Lambda(s)$. We note that for $s = \sigma + iT$,

$$\text{Re} \sum_{\rho} \frac{1}{s-\rho} = \frac{1}{s} \sum_{\rho} \frac{\sigma - \text{Re}(\rho)}{(\sigma - \text{Re}(\rho))^2 + (\text{Im}(\rho) - T)^2}$$

and this series is convergent since $\Lambda(s)$ is of order one. Taking the real part of the logarithmic derivative of $\Lambda(s)$, and applying the functional equation (4.28) to the right-hand side, we get

$$\text{Re}(B) + \text{Re} \sum_{\rho} \frac{1}{s-\rho} = -\text{Re}(B) - \text{Re} \sum_{\rho} \frac{1}{s-\rho} + \text{Re} \sum_{\rho} \frac{1}{s-(1-\overline{\rho})}.$$ 

Now observe that by (4.28), $\rho$ is a zero of $\Lambda(s)$ if and only if $1 - \overline{\rho}$ is a zero of $\Lambda(s)$. Hence the two sums involving $s$ are equal, as they run over the same set of zeros. It follows that

$$\text{Re}(B) = -\text{Re} \left( \sum_{\rho} \frac{1}{s-\rho} \right).$$

Together with (4.27) this leads to

$$\text{Re} \sum_{\rho} \frac{1}{s-\rho} = \text{Re} \frac{\Lambda'(s)}{\Lambda(s)} = \frac{a}{s} + \frac{1}{s-1} + \frac{1}{2} \log(D_E^{m_1 m_2} N(\pi_1 \times \pi_2))$$

$$+ \frac{L'(s, \pi_{1,\infty} \times \pi_{2,\infty})}{L(s, \pi_{1,\infty} \times \pi_{2,\infty})} + \frac{L'(s, \pi_{1,f} \times \pi_{2,f})}{L(s, \pi_{1,f} \times \pi_{2,f})}.$$
Let $\epsilon > 0$, and set $\sigma = 2 + \epsilon$. By Lemma 4.3, Lemma 4.4 and (4.29), there exists $C > 0$ such that
\[
\sum_{\rho} \frac{\sigma - \text{Re}(\rho)}{(\sigma - \text{Re}(\rho))^2 + (\text{Im}(\rho) - T)^2} \leq \frac{1}{2} \log(D_{E}^{m_{1}m_{2}N(\pi_{1} \times \pi_{2})})
+ C(1 + \log(|T| + c(\pi_{1} \times \pi_{2})))
\leq C_1 \log(|T| + \nu(\pi_{1} \times \pi_{2}))
\leq C_1 \log(|T| + \nu(\pi_{1} \times \pi_{2}))
\]
for all $T \in \mathbb{R}$ and $\pi_{i} \in \Pi_{\text{cus}}(\text{GL}_{m_{i}}, (A_{E}))$, $i = 1, 2$. Let $T > 0$. Then it follows from (4.31) that
\[
N(T + 1; \pi_{1}, \pi_{2}) - N(T; \pi_{1}, \pi_{2}) \leq 2(3 + \epsilon) \sum_{\rho} \frac{\sigma - \text{Re}(\rho)}{(\sigma - \text{Re}(\rho))^2 + (\text{Im}(\rho) - T)^2}
\leq C \log(|T| + \nu(\pi_{1} \times \pi_{2}))
\]
for all $\pi_{i} \in \Pi_{\text{cus}}(\text{GL}_{m_{i}}, (A_{E}))$, $i = 1, 2$. This implies the proposition. \hfill \square

5. Normalizing factors

In this section we consider the global normalizing factors of intertwining operators. Our main purpose is to estimate certain integrals involving the logarithmic derivatives of the normalizing factors. The behaviour of these integrals is crucial for the estimation of the spectral side. From now on we assume that the ground field is $\mathbb{Q}$. Denote by $\mathbb{A}$ the ring of ad`eles of $\mathbb{Q}$.

Let $M \in \mathcal{L}$. Then there exists a partition $(n_{1}, \ldots, n_{r})$ of $n$ such that
\[
M = \text{GL}_{n_{1}} \times \cdots \times \text{GL}_{n_{r}}.
\]
Let $Q, P \in \mathcal{P}(M)$. Let $v$ be a place of $\mathbb{Q}$ and let $\pi_{v} \in \Pi(M(Q_{v}))$. Associated to $P, Q$ and $\pi_{v}$ is the local intertwining operator
\[
J_{Q|P}(\pi_{v}, \lambda), \quad \lambda \in \mathfrak{a}_{M,C}^{*},
\]
between the induced representations $I_{P}(\pi_{v}, \lambda)$ and $I_{Q}(\pi_{v}, \lambda)$, which is defined by an integral over $N_{Q}(Q_{v}) \cap N_{P}(Q_{v})$, and hence depends upon a choice of a Haar measure on this group. By [A7] there exist meromorphic functions $r_{Q|P}(\pi_{v}, \lambda)$, $\lambda \in \mathfrak{a}_{M,C}^{*}$, such that the normalized local intertwining operators
\[
R_{Q|P}(\pi_{v}, \lambda) = r_{Q|P}(\pi_{v}, \lambda)^{-1}J_{Q|P}(\pi_{v}, \lambda)
\]
satisfy the conditions of Theorem 2.1 of [A7]. There is a general construction of normalizing factors which works for any group [A7], [CLL]. For $\text{GL}_{n}$, however, the intertwining operators can be normalized by $L$-functions [A7, §4], [AC, p. 87]. The normalizing factors are given as
\[
r_{Q|P}(\pi_{v}, \lambda) = \prod_{\alpha \in \sum_{P} \cap \sum_{Q}} r_{\alpha}(\pi_{v}, \lambda(\alpha)),
\]
where \( r_\alpha(\pi_v, s) \) is a meromorphic function of one complex variable and \( \Sigma_P \) (resp. \( \Sigma_Q \)) denotes the roots of \((P, A_M)\) (resp. \((Q, A_M)\)). Thus to define the normalizing factors, it is enough to define the functions \( r_\alpha(\pi_v, s) \) for any root \( \alpha \) of \((G, A_M)\) and any \( \pi_v \in \Pi(M(Q_v)) \). To this end note that \( \pi_v \) is equivalent to a representation \( \pi_{1,v} \otimes \cdots \otimes \pi_{r,v} \) with \( \pi_{i,v} \in \Pi(GL_{n_i}(Q_v)) \) and the root \( \alpha \) corresponds to an ordered pair \((i, j)\) of distinct integers between 1 and \( r \). Fix a nontrivial additive character \( \psi_v \) of \( Q_v \). Let \( L(s, \pi_{i,v} \times \overline{\pi}_{j,v}) \) and \( \epsilon(s, \pi_{i,v} \times \overline{\pi}_{j,v}, \psi_v) \) be the Rankin-Selberg \( L \)-function and the \( \epsilon \)-factor attached to \((\pi_{i,v}, \overline{\pi}_{j,v})\) and \( \psi_v \). Set
\[
(5.2) \quad r_\alpha(\pi_v, s) = \frac{L(s, \pi_{i,v} \times \overline{\pi}_{j,v})}{L(1 + s, \pi_{i,v} \times \overline{\pi}_{j,v}) \epsilon(s, \pi_{i,v} \times \overline{\pi}_{j,v}, \psi_v)}.
\]

It follows from Theorem 6.1 of [Sh1] that there are Haar measures on the group \( N_Q(Q_v) \cap N_P(Q_v) \), depending on \( \psi_v \), such that the normalizing factors (5.1) have all the right properties (see [A7, §4], [AC, p. 87]). Now suppose that \( \pi \in \Pi_{\text{dis}}(M(\mathbb{A})) \). Then the global normalizing factor \( r_Q|P(\pi, \lambda) \) is defined by the infinite product
\[
(5.3) \quad r_Q|P(\pi, \lambda) = \prod_{\alpha \in \Sigma_P \cap \Sigma_\pi} r_\alpha(\pi, \lambda(\alpha)).
\]

Let \( \pi = \pi_1 \otimes \cdots \otimes \pi_r \). If \( \alpha \) corresponds to \((i, j)\) then by (5.2) we have
\[
(5.4) \quad r_\alpha(\pi, s) = \frac{L(s, \pi_i \times \overline{\pi}_j)}{L(1 + s, \pi_i \times \overline{\pi}_j) \epsilon(s, \pi_i \times \overline{\pi}_j)},
\]
where \( L(s, \pi_i \times \overline{\pi}_j) \) and \( \epsilon(s, \pi_i \times \overline{\pi}_j) \) are the global \( L \)-function and the \( \epsilon \)-factor, respectively, considered in the previous section.

The main goal of this section is to study the multidimensional logarithmic derivatives of the normalizing factors that occur on the spectral side of the trace formula [A4]. By (5.3) this problem is reduced to the investigation of the logarithmic derivatives of the analytic functions \( r_\alpha(\pi, s) \). Furthermore, by (5.4) each \( r_\alpha(\pi, s) \) may be regarded as the normalizing factor attached to a standard maximal parabolic subgroup in \( GL_m \) with \( m \leq n \). So let \( m_1, m_2 \in \mathbb{N} \) with \( m_1 + m_2 \leq n \). Given \( \pi_i \in \Pi_{\text{dis}}(GL_{m_i}(\mathbb{A})), \; i = 1, 2 \), set
\[
(5.5) \quad r(\pi_1 \otimes \pi_2, s) = \frac{L(s, \pi_1 \times \overline{\pi}_2)}{L(1 + s, \pi_1 \times \overline{\pi}_2) \epsilon(s, \pi_1 \times \overline{\pi}_2)}.
\]

We shall now study the logarithmic derivatives of these functions. For this purpose we need some preparation. Suppose that \( \pi_i, \; i = 1, 2 \), is given in
the form (4.20) and assume that $k_1 \leq k_2$. Set $k = k_1 + k_2 - 2$. For $j = 0, \ldots , k$ let the integers $a_j$ be defined by

$$a_i = \begin{cases} 
  i + 1 & : i \leq k_1 - 1; \\
  k_1 & : k_1 - 1 \leq i \leq k_2 - 1; \\
  k - i + 1 & : i \geq k_2 - 1.
\end{cases}$$

Note that $a_i = a_{k-i}$, $i = 0, \ldots , k$. It follows from (4.20) that

$$L(s, \pi_1 \times \tilde{\pi}_2) = \prod_{i=0}^{k} L(s + k/2 - i, \delta_1 \times \tilde{\delta}_2)^{a_i}.$$  

(5.7)

Define a polynomial of one variable $x$ by

$$p(x) = \prod_{i=0}^{k} ((x + k/2 - i)(1 - x - k/2 + i))^{a_i}.$$  

Then $p(x)$ has real coefficients and satisfies $p(x) = p(1 - x)$. Let $a$ be the order of the pole of $L(s, \delta_1 \times \tilde{\delta}_2)$ at $s = 1$. Note that $a \leq 1$. Set

$$\Lambda(s) = p(s)^a N(\pi_1 \times \tilde{\pi}_2)^{s/2} L(s, \pi_1 \times \tilde{\pi}_2).$$  

(5.8)

Then $\Lambda(s)$ satisfies the functional equation

$$\Lambda(s) = W(\pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{1/2} \Lambda(1 - s).$$  

(5.9)

Furthermore $\Lambda(s)$ is an entire function of order 1. Therefore it can be written as a Weierstrass product of the form

$$\Lambda(s) = e^{A + Bs} \prod_{\rho} (1 - s/\rho) e^{s/\rho}$$  

with $A, B \in \mathbb{C}$ and $\rho$ runs over the zeros of $\Lambda(s)$. Taking the logarithmic derivative and applying the functional equation (5.9) to the right-hand side, we get

$$\left( \frac{\Lambda(s)}{\Lambda(s + 1)} \right)' \cdot \frac{\Lambda(s + 1)}{\Lambda(s)} = \frac{\Lambda'(s)}{\Lambda(s)} + \frac{\Lambda'(-s)}{\Lambda(-s)} = 2 \text{Re}(B) + 2 \text{Re} \sum_{\rho} \frac{1}{\rho} + \sum_{\rho} \left\{ \frac{1}{s - \rho} - \frac{1}{s + \rho} \right\}.$$  

By (4.30) it follows that the first two terms on the right-hand side cancel and hence we get

$$\left( \frac{\Lambda(s)}{\Lambda(s + 1)} \right)' \cdot \frac{\Lambda(s + 1)}{\Lambda(s)} = 2 \sum_{\rho} \frac{\text{Re}(\rho)}{(s - \rho)(s + \rho)}.$$  

(5.10)
Therefore, combining (4.18), (5.5) and (5.8), we obtain
\[
\frac{r'(\pi_1 \otimes \pi_2, s)}{r(\pi_1 \otimes \pi_2, s)} = \log N(\pi_1 \times \bar{\pi}_2) + a \left\{ \frac{p'(s + 1)}{p(s + 1)} - \frac{p'(s)}{p(s)} \right\} + 2 \sum_{\rho} \frac{\text{Re}(\rho)}{(s - \rho)(s + \beta)}.
\]

In particular, if \( s = i\lambda, \lambda \in \mathbb{R}, \) then it follows from the definition of \( p(s) \) that
\[
\frac{r'(\pi_1 \otimes \pi_2, i\lambda)}{r(\pi_1 \otimes \pi_2, i\lambda)} = \log N(\pi_1 \times \bar{\pi}_2) + 2 \sum_{\rho} \frac{\text{Re}(\rho)}{\text{Re}(\rho)^2 + (\text{Im}(\rho) - \lambda)^2}.
\]

**Proposition 5.1.** There exists \( C > 0 \) such that
\[
\int_{-T}^{T} \left| \frac{r'(\pi_1 \otimes \pi_2, i\lambda)}{r(\pi_1 \otimes \pi_2, i\lambda)} \right| d\lambda \leq CT \log(T + \nu(\pi_1 \times \bar{\pi}_2))
\]
for all \( T > 0 \) and \( \pi_i \in \Pi_{\text{dis}}(\text{GL}_{m_i}(\mathbb{A})), \ i = 1, 2. \)

**Proof.** By the above formula it suffices to estimate the integral
\[
\int_{-T}^{T} \sum_{\rho} \frac{|\text{Re}(\rho)|}{\text{Re}(\rho)^2 + (\text{Im}(\rho) - \lambda)^2} d\lambda.
\]

We split the series as follows
\[
\sum_{\rho} = \sum_{|\text{Im}(\rho)| \leq T + 1} + \sum_{|\text{Im}(\rho)| > T + 1}.
\]

To estimate the integral of the first sum, observe that for all \( \beta \in \mathbb{R}^+ \) and \( \gamma \in \mathbb{R} \) we have
\[
\int_{-T}^{T} \frac{\beta}{\beta^2 + (\gamma - \lambda)^2} d\lambda \leq \int_{-\infty}^{\infty} \frac{d\lambda}{1 + \lambda^2} = \pi.
\]

Hence by Proposition 4.5 we get
\[
\int_{-T}^{T} \sum_{|\text{Im}(\rho)| \leq T + 1} \frac{|\text{Re}(\rho)|}{\text{Re}(\rho)^2 + (\text{Im}(\rho) - \lambda)^2} d\lambda \leq \pi N(T + 1, \pi_1, \bar{\pi}_2)
\]
\[
\leq CT \log(T + \nu(\pi_1 \times \bar{\pi}_2)).
\]

It remains to consider the integral of the second sum. Observe that by (5.7) the zeros \( \rho \) of \( \Lambda(s) \) satisfy \( |\text{Re}(\rho)| \leq k/2 + 1. \) Set
\[
\sigma = k + 3, \quad C = 2(k + 2)^2.
\]
Then the following inequality holds for all $\lambda \in \mathbb{R}$ with $|\lambda| \leq T$, all $\beta \in \mathbb{R}^\times$ with $|\beta| \leq k/2 + 1$ and all $\gamma \in \mathbb{R}$ with $|\gamma| > T + 1$:

$$\frac{|\beta|^2}{\beta^2 + (\gamma - \lambda)^2} \leq C \left\{ \frac{\sigma - \beta}{(\sigma - \beta)^2 + (\gamma - T)^2} + \frac{\sigma - \beta}{(\sigma - \beta)^2 + (\gamma + T)^2} \right\}.$$ 

Thus we get

$$\sum_{|\text{Im}(\rho)| > T + 1} \frac{|\text{Re}(\rho)|}{\text{Re}(\rho)^2 + (\text{Im}(\rho) - \lambda)^2} \leq C \left\{ \sum_{\rho} \frac{\sigma - \text{Re}(\rho)}{(\sigma - \text{Re}(\rho))^2 + (\text{Im}(\rho) - T)^2} + \sum_{\rho} \frac{\sigma - \text{Re}(\rho)}{(\sigma - \text{Re}(\rho))^2 + (\text{Im}(\rho) + T)^2} \right\}.$$ 

Combining (5.7) and (4.31), we see that for $\sigma = k + 3$ there exists $C_1 > 0$ such that

$$\sum_{\rho} \frac{\sigma - \text{Re}(\rho)}{(\sigma - \text{Re}(\rho))^2 + (\text{Im}(\rho) - T)^2} \leq C_1 \log(|T| + \nu(\pi_1 \times \bar{\pi}_2))$$

for all $T \in \mathbb{R}$ and $\pi_i \in \Pi_{\text{dis}}(\text{GL}_{m_i}(\mathbb{A})), i = 1, 2$. Combining these observation we get

$$\int_{-T}^{T} \sum_{|\text{Im}(\rho)| > T + 1} \frac{|\text{Re}(\rho)|}{\text{Re}(\rho)^2 + (\text{Im}(\rho) - \lambda)^2} \leq CT \log(T + \nu(\pi_1 \times \bar{\pi}_2)).$$

This completes the proof of the proposition. 

The next proposition will be important for the determination of the asymptotic behaviour of the spectral side.

**Proposition 5.2.** There exists $C > 0$ such that

$$\int_{-\infty}^{\infty} |r'(\pi_1 \otimes \pi_2, i\lambda)r(\pi_1 \otimes \pi_2, i\lambda)^{-1}|e^{-tx^2} \, d\lambda$$

$$\leq C \log(1 + \nu(\pi_1 \times \bar{\pi}_2)) \frac{1 + |\log t|}{\sqrt{t}}$$

for all $0 < t \leq 1$ and $\pi_i \in \Pi_{\text{dis}}(\text{GL}_{m_i}(\mathbb{A})), i = 1, 2$.

**Proof.** By Proposition 5.1,

$$\int_{0}^{\lambda} |r'(\pi_1 \otimes \pi_2, iu)r(\pi_1 \otimes \pi_2, iu)^{-1}| \, du \leq C\lambda^2$$
as $|\lambda| \to \infty$. Hence, using integration by parts, we see that the integral on the left-hand side of the claimed inequality equals

$$2t \int_{-\infty}^{\lambda} \int_{0}^{\lambda} \left| r'(\pi_1 \otimes \pi_2, iu) r(\pi_1 \otimes \pi_2, iu)^{-1} \right| du \lambda e^{-t \lambda^2} d\lambda.$$  

Applying Proposition 5.1 we get

$$\int_{-\infty}^{\lambda} \left| r'(\pi_1 \otimes \pi_2, i\lambda) r(\pi_1 \otimes \pi_2, i\lambda)^{-1} e^{-t \lambda^2} d\lambda \right|$$

\leq C t \int_{-\infty}^{\infty} \log (|\lambda| + \nu(\pi_1 \times \pi_2)) \frac{1 + |\log t|}{\sqrt{t}} d\lambda$$

for all $0 < t \leq 1$ and $\pi_i \in \Pi_{\text{dis}}(\text{GL}_{m_i}(A)), i = 1, 2.$

Let $M \in \mathcal{L}$ and let $Q, P \in \mathcal{P}(M)$. Our next goal is to estimate the corresponding integrals involving the generalized logarithmic derivatives of the global normalizing factors $r_{Q|P}(\pi, \lambda)$. For this purpose we will use the notion of a $(G, M)$ family introduced by Arthur in Section 6 of [A5]. For the convenience of the reader we recall the definition of a $(G, M)$ family and explain some of its properties.

For each $P \in \mathcal{P}(M)$, let $c_P(\lambda)$ be a smooth function on $i\mathfrak{a}_M^*$. Then the set

$$\{c_P(\lambda) \mid P \in \mathcal{P}(M)\}$$

is called a $(G, M)$ family if the following holds: Let $P, P' \in \mathcal{P}(M)$ be adjacent parabolic groups and suppose that $\lambda$ belongs to the hyperplane spanned by the common wall of the chambers of $P$ and $P'$. Then

$$c_P(\lambda) = c_{P'}(\lambda).$$

Let

$$\theta_P(\lambda) = \text{vol} \left( \mathfrak{a}_P^G / \mathbb{Z}(\Delta_P^\vee) \right)^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee), \quad \lambda \in i\mathfrak{a}_P^*,$$

where $\mathbb{Z}(\Delta_P^\vee)$ is the lattice in $\mathfrak{a}_P^G$ generated by the co-roots

$$\{\alpha^\vee \mid \alpha \in \Delta_P\}.$$

Let $\{c_P(\lambda)\}$ be a $(G, M)$ family. Then by Lemma 6.2 of [A5], the function

$$c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c_P(\lambda) \theta_P(\lambda)^{-1}$$

extends to a smooth function on $i\mathfrak{a}_M^*$. The value of $c_M(\lambda)$ at $\lambda = 0$ is of particular importance in connection with the spectral side of the trace formula.
It can be computed as follows. Let \( p = \dim(A_M/A_G) \). Set \( \lambda = t\Lambda, \ t \in \mathbb{R}, \ \Lambda \in a_M^* \), and let \( t \) tend to 0. Then

\[
(5.13) \quad c_M(0) = \frac{1}{p!} \sum_{P \in \mathcal{P}(M)} \left( \lim_{t \to 0} \left( \frac{d}{dt} \right)^p c_P(t\Lambda) \right) \theta_P(\Lambda)^{-1}
\]

[A5, (6.5)]. This expression is of course independent of \( \Lambda \).

For any \((G, M)\) family \( \{c_P(\lambda) \mid P \in \mathcal{P}(M)\} \) and any \( L \in \mathcal{L}(M) \) there is associated a natural \((G, L)\) family which is defined as follows. Let \( Q \in \mathcal{P}(L) \) and suppose that \( P \subset Q \). The function

\[
\lambda \in ia_L^* \mapsto c_P(\lambda)
\]

depends only on \( Q \). We will denote it by \( c_Q(\lambda) \). Then

\[
\{c_Q(\lambda) \mid Q \in \mathcal{P}(L)\}
\]

is a \((G, L)\) family. We write

\[
c_L(\lambda) = \sum_{Q \in \mathcal{P}(L)} c_Q(\lambda)\theta_Q(\lambda)^{-1}
\]

for the corresponding function (5.12).

Let \( Q \in \mathcal{P}(L) \) be fixed. If \( R \in \mathcal{P}^L(M) \), then \( Q(R) \) is the unique group in \( \mathcal{P}(M) \) such that \( Q(R) \subset Q \) and \( Q(R) \cap L = R \). Let \( c_Q^L \) be the function on \( ia_M^* \) which is defined by

\[
c_Q^L(\lambda) = c_Q(R)(\lambda).
\]

Then \( \{c_Q^L(\lambda) \mid R \in \mathcal{P}^L(M)\} \) is an \((L, M)\) family. Let \( c_M^Q(\lambda) \) be the function (5.12) associated to this \((L, M)\) family.

We consider now special \((G, M)\) families defined by the global normalizing factors. Fix \( P \in \mathcal{P}(M), \ \pi \in \pi_{dis}(M(\mathbb{A})) \) and \( \lambda \in ia_M^* \). Define

\[
(5.14) \quad \nu_Q(P, \pi, \lambda, \Lambda) := r_Q|P(\pi, \lambda)^{-1} r_Q|P(\pi, \lambda + \Lambda), \quad Q \in \mathcal{P}(M).
\]

This set of functions is a \((G, M)\) family [A4, p. 1317]. It is of a special form. By (5.3) we have

\[
\nu_Q(P, \pi, \lambda, \Lambda) = \prod_{\alpha \in \Sigma_Q \cap \Sigma_P} r_\alpha(\pi, \lambda(\alpha^\vee))^{-1} r_\alpha(\pi, \lambda(\alpha^\vee) + \Lambda(\alpha^\vee)).
\]

Suppose that \( L \in \mathcal{L}(M), \ L_1 \in \mathcal{L}(L) \) and \( S \in \mathcal{P}(L_1) \). Let

\[\{\nu_Q^S(P, \pi, \lambda, \Lambda) \mid Q_1 \in \mathcal{P}^{L_1}(L)\}\]

be the associated \((L_1, L)\) family and let \( \nu_L^S(P, \pi, \lambda, \Lambda) \) be the function (5.12) defined by this family. Set

\[
\nu_L^S(P, \pi, \lambda) := \nu_L^S(P, \pi, \lambda, 0).
\]
If $\alpha$ is any root in $\Sigma(G, A_M)$, let $\alpha_L^\vee$ denote the projection of $\alpha^\vee$ onto $a_L$. If $F$ is a subset of $\Sigma(G, A_M)$, let $F_L^\vee$ be the disjoint union of all the vectors $\alpha_L^\vee$, $\alpha \in F$. Then by Proposition 7.5 of [A4] we have

$$\nu_L^S(P, \pi, \lambda) = \sum_F \text{vol}(a_L^{\alpha_L} / \mathbb{Z}(F_L^\vee))$$

(5.15)

\[ \cdot \left( \prod_{\alpha \in F} r_\alpha(\pi, \lambda(\alpha^\vee))^{-1} r'_{\alpha'}(\pi, \lambda(\alpha^\vee)) \right), \]

where $F$ runs over all subsets of $\Sigma(L_1, A_M)$ such that $F_L^\vee$ is a basis of $a_L^{\alpha_L}$. Let $t > 0$. Then by (5.15),

$$\int_{i a_L^{\alpha_L} / a_G^\vee} |\nu_L^S(P, \pi, \lambda)| e^{-i\|\lambda\|^2} \, d\lambda \leq \sum_F \text{vol}(a_L^{\alpha_L} / \mathbb{Z}(F_L^\vee))$$

\[ \cdot \int_{i a_L^{\alpha_L} / a_G^\vee} \prod_{\alpha \in F} \left| r_\alpha(\pi, \lambda(\alpha^\vee))^{-1} r'_{\alpha'}(\pi, \lambda(\alpha^\vee)) \right| e^{-i\|\lambda\|^2} \, d\lambda. \]

Fix any subset $F$ of $\Sigma(L_1, A_M)$ such that $F_L^\vee$ is a basis of $a_L^{\alpha_L}$. Let

$$\{\tilde{\omega}_\alpha \mid \alpha \in F\}$$

be the basis of $(a_L^{\alpha_L})^*$ which is dual to $F_L^\vee$. We can write $\lambda \in ia_L^{\alpha_L} / ia_G^\vee$ as

$$\lambda = \sum \alpha z_\alpha \tilde{\omega}_\alpha + \lambda_1, \quad z_\alpha \in i\mathbb{R}, \quad \lambda_1 \in ia_L^{\alpha_L} / ia_G^\vee.$$ 

Observe that $\lambda(\alpha^\vee) = z_\alpha$. Let $l_1 = \dim(A_{L_1}/A_G)$. Then there exists $C > 0$, independent of $\pi$, such that for all $t > 0$,

$$\int_{i a_L^{\alpha_L} / a_G^\vee} \prod_{\alpha \in F} \left| r_\alpha(\pi, \lambda(\alpha^\vee))^{-1} r'_{\alpha'}(\pi, \lambda(\alpha^\vee)) \right| e^{-i\|\lambda\|^2} \, d\lambda$$

(5.16)

$$\leq C t^{-l_1/2} \sum_{\alpha \in F} \int_{i \mathbb{R}} \left| r_\alpha(\pi, z_\alpha)^{-1} r'_{\alpha'}(\pi, z_\alpha) \right| e^{-t z_\alpha^2} \, dz_\alpha.$$

Suppose that $M = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_r}$. Then $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ with $\pi_i \in \Pi_{\text{dis}}(\text{GL}_{n_i}(\mathbb{A}))$. Now recall that a given root $\alpha \in \Sigma(G, A_M)$ corresponds to an ordered pair $(i, j)$ of distinct integers $i$ and $j$ between 1 and $r$. Then it follows from (5.4) and (5.5) that $r_\alpha(\pi, s) = r(\pi_i \otimes \pi_j, s)$. Let $l = \dim(A_{L_1}/A_G)$ and $k = \dim(A_{L_1}/A_{L_1})$. Then by Proposition 5.2 and (5.16), there exists $C > 0$ such that

$$\int_{i a_L^{\alpha_L} / a_G^\vee} |\nu_L^S(P, \pi, \lambda)| e^{-t\|\lambda\|^2} \, d\lambda$$

(5.17)

$$\leq C \prod_{i,j} \log(1 + \nu(\pi_i \times \pi_j)) \left( \frac{1 + |\log t|^k}{t^{l/2}} \right)$$

for all $0 < t \leq 1$ and all $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$. 
Next we shall estimate the numbers \( \nu(\pi_i \times \pi_j) \). For \( \pi_\infty \in \Pi(\text{GL}_m(\mathbb{R})) \), let the complex numbers \( \mu_j(\pi_\infty) \), \( j = 1, \ldots, m \), be defined by (4.14) and set
\[
c(\pi_\infty) = \left( \sum_{j=1}^{m} |\mu_j(\pi_\infty)|^2 \right)^{1/2}.
\]
Given an open compact subgroup \( K_f \) of \( \text{GL}_m(\mathbb{A}_f) \), set
\[
\Pi(\text{GL}_m(\mathbb{A}))_{K_f} := \{ \pi \in \Pi(\text{GL}_m(\mathbb{A})) \mid \mathcal{H}_{\pi_f} \neq 0 \},
\]
where \( \pi = \pi_\infty \otimes \pi_f \) and \( \mathcal{H}_{\pi_f} \) denotes the Hilbert space of the representation \( \pi_f \).

**Lemma 5.3.** Let \( K_{f,i} \subset \text{GL}_m(\mathbb{A}_f) \), \( i = 1, 2 \), be two open compact subgroups. There exists \( C > 0 \) such that
\[
\nu(\pi_1 \times \pi_2) \leq C(1 + c(\pi_1, \infty) + c(\pi_2, \infty))
\]
for all \( \pi_i \in \Pi(\text{GL}_m(\mathbb{A}))_{K_{f,i}}, i = 1, 2 \).

**Proof.** First consider \( c(\pi_1 \times \pi_2) \) which is defined by (4.21). It follows from Lemma 4.2 that there exists \( C > 0 \) such that
\[
c(\pi_1 \times \pi_2) \leq C(c(\pi_1, \infty) + c(\pi_2, \infty))
\]
for all \( \pi_i \in \Pi(\text{GL}_m(\mathbb{A})) \), \( i = 1, 2 \). It remains to estimate \( N(\pi_1 \times \pi_2) \). For this we first observe that, as the epsilon factor is a product of local epsilon factors, we can factor \( N(\pi_1 \times \pi_2) \) as
\[
N(\pi_1 \times \pi_2) = \prod_p N(\pi_{1,p} \times \pi_{2,p}),
\]
where \( p \) runs over the finite places of \( \mathbb{Q} \). This is a finite product. In fact, if \( p \) is unramified for both \( \pi_1 \) and \( \pi_2 \), then \( N(\pi_{1,p} \times \pi_{2,p}) = 1 \). Moreover there is an integer \( f(\pi_{1,p} \times \pi_{2,p}) \) such that
\[
N(\pi_{1,p} \times \pi_{2,p}) = p^{f(\pi_{1,p} \times \pi_{2,p})}
\]
(see e.g. [MS]). Since we fix the ramification, there is a finite set \( S \) of finite places of \( \mathbb{Q} \), such that
\[
N(\pi_1 \times \pi_2) = \prod_{p \in S} p^{f(\pi_{1,p} \times \pi_{2,p})}
\]
for all \( \pi_i \in \Pi(\text{GL}_m(\mathbb{A}))_{K_{f,i}}, i = 1, 2 \). This reduces our problem to the estimation of \( f(\pi_{1,p} \times \pi_{2,p}) \). Let \( f(\pi_{i,p}) \) be the conductor of \( \pi_{i,p}, i = 1, 2 \). Then \( f(\pi_{i,p}) \geq 0 \) and by Theorem 1 of [BH] and Corollary (6.5) of [BHK] we have
\[
(5.18) \quad 0 \leq f(\pi_{1,p} \times \pi_{2,p}) \leq m_1 f(\pi_{1,p}) + m_2 f(\pi_{2,p}).
\]
Let \( m \in \mathbb{N} \) and let \( K_p \) be an open compact subgroup of \( \text{GL}_m(\mathbb{Q}_p) \). By Lemma 2.2 of [MS] there exists \( C_p > 0 \) such that \( f(\pi_p) \leq C_p \) for all \( \pi_p \in \Pi(\text{GL}_m(\mathbb{Q}_p)) \).
with \( \pi_p^{K^*} \neq 0 \). Together with (5.18) this implies that there exists \( C > 0 \) such that

\[
N(\pi_1 \times \pi_2) \leq C
\]

for all \( \pi_i \in \Pi(\mathrm{GL}_m, (\mathbb{A}))_{K_i,i}, i = 1, 2 \). This completes the proof of the lemma.

We continue with the estimation of \( c(\pi_\infty) \). Given \( \pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}), \xi_0) \), let \( \lambda_{\pi_\infty} \) be the Casimir eigenvalue of the restriction of \( \pi_\infty \) to \( \mathrm{GL}_m(\mathbb{R}) \). Furthermore for \( \sigma \in \Pi(\mathrm{O}(m)) \) let \( \lambda_\sigma \) denote the Casimir eigenvalue of \( \sigma \). We note that if \( [\pi_\infty|_{\mathrm{O}(m)} : \sigma] > 0 \), then \(-\lambda_{\pi_\infty} + \lambda_\sigma \geq 0 \) [DH, Lemma 2.6].

**Lemma 5.4.** There exists \( C > 0 \) such that

\[
c(\pi_\infty) \leq C(1 - \lambda_{\pi_\infty} + \lambda_\sigma)^{1/2}
\]

for all \( \pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}), \xi_0) \) and \( \sigma \in \Pi(\mathrm{O}(m)) \) with \([\pi_\infty|_{\mathrm{O}(m)} : \sigma] > 0 \).

**Proof.** Write \( \pi_\infty \) as the Langlands quotient \( \pi_\infty = J_{R}^{\mathrm{GL}_m}(\tau, s) \), where \( \tau \) is a discrete series representation of \( M_R(\mathbb{R}) \) and the parameters \( s_1, \ldots, s_r \in \mathbb{C} \) satisfy \( \Re(s_1) \geq \Re(s_2) \geq \cdots \geq \Re(s_r) \). We may assume that the central character of \( \tau \) is trivial on \( A_R(\mathbb{R})^0 \) and hence, we can regard \( \tau \) as a discrete series representation of \( M_R(\mathbb{R}) \). Let \( \mathfrak{m}_R^1 \) denote the Lie algebra of \( M_R(\mathbb{R}) \). Note that \( \mathfrak{m}_R^1 \) is the direct sum of a finite number of copies of \( \mathfrak{sl}(2, \mathbb{R}) \). Let \( t \subset \mathfrak{m}_R^1 \) be the standard compact Cartan subalgebra equipped with the canonical norm. Then \( \mathfrak{h} = t \oplus \mathfrak{a}_R \) is a Cartan subalgebra of \( \mathfrak{gl}_m(\mathbb{R}) \). Let \( \Lambda_\tau \in \mathfrak{it}^* \) be the Harish-Chandra parameter of \( \tau \). It follows from the definition of the parameters \( \mu_j(\pi_\infty) \) in terms of the Langlands parameters that there exists \( C > 0 \) such that

\[
c(\pi_\infty)^2 \leq C(\|\Lambda_\tau\|^2 + \|s\|^2)
\]

for all \( \pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}), \xi_0) \). Let \( \gamma : Z(\mathfrak{gl}_m(\mathbb{C})) \to I(\mathfrak{h}\mathbb{C}) \) be the Harish-Chandra homomorphism. By Proposition 8.22 of [Kn] the infinitesimal character \( \chi \) of the induced representation \( I_{R}^{\mathrm{GL}_m}(\tau, s) \) with respect to \( \mathfrak{h} \) is given by \( \chi(Z) = (\Lambda_\tau + s)(\gamma(Z)) \), \( Z \in Z(\mathfrak{gl}_m(\mathbb{C})) \). Since \( \pi_\infty \) is an irreducible quotient of \( I_{R}^{\mathrm{GL}_m}(\tau, s) \), it has the same infinitesimal character. Let \( H_1, \ldots, H_r \) be an orthonormal basis of \( \mathfrak{a}_R \) and \( H_{r+1}, \ldots, H_m \) an orthonormal basis of \( t \). Then

\[
\gamma(\Omega) = \sum_{i=1}^{r} H_i^2 - \sum_{j=r+1}^{m} H_j^2 - \|\rho\|^2
\]

[Wa1, p. 168]. Hence, the Casimir eigenvalue \( \lambda_{\pi_\infty} \) of \( \pi_\infty \) is given by

\[
\lambda_{\pi_\infty} = (\Lambda_\tau + s)(\gamma(\Omega)) = \sum_{i=1}^{r} s_i^2 + \|\Lambda_\tau\|^2 - \|\rho\|^2.
\]
Since \( \pi_\infty \) is unitary, it follows from Theorem 3.3 of Chapter XI of [BW] that there exists \( C > 0 \), independent of \( \pi_\infty \), such that \( \| \text{Re}(s) \| \leq C \). Hence there exists \( C_1 > 0 \) such that
\[
\| \Lambda_f \|^2 + \| s \|^2 \leq C_1 - \lambda_\pi_\infty + \| \Lambda_f \|^2
\]
for all \( \pi_\infty \in \Pi(\text{GL}_m(\mathbb{R}), \xi_0) \). Now let \( \sigma \in \Pi(\text{O}(m)) \) and suppose that \( \pi_\infty \in \Pi(\text{GL}_m(\mathbb{R}), \xi_0) \) is such that \( \| \text{Re}(s) \| \sigma \| > 0 \). Since \( \sigma \) occurs in \( \pi_\infty \), it also occurs in \( I^\mathbb{R}_\text{GL}_m(\tau, s) \). Using Frobenius reciprocity as in [Kn, p. 208], we see that there exists \( \omega \in \Pi(\text{O}(m) \cap M_R(\mathbb{R})) \) such that
\[
| \tau |_{\text{O}(m) \cap M_R(\mathbb{R})} : \omega > 0 \quad \text{and} \quad | \sigma |_{\text{O}(m) \cap M_R(\mathbb{R})} : \omega > 0.
\]
Let \( \lambda_\sigma \) and \( \lambda_\omega \) denote the Casimir eigenvalues of \( \sigma \) and \( \omega \), respectively. By [Mu2, (5.15)], the second inequality implies \( \lambda_\omega \leq \lambda_\sigma \). On the other hand, by [Wa2, p. 398], the first inequality implies
\[
\| \Lambda_f \|^2 \leq \lambda_\omega + \| R \|^2.
\]
By combining our estimations the lemma follows. \( \square \)

Now let \( K_f \) be an open compact subgroup of \( G(\mathbb{A}_f) \). Set
\[
K_{M,f} = K_f \cap M(\mathbb{A}_f).
\]
Then \( K_{M,f} \) is an open compact subgroup of \( M(\mathbb{A}_f) \). There exist open compact subgroups \( K_{f,i} \) of \( \text{GL}_m(\mathbb{A}_f) \), \( i = 1, \ldots, r \), such that \( K_{f,1} \times \cdots \times K_{f,r} \) is a subgroup of finite index of \( K_f \). Set
\[
\Pi(M(\mathbb{A}), \xi_0)_{K_f} = \{ \pi \in \Pi(M(\mathbb{A}), \xi_0) \mid \mathcal{H}^{K_{M,f}}_{\pi} \neq \{0\} \},
\]
where \( \pi = \pi_\infty \otimes \pi_f \). Let \( \pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f} \). Then \( \pi = \pi_1 \otimes \cdots \otimes \pi_r \) and \( \pi_i \) belongs to \( \Pi(\text{GL}_{m_i}(\mathbb{A}), \xi_0)_{K_{f,i}} \), and by Lemma 5.3 it follows that there exists \( C > 0 \) such that
\[
\prod_{i,j} \log(1 + \nu(\pi_i \times \pi_j)) \leq C \prod_{i,j} \log(2 + c(\pi_{i,\infty}) + c(\pi_{j,\infty}))
\]
for all \( \pi = \pi_1 \otimes \cdots \otimes \pi_r \in \Pi(M(\mathbb{A}), \xi_0)_{K_f} \). Let \( K_{M,\infty} = \text{O}(n_1) \times \cdots \times \text{O}(n_r) \) be the standard maximal compact subgroup of \( \text{M}(\mathbb{R}) \). Let \( \sigma \in \Pi(\text{O}(n)) \). For \( \pi \in \Pi(M(\mathbb{A}), \xi_0) \) set
\[
[\pi_\infty : \sigma] = \sum_{\tau \in \Pi(K_{M,\infty})} [\pi_\infty |_{K_{M,\infty}} : \tau] [\sigma |_{K_{M,\infty}} : \tau] - \text{tr} \frac{1}{n} [\sigma |_{K_{M,\infty}} : \tau].
\]
Put
\[
\Pi(M(\mathbb{A}), \xi_0)_{K_f, \sigma} = \{ \pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f} \mid [\pi_\infty : \sigma] > 0 \}
\]
and
\[
\Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)_{K_f, \sigma} = \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0) \cap \Pi(M(\mathbb{A}), \xi_0)_{K_f, \sigma}.
\]
Suppose that $\pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f, \sigma}$. Let $\tau \in \Pi(K_{M, \infty})$ be such that $[\sigma|_{K_{M, \infty}} : \tau] > 0$ and $[\pi|_{K_{M, \infty}} : \tau] > 0$.

Let $\lambda_{\pi, \infty}$ and $\lambda_{\tau}$ denote the Casimir eigenvalues of the restriction of $\pi_{\infty}$ to $M(\mathbb{R})$ and of $\tau$, respectively. Note that $\lambda_{\pi, \infty} = \sum_i \lambda_{\pi, i, \infty}$ and $\lambda_{\tau} = \sum_i \lambda_{\tau, i}$, where $\tau = \otimes_i \tau_i$. Then it follows from (5.19) and Lemma 5.4 that there exists $C > 0$ such that
\[
\prod_{i,j} \log(1 + \nu(\pi_i \times \pi_j)) \leq C \prod_{i,j} \log(2 - \lambda_{\pi, i, \infty} + \lambda_{\tau, i} - \lambda_{\pi, j, \infty} + \lambda_{\tau, j}) \leq C (\log(2 - \lambda_{\pi, \infty} + \lambda_{\tau}))^r
\]
for all $\pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f, \sigma}$. Since there are only finitely many $\tau$ that occur in $\sigma|_{K_{M, \infty}}$, we get
\[
(5.20) \quad \prod_{i,j} \log(1 + \nu(\pi_i \times \pi_j)) \leq C_1 (\log(2 + |\lambda_{\pi, \infty}|))^r
\]
for all $\pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f, \sigma}$. Combining (5.17)–(5.20) we obtain

**Proposition 5.5.** Let $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$ and $P \in \mathcal{P}(M)$. Let $l = \dim(A_L/A_G)$. Let $K_f$ be an open compact subgroup of $\text{GL}_n(\mathbb{A}_f)$ and let $\sigma \in \Pi(O(n))$. There exists $C > 0$ such that
\[
\int_{a_{L,f}^* \backslash a_{G}^*} |\nu^S_{L}(P, \pi, \lambda)| e^{-t||\lambda||^2} d\lambda \leq C (\log(2 + |\lambda_{\pi, \infty}|))^{n_2} \frac{(1 + |\log t|)^l}{t^{l/2}}
\]
for all $0 < t \leq 1$ and $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_0)_{K_f, \sigma}$.

6. The spectral side

We shall use the noninvariant trace formula of Arthur [A1], [A2], applied to the heat kernel, to determine the growth of the discrete spectrum. To begin with, we explain the general structure of the spectral side of the Arthur trace formula. The spectral side is a sum of distributions
\[
\sum_{\chi \in \mathcal{X}} J_\chi(f), \quad \chi \in C_0^\infty(G(\mathbb{A})^1).
\]
Here $\mathcal{X}$ is the set of cuspidal data which consist of Weyl group orbits of pairs $(M_B, \rho_B)$, where $M_B$ is the Levi component of a parabolic subgroup and $\rho_B$ is a cuspidal automorphic representation of $M_B(\mathbb{A})$. The distributions $J_\chi$ are described by Theorem 8.2 of [A4]. Let $C^1(G(\mathbb{A})^1)$ be the space of integrable rapidly decreasing functions on $G(\mathbb{A})^1$ [MS, §1.3]. In [MS, Th. 0.1] it was proved that the spectral side of the trace formula for $\text{GL}_n$ is absolutely convergent for all $f \in C^1(G(\mathbb{A})^1)$. In this case the expression of the spectral side simplifies.
To describe this in more detail, we need to introduce some notation. Let $M \in \mathcal{L}$ and $P, Q \in \mathcal{P}(M)$. Let $\mathcal{A}^2(P)$ and $\mathcal{A}^2(Q)$ be the corresponding spaces of automorphic functions (see §1.5). Let $W(a_P, a_Q)$ be the set of all linear isomorphisms from $a_P$ to $a_Q$ which are restrictions of elements of the Weyl group $W(A_0)$. The theory of Eisenstein series associates to each $s \in W(a_P, a_Q)$ an intertwining operator $M_{Q|P}(s, \lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q)$, $\lambda \in a^*_{P;C}$, which, for $\text{Re}(\lambda)$ in a certain chamber, can be defined by an absolutely convergent integral and admits an analytic continuation to a meromorphic function of $\lambda \in a^*_{P;C}$ [La].

Fix $P \in \mathcal{P}(M)$ and $\lambda \in ia^*_M$. For $Q \in \mathcal{P}(M)$ and $\Lambda \in ia^*_M$ define

$$\mathcal{M}_Q(P, \lambda, \Lambda) = M_{Q|P}(\lambda) - 1 M_{Q|P}(\lambda + \Lambda) \prod_{\alpha \in \Delta} \Lambda(\alpha^\vee).$$

Then

$$\{ \mathcal{M}_Q(P, \lambda, \Lambda) \mid \Lambda \in ia^*_M, Q \in \mathcal{P}(M) \}$$

is a $(G, M)$ family with values in the space of operators on $\mathcal{A}^2(P)$ [A4, p. 1310]. Let $L \in \mathcal{L}(M)$. Then, as explained in the previous section, the $(G, M)$ family (6.1) has an associated $(G, L)$ family

$$\{ \mathcal{M}_Q(L, \lambda, \Lambda) \mid \Lambda \in ia^*_L, Q \in \mathcal{P}(L) \}$$

and

$$\mathcal{M}_L(P, \lambda, \Lambda) = \sum_{Q_1 \in \mathcal{P}(L)} \mathcal{M}_Q(P, \lambda, \Lambda) \theta_{Q_1}(\Lambda)^{-1}$$

extends to a smooth function on $ia^*_L$. Put

$$\mathcal{M}_L(P, \lambda) = \mathcal{M}_L(P, \lambda, 0).$$

This operator depends only on the intertwining operators. It equals

$$\mathcal{M}_L(P, \lambda) = \lim_{\Lambda \to 0} \left( \sum_{Q_1 \in \mathcal{P}(L)} \text{vol}(a^*_Q / \mathbb{Z}(\Delta^\vee)) M_{Q|P}(\lambda) - 1 M_{Q|P}(\lambda + \Lambda) \prod_{\alpha \in \Delta} \Lambda(\alpha^\vee) \right),$$

where $\lambda$ and $\Lambda$ are constrained to lie in $ia^*_L$, and for each $Q_1 \in \mathcal{P}(L)$, $Q$ is a group in $\mathcal{P}(M)$ which is contained in $Q_1$. Then $\mathcal{M}_L(P, \lambda)$ is an unbounded operator which acts on the Hilbert space $\mathcal{A}^2(P)$. For $\pi \in \Pi(M(\mathbb{A})^1)$ let $\mathcal{A}^2_\pi(P)$ be the subspace of $\mathcal{A}^2(P)$ determined by $\pi$ (see §1.5). Let $\rho_\pi(P, \lambda)$ be the
induced representation of $G(\mathbb{A})$ in $\mathcal{A}_1^2(P)$. Let $W^L(a_M)_{\text{reg}}$ be the set of elements $s \in W(a_M)$ such that \{ $H \in a_M$ | $sH = H$ \} = $a_L$. For any function $f \in \mathcal{C}^1(G(\mathbb{A})^1)$ and $s \in W^L(a_M)_{\text{reg}}$ set

$$J_{M,P}^L(f,s) = \sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1)} \int_{a_m^L/a_G^\mathbb{C}} \text{tr}(\mathcal{M}_L(P,\lambda)M_{P,P}(s,0)\rho_\pi(P,\lambda,f)) \, d\lambda.$$  

By Theorem 0.1 of [MS] this integral-series is absolutely convergent with respect to the trace norm. Furthermore for $M \in \mathcal{L}$ and $s \in W^L(a_M)_{\text{reg}}$ set

$$a_{M,s} = |\mathcal{P}(M)|^{-1}|W_0^M||W_0|^{-1}|\det(s-1)_{a_G^\mathbb{C}}|^{-1}.$$  

Then for any $f \in \mathcal{C}^1(G(\mathbb{A})^1)$, the spectral side $J_{\text{spec}}(f)$ of the Arthur trace formula is given by

$$J_{\text{spec}}(f) = \sum_{M \in \mathcal{L}} \sum_{L \in \mathcal{L}(M)} \sum_{P \in \mathcal{P}(M)} \sum_{s \in W^L(a_M)_{\text{reg}}} a_{M,s} J_{M,P}^L(f,s).$$  

Note that all sums in this expression are finite.

We shall now evaluate the spectral side at a function $\phi_t$, $t > 0$, which is given in terms of the heat kernel of a Bochner-Laplace operator. Then our main purpose is to determine the behaviour of $J_{\text{spec}}(\phi_t)$ as $t \to 0$.

Let $G(\mathbb{R})^1 = G(\mathbb{A})^1 \cap G(\mathbb{R})$. By definition $G(\mathbb{R})^1$ consists of all $g \in G(\mathbb{R})$ with $|\det(g)| = 1$. Hence $G(\mathbb{R})^1$ is semisimple and

$$G(\mathbb{R}) = G(\mathbb{R})^1 \cdot A_G(\mathbb{R})^0.$$  

Let

$$X = G(\mathbb{R})^1/K_\infty$$  

be the associated Riemannian symmetric space. Given $\sigma \in \Pi(K_\infty)$, let $\tilde{E}_\sigma \to X$ be the associated homogeneous vector bundle. Let $\Omega_{G(\mathbb{R})^1}$ be the Casimir element of $G(\mathbb{R})^1$ and let $\Delta_\sigma$ be the operator in $L^2(\tilde{E}_\sigma)$ which is induced by $-R(\Omega_{G(\mathbb{R})^1}) \otimes \text{Id}$. Let

$$H_t^\sigma \in (\mathcal{C}^1(G(\mathbb{R})^1) \otimes \text{End}(V_\sigma))^{K_\infty \times K_\infty}$$  

be the kernel of the heat operator $e^{-t\Delta_\sigma}$ where $\mathcal{C}^1(G(\mathbb{R}))$ is Harish-Chandra’s space of integrable rapidly decreasing functions. Set

$$h_t^\sigma = \text{tr} H_t^\sigma.$$  

We extend $h_t^\sigma$ to a function on $G(\mathbb{R})$ by

$$h_t^\sigma(g \cdot z) = h_t^\sigma(g), \quad g \in G(\mathbb{R})^1, \ z \in A_G(\mathbb{R})^0.$$  

Then $h_t^\sigma$ satisfies

$$h_t^\sigma(gz) = h_t^\sigma(g), \quad g \in G(\mathbb{R}), \ z \in A_G(\mathbb{R})^0.$$
Let \( \chi_\sigma \) be the character of \( \sigma \). Then \( h^\sigma_t \) also satisfies
\[
h^\sigma_t = \chi_\sigma * h^\sigma_0 * \overline{\chi_\sigma}.
\]
Let \( K_f \) be an open compact subgroup of \( G(\mathbb{A}) \) and let \( 1_{K_f} \) be the characteristic function of \( K_f \) in \( G(\mathbb{A}) \). Set
\[
\chi_{K_f} = \text{vol}(K_f)^{-1} 1_{K_f}.
\]
Define the function \( \phi_t \) on \( G(\mathbb{A}) \) by
\[
(6.5) \quad \phi_t(g) = h^\sigma_t(g_\infty)\chi_{K_f}(g_f)
\]
for any point
\[
g = g_\infty g_f, \quad g_\infty \in G(\mathbb{R}), \quad g_f \in G(\mathbb{A}),
\]
in \( G(\mathbb{A}) \). Then \( \phi_t \) satisfies \( \phi_t(gz) = \phi_t(g) \) for \( z \in \mathbb{A}_G(\mathbb{R})^0, \ g \in G(\mathbb{A}) \). It follows from (6.4) and the definition of \( C^1(\mathbb{A}) \) that the restriction \( \phi^1_t \) of \( \phi_t \) to \( G(\mathbb{A}) \) belongs to \( C^1(\mathbb{A}) \).

Let \( \pi \) be any unitary representation of \( G(\mathbb{A}) \) which is trivial on \( \mathbb{A}_G(\mathbb{R})^0 \). Then we can define
\[
\pi(\phi_t) = \int_{G(\mathbb{A})/\mathbb{A}_G(\mathbb{R})^0} \phi_t(g)\pi(g) \, dg.
\]
Suppose that \( \pi = \pi_\infty \otimes \pi_f \), where \( \pi_\infty \) and \( \pi_f \) are unitary representations of \( G(\mathbb{R}) \) and \( G(\mathbb{A}) \), respectively. Then \( \pi_\infty \) is trivial on \( \mathbb{A}_G(\mathbb{R})^0 \). So we can set
\[
\pi_\infty(\phi_t) = \int_{G(\mathbb{R})/\mathbb{A}_G(\mathbb{R})^0} \pi_\infty(g_\infty)h^\sigma_t(g_\infty) \, dg_\infty.
\]
Let \( \Pi_{K_f}^f \) denote the orthogonal projection of the Hilbert space \( \mathcal{H}_{\pi_f} \) of \( \pi_f \) onto the subspace \( \mathcal{H}_{\pi_f}^{K_f} \) of \( K_f \)-invariant vectors. Then
\[
\pi(\phi_t) = \pi_\infty(h^\sigma_t) \otimes \Pi_{K_f}.
\]
Now let \( \pi \in \Pi(M(\mathbb{A})^1) \). We identify \( \pi \) with a representation of \( M(\mathbb{A}) \) which is trivial on \( \mathbb{A}_M(\mathbb{R})^0 \). Let \( I^G_P(\pi_\lambda), \ \lambda \in \mathfrak{a}^*_M, \mathbb{C} \), be the induced representation of \( G(\mathbb{A}) \). Let \( \pi = \pi_\infty \otimes \pi_f \). Then
\[
I^G_P(\pi_\lambda) = I^G_P(\pi_{\infty,\lambda}) \otimes I^G_P(\pi_{f,\lambda}).
\]
Let \( \mathcal{H}_P(\pi_\infty) \) denote the \( \sigma \)-isotypical subspace of the Hilbert space \( \mathcal{H}_P(\pi_\infty) \) of the induced representation. Then \( \mathcal{H}_P(\pi_\infty) \) is an invariant subspace of \( I^G_P(\pi_{\infty,\lambda}, h^\sigma_t) \). Let \( \lambda_\pi \) be the Casimir eigenvalue of the restriction of \( \pi_\infty \) to \( M(\mathbb{R})^1 \). By Proposition 8.22 of [Kn] it follows that
\[
I^G_P(\pi_{\infty,\lambda}, h^\sigma_t) \downarrow \mathcal{H}_P(\pi_\infty) = e^{-t(-\lambda_\pi + ||\lambda||^2)} \text{Id}.
\]
Now observe that there is a canonical isomorphism
\[
j_P : \mathcal{H}_P(\pi) \otimes \text{Hom}_{\mathbb{M}(\mathbb{A})} \big( \pi, I^M_{\mathbb{M}(\mathbb{R})^0}(\xi_0) \big) \to \mathfrak{X}_{\pi}^2(P),
\]
which intertwines the induced representations. Let \( \Pi_{K_f, \sigma} \) denote the orthogonal projection of \( \mathcal{X}_\pi^2(P) \) onto \( \mathcal{A}_\pi^2(P)_{K_f, \sigma} \). Then it follows that

\[
\rho_\pi(P, \lambda, \phi_t) = e^{-(\lambda_\tau + \|\lambda\|^2)}\Pi_{K_f, \sigma}.
\]

Suppose that \( \lambda \in (a^*_P)^* \). Then \( \rho_\pi(P, \lambda, g) \) is trivial on \( A_G(\mathbb{R})^0 \). This implies \( \rho_\pi(P, \lambda, \phi_t) = \rho_\pi(P, \lambda, \phi^1) \), where \( \phi^1 \) is the restricticon of \( \phi_t \) to \( G(\mathbb{A})^1 \). Together with (6.6) we get

\[
J^{L}_{M,P}(\phi^1_t, s) = \sum_{\pi \in \Pi_{\text{disc}}(M(A)^\text{1})} e^{i\lambda_\tau} \\
\cdot \int_{\text{i}a^*_M/a^*_\mathbb{C}} e^{-\|\lambda\|^2} \text{tr}(\mathfrak{m}_L(P, \lambda)M_{P|P}(s, 0)\Pi_{K_f, \sigma}) \, d\lambda.
\]

To study this integral-series, we introduce the normalized intertwining operators

\[
N_{Q|P}(\pi, \lambda) := r_{Q|P}(\pi, \lambda)^{-1}M_{Q|P}(\pi, \lambda), \quad \lambda \in a^*_M, \mathbb{C},
\]

where \( r_{Q|P}(\pi, \lambda) \) are the global normalizing factors considered in the previous section. Let \( P \in \mathcal{P}(M) \) and \( \lambda \in ia^*_M \) be fixed. For \( Q \in \mathcal{P}(M) \) and \( \Lambda \in ia^*_M \) define

\[
\mathfrak{m}_Q(P, \pi, \lambda, \Lambda) = N_{Q|P}(\pi, \lambda)^{-1}N_{Q|P}(\pi, \lambda + \Lambda),
\]

Then as functions of \( \Lambda \in ia^*_M \),

\[
\{\mathfrak{m}_Q(P, \pi, \lambda, \Lambda) \mid Q \in \mathcal{P}(M)\}
\]

is a \((G, M)\) family. The verification is the same as in the case of the unnormalized intertwining operators [A4, p. 1310]. For \( L \in \mathcal{L}(M) \), let

\[
\{\mathfrak{m}_{Q_1}(P, \pi, \lambda, \Lambda) \mid \Lambda \in ia^*_L, \ Q_1 \in \mathcal{P}(L)\}
\]

be the associated \((G, L)\) family.

Let \( \mathfrak{m}_{Q_1}(P, \pi, \lambda, \Lambda) \) be the restriction of \( \mathfrak{m}_{Q_1}(P, \lambda, \Lambda) \) to \( \overline{\mathcal{X}_\pi^2(P)} \). Then by (6.8) and (5.14) it follows that

\[
\mathfrak{m}_{Q_1}(P, \pi, \lambda, \Lambda) = \mathfrak{m}_{Q_1}(P, \pi, \lambda, \Lambda)v_{Q_1}(P, \pi, \lambda, \Lambda)
\]

for all \( \Lambda \in ia^*_L \) and all \( Q_1 \in \mathcal{P}(L) \).

For \( Q \supset P \) let \( \hat{\mathcal{L}}_P^Q \subseteq a^*_P \) be the lattice generated by \( \{\hat{\omega}^\vee \mid \hat{\omega} \in \hat{\Delta}_P^Q\} \). Define

\[
\hat{\theta}_P^Q(\lambda) = \text{vol}(a^*_P/\hat{\mathcal{L}}_P^Q)^{-1}\prod_{\hat{\omega} \in \hat{\Delta}_P^Q} \lambda(\hat{\omega}^\vee).
\]

For \( S \in \mathcal{F}(L) \) put

\[
\mathfrak{m}_{S}(P, \pi, \lambda)
\]

\[
= \lim_{\Lambda \rightarrow 0} \sum_{\{R \mid R \supset S\}} (-1)^{\text{dim}(A_S/A_R)}\hat{\theta}_S^R(\lambda)^{-1}\mathfrak{m}_R(P, \pi, \lambda, \Lambda)\theta_R(\Lambda)^{-1}.
\]
Let $\mathcal{M}_L(P, \pi, \lambda)$ be the restriction of $\mathcal{M}_L(P, \lambda)$ to $\mathfrak{F}_\pi^2(P)$. Then by (6.10) and Lemma 6.3 of [A5] we get

$$\tag{6.12} \mathcal{M}_L(P, \pi, \lambda) = \sum_{S \in \mathcal{F}(L)} \mathfrak{N}'_S(P, \pi, \lambda) \nu_L^S(P, \pi, \lambda).$$

Let $\mathfrak{N}'_S(P, \pi, \lambda)_{K_f, \sigma}$ denote the restriction of $\mathfrak{N}'_S(P, \pi, \lambda)$ to $\mathcal{A}_\pi^2(P)_{K_f, \sigma}$. Then by (6.7),

$$\tag{6.13} J_{M,P}^L(\phi^1_t, s) = \sum_{\pi \in \Pi_{\text{dis}}(M(\A)^1)} e^{i \lambda_\pi} \cdot \sum_{S \in \mathcal{F}(L)} e^{-t||\lambda||^2} \nu_L^S(P, \pi, \lambda) \text{tr}(M_P|P(s,0)\mathfrak{N}'_S(P, \pi, \lambda)_{K_f, \sigma}) \, d\lambda.$$

Next we shall estimate the norm of $\mathfrak{N}'_S(P, \pi, \lambda)_{K_f, \sigma}$. For a given place $v$ of $\Q$ let $J_{Q,P}(\pi_v, \lambda)$ be the intertwining operator between the induced representations $I_{Q,P}^G(\pi_v, \lambda)$ and $I_{Q,P}^G(\pi_v, \lambda)$. Let

$$R_{Q,P}(\pi_v, \lambda) = r_{Q,P}(\pi_v, \lambda)^{-1} J_{Q,P}(\pi_v, \lambda), \quad \lambda \in \mathfrak{a}_M^* \cap \mathfrak{a}_C^*,$$

be the normalized local intertwining operator. These operators satisfy the conditions $(R_1) - (R_8)$ of Theorem 2.1 of [A7]. Assume that $K_f = \prod_{p < \infty} K_p$. For any place $v$ denote by $\mathcal{H}_P(\pi_v)$ the Hilbert space of the induced representation $I_{Q,P}^G(\pi_v)$. If $p < \infty$ let $R_{Q,P}(\pi_p, \lambda)_{K_p}$ be the restriction of $R_{Q,P}(\pi_p, \lambda)$ to the subspace of $K_p$-invariant vectors $\mathcal{H}_P(\pi_p)^{K_p}$ in $\mathcal{H}_P(\pi_p)$. Let $R_{Q,P}(\pi_\infty, \lambda)_\sigma$ denote the restriction of $R_{Q,P}(\pi_\infty, \lambda)$ to the $\sigma$-isotypical subspace of $I_{Q,P}^G(\pi_\infty)$ in $\mathcal{H}_P(\pi_\infty)$. It was proved in [Mu2, (6.24)] that there exist a finite set of places $S_0$, including the Archimedean one, and constants $C > 0$ and $q \in \N$, such that

$$||\mathfrak{N}'_S(P, \pi, \lambda)_{K_f, \sigma}|| \leq C \left( \sum_{p \in S_0 \setminus \{\infty\}} \sum_{k=1}^q \left\| D_k^q R_{Q,P}(\pi_p, \lambda)_{K_p} \right\| \right)^{1/2},$$

for all $\lambda \in i \mathfrak{a}_M^* \cap \Pi(K_\infty)$ and $\pi \in \Pi(M(\A))$. By Proposition 0.2 of [MS], there exists $C > 0$ such that

$$\tag{6.14} ||\mathfrak{N}'_S(P, \pi, \lambda)_{K_f, \sigma}|| \leq C$$

for all $\lambda \in i \mathfrak{a}_M^* \in \Pi_{\text{dis}}(M(\A)^1)$. Observe that $M_{P|P}(s,0)$ is unitary. Let

$$l = \dim(\mathcal{A}_L/\mathcal{A}_G).$$

Using (6.13), (6.14) and Proposition 5.5 it follows that there exists $C > 0$ such that

$$|J_{M,P}^L(\phi^1_t, s)| \leq C \left( \frac{(2 + |\log t|)^l}{t^{l/2}} \right) \cdot \sum_{\pi \in \Pi_{\text{dis}}(M(\A)^1)} \dim \mathcal{A}_\pi^2(P)_{K_f, \sigma} (\log(1 + |\lambda_\pi|))^{n_s}\, e^{i \lambda_\pi},$$

where $n_s = n_s(\pi_v)$. By (6.7),

$$J_{M,P}^L(\phi^1_t, s) = \sum_{\pi \in \Pi_{\text{dis}}(M(\A)^1)} e^{i \lambda_\pi} \cdot \sum_{S \in \mathcal{F}(L)} e^{-t||\lambda||^2} \nu_L^S(P, \pi, \lambda) \text{tr}(M_P|P(s,0)\mathfrak{N}'_S(P, \pi, \lambda)_{K_f, \sigma}) \, d\lambda.$$
for all \(0 < t \leq 1\). The series can be estimated using Proposition 3.5. Let \(X_M = M(\mathbb{R})/K_{M,\infty}\) and let \(m = \dim X_M\). It follows from Proposition 3.5 that for every \(\epsilon > 0\) there exists \(C > 0\) such that the series is bounded by \(C t^{-m/2-\epsilon}\) for \(0 < t \leq 1\). This together with (6.15) yields the following proposition.

**Proposition 6.1.** Let \(m = \dim X_M\) and \(l = \dim A_L/A_G\). For every \(\epsilon > 0\) there exists \(C > 0\) such that

\[
|J_{M,P}^l(\phi^1_t, s)| \leq C t^{-(m+l)/2-\epsilon}
\]

for all \(0 < t \leq 1\).

Now we distinguish two cases. First assume that \(M = G\). Then \(L = P = G\) and \(s = 1\). Let \(R^1_{\text{dis}}\) be the restriction of the regular representation \(R^1\) of \(G(\mathbb{A})^1\) in \(L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))\) to the discrete subspace. Then \(J_{G,G}^G(\phi^1_t, 1) = \text{Tr} R^1_{\text{dis}}(\phi^1_t)\). Let \(R_{\text{dis}}\) be the regular representation of \(G(\mathbb{A})\) in

\[
L^2_{\text{dis}}(AG(\mathbb{R})^0G(\mathbb{Q})\backslash G(\mathbb{A})).
\]

Then the operator \(R_{\text{dis}}(\phi_t)\) is isomorphic to \(R^1_{\text{dis}}(\phi^1_t)\). Thus

\[
J_{G,G}^G(\phi^1_t, 1) = \text{Tr} R_{\text{dis}}(\phi_t).
\]

Given \(\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)\), let \(m(\pi)\) denote the multiplicity with which \(\pi\) occurs in the regular representation of \(G(\mathbb{A})\) in \(L^2(A_G(\mathbb{R})^0G(\mathbb{Q})\backslash G(\mathbb{A}))\). Then using Corollary 2.2 in [BM] we get

\[
J_{G,G}^G(\phi^1_t, 1) = \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\pi) \dim(\mathcal{H}^{K_{\pi_f}}_{\pi_f}) \dim(\mathcal{H}_{\pi_\infty} \otimes V_\sigma)^{O(n)} e^{tl_\pi}.
\]

Now assume that \(M \neq G\) is a proper Levi subgroup. Let \(P = MN\). Let \(X = G(\mathbb{R})^1/K_{\infty}\). Then

\[
X \cong X_M \times A_M(\mathbb{R})^0 / A_G(\mathbb{R})^0 \times N(\mathbb{R}).
\]

Since \(l = \dim A_L/A_G \leq \dim A_M/A_G\), it follows that \(m + l \leq \dim X - 1\). Thus, using this together with Proposition 6.1, we get

**Theorem 6.2.** Let \(d = \dim X\). For every open compact subgroup \(K_f\) of \(G(A_f)\) and every \(\sigma \in \Pi(O(n))\) the spectral side of the trace formula, evaluated at \(\phi^1_t\), satisfies

\[
J_{\text{spec}}(\phi^1_t) = \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\pi) \dim(\mathcal{H}^{K_{\pi_f}}_{\pi_f}) \dim(\mathcal{H}_{\pi_\infty} \otimes V_\sigma)^{O(n)} e^{tl_\pi} + O(t^{-(d-1)/2})
\]

as \(t \to 0^+\).
This theorem can be restated in a slightly different way as follows. There exist arithmetic subgroups \( \Gamma_i \subset G(\mathbb{R}) \), \( i = 1, \ldots, m \), such that

\[
A_G(\mathbb{R})^0 G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f \cong \bigcap_{i=1}^{m} (\Gamma_i \backslash G(\mathbb{R}))^1
\]

(cf. [Mu1, §9]). Let \( \Delta_{\sigma,i} \) be the operator induced by the negative of the Casimir operator in \( C^\infty(\Gamma_i \backslash G(\mathbb{R})^1, \sigma) \), \( i = 1, \ldots, m \). Let

\[
\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots
\]

be the \( L^2 \)-eigenvalues of \( \Delta_\sigma = \bigoplus_{i=1}^{m} \Delta_{\sigma,i} \), where each eigenvalue is counted with its multiplicity. Let \( d = \dim X \). If we proceed in the same way as in the proof of Lemma 3.2, then it follows that (6.17) is equivalent to

\[
J_{\text{spec}}(\phi_1^f) = \sum_i e^{-t\lambda_i} + O(t^{-(d-1)/2})
\]
as \( t \to 0^+ \).

Let \( \Gamma(N) \subset \text{SL}_n(\mathbb{Z}) \) be the principal congruence subgroup of level \( N \). Let \( \mu_0 \leq \mu_1 \leq \cdots \) be the eigenvalues, counted with multiplicity, of \( \Delta_\sigma \) acting in \( L^2(\Gamma(N) \backslash \text{SL}_n(\mathbb{R}), \sigma) \). Then it follows from (6.18) and (3.10) that

\[
J_{\text{spec}}(\phi_1^f) = \varphi(N) \sum_i e^{-t\mu_i} + O(t^{-(d-1)/2})
\]
as \( t \to 0^+ \).

Our next purpose is to study \( J_{\text{spec}} \) as a functional on the Schwartz space. Let \( K_f \) be an open compact subgroup of \( G(\mathbb{A}_f) \) and let \( \sigma \in \Pi(K_\infty) \). Denote by \( C^1(G(\mathbb{A})^1; K_f, \sigma) \) the set of all \( h \in C^1(G(\mathbb{A})^1) \) which are bi-invariant under \( K_f \) and transform under \( K_\infty \) according to \( \sigma \). Let \( \Delta_G \) be the Laplace operator of \( G(\mathbb{R})^1 \). Then we have

**Proposition 6.3.** For every open compact subgroup \( K_f \) of \( G(\mathbb{A}_f) \) and every \( \sigma \in \Pi(K_\infty) \) there exist \( C > 0 \) and \( k \in \mathbb{N} \) such that

\[
|J_{\text{spec}}(f)| \leq C\|\text{Id} + \Delta_G\|^k f\|_{L^1(G(\mathbb{A})^1)}
\]

for all \( f \in C^1(G(\mathbb{A})^1; K_f, \sigma) \).

**Proof.** This follows essentially from the proof of Theorem 0.2 in [Mu2] combined with Proposition 0.2 of [MS]. We include some details. Let \( M \in \mathcal{L} \), \( L \in \mathcal{L}(M) \) and \( P \in \mathcal{P}(M) \). By (6.3) it suffices to estimate \( J_{M,L}^L(f,s) \). Since \( M_{\mathcal{P}}(s,0) \) is unitary, it follows from (6.2) that

\[
|J_{M,L}^L(f,s)| \leq \sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1)} \int_{\mathfrak{a}_\mathbb{Z}/\mathfrak{a}_0^*} \|\mathfrak{M}_L(P,\lambda)\rho_{\pi}(P,\lambda,f)\|_1d\lambda,
\]
where $\| \cdot \|_1$ denotes the trace norm for operators in the Hilbert space $\mathcal{F}_\pi^2(P)$. By (6.12) it follows that the right-hand side is bounded by

$$\sum_{\pi \in \Pi_{\text{dis}}(M(A))} \int_{i\mathfrak{a}^*_L/\mathfrak{a}^*_G} \| \mathcal{N}_S^\prime(P, \pi, \lambda) \rho_\pi(P, \lambda, f)\|_1 |\mathcal{N}_L^\prime(P, \pi, \lambda)| \, d\lambda.$$ 

The function $\mathcal{N}_L^\prime(P, \pi, \lambda)$ can be estimated by Theorem 5.4 of [Mu2]. This reduces our problem to the estimation of the trace norm of the operator $\mathcal{N}_S^\prime(P, \pi, \lambda) \rho_\pi(P, \lambda, f)$. Let $K_f$ be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\sigma \in \Pi(K_{\infty})$. Denote by $\Pi_{K_f, \sigma}$ the orthogonal projection of the Hilbert space $\mathcal{F}_\pi^2(P)$ onto the finite-dimensional subspace $\mathcal{A}_\pi^2(P)_{K_f, \sigma}$. Let $f \in C^1(G(\mathbb{A})^1; K_f, \sigma)$. Then

$$\rho_\pi(P, \lambda, f) = \Pi_{K_f, \sigma} \circ \rho_\pi(P, \lambda, f) \circ \Pi_{K_f, \sigma}$$

for all $\pi \in \Pi(M(A))$. Let

$$D = \text{Id} + \Delta_G.$$ 

For any $k \in \mathbb{N}$ let $\rho_\pi(P, \lambda, D^{2k})_{K_f, \sigma}$ denote the restriction of the operator $\rho_\pi(P, \lambda, D^{2k})$ to the subspace $\mathcal{A}_\pi^2(P)_{K_f, \sigma}$. Then

$$\| \mathcal{N}_S^\prime(P, \pi, \lambda) \rho_\pi(P, \lambda, f)\|_1 \leq \| \mathcal{N}_S^\prime(P, \pi, \lambda)_{K_f, \sigma}\| \cdot \| \rho_\pi(P, \lambda, D^{2k})^{-1}_{K_f, \sigma}\| \cdot \| \rho_\pi(P, \lambda, D^{2k} f)\|.$$ 

By (6.9) of [Mu2] we get

$$\| \rho_\pi(P, \lambda, D^{2k})^{-1}_{K_f, \sigma}\| \leq C \frac{\dim \mathcal{A}_\pi^2(P)_{K_f, \sigma}}{(1 + \|\lambda\|^2 + \lambda_\pi^2)^{k/2}},$$

and since $\rho_\pi(P, \lambda)$ is unitary, we have

$$\| \rho_\pi(P, \lambda, D^{2k} f)\| \leq \| D^{2k} f\|_{L^1(G(\mathbb{A}))^1}.$$ 

This, together with (6.14), gives $C > 0$ such that

$$\| \mathcal{N}_S^\prime(P, \pi, \lambda) \rho_\pi(P, \lambda, f)\|_1 \leq C \| D^{2k} f\|_{L^1(G(\mathbb{A}))^1} (1 + \|\lambda\|)^{-k/2} \frac{\dim \mathcal{A}_\pi^2(P)_{K_f, \sigma}}{(1 + \lambda_\pi^2)^{k/2}}$$

for all $\lambda \in i\mathfrak{a}^*_f$ and $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$. Let $d = \dim G(\mathbb{R})^1/\Gamma_\infty$. By Theorem 5.4 of [Mu2] there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ we have

$$\int_{i\mathfrak{a}^*_L/\mathfrak{a}^*_G} |\mathcal{N}_L^\prime(P, \pi, \lambda)|(1 + \|\lambda\|)^{-k/2} \, d\lambda \leq C_k (1 + \lambda_\pi^2)^{8d^2}$$

for all $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$ with $\mathcal{A}_\pi^2(P)_{K_f, \sigma} \neq 0$. Furthermore, by Proposition 3.4,

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \frac{\dim \mathcal{A}_\pi^2(P)_{K_f, \sigma}}{(1 + \lambda_\pi^2)^{k/2}} < \infty.$$
for $k > m/2 + 1$, where $m = \dim M(\mathbb{R})^1/K_{M,\infty}$. Combining (6.23)–(6.25), shows that for each $k > m/2 + 16d^2 + 1$ there exists $C_k > 0$ such that

$$
\sum_{\pi \in \Pi_{\text{dist}}(M(\lambda)^1)} \int_{i\alpha_L/i\alpha_G} \|\Pi^r_S(P, \pi, \lambda) \rho_x(P, \lambda, f)\|_1 \|\nu^{S}_L(P, \pi, \lambda)\| \, d\lambda 
\leq C_k \|D^{2k}f\|_{L^1(G(\lambda)^1)}.
$$

This completes the proof.

Now we return to the function $\phi_t$ defined by (6.5). It follows from the definition that the restriction $\phi^1_t$ of $\phi_t$ to $G(\mathbb{A})^1$ belongs to $C^1(G(\mathbb{A})^1, K_f, \sigma)$. We shall now modify $\phi_t$ in the following way. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(u) = 1$, if $|u| \leq 1/2$, and $\varphi(u) = 0$, if $|u| \geq 1$. Let $d(x, y)$ denote the geodesic distance of $x, y \in \mathcal{X}$ and set

$$
r(g_\infty) := d(g_\infty K_\infty, K_\infty).
$$

Given $t > 0$, let $\varphi_t \in C_0^\infty(G(\mathbb{R})^1)$ be defined by

$$
\varphi_t(g_\infty) = \varphi(r^2(g_\infty)/t^{1/2}).
$$

Then $\supp \varphi_t$ is contained in the set $\{g_\infty \in G(\mathbb{R})^1 \mid r(g_\infty) < t^{1/4}\}$. Extend $\varphi_t$ to $G(\mathbb{R})$ by

$$
\varphi_t(g_\infty z) = \varphi_t(g_\infty), \quad g_\infty \in G(\mathbb{R})^1, \quad z \in A_G(\mathbb{R})^0,
$$

and then to a function on $G(\mathbb{A})$ by multiplying $\varphi_t$ by the characteristic function of $K_f$. Put

$$
(6.26) \quad \tilde{\phi}_t(g) = \varphi_t(g) \phi_t(g), \quad g \in G(\mathbb{A}).
$$

Then the restriction $\tilde{\phi}^1_t$ of $\tilde{\phi}_t$ to $G(\mathbb{A})^1$ belongs to $C^c_c(G(\mathbb{A})^1)$.

**Proposition 6.4.** There exist $C, c > 0$ such that

$$
|J_{\text{spec}}(\phi^1_t) - J_{\text{spec}}(\tilde{\phi}^1_t)| \leq Ce^{-c/\sqrt{t}}
$$

for $0 < t \leq 1$.

**Proof.** Let $\psi_t = \phi_t - \tilde{\phi}_t$ and $f_t = 1 - \varphi_t$. Let $\psi^1_t$ denote the restriction of $\psi_t$ to $G(\mathbb{A})^1$. Then by Proposition 6.3 there exists $k \in \mathbb{N}$ such that

$$
|J_{\text{spec}}(\phi^1_t) - J_{\text{spec}}(\tilde{\phi}^1_t)| = |J_{\text{spec}}(\psi^1_t)| \leq C_k \|(\text{Id} + \Delta G)^k \psi^1_t\|_{L^1(G(\mathbb{A})^1)}.
$$

In order to estimate the $L^1$-norm of $\psi^1_t$, recall that by definition

$$
\psi_t(g_\infty g_f) = f_t(g_\infty) h^*_q(g_\infty) \chi_{K_f}(g_f).
$$

Hence

$$
\|(\text{Id} + \Delta G)^k \psi^1_t\|_{L^1(G(\mathbb{A})^1)} = \|(\text{Id} + \Delta G)^k (f_t h^*_q)\|_{L^1(G(\mathbb{R})^1)}.
$$

Let $\mathfrak{g}(\mathbb{R})^1$ be the Lie algebra of $G(\mathbb{R})^1$ and let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}(\mathbb{R})^1$. Then $\Delta_G = -\sum X_i^2$. Denote by $\nabla$ the canonical connection on $G(\mathbb{R})^1$. Then it follows that there exists $C > 0$ such that
\[
|(\text{Id} + \Delta_G)^k f(g)| \leq C \sum_{i=0}^{2k} \|\nabla^i f(g)\|, \quad g \in G(\mathbb{R})^1,
\]
for all $f \in C^\infty(G(\mathbb{R})^1)$. By Proposition 2.1 there exist constants $C, c > 0$ such that
\[
\|\nabla^j h_t^g(g)\| \leq Ct^{-a/2} e^{-cr^2(g)/t}, \quad g \in G(\mathbb{R})^1,
\]
for $j \leq 2k$ and $0 < t \leq 1$. Let $\chi_t$ be the characteristic function of the set $\mathbb{R} - (-t^{1/4}, t^{1/4})$. Recall that $f_t(g) = (1 - \varphi)(r^2(g)/t^{1/2})$ and $(1 - \varphi)(u)$ is constant for $|u| \geq 1$. This implies that there exist constants $C, c > 0$ such that
\[
\|\nabla^j f_t(g)\| \leq Ct^{-k}\chi_t(r(g)), \quad g \in G(\mathbb{R})^1,
\]
for $j \leq 2k$ and $0 < t \leq 1$. Combining (6.27) and (6.28) we obtain
\[
\sum_{l=0}^{2k} \|\nabla^l (f_t h_t^g(g))\| \leq C_1 t^{-a/2 - 2k} \chi_t(r(g)) e^{-cr^2(g)/t} \leq C_2 e^{-c_1/\sqrt{t}} e^{-cr^2(g)}
\]
for all $g \in G(\mathbb{R})^1$ and $0 < t \leq 1$. Finally note that for every $c > 0$, $e^{-cr^2(g)}$ is an integrable function on $G(\mathbb{R})^1$. This finishes the proof.

7. Proof of the main theorem

In this section we evaluate the geometric side of the trace formula at the function $\tilde{\phi}_t^1$ and investigate its asymptotic behaviour as $t \to 0$. Then we compare the geometric and the spectral sides and prove our main theorem.

Let us briefly recall the structure of the geometric side $J_{\text{geo}}$ of the trace formula [A1]. The coarse $\sigma$-expansion of $J_{\text{geo}}(f)$ is a sum of distributions
\[
J_{\text{geo}}(f) = \sum_{\sigma \in \mathcal{O}} J_\sigma(f), \quad f \in C^\infty_c(G(\mathbb{A})^1),
\]
which are parametrized by the set $\mathcal{O}$ of conjugacy classes of semisimple elements in $G(\mathbb{Q})$. The distributions $J_\sigma(f)$ are defined in [A1]. We shall use the fine $\sigma$-expansion of the spectral side [A10] which expresses the distributions $J_\sigma(f)$ in terms of weighted orbital integrals $J_M(\gamma, f)$. To describe the fine $\sigma$-expansion we have to introduce some notation. Suppose that $S$ is a finite set of valuations of $\mathbb{Q}$. Set
\[
G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(\mathbb{A})^1,
\]
where
\[ Q_S = \prod_{v \in S} Q_v. \]

Suppose that \( \omega \) is a compact neighborhood of 1 in \( G(\mathbb{A})^1 \). There is a finite set \( S \) of valuations of \( \mathbb{Q} \), which contains the Archimedean place, such that \( \omega \) is the product of a compact neighborhood of 1 in \( G(\mathbb{Q}_S)^1 \) with \( \prod_{v \in S} K_v \). Let \( S_\omega^0 \) be the minimal such set. Let \( C^\infty_\omega(G(\mathbb{A})^1) \) denote the space of functions in \( C^\infty(G(\mathbb{A})^1) \) which are supported on \( \omega \). For any finite set \( S \supset S_\omega^0 \) set
\[ C^\infty_\omega(G(\mathbb{Q}_S)^1) = C^\infty_\omega(G(\mathbb{A})^1) \cap C^\infty_\omega(G(\mathbb{Q}_S)^1). \]

Let us recall the notion of \((M,S)\)-equivalence [A10, p. 205]. For any \( \gamma \in M(\mathbb{Q}) \) denote by \( \gamma_s \) (resp. \( \gamma_u \)) the semisimple (resp. unipotent) Jordan component of \( \gamma \). Then two elements \( \gamma \) and \( \gamma' \) in \( M(\mathbb{Q}) \) are called \((M,S)\)-equivalent if there exists \( \delta \in M(\mathbb{Q}) \) with the following two properties.

(i) \( \gamma_s \) is also the semisimple Jordan component of \( \delta^{-1} \gamma \delta \).

(ii) \( \gamma_u \) and \( (\delta^{-1} \gamma \delta)_u \), regarded as unipotent elements in \( M(\gamma_u,\mathbb{Q}_S) \), are \( M(\gamma_u,\mathbb{Q}_S) \)-conjugate.

Denote by \( (M(\mathbb{Q}))_{M,S} \) the set of \((M,S)\)-equivalence classes in \( M(\mathbb{Q}) \). Note that \((M,S)\)-equivalent elements \( \gamma \) and \( \gamma' \) in \( M(\mathbb{Q}) \) are, in particular, \((M,S)\)-conjugate. Given \( \gamma \in M(\mathbb{Q}) \), let
\[ J_M(\gamma,f), \quad f \in C^\infty_\omega(G(\mathbb{Q}_S)^1), \]
be the weighted orbital integral associated to \( M \) and \( \gamma \) [A11]. We observe that \( J_M(\gamma,f) \) depends only on the \( M(\mathbb{Q}_S) \)-orbit of \( \gamma \). Then by Theorem 9.1 of [A10] there exists a finite set \( S_\omega \supset S_\omega^0 \) of valuations of \( \mathbb{Q} \) such that for all \( S \supset S_\omega \) and any \( f \in C^\infty_\omega(G(\mathbb{Q}_S)^1) \),
\[ J_{\text{geo}}(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))_{M,S}} a^M(S,\gamma) J_M(\gamma,f). \]

This is the fine \( \sigma \)-expansion of the geometric side of the trace formula. The interior sum is finite.

Recall that the restriction \( \tilde{\phi}_t^1 \) of \( \tilde{\phi}_t \) to \( G(\mathbb{A})^1 \) belongs to \( C^\infty_\omega(G(\mathbb{A})^1) \) and hence, \( J_{\text{geo}} \) can be evaluated at \( \tilde{\phi}_t^1 \). By construction of \( \tilde{\phi}_t^1 \) there exists a compact neighborhood \( \omega \) of 1 in \( G(\mathbb{A})^1 \) and a finite set \( S \supset S_\omega \) of valuations of \( \mathbb{Q} \) such that
\[ \tilde{\phi}_t^1 \in C^\infty_\omega(G(\mathbb{Q}_S)^1), \quad 0 < t \leq 1. \]

Hence we can apply (7.1) to evaluate \( J_{\text{geo}}(\tilde{\phi}_t^1) \). In this way our problem is reduced to the investigation of the weighted orbital integrals \( J_M(\gamma,\tilde{\phi}_t^1) \). Actually for \( \gamma \in M(\mathbb{Q}) \) we may replace \( \tilde{\phi}_t^1 \) by \( \tilde{\phi}_t \).
To begin with we establish some auxiliary results. Given \( h \in G(\mathbb{R}) \), let
\[
C_h = \{ g^{-1}hg \mid g \in G(\mathbb{R}) \}
\]
be the conjugacy class of \( h \) in \( G(\mathbb{R}) \).

**Lemma 7.1.** Let \( k \in K_\infty \). Then \( C_k \cap K_\infty \) is the \( K_\infty \)-conjugacy class of \( k \).

**Proof.** Let \( g \) and \( \mathfrak{k} \) denote the Lie algebras of \( G(\mathbb{R}) \) and \( K_\infty \), respectively. Let \( \theta \) be a Cartan involution of \( g \) with fixed point set \( \mathfrak{k} \) and let \( \mathfrak{p} \) be the \((-1)\)-eigenspace of \( \theta \). Then the map
\[
(k', X) \in K_\infty \times \mathfrak{p} \longrightarrow k' \exp(X) \in G(\mathbb{R})
\]
is an analytic isomorphism of analytic manifolds. If \( k_1 \in K_\infty \), then \( k_1 \) is a \( \theta \)-invariant semisimple element. Therefore, its centralizer \( G_{k_1} \) is a reductive subgroup and the restriction of \( \theta \) to \( G_{k_1} \) is a Cartan involution. Thus the restriction of the above Cartan decomposition to the centralizer of \( k_1 \) yields a Cartan decomposition of \( G_{k_1}(\mathbb{R}) \). Let \( g \in G(\mathbb{R}) \) such that \( g^{-1}kg \in K_\infty \). Write \( g = k' \exp(X) \) with \( k' \in K_\infty \) and \( X \in \mathfrak{p} \). Since \( g^{-1}kg \) is \( \theta \)-invariant, we get
\[
\exp(-X)k'^{-1}kk' \exp(X) = \exp(X)k'^{-1}k' \exp(-X).
\]
Hence \( \exp(2X) \in G_{k'^{-1}kk'}(\mathbb{R}) \). From the Cartan decomposition of the latter group we conclude that \( \exp(2X) = \exp(Y) \) for some \( Y \in \mathfrak{p} \), and hence \( X \in \mathfrak{p} \). This implies that \( g^{-1}kg = k'^{-1}kk' \).

It follows from Lemma 7.1 that \( C_k \cap K_\infty \) is a submanifold of \( C_k \).

**Lemma 7.2.** Let \( k \in K_\infty - \{ \pm 1 \} \). Then \( C_k \cap K_\infty \) is a proper submanifold of \( C_k \).

**Proof.** Let the notation be as in the previous lemma. First note that the tangent space of \( C_k \) at \( k \) is given by
\[
T_k C_k \cong (\text{Ad}(k) - \text{Id})(\mathfrak{g}).
\]
Furthermore
\[
\text{Ad}(k)(\mathfrak{k}) \subset \mathfrak{k}, \quad \text{Ad}(k)(\mathfrak{p}) \subset \mathfrak{p}.
\]
Hence we get
\[
T_k(C_k \cap K_\infty) = T_k C_k \cap \mathfrak{k} = (\text{Ad}(k) - \text{Id})(\mathfrak{k}),
\]
and so the normal space \( N_k \) to \( C_k \cap K_\infty \) in \( C_k \) at \( k \) is given by
\[
N_k \cong (\text{Ad}(k) - \text{Id})(\mathfrak{p}).
\]
Suppose that \( \text{Ad}(k) = \text{Id} \) on \( \mathfrak{p} \). Since \( \mathfrak{k} = [\mathfrak{p}, \mathfrak{p}] \), it follows that \( \text{Ad}(k) = \text{Id} \) on \( \mathfrak{g} \). Hence \( k \) belongs to the center of \( G^0 \), which implies that \( k = \pm 1 \). Thus if \( k \neq \pm 1 \), we have \( \dim N_k > 0 \). \( \square \)
Next we recall the notion of an induced space of orbits [A11, p. 255].

Given an element \( \gamma \in M(\mathbb{Q}_S) \), let \( \gamma^G \) be the union of those conjugacy classes in \( G(\mathbb{Q}_S) \) which for any \( P \in \mathcal{P}(M) \) intersect \( \gamma N_P(\mathbb{Q}_S) \) in an open set. There are only finitely many such conjugacy classes.

**Proposition 7.3.** Let \( d = \dim G(\mathbb{R})/K_\infty \). Let \( M \in \mathcal{L} \) and \( \gamma \in M(\mathbb{Q}) \). Then

\[
\lim_{t \to 0} t^{d/2} J_M(\gamma, \widetilde{\phi}_t) = 0
\]

if either \( M \neq G \), or \( M = G \) and \( \gamma \neq \pm 1 \).

**Proof.** By Corollary 6.2 of [A11] the distribution \( J_M(\gamma, \widetilde{\phi}_t) \) is given by the integral of \( \widehat{\phi}_t \) over \( \gamma^G \) with respect to a measure \( d\mu \) on \( \gamma^G \) which is absolutely continuous with respect to the invariant measure class. Thus \( J_M(\gamma, \widetilde{\phi}_t) \) is equal to a finite sum of integrals of the form

\[
\int_{G_{\gamma_n}(\mathbb{Q}_S)) \setminus G(\mathbb{Q}_S)} \widetilde{\phi}_t(g^{-1}\gamma ng) d\mu(g),
\]

where \( n \in N_P(\mathbb{Q}_S) \) for some \( P \in \mathcal{P}(M) \). Now recall that by (6.5) and (6.26), \( \widetilde{\phi}_t(g) \) is the product of \( \varphi_t(g_\infty) h_t^g(g_\infty) \) with \( \chi_{K_f}(g_f) \) for any \( g = g_\infty g_f \). Hence our problem is reduced to the investigation of the integral

\[
\int_{G_{\gamma_n}(\mathbb{R}) \setminus G(\mathbb{R})} (\varphi_t h_t^g)(g^{-1}_\infty\gamma n_\infty g_\infty) d\mu(g_\infty).
\]

Furthermore, by Proposition 2.1 there exists \( C > 0 \) such that

\[
|h_t^g(g_\infty)| \leq C t^{-d/2}, \quad 0 < t \leq 1.
\]

Hence it suffices to show that

\[
\lim_{t \to 0} \int_{G_{\gamma_n}(\mathbb{R}) \setminus G(\mathbb{R})} \varphi_t(g^{-1}_\infty\gamma n_\infty g_\infty) d\mu(g_\infty) = 0
\]

if either \( M \neq G \), or \( M = G \) and \( \gamma \neq \pm 1 \).

By definition of \( \gamma^G \), the conjugacy class of \( \gamma n \) in \( G(\mathbb{Q}_S) \) has to intersect \( \gamma N_P(\mathbb{Q}_S) \) in an open subset. This implies that \( \gamma n_\infty \neq \pm 1 \), if either \( M \neq G \), or \( M = G \) and \( \gamma \neq \pm 1 \). Then it follows from Lemma 7.2 that \( C_{\gamma_n_\infty} \subset K_\infty \) is a proper submanifold of \( C_{\gamma_n} \). Being a proper submanifold, \( C_{\gamma_n_\infty} \setminus K_\infty \) is a subset of \( C_{\gamma_n} \) with measure zero with respect to \( dg \) and therefore, also with respect to \( d\mu \). Next observe that

\[
\int_{G_{\gamma_n}(\mathbb{R}) \setminus G(\mathbb{R})} \varphi_t(g^{-1}_\infty\gamma n_\infty g_\infty) |f(g_\infty)| \, dg_\infty < \infty.
\]

Since \( \text{supp} \varphi_t \subset \text{supp} \varphi_t \) for \( t' < t \), and \( 0 \leq \varphi_t \leq 1 \) for all \( t > 0 \), there exists \( C > 0 \) such that

\[
\left| \int_{G_{\gamma_n}(\mathbb{R}) \setminus G(\mathbb{R})} \varphi_t(g^{-1}_\infty\gamma n_\infty g_\infty) d\mu(g_\infty) \right| \leq C
\]
for all $0 < t \leq 1$. Furthermore by definition of $\varphi_t$ we have
\[
\lim_{t \to 0} \varphi_t(x) = 0
\]
for all $x \in C_{\gamma_{n,\infty}} - (C_{\gamma_{n,\infty}} \cap K_{\infty})$. Since $C_{\gamma_{n,\infty}} \cap K_{\infty}$ has measure zero with respect to $d\mu$, (7.2) follows by the dominated convergence theorem. 

We can now state the main result of this section.

**Theorem 7.4.** Let $d = \dim G(\mathbb{R})^1 / K_{\infty}$, let $K_f$ be an open compact subgroup of $G(\mathbb{A})$ and let $\sigma \in \Pi(O(n))$ such that $\sigma(-1) = \text{Id}$ if $-1 \in K_f$. Then
\[
\lim_{t \to 0} t^{d/2} J_{\text{geo}}(\tilde{\phi}_t^1) = \frac{\dim(\sigma)}{(4\pi)^{d/2}} \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1) (1 + 1_{K_f}(-1)).
\]

**Proof.** By (7.1) and Proposition 7.3 it follows that
\[
\lim_{t \to 0} t^{d/2} J_{\text{geo}}(\tilde{\phi}_t^1) = \lim_{t \to 0} t^{d/2} (a^G(S, 1) \tilde{\phi}_t^1(1) + a^G(S, -1) \tilde{\phi}_t^1(-1)).
\]
By Theorem 8.2 of [A10] we have
\[
a^G(S, \pm 1) = \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1).
\]
Furthermore
\[
\tilde{\phi}_t^1(\pm 1) = h_t^\sigma(\pm 1) \chi_{K_f}(\pm 1).
\]
Since $\sigma$ satisfies $\sigma(-1) = \text{Id}$, if $-1 \in K_f$, it follows from (2.5) that $h_t^\sigma(-1) = h_t^\sigma(1)$. Finally, by Lemma 2.3 we have
\[
h_t^\sigma(\pm 1) = \frac{\dim(\sigma)}{(4\pi)^{d/2}} t^{-d/2} + O(t^{-(d-1)/2})
\]
as $t \to 0$. This combined with $\chi_{K_f}(\pm 1) = 1_{K_f}(\pm 1) \text{vol}(K_f)^{-1}$, proves the theorem. 

We shall now use the trace formula to prove the main results of this paper. Recall that the coarse trace formula is the identity
\[
J_{\text{spec}}(f) = J_{\text{geo}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),
\]
between distributions on $G(\mathbb{A})^1$ [A1]. Applied to $\tilde{\phi}_t^1$ we get the equality
\[
J_{\text{spec}}(\tilde{\phi}_t^1) = J_{\text{geo}}(\tilde{\phi}_t^1), \quad t > 0.
\]
Put $\varepsilon_{K_f} = 1$, if $-1 \in K_f$ and $\varepsilon_{K_f} = 0$ otherwise. Combining Theorem 6.2, Proposition 6.4 and Theorem 7.4, we obtain
\[
\sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\pi) \dim(\mathcal{H}_{K_f}^\pi) \dim(\mathcal{H}_{\pi_{\infty}} \otimes V_{\sigma}) O(n) e^{i\lambda_\pi}
\]

(7.3)

\[
\sim \frac{\dim(\sigma)}{(4\pi)^{d/2}} \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1 / K_f) (1 + \varepsilon_{K_f}) t^{-d/2}
\]
as $t \to 0$. Applying Karamat’s theorem [Fe, p. 446], we obtain

$$
\sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\pi) \dim(\mathcal{H}^{K_f}_\pi) \dim(\mathcal{H}_\pi \otimes V_\sigma)^{O(n)} \sim \dim(\mathcal{O}) \cdot \dim(\sigma) \cdot \frac{\text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1 / K_f)}{(4\pi)^{d/2} \Gamma(d/2 + 1)} \lambda^{d/2}
$$

as $\lambda \to \infty$. By Lemma 3.3 it follows that this asymptotic formula continues to hold if we replace the sum over $\Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)$ by the sum over $\Pi_{\text{cus}}(G(\mathbb{A}), \xi_0)$. Finally note that by [Sk] we have $m(\pi) = 1$ for all $\pi \in \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0)$. This completes the proof of Theorem 0.2.

Now suppose that $K_f$ is the congruence subgroup $K(N)$ and $\Gamma(N) \subset \text{SL}_n(\mathbb{Z})$ the principal congruence subgroup of level $N$. Then by (3.10) we have

$$
\text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1 / K(N)) = \varphi(N) \text{vol}(\Gamma(N) \backslash \text{SL}_n(\mathbb{R})).
$$

Furthermore, $\varepsilon_{K(N)} = 1$ if and only if $-1 \in \Gamma(N)$. If $-1$ is contained in $\Gamma(N)$, then the fibre of the canonical map

$$
\Gamma(N) \backslash \text{SL}_n(\mathbb{R}) \to \Gamma(N) \backslash \text{SL}_n(\mathbb{R}) / \text{SO}(n)
$$

is equal to $\text{SO}(n)/\{\pm 1\}$. Otherwise the fibre is equal to $\text{SO}(n)$. We normalize the Haar measure on $\text{SL}_n(\mathbb{R})$ so that $\text{vol}(\text{SO}(n)) = 1$. Then in either case we have

$$
\text{vol}(\Gamma(N) \backslash \text{SL}_n(\mathbb{R}))(1 + \varepsilon_{K(N)}) = \text{vol}(\Gamma(N) \backslash \text{SL}_n(\mathbb{R}) / \text{SO}(n)).
$$

Let $X = \text{SL}_n(\mathbb{R}) / \text{SO}(n)$ and let $\lambda_0 \leq \lambda_1 \leq \cdots$ be the eigenvalues, counted with multiplicity, of the Bochner-Laplace operator $\Delta_\sigma$ acting in $L^2(\Gamma(N) \backslash \text{SL}_n(\mathbb{R}), \sigma)$. Combining (6.18), Proposition 6.4, Theorem 7.4 and the above observations, we get

$$
\sum_i e^{-t\lambda_i} = \dim(\sigma) \frac{\text{vol}(\Gamma(N) \backslash X)}{(4\pi)^{d/2}} t^{-d/2} + o(t^{-d/2})
$$

as $t \to 0$. Using again Karamata’s theorem [Fe, p. 446], we get

$$
N_{\text{dis}}^{\Gamma(N)}(\lambda, \sigma) = \dim(\sigma) \frac{\text{vol}(\Gamma(N) \backslash X)}{(4\pi)^{d/2} \Gamma(d/2 + 1)} \lambda^{d/2} + o(\lambda^{d/2})
$$

as $\lambda \to \infty$. By Proposition 3.6 it follows that the same asymptotic formula holds if we replace $N_{\text{dis}}^{\Gamma(N)}(\lambda, \sigma)$ by $N_{\text{cus}}^{\Gamma(N)}(\lambda, \sigma)$. This is exactly the statement of Theorem 0.1.
References


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