Annals of Mathematics, **165** (2007), 55–88

Bounds for polynomials with a unit discrete norm

By E. A. Rakhmanov*

Abstract

Let E be the set of N equidistant points in $(-1,1)$ and $\mathbb{P}_n(E)$ be the set of all polynomials P of degree $\le n$ with $\max\{|P(\zeta)|, \zeta \in E\} \le 1$. We prove that

$$K_{n,N}(x) = \max_{P\in\mathbb{P}_n(E)} |P(x)| \le C \log \frac{\pi}{\arctan\left(\frac{N}{n}\sqrt{r^2-x^2}\right)},$$
$$|x| \le r := \sqrt{1-n^2/N^2}$$

where $n < N$ and C is an absolute constant. The result is essentially sharp. Bounds for $K_{n,N}(z)$, $z \in \mathbb{C}$, uniform for $n < N$, are also obtained.

The method of proof of those results is a general one. It allows one to obtain sharp, or sharp up to a $\log N$ factor, bounds for $K_{n,N}$ under rather general assumptions on E ($\#E = N$). A "model" result is announced for a class of sets E. Main components of the method are discussed in some detail in the process of investigating the case of equally spaced points.

1. Introduction

Let N be a natural number, E be the set of N equidistant points in $\Delta = [-1, 1]$; that is

$$E = \{\zeta_k = -1 + (2k-1)/N, k = 1, 2, \ldots, N\}\,. \tag{1.1}$$

Let, further, $n < N$ and $\mathbb{P}_n$ be the set of all polynomials of degree $\le n$. We denote $\|f\|_E = \max_{\zeta\in E} |f(\zeta)|$ and, then,

$$K_{n,N}(x) = \max_{P\in\mathbb{P}_n} \frac{|P(x)|}{\|P\|_E}, \quad x \in \mathbb{C}. \tag{1.2}$$

The main purpose of the paper is to derive "optimal" estimates for $K_{n,N}(x)$ and to discuss in some detail our method in more general settings.

*Research supported by U.S. National Science Foundation under grant DMS-9801677.

1.1. *Main results of the paper.* We note that estimates for $K_{n,N}(x)$ may potentially have a large circle of applications. We single out one immediate application in approximation theory.

Suppose one wants to recover a smooth function $f(x)$, $x \in \Delta$, from its values $f(\zeta_k)$, $k = 1, \ldots, N$, using a polynomial $P(x) \in \mathbb{P}_n$ of the best (say, discrete uniform or least square) approximation to $f|_E$. Then such a polynomial $P_n(x)$ will be close to $f(x)$ at all points $x \in \Delta$ where $K_{n,N}(x)$ is not large. Thus, the function $K_{n,N}(x)$ plays a role similar to the one the Lebesgue function plays in interpolation. It is important, then, for a given $x \in \Delta$ to find the conditions on n, N under which $K_{n,N}(x)$ is bounded. Clearly, n has to be, in a sense, small with respect to N.

The criterion is known of the boundedness of a related quantity

$$K_{n,N} = \|K_{n,N}(x)\|_\Delta = \max_{x \in \Delta} |K_{n,N}(x)|$$

which is similar to the Lebesgue constant. Namely, $K_{n,N}$ is bounded if and only if n^2/N is bounded. We mention briefly the main steps towards the proof of this criterion. First, if $n^2/N \le 1 - \epsilon$, $\epsilon > 0$, then $K_{n,N} \le 1/\epsilon$ and this fact is a direct corollary of Markov's inequality $\|P'\|_\Delta \le n^2 \|P\|_\Delta$ for $P \in \mathbb{P}_n$. It turned out that each following refinement of this fact required significant efforts. Schönhage [15] proved that $K_{n,N}$ remains bounded if $n^2/N < 1$. Ehlich and Zeller [6] showed that the condition $n^2/N \le \sqrt{6}$ is still sufficient for the boundedness of $K_{n,N}$; in [7] they relaxed this condition to $n^2/N \le \pi^2/2$. On the other side, Ehlich [5] proved that $n^2/N \to \infty$ implies $K_{n,N} \to \infty$ as $n, N \to \infty$. Finally, Coppersmith and Rivlin [2] proved the two-sided estimate

$$e^{c_1 n^2/N} \le K_{n,N} \le e^{c_2 n^2/N} \tag{1.3}$$

with some absolute constants $c_1, c_2 > 0$ which proves, in particular, the criterion mentioned above.

In this paper we present a new method which allows us to obtain pointwise estimates for $K_{n,N}(x)$. It may also help to better understand the nature of the problem. The main result of the paper asserts, roughly speaking, that for any $n < N$ the function $K_{n,N}(x)$ is uniformly bounded "inside" the interval $(-r, r)$ where

$$r = r_{n,N} = \sqrt{1 - n^2/N^2} \tag{1.4}$$

and $(-r, r)$ is the "maximal" subinterval with this property.

More exactly, we will prove, first, the following:

THEOREM 1. *With r defined, in (1.4) for $n < N$,*

$$K_{n,N}(x) \le C \log \frac{\pi}{\tan^{-1}\left(\frac{N}{n}\sqrt{r^2 - x^2}\right)}, \quad |x| < r, \tag{1.5a}$$

$$K_{n,N}(x) \le C \log \left(2 + \frac{n^2}{Nr}\right), \quad |x| \le r, \tag{1.5b}$$

where C is an absolute constant.

In a somewhat weaker form the result has been announced in [8].

Inequality (1.5a) implies that $K_{n,N}(x)$ is bounded in any compact subinterval in $(-r, r)$; for any $\delta > 0$ we have

$$K_{n,N}(x) \le C \log \frac{1}{\tan^{-1}\delta}, \quad |x| \le \sqrt{1 - (1+\delta)n^2/N^2}. \tag{1.6}$$

In particular, if $n/N \to 0$ as $n, N \to \infty$, then $K_{n,N}(x)$ is bounded in any interval $[-1+\epsilon, 1-\epsilon]$, $\epsilon > 0$. Moreover, under the same assumption $n/N \to 0$, combining (1.6) and the Bernstein inequality $|P'(x)| \le n\left(\rho^2 - x^2\right)^{-1/2} \|P\|_{[-\rho,\rho]}$ where $P \in \mathbb{P}_n$, $\rho \in (0,1)$, $|x| < \rho$, one would easily obtain that

$$K_{n,N}(x) \le 1 + \epsilon, \quad |x| \le 1 - \epsilon, \quad n \ge n(\epsilon).$$

Note that, if we simultaneously have $n^2/N \to \infty$, then $K_{n,N}(x) \to \infty$ "near" endpoints of Δ according to (1.3).

Next, we will show that for $x \in \Delta \setminus [-r, r]$ the magnitude of $K_{n,N}(x)$ is characterized by the function

$$W_{n,N}(x) = \exp\left\{N \int_r^{|x|} \int_y^{|x|} \frac{dt}{\sqrt{t^2 - y^2}} \frac{y\,dy}{\sqrt{1-y^2}}\right\}, \quad |x| \in [r, 1], \tag{1.7}$$

where $r = r_{n,N}$ is as defined in (1.4). In §2.3 below, we prove that for any $\delta > 0$ we have for $n \le (1-\delta)N$, $n \le N - 2$,

$$K_{n,N}(x) \le \frac{C}{\sqrt{\delta}} \cdot \log N \cdot W_{n,N}(x), \quad |x| \in [r, 1]. \tag{1.8}$$

On the other hand, with some other constant C we have for any $n < N$

$$K_{n,N}(x) \ge C \left|\cos\left(\frac{\pi}{2} N(1-x)\right)\right| \cdot W_{n,N}(x), \quad x \in [r, 1]. \tag{1.9}$$

The proof of (1.9) will be outlined in §4.2.

Since $\|W_{n,N}\|_\Delta = W_{n,N}(1)$ (assuming that $W_{n,N} \equiv 1$ on $[-r, r]$), inequalities (1.8), (1.9) allow us to find "the optimal" values for constants c_1, c_2 in Coppersmith-Rivlin's estimates (1.3). In particular, the elementary estimate $W_{n,N}(1) \le \exp\left\{2\sqrt{2r}\,(1+\sqrt{r})^{-1} n^2/N\right\}$ which is sharp as $r = r_{n,N} \to 1$ makes $K_{n,N} \le C \log N \cdot \exp\left\{2n^2/N\right\}$ so that the upper bound in (1.3) holds with any $c_2 > 2$ for $n \ge n(c_2)$.

We also note (without proof) that $\log N$ in the estimate above and in (1.8) may be replaced with $\log\left(2+n^2/N\right)$ which is somewhat better if n/N is small. Moreover, it is possible to prove that this logarithmic factor may be in effect only in a small neighborhood of points $-r$ and r (as in Theorem 1). However, we do not know if (1.8) holds true without any logarithmic factor. Our conjecture is that the answer is negative and, furthermore, estimates in Theorem 1 are sharp. The problem of the logarithmic factor is, in fact, connected with the problems discussed in §1.3 below.

Finally, we will discuss bounds for $K_{n,N}(z)$, $z \in \mathbb{C} \setminus [-1,1]$, which are obtainable as easy corollaries of corresponding results for $z \in [-1,1]$ (Remark 1, §2.3 below).

1.2. *Outline of the method.* The proofs of Theorem 1 and related estimates (1.8) and (1.9) are based on a rather general method which may be described in a few words as follows.

Suppose a set of points $E = \{\zeta_1, \dots, \zeta_N\} \subset [-1,1]$ is defined by a measure σ. In other words, we are given originally a positive and absolutely continuous measure $d\sigma(x) = \sigma'(x)dx$ in $[-1,1]$ with $|\sigma| = \sigma([-1,1]) = N \in \mathbb{N}$ and points ζ_K are then defined as uniformly distributed with respect to σ; that is,

$$\sigma\left([\zeta_K, \zeta_{K+1}]\right) = 1,\ K = 1, \dots, N-1;\ \sigma\left([-1,\zeta_1]\right) = \sigma\left([\zeta_N, 1]\right) = \frac{1}{2} \tag{1.10}$$

(note that points (1.1) are produced by the measure $d\sigma(x) = \dfrac{N}{2}dx$). Denote

$$T(x,\sigma) = \prod_{K=1}^{n}\left(x - \zeta_K\right), \quad V(x,\sigma) = \int_{-1}^{1} \log\frac{1}{|x-t|}\, d\sigma(t).$$

In other words, we consider a polynomial $T(x,\sigma)$ whose zeros are distributed with a given density $\sigma'(x)$, $x \in [-1,1]$.

Subsequent analysis is technically based on the following representation for such a polynomial:

$$T(x,\sigma) = C(x,\sigma)e^{-V(x,\sigma)}\cos\left(\pi\int_x^1 d\sigma(t)\right), \tag{1.11}$$

where $C(x,\sigma)$ is a positive function defined by σ. In the case when $\sigma'(x)$ is analytic and positive in $(-1,1)$, an integral representation for $C(x,\sigma)$ has been found in [12] which allows us to effectively estimate this function (see Theorem 2, §3). We note that "under normal circumstances" $C(x,\sigma)$ is close to 2 when $|\sigma|$ is large in most of $(-1,1)$. For the purposes of this paper, we need an estimate for $C(\sigma) = \max\limits_{[-1,1]} C(x,\sigma)/\min\limits_{[-1,1]} C(x,\sigma)$ and, in particular, will prove that $C(N\,dx/2) \le 2$, $N \in \mathbb{N}$.

Next, using (1.11) and the estimate above, we derive an inequality connecting the original extremal problem and a dual one with the weight $U(x) =$

$\exp\{V(x,\sigma)\}$. In a simplified form it may be written as follows

$$\sup_{P\in\mathbb{P}_n}\frac{|P(x)|}{\|P\|_E}\cdot\sup_{Q\in\mathbb{P}_{N-n-1}}\frac{|Q(x)U(x)|}{\|QU\|_E}\le C\log N. \tag{1.12}$$

(The second sup has to be modified to get rid of the $\log N$ on the right-hand side; see (2.20) below.)

Now, low bounds for each of the two suprema above may be obtained by construction near extremal polynomials $P_n\in\mathbb{P}_n$ and $Q_{N-n-1}\in\mathbb{P}_{N-n-1}$. Then, (1.12) will provide us with upper bounds for both of them.

Then we construct the required polynomials using the potential theoretic nature of the two extremal problems in (1.12). A closely related problem on asymptotics for discrete orthogonal polynomials with the same potential theoretic background has been considered in [13] and all the technical details may be taken from this paper (see §4 for detailed references and remarks). In short, there are two dual equilibrium problems associated with σ; namely, the equilibrium in the external field $-V(x,\sigma)$ and the equilibrium with the upper constraint σ. Let $n<N$ and μ and λ be solutions of those two problems normalized by $|\mu|=N-n$ and $|\lambda|=n$ (then $\sigma=\mu+\lambda$).

Measures λ and μ may be, in fact, regarded as solutions of two extremal problems which present continuous versions of the two extremal problems in (1.12). Thus, they represent the distribution of zeros of corresponding extremal polynomials. Conversely, polynomials $P_n(x)=T(x,\lambda)$ and $Q_{N-n}=T(x,\mu)$, whose zeros are uniformly distributed with respect to λ and μ in the sense of (1.10) are, indeed, close to the extremal polynomials in (1.12). Using for those polynomials, P_n and Q_{N-n}, representation (1.11), we will obtain fairly good low estimates for the two extrema in (1.12) (one certain zero of Q_{N-n} must be dropped for technical reasons; see (2.21) in §2.2 below).

Following the method outlined above, one would come to the conclusion that under certain restrictions on σ we have the estimate

$$K_{n,N}(x;\sigma)=\max_{P\in\mathbb{P}_n}\frac{|P(x)|}{\|P\|_{E(\sigma)}}\le C\log\frac{2\sigma'(x)}{\mu'(x)},\quad x\in\operatorname{supp}(\mu) \tag{1.13}$$

where $E(\sigma)=\{\zeta_k\}_{k=1}^N$ is defined by (1.10) and μ is the equilibrium measure with $|\mu|=N-n$ in the external field $-V(x,\sigma)$ on Δ. Thus, the problem is reduced to the investigation of the equilibrium measure μ which is uniquely defined by σ and n. Normally, no further restrictions on σ is required to prove that $\sigma'(x)/\mu'(x)$ is bounded "inside" $\operatorname{supp}(\mu)$.

To keep the length of the paper reasonable, we present the detailed proofs only for the case $d\sigma=(N/2)\,dx$ which is, probably, one of the most interesting cases in applications. In this case we have $\operatorname{supp}(\mu)=[-r,r]$ with r from (1.4), $\mu'(x)=\dfrac{N}{\pi}\tan^{-1}\left(\dfrac{N}{n}\sqrt{r^2-x^2}\right)$ and, thus, (1.5a) coincides with (1.13).

However in Sections 3 and 4 below, we discuss the main components of the method under general assumptions.

We also announce the following "model" generalization of Theorem 1 which will help us, in particular, to discuss some open problems in §1.3 of this introduction.

THEOREM 1 (a). *Let the measure* $\sigma = \sigma_{N,\beta}$ *be defined by* $d\sigma(x) = \sigma'(x)dx$, $x \in \Delta$,

$$\sigma'(x) = C_\beta N \left(1 - x^2\right)^\beta \text{ where } C_\beta^{-1} = \int_\Delta \left(1 - x^2\right)^\beta dx \tag{1.14}$$

(*thus*, $\int_\Delta d\sigma = N$). *Then inequality* (1.13) *holds true and*, *furthermore*:

(i) *If* $\beta > -\dfrac{1}{2}$, *then* $\operatorname{supp}(\mu) = [-r, r]$ *where* $r^2 = 1-(n/N)^\alpha$, $\alpha = 2/(2\beta+1)$ *and*, *further*,

$$\mu'(x) = \sigma'(x) \int_0^{q(x)} \left(1 - t^2\right)^{\beta - 1/2} dt \text{ where } q(x) = \sqrt{\frac{r^2 - x^2}{1 - x^2}}.$$

Consequently, $K_n(x; \sigma)$ *is bounded on compact subintervals in* $(-r, r)$.

(ii) *If* $\beta = -\dfrac{1}{2}$ *then* $\operatorname{supp}(\mu) = \Delta$, $\mu'(x) = (N - n)/\left(\pi\sqrt{1 - x^2}\right)$.

(iii) *If* $\beta \in (-1, -1/2)$, *then* $\operatorname{supp}(\mu) = [-1, -r] \cup [r, 1]$ *where* $r = \sqrt{1 - x}$ *and* x *is the root of the equation* $NF(x) = nF(0)$ *with*

$$F(x) = \int_0^1 (t + x(1 - t))^{\beta + 1/2} \frac{dt}{\sqrt{t(1 - t)}}.$$

We note that at least some kind of smoothness of $\sigma'(x)$ is required to prove (1.13). In fact, some additional structural conditions may also be necessary. Weaker results on the asymptotics for $K_n^{1/n}(x; \sigma_n)$ may be obtained under more general assumptions on the sequence $\{\sigma_n\}$; see [1], [3], [4], [8], [9].

1.3. *Some related open problems for interval and circle.* For $F = \Delta = [-1, 1]$ or $F = T = \{z : |z| = 1\}$ and a finite subset $E \subset F$ we define

$$K_n(E) = \max\left(\|P\|_F / \|P\|_E\right) \tag{1.15}$$

where $N = \operatorname{card}(E) > n$, and

$$K_{n,N} = \min_{\operatorname{card}(E) = N} K_n(E). \tag{1.16}$$

We mention in this subsection a few open problems related to estimates for $K_{n,N}$ and a "dual" quantity $\tilde{K}_{N,n}$ (see (1.15a), (1.16a) below). We are also concerned with the extremal subset E in (1.16).

First, let $F = \Delta$, $E_{N,\beta} = E(\sigma_{N,\beta})$ where $\sigma_{N,\beta} = \sigma$ is the measure defined in (1.14) in Theorem 1(a). We introduce the special notation $E_N = E_{N,-1/2}$ for the case $\beta = -1/2$ (points E_N are uniformly distributed with respect to the measure $d\sigma = N dx/\left(\pi\sqrt{1-x^2}\right)$ and, thus, are roots of the Tchebyshev polynomial of order N). Theorem 1(a) shows that the value $\beta = -1/2$ is an exceptional one: by the assertion (ii) of the theorem we have

$$K_{n,N} \leq K_n(E_N) \leq C \log \frac{N}{N-n}. \tag{1.17}$$

For $\beta \neq -1/2$ we have, in fact, an exponential growth of $K_n(E_{N,\beta})$ for $N/n \leq C$ (case $\beta = 0$ presents a typical example; see (1.3)). It is not surprising that the value $\beta = -1/2$ is outstanding, since associated points E_N are uniformly distributed with respect to the Roben measure of Δ. In view of the potential theoretic backgrounds of the problem, those points must be at least "near optimal" in the extremal problem (1.16) in the sense that $K_n(E_N) \leq C K_{n,N}$.

It turns out that this natural conjecture presents an open problem. Moreover, there is a problem even with a particular set E_N. More precisely, we have

Problem 1.[1] Prove that

$$K_n(E_N) \geq C \log \frac{N}{N-n}. \tag{1.18}$$

Problem 2. Prove that

$$K_{n,N} \geq C \log \frac{N}{N-n} \tag{1.19}$$

(where C is a positive constant not necessarily the same in different inequalities).

The inequality in (1.19) is much stronger than the one in (1.18) and may present a difficult problem.

If (1.19) is, indeed, valid, then it follows in combination with (1.17) that E_N is, indeed, near optimal in problem (1.16). The answer to the next question is not clear.

Problem 3. Is it true that E_N provides the exact minimum in (1.16)?

Similar problems are open also in the case of the circle which is somewhat better investigated.

Let, now, $F = T$ and $E_N = \left\{e^{2\pi i k/N}, k = 1, 2, \ldots, N\right\}$.

Then the upper bound in (1.17) remains true. In both cases Δ and T it may be easily proved by the methods of the present paper. Actually, the

[1] The problem was recently solved in E. Rakhmanov and B. Shekhtman, On discrete norms of polynomials, *J. Approx. Theory* **139** (2006), 2–7.

two cases connected with sets E_N allow significant simplification and the corresponding proof may be made rather short.

Next, Problem 2 above remains open for $F = T$. All three relations (1.17)–(1.19) were conjectured by Shekhtman [16]; his paper also contains the following result related to Problem 2:

$$\left\{N - n = 0\left(\log^2 n\right)\right\} \Rightarrow \{K_{n,N} \to \infty\}.$$

We note that his proof uses methods of the operator theory in Banach spaces which were never used before in the problems under consideration.

The common and natural conjecture for Problem 3 is that the answer is positive for $F = T$, but it is still an open problem.

It was also pointed out in [16] that there is an apparent "duality" between results and conjectures related to the problems (1.15)–(1.16) and results and conjectures related to another problem on interpolation which we shall shortly describe below.

As everywhere above, we assume that $n < N$ but now we switch the meaning of those parameters. That is, n will stand for a number of points in a discrete set $E \subset T$ while N will denote the degree of a polynomial.

So, for $E \subset T$, $\operatorname{card}(E) = n$ and for a function $f : E \to \mathbb{C}$ we define

$$\mathbb{P}_N(f, E) = \{P \in \mathbb{P}_N : P(\zeta) = f(\zeta), \zeta \in E\}$$

and define

$$\tilde{K}_N(E_n) = \max_{\|f\|_E \le 1} \min_{P \in \mathbb{P}_N(f,E)} \|P\|_T, \tag{1.15a}$$

$$\tilde{K}_{N,n} = \min_{\operatorname{card}(E)=n} \tilde{K}_N(E). \tag{1.16a}$$

It was proved by Szabados [17] that

$$\tilde{K}_{N,n} \le \tilde{K}_{N,n}(E_n) \le C \log \frac{N}{N-n} \tag{1.17a}$$

and

$$\tilde{K}_N(E_n) \ge C \log \frac{N}{N-n}. \tag{1.18a}$$

The last inequality solves the tilde-version of Problem 1. The corresponding version of Problem 2 remains open; the conjecture

$$\tilde{K}_{N,n} \ge C \log \frac{N}{N-n} \tag{1.19a}$$

belongs to Erdős and Szabados; see [17].

Problem 3, related to the extremal problem (1.16a), is open.

It would be interesting to figure out if there is any deeper connection between the extremal problems (1.16) and (1.16a) than a simple coincidence of inequalities indicated above.

2. Proof of Theorem 1

In this section, we reduce the proof of Theorem 1 to three auxiliary lemmas (Lemmas 1, 2 and 3 below). Proofs of those lemmas will be presented in Sections 3 and 4.

2.1. *Auxiliary results.* We denote for a natural N

$$V_1(x) = \frac{N}{2}\int_{-1}^{1} \log\frac{1}{|x-t|}\,dt, \tag{2.1}$$

$$\phi_1(x) = \pi\int_{x}^{1} \frac{N}{2}\,dt = \frac{\pi N}{2}\,(1-x), \quad |x| \le 1, \tag{2.2}$$

$$T(x) = \prod_{K=1}^{N}\left(x-\zeta_K\right), \tag{2.3}$$

where ζ_K, $K = 1, 2, \ldots, N$ are as defined in (1.1).

Lemma 1. *There exists the following representation*

$$T(x) = C_1(x)e^{-V_1(x)}\cos\phi_1(x), \quad |x| \le 1 \tag{2.4}$$

where $C_1(x)$ is a positive continuous function (*depending on N*) *with*

$$\max_{|x|\le 1} C_1(x) / \min_{|x|\le 1} C_1(x) \le 2, \quad N \subset \mathbb{N}. \tag{2.5}$$

An immediate corollary of (2.4) and (2.2) is the representation for the derivative of T at zeros ζ_K of this polynomial

$$\left|T'\left(\zeta_K\right)\right| = \frac{\pi N}{2}C_1\left(\zeta_K\right)e^{-V_1(\zeta_K)}. \tag{2.6}$$

For a pair of natural numbers $n < N$ we further denote,

$$r = \sqrt{1 - n^2/N^2}, \tag{2.7}$$

$$\mu'(x) = \frac{N}{\pi}\tan^{-1}\left(\frac{N}{n}\sqrt{r^2-x^2}\right), \quad |x| \le r, \tag{2.8}$$

$\mu'(x) = 0$, $|x| \in [r, 1]$. It is convenient to consider μ' as the derivative with respect to Lebesgue measure dx of an absolutely continuous measure $d\mu(x) = \mu'(x)\,dx$ supported on $[-r, r]$. We define

$$V_2(x) = \int_{-1}^{1} \log\frac{1}{|x-t|}\,d\mu(t), \tag{2.9}$$

$$\phi_2(x) = \pi\int_{x}^{1} d\mu(t), \quad |x| \le 1, \tag{2.10}$$

$$S(x) = \prod_{i=1}^{N-n}\left(x-y_i\right), \tag{2.11}$$

where $N-n$ points $-r<y_1<y_2<\cdots<y_{N-n}<r$ are defined by

$$\cos\phi_2\left(y_i\right)=0,\quad i=1,2,\ldots,N-n \tag{2.12}$$

or, equivalently, by

$$\begin{gathered}\mu\left([-r,y_1]\right)=\mu\left([y_{N-n},r]\right)=\frac{1}{2},\\ \mu\left([y_i,y_{i+1}]\right)=1,\quad i=1,2,\ldots,N-n-1.\end{gathered} \tag{2.13}$$

Note that $|\mu|=\mu([-1,1])=N-n\in\mathbb{N}$ by (2.16) in Lemma 3 below so that conditions (2.13) are consistent and $S(x)$ is equivalently defined as $S(x)=T(x,\mu)$. Similar points ζ_K in (1.1) are uniformly distributed with respect to the measure $d\sigma=\dfrac{N}{2}\,dx$ in the sense of (1.10) and for T in Lemma 1 we have $T(x)=T(x,\sigma)$.

We note also that $|\cos\phi_2(x)|=1$ for $|x|\in[r,1]$. Now we have

Lemma 2. *The following representation holds true in* $[-1,1]$:

$$S(x)=C_2(x)e^{-V_2(x)}\cos\phi_2(x) \tag{2.14}$$

where $C_2(x)$ *is a positive continuous function in* $[-1,1]$ (*depending on* n,N) *with*

$$\max_{|x|\le 1}C_2(x)/\min_{|x|\le 1}C_2(x)\le 12. \tag{2.15}$$

Proofs of Lemmas 1 and 2 are presented in Section 3 below.

Finally, the following lemma provides a connection between $V_1(x)$ and $V_2(x)$.

Lemma 3. *There exist the following relations*

$$|\mu|=\int_{-r}^{r}d\mu(t)=N-n; \tag{2.16}$$

$$\begin{aligned}&V_2(x)-V_1(x)=w,\quad x\in[-r,r]=\operatorname{supp}(\mu),\\ &V_2(x)-V_1(x)\ge w,\quad x\in[-1,1],\end{aligned} \tag{2.17}$$

where $w=w_{n,N}$ *is constant and*

$$\begin{aligned}W(x):&=\exp\left\{V_2(x)-V_1(x)-w\right\}\\ &=\exp\left\{N\int_r^x\int_y^x\frac{dt}{\sqrt{t^2-y^2}}\,\frac{y\,dy}{\sqrt{1-y^2}}\right\},\ x\in[r,1].\end{aligned} \tag{2.18}$$

For a proof of this lemma, see Section 4 below.

We note that all the functions and constants introduced above depend on n or N or both. We drop this dependence from the notation to make the statement shorter.

2.2. *Proof of Theorem* 1. For any $P \in \mathbb{P}_n$ satisfying $|P(\zeta)| \le 1$, $\zeta \in E$ and any $Q \in \mathbb{P}_{N-n-1}$ we have

$$R(x) := \frac{P(x)Q(x)}{T(x)} = \sum_{\zeta \in E} \frac{c(\zeta)}{x-\zeta}$$

where

$$c(\zeta) = P(\zeta)Q(\zeta)/T'(\zeta).$$

Since $|c(\zeta)| \le |Q(\zeta)|/\,|T'(\zeta)|$ it follows that

$$|P(x)| = \frac{|T(x)|}{|Q(x)|}|R(x)| \le \sum_{\zeta \in E} \frac{|T(x)|}{|T'(\zeta)|} \cdot \frac{|Q(\zeta)|}{|Q(x)|}\, \frac{1}{|x-\zeta|}.$$

Since $P \in \mathbb{P}_n$ is an arbitrary polynomial with normalization $|P(\zeta)| \le 1$, $\zeta \in E$,

$$K_{n,N}(x) \le \sum_{\zeta \in E} \frac{|T(x)|}{|T'(\zeta)|}\, \frac{|Q(\zeta)|}{|Q(x)|}\, \frac{1}{|x-\zeta|}. \tag{2.19}$$

Using Lemma 1 and (2.6), we obtain

$$\frac{|T(x)|}{|T'(\zeta)|} \le \frac{4}{\pi N}\, |\cos\phi_1(x)|\, e^{V_1(\zeta)-V_1(x)}.$$

Together with (2.19) this yields for $x \in [-r, r]$

$$K_{n,N}(x) \le \frac{4}{\pi N} \sum_{\zeta \in E} \frac{|Q(\zeta)| e^{V_1(\zeta)}}{|Q(x)| e^{V_1(x)}}\, \frac{|\cos\phi_1(x)|}{|x-\zeta|}. \tag{2.20}$$

Next, for a fixed $x \in [-r, r]$ we select a convenient polynomial Q. Let $y = y(x)$ be a root of S in (2.11) minimizing the total mass of the measure μ of the interval $[x, y]$ between x and y. Equivalently, y is defined by

$$\mu([x, y]) \le \frac{1}{2}, \quad S(y) = 0.$$

If there are two roots of S with this property we select any one of them. Then we define

$$Q(z) = S(z)/(z-y) \tag{2.21}$$

which is clearly a polynomial of degree $N-n-1$, and which depends also on x.

By Lemma 2,

$$\frac{|Q(\zeta)|}{|Q(x)|} \le 12 e^{V_2(x)-V_2(\zeta)}\, \frac{|\cos\phi_2(\zeta)|}{|\zeta-y|} \cdot \frac{|x-y|}{|\cos\phi_2(x)|}. \tag{2.22}$$

Next we estimate the last two terms on the right-hand side of (2.22) using a method based on the fact that $\mu'(x)$ is a concave function in $[-r,r]$. Concavity of μ' implies that for any interval $\Delta \subset [-r,r]$ we have

$$1 \le \frac{\max_{t\in\Delta} \mu'(t)\cdot|\Delta|}{\mu(\Delta)} \le 2 \tag{2.23}$$

where $|\Delta|$ is the length of Δ.

Let $\Delta_0 = [x,y]$ and $M_0 = \max_{t\in\Delta_0} \mu'(t)$. Then

$$|\cos\phi_2(x)|\,/|x-y| \ge \frac{\pi}{4}\,M_0. \tag{2.24}$$

Indeed, in the case where $\mu\left(\Delta_0\right) \ge \frac{1}{3}$ we have $|\cos\phi_2(x)| \ge \sqrt{3}/2$ (note that $\mu\left(\Delta_0\right) \le \frac{1}{2}$ and $\cos\phi_2(y)=0$). At the same time by (2.23) $|x-y| = |\Delta_0| \le 2\mu\left(\Delta_0\right)/M_0 \le 1/M_0$ and (2.24) follows. In the opposite case where $\mu\left(\Delta_0\right) < 1/3$ we have $|\sin\phi_2(t)| \ge \frac{1}{2}$, $t\in\Delta_0$ and, subsequently,

$$|\cos\phi_2(x)| = |\cos\phi_2(x)-\cos\phi_2(y)| = \pi\int_{[x,y]} \mu'(t)\,|\sin\phi_2(t)|\;d\mu \ge \frac{\pi}{2}\,\mu\left(\Delta_0\right),$$

$$\frac{|\cos\phi_2(x)|}{|x-y|} \ge \frac{\pi}{2}\,\frac{\mu\left(\Delta_0\right)}{|\Delta_0|} \ge \frac{\pi}{4}\,M_0$$

so that (2.24) holds true in both cases.

Next we introduce one more interval $\Delta = [\alpha,\beta] \supset \Delta_0$ with α,β defined by

$$\mu([\alpha,y]) = \mu([y,\beta]) = \frac{1}{2} \tag{2.25}$$

and prove that

$$M := \max_{t\in\Delta} \mu'(t) \le 2M_0. \tag{2.26}$$

Without loss of generality, we may assume that $y \le 0$. Let β' be the maximum point of μ' on Δ. Since μ' is increasing in $[-r,0]$, we have $\beta' \in [y,\beta]$ and $\mu\left([y,\beta']\right) \le \mu([\alpha,y])$. Hence, there exists $\alpha' \in [\alpha,y]$ with $\beta'-y = y-\alpha'$. Since μ' is concave

$$2M_0 \ge 2\mu'(y) \ge \mu'\left(\alpha'\right) + \mu'\left(\beta'\right) \ge \mu'\left(\beta'\right) = M$$

and (2.26) follows.

Next, we show that

$$\frac{|\cos\phi_2(\zeta)|}{|\zeta-y|} \le \min\left\{\pi M, \frac{1}{|\zeta-y|}\right\}, \quad \zeta\in E. \tag{2.27}$$

Indeed, for $\zeta \in \Delta$ by the midvalue theorem

$$\left|\frac{\cos\phi_2(\zeta)}{\zeta - y}\right| = \left|\frac{\cos\phi_2(\zeta) - \cos\phi_2(y)}{\zeta - y}\right| = \pi\mu'(t)\,|\sin\phi_2(t)|$$

with some $t \in [\zeta, y]$. Thus, the left-hand side of (2.27) does not exceed πM for $\zeta \in \Delta$. If $\zeta \notin \Delta$ then we have $\zeta < \alpha$ or $\zeta > \beta$. In the first case $|\zeta - y| > |\alpha - y| \geq 1/2M$ by (2.23). In the second one we have $|\zeta - y| > |\beta - y| \geq 1/2M$ by (2.23). Thus, (2.27) holds true for any $\zeta \in E$.

Similarly, we have for $x, \zeta \in [-1, 1]$

$$\frac{|\cos\phi_1(x)|}{|x - \zeta|} \leq \min\left\{\frac{\pi N}{2}, \frac{1}{|x - \zeta|}\right\}. \tag{2.28}$$

Finally, it follows from (2.17) in Lemma 3 that

$$\frac{e^{V_2(x) - V_1(x)}}{e^{V_2(\zeta) - V_1(\zeta)}} \leq 1, \quad x \in [-r, r]. \tag{2.29}$$

Now, using (2.20) and taking into account (2.22), (2.24), (2.26), (2.27), (2.28), and (2.29), we obtain the basic estimate

$$K_{n,N}(x) \leq C\frac{1}{NM}\sum_{\zeta \in E}\min\left\{\frac{\pi N}{2}, \frac{1}{|\zeta - x|}\right\} \cdot \min\left\{\pi M, \frac{1}{|\zeta - y|}\right\} \tag{2.30}$$

where $x \in [-r, r]$, $C = 384/\pi^2$.

In the conclusion of the proof below, we use the abbreviation $\sum$ for the part of the sum on the right-hand side in (2.30) over a subset $A \subseteq E$ with the coefficient $1/NM$ (without C).

Let $E_1 = \{\zeta \in E : |\zeta - x| < 2/N\}$. This set contains at most two points and

$$\sum_{\zeta \in E_1} \leq \frac{2}{MN} \cdot \frac{\pi N}{2} \cdot \pi M \leq \pi^2. \tag{2.31}$$

Let $E_2 = \left\{\zeta \in E \setminus E_1 : |\zeta - y| \leq \dfrac{1}{M}\right\}$. We note that $|x - y| < \dfrac{1}{M}$ by (2.23) and further define $E_2^+ = \{\zeta \in E_2 : \zeta > x\}$, $E_2^- = \{\zeta \in E_2 : \zeta < x\}$. We numerate points $\zeta \in E_2^+$ from the left to the right so that $E_2^+ = \{\zeta_1, \zeta_2, \ldots, \zeta_K\}$ where $K \leq \dfrac{N}{M}$ (= the total number of points in E_2^+); we have $|\zeta_j - x| \geq 2j/N$, $j = 1, 2, \ldots, K$ and, therefore,

$$\sum_{\zeta \in E_2^+} \leq \frac{1}{MN}\sum_{j=1}^{K}\pi M \cdot \frac{1}{2j/N} \leq \frac{\pi}{2}\sum_{j=1}^{K}\frac{1}{j} \leq \frac{\pi}{2}\left(1 + \log\frac{N}{M}\right).$$

The same estimate clearly holds true for $\sum_{\zeta \in E_2^-}$ so that, totally, we obtain

$$\sum_{\zeta \in E_2} \leq \pi\left(1 + \log\frac{N}{M}\right). \tag{2.32}$$

At last, let $E_3 = \{\zeta \in E \setminus E_1 : |\zeta - y| > 1/M\}$ and E_2^+, E_2^- be subsets of E_3 subsequently to the right and to the left from x. Let $E_3^+ = \{\zeta_1 < \zeta_2 < \cdots\}$ be the numeration of points in E_3^+ from the left to the right. We have $|\zeta_j - x| \geq 2j/N$, $|\zeta_j - y| \geq 1/M + 2(j-1)/N$; thus

$$\sum_{\zeta \in E_3^+} \leq \frac{1}{NM} \sum_{j=1}^{\infty} \frac{1}{\frac{2j}{N}\left(\frac{1}{M} + \frac{2(j-1)}{N}\right)}.$$

We break the last sum into two parts and estimate them as follows

$$\frac{1}{NM} \sum_{j \leq 1+N/M} \leq \frac{1}{2} \sum_{j \leq 1+N/M} \frac{1}{j} \leq \frac{1}{2}\left(1 + \log\left(1 + \frac{N}{M}\right)\right),$$
$$\frac{1}{NM} \sum_{j > 1+N/M} \leq \frac{N}{4M} \sum_{j > 1+N/M} \frac{1}{j(j-1)} = \frac{N}{4M} \cdot \frac{1}{N/M} = \frac{1}{4}.$$

Totally,

$$\sum_{\zeta \in E_3} \leq 2 \sum_{\zeta \in E_3^+} \leq \frac{3}{2} + \log\left(1 + \frac{N}{M}\right) \leq \frac{5}{2} + \log\frac{N}{M}. \tag{2.33}$$

Now, (2.30), combined with (2.31)–(2.33), yields

$$\begin{aligned} K_{n,N}(x) &\leq C\left\{\left(\pi^2 + \pi + \frac{5}{2}\right) + (\pi + 1)\log\frac{N}{M}\right\} \\ &\leq C_1 + C_2 \log\frac{N}{M}, \quad |x| \leq r. \end{aligned} \tag{2.34}$$

Since $M = M(x) \geq \mu'(x)$ and $N(\mu'(x)) \geq 2$, $|x| \leq r$,

$$K_{n,N}(x) \leq C_1 + C_2 \log\frac{N}{\mu'(x)} \leq C \log\frac{N}{\mu'(x)} \tag{2.35}$$

and (1.5a) in Theorem 1 is proved.

To prove (1.5b) we denote by x_0 the root of

$$\int_{-r}^{x_0} \mu'(t)\, dt = 1. \tag{2.36}$$

We will assume that $n \leq N - 2$ so that $x_0 \leq 0$ (otherwise (1.5b) is a trivial consequence of (2.34)). For any $x \in [-r, r]$, the associated value of $M = M(x)$ in (2.26) satisfies $M \geq \mu'(x_0)$; thus, it is enough to prove the inequality

$$\log\frac{N}{\mu'(x_0)} \leq 3 \log\left(2 + \frac{n^2}{Nr}\right). \tag{2.37}$$

We consider separately the case when $n^2 \geq Nr$ and $n^2 < Nr$. First, let $n^2 \geq Nr$. By (2.23) the equation (2.36) is equivalent to

$$Nt_0 \tan^{-1}\left(\frac{N}{n}\sqrt{t_0(2r - t_0)}\right) = \theta\pi, \quad \theta \in [1, 2], \tag{2.38}$$

where $t_0 = x_0 + r$. Since $2r - t_0 \le 2r$ and $\tan^{-1} x \le x$ it follows that $\left(N^2/n\right) t_0^{3/2}\sqrt{r} \ge \pi/\sqrt{2} > 1$, so that $t_0 \ge \left(n/N^2\sqrt{r}\right)^{3/2}$. From here, using the inequalities $2r - t_0 \ge r$, $\tan^{-1} x \ge \frac{\pi}{4} x$, $x \ge 1$ and $n^2 \ge Nr$, we come to

$$\mu'(t_0) \ge \frac{N}{\pi}\tan^{-1}\left(\frac{N}{n}\sqrt{rt_0}\right) \ge \frac{N}{\pi}\tan^{-1}\left(\left(\frac{Nr}{n^2}\right)^{1/3}\right) \ge \frac{N}{4}\left(\frac{Nr}{n^2}\right)^{1/3}$$

and, subsequently,

$$\log\frac{N}{\mu'(x_0)} \le \log\left(4\left(\frac{n^2}{Nr}\right)^{1/3}\right) \le 2\log 2 + \log\left(2+\frac{n^2}{Nr}\right) \le 3\log\left(2+\frac{n^2}{Nr}\right).$$

In the case where $n^2 < Nr$ we again use (2.38) and $\tan^{-1} x \le \pi/2$, $x \ge 0$. This yields $Nt_0 \cdot \pi/2 \ge \pi$.

Now, $t_0 \ge 2/N$ and, further,

$$\mu'(t_0) \ge \frac{N}{\pi}\tan^{-1}\left(\frac{N}{n}\sqrt{rt_0}\right) \ge \frac{N}{\pi}\tan^{-1}\left(\frac{\sqrt{Nr}}{n}\right) \ge \frac{N}{4}.$$

Hence, $\log\dfrac{N}{\mu'(x_0)} \le 2\log 2$ and (2.37) follows. This completes the proof of (1.5b) and Theorem 1.

2.3. *Proof of the estimate* (1.8) *and its generalization for* $x \in \mathbb{C}$. First we'll prove that for any n, N

$$K_{n-1,N}(x) \le C\log N \cdot W_{n,N}(x), \quad |x| \in [r,1]. \tag{2.39}$$

The proof is based on a significantly simplified version of the method of Section 2.2 above.

We note that inequality (2.20) with $K_{n-1,N}(x)$ on the left-hand side holds true for any $Q \in \mathbb{P}_{N-n}$. Now we select $Q(x) = S(x)$ instead of (2.21) and obtain from Lemma 2

$$|Q(\zeta)|/|Q(x)| \le 12e^{V_2(x)-V_2(\zeta)}, \quad |x| \in [r,1], \quad \zeta \in E$$

in place of (2.22). Next, the function $W(x) = W_{n,N}(x)$ defined in (2.18) satisfies $W(x) = 1$, $x \in [-r,r]$; $W(x) \ge 1$, $|x| \in [r,1]$. Accordingly, the estimate (2.29) is replaced by

$$\frac{e^{V_2(x)-V_1(x)}}{e^{V_2(\zeta)-V_1(\zeta)}} = \frac{W(x)}{W(\zeta)} \le W(x), \quad |x| \in [r,1].$$

Comparing this with (2.30)), we get

$$K_{n-1,N}(x) \le \frac{C}{N}W_{n,N}(x)\sum_{\zeta\in E}\min\left\{\frac{\pi N}{2}, \frac{1}{|\zeta - x|}\right\}. \tag{2.40}$$

Together with a trivial estimate

$$\frac{1}{N}\sum_{\zeta\in E}\min\left\{\frac{\pi N}{2},\frac{1}{|\zeta-x|}\right\}\le \pi+1+\log N$$

this implies (2.39).

Next, we rewrite (2.39) as follows,

$$K_{n,N}(x)\le C\log N\cdot W_{n+1,N}(x),\quad |x|\in[r,1], \tag{2.41}$$

and use the simple auxiliary estimate (2.42) in Lemma 4 below. This completes the proof of (1.8).

LEMMA 4. *For any* $\delta>0$, $n\le(1-\delta)N$, $n\le N-2$,

$$W_{n+1,N}(x)/W_{n,N}(x)\le 4/\sqrt{\delta},\quad |x|\le 1. \tag{2.42}$$

Proof. We have by (2.18)

$$\log\frac{W_{n+1,N}(x)}{W_{n,N}(x)}=N\int_{r'}^{r}\int_{y}^{|x|}\frac{dt}{\sqrt{t^2-y^2}}\,\frac{y\,dy}{\sqrt{l-y^2}},\quad |x|\le 1 \tag{2.43}$$

where $r=\sqrt{1-n^2/N^2}$, $r'=\sqrt{1-(n+1)^2/N^2}$. Note that (2.43) indeed holds over the whole interval $x\in[-1,1]$ if we define

$$\int_{y}^{|x|}\frac{dt}{\sqrt{t^2-y^2}}:=0 \text{ for } |x|\le y.$$

For $|x|>y$ we have, since $|x|\le 1$, $y\ge 1/r'$,

$$\int_{y}^{|x|}\frac{dt}{\sqrt{t^2-y^2}}=\int_{1}^{|x|/y}\frac{dt}{\sqrt{t^2-1}}\le\int_{1}^{1/r'}\frac{dt}{\sqrt{t^2-1}}$$
$$=\log\left(\frac{1}{r'}+\sqrt{\frac{1}{(r')^2}-1}\right)\le\log\frac{2}{r'}.$$

At the same time

$$\int_{r'}^{r}\frac{y\,dy}{\sqrt{1-y^2}}=\sqrt{1-(r')^2}-\sqrt{1-r^2}=\frac{1}{N}.$$

At last, we note that $r/r'=\sqrt{\dfrac{N^2-n^2}{N^2-(n+1)^2}}<\sqrt{\dfrac{N-n}{N-n-1}}<\sqrt{2}$ for $n\le N-2$. With these remarks, gives (2.43)

$$\log\frac{W_{n+1,N}(x)}{W_{n,N}(x)}\le\log\frac{2}{r'}<\log\frac{2\sqrt{2}}{r},\quad |x|\le 1.$$

To complete the proof of Lemma 4, it remains to notice that $r\ge\sqrt{\delta}$ for $n\le(1-\delta)N$. □

With $W(x) = W_{n,N}(x)$ as defined in (2.18), inequality (1.8) holds for $x \in [-1,1]$ (for $x \in [-r,r]$ it follows by (1.5b)). The next remark extends (1.8) to the whole complex plane.

Remark 1. For any $\delta > 0$, n, N with $n \le N-2$, $n \le (1-\delta)N$, and $z \in \mathbb{C}$ we have

$$K_{n,N}(z) \le \frac{C}{\sqrt{\delta}} (\log N)\, W_{n,N}(z). \tag{2.44}$$

Indeed, it follows from (1.5b), (1.8) and (2.18) that for any polynomial $P \in \mathbb{P}_n$ with $|P(\zeta)| \le 1$, $\zeta \in E$ we have for $z \in [-1,1]$ the inequality

$$u(z) := \log \frac{|P(z)|}{W_{n,N}(z)} = \log |P(z)| + (V_1 - V_2)(z) + w \le \log \left(\frac{C}{\delta} \log N \right).$$

For $z \in \bar{\mathbb{C}} \setminus [-1,1]$ the function $u(z)$ above is subharmonic. Therefore, by the maximum principle the inequality above is valid for any $z \in \bar{\mathbb{C}}$, and (2.44) follows.

We note that the problem in general is significantly simpler for points z separated from $\Delta = [-1,1]$ than for the case $z \in \Delta$. For example, it may be proved comparatively easily that for any ρ, δ, $M > 0$ and then for any measure σ on Δ, $|\sigma| = N \in \mathbb{N}$ with $\sigma'(x) \ge \delta N$, $\sigma''(x) \le MN$, we have for the extremal quantity in (1.12),

$$C_1 \le K_n(z,\sigma)/W_n(z,\sigma) \le C_2; \quad \operatorname{dist}(z,\Delta) \ge \rho,$$

where $\log W_n(z,\sigma) = V(z,\sigma) - V(z,\mu) - w$; $\mu = \mu_{t,\varphi}$, $w = W_{t,\varphi}$ for $t = N-n$, $\varphi(x) = -V(x,\sigma)$ (see §4.1 for definitions). Constants $C_1 C_2$ depend on ρ, δ, M but not on n, N (of course $n < N$).

In our case, $d\sigma = \dfrac{N}{2}\, dx$ we have $W_n(x,\sigma) = W_{n,N}(x)$ and the estimate above suggests that the $\log N$ factor is not in effect in (2.44) for $\operatorname{dist}(z,\Delta) \ge \rho$ if we allow the constant C to depend on ρ. That is indeed true and remains true for $\operatorname{dist}(z, \{-r, r\}) \ge \rho$.

Finally, the proof of the low bound associated with (2.44),

$$K_{n,N}(z) \ge C(\rho) W_{n,N}(z), \quad \operatorname{dist}(z, [-1,-r] \cup [r,1]) \ge \rho,$$

will be clearly outlined in §4.2 when we prove related versions of this low bound for $z \in [-1,1] \cup [r,1]$.

Remark 2. We mention also the following version of (1.8). For $s > 1$ we define

$$\|P\|_{S,E} = \left(\frac{1}{N} \sum_{\zeta \in E} |P(\zeta)|^s \right)^{1/s}. \tag{2.45}$$

Then for any $\delta > 0$ and $n \le (1-\delta)N$, $n \le N-2$ we have

$$\max_{P\in\mathbb{P}_n} \frac{|P(x)|}{\|P\|_{s,E}} \le \frac{C(s)}{\sqrt{\delta}} N^{\frac{1}{s}} W_{n,N}(x), \quad x \in \mathbb{C}, \tag{2.46}$$

with a constant $C(s)$ depending only on s.

To prove (2.46) for $|x| \in [-1,1]$ we return to the beginning of the proof of (2.40) and obtain, using the same arguments, the following version of (2.40):

$$|P(x)| \le \frac{C}{N} W(x) \sum_{\zeta\in E} |P(\zeta)| m(\zeta, x), \quad P \in \mathbb{P}_{n-1}, \tag{2.47}$$

where $m(\zeta, x) = \min\left\{N\pi/2, \dfrac{1}{|\zeta - x|}\right\}$, $W = W_{n,N}$, $|x| \le 1$. Then we apply the Hölder inequality to the sum in (2.47) above; this makes

$$|P(x)| \le CN^{1/s-1} W(x) \|P\|_{s,E} M(\zeta, x), \quad P \in \mathbb{P}_{n-1},$$

where

$$M(s,x) = \left(\sum_{\zeta\in E} m(\zeta,x)^{s'}\right)^{1/s'}, \quad \frac{1}{s'} + \frac{1}{s} = 1.$$

Now, straightforward computations show that

$$M(\zeta, x) < C_0(s)N, \text{ where } C_0(s)^{1/s'} = \pi^{s'} + 2\sum_{K=1}^{\infty} \frac{1}{K^{s'}}.$$

Thus, $|P(x)| \le C_1(s) N^s W_{n,N}(x) \|P\|_{s,E}$, $|x| \le 1$, $P \in \mathbb{P}_{n-1}$. Finally we apply the last inequality above with n replaced by $n+1$ and use Lemma 4 to reduce $W_{n+1,N}$ to $W_{n,N}$; (2.46) follows for $|x| \le 1$. To extend this inequality to the whole plane we use the maximum principle for subharmonic functions as in the proof of (2.44) above.

3. Proofs of Lemma 1 and Lemma 2

In the next subsection, 3.1, we present a short discussion of the representation (1.11). It seems that this construction may potentially have a large field of applications in approximation theory and beyond. For earlier applications see, for example, [10]–[12]. See also [14]–[18] for applications of a closely-related "center mass" version. Subsequently, in §3.1 and §3.2, we will apply Theorem 2 of §3.1 in two particular situations. Analysis of those situations will help us understand how the method works in general (see also Lemma 7 in §4.2 below).

3.1. *Phase-Amplitude representation of a polynomial with real zeros.* Let $d\sigma(x) = \sigma'(x)\,dx$ be a positive absolutely continuous measure on the interval

$\Delta = [-\beta, \beta]$ whose norm $|\sigma| = \int_{-\beta}^{\beta} \sigma'(t)\, dt = N$ is a natural number. We also assume that $\sigma'(t)$ is positive and continuous in $(-\beta, \beta)$. We denote

$$\phi(x) = \phi(x, \sigma) = \pi \int_x^\beta d\sigma(t), \tag{3.1}$$

$$V(x) = V(x, \sigma) = \int \log \frac{1}{|x-t|}\, d\sigma(t). \tag{3.2}$$

Then we define N points $-\beta < t_1 < t_2 < \cdots < t_N < \beta$ by

$$\cos\phi\,(t_k) = 0, \quad k = 1, 2, \ldots, N \tag{3.3}$$

and the polynomial

$$T(x) = T(x, \sigma) = \prod_{k=1}^{N} (x - t_k)\,. \tag{3.4}$$

We note that $\phi(x)$ is decreasing from πn to 0 in $(-\beta, \beta)$, so the equation (3.3) has indeed n zeros in this interval whose roots may be equivalently defined by

$$\sigma\,([-\beta, t_1]) = \sigma\,([t_n, \beta]) = 1/2, \tag{3.5a}$$

$$\sigma\,([t_k, t_{k+1}]) = 1, \quad k = 1, 2, \ldots, N, \tag{3.5b}$$

which conicides with (1.10) if $\beta = 1$. Now we define, for $|x| \le \beta$,

$$\eta(x) = \eta(x, \sigma) = \log \frac{T(x)}{2\cos\phi(x)} + V(x). \tag{3.6}$$

In an equivalent form, (3.6) may be written as

$$T(x) = 2e^{\eta(x) - V(x)} \cos\phi(x). \tag{3.7}$$

In other words, the function $C(x)$ in (1.11) is now represented in the form $C(x) = 2e^{\eta(x)}$. The reason is that under "normal circumstances" the new function $\eta(x)$ is small at least for, at most, part of the points $x \in \Delta$. (Consequently, $C(x, \sigma)$ is normally close to 2; see Remark 4 in §3.3 below.)

For the purposes of this paper we need only to prove boundedness of $\eta(x)$ for three particular piecewise analytic functions $\sigma'(x)$. We will do this using an integral formula (Theorem 2 below) which, by the way, allows us to prove that $\eta(x)$ is, indeed, small at regular points $x \in \Delta$. To make things simpler, we consider in detail the basic case when $\sigma'(x)$ is analytic in the whole interval $(-\beta, \beta)$ which is enough to give complete proofs of Lemmas 1 and 2. A remark on the piecewise version needed in Lemma 7, Section 4, is presented at the end of this section.

Let $\sigma'(x)$ be positive and analytic in $(-\beta, \beta)$. We denote by $\Omega(\sigma)$ the maximal domain of analyticity of σ which is convex in the direction of the imaginary axis. Thus, $\sigma \in H(\Omega(\sigma))$ and $\Omega(\sigma) \cap \{\operatorname{Re} z = x\}$ is an interval.

Then $\phi(z) = \int_z^\beta \sigma'(\zeta)\,d\zeta \in H(\Omega(\sigma))$ and integrating subsequently along the two segments from $x+iy$ to x and then from x to β we come to the formula

$$\phi(x+iy) = \pi \int_{x+iy}^{x} \sigma'(\zeta)\,d\zeta + \pi \int_x^\beta \sigma'(\zeta)\,d\zeta = -\pi i \int_0^y \sigma'(x+it)\,dt + \phi(x),$$

$x+iy \in \Omega_\sigma$. Since $\sigma'(x) > 0$, $|x| < \beta$ it follows that the function

$$\operatorname{Im}\phi(x+iy) = -\pi \int_0^y \operatorname{Re}\left(\sigma'(x+it)\right)\,dt \tag{3.8}$$

is negative for $0 < y < y(x)$, $|x| < \beta$, where $y(x)$ is some positive function.

Definition 1. A piecewise smooth curve Γ with endpoints $-\beta$ and β is called admissible if $\Gamma \smallsetminus \{-\beta, \beta\}$ belongs to $\Omega(\sigma) \cap \{\operatorname{Im} z > 0\}$ and $\operatorname{Im}\phi(z) < 0$ in the domain Ω_Γ bounded by $\Gamma \cup \Delta$.

In view of the remark above, any curve in the part of $\Omega(\sigma)$ in the upper half-plane which is close enough to Δ is admissible. Moreover, we have

$$\Lambda(z) = \log\left(1 + e^{-2i\phi(z)}\right) \in H\left(\Omega_\Gamma\right) \tag{3.9}$$

where $\log\zeta$ is the principle branch in the right half-plane (note that $|\exp(-2i\phi(z))| < 1$, $z \in \Omega_\Gamma$).

Theorem 2. *For any admissible curve* Γ *oriented from* β *to* $-\beta$,

$$\eta(x) = \frac{1}{\pi}\operatorname{Im}\int_\Gamma \frac{\Lambda(\zeta)d\zeta}{x-\zeta}, \quad x \in \mathbb{R} \smallsetminus \{-\beta, \beta\},$$

where $\eta(x) = \eta(x,\sigma)$ *is as defined by* (3.6) *for* $|x| < \beta$ *and otherwise defined by*

$$\eta(x) = \log|T(x)| + V(x), \quad |x| > \beta. \tag{3.10}$$

The theorem was proved in [12, Th. 3] for a slightly-modified case when $|\sigma| = N + \frac{1}{2}$ and subsequently $\frac{1}{2}$ is replaced by $\frac{3}{4}$ on the right-hand side of (3.5a) defining zeros of T (this case is immediately related to orthogonal polynomials). Then the definition of $\eta(x)$ has to be changed by adding $\frac{1}{4}\log\left(\beta^2 - x^2\right)$ to the right-hand side of (3.6) and $\phi(x)$ in (3.1) is replaced by $\phi(x) - \pi/4$. The proof of Theorem 2 above is, after that, identical to the proof of Theorem 3 in [12] (see pp. 85–87).

The following simple remark is often useful in applications of Theorem 2. Denote

$$\eta(x,\gamma) = \frac{1}{\pi}\operatorname{Im}\int_\gamma \frac{\Lambda(\zeta)d\zeta}{x-\zeta}.$$

If γ^+ and γ^- are two subarcs of an admissible curve Γ symmetric to each other with respect to the imaginary axis and $\sigma'(x) = \sigma'(-x)$, $x \in \Delta$, then

$$\eta\left(x, \gamma^+\right) = \eta\left(-x, \gamma^-\right), \quad x \in \mathbb{R} \setminus \{-\beta, \beta\}, \tag{3.11}$$

(see (4.32) in [12, p. 87]).

Remark 3. Suppose that the function $\sigma'(x)$ is piecewise positive and piecewise analytic in $(-\beta, \beta)$. In other words there is a finite number of points

$$\beta_0 = -\beta < \beta_1 < \cdots < \beta_p < \beta = \beta_{p+1}$$

such that $\sigma'(x)$ is positive and analytic in each interval $\Delta_k = (\beta_{k-1}, \beta_k)$, $k = 1, \ldots, p+1$.

For each interval Δ_k we define admissible curve Γ_k in exactly the same way as we did in Definition 1 for the whole interval $(-\beta, \beta)$. Subsequently, the definition of the admissible curve for σ now takes the following form.

Definition 2. A curve $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_p$ is called admissible (for σ) if for each k the curve Γ_k is admissible for $\sigma|_{\Delta_k}$.

With this modification, Theorem 2 remains valid for piecewise analytic $\sigma'(x)$. Moreover, its proof does not require any modifications. We note that the sum $\Gamma = \Gamma_1 + \cdots + \Gamma_p$ has to be understood as a formal sum meaning $\displaystyle\int_\Gamma f d\zeta = \sum_{k=1}^{P} \int_{\Gamma_k} f(\zeta)\, d\zeta$; see comments to Lemma 7, §4.2 below.

3.2. *Proof of Lemma* 1. By definitions (2.1)–(2.3) compared with (3.1)–(3.4) the polynomial T in (2.3) may be written as

$$T(x) = T(x, \sigma), \quad d\sigma = \frac{N}{2}\, dx, \quad x \in [-1, 1],$$

and further $V_1(x) = V(x, \sigma)$, $\phi_1(x) = \phi(x, \sigma)$. The conditions of Theorem 2 are clearly satisfied. We have for $z = x + iy$,

$$\Lambda(z) = \log\left(1 + e^{-2i\phi_1(z)}\right) = \log\left(1 + e^{-\pi N y - i\pi N(1-x)}\right).$$

In particular, $\operatorname{Im}\phi(z) = -\dfrac{\pi}{2} N y$, so that any curve in the upper half-plane joining 1 and -1 is admissible. For $R > 0$ we define $\Gamma(R) = \Gamma^+(R) + \Gamma_0(R) + \Gamma^-(R)$ where

$$\Gamma^\pm(R) = \{\zeta = \pm 1 + it, t \in [0, R]\},$$
$$\Gamma_0(R) = \{\zeta = x + iR, x \in [-1, 1]\},$$

and represent the function $\eta(x)$ in (3.6) by Theorem 2 with $\Gamma = \Gamma(R)$. Since

$$\max_{\zeta \in \Gamma_0(R)} |\Lambda(\zeta)| \leq \log\left(1 - e^{\pi N R}\right)$$

we have $\max_{[-1,1]} |\eta(x, \Gamma_0(R))| \to 0$ as $R \to \infty$ and therefore Theorem 2 may be used with

$$\Gamma = \Gamma^+ + \Gamma^-, \quad \Gamma^\pm = \Gamma^\pm(\infty).$$

For the part of the integer related to Γ^+,

$$\eta^+(x) = \eta\left(x, \Gamma^+\right) = \frac{1}{\pi} \operatorname{Im} \int_{\Gamma^+} \Lambda(\zeta) \frac{d\zeta}{x-\zeta},$$

we make substitution $\zeta = 1 + it$, which reduces the integral to the following:

$$\eta^+(x) = -\frac{1}{\pi} \int_0^\infty \log\left(1 + e^{-\pi N t}\right) \frac{y\, dt}{y^2 + t^2}, \quad y = 1 - x.$$

From here $\eta^+(x) < 0$ and

$$\left|\eta^+(x)\right| \le \log 2 \frac{1}{\pi} \int_0^\infty \frac{y\, dt}{y^2 + t^2} = \frac{\log 2}{2}, \quad |x| \le 1.$$

On the other hand, $\eta(x) = \eta^+(x) + \eta^+(-x)$ by (3.11). Thus, $-\log 2 \le \eta(x) \le 0$, $x \in [-1, 1]$. Since $C_1(x) = 2e^{\eta(x)}$, Lemma 1 follows.

Remark 4. Making the substitution $t = N\tau$ in the integral representing $\eta^+(x)$ above we reduce it to the form

$$\eta^+(x) = -\frac{1}{\pi} \int_0^\infty \log\left(1 + e^{-\pi\tau}\right) \frac{y_N\, d\tau}{y_N^2 + \tau^2}, \quad y_N = N(1 - x) \ge 0,$$

which immediately gives for η^+ the asymptotic expansion

$$\sim \eta^+(x) \simeq \sum_{k=0}^\infty (-1)^k \frac{c_k}{y_k^{2k+1}}, \quad c_k = \frac{1}{\pi} \int_0^\infty t^{2k} \log\left(1 + e^{-\pi t}\right) dt$$

as $y_N \to \infty$. It is practically convenient to use the first term (case $k = 0$) of the series slightly modified to keep it reasonably close to $-\eta^+$ on the whole segment $[-1, 1]$. Taking into account the corresponding term for $\eta^-(x) = \eta^+(-x)$ we obtain the asymptotic representation

$$-\eta(x) \simeq \frac{2c_0}{N\left(1 - x^2\right) + c_0/\log 2}, \quad |x| \le 1$$

meaning that the ratio $\Gamma_N(x)$ of the two functions above is reasonably (but not infinitely as $N \to \infty$) close to 1 on the whole interval $[-1, 1]$ and
$\Gamma_N(x) - 1 = O\left(N^{-3}\left(1 - x^2\right)^{-3}\right)$ if $N\left(1 - x^2\right)$ is large.

Now, it may be easily verified using the construction in §4.1 with a finite $R > 0$ that exactly the same asymptotic formula for $-\eta(x)$ holds true for an arbitrary σ with large enough $N = |\sigma|$ under (for example) the following conditions: $\sigma(x) = \sigma(-x)$, $|x| \le 1$. Then, for some C_1, C_2, C_3 we have $\sigma' \in H\left(\bar{\Omega}\right)$ where $\Omega = \{x + iy, |x| < 1, 0 < y < C_1\}$. Next, $\operatorname{Re} \sigma'(z) \ge C_2 N$, $z \in \Omega$;

then, $|\sigma''(1+it)/\sigma'(1+it)| \le C_3,\ t \in [0, C_1]$ and $\sigma'(1) = N/2$. If the last condition is not satisfied one has to replace the formula above with the following one

$$-\eta(x) \simeq \frac{2C_0}{2\sigma'(1)\,(1-x^2) + C_0/\log 2}, \quad |x| \le 1,$$

with the same meaning and the same C_0. This leads to the important conclusion that asymptotics for $\eta(x)$ depend only on σ' at the end points of the interval (remember: $\sigma(x) = \sigma(-x)$; actually we still need some regularity of σ' near endpoints as in the conditions above).

If σ' has algebraic zeros or singularities at the endpoints, that is

$$\sigma'(x) = \left(1-x^2\right)^{\alpha} \sigma_0(x), \quad |x| \le 1, \quad \alpha > -1,$$

and σ_0 satisfies the same conditions which were earlier assumed for σ in case $\alpha = 0$ then we have

$$-\eta(x) = \frac{A}{\sigma_0(1)^{\beta}\,(1-x^2) + B}, \quad x \le 1; \quad \beta = \frac{1}{1+\alpha}, \quad \alpha \ne -\frac{1}{2}$$

where A, B depend on α. In the exceptional case $\alpha = -\frac{1}{2}|\eta(x)|$ is exponentially small for large $|\sigma|$. In the ideal case $\sigma'(x) = \frac{1}{\pi} N \left(1-x^2\right)^{-1/2}$ we have $\eta(x) \equiv 0$. ($T(x,\sigma)$ is the Chebyshev polynomial in this case.)

Case $\alpha = 1/2$ related to orthogonal polynomials on $\mathbb{R}$ was particularly investigated in [12]. In the following proof of Lemma 3 we deal exactly with this situation: $\sigma = \mu$ is defined by (2.8). For certain technical reasons we will give the direct proof of Lemma 3 based on Theorem 2.

3.3. *Proof of Lemma* 2. Here we have $S(x) = T(x, \mu)$ with μ as defined in (2.8), $V_2(x) = V(x, \mu)$, $\phi_2(x) = \phi(x, \mu)$, $\beta = r$. Let Ω be the equilateral triangle in the upper half-plane based on $[-r, r]$. Vertices of the triangle are $-r,\ r,\ ri\sqrt{3}$. Let

$$\Gamma = \Gamma^+ + \Gamma^-$$

be the part of the boundary of Ω in the upper half-plane that is, with $\sigma = e^{\pi i/3}$,

$$\Gamma^+ = \left\{\zeta = r\left(1 - \bar{\sigma} t\right), 0 \le t \le 2\right\},$$
$$\Gamma^- = \left\{\zeta = -r(1 + \sigma t), -2 \le t \le 0\right\}.$$

It follows immediately from geometry that

$$\left|\arg\sqrt{r^2 - z^2}\right| \le \pi/6, \quad z \in \Omega. \tag{3.12}$$

We note also that for the standard branch of $\tan^{-1} z$ in the right half-plane $\{z : |\arg z| < \pi/2\}$,

$$\left|\arg \tan^{-1} z\right| \le |\arg z|. \tag{3.13}$$

Indeed, in the case where $\arg z \geq 0$ it follows from simple geometry that $\left|\arg\left(d\zeta/\left(1+\zeta^2\right)\right)\right| \leq \arg z$ where $d\zeta$ is the differential in $\tan^{-1} z = \int\limits_0^z d\zeta/(1+\zeta^2)$ and integration goes along the segment from 0 to z; (3.13) follows. The case $\arg z < 0$ is similar. It is also clear that $\arg z$ and $\arg\left(\tan^{-1} z\right)$ have the same sign.

It follows by (3.12) and (3.13) that

$$\arg \mu'(z) = \arg\left\{\frac{N}{\pi}\tan^{-1}\frac{N}{n}\sqrt{r^2-z^2}\right\} \in \left[-\frac{\pi}{6},\frac{\pi}{6}\right], \quad z \in \Omega.$$

From here

$$\operatorname{Im}\phi(x+iy) = -\int_0^y \operatorname{Re}\mu'(x+it)\,dt < 0, z \in \Omega$$

and, therefore, the curve Γ defined above is admissible. Next, we estimate the integral

$$\eta^+(x) = \frac{1}{\pi}\operatorname{Im}\int_{\Gamma^+}\frac{\Lambda(\zeta)\,d\zeta}{x-\zeta}. \tag{3.14}$$

Denote $\sigma = e^{i\pi/3}$, $y = 1 - x/r$, $\zeta(t) = r\,(1-\bar\sigma t)$ and, further,

$$\Lambda_1(\zeta(t)) = \operatorname{Re}(\Lambda(\zeta(t)), \quad \Lambda_2(t) = \operatorname{Im}(\Lambda(\zeta(t))). \tag{3.15}$$

Making substitution $\zeta = \zeta(t)$ in the integral in (3.14) and observing that

$$\frac{d\zeta(t)}{x-\zeta(t)} = \frac{1-i\sqrt{3}}{2}\,\frac{y\,dt}{|y-\bar\sigma t|^2} - \frac{t\,dt}{|y-\bar\sigma t|^2}$$

we rewrite (3.14) as follows

$$\eta^+(x) = \frac{1}{2\pi}\int_0^2\left(\Lambda_2(t)-\sqrt{3}\Lambda_1(t)\right)\frac{y\,dt}{|y-\bar\sigma t|^2} - \frac{1}{\pi}\int_0^2 \Lambda_2(t)\frac{t\,dt}{|y-\bar\sigma t|^2}. \tag{3.16}$$

With (3.9) we may represent Λ_1, Λ_2 in (3.15) as follows:

$$\Lambda_1(t) = \frac{1}{2}\log\left(2+2e^{-R(t)}\cos\mathfrak{J}(t)\right), \tag{3.17}$$

$$\Lambda_2(t) = \tan^{-1}\frac{e^{-R(t)}\sin\mathfrak{J}(t)}{1+e^{-R(t)}\cos\mathfrak{J}(t)} \tag{3.18}$$

where

$$R(t) = \operatorname{Re}(2i\phi(\zeta(t)), \quad \mathfrak{J}(t) = \operatorname{Im}(-2i\phi(\zeta(t))). \tag{3.19}$$

LEMMA 5. *The function $R(t)$ is positive, increasing and convex in $(0,2]$. The following inequalities are valid for $t \in [0,2]$:*

$$R(t) \geq \sqrt{3}|\mathfrak{J}(t)|, \tag{3.20}$$

$$0 \leq \Lambda_1(t) \leq \log 2, \tag{3.21}$$

$$|\Lambda_2(t)| \leq \frac{1}{\sqrt{3}}R(t)e^{-R(t)}. \tag{3.22}$$

Proof. Making the substitution $z = r(1-\bar{\sigma}\tau)$, $\tau \in [0,2]$, in the integral

$$2i\phi(\zeta(t)) = 2\pi i \int_{\zeta(t)}^{r} \mu'(z)\,dz$$

where μ' is defined by (2.8) we reduce the integral to the following

$$2i\phi(\zeta(t)) = 2Nr\bar{\sigma}i \int_0^t \tan^{-1}\left(\frac{Nr}{n}\sqrt{S(\tau)}\right) d\tau, \tag{3.23}$$

where

$$S(\tau) = \bar{\sigma}\tau\,(2-\bar{\sigma}\tau)\,, \quad \sigma = e^{\pi i/3}. \tag{3.24}$$

We note that $\arg S(\tau) \in [-\pi/3, 0]$, $\tau \in [0,2]$. With (3.13) this gives

$$\arg\left(\tan^{-1}\left(\frac{Nr}{n}\sqrt{S(\tau)}\right)\right) \in [-\pi/6, 0];$$

the same is true for the integral in (3.23) and, therefore,

$$\arg(2i\phi(\zeta(t))) \in [0, \pi/6], \quad t \in [0,2].$$

This proves that R is positive and that (3.20) holds true. For the same reasons, we have

$$R''(t) = 2\frac{(Nr)^2}{n\sqrt{t}}\operatorname{Re}(f(t))$$

where

$$f(t) = \frac{1-\bar{\sigma}t}{\sqrt{2-\bar{\sigma}t}\,(1+(Nr/n)^2 S(t))}.$$

Now, it will be enough to prove that

$$\arg f(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad t \in [0,2]. \tag{3.25}$$

We have $\arg\left(1+\left(\frac{Nr}{n}\right)^2 S(t)\right) = \theta \arg S(t)$ for some $\theta \in [0,1]$ and, therefore,

$$\begin{aligned}\arg f(t) &= \arg\frac{1-\bar{\sigma}t}{\sqrt{2-\bar{\sigma}t}} - \theta\arg S(t)\\ &= \arg\frac{1-\bar{\sigma}t}{\sqrt{2-\bar{\sigma}t}} + \theta\left(\frac{\pi}{3} - \arg(2-\bar{\sigma}t)\right).\end{aligned}$$

It follows from geometry that both terms in the right-hand side above are nonnegative for $t \in [0,2]$ so that $\theta = 1$ gives the upper bound for $\arg f$. Thus,

$$\begin{aligned}0 \le \arg f(t) \le a(t) &:= \frac{\pi}{3} + \arg\frac{1-\bar{\sigma}t}{(2-\bar{\sigma}t)^{3/2}}\\ &= \frac{\pi}{3} + \tan^{-1}\frac{t\sqrt{3}}{2-t} - \frac{3}{2}\tan^{-1}\frac{t\sqrt{3}}{4-t}.\end{aligned}$$

Differentiating $a(t)$ we come to

$$a'(t) = \frac{8\sqrt{3}\left(1+t-2t^2\right)}{\left(3t^2+(t-2)^2\right)\left(3t^2+(t-4)^2\right)}.$$

Since $1+t-2t^2 = (2t+1)(1-t)$ we obtain

$$\max_{[0,2]} a(t) = a(1) = \frac{\pi}{3} + \tan^{-1}\sqrt{3} - \frac{2}{3}\tan^{-1}\frac{1}{\sqrt{3}} = \frac{5\pi}{12}.$$

This proves (3.25) and the convexity of $R(t)$ follows.

It remains to prove (3.21) and (3.22). The upper bound in (3.21) follows from (3.17) and the positivity of R. To prove the related low bound we consider two cases. If $|\mathfrak{J}(t)| \le \pi/2$ then $\Lambda_1(t) \ge \frac{1}{2}\log 2 > 0$. If $|\mathfrak{J}(t)| \ge \pi/2$ then $R(t) \ge \pi\sqrt{3}/2$ and

$$\Lambda_1(t) \ge \frac{1}{2}\log\left(2 - 2e^{-\pi\sqrt{3}/2}\right) > 0.$$

To prove (3.22) we use the same method. If $|\mathfrak{J}(t)| \le \pi/2$, $|\sin\mathfrak{J}(t)| \le |\mathfrak{J}(t)| \le R(t)/\sqrt{3}$, $1 + e^{-R}\cos\mathfrak{J} \ge 1$ and (3.22) follows by (3.18).

If $|\mathfrak{J}(t)| \ge \pi/2$ then $|\sin\mathfrak{J}(t)| \le \dfrac{2}{\pi}|\mathfrak{J}(t)| \le \dfrac{2R(t)}{\pi\sqrt{3}}$ and $1 + e^{-R}\cos\mathfrak{J} \ge 1 - e^{-\pi\sqrt{3}/2}$. Combining the two inequalities above we obtain (3.22). The proof of Lemma 5 is completed. □

Now we are ready to estimate integrals in the right-hand side of (3.16). First, we will consider the part defined by

$$\eta_1^+(x) := -\frac{\sqrt{3}}{2\pi}\int_0^2 \Lambda_1(t)\frac{y\,dt}{|y-\bar{\sigma}t|^2}, \quad y = 1 - \frac{x}{r} \tag{3.26}$$

and prove that the following inequalities are satisfied

$$-\frac{2}{3}\log 2 < \eta_1^+(x) < 0, \quad x < r, \tag{3.27a}$$

$$0 < \eta_1^+(x) < \frac{1}{3}\log 2, \quad x > r. \tag{3.27b}$$

Let $x < r$, so that $y = 1 - x/r > 0$. We have

$$\begin{aligned} 0 < I(y) &= \int_0^2 \frac{y\,dt}{|y-\bar{\sigma}t|^2} = \int_0^2 \frac{y\,dt}{y^2 - yt + t^2} \\ &= \int_0^{2/y} \frac{dt}{t^2-t+1} < \int_0^{\infty}\frac{dt}{t^2-t+1} \\ &= \int_{-1/2}^{\infty}\frac{dt}{t^2+3/4} = \frac{4\pi}{3\sqrt{3}}. \end{aligned} \tag{3.28}$$

Combined with (3.21) this proves (3.27a). For $x > r$ $(y < 0)$ we similarly obtain

(3.29)
$$0 > I(y) = -\int_0^2 \frac{|y|dt}{y^2+|y|t+t^2} = -\int_0^{2/|y|} \frac{dt}{t^2+t+1} > -\int_0^\infty \frac{dt}{t^2+t+1}$$
$$= -\int_{1/2}^\infty \frac{dt}{t^2+3/4} = -\frac{2\pi}{3\sqrt{3}}.$$

With (3.21) this gives (3.27b). Next, it follows by (3.22) that $|\Lambda_2(t)| \le 1/e\sqrt{3}$, $t \in [0,2]$. Using (3.28) and (3.29) we now obtain

(3.30)
$$\left|\eta_2^+(x)\right| = \left|\frac{1}{2\pi}\int_0^2 \Lambda_2(t)\frac{y\,dt}{|y-\bar{\sigma}t|^2}\right| \le \frac{2}{9e}.$$

Finally, for
$$\eta_3^+(x) = -\frac{1}{\pi}\int_0^2 \Lambda_2(t)\frac{t\,dt}{|y-\bar{\sigma}t|^2}$$

we use (3.22) and the convexity of $R(t)$ (Lemma 4) which implies that $R(t)/t \le R'(t)$, $t > 0$. We note also that $|y-\bar{\sigma}t|^2 = (y-t/2)^2 + 3/4t^2 \ge 3/4t^2$. Now,

(3.31)
$$\left|\eta_3^+(x)\right| \le \frac{4}{3\pi\sqrt{3}}\int_0^2 e^{-R(t)}\frac{R(t)}{t}\,dt \le \frac{4}{3\pi\sqrt{3}}\int_0^2 e^{-R(t)}R'(t)\,dt$$
$$\le \frac{4}{3\pi\sqrt{3}}\int_0^\infty e^{-x}\,dx = \frac{4}{3\pi\sqrt{3}}.$$

According to the definition of $\eta(x)$ in (3.6) and (3.10), the function $C_2(x)$ in Lemma 2 is represented as follows:

(3.32)
$$C_2(x) = \begin{cases} 2e^{\eta(x)}, & |x| < r \\ e^{\eta(x)}, & |x| \in (r,1) \end{cases}.$$

We note that this function C_2 is continuous, it is $\eta(x)$ that makes jumps of $\pm\log 2$ at $x = \pm r$. Next, we have by (3.11)

(3.33)
$$\eta(x) = \eta_1(x) + \eta_2(x) + \eta_3(x),$$

(3.34)
$$\eta_i(x) = \eta_i^+(x) + \eta_i^+(-x), \quad i = 1,2,3.$$

Let
$$\tilde{\eta}_1(x) = \begin{cases} \eta_1(x) + \log 2, & |x| < r \\ \eta_1(x), & |x| \in (r,1) \end{cases}.$$

From (3.33) and (3.34),

(3.35)
$$\max_{[-1,1]} \tilde{\eta}_1(x) - \min_{[-1,1]} \tilde{\eta}_1(x) \le \frac{5}{3}\log 2.$$

At the same time we have by (3.30) and (3.31)

$$\max_{[-1,1]} (\eta_2 + \eta_3)(x) - \min_{[-1,1]} (\eta_2 + \eta_3)(x) \le 4\left(\frac{2}{9e} + \frac{4}{3\pi\sqrt{3}}\right). \tag{3.36}$$

Now we have $C_2(x) = \exp\{(\tilde{\eta}_1 + \eta_2 + \eta_3)(x)\}$, $|x| \le 1$ and

$$\max_{[-1,1]} C_2(x) / \min_{[-1,1]} C_2(x) \le 2^{5/3} \exp\left\{\frac{8}{9e} + \frac{16}{3\pi\sqrt{3}}\right\} < 12.$$

The proof of Lemma 2 is completed. □

4. Equilibrium problems related to Theorem 1

We have included a definition and a few comments in the following text to make it connected and understandable for a reader not experienced in potential theory.

4.1. *Equilibrium in the external field. Proof of Lemma* 3. Let φ be a continuous function (external field) on $[-1,1]$ and t. The measure $\mu_t = \mu_{t,\varphi}$ with $|\mu_t| = t$ satisfying the following (equilibrium) conditions

$$V(x,\mu_t) + \varphi(x) = w_t, \quad x \in \operatorname{supp}(\mu_t), \tag{4.1a}$$

$$V(x,\mu_t) + \varphi(x) \ge w_t, \quad x \in [-1,1] \tag{4.1b}$$

with a constant w_t is called the equilibrium measure on $[-1,1]$ in the (external) field φ and with norm t.

Conditions (4.1 a,b) with $|\mu_t| = t$ uniquely define both μ_t and the equilibrium constant w_t. Many basic extremal problems for polynomials may be explicitly solved in terms of $\mu_{t,\varphi}$, $w_{t,\varphi}$; for this and other reasons the concept of the equilibrium in the external field has been intensively studied in the last two decades; see [14] for general information and references.

By comparing conditions (4.1) above with (2.17) in Lemma 3 and associated normalization conditions we conclude that conditions (2.16)–(2.17) uniquely define μ and w and $\mu = \mu_{t,\varphi}$ where $t = N - n$, $\varphi(x) = -V(x,\sigma)$, $\sigma = \dfrac{N}{2}\,dx$. So, in other words, our problem is to prove formula (2.8) for the density of $\mu_{t,\varphi}$ and formula (2.18) for its equilibrium potential $V(x,\mu_{t,\varphi}) + \varphi(x)$. The problem was essentially solved in [13] where different normalization was used and formula (2.18) was not explicitly presented. To make clear connections and complete the proof of Lemma 3 we will reproduce some general results from [13] as Lemma 6 below. This lemma may also be useful for the analysis of the generalized problem (1.12).

We note that to find $\mu_{t,\varphi}$ explicitly is in general a difficult problem even for very smooth φ. Actually, most difficult is to find $\operatorname{supp}(\mu_{t,\varphi}) = S_{t,\varphi}$. On the other hand, the problem is easily solvable when it is known in advance that

$S_{t,\varphi}$ is a segment. Thus, it is important to know under what conditions on φ the support $S_{t,\varphi}$ of related equilibrium measure is a segment. The next lemma gives an answer to this question for the symmetric case $\varphi(x) = \varphi(-x)$. (See [14] for earlier results.)

LEMMA 6. *Let φ be an even absolutely continuous function in $[-1, 1]$ such that the corresponding function*

$$\nu(x) = \frac{2}{\pi}\int_0^x \frac{t\varphi'(t)}{\sqrt{x^2-t^2}}\,dt, \quad x \in [0,1] \tag{4.2}$$

is increasing in $[0, 1]$ from 0 to $T \in [0, +\infty]$. Denote by β_t, $t \in [0, T]$ the corresponding inverse function (e.g., $\nu(\beta_t) = \beta_{\nu(t)} = t$). Then for the equilibrium measure $\mu_t = \mu_{t,\varphi}$, for $t \le T$,

$$\operatorname{supp}(\mu_t) = [-\beta_t, \beta_t], \quad 0 < t \le T, \tag{4.3}$$

$$\mu_t'(x) = \frac{1}{\pi}\int_{|x|}^{\beta_t} \frac{\nu'(\tau)\,d\tau}{\sqrt{\tau^2-x^2}}, \quad |x| \le \beta_t, \tag{4.4}$$

$$V(x,\mu_t) + \varphi(x) - w_t = \int_t^\infty g_\tau(x)\,d\tau, \quad x \in [-1,1], \tag{4.5}$$

where $g_\tau(x)$ is the Green function for $\mathbb{C} \setminus [-\beta_\tau, \beta_\tau]$ with the pole at ∞; that is $g_\tau(x) = 0$, $|x| \le \beta_\tau$ and

$$g_\tau(x) = \int_{\beta_\tau}^{|x|} \frac{d\zeta}{\sqrt{\zeta^2-\beta_\tau^2}}, \quad |x| \ge \beta_\tau. \tag{4.6}$$

Proofs of (4.3) and (4.4) are presented in [13, Th. 4, p. 1224], proof of (4.5) is in [13, Lemma 5.3, p. 1226].

We note that if $\nu(x)$ in (4.2) is nondecreasing and $T > 0$ then assertions of Lemma 6 remain valid if we properly define the inverse function β_t at its points of discontinuity. If $\nu(x)$ is not nondecreasing then there exist $t \in (0, T)$ such that $\operatorname{supp}(\mu_{t,\varphi})$ is disconnected.

Next, simple computations of expressions in (4.2)–(4.4) for the external field

$$\varphi(x) = \frac{1}{2c}\int_{-1}^{1} \log|x-t|\,dt, \quad x \in [-1,1] \tag{4.7}$$

with a parameter $c \in (0, 1)$ were performed in [13, pp. 1227–1228] with the following results

$$\begin{aligned} \nu(x) &= \frac{1}{c}\left(1-\sqrt{1-x^2}\right), \\ \operatorname{supp}(\mu_t) &= [-r,r], \quad r = \sqrt{1-c^2}, \end{aligned} \tag{4.8}$$

$$\mu_t'(x) = \frac{1}{\pi c}\tan^{-1}\left(\frac{1}{c}\sqrt{r^2-x^2}\right), \quad |x| \le r \tag{4.9}$$

where $\mu_t = \mu_{t,\varphi}$ is the equilibrium measure with the norm

$$t = \frac{1}{c} - 1. \tag{4.10}$$

To connect this case to the case in Lemma 3 we set $c = n/N$. Then φ in (4.7) and V_1 in (2.1) are related by $-V_1(x) = n\varphi(x)$. Now the measure $n\mu_t$ has the norm $n\left(\frac{1}{c} - 1\right) = N - n$ and its equilibrium conditions (4.1) coincide with conditions (2.17). Therefore, the measure μ which is uniquely defined by (2.16), (2.17) is equal to $n\mu_{t,\varphi}$. Then (2.8) follows from (4.9).

To prove (2.18) we use (4.5) and (4.6) in the same case when φ is defined by (4.8) and $t = 1/c - 1 = N/n - 1$. Taking into account that $g_\tau(x) = 0$ for $\tau > \nu(x)$ (note that the last inequality is equivalent to $x < \beta_\tau$) we obtain for $\beta_\tau = r \le x < 1$,

$$V(x, \mu_t) + \varphi(x) - w_t = \int_t^{\nu(x)} \left(\int_{\beta_\tau}^{|x|} \frac{d\zeta}{\sqrt{\zeta^2 - \beta_\tau^2}} \right) d\tau.$$

Substituting $\tau = \nu(y)$ in the integral above yields

$$V(x, \mu_t) + \varphi(x) - w_t = \frac{1}{c} \int_r^{|x|} \int_y^{|x|} \frac{d\zeta}{\sqrt{\zeta^2 - y^2}} \frac{y\, dy}{\sqrt{1 - y^2}}.$$

Multiplying by n and taking into account that $\mu = n\mu_t$, $c = n/N$ we come to (2.18). The proof of Lemma 3 is completed. □

4.2. *Constrained equilibrium problem. Outline of the proof of the low bound* (2.19). The concept of the equilibrium with an upper constraint has been introduced in a recent paper [13] (see also [1], [3], [4], [8], [9] for subsequent developments) and methods based on this concept are not widely known. To illustrate the direct connection between the original problem (1.12) and the problem of constrained equilibrium, we include the following informal remark. More exactly, we will (roughly speaking) explain why the distribution of zeros of the extremal polynomial in (1.12) is characterized by the measure λ defined as a solution of certain extremal problems for logarithmic potentials related to (1.12).

Remark 5. Suppose that $P \in \mathbb{P}_n$ is an extremal polynomial in (1.12) and a measure λ on $\Delta = [-1, 1]$ with $|\lambda| = n$ represents the distribution of zeros of P in the sense of (1.10), e.g., $P(x) = T(x, \lambda)$ (if P has some zeros in $\mathbb{C} \setminus \Delta$ we begin with the balance of their counting measures onto Δ). Then there is a good reason to expect that $\lambda'(x) \le \sigma'(x)$. Indeed, suppose there exists, say, a subinterval $\tilde{\Delta} \subset \Delta$ where $\lambda'(x) > \sigma'(x)$. Then one could construct a better polynomial $\tilde{P}(x) = T\left(x, \tilde{\lambda}\right)$ as follows. First, we substitute the set of zeros of P in $\tilde{\Delta}$ by $E \cap \tilde{\Delta}$, in other words we define $\tilde{\lambda}\Big|\tilde{\Delta} = \sigma\Big|\tilde{\Delta}$; this gives $\|\tilde{P}\|_{E\cap\tilde{\Delta}} = 0$.

Then we define $\tilde{\lambda}|_{\Delta\setminus\tilde{\Delta}}$ as the sum of $\lambda|_{\Delta\setminus\tilde{\Delta}}$ and the balayage of $(\lambda-\sigma)|_{\tilde{\Delta}}$ onto $\Delta\setminus\tilde{\Delta}$. It makes $\|\tilde{P}\|_E < \|P\|_E$. To keep the value $|\tilde{P}(x)|$ at a fixed point x large we also need a slight modification of conditions (1.10) defining distribution of zeros of $\tilde{P}(x) = T(x,\tilde{\lambda})$ with respect to $\tilde{\lambda}$. Then $|\tilde{P}(x)|/\|\tilde{P}\|_E > |P(x)|/\|P\|_E$ which contradicts the definition of P. Thus, a distribution measure for zeros of an extremal polynomial in (1.12) is likely to be an element of the following class of measures

$$\mathcal{M}_n^\sigma = \{\mu : |\mu| = n, \qquad \operatorname{supp}(\mu) \subset \Delta, \mu \le \sigma\}. \tag{4.11}$$

Next, let $P(x) = T(x,\mu)$, $\mu \in \mathcal{M}_n^\sigma$. On the set $\Delta \setminus \operatorname{supp}(\sigma-\mu)$ where $\mu'(x) = \sigma(x)$ zeros of P and points from E mutually separate each other. So, there is a polynomial $\tilde{P}$ close to P with $P(\zeta) = 0$, $\zeta \in E \setminus \operatorname{supp}(\sigma-\mu)$. There is the reason to expect that $\|P\|_{E\cap\operatorname{supp}(\sigma-\mu)} \le \|P\|_{\operatorname{supp}(\sigma-\mu)}$ is in a certain sense the main part $\|P\|_E$.

Finally, taking into account that $|P(x,\mu)|$ is essentially represented by $\exp(-V(x,\mu))$ we come to the conclusion that the extremal problem (1.12) in its essentials is close to the extremal problem

$$\tilde{K}_{n_1}(x,\sigma) = \sup_{\mu\in\mathcal{M}_n^\sigma} \frac{e^{-V(x,\mu)}}{\left\|e^{-V(x,\mu)}\right\|_{\operatorname{supp}(\sigma-\mu)}}.$$

This is what we called in Section 1 the continuous version of (1.12). After simple transformation it may be equivalently written as

$$\log \tilde{K}_{n_1}(x,\sigma) = \sup_{\mu\in\mathcal{M}_n^\sigma} \left\{ \min_{t\in\operatorname{supp}(\sigma-\mu)} V(t,\mu) - V(x,\mu) \right\}. \tag{4.12}$$

The extremal problem (4.12) itself depends on a fixed point $x \in \mathbb{C}$. It turns out that there exists a unique extremal measure $\lambda = \lambda_n^\sigma \in \mathcal{M}_n^\sigma$ of the problem (4.12) and, most important, that this measure does not depend on x; we call λ the equilibrium measure on Δ with the upper constraint σ and norm n. The measure $\lambda = \lambda_n^\sigma$ is also uniquely defined by $|\lambda| = n$ and the folllowing inequalities (equilibrium conditions):

$$\begin{aligned} V(x,\lambda) &\le w, \quad x \in \operatorname{supp}(\lambda);\\ V(x,\lambda) &\ge w, \quad x \in \operatorname{supp}(\sigma-\lambda). \end{aligned} \tag{4.13}$$

The measure λ_n^σ also minimizes the energy integral in class $\mathcal{M}_n^\sigma$. See [13, Theorem 3, p. 1217] for a proof under the assumption of continuity $V(x,\sigma)$; see also [1], [3], [4], [8], [9] for further generalizations.

There exists the duality between constrained equilibrium measure and the equilibrium measure in the field of the negative potential of the constraining measure, namely

$$\lambda_n^\sigma + \mu_{t,\varphi} = \sigma \text{ where } \varphi(x) = -V(x,\sigma), \quad t + n = |\sigma| \tag{4.14}$$

(see [8, Th. 5.1]). This allows us to obtain explicit formulas for λ_n^σ at least in the case when supp $(\mu_{t,\varphi})$ is a segment. In particular, the following formulas

$$\lambda'(x) = \frac{N}{\pi}\tan\left(\frac{n}{N\tan^{-1}\sqrt{r^2-x^2}}\right), \quad |x|\le r, \tag{4.15a}$$

$$\lambda'(x) = \frac{N}{2}, \quad |x|\in[r,1], \quad r=\sqrt{1-n^2/N^2}, \tag{4.15b}$$

for $\lambda=\lambda_n^\sigma$ with $d\sigma = \dfrac{N}{2}\,dx$ were obtained in [13, p. 1228] (in a renormalized form).

We mentioned above that the polynomial $P = T(x, \lambda_n^\sigma)$ must be, in general, close to the extremal polynomial in (1.12). Thus, one may expect that $|P(x)|/\|P\|_E$ will give us a fairly good low bound for $K_n(x,\sigma)$. Using this method in the case where $d\sigma(x) = \dfrac{N}{2}\,dx$ we will next prove the low bound (1.19) for $K_{n,N}(x)$.

With $d\lambda = \lambda' dx$ in (4.15) we denote $V_3(x) = V(x,\lambda)$, $\phi_3(x) = \pi\int_x^1 d\lambda(t)$, $P(x) = T(x,\lambda)$.

Lemma 7. $P(x) = C_3(x)e^{-V_3(x)}\cos\phi_3(x)$, $x\in[-1,1]$, *where* $C_3(x)$ *is a positive function satisfying*

$$\max_{[-1,1]} C_3(x)/\min_{[-1,1]} C_3(x) \le C_3$$

and C_3 *is an absolute constant.*

The method of the proof of Lemma 7 is essentially the same method which used in Section 3 to prove Lemmas 1 and 2: we write $C_3(x)$ in the form $C(x) = 2e^{\eta(x)}$ and then use the integral representation of Theorem 2 for $\eta(x)$. Measure σ is related to the case is λ in (4.15) and $\beta=1$. The density $\lambda'(x)$ is piecewise analytic in $(-1,1)$ and we use Definition 2 for an admissible curve. There are three subintervals $\Delta_1 = (r,1)$, $\Delta_2 = (-r,r)$, and $\Delta_3 = (-1,-r)$ where λ' is analytic. For a fixed $R>0$ let $\Omega_k = \Omega_k(R)$ be the rectangle in the upper half-plane based on the interval Δ_k with the height R $(k=1,2,3)$. Let $\tilde\Gamma_k = \partial\Omega_k$ be the (positively oriented) boundary of Ω_k. We define $\Gamma_k = \tilde\Gamma_k \setminus \Delta_k$ and use Theorem 2 with

$$\Gamma = \Gamma_1+\Gamma_2+\Gamma_3$$

and a large R (actually $R=\infty$). We note that, for instance, Γ_1 and Γ_2 have a common interval I_2 with end points r and $r+iR$. We have to consider Γ as the sum of parametric curves Γ_k which contains both $I_2^+\subset\Gamma_2$ and $I_2^-\subset\Gamma_1$ where I_2^+ is I_2 oriented upward and I_2^- is I_2 with the opposite orientation. The reason is that the function $\Lambda(z)$ in (3.9) related to the case consisting of the three different analytic functions $\Lambda_k\in H(\Omega_k)$ in the three disjoint domains Ω_k, $k=1,2,3$.

In particular, $\Lambda(\zeta) = \Lambda_1(\zeta)$, $\zeta \in I_2^0$, and $\Lambda(\zeta) = \Lambda_2(\zeta)$, $\zeta \in I_2^+$. Totally, the part of integral for $\eta(x)$ related to I_2 with $R = +\infty$,

$$\eta_2 = \frac{1}{\pi} Im \int_0^{+\infty} (\Lambda_1(r+it) - \Lambda_2(r+it)) \frac{i\,dt}{x - r + it}. \tag{4.16}$$

The whole function η will be represented as $\eta(x) = \sum_{j=1}^{4} \eta_j(x)$ where $\eta_j(x)$ is defined by the integral in (4.16) with $\Lambda_1 - \Lambda_2$ replaced by $\Lambda_{j-1} - \Lambda_j$, $j = 1, \dots, 4$; we assume $\Lambda_0 = \Lambda_4 \equiv 0$. Explicit computations of those integrals and their estimates are rather trivial (needed remarks on $\tan^{-1} z$, $\operatorname{Re} z > 0$, are made in the proof of Lemma 2) but they would take considerable space to present them in detail. An interested reader will easily make the calculations required to show that $\|\eta(x)\|_{[-1,1]} \le C$ which is enough to prove Lemma 7. (Actually $\eta(x)$ is small if $n < N$ are large and x is not too close to one of the endpoints ± 1.)

To complete the proof of (1.9) we note that λ in (4.15) and μ in (2.8) are related by

$$\lambda + \mu = \sigma = \frac{N}{2}\, dx$$

(see (4.14) or use explicit formulas for a direct verification). Therefore, functions $V_3(x) = V(x, \lambda)$ above, $V_1(x) = V(x, \sigma)$ in (2.1) and $V_2(x) = V(x, \mu)$ in (2.9) satisfy $V_1(x) = V_2(x) + V_3(x)$. It follows that (2.17) and (2.18) may be equivalently written as

$$\begin{aligned} V_3(x) &= -w, \quad |x| \le r; \\ W(x) &= W_{n,N}(x) = e^{-w - V_3(x)}. \end{aligned}$$

Combining these equalities with Lemma 7 we obtain for $P(x) = T(x, \lambda)$

$$\begin{aligned} |P(x)| &\ge \min_{[-1,1]} C_3(x) e^w W_{n,N}(x) \cdot |\cos \phi_3(x)|, \\ \max_{\zeta \in E} |P(\zeta)| &\le \max_{|x| \le r} |P(x)| \le \max_{[-1,1]} C_3(x) e^w \end{aligned}$$

(note that $P(\zeta) = 0$ for $\zeta \in E$, $|\zeta| \in [r, 1]$ since $\cos \phi_3(\zeta) = 0$). The two estimates above make

$$K_{n,N}(x) \ge \frac{|P(x)|}{\|P\|_E} \ge \frac{1}{C_3} W_{n,N}(x) \, |\cos \phi_3(x)|, \quad |x| \in [r, 1].$$

Since $|\cos \phi_3(x)| = \left| \cos \frac{\pi N}{2}(1 - x) \right|$, $|x| \in [r, 1]$, the estimate (1.9) follows.

Acknowledgement. The author thanks the referee for his valuable remarks directed toward improvement of the text.

University of South Florida, Tampa, FL
E-mail address: rakhmano@math.usf.edu

References

[1] B. Beckerman, On a conjecture of E. A. Rakhmanov, *Constr. Approx.* **16** (2000), 427–448.

[2] D. Coppersmith and T. Rivlin, The growth of polynomials bounded at equally spaced points, *SIAM J. Math. Anal.* **23** (1992), 970–983.

[3] S. Damelin, Weighted polynomial approximation on discrete sets, *Montash. Math.* **138** (2003), 111–131.

[4] P. Dragnev and E. Saff, Constrained energy problems with applications to orthogonal polynomials of a discrete variable, *J. Anal. Math.* **72** (1997), 223–259.

[5] H. Ehlich, Polynome zwischen Gitterpunkten, *Math. Z.* **93** (1966), 144–153.

[6] H. Ehlich and K. Zeller, Schwankung von Polynomen zwischen Gitterpunkten, *Math. Z.* **86** (1964), 41–44.

[7] ———, Numerishe Abschätzung von Polynomen, *Z. Angew. Math. Mech.* **45** (1965), T20–T22.

[8] A. Kuijlaars and E. Rakhmanov, Zero distributions for discrete orthogonal polynomials, *J. Comp. Appl. Math.* **99** (1998), 255–274.

[9] A. Kuijlaars and W. Van Assche, Extremal polynomials on discrete sets, *Proc. London Math. Soc.*˙**79** (1999), 191–221.

[10] G. Lopez and E. Rakhmanov, Rational approximations, orthogonal polynomials and equilibrium distributions, in *Orthogonal Polynomials and their Applications* (Alfaro, et al.k, eds.), *Lecture Notes in Math.* **1329**, 125–156, Springer-Verlag, New York (1998).

[11] E. Rakhmanov, n asymptotic properties of polynomials orthogonal on the real axis, *Mat. Sb.* **119** (1982), 163–203; English transl. in *Math. USSR-Sb.* **47** (1984), 155–193.

[12] ———, Strong asymptotics for orthogonal polynomials associated with exponential weights on R, in *Methods of Approximation Theory in Complex Analysis and Mathematical Physics* (A. A. Gonchar and E. B. Saff, eds.), Nauka, Moscow, 1992, 71–97.

[13] ———, Equilibrium measure and the distribution of zeros of the extremal polynomials of a discrete variable, *Mat. Sb.* **187** (1996), 109–124; English transl. in *Sb. Math.* **187** (1996), 1213–1228.

[14] E. Saff and V. Totik, *Logarithmic Potentials With External Fields*, Springer-Verlag, New York, 1997.

[15] A. Schönhage, Fehlerfort pflantzung bei Interpolation, *Numer. Math.* **3** (1961), 62–71.

[16] B. Shekhtman, On the discrete norms of polynomials, in *Approximation Theory* IX, Vol. 1: *Theoretical Aspects* (C. K. Chui and L. L. Schumaker, eds.), Vanderbilt Univ. Press, Nasvhille, TN, 1999, 303–307.

[17] J. Szabados and P. Vértesi, *Interpolation of Functions*, World Sci. Publ. Co. Inc., Teaneck, NJ, 1990.

[18] V. Totik, *Weighted Approximation With Varying Weights*, *Lecture Notes in Math.* **1300**, Springer-Verlag, New York, 1994.

(Received November 22, 2002)
(Revised October 3, 2004)