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Quantum Riemann-Roch, Lefschetz and Serre

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To Vladimir Arnold on the occassion of his 70^{th} *birthday*

Abstract

Given a holomorphic vector bundle E over a compact Kähler manifold X, one defines twisted Gromov-Witten invariants of X to be intersection numbers in moduli spaces of stable maps $f : \Sigma \to X$ with the cap product of the virtual fundamental class and a chosen multiplicative invertible characteristic class of the virtual vector bundle $H^0(\Sigma, f^*E) \ominus H^1(\Sigma, f^*E)$. Using the formalism of quantized quadratic Hamiltonians [25], we express the descendant potential for the twisted theory in terms of that for X. This result (Theorem 1) is a consequence of Mumford's Grothendieck-Riemann-Roch theorem applied to the universal family over the moduli space of stable maps. It determines all twisted Gromov-Witten invariants, of all genera, in terms of untwisted invariants.

When E is concave and the $\mathbb{C}^\times$-equivariant inverse Euler class is chosen as the characteristic class, the twisted invariants of X give Gromov-Witten invariants of the total space of E. "Nonlinear Serre duality" [21], [23] expresses Gromov-Witten invariants of E in terms of those of the super-manifold ΠE: it relates Gromov-Witten invariants of X twisted by the inverse Euler class and E to Gromov-Witten invariants of X twisted by the Euler class and E^*. We derive from Theorem 1 nonlinear Serre duality in a very general form (Corollary 2).

When the bundle E is convex and a submanifold $Y \subset X$ is defined by a global section of E, the genus-zero Gromov-Witten invariants of ΠE coincide with those of Y. We establish a "quantum Lefschetz hyperplane section principle" (Theorem 2) expressing genus-zero Gromov-Witten invariants of a complete intersection Y in terms of those of X. This extends earlier results [4], [9], [18], [29], [33] and yields most of the known mirror formulas for toric complete intersections.

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Introduction

The mirror formula of Candelas *et al.* [10] for the virtual numbers n_d of degree $d = 1, 2, 3, \dots$ holomorphic spheres on a quintic 3-fold $Y \subset X = \mathbb{C}P^4$ can be stated [20] as the coincidence of the 2-dimensional cones over the following two curves in $H^{\mathrm{even}}(Y;\mathbb{Q}) = \mathbb{Q}[P]/(P^4)$:

$$J_Y(\tau) = e^{P\tau} + \frac{P^2}{5}\sum_{d>0} n_d\, d^3 \sum_{k>0} \frac{e^{(P+kd)\tau}}{(P+kd)^2}$$

and

$$I_Y(t) = \sum_{d\geq 0} e^{(P+d)t} \frac{(5P+1)(5P+2)\dots(5P+5d)}{(P+1)^5(P+2)^5\dots(P+d)^5}.$$

The new proof given in this paper shares with earlier work [9], [18], [21], [29], [33], [35] the formulation of sphere-counting in a hypersurface $Y \subset X$ as a problem in the Gromov-Witten theory of X.

Gromov-Witten invariants of a compact almost-Kähler manifold X are defined as intersection numbers in moduli spaces $X_{g,n,d}$ of stable pseudo-holomorphic maps $f : \Sigma \to X$. Most results in this paper can be stated and hold true in this generality (see Appendix 2 in [11]): the only exceptions are those discussed in Sections 9 and 10 which depend on equation (19). We prefer however to stay on the firmer ground of algebraic geometry, where the majority of applications belong.

Given a holomorphic vector bundle E over a compact projective complex manifold X and an invertible multiplicative characteristic class $\mathbf{c}$ of complex vector bundles, we introduce *twisted* Gromov-Witten invariants as intersection indices in $X_{g,n,d}$ with the characteristic classes $\mathbf{c}(E_{g,n,d})$ of the virtual bundles $E_{g,n,d} =$ "$H^0(\Sigma, f^*E)\ominus H^1(\Sigma, f^*E)$". The "quantum Riemann-Roch theorem" (Theorem 1) expresses twisted Gromov-Witten invariants (of any genus) and their gravitational descendants via untwisted ones.

The totality of gravitational descendants in the genus-zero Gromov-Witten theory of X can be encoded by a semi-infinite cone $\mathcal{L}_X$ in the cohomology algebra of X with coefficients in the field of Laurent series in $1/z$ (see §6). Another such cone corresponds to each twisted theory. Let $\mathcal{L}_E$ be the cone corresponding to the total Chern class

$$\mathbf{c}(\cdot) = \lambda^{\dim(\cdot)} + c_1(\cdot)\lambda^{\dim(\cdot)-1} + \dots + c_{\dim(\cdot)}(\cdot).$$

Theorem 1 specialized to this case says that the cones $\mathcal{L}_X$ and $\mathcal{L}_E$ are related by a linear transformation. It is described in terms of the stationary phase asymptotics $a_\rho(z)$ of the oscillating integral

$$\frac{1}{\sqrt{2\pi z}} \int_0^\infty e^{\frac{-x+(\lambda+\rho)\ln x}{z}} dx$$

as multiplication in the cohomology algebra by $\prod_i a_{\rho_i}(z)$, where ρ_i are the Chern roots of E.

Assuming E to be a line bundle, we derive a "quantum hyperplane section theorem" (Theorem 2). It is more general than the earlier versions [4], [29], [18], [33] in the sense that the restrictions $t \in H^{\leq 2}(X;\mathbb{Q})$ on the space of parameters and $c_1(E) \leq c_1(X)$ on the Fano index are removed.

In the quintic case when $X = \mathbb{C}P^4$ and $\rho = 5P$, the cone $\mathcal{L}_X$ is known to contain the curve

$$J_X(t) = \sum_{d\geq 0} \frac{e^{(P+zd)t/z}}{(P+z)^5 \dots (P+zd)^5},$$

and Theorem 2 implies that the cone $\mathcal{L}_E$ contains the curve

$$I_E(t) = \sum_{d\geq 0} e^{(P+zd)t/z} \frac{(\lambda+5P+z)\dots(\lambda+5P+5dz)}{(P+z)^5 \dots (P+dz)^5}.$$

One obtains the quintic mirror formula by passing to the limit $\lambda = 0$.

The idea of deriving mirror formulas by applying the Grothendieck-Riemann-Roch theorem to universal stable maps is not new. Apparently this was the initial plan of M. Kontsevich back in 1993. In 2000, we had a chance to discuss a similar proposal with R. Pandharipande. We would like to thank these authors as well as A. Barnard and A. Knutson for helpful conversations, and the referee for many useful suggestions.

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1. Generating functions

Let X be a compact projective complex manifold of complex dimension D. Denote by $X_{g,n,d}$ the moduli orbispace of genus-g, n-pointed stable maps [7], [31] to X of degree d, where $d \in H_2(X;\mathbb{Z})$. The moduli space is compact and can be equipped [8], [34], [38] with a (rational-coefficient) *virtual fundamental cycle* $[X_{g,n,d}]$ of complex dimension $n + (1-g)(D-3) + \int_d c_1(TX)$.

The *total descendant potential* of X is a generating function for Gromov-Witten invariants. It is defined as

$$\mathcal{D}_X(t_0, t_1, \dots) := \exp\left(\sum_{g\geq 0} \hbar^{g-1} \mathcal{F}_X^g(t_0, t_1, \dots)\right), \tag{1}$$

where $\mathcal{F}^g_X$ is the *genus-g descendant potential*,

$$(2)\quad \mathcal{F}^g_X(t_0, t_1, \dots) := \sum_{n,d} \frac{Q^d}{n!} \int_{[X_{g,n,d}]} \Big(\sum_{k_1=0}^{\infty} (\mathrm{ev}_1^* t_{k_1}) \psi_1^{k_1}\Big) \dots \Big(\sum_{k_n=0}^{\infty} (\mathrm{ev}_n^* t_{k_n}) \psi_n^{k_n}\Big).$$

Here ψ_i is the first Chern class of the universal cotangent line bundle over $X_{g,n,d}$ corresponding to the i^{th} marked point, the map $\mathrm{ev}_i : X_{g,n,d} \to X$ is evaluation at the i^{th} marked point, $t_0, t_1, \ldots \in H^*(X; \mathbb{Q})$ are cohomology classes, and Q^d is the representative of $d \in H_2(X; \mathbb{Z})$ in the semigroup ring of degrees of holomorphic curves in X.

Let E be a holomorphic vector bundle over X. We associate to it an element $E_{g,n,d}$ in the Grothendieck group $K^0(X_{g,n,d})$ of orbibundles[1] over $X_{g,n,d}$ as follows. Consider the *universal stable map*

$$\begin{array}{ccc} X_{g,n+1,d} & \xrightarrow{\mathrm{ev}_{n+1}} & X \\ \pi \big\downarrow & & \\ X_{g,n,d} & & \end{array}$$

formed by the operations of forgetting and evaluation at the last marked point. We pull E back to the universal family and then apply the K-theoretic push-forward to $X_{g,n,d}$. This means the following: there is a complex $0 \to E^0_{g,n,d} \to E^1_{g,n,d} \to 0$ of bundles on $X_{g,n,d}$ with cohomology sheaves equal to $R^0\pi_*(\mathrm{ev}_{n+1}^* E)$ and $R^1\pi_*(\mathrm{ev}_{n+1}^* E)$ respectively. Moreover, the difference

$$E_{g,n,d} := [E^0_{g,n,d}] - [E^1_{g,n,d}]$$

in the Grothendieck group of bundles does not depend on the choice of the complex. These facts are based on some standard results about *local complete intersection morphisms*, and are discussed further in Appendix 1.

A rational invertible multiplicative characteristic class of complex vector bundles takes the form

$$(3)\qquad \mathbf{c}(\cdot) = \exp\left(\sum_{k=0}^{\infty} s_k \, \mathrm{ch}_k(\cdot)\right)$$

where ch_k are components of the Chern character and $s_0, s_1, s_2, \ldots$ are arbitrary coefficients or indeterminates. Given such a class and a holomorphic vector bundle $E \in K^0(X)$ over X, we define the *$(\mathbf{c}, E)$-twisted descendant potentials* $\mathcal{D}_{\mathbf{c},E}$ and $\mathcal{F}^g_{\mathbf{c},E}$ by replacing the virtual fundamental cycles $[X_{g,n,d}]$ in (1) and (2) with the cap-products $\mathbf{c}(E_{g,n,d}) \cap [X_{g,n,d}]$. For example, the Poincaré intersection pairing arises in Gromov-Witten theory as an intersection index in $X_{0,3,0} = X$, and in the twisted theory therefore takes on the

[1] We will usually omit the prefix *orbi*.

form

$$(a,b)_{\mathbf{c}(E)} := \int_{[X_{0,3,0}]} \mathbf{c}(E_{0,3,0})\,\mathrm{ev}_1^*(a)\,\mathrm{ev}_2^*(1)\,\mathrm{ev}_3^*(b) = \int_X \mathbf{c}(E)\ a\ b. \tag{4}$$

We will often assume that all vector bundles carry the S^1-action given by fiberwise multiplication by the unitary scalars. In this case the ch_k should be understood as S^1-equivariant characteristic classes, and all Gromov-Witten invariants take values in the coefficient ring of S^1-equivariant cohomology theory. We will always identify this ring $H^*(BS^1;\mathbb{Q})$ with $\mathbb{Q}[\lambda]$, where λ is the first Chern class of the line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^\infty$.

2. Quantization formalism

Theorem 1 below expresses $\mathcal{D}_{\mathbf{c},E}$ in terms of $\mathcal{D}_X$ via the formalism of quantized quadratic Hamiltonians [25], which we now outline. Consider $H = H^*(X;\mathbb{Q})$ as a super-space equipped with the nondegenerate symmetric bilinear form defined by the Poincaré intersection pairing $(a,b) = \int_X ab$. Let $\mathcal{H} = H(\!(z^{-1})\!)$ denote the super-space of Laurent polynomials in $1/z$ with coefficients in H, where the indeterminate z is regarded as even. We equip $\mathcal{H}$ with the even symplectic form

$$\begin{aligned}\Omega(\mathbf{f},\mathbf{g}) &:= \frac{1}{2\pi i}\oint (\mathbf{f}(-z),\mathbf{g}(z))\ dz \\ &= -(-1)^{\bar{\mathbf{f}}\bar{\mathbf{g}}}\Omega(\mathbf{g},\mathbf{f}).\end{aligned}$$

The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ defined by the Lagrangian subspaces $\mathcal{H}_+ = H[z]$, $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$ identifies $(\mathcal{H},\Omega)$ with the cotangent bundle $T^*\mathcal{H}_+$.

The standard quantization convention associates to quadratic Hamiltonians G on $(\mathcal{H},\Omega)$ differential operators $\hat{G}$ of order ≤ 2 acting on functions on $\mathcal{H}_+$. More precisely, let $\{q_\alpha\}$ be a $\mathbb{Z}_2$-graded coordinate system on $\mathcal{H}_+$ and $\{p_\alpha\}$ be the dual coordinate system on $\mathcal{H}_-$, so that the symplectic structure in these coordinates assumes the Darboux form

$$\Omega(\mathbf{f},\mathbf{g}) = \sum_\alpha [p_\alpha(\mathbf{f})q_\alpha(\mathbf{g}) - (-1)^{\bar{p}_\alpha\bar{q}_\alpha}q_\alpha(\mathbf{f})p_\alpha(\mathbf{g})].$$

For example, when H is the standard one-dimensional Euclidean space then $\mathbf{f} = \sum q_k z^k + \sum p_k(-z)^{-1-k}$ is such a coordinate system. In a Darboux coordinate system the quantization convention reads

$$(q_\alpha q_\beta)\hat{}\ := \frac{q_\alpha q_\beta}{\hbar},\quad (q_\alpha p_\beta)\hat{}\ := q_\alpha\frac{\partial}{\partial q_\beta},\quad (p_\alpha p_\beta)\hat{}\ := \hbar\frac{\partial^2}{\partial q_\alpha \partial q_\beta}.$$

The quantization gives only a *projective* representation of the Lie algebra of quadratic Hamiltonians on $\mathcal{H}$ as differential operators. For quadratic Hamiltonians F and G we have

$$[\hat{F},\hat{G}] = \{F,G\}\hat{}\ + \mathcal{C}(F,G),$$

where $\{\cdot,\cdot\}$ is the Poisson bracket, $[\cdot,\cdot]$ is the super-commutator, and $\mathcal{C}$ is the cocycle

$$\mathcal{C}(p_\alpha p_\beta, q_\alpha q_\beta) = \begin{cases} (-1)^{\bar{q}_\alpha \bar{p}_\beta} & \text{if } \alpha \neq \beta, \\ 1 + (-1)^{\bar{q}_\alpha \bar{p}_\alpha} & \text{if } \alpha = \beta, \end{cases}$$

$$\mathcal{C} = 0 \text{ on any other pair of quadratic Darboux monomials.}$$

We associate the quadratic Hamiltonian $h_T(\mathbf{f}) = \Omega(T\mathbf{f}, \mathbf{f})/2$ to an infinitesimal symplectic transformation T, and write $\hat{T}$ for the quantization $\hat{h_T}$. If A and B are self-adjoint operators on H then the operators $\mathbf{f} \mapsto (A/z)\mathbf{f}$ and $\mathbf{f} \mapsto (Bz)\mathbf{f}$ on $\mathcal{H}$ are infinitesimal symplectic transformations, and

$$\mathcal{C}(h_{A/z}, h_{Bz}) = \operatorname{str}(AB)/2.$$

In what follows, we will often apply symplectic transformations $\exp T$ in the *quantized* form $\exp \hat{h_T}$ to various generating functions for Gromov-Witten invariants — that is to certain formal functions of $\mathbf{q} = q_0 + q_1 z + q_2 z^2 + \ldots \in \mathcal{H}_+$ and $\hbar$ — which we refer to as *asymptotic elements of the Fock space*. In fact the quantized symplectic transformations that we will use do not have a convenient common domain that includes all the formal functions which we will need. We will therefore not describe any "Fock space", but instead regularly indicate those special circumstances that make the application of particular quantized symplectic transformations to particular generating functions well-defined. Such special circumstances usually involve *-adic* convergence with respect to some auxiliary formal parameters (such as s_k in Corollary 3, $1/\lambda$ in (12), Q in (13), etc.). The key point here is that our formulas provide unambiguous rules for transforming generating functions (and their coefficients): the description of these rules as symplectic transformations or their quantizations remains merely a convenient interpretation[2].

Let us begin by setting up notation for such an interpretation. We will assume that the ground field $\mathbb{Q}$ of constants is extended to the Novikov ring $\mathbb{Q}[[Q]]$, or to $\mathbb{Q}[[Q]] \otimes \mathbb{Q}(\lambda)$ in the S^1-equivariant setting, and will denote the ground ring by Λ. The potentials $\mathcal{F}_X^g(t_0, t_1, \ldots)$ are naturally defined as formal functions on the space of vector polynomials $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \ldots$ where $t_0, t_1, t_2, \ldots \in H$. The total descendant potential $\mathcal{D}_X$ is simply the *formal expression* $\exp \sum \hbar^{g-1} \mathcal{F}_X^g$ defined by these formal functions. It cannot be viewed as a formal function of $\hbar$ and $\mathbf{t}$ because of the presence of $\hbar^{-1}$- and $\hbar^0$-terms in the exponent. The reader uncomfortable with this situation could note that the formal functions $\mathcal{F}_X^0$ and $\mathcal{F}_X^1$ when reduced modulo Q contain only terms which are respectively at-least-cubic and at-least-linear in

[2]This approach, somewhat resembling the terminology in the theory of formal groups, is not the only one possible. We refer to Section 8 in [26] where the class of *tame asymptotic functions* (convenient for the purposes of that paper) is introduced.

the variables t_i, and that $\mathcal{D}_X$ can therefore be considered as a formal function of $\hbar, \mathbf{t}/\hbar$ and $Q/\hbar$. This point of view will, however, play no role in what follows.

We regard the total descendant potential (1) as an asymptotic element of the Fock space via the identification

$$\mathbf{q}(z) = \mathbf{t}(z) - z \tag{5}$$

which we call the *dilaton shift*. The twisted descendant potentials $\mathcal{D}_{\mathbf{c},E}$ can be similarly considered as asymptotic elements of Fock spaces corresponding to the super-space H equipped with the twisted inner products (4). Alternatively, we can identify the inner product spaces $(H, (\cdot,\cdot)_{\mathbf{c}(E)})$ with $(H, (\cdot,\cdot))$ by means of the maps $a \mapsto a\sqrt{\mathbf{c}(E)}$, hence considering the twisted descendant potentials $\mathcal{D}_{\mathbf{c},E}$ as asymptotic elements of the original Fock space via the *twisted dilaton shift*:

$$\mathbf{q}(z) = \sqrt{\mathbf{c}(E)}\,(\mathbf{t}(z) - z). \tag{6}$$

We thus obtain a formal family $\mathcal{D}_{\mathbf{s}} := \mathcal{D}_{\mathbf{c},E}$ of asymptotic elements of the Fock space depending on the parameters $\mathbf{s} = (s_0, s_1, s_2, \dots)$ from (3). Note that, due to the dilaton shift, $\mathcal{D}_{\mathbf{s}}$ is a formal function of $\mathbf{q}$ defined near the shifted origin $\mathbf{q}(z) = -\sqrt{\mathbf{c}(E)}z$, which varies with $\mathbf{s}$.

3. Quantum Riemann-Roch

Let us identify z with the first Chern class of the universal line bundle L and denote by $\langle\cdot\rangle$ the one-dimensional subspace spanned by the asymptotic element "$\cdot$" of the Fock space.

Theorem 1.

$$\langle\mathcal{D}_{\mathbf{c},E}\rangle = \hat{\triangle}\,\langle\mathcal{D}_X\rangle\,,$$

where $\triangle : \mathcal{H} \to \mathcal{H}$ is the linear symplectic transformation defined by the asymptotic expansion of

$$\sqrt{\mathbf{c}(E)}\prod_{m=1}^{\infty}\mathbf{c}(E\otimes L^{-m}).$$

This should be interpreted as follows. Let $\rho_1, \dots \rho_r$ be the Chern roots of E, and let

$$s(\cdot) = \sum_{k\geq 0} s_k\,\mathrm{ch}_k(\cdot).$$

Then

$$\ln\left(\sqrt{\mathbf{c}(E)}\prod_{m=1}^{\infty}\mathbf{c}(E\otimes L^{-m})\right)=\sum_{i=1}^{r}\left(\frac{s(\rho_i)}{2}+\sum_{m=1}^{\infty}s(\rho_i-mz)\right)$$
$$=\sum_{i=1}^{r}\left[\frac{1}{1-e^{-z\partial_x}}-\frac{1}{2}\right]s(x)\Big|_{x=\rho_i}$$
$$\sim\sum_{i=1}^{r}\left[\sum_{m\geq 0}\frac{B_{2m}}{(2m)!}(z\partial_x)^{2m-1}s(x)\right]\Bigg|_{x=\rho_i}$$
$$=\sum_{m\geq 0}\sum_{l\geq 0}\frac{B_{2m}}{(2m)!}s_{l+2m-1}\operatorname{ch}_l(E)z^{2m-1}.$$

Here the B_{2m} are Bernoulli numbers:

$$\frac{t}{1-e^{-t}}=\frac{t}{2}+\sum_{m\geq 0}\frac{B_{2m}}{(2m)!}t^{2m}.$$

The operator of multiplication by $\operatorname{ch}_l(E)$ in the cohomology algebra H of X is self-adjoint with respect to the Poincaré pairing. Consequently, the operator of multiplication by $\operatorname{ch}_l(E)z^{2m-1}$ in the algebra $\mathcal{H}$ is an infinitesimal symplectic transformation of $\mathcal{H}$ and so is $\ln\triangle$. Theorem 1 therefore is derived from the following more precise version.

Theorem 1′.

$$(7)\quad \exp\left(-\frac{1}{24}\sum_{l>0}s_{l-1}\int_X\operatorname{ch}_l(E)c_{D-1}(T_X)\right)\left(\operatorname{sdet}\sqrt{\mathbf{c}(E)}\right)^{-\frac{1}{24}}\mathcal{D}_{\mathbf{c},E}$$
$$=\exp\left(\sum_{m>0}\sum_{l\geq 0}s_{2m-1+l}\frac{B_{2m}}{(2m)!}(\operatorname{ch}_l(E)z^{2m-1})\hat{}\right)\exp\left(\sum_{l>0}s_{l-1}(\operatorname{ch}_l(E)/z)\hat{}\right)\mathcal{D}_X.$$

Here $\operatorname{sdet}(\cdot)=\exp\operatorname{str}\ln(\cdot)$ is the Berezinian.

Remarks. (1) The variable s_0 is present on the RHS of (7) only in the form $\exp(s_0\rho/z)\hat{}$ where $\rho=\operatorname{ch}_1(E)$. For any $\rho\in H^2(X)$ the operator $(\rho/z)\hat{}$ is in fact a *divisor* operator: the total descendant potential satisfies the *divisor equation*

$$(8)\qquad \left(\frac{\rho}{z}\right)\hat{}\,\mathcal{D}_X=\sum\rho_iQ_i\frac{\partial}{\partial Q_i}\mathcal{D}_X-\frac{1}{24}\int_X\rho\,c_{D-1}(T_X)\,\mathcal{D}_X.$$

Here Q_i are generators in the Novikov ring corresponding to a choice of a basis in $H_2(X)$ and ρ_i are coordinates of ρ with respect to the dual basis. For $\rho=\operatorname{ch}_1(E)$ the c_{D-1}-term cancels with the s_0-term on the LHS of (7). Thus the action of the s_0-flow reduces to the change $Q^d\mapsto Q^d\exp(s_0\int_d\rho)$ in the descendant potential $\mathcal{D}_X$ combined with the multiplication by the factor $\exp(s_0(\dim E)\chi(X)/48)$ coming from the super-determinant.

(2) If $E = \mathbb{C}$ then $E_{g,n,d} = \mathbb{C} - \mathbf{E}_g^*$, where $\mathbf{E}_g$ is the *Hodge bundle*. The Hodge bundles satisfy $\mathrm{ch}_k(\mathbf{E}_g) = -\mathrm{ch}_k(\mathbf{E}_g^*)$. In view of this, Theorem 1 in this case turns into Theorem 4.1 in [25] and is a reformulation in terms of the formalism explained in Section 2 of the results of Mumford [36] and Faber-Pandharipande [16]. The proof of Theorem 1 is based on a similar application of Mumford's Grothendieck-Riemann-Roch argument to our somewhat more general situation. The argument was doubtless known to the authors of [16]. The main new observation here is that the combinatorics of the resulting formula, which appears at first sight rather complicated, fits nicely with the formalism of quantized quadratic Hamiltonians. A verification of this — somewhat tedious but straightforward — is presented in Appendix 1.

4. The Euler class

The S^1-equivariant Euler class of E is written in terms of the (nonequivariant) Chern roots ρ_i as

$$\mathbf{e}(E) = \prod_i (\lambda + \rho_i).$$

Using the identity $(\lambda + x) = \exp(\ln\lambda - \sum_k (-x)^k/k\lambda^k)$ we can express it via the components of the nonequivariant Chern character:

$$\mathbf{e}(E) = \exp\left(\mathrm{ch}_0(E)\ln\lambda + \sum_{k>0} \mathrm{ch}_k(E)\frac{(-1)^{k-1}(k-1)!}{\lambda^k} \right). \tag{9}$$

Denote by $\mathcal{D}_{\mathbf{e}}$ the asymptotic element $\mathcal{D}_{\mathbf{s}}$ of the Fock space corresponding to

$$s_k = \begin{cases} \ln\lambda & k = 0 \\ \frac{(-1)^{k-1}(k-1)!}{\lambda^k} & k > 0. \end{cases} \tag{10}$$

Substituting these values of s_k into (7), replacing $\mathrm{ch}_l(E)$ by $\sum \rho_i^l/l!$ and using the binomial formula

$$(1+x)^{1-2m} = \sum_{l\geq 0} \frac{(-1)^l(2m-2+l)!}{(2m-2)!\; l!}\; x^l \qquad \text{for } m > 0.$$

we arrive at the following conclusion.

COROLLARY 1.

$$\mathcal{D}_{\mathbf{e}} = \prod_i \left(\operatorname{sdet}\sqrt{\lambda+\rho_i}\right)^{\frac{1}{24}}$$
$$\prod_i \exp\left(\frac{1}{24}\int_X [(\lambda+\rho_i)\ln(\lambda+\rho_i)-(\lambda+\rho_i)]\; c_{D-1}(T_X)\right)$$
$$\prod_i \exp\left(\sum_{m>0}\frac{B_{2m}}{2m(2m-1)}\left(\left(\frac{z}{\lambda+\rho_i}\right)^{2m-1}\right)^{\wedge}\right)$$
$$\prod_i \exp\left(\left(\frac{(\lambda+\rho_i)\ln(\lambda+\rho_i)-(\lambda+\rho_i)}{z}\right)^{\wedge}\right)\mathcal{D}_X.$$

Remark. The $1/z$-term in this formula actually arises in the form

$$\rho\ln\lambda+\sum\frac{(-1)^{k-1}\rho^{k+1}}{k(k+1)\lambda^k} = \int_0^\rho \ln(\lambda+x)dx = [(\lambda+x)\ln(\lambda+x)-(\lambda+x)]\;|_0^\rho.$$

It has positive cohomological degree and is small in this sense. The constant term $(\lambda\ln\lambda-\lambda)/z$ is thrown away on the following grounds. According to [25], $(1/z)^{\wedge}$ is the *string operator* and annihilates the descendant potential $\mathcal{D}_X$. Thus the operators $\exp((\lambda\ln\lambda-\lambda)/z)^{\wedge})$ do not change $\mathcal{D}_X$. The rest of the series in the exponent converges in the $1/\lambda$-adic topology.

5. Quantum Serre

Introduce the multiplicative characteristic class

$$\mathbf{c}^*(\cdot) = \exp\left(\sum(-1)^{k+1}s_k\operatorname{ch}_k(\cdot)\right).$$

Since $\operatorname{ch}_l(E^*) = (-1)^l\operatorname{ch}_l(E)$ we have

$$\mathbf{c}^*(E^*) = \frac{1}{\mathbf{c}(E)}.$$

There is no obvious relationship between $\mathbf{c}^*((E^*)_{g,n,d})$ and $\mathbf{c}(E_{g,n,d})$, but nonetheless the twisted descendant potentials $\mathcal{D}_{\mathbf{s}} = \mathcal{D}_{\mathbf{c},E}$ and $\mathcal{D}^*_{\mathbf{s}} := \mathcal{D}_{\mathbf{c}^*,E^*}$ are closely related.

COROLLARY 2. *We have* $\mathcal{D}^*_{\mathbf{s}} = (\operatorname{sdet}\mathbf{c}(E))^{-1/24}\,\mathcal{D}_{\mathbf{s}}$. *More explicitly,*

$$\mathcal{D}_{\mathbf{c}^*,E^*}(\mathbf{t}^*) = (\operatorname{sdet}\mathbf{c}(E))^{-\frac{1}{24}}\,\mathcal{D}_{\mathbf{c},E}(\mathbf{t}),$$

where

$$\mathbf{t}^*(z) = \mathbf{c}(E)\mathbf{t}(z) + (1-\mathbf{c}(E))z.$$

Proof. Replacing $\operatorname{ch}_l(E)$ with $(-1)^l\operatorname{ch}_l(E)$, and s_k with $(-1)^{k+1}s_k$ in (7) preserves all terms except the super-determinant. ∎

Corollary 3. *Consider the dual bundle E^* equipped with the dual S^1-action, and the S^1-equivariant inverse Euler class $\mathbf{e}^{-1}$. Put*

$$\mathbf{t}^*(z) = z + (-1)^{\dim E/2}\mathbf{e}(E)(\mathbf{t}(z) - z)$$

and introduce the change $\pm : Q^d \mapsto Q^d(-1)^{\int_d \mathrm{ch}_1(E)}$ in the Novikov ring. With this notation

$$\mathcal{D}_{\mathbf{e}^{-1},E^*}(\mathbf{t}^*, Q) = \mathrm{sdet}[(-1)^{\frac{\dim E}{2}}\mathbf{e}(E)]^{-\frac{1}{24}}\,\mathcal{D}_{\mathbf{e},E}(\mathbf{t}, \pm Q).$$

Proof. Let s_k be as in (10). We have $\mathbf{e}^{-1}(E^*) = \prod_i(-\lambda - \rho_i)^{-1}$. Since

$$(-\lambda + x)^{-1} = \exp\left(-\ln(-\lambda) + \sum_k \frac{x^k}{k\lambda^k}\right)$$

we find that $\mathbf{e}^{-1}(\cdot) = \exp\sum s_k^* \,\mathrm{ch}_k(\cdot)$ where $s_k^* = (k-1)!/\lambda^k$ for $k > 0$ and $s_0^* = -\ln(-\lambda)$. For $k > 0$, $s_k^* = (-1)^{k+1}s_k$ as in the situation of Corollary 2. However, $s_0^* = -s_0 - \pi\sqrt{-1}$. We compensate for the discrepancy $-\pi\sqrt{-1}$ using the divisor equation (8) described in Remark 1 following Theorem 1′. □

6. The genus-zero picture

The genus-zero descendant potential $\mathcal{F}_X^0$ can be recovered from the so-called "J-function" of finitely many variables due to a reconstruction theorem essentially due to Dubrovin [14] and going back to Dijkgraaf and Witten [13]. The *J-function* is a formal function of $t \in H$ and $1/z$ with vector coefficients in H defined by

(11)

$$\forall a \in H, \quad (J_X(t,z), a) := (z + t, a) + \sum_{d,n} \frac{Q^d}{n!}\int_{[X_{0,n+1,d}]} \bigwedge_{i=1}^{n} \mathrm{ev}_i^*\, t\ \frac{\mathrm{ev}_{n+1}^*\, a}{z - \psi_{n+1}}.$$

We need the following reformulation of the reconstruction theorem in terms of the geometry of the symplectic space $(\mathcal{H}, \Omega)$.

The genus-zero descendant potential $\mathcal{F}_X^0$ considered as a formal function of $\mathbf{q} \in \mathcal{H}_+$ via the dilaton shift (5) generates (the germ of) a Lagrangian section $\mathcal{L}_X \subset \mathcal{H} = T^*\mathcal{H}_+$. In Darboux coordinates

$$\mathcal{L}_X = \{(\mathbf{p}, \mathbf{q}) : \ \mathbf{p} = d_{\mathbf{q}}\mathcal{F}_X^0\}.$$

Proposition 1. *$\mathcal{L}_X$ is a homogeneous Lagrangian cone swept by a moving semi-infinite isotropic subspace depending on* $\dim H$ *parameters. More precisely,*

(i) *tangent spaces $L \subset \mathcal{H}$ to $\mathcal{L}_X$ are tangent to $\mathcal{L}_X$ along zL and*, vice versa, *if $L = T_{\mathbf{f}}\mathcal{L}$ is a tangent space to $\mathcal{L}$ then $\mathbf{f}$ is contained in zL;*

(ii) *$J_X(t, -z) \in \mathcal{H}$ is the intersection of $\mathcal{L}_X$ with $(t - z) + \mathcal{H}_-$.*

Remarks. (1) $\mathcal{L}_X \subset T^*\mathcal{H}_+$ is a formal germ of a Lagrangian section defined near $\mathbf{q} = -z$. All geometric statements about $\mathcal{L}_X$ should be understood in the sense of formal geometry.

(2) Part (i) of the proposition implies that the tangent spaces L are Lagrangian subspaces satisfying $zL \subset L$ (as well as $zL \subset \mathcal{L}_X$). They consequently belong to the loop group Grassmannian of the "twisted" series $A^{(2)}$, or to its super-version.

(3) Part (i) of the proposition means that the tangent spaces L actually depend only on $\dim H$ parameters and form a *semi-infinite variation of Hodge structure* in the sense of [3].

(4) Part (i) follows easily from Dubrovin's reconstruction formula (see [14, Th. 6.1]) in the axiomatic theory of Frobenius structures. We refer to [27] for details. In Appendix 2 we give another, more direct proof applicable in Gromov-Witten theory. It is based on Theorem 5.1 stated in [25] which relates gravitational descendants with *ancestors*.

(5) Part (ii) of the proposition follows immediately from the definitions of J_X and $\mathcal{L}_X$. Together with part (i) it shows how to reconstruct the cone $\mathcal{L}_X$ from the J-function. Namely, the first t-derivatives of $J_X(t,z)$ form a basis in the intersection of the tangent space L to $\mathcal{L}_X$ with $z\mathcal{H}_-$ and therefore form a basis of L as a free $\Lambda[z]$-module. We describe this reconstruction procedure in more detail in Section 8.

In the quasi-classical limit $\hbar \to 0$, quantized symplectic transformations $\exp \hat{A}$ of Theorem 1′ acting on the total potentials $\mathcal{D}_\mathbf{s}$ considered as asymptotic elements of the Fock space turn into "unquantized" symplectic transformations acting by $\mathcal{L}_\mathbf{s} \mapsto (\exp A)\mathcal{L}_\mathbf{s}$ on the Lagrangian cones $\mathcal{L}_\mathbf{s}$ generated by the genus-zero potentials $\mathcal{F}^0_{\mathbf{c},E}$.

Corollary 4.

$$\mathcal{L}_\mathbf{s} = \triangle\, \mathcal{L}_X = \exp\left(\sum_{m\geq 0} \sum_{l \geq 0} s_{2m-1+l} \frac{B_{2m}}{(2m)!} \operatorname{ch}_l(E) z^{2m-1} \right) \mathcal{L}_X.$$

7. Quantum Lefschetz

In the case of genus-zero Gromov-Witten theory twisted by the Euler class $\mathbf{e}(E)$, the corresponding Lagrangian cone $\mathcal{L}_\mathbf{e}$ is obtained from $\mathcal{L}_X$ by multiplication in $\mathcal{H}$ by the product over the Chern roots ρ of the series

(12)
$$b_\rho(z) = \exp\left(\frac{(\lambda+\rho)\ln(\lambda+\rho) - (\lambda+\rho)}{z} + \sum_{m>0} \frac{B_{2m}}{2m(2m-1)} (\frac{z}{\lambda+\rho})^{2m-1} \right).$$

The series (12) is well-known [28] in connection with the asymptotic expansion of the gamma function $\Gamma((\lambda+\rho)/z)$. More precisely, (12) coincides with the stationary phase asymptotics of the integral

$$\frac{1}{\sqrt{2\pi z(\lambda+\rho)}}\int_0^\infty e^{\frac{-x+(\lambda+\rho)\ln x}{z}}\,dx$$

near the critical point $x=\lambda+\rho$ of the phase function.

Let us assume now that E is the direct sum of r line bundles with first Chern classes ρ_i — in what follows we will need the Chern roots to be defined over $\mathbb{Z}$ — and consider the J-function $J_X(t,z)=\sum_d J_d(t,z)Q^d$. Put $\rho_i(d)=\int_d \rho_i$ and introduce the following *hypergeometric modification* of J_X:

$$I_E(t,z)=\sum_d J_d(t,z)Q^d\prod_{i=1}^r \frac{\prod_{k=-\infty}^{\rho_i(d)}(\lambda+\rho_i+kz)}{\prod_{k=-\infty}^{0}(\lambda+\rho_i+kz)}. \tag{13}$$

THEOREM 2. *The hypergeometric modification I_E, considered as a family $t\mapsto I_E(t,-z)$ of vectors in the symplectic space $(\mathcal{H},\Omega_{\mathbf{e}(E)})$ corresponding to the twisted inner product $(a,b)_{\mathbf{e}(E)}=\int_X \mathbf{e}(E)ab$ on H, is situated on the Lagrangian section $\mathcal{L}_{\mathbf{e},E}\subset(\mathcal{H},\Omega_{\mathbf{e}(E)})$ defined by the differential of the twisted genus-zero descendant potential $\mathcal{F}^0_{\mathbf{e},E}$.*

Note that in defining $\mathcal{L}_{\mathbf{e},E}$ we regard $\mathcal{F}^0_{\mathbf{e},E}$ as a formal function of $\mathbf{q}$ via the (untwisted) dilaton shift. Also, the following comment is in order. The series I_E does not necessarily belong to $H((z^{-1}))$ because of possible unbounded growth of the numbers $\rho_i(d)$. However the coefficients of each particular monomial Q^d do. Similarly, multiplication by the series (12) moves the cone $\mathcal{L}_X$ out of the space $H((z^{-1}))$. However modulo each particular power of $1/\lambda$ it does not (the invariance of the cone with respect to the string flow $\exp(\lambda\ln\lambda-\lambda)/z$ is once again essential here). In fact all our formulas make sense as operations with generating functions (i.e. give rise to legitimate operations with their coefficients) because of the presence of suitable auxiliary variables — s_k in Corollary 3, $1/\lambda$ in (12), Q in (13). More formally, this means the following. We replace the ground ring Λ in $H=H^*(X,\Lambda)$ with its completion (which we will still denote Λ) in the appropriate ($\mathbf{s}$-adic, $1/\lambda$-adic, Q-adic) topology. In the role of the symplectic space $\mathcal{H}$, we should take the space (we will denote it $H\{\{z^{-1}\}\}$) of Laurent series $\sum_{k\in\mathbb{Z}} h_k z^k$ possibly infinite in both directions but satisfying the following convergence condition: as $k\to+\infty$, $h_k\to 0$ in the topology of Λ. In the following proof we will have to similarly replace $\Lambda[z]$ by $\Lambda\{z\}$, and the ring Λ should also be extended by $\sqrt{\lambda}$.

8. Proof of Theorem 2

Due to the equivariance properties (see [21, §6]) of J-functions with respect to the string and divisor flows (8) we have

$$J_X(t+\sum(\lambda+\rho_i)\ln x_i) = e^{\frac{\sum(\lambda+\rho_i)\ln x_i}{z}} \sum_d J_d(t,z) Q^d \prod_i x_i^{\rho_i(d)}. \tag{14}$$

Integrating by parts (as in the proof of the identity $\Gamma(x+1) = x\Gamma(x)$) we find

$$\begin{aligned}(2\pi z)^{-\frac{r}{2}} \int_0^\infty dx_1 \ldots \int_0^\infty dx_r \; e^{-\sum x_i/z} J_X(t+\sum(\lambda+\rho_i)\ln x_i) \\ = I_E(t,z)\sqrt{\mathbf{e}(E)} \prod_i \frac{1}{\sqrt{2\pi z(\lambda+\rho_i)}} \int_0^\infty e^{\frac{-x_i+(\lambda+\rho_i)\ln x_i}{z}} dx_i\,.\end{aligned} \tag{15}$$

We conclude that the asymptotic expansion of the integral (15) coincides with $I_E(t,z)\sqrt{\mathbf{e}(E)}\prod_i b_{\rho_i}(z)$.

The multiplication by $\sqrt{\mathbf{e}(E)}$ identifies the Lagrangian cone $\mathcal{L}_{\mathbf{e},E} \subset (\mathcal{H}, \Omega_{\mathbf{e}(E)})$ with its normalized incarnation $\mathcal{L}_{\mathbf{e}} \subset (\mathcal{H}, \Omega)$. Therefore Theorem 2 is equivalent to the inclusion

$$I_E(t,-z)\sqrt{\mathbf{e}(E)} \in \mathcal{L}_{\mathbf{e}}$$

and hence

$$I_E(t,-z)\sqrt{\mathbf{e}(E)}\prod b_{\rho_i}(-z) \in \mathcal{L}_X$$

due to Corollary 4. It remains to show therefore that the asymptotic expansion of the integral (15) belongs to the cone determined by the J-function $J_X(t,z)$. In fact we will prove the following.

Lemma. *For each t, the asymptotic expansion of the integral* (15) *differs from $\lambda^{\dim E/2} J_X(t^*,z)$ (at some other point $t^*(t)$) by a linear combination of the first t-derivatives of J_X at t^* with coefficients in $z\Lambda\{z\}$.*

For this, we are going to use another property of the J-function J_X well-known in quantum cohomology theory and in the theory of Frobenius structures (see for instance [21, §6] and [14]). The first derivatives $\partial J_X/\partial t^\alpha$ satisfy the system of linear PDEs

$$z\frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial t^\beta} J_X(t,z) = \sum_\gamma A_{\alpha\beta}^\gamma(t) \frac{\partial}{\partial t^\gamma} J_X(t,z), \tag{16}$$

where we use a coordinate system $t = \sum t^\alpha \phi_\alpha$ on H. Indeed, we can argue as in [3]. The first t-derivatives of J_X form a basis in the intersection of the tangent space L to the cone $\mathcal{L}_X$ with $z\mathcal{H}_-$. The LHS of (16) belongs to this intersection: it is in L since infinitesimal t-variations of zL are in L, and it is in $z\mathcal{H}_-$ since $J_X \in z + t + \mathcal{H}_-$.

Further analysis reveals that $A_{\alpha\beta}^{\gamma}$ are structure constants of the quantum cohomology algebra $\phi_\alpha \bullet \phi_\beta = \sum A_{\alpha\beta}^{\gamma}\phi_\alpha$. In particular, $z\partial_1 J_X = J_X$ since $1\bullet = \mathrm{id}$. We use the notation ∂_v for the directional derivative in the direction of $v \in H$ and take here $v = 1$.

We can interpret (16) as the relations defining the D-module generated by J_X, i.e. obtained from it by application of all differential operators. Using Taylor's formula $J_X(t + y\rho) = \exp(y\partial_\rho)J_X(t)$ we now view (15) as the asymptotic expansion of the oscillating integral taking values in this D-module:

$$(17)\quad (2\pi z)^{-\frac{r}{2}} \int_0^\infty dx_1 \dots \int_0^\infty dx_r\ e^{\frac{-\sum x_i + \sum(\lambda + z\partial_{\rho_i})\ln x_i}{z}}\ J_X(t,z)$$
$$\sim \prod_i e^{\frac{(\lambda+z\partial_{\rho_i})\ln(\lambda+z\partial_{\rho_i}) - (\lambda+z\partial_{\rho_i})}{z} + \frac{1}{2}\ln(\lambda+z\partial_{\rho_i}) + \sum_{m>0}\frac{B_{2m}}{2m(2m-1)}\frac{z^{2m-1}}{(\lambda+z\partial_{\rho_i})^{2m-1}}}\ J_X(t,z).$$

The exact form of the series is not relevant here. What matters is that the relations (16) in the D-module allow us to rewrite any high order derivation as a differential operator of first order and that composition of derivations coincides with the quantum cup-product $\bullet$ modulo higher order terms in z:

$$z\partial_{v_1} \dots z\partial_{v_N} = z\partial_{v_1\bullet\dots\bullet v_N} + o(z),$$

where $o(z)$ stands for a linear combination of $z\partial_{\phi_\alpha}$ with coefficients in $z\Lambda\{z\}$. Using this (and also the relation $\lambda J_X = z\partial_{\lambda\cdot 1}J_X$ mentioned earlier) we see that

$$\prod_i e^{\frac{(\lambda+z\partial_{\rho_i})\ln(\lambda+z\partial_{\rho_i}) - (\lambda+z\partial_{\rho_i})}{z}} J_X(t,z) = \prod_i e^{\partial_{[(\lambda+\rho_i\bullet)\ln(\lambda+\rho_i\bullet) - (\lambda+\rho_i\bullet)]1} + \frac{o(z)}{z}} J_X(t,z)$$
$$= e^{\frac{o(z)}{z}} J_X(t^*, z)$$

where $t^*(t) = t + [\ \sum(\lambda+\rho_i\bullet)\ln(\lambda+\rho_i\bullet) - (\lambda+\rho_i\bullet)\ ]\,1$. Processing next the factor $e^{\frac{1}{2}\ln(\lambda+z\partial_{\rho_i})}$, we take out $\sqrt{\lambda}$. The remaining factor $e^{\frac{1}{2}\ln(1+z\partial_{\rho_i}/\lambda)}$ together with the rest of the exponent in the asymptotic expansion (17) yields an expression of the type $e^{o(z)/z}J_X(t^*,z)$ too. We conclude that the expansion (17) assumes the form

$$\lambda^{\frac{\dim E}{2}} J_X(t^*,z) + \sum_\alpha C_\alpha(t^*,z)\ z\partial_{\phi_\alpha} J_X(t^*,z),$$

where the coefficients $C_\alpha(t^*,\cdot)$ are in $\Lambda\{z\}$ as required.

Remark. The proof of the lemma actually shows that given a family $\Phi(x,p)$ of phase functions parametrized by $p \in H^*$ the asymptotic expansion of the oscillating integral $\int dx\ e^{\Phi(x,z\partial)/z} J_X(t,z)$ belongs to the same cone as J_X.

Thus we have proved that the vector $I_E(t,-z)$ is situated on the Lagrangian cone $\mathcal{L}_{\mathbf{e},E}$. It therefore differs from the value of the corresponding J-function $J_{\mathbf{e},E}(\tau,-z)$ at a suitable point $\tau = \tau(t)$ by a linear combination of the derivatives $\partial J_{\mathbf{e},E}/\partial\tau^\alpha$ with coefficients in $z\Lambda\{z\}$. Moreover, these derivatives form a basis in the tangent space L to $\mathcal{L}_{\mathbf{e},E}$ considered as a free $\Lambda\{z\}$-module, and so the derivatives $\partial I_E/\partial t^\alpha \in L$ are expressible as their linear combinations. The last statement is equivalent to the *Birkhoff factorization* $U(z,z^{-1}) = V(z^{-1})W(z)$ where the columns of the matrix U are the derivatives of I_E, and those of V are the derivatives of $J_{\mathbf{e},E}$.

Let us use now the obvious fact that modulo the Novikov variables Q the functions I_E and $J_{\mathbf{e},E}$ coincide (at $t=\tau$) and hence $W(z)$ turns into the identity matrix in this specialization. Thus $\det W \in 1 + Q\,\Lambda\{z\}$ is invertible in $\Lambda\{z\}$ and therefore we can write $V = UW^{-1}$. Together with the expression $J_{\mathbf{e},E} = z\partial_1 J_{\mathbf{e},E}$ of the function $z^{-1}J_{\mathbf{e},E}$ as one of the columns of the matrix V this proves existence of the representation (18) in the following corollary.

Corollary 5. *Let $\mathcal{L}_{\mathbf{e},E} \subset (\mathcal{H}, \Omega_{\mathbf{e}(E)})$ be the Lagrangian cone determined by the* J*-function $J_{\mathbf{e},E}$ corresponding to $(\mathbf{e}, E)$-twisted Gromov-Witten theory, and let L_t be the tangent space to $\mathcal{L}_{\mathbf{e},E}$ at the point $I_E(t,-z)$. Then the intersection (unique due to some transversality property) of zL_t with the affine subspace $-z + z\mathcal{H}_-$ coincides with the value $J_{\mathbf{e},E}(\tau,-z) \in -z + \tau(t) + \mathcal{H}_-$ of the* J*-function. In other words,*

$$J_{\mathbf{e},E}(\tau,z) = I_E(t,z) + \sum_\alpha c_\alpha(t,z)\, z\, \partial_{\phi_\alpha} I_E(t,z), \ \ \textit{where } c_\alpha(t,\cdot) \in \Lambda\{z\}, \tag{18}$$

and $\tau(t)$ is determined as the z^0-mode of the RHS.

Remark. A by-product of Corollary 5 is a geometrical description of the "mirror map" $t \mapsto \tau$: the J-function obtained as the intersection $L_t \cap (-z + z\mathcal{H}_-)$ comes naturally parametrized by t which may have little in common with the projections $\tau - z$ of the intersection points along $\mathcal{H}_-$.

9. Mirror formulas

Let us assume now that the bundle E (which is still the sum of line bundles with first Chern classes ρ_i) is *convex*, i.e. spanned fiberwise by global sections, and apply the above results to the genus-zero Gromov-Witten theory of a complete intersection $j : Y \hookrightarrow X$ defined by a global section. While the above proof of Theorem 2 fails miserably in the limit $\lambda = 0$, the definition of the series $J_{\mathbf{e},E}$ and I_E and the relation between them described by Corollary 5 survive the nonequivariant specialization. Namely, at $\lambda = 0$ the J-function $J_{\mathbf{e},E}$ degenerates into

$$J_{X,Y}(t,z) = z + t + \sum_{d,n} \frac{Q^d}{n!}\, (\mathrm{ev}_{n+1})_* \left[\frac{\mathbf{e}(E'_{0,n+1,d})}{z - \psi_{n+1}} \bigwedge_{i=1}^n \mathrm{ev}_i^*\, t \right],$$

where $(\mathrm{ev}_{n+1})_*$ is the cohomological push-forward along the evaluation map $\mathrm{ev}_{n+1} : X_{0,n+1,d} \to X$ and $\mathbf{e}$ is the (nonequivariant!) Euler class. Here

$E'_{0,n+1,d} \subset E_{0,n+1,d}$ is the subbundle defined as the kernel of the evaluation map $E_{0,n+1,d} \to \mathrm{ev}_{n+1}^* E$ of sections (from $H^0(\Sigma, f^*E)$) at the $(n+1)$st marked point.

The function $J_{X,Y}$ is related to the Gromov-Witten invariants of Y by

$$\mathbf{e}(E) J_{X,Y}(u,z) =_{H_2(Y)\to H_2(X)} j_* J_Y(j^*u, z), \tag{19}$$

since $[Y_{0,n+1,d}] = \mathbf{e}(E_{0,n+1,d}) \cap [X_{0,n+1,d}]$ (see for instance [30]). The long subscript here is to remind us that the corresponding homomorphism between Novikov rings should be applied to the RHS.

On the other hand, the series I_E in the limit $\lambda = 0$ specializes to

$$I_{X,Y}(t,z) = \sum_d J_d(t,z) Q^d \prod_i \prod_{k=1}^{\rho_i(d)} (\rho_i + kz)$$

since $\rho_i(d) \geq 0$ for all degrees d of holomorphic curves. Passing to the limit $\lambda = 0$ in Theorem 2 and Corollary 5 we obtain the following "mirror theorem".

Corollary 6. *The series $I_{X,Y}(t,-z)$ and $J_{X,Y}(\tau,-z)$ determine the same cone. In particular, the series $J_{X,Y}$ related to the* J*-function of Y by* (19) *is recovered from $I_{X,Y}$ via the Birkhoff factorization procedure followed by the mirror map $t \mapsto \tau$ as described in Corollary* 5.

Remark. Corollary 6 is more general than the (otherwise similar) quantum Lefschetz hyperplane section theorems by Bertram and Lee [9], [33] and Gathmann [18] for

(i) it is applicable to arbitrary complete intersections Y without the restriction $c_1(Y) \geq 0$, and

(ii) it describes the J-functions not only over the *small* space of parameters $t \in H^{\leq 2}(X, \Lambda)$ but over the entire Frobenius manifold $H^*(X,\Lambda)$.

In fact the results of [18] allow one to deal with both generalizations and to compute recursively the corresponding Gromov-Witten invariants one at a time. What has been missing so far is the part that Birkhoff factorization plays in the formulations.

Now restricting $J_{X,Y}$ and $I_{X,Y}$ to the small parameter space $H^{\leq 2}(X,\Lambda)$ and assuming that $c_1(E) \leq c_1(X)$ we can derive the quantum Lefschetz theorems of [4], [9], [18], [29], [33]. A dimensional argument shows that the series $I_{X,Y}$ on the small parameter space has the form

$$I_{X,Y}(t,z) = zF(t) + \sum G^i(t)\phi_i + \mathrm{O}(z^{-1}),$$

where $\{\phi_i\}$ is a basis in $H^{\leq 2}(X,\Lambda)$, G^i and F are scalar formal functions, and F is invertible (we have $F = 1$ and $G^i = t^i$ when the Fano index is not too small).

Corollary 7. *When $c_1(E) \le c_1(X)$ the restriction of $J_{X,Y}$ to the small parameter space $\tau \in H^2(X,\Lambda)$ is given by*

$$J_{X,Y}(\tau, z) = \frac{I_{X,Y}(t,z)}{F(t)}, \ \textit{where } \tau = \sum \frac{G^i(t)}{F(t)}\phi_i.$$

The J-function of $X = \mathbb{C}P^{n-1}$ restricted to the small parameter plane $t_0 + tP$ (where P is the hyperplane class generating the algebra $H^*(X,\Lambda) = \Lambda[P]/(P^n)$) takes the form

$$J_X(t_0 + tP, z) = z\ e^{(t_0+Pt)/z} \sum_{d \ge 0} \frac{Q^d e^{dt}}{\prod_{k=1}^d (P+kz)^n}. \tag{20}$$

For a hypersurface Y of degree l in $\mathbb{C}P^{n-1}$ we then have

$$I_{X,Y}(t_0 + tP, z) = z e^{(t_0+Pt)/z} \sum_{d \ge 0} Q^d e^{dt} \frac{\prod_{k=1}^{ld}(lP+kz)}{\prod_{k=1}^d (P+kz)^n}. \tag{21}$$

Corollary 8. *On the small parameter space*

(i) *when $l < n-1$,*

$$J_{X,Y}(t_0 + tP, z) = I_{X,Y}(t_0 + tP, z);$$

(ii) *when $l = n-1$,*

$$J_{X,Y}(\tau_0 + tP, z) = I_{X,Y}(t_0 + tP, z)$$

where $\tau_0 = t_0 + l!Qe^t$;

(iii) *when $l = n$,*

$$J_{X,Y}(t_0 + \tau P, z) = I_{X,Y}(t_0 + tP, z)/F(t)$$

where $\tau = G(t)/F(t)$ and the series F and G are found from the expansion $I_{X,Y} = \exp(t_0/z)[zF(t) + G(t)P + \mathrm{O}(z^{-1})]$.

Projecting $J_{X,Y}$ by j^* onto the cohomology algebra $\Lambda[P]/(P^{n-1}) \subset H^*(Y,\Lambda)$ we recover the mirror theorem of [25] and, in the case $l = n = 5$, the quintic mirror formula of Candelas *et al.* [10].

10. Serre duality in genus zero

Let $\mathcal{L}_{\mathbf{c},E}$ be the Lagrangian cone in the symplectic space $(\mathcal{H}, \Omega_{\mathbf{c}(E)})$ defined by the genus-zero descendant potential $\mathcal{F}^0_{\mathbf{c},E}$, and $\mathcal{L}_{\mathbf{c}^*,E^*}$ be the Lagrangian cone in the symplectic space $(\mathcal{H}, \Omega_{\mathbf{c}^*(E^*)})$ defined by the genus-zero descendant potential $\mathcal{F}^0_{\mathbf{c}^*,E^*}$. Let $J_{\mathbf{c},E}$ and $J_{\mathbf{c}^*,E^*}$ denote the J-functions of the cones $\mathcal{L}_{\mathbf{c},E}$ and $\mathcal{L}_{\mathbf{c}^*,E^*}$ respectively:

$$J_{\mathbf{c},E}(\tau,-z) := \mathcal{L}_{\mathbf{c},E} \cap (-z+\tau+\mathcal{H}_-),$$
$$J_{\mathbf{c}^*,E^*}(\tau^*,-z) := \mathcal{L}_{\mathbf{c}^*,E^*} \cap (-z+\tau^*+\mathcal{H}_-).$$

The following result is obtained from Corollary 2 by passing to the quasi-classical limit $\hbar \to 0$.

Corollary 9. *The isomorphism*

$$(\mathcal{H}, \Omega_{\mathbf{c}^*(E^*)}) \to (\mathcal{H}, \Omega_{\mathbf{c}(E)}) \ : \ \mathbf{f} \mapsto \mathbf{c}^*(E^*)\mathbf{f}$$

of linear symplectic spaces identifies $\mathcal{L}_{\mathbf{c}^*,E^*}$ with $\mathcal{L}_{\mathbf{c},E}$.

In particular, the family

$$H \to \mathcal{H} : \ \tau \mapsto \mathbf{c}^*(E^*) J_{\mathbf{c}^*,E^*}(\tau,-z)$$

generates the cone $\mathcal{L}_{\mathbf{c},E}$. We can therefore recover the J-function $J_{\mathbf{c},E}$ from $J_{\mathbf{c}^*,E^*}$.

Corollary 10. $J_{\mathbf{c},E}(\tau,z) = z\mathbf{c}^*(E^*)\partial_{\mathbf{c}(E)} J_{\mathbf{c}^*,E^*}(\tau^*,z)$, *where*

$$(\tau,\phi) = \partial_\phi \partial_{\mathbf{c}(E)} \mathcal{F}^0_{\mathbf{c}^*,E^*}\big|_{t_0=\tau^*,t_1=t_2=\ldots=0} \quad \forall \phi \in H.$$

Proof. To simplify the notation put $J := J_{\mathbf{c},E}$, $J^* := J_{\mathbf{c}^*,E^*}$, $c := \mathbf{c}(E)$, $c^* := \mathbf{c}^*(E^*) = c^{-1}$. There exist coefficients C^α (which could *a priori* be polynomial in z and depend on τ^* but turn out here to be constant) and a change of variables $\tau = \tau(\tau^*)$, such that

$$c^* J^*(\tau^*,z) + z \sum C^\alpha \partial_{\phi_\alpha} c^* J^*(z,\tau^*) = z + \tau + \mathrm{O}(1/z).$$

The left-hand side therefore coincides with $J(\tau,z)$. Comparing the z-terms, we find $1 = c^* + c^*(\sum C^\alpha \phi_\alpha)$, i.e. $\sum C^\alpha \phi_\alpha = c - 1$. Together with $z\partial_{\mathbf{1}} J^* = J^*$ this implies that $J(\tau,z) = zc^*\partial_c J^*(\tau^*,z)$. Comparing the z^0-terms we find

$$(\tau,\phi) = \frac{1}{2\pi i}\oint (J(\tau,z),\phi)\frac{dz}{z} = \partial_c \Omega_{c^*}(J^*(\tau^*,-z),\phi)$$
$$= \partial_c \partial_\phi \mathcal{F}^0_{\mathbf{c}^*,E^*}|_{t_0=\tau^*,t_1=t_2=\ldots=0}. \qquad \square$$

We can repeat the above arguments in the situation of Corollary 3 where $\mathbf{c} = \mathbf{e}$ is the S^1-equivariant Euler class.

Corollary 11. *The map $\mathbf{f} \mapsto (-1)^{\dim E}\mathbf{e}^{-1}(E)\mathbf{f}$, $Q \mapsto \pm Q$ identifies $\mathcal{L}_{\mathbf{e}^{-1},E^*}$ with $\mathcal{L}_{\mathbf{e},E}$. Furthermore,*

$$\mathbf{e}(E) J_{\mathbf{e},E}(\tau,z;Q) = z(-1)^{\dim E}\partial_{\mathbf{e}(E)} J_{\mathbf{e}^{-1},E^*}(\tau^*,z;\pm Q),$$

where for all $\phi \in H$ we have $(\tau,\phi) = \partial_\phi \partial_{\mathbf{e}(E)} \mathcal{F}^0_{\mathbf{e}^{-1},E^*}(\tau^*,0,0,\ldots;\pm Q)$.

Remark. Corollary 11 generalizes the "nonlinear Serre duality" from [23] (Theorem 5.2) obtained there by fixed point localization and applicable to torus-equivariant bundles E with isolated fixed points.

Theorem 2, Corollary 9 and the mirror formulas of Section 9 have Serre-dual partners obtained by replacing $\mathbf{e}$ and E with with $\mathbf{e}^{-1}$ and E^*. We assume that the bundle E has integer Chern roots $\rho_1, ..., \rho_r$, and thus that $\mathbf{e}^{-1}(E^*) = \prod(-\lambda - \rho_i)^{-1} = (-1)^r \mathbf{e}^{-1}(E)$. We put

$$\begin{aligned} I^*_{E^*}(t,z) &:= \sum_d Q^d J_d(t,z) \, \prod_i \frac{\prod_{k=-\infty}^{0}(-\lambda - \rho_i + kz)}{\prod_{k=-\infty}^{-\rho_i(d)}(-\lambda - \rho_i + kz)} \\ &= \sum_d (\pm Q)^d J_d(t,z) \, \prod_i \frac{\prod_{k=-\infty}^{\rho_i(d)-1}(\lambda + \rho_i + kz)}{\prod_{k=-\infty}^{-1}(\lambda + \rho_i + kz)}. \end{aligned}$$

THEOREM $2'$. *The series $I^*_{E^*}$, considered as a family $t \mapsto I^*_{E^*}(t,-z)$ of vectors in the symplectic space $(\mathcal{H}, \Omega_{\mathbf{e}^{-1}(E^*)})$ corresponding to the twisted inner product $(a,b)_{\mathbf{e}^{-1}(E^*)} = \int_X \mathbf{e}^{-1}(E^*)ab$ on H, is situated on the Lagrangian section $\mathcal{L}_{\mathbf{e}^{-1},E^*} \subset (\mathcal{H}, \Omega_{\mathbf{e}^{-1}(E^*)})$ defined by the differential of the twisted genus-zero descendant potential $\mathcal{F}^0_{\mathbf{e}^{-1},E^*}$.*

Theorem $2'$ can be proved by repeating arguments from Section 8. In fact Theorem 2 follows easily from Theorem $2'$ and Corollary 11. Namely, the divisor equation (14) yields

$$\mathbf{e}(E) I_E(t,z;Q) = (-1)^r z \partial_{\mathbf{e}(E)} I^*_{E^*}(t,z;\pm Q). \tag{22}$$

Theorem $2'$ implies that the family $t \mapsto I^*_{E^*}(t,-z;Q)$ lies in $\mathcal{L}_{\mathbf{e}^{-1},E^*}$, and therefore that the family $t \mapsto -z\partial_{\mathbf{e}(E)} I^*_{E^*}(t,-z;Q)$ does too since the cone $\mathcal{L}_{\mathbf{e}^{-1},E^*}$ meets each of its tangent spaces L along zL. Then the first statement in Corollary 11 implies that the family $t \mapsto I_E(t,-z;Q)$ lies in $\mathcal{L}_{\mathbf{e},E}$.

When the classes ρ_i are positive, the bundle E^* is *concave* in the sense that $H^0(\Sigma, f^*E^*) = 0$ for all nonconstant maps $f : \Sigma \to X$ of compact connected complex curves Σ. Then $(\mathbf{e}^{-1}, E^*)$-twisted Gromov-Witten invariants of positive degrees $d \neq 0$ admit the nonequivariant specialization $\lambda = 0$. The reader can check that in the case of toric manifolds X the results of this section reproduce genus-zero mirror theorems (Theorem 4.2 and Corollary 5.1) from [23].

We illustrate some results of the present section in an example where $X = \mathbb{C}P^{n-1}$, E is a line bundle of degree $0 < l \leq n$, and the J-functions and their hypergeometric modifications are restricted to the small parameter plane

$t_0 + tP$. Then

$$J_X(t_0+tP,z;Q) = z\, e^{(t_0+Pt)/z} \sum_{d\geq 0} \frac{Q^d e^{dt}}{\prod_{k=1}^d (P+kz)^n},$$

$$I_E(t_0+tP,z;Q) = z\, e^{(t_0+Pt)/z} \sum_{d\geq 0} Q^d e^{dt} \frac{\prod_{k=1}^{ld}(\lambda+lP+kz)}{\prod_{k=1}^d (P+kz)^n},$$

$$I^*_{E^*}(t_0+tP,z;Q) = z\, e^{(t_0+Pt)/z} \sum_{d\geq 1} (-1)^{ld} Q^d e^{dt} \frac{\prod_{k=0}^{ld-1}(\lambda+lP+kz)}{\prod_{k=1}^d (P+kz)^n}.$$

The factor $\lambda + lP$ in $I^*_{E^*}$ (corresponding to $k=0$ in the numerators) contains no z. As a result, the expansion $I^*_{E^*}(t_0+tP,z;Q) = z + (t_0+tP) + \mathrm{O}(1/z)$ is valid for $l < n$. Thus $J_{\mathbf{e}^{-1},E^*}(t_0+tP,z;Q)$ and $I^*_{E^*}(t_0+tP,z;Q)$ coincide when $l < n$.

Trying to compute $J_{\mathbf{e},E}$ using Corollary 11, we see from (22) that

$$\mathbf{e}^{-1}(E^*) z \partial_{\mathbf{e}(E)} J_{\mathbf{e}^{-1},E^*}(t_0+tP,z;\pm Q) = I_E(t_0+tP,z;Q) \quad \text{when } l<n.$$

This results in the same formulas for $J_{\mathbf{e},E}$ as in Corollary 8, both equivariantly and in the nonequivariant limit: $J_{\mathbf{e},E}(t_0+tP,z;Q) = I_E(t_0+tP,z;Q)$ when $l < n-1$, and $J_{\mathbf{e},E}(\tau_0+tP,z;Q) = I_E(t_0+tP,z;Q)$, $\tau_0 = t_0 + l!Qe^t$, when $l = n-1$.

When $l = n$, we have the following expansion:

$$I^*_{E^*}(t_0+tP,z;Q) = z + t_0 + Pt + (\lambda+nP)\Phi(t;Q) + \mathrm{O}(1/z),$$

where

$$\Phi(t,Q) = \sum_{d=1}^{\infty} (-1)^{nd} \frac{(nd-1)!}{(d!)^n} Q^d e^{dt}.$$

Theorem $2'$ implies that $J_{\mathbf{e}^{-1},E^*}(\tau_0^* + P\tau^*, z;Q) = I^*_{E^*}(t_0+Pt,z;Q)$, where $\tau_0^* + P\tau^* = t_0 + Pt + (\lambda+nP)\Phi(t,Q)$. From this change of variables we derive

$$\lambda\frac{\partial}{\partial t_0} + n\frac{\partial}{\partial t} = \left(1 + n\frac{d\Phi}{dt}(t,Q)\right)\left(\lambda\frac{\partial}{\partial \tau_0^*} + n\frac{\partial}{\partial \tau^*}\right).$$

Therefore

$$\mathbf{e}^{-1}(E^*) z \partial_{\mathbf{e}(E)} J_{\mathbf{e}^{-1},E^*}(\tau_0^* + \tau^* P,z;\pm Q) = I_E(t_0+tP,z;Q)/F(t,Q),$$

where

$$F(t,Q) = 1 + n\frac{d\Phi}{dt}(t,\pm Q) = \sum_{d=0}^{\infty} \frac{(nd)!}{(d!)^n} Q^d e^{dt}.$$

Thus, trying to compute $J_{\mathbf{e},E}$ from $J_{\mathbf{e}^{-1},E^*}$ in the case $l = n$ using Corollary 11, we in fact arrive at the same result as in Corollary 8:

$$J_{\mathbf{e},E}(\tau_0+\tau P,z;Q) = I_E(t_0+tP,z;Q)/F(t,Q),$$

where the change of variables $(t_0, t) \mapsto (\tau_0, \tau)$ is determined by expanding this identity:

$$z + \tau_0 + P\tau + \mathrm{O}(1/z) = z + t_0 + PG(t)/F(t) + \mathrm{O}(1/z).$$

11. Further comments

On quantum Riemann-Roch. The operators $\mathrm{ch}_l(E)z^{2m-1}$ commute. In the nonequivariant setting this property is preserved under quantization for the operators with $m \geq 0$ which occur in Theorem 1. This is due to the nilpotence of $\mathrm{ch}_l(E)$ with $l > 0$. Also, the summand with $l = 1$ on the LHS of (7) is the only one left in this case. Thus formula (7) simplifies in the nonequivariant case:

$$\mathcal{D}_{\mathbf{s}} = \left(e^{s_0(c_1(E), c_{D-1}(T_X))} e^{\frac{1}{2}s_0\chi(X)\dim E}\right)^{\frac{1}{24}} \; e^{\sum_{m\geq 0}\sum_{l\geq 0} s_{2m-1+l}\frac{B_{2m}}{(2m)!}\widehat{\mathrm{ch}_l(E)z^{2m-1}}} \; \mathcal{D}_X.$$

The formula defines a formal group homomorphism from the group of invertible multiplicative characteristic classes to invertible operators acting on elements of the Fock space. It would be interesting to find a quantum-mechanical interpretation of the normalizing factor in this formula. Since the Fock space should consist of top-degree forms on $\mathcal{H}_+$ rather than functions, the contribution $e^{s_0\chi(X)\dim E/48}$ from the super-determinant probably takes on the role of the Jacobian of our "bare hands" identification $\mathbf{q} \mapsto \sqrt{\mathbf{c}(E)}\mathbf{q}$. We do not have however a plausible physical interpretation for the other factor.

On the Lagrangian cones. In the case of genus-zero Gromov-Witten theory of $X = pt$ the cone $\mathcal{L}_{pt}$ is generated by the family of functions in one variable x:

$$F(x, \mathbf{q}) := \frac{1}{2}\int_0^x Q^2(u)\, du, \quad \text{where } Q(x) = \sum q_k \frac{x^k}{k!}.$$

In particular, under analytic continuation in Q (from the formal neighborhood of the function $Q(x) = x$ to the space of all functions Q, say, polynomial in x) the cone $\mathcal{L}_X$ acquires singularities studied in geometrical optics on manifolds with boundary (see for instance [2], [19], [37]) and called *open swallowtails*. It would be interesting to study singularities of $\mathcal{L}_X$ under analytic continuation and to understand the significance of the relationship with geometrical optics.

According to some results and conjectures of [14] and [25], the Lagrangian cones $\mathcal{L}$ corresponding to semisimple Frobenius structures are each linearly isomorphic to a closure of the Cartesian product of $\dim H$ copies of $\mathcal{L}_{pt}$, and various models in genus-zero Gromov-Witten theory differ only by the position of this product with respect to the polarization. The same is true for the cones $\mathcal{L}_{\mathbf{s}}$ corresponding to the different twisted theories on the same X: according to Corollary 4 they are obtained from each other by linear symplectic transformations. These transformations form the multiplicative group $\exp(\sum \tau_m z^{2m-1})$

where τ_m are even elements of the *algebra* H. The action of this group on the semi-infinite Grassmannian resembles the Grassmannian interpretation of the KdV hierarchy. It would be interesting to further this analogy.

On the mirror theory. When $X = pt$, the function $J_X = z\exp(t_0/z)$. When $E = \mathbb{C}^n$ is the trivial bundle over a point, the integral (15) turns into

$$\int_0^\infty \dots \int_0^\infty e^{-\frac{x_1+\dots+x_n}{z}} (x_1\dots x_n)^{\frac{\lambda}{z}}\ dx_1\wedge\dots\wedge dx_n\ .$$

It would be interesting to find a "quantum symplectic reduction theorem" which would explain how this integral is related to the J-functions of toric manifolds X (see [22]) obtained by symplectic reduction from $\mathbb{C}^n$. For example, when $X = \mathbb{C}P^{n-1} = \mathbb{C}^n /\!\!/ S^1$, components of the J-function (20) coincide with complex oscillating integrals

$$J_X(t) = z\ \int_{\gamma\subset\{u_1\dots u_n=e^t\}} e^{\frac{u_1+\dots+u_n}{z}} \frac{d\ln u_1\wedge\dots\wedge d\ln u_n}{dt} \tag{23}$$

over suitable cycles γ. For a degree $l \le n$ hypersurface $Y \subset X$, this yields integral representations for $I_{X,Y}$ and I_Y. Indeed the I-function (21) is proportional to the convolution (15)

$$\int_0^\infty dv\ e^{-\frac{v}{z}} J_X(t + l\ \ln v) = \int_{\{u_1\dots u_n=v^l e^t\}} e^{\frac{u_1+\dots+u_n-v}{z}} \frac{dv\wedge d\ln u_1\wedge\dots\wedge d\ln u_n}{dt}.$$

Using the change $u_i \mapsto u_i v$ for $i = 1,\dots, l \le n$, we transform it to the "mirror partner" of Y:

$$\frac{1}{2\pi i}\int_{\{u_1\dots u_n=e^t\}} e^{(u_{l+1}+\dots+u_n)/z} \frac{d\ln u_1\wedge\dots\wedge d\ln u_n}{(1-u_1-\dots-u_l)\ dt}$$
$$= \int_{\{u_1\dots u_n=e^t;\ u_1+\dots+u_l=1\}} e^{\frac{u_{l+1}+\dots+u_n}{z}} \frac{d\ln u_1\wedge\dots\wedge d\ln u_n}{d(1-u_1-\dots-u_l)\wedge dt}.$$

Another question. According to the physics literature [40], the mirror maps $t \mapsto \tau$ arise from the mysterious *renormalization.* According to [12] the mathematical content of some important examples of renormalization in quantum field theory is Birkhoff factorization in suitable infinite-dimensional groups. Are *renormalization* and *Birkhoff factorization* synonymous?

On Serre duality. In the genus-zero theory, when E is convex and E^* is concave, the sheaves $E_{0,n,d}$ and $-E^*_{0,n,d}$ are vector bundles with fibers $H^0(\Sigma, f^*E)$ and $H^1(\Sigma, f^*E^*)$ respectively. Twisting by the Euler class of $E^*_{0,n,d}$ one obtains Gromov-Witten invariants of the noncompact total space of the bundle E^*. Genus-zero invariants twisted by the Euler class of $E_{0,n,d}$ can be interpreted as Gromov-Witten invariants of the super-manifold (ΠE), i.e. of the total space of the bundle E with the parity of the fibers reversed. The "nonlinear Serre duality" phenomenon emerged in [21], [23] in the context of fixed point localization for genus-zero Gromov-Witten invariants of (ΠE) and E^*. The duality was formulated as an identification (modulo minor adjustments such as $\lambda \mapsto -\lambda$, $Q \mapsto \pm Q$) of certain genus-zero potentials *written in Dubrovin's canonical coordinates* of the semi-simple Frobenius structures associated with the two theories. According to [24], [25] the total descendant potential of a semi-simple Frobenius structure can be described in terms of genus-zero data presented in canonical coordinates. This implies a higher genus version of the nonlinear Serre duality principle whenever the fixed point localization technique [24] applies. Corollaries 2 and 3, as well as their genus-zero counterparts Corollaries 10 and 11, assert the principle in much greater generality and show that both the localization technique and the reference to semi-simplicity and canonical coordinates in this matter are redundant.

Appendix 1. The proof of Theorem 1

We will begin by applying the Grothendieck-Riemann-Roch theorem to the bundle $\mathrm{ev}_{n+1}^*(E)$ over the universal family of stable maps $\pi : X_{g,n+1,d} \to X_{g,n,d}$. This yields

$$[X_{g,n,d}] \cap \mathrm{ch}_k(E_{g,n,d}) = \pi_* \left(\sum_{\substack{r+l=k+1 \\ r,l \geq 0}} \frac{B_r}{r!} \, \mathrm{ch}_l(\mathrm{ev}_{n+1}^*(E)) \cdot \Psi(r) \right) \tag{24}$$

where

$$\begin{aligned} \Psi(r) = {} & \psi_{n+1}^r \cap [X_{g,n+1,d}] - \sum_{i=1}^n (\sigma_i)_* \left(\psi_i^{r-1} \cap [X_{g,n,d}] \right) \\ & + \frac{1}{2} j_* \left(\sum_{\substack{a+b=r-2 \\ a,b \geq 0}} (-1)^a \psi_+^a \psi_-^b \cap [\tilde{Z}_{g,n+1,d}] \right). \end{aligned}$$

Here $\sigma_i : X_{g,n,d} \to X_{g,n+1,d}$ is the section of the universal family defined by the i^{th} marked point, $Z_{g,n+1,d}$ is the locus of virtual codimension two formed by nodes of the fibers of π, $\tilde{Z}_{g,n+1,d}$ is its double cover given by a choice of one of the branches of the curve at the node, $[\tilde{Z}_{g,n+1,d}]$ is the virtual fundamental class (which is described explicitly in note (iii) below), $j : \tilde{Z}_{g,n+1,d} \to X_{g,n+1,d}$ is the natural map, ψ_+ and ψ_- are the first Chern classes of the bundles L_+ and L_- over $\tilde{Z}_{g,n+1,d}$ formed by the cotangent lines at the nodes, and the K-theoretic push-forward $E_{g,n,d} = \pi_* \, \mathrm{ev}_{n+1}^*(E)$ is defined as follows.

A holomorphic vector bundle E over X can be represented as the quotient A/B of two concave bundles. For this, pick a positive line bundle L and let the exact sequence $0 \to \mathrm{Ker} \to H^0(X; E \otimes L^N) \otimes L^{-N} \to E \to 0$ take on the role of $0 \to B \to A \to E \to 0$. Then $H^0(\Sigma; f^*A)$ and $H^0(\Sigma; f^*B)$ vanish for any nonconstant map $f:\Sigma\to X$ and sufficiently large N, so that the following sequence is exact:

$$0\to H^0(\Sigma;f^*E)\to H^1(\Sigma;f^*B)\to H^1(\Sigma;f^*A)\to H^1(\Sigma;f^*E)\to 0.$$

This construction applied to the universal family over the moduli space of stable maps yields, at least[3] in degree $d\neq 0$, a locally free resolution $0\to R^1\pi_*(\mathrm{ev}^*B)\to R^1\pi_*(\mathrm{ev}^*A)\to 0$ for $R^0\pi_*(\mathrm{ev}^*E)\ominus R^1\pi_*(\mathrm{ev}^*E)$. By definition,

$$E_{g,n,d}:=[R^1\pi_*(\mathrm{ev}^*B)]\ominus[R^1\pi_*(\mathrm{ev}^*A)]\ \in\ K^0(X_{g,n,d}). \tag{25}$$

In fact this definition is an example of a general construction applicable to families of nodal curves or, even more generally, to local complete intersection morphisms.

A map $p:Y\to B$ is called a *local complete intersection* (l.c.i.) *morphism* if for some (and hence for any) factorization $p=q\circ i$ with $i:Y\to P$ a closed embedding and $q:P\to B$ smooth (i.e. a submersion), i is in fact a regular embedding. The latter means that the normal sheaf $N_{Y/P}$ is locally free, and therefore that the relative tangent bundle $T_p:=[i^*T_{P/B}]-[N_{Y/P}]\in K^0(Y)$ is well-defined. According to [1], for any bundle V on Y and any proper l.c.i. morphism $p:Y\to B$ there exists a resolution of $R^\bullet p_*(V)$, i.e. a complex $0\to A^0\to\ldots\to A^N\to 0$ of vector bundles on B with cohomology sheaves equal to $R^\bullet p_*V$. Moreover the K-theoretic push-forward p_*V, defined to be the element $[A^0]-[A^1]+\ldots+(-1)^N[A^N]\in K^0(B)$, does not depend on the choice of resolution.

According to [5], the Grothendieck-Riemann-Roch formula

$$\mathrm{ch}(p_*V)=p_*(\mathrm{ch}(V)\cdot\mathrm{Td}\,T_p) \tag{26}$$

holds for any proper l.c.i. morphism of schemes which admits a factorization $p=q\circ i$ as above with q proper. To extend (26) to our orbispace/orbibundle situation we may assume that the map $p:Y\to B$ arises from a G-equivariant map $p':Y'\to B'$ between G-spaces Y', B' with almost free actions of an algebraic Lie group G such that $Y=Y'/G$, $B=B'/G$. We lift the orbibundle V to a G-equivariant bundle over Y' and apply (26) G-equivariantly. More concretely, one can find a line orbibundle on the total space $Y=X_{g,n+1,d}$ of the universal family over $B=X_{g,n,d}$ such that its restriction to each fiber of the universal family is very ample (see e.g. [16], where a suitable construction is given in terms of Hilbert schemes of projective curves). Then spaces of global sections over the curves form a vector orbibundle W over B. Take B' to be the total space of the frame bundle associated to this vector bundle.

[3]When $d=0$, $H^0(\Sigma;f^*A)$ and $H^0(\Sigma;f^*B)$ are nonzero but also have constant rank, so that the construction of $E_{g,n,d}$ easily extends to this case as well.

The universal family of stable maps lifts naturally to form $G = GL_{\dim W}$-equivariant families $p' : Y' \to B'$ and $\mathrm{ev} : Y' \to X$ of nodal curves and their stable maps. The map p' is an l.c.i. morphism of schemes, and factors properly through the projectivization of the lift of W^* to B'. The classifying space BG admits finite-dimensional algebraic approximations BG_N. The corresponding Y'- and B'-bundles over BG_N form finite-dimensional approximations Y'_N and B'_N to the homotopy quotients $Y' \times_G EG$ and $B' \times_G EG$ respectively. The orbibundle version of the Grothendieck-Riemann-Roch formula (26) is obtained by applying (26) to the properly factorizable l.c.i. morphisms $p'_N : Y'_N \to B'_N$ (cf. [15]).

Applying (26) as directed, we find that

$$\begin{aligned} \mathrm{ch}(E_{g,n,d}) &= \mathrm{ch}(\pi_* \mathrm{ev}_{n+1}^* E) \\ &= \pi_*(\mathrm{ch}(\mathrm{ev}_{n+1}^* E) \cdot \mathrm{Td}^\vee \Omega_\pi) \end{aligned} \tag{27}$$

where Ω_π is the sheaf of relative differentials of $\pi : X_{g,n+1,d} \to X_{g,n,d}$ and $\mathrm{Td}^\vee$ is the dual Todd class. To derive (24) from (27) we follow Mumford [36] and Faber-Pandharipande [16]. We begin by expressing the sheaf Ω_π of relative differentials in terms of universal cotangent lines. Assume first that $X_{g,n,d}$, $X_{g,n+1,d}$, and $Z_{g,n+1,d}$ are smooth and of the expected dimension, and that the image $\pi(Z_{g,n+1,d})$ of the nodal locus forms a divisor with normal crossings in $X_{g,n,d}$. Then there are exact sequences of sheaves on $X_{g,n+1,d}$

$$0 \to \Omega_\pi \to \omega_\pi \to \mathcal{O}_{Z_{g,n+1,d}} \to 0$$

and

$$0 \to \omega_\pi \to L_{n+1} \to \oplus_{i=1}^n \mathcal{O}_{D_i} \to 0,$$

where ω_π is the relative dualizing sheaf of the universal family $\pi : X_{g,n+1,d} \to X_{g,n,d}$ and D_i is the divisor $\sigma_i(X_{g,n,d})$. Thus

$$\Omega_\pi = L_{n+1} - \sum_{i=1}^n \mathcal{O}_{D_i} - \mathcal{O}_{Z_{g,n+1,d}}$$

in $K^0(X_{g,n+1,d})$, and so

$$\mathrm{Td}^\vee(\Omega_\pi) = \mathrm{Td}^\vee(L_{n+1}) \cdot \left(\prod_{i=1}^n \mathrm{Td}^\vee(-\mathcal{O}_{D_i}) \right) \cdot \mathrm{Td}^\vee\left(-\mathcal{O}_{Z_{g,n+1,d}}\right). \tag{28}$$

The class $c_1(L_{n+1}) = \psi_{n+1}$ restricts to zero on the pairwise disjoint strata D_1, $\ldots$, D_n and $Z_{g,n+1,d}$. This translates the multiplicativity of the dual Todd class $\mathrm{Td}^\vee(\cdot)$ into additivity of $\mathrm{Td}^\vee(\cdot) - 1$:

$$\mathrm{Td}^\vee(\Omega_\pi) = 1 + \left[\mathrm{Td}^\vee(L_{n+1}) - 1\right] + \sum_{i=1}^{n}\left[\frac{1}{\mathrm{Td}^\vee(\mathcal{O}_{D_i})} - 1\right] + \left[\frac{1}{\mathrm{Td}^\vee(\mathcal{O}_{Z_{g,n+1,d}})} - 1\right]. \tag{29}$$

The first two terms yield

$$\mathrm{Td}^\vee(L_{n+1}) = \frac{\psi_{n+1}}{\exp\psi_{n+1} - 1} = \sum_{r\geq 0}\frac{B_r}{r!}\psi_{n+1}^r.$$

Using $\sigma_i^*(-D_i) = \psi_i$ and the exact sequence $0 \to \mathcal{O}(-D_i) \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0$, we find

$$\begin{aligned}\frac{1}{\mathrm{Td}^\vee(\mathcal{O}_{D_i})} - 1 &= \mathrm{Td}^\vee(\mathcal{O}(-D_i)) - 1\\ &= \sum_{r\geq 1}\frac{B_r}{r!}(-D_i)^r\\ &= -(\sigma_i)_*\sum_{r\geq 1}\frac{B_r}{r!}\psi_i^{r-1}.\end{aligned}$$

The codimension-2 summand in (29), which is supported in the neighborhood of $Z_{g,n+1,d}$, is processed using the inclusion-exclusion formula for the bi-graded Poincaré polynomial of $\mathbb{C}[x,y]/(xy)$:

$$\frac{1-uv}{(1-u)(1-v)} = \frac{1}{1-u} + \frac{1}{1-v} - 1.$$

The pull-back to the double cover $\tilde{Z}_{g,n+1,d}$ of the normal bundle to $Z_{g,n+1,d}$ in $X_{g,n+1,d}$ is $L_+^{-1}\oplus L_-^{-1}$, and there is a map from a neighbourhood of $\tilde{Z}_{g,n+1,d}$ in the total space of this bundle to a neighbourhood of $Z_{g,n+1,d}$ in $X_{g,n+1,d}$ which is generically two-to-one. We see from the pull-back

$$0 \to L_+\otimes L_- \to L_+\oplus L_- \to \mathcal{O} \to \mathcal{O}_{Z_{g,n+1,d}} \to 0$$

of the Koszul complex that in the neighborhood of $\tilde{Z}_{g,n+1,d}$ in $L_+^{-1}\oplus L_-^{-1}$,

$$\begin{aligned}\frac{1}{\mathrm{Td}^\vee(\mathcal{O}_{Z_{g,n+1,d}})} - 1 &= \frac{1-e^{-\psi_+-\psi_-}}{\psi_++\psi_-}\frac{\psi_+}{1-e^{-\psi_+}}\frac{\psi_-}{1-e^{-\psi_-}} - 1\\ &= \frac{\psi_+\psi_-}{\psi_++\psi_-}\left(\frac{1}{1-e^{-\psi_+}} + \frac{1}{1-e^{-\psi_-}} - 1 - \frac{1}{\psi_+} - \frac{1}{\psi_-}\right)\end{aligned}$$

and so on $X_{g,n+1,d}$,

$$\begin{aligned}\frac{1}{\mathrm{Td}^\vee(\mathcal{O}_{Z_{g,n+1,d}})} - 1 &= \frac{1}{2}\,j_*\left[\sum_{r\geq 2}\frac{B_r}{r!}\frac{\psi_+^{r-1}+\psi_-^{r-1}}{\psi_++\psi_-}\right]\\ &= \frac{1}{2}\,j_*\left[\sum_{r\geq 2}\frac{B_r}{r!}\sum_{a+b=r-2}(-1)^a\psi_+^a\psi_-^b\right].\end{aligned}$$

Combining the formulas for $\mathrm{Td}^\vee(\Omega_\pi)$ with (27) we arrive at (24).

In the general case, where $X_{g,n,d}$, $X_{g,n+1,d}$, and $Z_{g,n+1,d}$ need not be smooth and $\pi(Z_{g,n+1,d})$ need not be a divisor with normal crossings, this argument remains "virtually correct": we can find a flat family $\tilde{\pi} : \mathcal{C} \to \mathcal{M}$ of pre-stable curves and an embedding $X_{g,n,d} \to \mathcal{M}$ such that

- the family $\mathcal{C} \to \mathcal{M}$ restricts to the universal family over $X_{g,n,d}$:
$$\begin{array}{ccc} X_{g,n+1,d} & \longrightarrow & \mathcal{C} \\ \pi \big\downarrow & & \big\downarrow \tilde{\pi} \\ X_{g,n,d} & \longrightarrow & \mathcal{M}\ ; \end{array}$$
- the bundle $\mathrm{ev}_{n+1}^*(E)$ over $X_{g,n+1,d}$ is the pull-back of a bundle over $\mathcal{C}$;
- $\mathcal{C}$ and $\mathcal{M}$ are smooth;
- the locus $\mathcal{Z}$ of nodes of the fibers of $\tilde{\pi}$ is smooth, and $\tilde{\pi}(\mathcal{Z}) \subset \mathcal{M}$ is a divisor with normal crossings;
- there are a double cover $\tilde{\mathcal{Z}} \to \mathcal{Z}$ corresponding to the choice of one of the branches of the curve at the node, and line bundles $\mathcal{L}_+$ and $\mathcal{L}_-$ over $\tilde{\mathcal{Z}}$ with fibers given by the cotangent lines at the node;
- the pull-back to the double cover $\tilde{\mathcal{Z}}$ of the normal bundle $N_{\mathcal{Z}/\mathcal{C}}$ is isomorphic to $\mathcal{L}_+ \oplus \mathcal{L}_-$.

Such a family $\tilde{\pi} : \mathcal{C} \to \mathcal{M}$ was constructed in [16]. Since our argument in the smooth case used only the latter four properties above, it also proves the analogous statement for the family $\tilde{\pi} : \mathcal{C} \to \mathcal{M}$. We recover (24) by capping the result for $\tilde{\pi} : \mathcal{C} \to \mathcal{M}$ with the virtual fundamental class $[X_{g,n,d}]$.

In deriving Theorem 1 from (24), we will need also the following facts.

(i) The comparison formula for cotangent line classes:
$$\psi_i - \pi^*(\psi_i) = D_i.$$

(ii) The naturality of the virtual fundamental class under the flat morphism π:
$$\pi^*[X_{g,n,d}] = [X_{g,n+1,d}].$$

(iii) The composition rule: recall that the double cover $\tilde{Z}_{g,n+1,d}$ of the nodal locus coincides with the total range of the *gluing maps*

$$X_{g-1,n+\bullet+\circ} \times_{X \times X} X_{0,1+\bullet+\circ,0} \xrightarrow{\gamma_{\mathrm{irr}}} \tilde{Z}_{g,n+1,d} \tag{30}$$

and

$$X_{g_+,n_++\bullet,d_+} \times_X X_{0,1+\bullet+\circ,0} \times_X X_{g_-,n_-+\circ,d_-} \xrightarrow{\gamma_{\mathrm{red}}} \tilde{Z}_{g,n+1,d}, \tag{31}$$

where $g_+ + g_- = g$, $n_+ + n_- = n$, and $d_+ + d_- = d$. The composition rule says that images of the virtual fundamental classes under the gluing maps add up to the virtual fundamental class $[\tilde{Z}_{g,n+1,d}]$.

Properties (ii) and (iii) are part of the axioms in [7] proved in [6], and (i) is well known too — see for instance [39].

Next we need similar results about the elements $E_{g,n,d} \in K^0(X_{g,n,d})$:

(iv) $\pi^* E_{g,n,d} = E_{g,n+1,d}$.

(v) $\gamma_{\text{irr}}^* j^* E_{g,n+1,d} = \text{pr}^* E_{g-1,n+\bullet+\circ,d} - \text{ev}_\Delta^* E$ where γ_{irr} is the gluing map (30), pr is the projection to the first factor of the fiber product on the LHS of (30), and $\text{ev}_\Delta = \text{ev}_\bullet = \text{ev}_\circ$ denotes evaluation at the point of gluing.

(vi) $\gamma_{\text{red}}^* j^* E_{g,n+1,d} = \text{pr}_+^* E_{g_+,n_++\bullet,d_+} + \text{pr}_-^* E_{g_-,n_-+\circ,d_-} - \text{ev}_\Delta^* E$, where γ_{red} is the gluing map (31), $\text{pr}_\pm$ are projections to the first and last factors of the fiber product on the LHS of (31), and ev_Δ is as above.

In view of our construction (25) of $E_{g,n,d}$ and its resolution by vector bundles on $X_{g,n,d}$, it suffices to verify these properties either when $R^0\pi_* \text{ev}_{n+1}^* E = 0$ and $E_{g,n,d} = R^1\pi_* \text{ev}_{n+1}^* E$ is a vector bundle, or when $d = 0$. In each case they are straightforward consequences of Serre duality, which identifies $H^1(\Sigma, f^*E)$ with f^*E-valued meromorphic differential forms on Σ allowed poles only at the nodes subject to the condition that the sum of residues at each node is zero. It is this residue-matching condition which gives rise to the factors $\text{ev}_\Delta^* E$ above.

Finally, we will need three integrals over low-genus moduli spaces. Introduce correlator notation for Gromov-Witten invariants: for polynomials in ψ

$$\mathbf{a}^i(\psi) = a_0^i + a_1^i\psi + \dots, \qquad i = 1, \dots, n,$$

with coefficients in $H^*(X;\Lambda)$ and an element $\beta \in H^*(X_{g,n,d};\Lambda)$; define

$$\langle\!\langle \mathbf{a}^1(\psi), \dots, \mathbf{a}^n(\psi); \beta \rangle\!\rangle_{g,n,d} := \int_{[X_{g,n,d}]} \left(\bigwedge_{i=1}^{i=n} \sum_{j\geq 0} \text{ev}_i^*(a_j^i)\, \psi_i^j \right) \wedge \beta$$

Using this notation, we have

(vii) $$\langle\!\langle \mathbf{t}(\psi), \mathbf{t}(\psi), \text{ch}_{k+1}(E); \mathbf{c}(E_{0,3,0}) \rangle\!\rangle_{0,3,0} = \int_X t_0 \wedge t_0 \wedge \text{ch}_{k+1}(E) \wedge \mathbf{c}(E);$$

(viii) $$\langle\!\langle \text{ch}_k(E)\psi; \mathbf{c}(E_{1,1,0}) \rangle\!\rangle_{1,1,0} = \frac{1}{24}\int_X \text{ch}_k(E) \wedge \mathbf{e}(X);$$

(ix) $$\langle\!\langle \text{ch}_{k+1}(E); \mathbf{c}(E_{1,1,0}) \rangle\!\rangle_{1,1,0} = \frac{1}{24}\int_X \text{ch}_{k+1}(E) \wedge \mathbf{e}(X) \wedge \sum_{j\geq 1} s_j \,\text{ch}_{j-1}(E) - \frac{1}{24}\int_X \text{ch}_{k+1}(E) \wedge c_{D-1}(TX).$$

The equality (vii) is obvious since $X_{0,3,0} = X$ and $[X_{0,3,0}]$ is the usual fundamental class of X; (viii) and (ix) follow from the well-known facts:

- $X_{1,1,0} = X \times \overline{\mathcal{M}}_{1,1}$.
- $[X_{1,1,0}]$ is the cap product of the fundamental class of $X \times \overline{\mathcal{M}}_{1,1}$ with $\mathbf{e}(p_1^* T_X \otimes p_2^* \mathbf{E}_1^{-1})$. Here p_1 and p_2 are the projections to the first and second factors of $X \times \overline{\mathcal{M}}_{1,1}$ respectively and $\mathbf{E}_1$ is the Hodge bundle over $\overline{\mathcal{M}}_{1,1}$.
- $E_{1,1,0} = p_1^* E \otimes (1 \ominus p_2^* \mathbf{E}_1^{-1})$;
- $c_1(\mathbf{E}_1) = \psi_1$ on $\overline{\mathcal{M}}_{1,1}$.
- $\int_{[\overline{\mathcal{M}}_{1,1}]} \psi_1 = 1/24$.

Using (24) and the properties (i–ix) we now derive Theorem 1′. At $\mathbf{s} = (0, 0, \dots)$ Theorem 1′ holds trivially, so it suffices to prove the infinitesimal version

$$\frac{\partial}{\partial s_k} \mathcal{D}_{\mathbf{s}} = \left(\sum_{\substack{2m+r=k+1 \\ r,m \geq 0}} \frac{B_{2m}}{(2m)!} (\mathrm{ch}_r(E) z^{2m-1})\hat{} \right) \mathcal{D}_{\mathbf{s}} + \left(\begin{array}{c} \frac{1}{24} \int_X c_{D-1}(X) \wedge \mathrm{ch}_{k+1}(E) + \frac{1}{48} \int_X \mathbf{e}(X) \wedge \mathrm{ch}_k(E) \\ -\frac{1}{24} \int_X \mathbf{e}(X) \wedge \mathrm{ch}_{k+1}(E) \wedge (\sum_l s_{l+1} \mathrm{ch}_l(E)) \end{array} \right) \mathcal{D}_{\mathbf{s}}. \tag{32}$$

Here the first two exceptional terms come from the factors on the LHS of (7); in particular the second one is due to

$$\left(\mathrm{sdet} \sqrt{\mathbf{c}(E)} \right) = \exp \left(\mathrm{str} \ln \sqrt{\mathbf{c}(E)} \right) = \exp \left(\int_X \mathbf{e}(X) \wedge \frac{1}{2} \sum_{j \geq 0} s_j \, \mathrm{ch}_j(E) \right).$$

The third exceptional term is the cocycle value

$$\mathcal{C} \left(\frac{B_2}{2} \sum_{l \geq 0} s_{l+1} (\mathrm{ch}_l(E) z)\hat{}, \left(\frac{\mathrm{ch}_{k+1}(E)}{z} \right)\hat{} \right) = -\frac{1}{24} \, \mathrm{str} \left(\mathrm{ch}_{k+1}(E) \cdot \sum_{l \geq 0} s_{l+1} \, \mathrm{ch}_l(E) \right)$$

which arises from commuting the derivative of the $\frac{1}{z}$ terms on the RHS in (7) past the terms involving z.

In the above correlator notation,

$$\mathcal{D}_{\mathbf{s}} = \exp \left(\sum_{g \geq 0} \sum_{n \geq 0} \sum_{d \in H_2(X;\mathbb{Z})} \frac{\hbar^{g-1} Q^d}{n!} \langle\!\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi); \mathbf{c}(E_{g,n,d}) \rangle\!\rangle_{g,n,d} \right)$$

and

$$\mathcal{D}_{\mathbf{s}}^{-1}\frac{\partial}{\partial s_k}\mathcal{D}_{\mathbf{s}} = \sum_{g,n,d} \frac{\hbar^{g-1}Q^d}{n!}\left\langle\!\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi); \mathrm{ch}_k(E_{g,n,d})\,\mathbf{c}(E_{g,n,d})\right\rangle\!\right\rangle_{g,n,d} + \sum_{g,n,d}\frac{\hbar^{g-1}Q^d}{(n-1)!}\left\langle\!\!\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi),\frac{\partial\mathbf{t}}{\partial s_k}; \mathbf{c}(E_{g,n,d})\right\rangle\!\!\right\rangle_{g,n,d}. \tag{33}$$

We apply our expression (24) for $[X_{g,n,d}]\cap \mathrm{ch}_k(E_{g,n,d})$ and compare the result with (32) by extracting terms involving the same Bernoulli numbers.

We begin with $B_0 = 1$. The relevant part of (33) is

$$\sum_{g,n,d}\frac{\hbar^{g-1}Q^d}{n!}\left\langle\!\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi); \pi_*\left[B_0\,\mathrm{ev}_{n+1}^*\,\mathrm{ch}_{k+1}(E)\right]\mathbf{c}(E_{g,n,d})\right\rangle\!\right\rangle_{g,n,d}. \tag{34}$$

We compute $\left\langle\!\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi); \pi_*\left[B_0\,\mathrm{ev}_{n+1}^*\,\mathrm{ch}_{k+1}(E)\right]\mathbf{c}(E_{g,n,d})\right\rangle\!\right\rangle_{g,n,d}$ by pulling back to $X_{g,n+1,d}$ via π. Using the comparison formula (i) and the naturality (iv) of $\mathbf{c}(E_{g,n,d})$ under π^*, we find that

$$\begin{aligned}
&\left\langle\!\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi); \pi_*\left[\mathrm{ev}_{n+1}^*\,\mathrm{ch}_{k+1}(E)\right]\mathbf{c}(E_{g,n,d})\right\rangle\!\right\rangle_{g,n,d}\\
&= -\sum_{i=1}^{i=n}\left\langle\!\!\left\langle \underbrace{\mathbf{t}(\psi),\dots,\mathbf{t}(\psi)}_{i-1}, \mathrm{ch}_{k+1}(E)\left[\frac{\mathbf{t}(\psi)}{\psi}\right]_+, \underbrace{\mathbf{t}(\psi),\dots,\mathbf{t}(\psi)}_{n-i}; \mathbf{c}(E_{g,n,d})\right\rangle\!\!\right\rangle_{g,n,d}\\
&\qquad + \left\langle\!\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi),\mathrm{ch}_{k+1}(E); \mathbf{c}(E_{g,n+1,d})\right\rangle\!\right\rangle_{g,n+1,d}.
\end{aligned}$$

Thus (34) becomes

$$\begin{aligned}
&-\sum_{g,n,d}\frac{\hbar^{g-1}Q^d}{(n-1)!}\left\langle\!\!\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi),\mathrm{ch}_{k+1}(E)\left[\frac{\mathbf{t}(\psi)-\psi}{\psi}\right]_+; \mathbf{c}(E_{g,n,d})\right\rangle\!\!\right\rangle_{g,n,d}\\
&\quad -\frac{1}{2\hbar}\left\langle\!\left\langle \mathbf{t}(\psi),\mathbf{t}(\psi),\mathrm{ch}_{k+1}(E); \mathbf{c}(E_{0,3,0})\right\rangle\!\right\rangle_{0,3,0} - \left\langle\!\left\langle \mathrm{ch}_{k+1}(E); \mathbf{c}(E_{1,1,0})\right\rangle\!\right\rangle_{1,1,0}.
\end{aligned} \tag{35}$$

Here the exceptional terms arise from the fact that the moduli spaces $X_{0,2,0}$ and $X_{1,0,0}$ are empty and therefore $X_{0,3,0}$ and $X_{1,1,0}$ cannot be interpreted as universal curves.

The first two summands in (35) add up to $\mathcal{D}_{\mathbf{s}}^{-1}(\mathrm{ch}_{k+1}(E)/z)\hat{}\,\mathcal{D}_{\mathbf{s}}$. The quadratic Hamiltonian corresponding to $\mathrm{ch}_{k+1}(E)/z$ has pq- and q^2-terms but no p^2-terms. Quantization of the pq-terms yields the linear vector field associated to the linear map $\mathbf{q}(z)\mapsto -[\mathrm{ch}_{k+1}(E)\mathbf{q}(z)/z]_+$, whilst the q^2-term $-(\mathrm{ch}_{k+1}(E)q_0, q_0)/2$ matches the second summand in (35) due to (vii) and (6). Evaluating the third summand using (ix) we conclude that the terms in (33) involving B_0 can be written as

$$\mathcal{D}_{\mathbf{s}}^{-1}\begin{pmatrix} \left(\frac{\mathrm{ch}_{k+1}(E)}{z}\right)^{\wedge} + \frac{1}{24}\int_X c_{D-1}(X)\wedge \mathrm{ch}_{k+1}(E) \\ -\frac{1}{24}\int_X \mathbf{e}(X)\wedge\mathrm{ch}_{k+1}(E)\wedge\left(\sum_j s_j\,\mathrm{ch}_{j-1}(E)\right)\end{pmatrix}\mathcal{D}_{\mathbf{s}}. \tag{36}$$

The part of (33) involving $B_1 = -\frac{1}{2}$ is

$$\sum_{g,n,d} \frac{\hbar^{g-1}Q^d}{n!} \langle\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi); \\ \pi_* \left[B_1 \operatorname{ev}_{n+1}^* \operatorname{ch}_k(E) \left(\psi_{n+1} - \sum_{i=1}^{i=n} D_i \right) \right] \mathbf{c}(E_{g,n,d}) \rangle\rangle_{g,n,d}.$$

Processing this as above and using the fact that $\sigma_i^* \psi_{n+1} = 0$ we find that it is equal to

$$\frac{1}{2} \sum_{g,n,d} \frac{\hbar^{g-1}Q^d}{(n-1)!} \langle\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi), \operatorname{ch}_k(E)(\mathbf{t}(\psi) - \psi); \mathbf{c}(E_{g,n,d}) \rangle\rangle_{g,n,d} \\ + \frac{1}{2} \langle\langle \operatorname{ch}_k(E)\psi; \mathbf{c}(E_{1,1,0}) \rangle\rangle_{1,1,0}.$$

In view of (viii) and the twisted dilaton shift (6), this coincides with

(37)
$$-\sum_{g,n,d} \frac{\hbar^{g-1}Q^d}{(n-1)!} \langle\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi), \frac{\partial \mathbf{t}}{\partial s_k}; \mathbf{c}(E_{g,n,d}) \rangle\rangle_{g,n,d} + \frac{1}{48} \int_X \mathbf{e}(X) \wedge \operatorname{ch}_k(E).$$

Finally, it remains to check the equality of the B_{2m}-terms with $m > 0$. These split into three parts, corresponding to the decomposition (from (24))

$$\Psi(2m) = \psi_{n+1}^{2m} \cap [X_{g,n+1,d}] - \sum_{i=1}^{n} (\sigma_i)_* \left(\psi_i^{2m-1} \cap [X_{g,n,d}] \right) \\ + \frac{1}{2} j_* \left(\sum_{\substack{a+b=2m-2 \\ a,b \geq 0}} (-1)^a \psi_+^a \psi_-^b \cap [\tilde{Z}_{g,n+1,d}] \right)$$

of $\Psi(2m)$ as the sum of a term supported in the bulk of $X_{g,n+1,d}$, a term supported on the divisors D_i, and a term supported on the singular locus $Z_{g,n+1,d}$. We will call the three parts the codimension-0, codimension-1, and codimension-2 terms respectively. Processing the codimension-0 and codimension-1 terms as before, we find that they contribute

(38)
$$-\sum_{g,n,d} \frac{\hbar^{g-1}Q^d}{(n-1)!} \\ \cdot \left\langle\!\!\left\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi), \frac{B_{2m}}{(2m)!} \operatorname{ch}_{k+1-2m}(E) \psi^{2m-1} (\mathbf{t}(\psi) - \psi); \mathbf{c}(E_{g,n,d}) \right\rangle\!\!\right\rangle_{g,n,d}.$$

We can analyze the codimension-2 terms using the composition laws (iii), (v), and (vi); they yield

(39)

$$\sum_{\substack{g_1,n_1,d_1\\g_2,n_2,d_2}} \frac{\hbar^{g_1+g_2-1}Q^{d_1+d_2}}{n_1!n_2!}\frac{B_{2m}}{(2m)!}\sum_{\substack{a+b=2m-2\\a,b\geq 0}}(-1)^a g^{\alpha\beta}$$
$$\times\left\langle\!\!\left\langle \mathbf{t}(\psi),\ldots,\mathbf{t}(\psi),\mathrm{ch}_{k+1-2m}(E)\phi_\alpha\psi^a;\frac{\mathbf{c}(E_{g_1,n_1+1,d_1})}{\sqrt{\mathbf{c}\left(\mathrm{ev}^*_{n_1+1}E\right)}}\right\rangle\!\!\right\rangle_{g_1,n_1+1,d_1}$$
$$\times\left\langle\!\!\left\langle \mathbf{t}(\psi),\ldots,\mathbf{t}(\psi),\phi_\beta\psi^b;\frac{\mathbf{c}(E_{g_2,n_2+1,d_2})}{\sqrt{\mathbf{c}\left(\mathrm{ev}^*_{n_2+1}E\right)}}\right\rangle\!\!\right\rangle_{g_2,n_2+1,d_2}$$
$$+\sum_{g,n,d}\frac{\hbar^{g-1}Q^d}{n!}\frac{B_{2m}}{(2m)!}\sum_{\substack{a+b=2m-2\\a,b\geq 0}}(-1)^a g^{\alpha\beta}$$
$$\times\left\langle\!\!\left\langle \mathbf{t}(\psi),\ldots,\mathbf{t}(\psi),\mathrm{ch}_{k+1-2m}(E)\phi_\alpha\psi^a,\phi_\beta\psi^b;\frac{\mathbf{c}(E_{g-1,n+2,d})}{\sqrt{\mathbf{c}\left(\mathrm{ev}^*_{n+1}E\right)}\sqrt{\mathbf{c}\left(\mathrm{ev}^*_{n+2}E\right)}}\right\rangle\!\!\right\rangle_{g-1,n+2,d}$$

where $\{\phi_\mu\}$ is a basis for $H^*(X)$, $g_{\mu\nu}=(\phi_\mu,\phi_\nu)$, and $g^{\mu\nu}$ is the (μ,ν) entry of the matrix inverse to that with (μ,ν) entry $g_{\mu\nu}$. We use the summation convention here, summing over the repeated indices α and β. Equations (38) and (39) together add up to

$$\mathcal{D}_{\mathbf{s}}^{-1}\left(\frac{B_{2m}}{(2m)!}\left(\mathrm{ch}_{k+1-2m}(E)\,z^{2m-1}\right)^{\wedge}\right)\mathcal{D}_{\mathbf{s}}. \tag{40}$$

Indeed, the quadratic Hamiltonian corresponding to $\mathrm{ch}_{k+1-2m}(E)\,z^{2m-1}$ contains pq- and p^2-terms but no q^2-terms. Quantization of the pq-terms gives the linear vector field associated to the linear map

$$\mathbf{q}(z)\mapsto-\mathrm{ch}_{k+1-2m}(E)z^{2m-1}\mathbf{q}(z);$$

this matches with (38). Quantization of the p^2-terms yields a bivector field which when applied to $\mathcal{D}_{\mathbf{s}}$ brings down (39). In particular the factors of $\mathbf{c}(\mathrm{ev}^*_\Delta(E))$ in (39), which arise from the matching condition at the node in (v) and (vi) and which we have split up here into pairs of square roots, are absorbed by the difference between derivatives in $\mathbf{t}$ and derivatives in $\mathbf{q}$ coming from the twisted dilaton shift (6).

Combining (36), (37), and (40) with (32) and (33) yields Theorem 1′. The proof of Theorem 1 is now complete.

Appendix 2. Descendants and ancestors

In this appendix we will establish part (i) of Proposition 1 in Section 6, which describes properties of the genus-zero descendant potential $\mathcal{F}^0_X$ in terms of the geometry of the symplectic space $(\mathcal{H},\Omega)$. In fact we will do more: we will derive the proposition from a relationship between gravitational descendants of any genus and the corresponding cohomology classes pulled back from Deligne-Mumford spaces, which we call *ancestors*[4], expressed in terms of the quantization formalism of Section 2. The theorem in question, which is a reformulation of a result of Kontsevich and Manin [32], has been announced in [25]. We recall the formulation and furnish a proof below.

Consider the morphism $X_{g,m+l,d} \to \overline{\mathcal{M}}_{g,m}$ given by forgetting the map and the last l marked points and contracting any unstable components of the resulting marked curve. Denote by $\bar{\psi}_{m,l;i}$ the pull-back from Deligne-Mumford space $\overline{\mathcal{M}}_{g,m}$ of the first Chern class of the i^{th} universal cotangent line bundle. This differs from the descendant class ψ_i on $X_{g,m+l,d}$. Following [25], introduce the *genus-g ancestor potential*

(41)

$$\bar{\mathcal{F}}^g_X(\bar{t}_0, \bar{t}_1, \dots ; \tau) := \sum_{d,m,l} \frac{Q^d}{m!l!} \int_{[X_{g,m+l,d}]} \bigwedge_{i=1}^{m} \left[\sum_{k\geq 0} (\mathrm{ev}_i^*\, \bar{t}_k)\ \bar{\psi}^k_{m,l;i} \right] \bigwedge_{i=m+1}^{m+l} \mathrm{ev}_i^*\, \tau,$$

which is a formal function of the cohomology classes $\bar{t}_0, \bar{t}_1, \dots \in H$ and $\tau \in H$. The *total ancestor potential* is defined as

$$\mathcal{A}_\tau(\bar{t}_0, \bar{t}_1, \dots) = \exp\left(\sum_{g\geq 0} \hbar^{g-1} \bar{\mathcal{F}}^g_X(\bar{t}_0, \bar{t}_1, \dots ; \tau) \right).$$

This can be regarded as a formal function on the space of cohomology-valued polynomials $\bar{\mathbf{t}}(z) = \bar{t}_0 + \bar{t}_1 z + \dots$ which depends on the parameter τ. We regard it as an asymptotic element of the Fock space, depending on $\tau \in H$, via the dilaton shift $\mathbf{q}(z) = \bar{\mathbf{t}}(z) - z$.

We will use the abbreviated correlator notation

$$\begin{aligned}\langle\!\langle \mathbf{a}_1(\psi, \bar{\psi}), \dots, \mathbf{a}_m(\psi, \bar{\psi}) \rangle\!\rangle_{g,m}(\tau) \\ := \sum_{l,d} \frac{Q^d}{l!} \langle\!\langle \mathbf{a}_1(\psi, \bar{\psi}), \dots, \mathbf{a}_m(\psi, \bar{\psi}), \tau, \dots, \tau; 1 \rangle\!\rangle_{g,m+l,d}\end{aligned}$$

for Taylor series in τ with coefficients possibly mixing descendant and ancestor classes, so that for instance

$$\bar{\mathcal{F}}^g_X(\bar{t}_0, \bar{t}_1, \dots ; \tau) = \sum_m \frac{1}{m!} \langle\!\langle \bar{\mathbf{t}}(\bar{\psi}), \dots, \bar{\mathbf{t}}(\bar{\psi}) \rangle\!\rangle_{g,m}(\tau).$$

[4]We are thankful to E. Getzler for teaching us this relationship.

Introduce the operator series $S_\tau(z^{-1}) = 1 + S_1 z^{-1} + S_2 z^{-2} + \dots$ acting on the space $\mathcal{H} = H((z^{-1}))$ and defined in terms of genus-zero descendants by

$$\big(S_\tau(z^{-1})\, u, v\big) := (u, v) + \left\langle\!\!\left\langle \frac{u}{z-\psi}, v \right\rangle\!\!\right\rangle_{0,2}(\tau), \tag{42}$$

where we expand $1/(z-\psi)$ as a power series in $1/z$. The series S_τ depends on the parameter $\tau \in H$. According to [21], [23] it satisfies the identity $S^*_\tau(-z^{-1}) S_\tau(z^{-1}) = 1$ and consequently defines a symplectic transformation of $(\mathcal{H}, \Omega)$. We denote the quantization $\exp\big((\ln S_\tau^{-1})\hat{}\,\big)$ of the inverse of this symplectic transformation by $\hat{S}_\tau^{-1}$.

The action of the operator $\hat{S}_\tau^{-1}$ on an asymptotic element $\mathcal{G}$ of the Fock space is described by the formula

$$(\hat{S}_\tau^{-1}\mathcal{G})(\mathbf{q}) = e^{\langle\!\langle \mathbf{q}(\psi), \mathbf{q}(\psi)\rangle\!\rangle_{0,2}(\tau)/2\hbar}\mathcal{G}([S_\tau \mathbf{q}]_+)$$

where $[S_\tau \mathbf{q}]_+$ is the power series truncation of $S_\tau(z^{-1})\mathbf{q}(z)$. This is essentially an instance of Proposition 5.3 in [25], which describes the action on the Fock space of operators $\exp \epsilon \hat{A}$ where A is an $\mathrm{End}(H)$-valued power series in $1/z$. The quadratic Hamiltonian corresponding to such an A has no p^2-terms. This reduces computation of $(\exp \epsilon \hat{A})\mathcal{G}$ to solving the first order linear PDE $df/d\epsilon = \hat{A} f,\ f|_{\epsilon=0} = \mathcal{G}$, which can be achieved by the method of characteristics. The pq-terms give rise to the linear change of variables $\mathbf{q} \mapsto [\exp(\epsilon A)\mathbf{q}]_+$. To verify that in the case $\epsilon A = -\ln S_\tau$, the exponential factor in the above formula agrees with the one in [25, Prop. 5.3] one can use the WDVV-like identity

$$\langle\!\langle \mathbf{q}(\psi), 1, \mathbf{q}(\psi)\rangle\!\rangle_{0,3}(\tau) = \langle\!\langle \mathbf{q}(\psi), 1, \phi_\alpha\rangle\!\rangle_{0,3}(\tau)\, g^{\alpha\beta}\, \langle\!\langle \phi_\beta, 1, \mathbf{q}(\psi)\rangle\!\rangle_{0,3}(\tau),$$

where $g^{\alpha\beta}$ is as in (39) and we again use the summation convention, together with the string equation. We leave some details here to the reader.

Let $F^1(\tau) := \langle\!\langle\ \rangle\!\rangle_{1,0}(\tau) = \mathcal{F}^1_X(\tau, 0, 0, \dots)$ denote the genus-one nondescendant Gromov-Witten potential of X. Recall that the descendant potential $\mathcal{D}_X$ is identified with an asymptotic element of the Fock space via the dilaton shift $\mathbf{q}(z) = \mathbf{t}(z) - z$.

THEOREM. $\mathcal{D}_X = e^{F^1(\tau)} \hat{S}_\tau^{-1} \mathcal{A}_\tau$.

Proof. Let L be one of the universal cotangent line bundles over $X_{g,m+l,d}$ and $\bar{L}$ be its counterpart pulled back from $\overline{\mathcal{M}}_{g,m}$ and corresponding to the marked point with the same index (let it be 1). Let $\psi = c_1(L)$ and $\bar{\psi} = c_1(\bar{L})$. There is a section of $\mathrm{Hom}(\bar{L}, L)$ which is regular outside the virtual divisor D consisting of stable maps such that the first marked point $\mathbf{1}$ is situated on a component of the curve which gets contracted by the map $X_{g,m+l,d} \to \overline{\mathcal{M}}_{g,m}$. It is easy to see that D is the total range of the gluing maps

$$X_{0,\mathbf{1}+\bullet+l',d'} \times_X X_{g,m-\mathbf{1}+\circ+l'',d''} \to X_{g,m+l,d}$$

over all splittings $l'+l''=l, d'+d''=d$. At a generic point of D the virtual normal bundle to D is canonically identified with $\mathrm{Hom}(\bar{L}, L)$. This implies that $\psi-\bar{\psi}$ is virtually Poincaré-dual to D: $[X_{g,m+l,d}]\cap(\psi-\bar{\psi})=[D]$. Thus

$$\begin{aligned}&\langle\!\langle u\psi^{a+1}\bar{\psi}^b,\dots\rangle\!\rangle_{g,m}(\tau)\\ &\qquad=\langle\!\langle u\psi^{a}\bar{\psi}^{b+1},\dots\rangle\!\rangle_{g,m}(\tau)+\langle\!\langle u\psi^a,\phi_\alpha\rangle\!\rangle_{0,2}(\tau)\,g^{\alpha\beta}\,\langle\!\langle \phi_\beta\bar{\psi}^b,\dots\rangle\!\rangle_{g,m}(\tau),\end{aligned}$$

where "…" stands for the descendant and/or ancestor content of the other marked points (to be the same in all three places). Applying this identity inductively to express descendants in terms of ancestors we conclude that the descendant correlators $\langle\!\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi)\rangle\!\rangle_{g,m}(\tau)$ are obtained from the corresponding ancestor correlators $\langle\!\langle \bar{\mathbf{t}}(\bar{\psi}),\dots,\bar{\mathbf{t}}(\bar{\psi})\rangle\!\rangle_{g,m}(\tau)$ by the substitution $\bar{\mathbf{t}}(z)=[S_\tau(z^{-1})\mathbf{t}(z)]_+$. This is essentially the result from [32].

Let us compare this conclusion with the statement of the theorem. Noting the presence of the similar change $\mathbf{q}\mapsto[S_\tau\mathbf{q}]_+$ in the explicit description of the operator $\hat{S}_\tau^{-1}$ we should also observe that $\mathbf{q}(z)$ and $\mathbf{t}(z)$ are not the same: $\mathbf{q}(z)=\mathbf{t}(z)-z$. This gives rise to a discrepancy of $[z-S_\tau z]_+$. Since

$$\begin{aligned}([z-S_\tau z]_+,v)&=\left[-\left\langle\!\!\left\langle \frac{z1}{z-\psi},v\right\rangle\!\!\right\rangle_{0,2}(\tau)\right]_+\\ &=-\langle\!\langle 1,v,\tau\rangle\!\rangle_{0,3,0}\\ &=-(\tau,v)\end{aligned}$$

we find that the discrepancy is equal to $-\tau$. Thus the change $\mathbf{q}\mapsto[S_\tau\mathbf{q}]_+$ is equivalent to the change

$$\mathbf{t}\mapsto\bar{\mathbf{t}}=[S_\tau\mathbf{t}]_+-\tau=[S_\tau(\mathbf{t}-\tau)]_+.$$

By Taylor's formula, we have

$$\begin{aligned}\mathcal{F}_X^g(\mathbf{t})&=\sum_{m=0}^\infty\frac{1}{m!}\langle\!\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi)\rangle\!\rangle_{g,m}(0)\\ &=\sum_{m=0}^\infty\frac{1}{m!}\langle\!\langle \mathbf{t}(\psi)-\tau,\dots,\mathbf{t}(\psi)-\tau\rangle\!\rangle_{g,m}(\tau).\end{aligned}$$

We conclude that for $g>1$ the descendant potentials $\mathcal{F}_X^g$ (which do not depend on τ) are obtained from the ancestor potentials $\bar{\mathcal{F}}_X^g$ (which do depend on τ) by the substitution $\mathbf{q}(z)\mapsto[S_\tau(z^{-1})\mathbf{q}(z)]_+$. In order to make the same true for $g=0$ and $g=1$ we have to take account of the terms corresponding to the unstable indices $(g,m)=(0,0)$, $(0,1)$, $(0,2)$, and $(1,0)$ which are missing from the ancestor potentials. The first three of them give rise to the factor

$\exp\left(\langle\langle \mathbf{q}(\psi), \mathbf{q}(\psi)\rangle\rangle_{0,2}(\tau)/2\hbar\right)$: the equality

$$\frac{1}{2}\langle\langle \mathbf{t}(\psi)-\psi, \mathbf{t}(\psi)-\psi\rangle\rangle_{0,2}(\tau)$$
$$= \langle\langle\ \rangle\rangle_{0,0}(\tau) + \langle\langle \mathbf{t}(\psi)-\tau\rangle\rangle_{0,1}(\tau) + \frac{1}{2}\langle\langle \mathbf{t}(\psi)-\tau, \mathbf{t}(\psi)-\tau\rangle\rangle_{0,2}(\tau)$$

follows easily from the *dilaton equation* $\langle\langle \psi, \dots\rangle\rangle_{g,n+1,d} = (2g-2+n)\langle\langle \dots\rangle\rangle_{g,n,d}$ applied with $g = 0$. Finally, the missing summand $\langle\langle\ \rangle\rangle_{1,0}(\tau)$ coincides with $F^1(\tau)$. The theorem follows. □

Passing to the quasi-classical limit $\hbar \to 0$ we obtain the following result.

COROLLARY. *The Lagrangian sections $\mathcal{L}_X$ and $\bar{\mathcal{L}}_\tau$ which represent respectively the differentials of the genus-zero descendant potential $\mathcal{F}^0_X$ and genus-zero ancestor potential $\bar{\mathcal{F}}^0_X(\dots;\tau)$ are related by the symplectic transformation S_τ:*

$$\bar{\mathcal{L}}_\tau = S_\tau \mathcal{L}_X.$$

Finally, we derive part (i) of Proposition 1. When $\mathbf{q}(z) \in z\mathcal{H}_+$, so that $\bar{t}_0 = 0$, the genus-zero ancestor potential $\bar{\mathcal{F}}^0_X(\dots;\tau)$ has identically zero 2-jet at $\mathbf{q}(z)$. This follows from the fact that $\dim \overline{\mathcal{M}}_{0,m+2} < m$. Thus

(a) the cone $\bar{\mathcal{L}}_\tau$ contains the isotropic subspace $z\mathcal{H}_+$;

(b) at any point $\bar{\mathbf{q}} \in z\mathcal{H}_+$ the tangent space L to $\bar{\mathcal{L}}_\tau$ at $\bar{\mathbf{q}}$ is equal to $\mathcal{H}_+$.

Applying the symplectic transformation S_τ^{-1} we see that the tangent space $L_{\mathbf{f}}$ to $\mathcal{L}$ at $\mathbf{f} = S_\tau^{-1}\bar{\mathbf{q}}$ meets $\mathcal{L}$ along $zL_{\mathbf{f}}$ provided that $\bar{\mathbf{q}} \in z\mathcal{H}_+$. The condition $S_\tau \mathbf{f} \in z\mathcal{H}_+$ on $\mathbf{f} = (\mathbf{p}, \mathbf{q}) \in \mathcal{L}$ is equivalent to the system of equations

$$\langle\langle 1, \mathbf{q}(\psi), v\rangle\rangle_{0,3}(\tau) = 0 \quad \text{for all } v \in H.$$

In other words, τ must be a critical point of $\langle\langle 1, \mathbf{q}(\psi)\rangle\rangle_{0,2}(\tau)$ considered as a function of $\tau \in H$ (depending on the parameter $\mathbf{q} \in \mathcal{H}_+$). When $\mathbf{q}(z) = q_0 - z$, the function turns into $(q_0, \tau) - (\tau, \tau)/2$ and has the unique nondegenerate critical point $\tau = q_0$. This guarantees existence of a unique critical point $\tau(\mathbf{q})$ in a formal neighborhood of $\mathbf{q} = -z$. The result follows.

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