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Non-quasi-projective moduli spaces

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Abstract

We show that every smooth toric variety (and many other algebraic spaces as well) can be realized as a moduli space for smooth, projective, polarized varieties. Some of these are not quasi-projective. This contradicts a recent paper (Quasi-projectivity of moduli spaces of polarized varieties, *Ann. of Math.* **159** (2004) 597–639.).

A polarized variety is a pair (X, H) consisting of a smooth projective variety X and a linear equivalence class of ample divisors H on X. For simplicity, we look at the case when X is smooth, numerical and linear equivalence coincide for divisors on X, H is very ample and $H^i(X, \mathcal{O}_X(mH)) = 0$ for $i, m > 0$. A well established route to construct moduli spaces of such pairs is to embed X into $\mathbb{P}^N$ by $|H|$. The pair (X, H) and the embedding $X \hookrightarrow \mathbb{P}^N$ determine each other up to the action of $\mathrm{PGL}(N+1)$. Deformations of (X, H) cover an open subset $U(X, H)$ of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^N)$ with Hilbert polynomial $\chi(X, \mathcal{O}_X(mH))$. One can then view the quotient $U(X, H)/\mathrm{PGL}(N+1)$ as the moduli space of the pairs (X, H). (See [MF82, App. 5] or [Vie95, Ch. 1] for general introductions to moduli problems.)

The action of $\mathrm{PGL}(N+1)$ can be bad along some orbits, and therefore one has to make additional assumptions to ensure that the quotient $U(X, H)/\mathrm{PGL}(N+1)$ is reasonable. The optimal condition seems to be to require that the action be *proper*. This is equivalent to assuming that $U(X, H)/\mathrm{PGL}(N+1)$ exists as a separated complex space or as a separated algebraic space [Kol97], [KM97].

A difficult result of Viehweg (cf. [Vie95]) shows that if the canonical class K_X is assumed nef then $U(X, H)/\mathrm{PGL}(N+1)$ is a quasi-projective scheme.

A recent paper [ST04] asserts the quasi-projectivity of moduli spaces of polarized varieties for arbitrary K_X, whenever the quotient $U(X, H)/\mathrm{PGL}(N+1)$ exists as a separated algebraic space.

The aim of the present note is to confute this claim. The examples (9) and (29) show that the quotients $U(X, H)/\mathrm{PGL}(N+1)$ can contain smooth, proper subschemes which are not projective.

In the examples X is always a rational variety, but there are many more such cases as long as X is ruled. This leaves open the question of quasi-projectivity of the quotients $U(X,H)/\operatorname{PGL}(N+1)$ when X is not uniruled but K_X is not nef.

We work over an algebraically closed field of characteristic zero, though some of the examples apply in any characteristic.

1. First examples

1 (Versions of quasi-projectivity for moduli functors). In asserting that certain moduli spaces are quasi-projective, one hopes to show that an algebraic space S is quasi-projective if S "corresponds" to a family of pairs (X,H) in our class. There are at least three ways to formulate a precise meaning of "corresponds". (In order to avoid scheme theoretic complications, let us assume that S is normal.)

(1.1. There is a family over S.) That is, there is a smooth, proper morphism of algebraic spaces $f: U \to S$ and an f-ample Cartier divisor H such that every fiber $(U_s, H|_{U_s})$ is in our class and $(U_s, H|_{U_s}) \cong (U_{s'}, H|_{U_{s'}})$ if and only if $s = s'$.

(1.2. There is a family over some scheme over S.) That is, there are a surjective and open morphism $h: T \to S$, a smooth, proper morphism of algebraic spaces $f: U \to T$ and an f-ample Cartier divisor H such that every fiber $(U_t, H|_{U_t})$ is in our class and $(U_t, H|_{U_t}) \cong (U_{t'}, H|_{U_{t'}})$ if and only if $h(t) = h(t')$. (One can always reduce to the case when $h: T \to S$ is the geometric quotient by a PGL-action, but in many constructions quotients by smaller groups appear naturally.)

(1.3. There is a universal family over some scheme over S.) That is, we have $h: T \to S$ and $f: U \to T$ as in (1.2) but we also assume that every local deformation of a polarized fiber $(U_t, H|_{U_t})$ is induced from $f: U \to T$.

All the approaches to quasi-projectivity of quotients that I know of work equally well for any of the three cases. (For instance, although the main assertion [ST04, Thm. 1] explicity assumes local versality as in (1.3), the key technical steps [ST04, Thms. 4, 5] assume only the more general setting of (1.2).) Nonetheless, a counterexample to the variant (1.2) need not yield automatically a counterexample in the setting of (1.3).

I start with examples as in (1.2) where quasi-projectivity fails; these are the weak examples (2). Then we analyze deformations of some of these polarized pairs to show that quasi-projectivity also fails under the assumptions of (1.3). These examples are given in Section 4.

2 (Weak examples). Let W^0 be a smooth, quasi-projective variety of dimension at least 2 and G a reductive algebraic group acting on W^0. Let $W \supset W^0$ be a G-equivariant compactification of W^0.

The moduli space of the pairs (w, W) consisting of W (thinking of it as fixed) and a variable point $w \in W^0$ is naturally a quotient of W^0/G. These pairs can also be identified with pairs (B_wW, E) where $E \subset B_wW$ is the exceptional divisor of the blow up $\pi_w : B_wW \to W$ of $w \in W$. Fix a sufficiently ample G-invariant linear equivalence class of divisors H on W. Then $H_w = \pi^*H - E$ is ample on B_wW and (B_wW, H_w) uniquely determines (B_wW, E) (cf. (22).

Thus we obtain a G-equivariant morphism of W^0 to the moduli space of the polarized pairs (B_wW, H_w).

Assume now that in the above example the following conditions are satisfied:

(2.1) the G-action is proper on W^0,

(2.2) W^0/G is not quasi-projective, and

(2.3) $\mathrm{Aut}(W) = G$.

The quotient W^0/G exists as an algebraic space by the general quotient results of [Kol97], [KM97]. In (1.2) set $S = W^0/G$ and $T = W^0$. The pairs (B_wW, H_w) give a family of polarized varieties over W^0. Furthermore, isomorphisms between two polarized pairs (B_wW, H_w) and $(B_{w'}W, H_{w'})$ correspond to isomorphisms between the pairs (w, W) and (w', W), and by (2.3), these in turn are given by those elements of G that map w to w'. In particular, two polarized pairs (B_wW, H_w) and $(B_{w'}W, H_{w'})$ are isomorphic if and only if w and w' are in the same G-orbit.

Thus we have realized the non-quasi-projective algebraic space W^0/G as a moduli space of smooth, polarized varieties in the sense of (1.2).

Now we must find examples where the three conditions of (2) are satisfied. We start by reviewing some of the known examples of proper G-actions with non-quasi-projective quotient. The condition $\mathrm{Aut}(W) = G$ should hold for most G-equivariant compactifications, but it will take some effort to prove that such a W exists in many cases.

3 (Examples of non-quasi-projective quotients). There are many examples of $G = \mathrm{PGL}$ or a torus $G = (\mathbb{C}^*)^m$ acting properly on a smooth quasi-projective variety W^0 such that W^0/G is not quasi-projective.

Here we show two examples where a torus or $\mathrm{PGL}(n)$ acts properly on an open subset of projective space and the quotient is smooth, proper but not projective in the torus case and a smooth algebraic space which is not a scheme in the $\mathrm{PGL}(n)$ case.

(3.1) By a result of [Cox95, Thm. 2.1], every smooth toric variety can be written as the geometric quotient of an open subset $U \subset \mathbb{C}^N \setminus \{0\}$ by a suitable subtorus of $(\mathbb{C}^*)^N$. There are many proper but nonprojective toric varieties (see, for instance, [Oda88, §2.3]), and so we have our first set of examples.

(3.2) Here we work with PGL(3), but the construction can be generalized to any PGL(n) for $n \geq 3$.

Fix d and let $U_d \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ be the open set consisting of curves C such that

(i) C is smooth, irreducible and the genus of its normalization is $> \frac{1}{2}\binom{d-1}{2}$.

(ii) C is not fixed by any of the automorphisms of $\mathbb{P}^2$.

We claim that $\operatorname{Aut}(\mathbb{P}^2)$ operates properly and freely on U_d. Indeed, the action is set theoretically free by (ii). Properness is equivalent to uniquenes of specialization:

Claim 4. Let S be the spectrum of a DVR. A family of smooth plane curves of degree d over the generic point $S^* \subset S$ has at most one extension to a family over S where the central fiber is in U_d.

Proof. Assume that we have a family $X^* \to S^*$ and two extensions $X_1, X_2 \to S$ with central fibers C_1, C_2. If the natural map $X_1 \dashrightarrow X_2$ is an isomorphism at the generic point of C_1, then the two families are isomorphic by (12).

Otherwise, let $Y \to S$ be the normalization of the main component of the fiber product $X_1 \times_S X_2$. The central fiber of $Y \to S$ dominates both C_1, C_2, hence it has two irreducible components, both of geometric genus $> \frac{1}{2}\binom{d-1}{2}$. Thus the sum of the geometric genera of the irreducible components of the central fiber is bigger than the geometric genus of the generic fiber, a contradiction. □

Let us consider a general curve $C \subset \mathbb{P}^2$ which has multiplicity $\geq m$ at a given point $p \in \mathbb{P}^2$. Our condition for the geometric genus is

$$\binom{d-1}{2} - \binom{m}{2} > \frac{1}{2}\binom{d-1}{2},$$

which is asymptotically equivalent to $m < d/\sqrt{2}$.

On the other hand, if $m > 2d/3$ and $p = (0:0:1)$ then the subgroup (t, t, t^{-2}) shows that $[C] \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ is unstable. Since $2/3 < 1/\sqrt{2}$, we obtain:

Claim 5. For large d, there are curves C with $[C] \in U_d$ such that $[C]$ is unstable. □

COROLLARY 6. *For large d, the quotient $U_d/\operatorname{Aut}(\mathbb{P}^2)$ is a smooth algebraic space which is not a scheme.*

Proof. The quotient $U_d/\operatorname{Aut}(\mathbb{P}^2)$ is a smooth algebraic space by [Kol97], [KM97]. Let $\pi : U_d \to U_d/\operatorname{Aut}(\mathbb{P}^2)$ denote the quotient map.

Pick a curve $[C] \in U_d$ such that $[C]$ is unstable. We claim that $[C] \in U_d/\operatorname{Aut}(\mathbb{P}^2)$ has no neighborhood which is affine. Indeed, if $W \subset U_d/\operatorname{Aut}(\mathbb{P}^2)$ is any quasi-projective subset, then by [MF82, Converse 1.12], its preimage $\pi^{-1}(W) \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ consists of semi-stable points with respect to some polarization on $|\mathcal{O}_{\mathbb{P}^2}(d)| \cong \mathbb{P}^N$. Since $|\mathcal{O}_{\mathbb{P}^2}(d)|$ is a projective space and $\operatorname{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3)$ has no nontrivial homomorphisms to $\mathbb{C}^*$, up to powers one has only the standard polarization, and so $\pi^{-1}(W) \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ consists of semi-stable points with respect to the usual polarization. Thus $\pi^{-1}(W)$ cannot contain $[C]$ since C is unstable. $\square$

The third requirement (2.3) is to find a compactification of a G-variety whose automorphism group is exactly G. Thus we need to consider the following general problem.

Question 7. Let G be an algebraic group acting on a quasi-projective variety W^0. When can one find a projective compactification $W^0 \subset W$ such that $\operatorname{Aut}(W) = G$?

There are some cases when this cannot be done. The simplest counterexample occurs when W^0 is projective; here we have no choices for W. The answer can be negative even if W^0 is affine. For instance, consider the action of $\mathrm{O}(n)$ on $W = \mathbb{P}^{n-1}$. Here there are only two orbits; let W^0 be the open one. As the complement $W \setminus W^0$ is a single codimension 1 orbit, there are no $\mathrm{O}(n)$-equivariant blow ups to make, so $W = \mathbb{P}^{n-1}$ is the unique $\mathrm{O}(n)$-equivariant compactification of W^0 and $\operatorname{Aut}(W) = \mathrm{PGL}(n)$ is bigger than $\mathrm{O}(n)$.

The question becomes more reasonable if we assume that G acts properly on W^0. There is still an easy negative example, $G = W^0 = \mathbb{C}^*$, but there may not be any others where G is reductive. In the next two sections, we prove the following partial result.

Proposition 8. *Let G be either a torus $(\mathbb{C}^*)^n$ or $\mathrm{PGL}(n)$. Let W^0 be a smooth variety with a generically free and proper G-action such that $\rho(W^0) = 0$, that is,* $\operatorname{Pic}(W^0)$ *is a torsion group. Assume that there is a (not necessarily G-equivariant) smooth compactification $W^0 \subset W^*$ such that its Néron-Severi group $NS(W^*)$ is $\mathbb{Z}$.*

Then there is a smooth G-equivariant compactification $W \supset W^0$ and an ample divisor class H such that $\operatorname{Aut}(W, H) = G$.

Moreover, if $W' \to W$ is any other G-equivariant compactification dominating W then there is an ample divisor class H' such that $\operatorname{Aut}(W', H') = G$.

Putting this together with (3.1) we obtain the following:

Corollary 9. *Every smooth toric variety can be written as a moduli space of smooth, polarized varieties as in* (1.2). $\square$

By a theorem of [Wło93], a smooth proper variety X can be embedded into a smooth toric variety if and only if every two points of X are contained in an open affine subset. Thus (9) implies that a smooth proper variety X can be written as a moduli space of smooth, polarized varieties as in (1.2) provided every two points of X are contained in an open affine subset.

In the next section we start the proof of of (8) by finding W such that the connected component of $\mathrm{Aut}(W)$ is G. After that we choose the polarization H such that $\mathrm{Aut}(W, H)$ equals the connected component of $\mathrm{Aut}(W)$.

2. Rigidifying by compactification

Definition 10. Let X be a proper variety and $\mathrm{NS}(X)$ its Néron-Severi group. The automorphism group $\mathrm{Aut}(X)$ acts on $\mathrm{NS}(X)/(\text{torsion})$; let $\mathrm{Aut}^0(X)$ denote the kernel of this action.

LEMMA 11. *Let $f : Y \to X$ be a proper, birational morphism between smooth projective varieties. Then* $\mathrm{Aut}^0(Y) \subset \mathrm{Aut}^0(X)$.

Proof. The exceptional set $\mathrm{Ex}(f)$ is a union of divisors and an exceptional divisor is not linearly equivalent to any other effective divisor. Thus $\mathrm{Aut}^0(Y)$ stabilizes $\mathrm{Ex}(f)$ and so every $\sigma \in \mathrm{Aut}^0(Y)$ descends to an automorphism σ_X of $X \setminus f(\mathrm{Ex}(f))$. Since $f(\mathrm{Ex}(f))$ has codimension at least 2 and σ_X fixes an ample divisor, $\sigma_X \in \mathrm{Aut}(X)$ by (12). □

LEMMA 12 ([MM64]). *Let X, X' be normal, projective varieties and $Z \subset X$, $Z' \subset X'$ closed subsets of codimension ≥ 2. Let $\phi : X \setminus Z \to X' \setminus Z'$ be an isomorphism. Assume that there are ample divisors H on X and H' on X' such that $\phi^{-1}(H') = H$. Then ϕ extends to an isomorphism $\Phi : X \to X'$.* □

We deal with the difference between $\mathrm{Aut}^0(X)$ and $\mathrm{Aut}(X)$ later. Now we concentrate on answering (7) for certain cases that are of special interest in moduli constructions. To this end we introduce another subgroup of Aut.

Definition 13. Let W^0 be a variety with a G-action. and $W \supset W^0$ a G-equivariant compactification. Let $\mathrm{Aut}_\partial(W) \subset \mathrm{Aut}(W)$ be the subgroup consisting of all automorphisms which stabilize every G orbit in $W \setminus W^0$.

LEMMA 14. *Let W^0 be a variety with a G-action, G connected. Let $W \supset W^0$ be a G-equivariant smooth compactification. If $\rho(W^0) = 0$ then* $\mathrm{Aut}_\partial(W) \subset \mathrm{Aut}^0(W)$.

Proof. Since $\rho(W^0) = 0$, the divisorial irreducible components of $W \setminus W^0$ generate $\mathrm{NS}(W)_{\mathbb{Q}}$. Since G is connected, each irreducible component of

$W \backslash W^0$ is fixed by G, hence by $\mathrm{Aut}_\partial(W)$. Thus $\mathrm{Aut}_\partial(W)$ acts trivially on $\mathrm{NS}(X)/(\text{torsion})$. □

Corollary 15. *Let W^0 be a variety with a G-action, G connected. Let $W_i \supset W^0$ be G-equivariant smooth compactifications and $W_1 \to W_2$ a proper, birational G-equivariant morphism. If $\rho(W^0) = 0$ then $\mathrm{Aut}_\partial(W_1) \subset \mathrm{Aut}_\partial(W_2)$.*

Proof. From (14) we know that $\mathrm{Aut}_\partial(W_1) \subset \mathrm{Aut}^0(W_1)$ and $\mathrm{Aut}^0(W_1) \subset \mathrm{Aut}^0(W_2)$ by (11). Since every G-orbit in W_2 is the image of a G-orbit in W_1, the inclusion $\mathrm{Aut}_\partial(W_1) \subset \mathrm{Aut}_\partial(W_2)$ follows. □

Example 16. It is worth noting that (15) can fail if $\rho(W^0) > 0$. Start with the $\mathrm{O}(4)$ action on $W^0 = (xy - uv = 0) \setminus \{(0,0,0,0)\} \subset \mathbb{A}^4$. Let $W \subset W^0$ be its closure in $\mathbb{P}^4$. Let $W_1 \to W$ be the blow up of the origin and $W_2 \to W$ the blow up of $(x = u = 0)$. The induced map $W_1 \to W_2$ is a blow up of a single smooth rational curve. $\mathrm{O}(4)$ acts on W_1 but only $\mathrm{SO}(4)$ acts by automorphisms on W_2. The involution $(x,y,u,v) \mapsto (x,y,v,u)$ lifts to a birational involution on W_2 which is not an automorphism.

Proposition 17. *Let G be a connected algebraic group and W^0 a smooth variety with a G-action such that $\rho(W^0) = 0$ and $\dim G \le \dim W^0 - 2$. Then there is a smooth G-equivariant compactification $W \supset W^0$ such that $\mathrm{Aut}_\partial(W) = \mathrm{Aut}^0(W)$.*

Moreover, if $W' \to W$ is any other G-equivariant compactification dominating W then $\mathrm{Aut}_\partial(W') = \mathrm{Aut}^0(W')$.

Proof. Let us start with any smooth G-equivariant compactification $W_1 \supset W^0$. As Aut^0 can only decrease under further blow ups, we can assume that it is already minimal. That is, if $W' \to W$ is any other G-equivariant compactification then $\mathrm{Aut}^0(W') = \mathrm{Aut}^0(W)$.

Assume now that $\mathrm{Aut}_\partial(W) \ne \mathrm{Aut}^0(W)$. Then there are a $\sigma \in \mathrm{Aut}^0(W)$ and a G-orbit $Z \subset W \setminus W^0$ such that $\sigma(Z) \ne Z$. After some preliminary G-blow ups we can blow up Z to get $W_Z \to W$. Since $\dim G \le \dim W^0 - 2$, this blow up is nontrivial and the preimage of Z is an exceptional divisor E_Z. We also know that E_Z is not numerically equivalent to any other effective divisor and it is not stabilized by σ. Thus $\mathrm{Aut}^0(W_Z) \ne \mathrm{Aut}^0(W)$, a contradiction. □

18 (First examples with $G = \mathrm{Aut}_\partial(W)$). (18.1) Let $G = (\mathbb{C}^*)^n$ be the torus with its left action on itself. A natural compactification is $W = \mathbb{P}^n$. The coordinate "vertices" are fixed by G and by no other automorphism of W. Thus $G = \mathrm{Aut}_\partial(W)$. Moreover, if $W' \to W$ is any other G-equivariant compactification dominating W then $G \subset \mathrm{Aut}_\partial(W') \subset \mathrm{Aut}_\partial(W)$; hence $G = \mathrm{Aut}_\partial(W')$.

(18.2) Let $G = \mathrm{PGL}(n)$ with its left action on itself. A natural compactification is $W = \mathbb{P}(M_n)$ coming from the $\mathrm{GL}(n)$ action on $n \times n$-matrices by left multiplication. The $(n-1)$-dimensional G-orbits are of the form $\mathbb{P}^{n-1} \times (a_1, \ldots, a_n)$ where we think of the points in $\mathbb{P}^{n-1}$ as column vectors. The union of all $(n-1)$-dimensional G-orbits is $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ under the Segre embedding. From this we conclude that $\mathrm{Aut}_\partial(W)$ acts on $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ as multiplication on the first factor. Since the image of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ under the Segre embedding is not contained in any hyperplane, this implies that $\mathrm{Aut}_\partial(W) = \mathrm{PGL}(n)$.

As before, if $W' \to W$ is any other G-equivariant compactification dominating W then $\mathrm{Aut}_\partial(W') = \mathrm{PGL}(n)$ as well.

We are now ready to to answer (7) for $(\mathbb{C}^*)^n$ and for $\mathrm{PGL}(n)$.

Proposition 19. *Let G be either $(\mathbb{C}^*)^n$ or $\mathrm{PGL}(n)$. Let W^0 be a smooth variety with a generically free and proper G-action such that $\rho(W^0) = 0$ and $\dim G \leq \dim W^0 - 2$. Then there is a smooth G-equivariant compactification $W \supset W^0$ such that $\mathrm{Aut}^0(W) = G$.*

Moreover, if $W' \to W$ is any other G-equivariant compactification dominating W then $\mathrm{Aut}^0(W') = G$.

Proof. Let $Z \supset W^0/G$ be any compactification and choose any compactification $W_1 \supset W^0$ such that there is a morphism $h : W_1 \to Z$. By further G-equivariant blow ups in $W_1 \setminus W^0$, (17) gives $W_2 \supset W^0$ such that $\mathrm{Aut}_\partial(W_2) = \mathrm{Aut}^0(W_2)$ and neither of these groups changes under further G-equivariant blow ups in $W_2 \setminus W^0$.

Pick a big linear system of Weil divisors $|B|$ on Z and let $|M|$ be the moving part of the linear system given by a pull back of the general member of $|B|$. Then $|M|$ is a linear system which gives the map $h : W_2 \to Z$ over some open subset of Z. (Z is not projective in general, and may not even have any Cartier divisors. That is why we have to find $|M|$ in this roundabout way.)

Any element of $\mathrm{Aut}^0(W_2)$ sends $|M|$ to itself, hence $h : W_2 \to Z$ is $\mathrm{Aut}^0(W_2)$-equivariant.

General fibers of h contain a G-orbit which is in $W_2 \setminus W^0$, and so every general fiber of h is $\mathrm{Aut}^0(W_2)$-stable since $\mathrm{Aut}_\partial(W_2) = \mathrm{Aut}^0(W_2)$.

Pick any $\sigma \in \mathrm{Aut}^0(W_2)$ and look at its action σ_z on $h^{-1}(z)$ for general $z \in Z$.

Since z is general, the fiber $h^{-1}(z)$ is a smooth projective G-equivariant compactification of G acting on itself. We claim that $\sigma_z = g(z, \sigma)$ for some $g(z, \sigma) \in G$. This follows from (18) if $h^{-1}(z)$ dominates the compactifications considered there. Otherwise, by further blow ups we could get $W_3 \to W_2$ such that the birational transform of $h^{-1}(z)$ dominates the standard compactifications considered in (18). This would, however, mean that σ does not lift to

$\mathrm{Aut}^0(W_3)$, a contradiction to our assumption that $\mathrm{Aut}^0(W_2)$ does not change under further G-equivariant blow ups.

Thus we conclude that G and $G' := \mathrm{Aut}^0(W_2)$ both act on W_2 in such a way that for a general $w \in W^0$,

(1) $Gw = G'w$, and

(2) the G'-action on Gw is via a homomorphism $\rho_w : G' \to G$.

Let $H'_w \subset G'$ be the kernel of ρ_w. Since G is reductive, H'_w contains the unipotent radical $U' \subset G'$. The quotients H'_w/U' are normal subgroups of the reductive group G'/U', and they depend continuously on w over an open set of W (21). A continuously varying family of normal subgroups would give a continuously varying family of finite dimensional representations, but a reductive group has only discrete series representations in finite dimensions. This implies that H'_w is independent of w for general $w \in W$ and so the H'_w-action is trivial on W_2. But $H'_w \subset \mathrm{Aut}^0(W_2)$, thus H'_w is the trivial group and so $G = \mathrm{Aut}^0(W_2)$. $\square$

Example 20. The example $G' = \mathbb{C}^2_{x,y}$ acting on $\mathbb{C}^2_{u,v}$ as

$$(u, v) \mapsto (u, v + x - uy)$$

shows that the above argument does not work if G is not reductive.

Remark 21. Let G be an algebraic group acting on a variety X. The stabilizer subgroups G_x of points $x \in X$ are the same as the fibers of $G \times X \to X \times X$ over the diagonal. Thus we see that

(1) the dimension of G_x is a constructible function on X,

(2) the number of connected components of G_x is a constructible function on X,

(3) the subgroups $G_x \subset G$ depend continuously on x for x in a suitable open subset of X.

3. Rigidifying using polarizations

Let X be a proper variety and H an ample divisor on X. Then $\mathrm{Aut}(X, H)$ can be viewed as a closed subgroup of $\mathbb{P}(H^0(X, \mathcal{O}_X(mH)))$ for $m \gg 1$. Hence $\mathrm{Aut}(X, H)$ is an algebraic group and so it has only finitely many connected components. This implies that the action of $\mathrm{Aut}(X, H)$ on $\mathrm{NS}(X)$ is through a finite group.

While not crucial, it will be convenient for us to choose a polarization such that $\mathrm{Aut}(X, H)$ acts trivially on $\mathrm{NS}(X)$. In particular, $\mathrm{Aut}(X, H) = \mathrm{Aut}^0(X)$.

LEMMA 22. *Let $g : Y \to X$ be a birational morphism between smooth, projective varieties. Assume that* $\mathrm{NS}(X) \cong \mathbb{Z}$. *Then there is an ample divisor H^* on Y such that* $\mathrm{Aut}(Y, H^*) = \mathrm{Aut}^0(Y)$.

Proof. Let $g : Y \to X$ be a birational morphism between smooth, projective varieties with exceptional divisors E_i. Let H be any ample divisor on X.

Let H_Y be ample on Y. Then we can write $H_Y = g^*(g_*H_Y) - \sum a_i E_i$ for some $a_i > 0$. For reasons that will become clear soon, let us change the a_i a little so that we get an ample $\mathbb{Q}$-divisor $H'_Y := g^*(g_*H_Y) - \sum a'_i E_i$ where the a'_i are different from each other. Choose m such that $mH - g_*H'_Y$ is ample on X. Then

$$mg^*H - \sum a'_i E_i = g^*(mH - g_*H'_Y) + H'_Y \quad \text{is ample on } Y.$$

Let us multiply through with the common denominator of the a'_i to get natural numbers b_i and m_0 such that

$$H_m := mg^*H - \sum b_i E_i \quad \text{is ample for } m \geq m_0,$$

and the b_i are different from each other.

Write $K_Y = f^*K_X + \sum e_i E_i$, where $e_i > 0$ for every i. Choose a natural number c such that $ce_i - b_i \geq 0$ for every i. Finally, choose m_1 such that $mH + cK_X$ is very ample on X for $m \geq m_1$.

Claim 23. For $m \geq \max\{m_0, m_1\}$, the polarized variety (Y, H_m) uniquely determines $f : Y \to X$ and also $\sum b_i E_i$.

Proof. Given H_m, we consider the linear system

$$\begin{aligned} |H_m + cK_Y| &= |mg^*H - \textstyle\sum b_i E_i + cg^*K_X + \sum ce_i E_i| \\ &= g^*|mH + cK_X| + \textstyle\sum (ce_i - b_i) E_i, \end{aligned}$$

where the second equality holds since an effective exceptional divisor does not increase a linear system that is pulled back from the base. As $mH + cK_X$ is very ample by assumption, we see that we recover $g : Y \to X$ as given by $|H_m + cK_Y|$. Furthermore, since the fixed part $\sum (ce_i - b_i) E_i$ is also determined by $|H_m + cK_Y|$, we also recover $\sum b_i E_i$. □

Now we use the fact that all the b_i are different from each other. This implies that every automorphism of (Y, H_m) maps each E_i to itself. Furthermore, g^*H is also mapped to itself. Since X has Picard number 1, these together generate a finite index subgroup of the free abelian group $\mathrm{NS}(Y)$. Thus $\mathrm{Aut}(Y, H_m)$ acts trivially on $\mathrm{NS}(Y)$. □

4. Locally versal examples

Start with $\mathbb{A}^n$ with the standard $(\mathbb{C}^*)^n$-action. Let $T \subset (\mathbb{C}^*)^n$ be a subtorus and $U \subset \mathbb{A}^n$ a $(\mathbb{C}^*)^n$-invariant open set on which T acts properly. In (2) we showed how to construct a moduli problem for smooth polarized varieties whose moduli space is U/T. These give examples of moduli spaces as in (1.2), but in general the local versality condition of (1.3) fails.

In this section we present a version of the construction where we can control local versality as well. The key point is to get a rather explicit series of examples as in (2) where we can describe all deformations in a uniform way.

This should be possible to do in most cases, but the combinatorial aspects of finding explicit resolutions and describing their deformations seem rather daunting. So here I consider a class of special examples, where the ancillary problems are easy to handle.

24 (Conditions on T). We assume from now on that our torus T and its action on affine space is of the following form.

Start with $\mathbb{A}^{s+t}$ with the standard $(\mathbb{C}^*)^{s+t}$-action. Fix positive integers c_{ij} and let $T = T(c_{ij}) = \operatorname{im}[(\mathbb{C}^*)^t \to (\mathbb{C}^*)^{s+t}]$ where the map is given by

$$(\lambda_1, \dots, \lambda_t) \mapsto \left(\prod_j \lambda_j^{c_{1j}}, \cdots, \prod_j \lambda_j^{c_{sj}}, \lambda_1, \dots, \lambda_t\right).$$

Let $U \subset \mathbb{A}^{s+t}$ be a $(\mathbb{C}^*)^{s+t}$-invariant open set on which T acts properly.

25 (Choosing singular moduli problems). Take $\mathbb{P}^n \supset \mathbb{A}^n$ with coordinate vertices $p_0 = \mathbf{0} \in \mathbb{A}^n$ and $p_1, \dots, p_n$ at infinity.

Given $n = s+t$ as in (24) and a positive integer d, let $\mathcal{X} = \mathcal{X}(s,t,d)$ be the set of all varieties X that are obtained as $f : X \to \mathbb{P}^n$ from $\mathbb{P}^n$ by performing

(1) a weighted blow up (see (32)) with weights $(d^s, 1^t)$ at p_0,

(2) ordinary blow ups at the n points $p_1, \dots, p_n$ and at a further point $q \in U$.

Let $E_0, \dots, E_n, E_{n+1}$ be the corresponding exceptional divisors. As in (22) choose a polarization H on X of the form $f^*\mathcal{O}_{\mathbb{P}^n}(m) - \sum b_i E_i$ such that the map $f : X \to \mathbb{P}^n$ and the E_i are determined by the pair (X, H). We obtain the set of polarized pairs $(\mathcal{X}, \mathcal{H})$

Weighted blow ups depend on the choice of a local coordinate system, and for weights $(d^s, 1^t)$ we show that they correspond uniquely to certain ideals $I_d \subset \mathcal{O}_{\mathbf{0},\mathbb{A}^{s+t}}$. Thus $\mathcal{X}$ has a natural scheme structure as a subset of the Hilbert scheme of points $\operatorname{Hilb}(\mathbb{A}^{s+t})$ corresponding to the union of $q \in U$ and $\mathcal{O}_{\mathbf{0},\mathbb{A}^{s+t}}/I_d$.

Proposition 26 (Notation as above). *Assume that $t \geq 3$. Then $(\mathcal{X}, \mathcal{H})$ is locally versal and the isomorphisn classes of the polarized pairs $(X, H) \in (\mathcal{X}, \mathcal{H})$ are in one-to-one correspondence with the $(\mathbb{C}^*)^{s+t}$-orbits on $\mathcal{X}$.*

Proof. Every defomation of a smooth point blow up is again a smooth point blow up, and we prove in (39) that every deformation of a weighted point blow up is again a weighted point blow up if $t \geq 3$. Thus every deformation of a variety X in $\mathcal{X}$ is obtained by deforming the points $p_0, \dots, p_n, q \in \mathbb{P}^n$ and also the local coordinate system used for the weighted blow up at p_0. Since $p_0, \dots, p_n \in \mathbb{P}^n$ are in general position, we can move their deformations back to the coordinate vertices by $\operatorname{Aut}(\mathbb{P}^n)$; hence we can assume that the points $p_0, \dots, p_n \in \mathbb{P}^n$ stay fixed in any deformation.

The point q and the local coordinate system used for the weighted blow up at p_0 however can deform nontrivially. With these choices, only the $(\mathbb{C}^*)^{s+t}$-action remains of $\operatorname{Aut}(\mathbb{P}^n)$. □

27 (Choosing smooth moduli problems). Using the explicit description of the weighted blow ups given in (32) we immediately obtain:

Claim 28. For every $X \in \mathcal{X}(s,t,d)$ the singular set $\operatorname{Sing} X$ is isomorphic to $\mathbb{P}^{s-1}$ and (Zariski locally) X along $\operatorname{Sing} X$ is isomorphic to

$$\mathbb{A}^{s-1} \times \mathbb{A}^{t+1}/\tfrac{1}{d}(1,(-1)^t). \qquad \square$$

These singularities are simple enough that one can write down an explicit resolution for them, giving "canonical" resolutions $X^* \to X$ for every $X \in \mathcal{X}(s,t,d)$. We do this in (40). Moreover, we prove that the local deformation theory of X^* is identical to the local deformation theory of X.

By a suitable choice of the polarization (X^*, H^*) we get a smooth polarized moduli problem $(\mathcal{X}^*(s,t,d), \mathcal{H}^*)$, where the contraction $X^* \to X$ induces an isomorphism of the moduli spaces $\mathcal{X}^*(s,t,d) \cong \mathcal{X}(s,t,d)$.

Proposition 29. *Notation and assumptions are as in* (24) *and* (25). *For* $d \gg 1$, *there is an open subset*

$$\mathcal{X}^0(s,t,d) \subset \mathcal{X}(s,t,d) \cong \mathcal{X}^*(s,t,d)$$

such that the $(\mathbb{C}^*)^{s+t}$*-action is proper on* $\mathcal{X}^0(s,t,d)$ *and* U/T *is isomorphic to a closed subscheme of the quotient* $U/T \hookrightarrow \mathcal{X}^0(s,t,d)/(\mathbb{C}^*)^{s+t}$.

Thus $\mathcal{X}^0(s,t,d)/(\mathbb{C}^*)^{s+t}$ *is a versal moduli problem for smooth, polarized varieties as in* (1.3) *which contains* U/T *as a closed subscheme.* □

All that remains is to find examples satisfying (24) where U/T is not quasi-projective.

Example 30. Consider $\mathbb{A}^{2t}$ with coodinates $y_1, \dots, y_t, x_1, \dots, x_t$. Let T be the torus $(\mathbb{C}^*)^t$ acting by

$$y_i \mapsto \lambda_i \lambda_{i+1}^2 y_i \quad \text{(with } t+1=1\text{), and} \quad x_i \mapsto \lambda_i x_i.$$

Set $U_i := (y_i \prod_{j \neq i} x_j \neq 0)$ and $U = \cup_i U_i$. The T action is free on U.

A polarization consists of an ample line bundle on $\mathbb{P}^{2t}$, together with a linearization, that is, a choice of the lifting of the T-action. These correspond to characters

$$\chi(b_1,\dots,b_t):(\lambda_1,\dots,\lambda_t)\mapsto \lambda_1^{b_1}\cdots\lambda_t^{b_t}.$$

The T-equivariant monomials under this polarization are of the form

$$\Bigl(\frac{y_1}{x_1x_2^2}\Bigr)^{a_1}\cdots\Bigl(\frac{y_t}{x_tx_1^2}\Bigr)^{a_t}\cdot\bigl(x_1^{b_1}\cdots x_t^{b_t}\bigr)^m.$$

A T-orbit is semistable in the polarization given by $\chi(b_1,\dots,b_t)$ if and only if there is a monomial as above which is nonzero on the orbit.

Consider the orbit $C_i:=(x_i=0, y_j=0\ \forall\ j\neq i)$. A monomial nonzero on C_i can involve only y_i and the x_j for $j\neq i$. Thus, in the above form, $a_j=0$ for $j\neq i$ and we have a monomial of the form

$$\Bigl(\frac{y_i}{x_ix_{i+1}^2}\Bigr)^{a_i}\cdot\bigl(x_1^{b_1}\cdots x_t^{b_t}\bigr)^m$$

which does not contain x_i. Thus $a_i=mb_i$ and $2a_i\le mb_{i+1}$, which is only possible if $b_{i+1}\ge 2b_i$.

Any collection of $t-1$ such inequalities has a common nonzero solution, but all t of them together lead to $b_1=\cdots=b_t=0$.

Thus we conclude that any $t-1$ orbits in U/T are contained in a quasi-projective open subset but the t orbits $C_1,\dots,C_t$ are not contained in a quasi-projective open subset of U/T. For $t\ge 3$ any 2 orbits are simultaneously stable with respect to some polarization, so the quotient is separated. Thus U/T is a variety which is not quasi-projective. (For $t=2$ we get a nonseparated quotient.)

Example 31. The simplest proper but nonprojective toric variety Y was found by Miyake and Oda, see [Oda88, §2.3]. By [Cox95], this can also be obtained as the quotient of an open subset of $\mathbb{A}^7$ by a $(\mathbb{C}^*)^4$-action. I thank H. Thompson for providing the following explicit description.

Consider $\mathbb{A}^7$ with coodinates $y_1,y_2,y_3,x_1,x_2,x_3,x_4$. Let T be the 4-torus $(\mathbb{C}^*)^4$ acting by

$$(y_1,y_2,y_3,x_1,x_2,x_3,x_4)\\ \mapsto(\lambda_1\lambda_2\lambda_4y_1,\lambda_1\lambda_2\lambda_3y_2,\lambda_1\lambda_3\lambda_4y_3,\lambda_1x_1,\lambda_2x_2,\lambda_3x_3,\lambda_4x_4).$$

Let $U=\mathbb{A}^7\setminus Z$ where Z is the subscheme corresponding to the ideal

$$(y_1,x_1)\cap(y_1,x_4)\cap(y_2,x_1)\cap(y_2,x_2)\cap(y_3,x_1)\cap(y_3,x_3)\cap(x_2,x_3,x_4).$$

Then U/T is isomorphic to the Miyake-Oda example.

It is rather straightforward, though somewhat tedious, to check directly that U/T is not projective by looking at the set of stable points under all possible polarizations.

5. Weighted blow ups

Definition 32. Let $x \in X$ be a smooth point on a variety of dimension n and $(u_1, \dots, u_n)$ local coordinates. Choose positive integers $(a_1, \dots, a_n)$, called weights. This assigns weights to any monomial by the rule

$$w(u_1^{m_1} \cdots u_n^{m_n}) = m_1 a_1 + \cdots + m_n a_n.$$

Let $I_c \subset \mathcal{O}_{x,X}$ be the ideal generated by all monomials of weight at least c. We can also view I_c as an ideal sheaf on X. The scheme

$$B_{\mathbf{u},\mathbf{a}} X := \operatorname{Proj}_X \bigl(\textstyle\sum_{c=0}^{\infty} I_c\bigr)$$

is called the *weighted blow up* of X with coordinates $\mathbf{u} = (u_1, \dots, u_n)$ and weights $\mathbf{a} = (a_1, \dots, a_n)$.

In order to describe the local coordinate charts, we use the notation

$$\mathbb{A}^n(u_1, \dots, u_n)/\tfrac{1}{d}(b_1, \dots, b_n)$$

to denote the quotient of $\mathbb{A}^n$ with coordinates $u_1, \dots, u_n$ by the cyclic group of d^{th} roots of unity μ_d acting as

$$\rho(\epsilon) : (u_1, \dots, u_n) \mapsto (\epsilon^{b_1} u_1, \dots, \epsilon^{b_n} u_n).$$

As a further shorthand,

$$\mathbb{A}^{s+t}/\tfrac{1}{d}(d^s, 1^t)$$

indicates that s of the b_i are d, and t of the b_i are 1.

With these conventions, (étale) local coordinate charts on $B_{\mathbf{u},\mathbf{a}} X$ are given by the quotients

$$\mathbb{A}^n(x_{1,i}, \dots, x_{n,i})/\tfrac{1}{a_i}(-a_1, \dots, -a_{i-1}, 1, -a_{i+1}, \dots, -a_n).$$

The projection map is given by

$$\begin{gathered} u_1 = x_{1,i} x_{i,i}^{a_1}, \dots, u_{i-1} = x_{i-1,i} x_{i,i}^{a_{i-1}}, \\ u_i = x_{i,i}^{a_i}, u_{i+1} = x_{i+1,i} x_{i,i}^{a_{i+1}}, \dots, u_n = x_{n,i} x_{i,i}^{a_n}. \end{gathered}$$

Let us now consider the special case when $n = s + t$ and $(a_1, \dots, a_n) = (d^s, 1^t)$. Then the singular charts on $B_{\mathbf{u},\mathbf{a}} X$ are of the form

$$\mathbb{A}^{s+t}/\tfrac{1}{d}(1, (-1)^t, 0^{s-1}),$$

proving (28).

For weights $(d^s, 1^t)$, we get that

$$I_c = (u_1, \dots, u_s) + m_x^c \quad \text{for } c \le d,$$

and the ideals I_c are all determined by I_d. This in turn is determined by the ideal $(u_1, \dots, u_s)$ modulo m_x^d. Thus we conclude:

Claim 33. The space $\mathcal{W}(s,t,d)$ of all weighted blow ups of weight $(d^s, 1^t)$ centered at a smooth point $x \in X$ can be identified with the subscheme of the Hilbert scheme of points on X parametrizing the quotients $\mathcal{O}_{x,X}/I_d$. □

Assume now that $X = \mathbb{A}^{s+t}$ with coordinates $(y_1, \ldots, y_s, x_1, \ldots, x_t)$.

It is an open condition on $\mathcal{W}(s,t,d)$ to assume that the y-terms in the linear parts of $u_1, \ldots, u_s$ are linearly independent. If this holds then by a linear change of coordinates we can get a different generating set $u_1', \ldots, u_s'$ of the ideal $(u_1, \ldots, u_s)$ such that

$$u_i' = y_i + (\text{linear in } x) + (\text{higher terms in } x, y).$$

Then by a nonlinear coordinate change we can get a final generating set $u_1^*, \ldots, u_s^*$ such that

$$(*) \qquad u_i^* = y_i + h_i(x_1, \ldots, x_t) \quad \text{where } h(\mathbf{0}) = 0 \text{ and } \deg h_i \le d-1.$$

The generators $u_1^*, \ldots, u_s^*$ are uniquely determined by the ideal $(u_1, \ldots, u_s)$ modulo $m_{\mathbf{0}}^d$ and hence by the weighted blow up. So we conclude:

Claim 34. There is an open subset $\mathcal{W}^1(s,t,d) \subset \mathcal{W}(s,t,d)$ such that every weighted blow up in $\mathcal{W}^1(s,t,d)$ can be uniquely given by local coordinates as in (*). □

The $(\mathbb{C}^*)^{s+t}$ action on $\mathcal{W}^1(s,t,d)$ is still pretty bad. To remedy this, we look at a subset $\mathcal{W}^0(s,t,d) \subset \mathcal{W}^1(s,t,d)$ consisting of those local coordinate systems $u_i^* = y_i + h_i(x_1, \ldots, x_t)$ such that $h_i(x_1, \ldots, x_t)$ contains $\prod_j x_j^{c_{ij}}$ with nonzero coefficient for every i where the c_{ij} are as in (24).

LEMMA 35. *With notation as above, for* $d > \max_i\{\sum_j c_{ij}\}$,

(1) $\mathcal{W}^0(s,t,d)$ *is* $(\mathbb{C}^*)^{s+t}$*-invariant,*

(2) *the diagonal* $(\mathbb{C}^*)^{s+t}$*-action on* $\mathcal{W}^0(s,t,d) \times U$ *is proper, and*

(3) *the local coordinate system* $u_i^* = y_i + \prod_j x_j^{c_{ij}}$ *is invariant under* T *but not invariant under any other element of* $(\mathbb{C}^*)^{s+t}$.

Proof. The action of $(\mu_1, \ldots, \mu_s, \lambda_1, \cdots, \lambda_t) \in (\mathbb{C}^*)^{s+t}$ is given by

$$y_i + h_i(x_1, \ldots, x_t) \mapsto \mu_i y_i + h_i(\lambda_1 x_1, \ldots, \lambda_t x_t) \mapsto y_i + \mu_i^{-1} h_i(\lambda_1 x_1, \ldots, \lambda_t x_t).$$

In particular, for $u_i^* = y_i + \prod_j x_j^{c_{ij}}$ we get the action.

$$y_i + \textstyle\prod_j x_j^{c_{ij}} \mapsto \mu_i y_i + \left(\prod_j \lambda_j^{c_{ij}}\right)\left(\prod_j x_j^{c_{ij}}\right) \mapsto y_i + \left(\mu_i^{-1} \prod_j \lambda_j^{c_{ij}}\right)\left(\prod_j x_j^{c_{ij}}\right).$$

Thus we have invariance if and only if $\mu_i = \prod_j \lambda_j^{c_{ij}}$ for every i, that is, only for elements of T.

Finally, the properness of the $(\mathbb{C}^*)^{s+t}$-action is established in two steps. First, note that $(\mathbb{C}^*)^{s+t} = T' + T$ where

$$T' := \{(\mu_1, \dots, \mu_s, 1, \dots, 1) \in (\mathbb{C}^*)^{s+t}\}.$$

Using the T' action we can uniquely normalize every coordinate system $u_i^* = y_i + h_i(x_1, \dots, x_t)$ in $\mathcal{W}^0(s,t,d)$ to the form

$$u_i^* = y_i + h_i(x_1, \dots, x_t) \text{ where } \prod_j x_j^{c_{ij}} \text{ appears with coefficient 1.}$$

Such coordinate systems form a closed subset $\mathcal{W}_1^0(s,t,d) \subset \mathcal{W}^0(s,t,d)$, and

$$\big(\mathcal{W}^0(s,t,d) \times U\big)/(\mathbb{C}^*)^{s+t} \cong \big(\mathcal{W}_1^0(s,t,d) \times U\big)/T.$$

Since the T action on U is already proper by assumption, it is also proper on $\mathcal{W}_1^0(s,t,d) \times U$. $\square$

6. Deformation and resolution of weighted blow ups

36 (Deformation of weighted projective spaces). For an introduction to weighted projective spaces, see [Dol82].

The infinitesimal deformation space of a weighted projective space can be quite large. The situation is, however, much better if we assume that every singularity is a quotient singularity and the singular set has codimension ≥ 3. In this case, by [Sch71], the local deformations at every singular point are trivial, hence the first order global deformations are classified by $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$. On the weighted projective space $X = \mathbb{P}(a_0, \dots, a_n)$ there is an exact sequence

$$0 \to \Omega_X \to \sum \mathcal{O}_X(-a_i) \to \mathcal{O}_X \to 0,$$

and for $n \geq 3$ we obtain that $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) = 0$. Hence we conclude:

Claim 37. If the singular set of $\mathbb{P}(a_0, \dots, a_n)$ has codimension at least 3, then every local deformation of $\mathbb{P}(a_0, \dots, a_n)$ is trivial. In particular, every local deformation of $\mathbb{P}(1,1,1,a_3, \dots, a_n)$ is trivial. $\square$

38 (Deformation of weighted blow ups). Let $X \subset Y$ be a proper subscheme of Y. If deformations of $X \subset Y$ are locally unobstructed, then the obstructions to deforming X in the Hilbert scheme $\mathrm{Hilb}(Y)$ lie in

$$H^1(X, \mathrm{Hom}(I_X, \mathcal{O}_X)),$$

where $I_X \subset \mathcal{O}_Y$ is the ideal sheaf of X. If every singularity of Y is a quotient singularity, X is a divisor and the singular set has codimension ≥ 3, then the deformations are locally unobstructed.

For the weighted blow up

$$B_{(a_1,\dots,a_n)}X \to X \quad \text{with exceptional divisor} \quad \mathbb{P}(a_1, \dots, a_n) \cong E \subset B_{(a_1,\dots,a_n)}X$$

the normal bundle is $\mathcal{O}(\sum a_i)$, hence there are no obstructions for $n \geq 3$. Thus E deforms with any deformation of $B_{(a_1,\dots,a_n)}X$, and the induced deformation of E is trivial if $a_1 = a_2 = a_3 = 1$. The exceptional divisor E, its normal bundle and the local structure of $B_{(a_1,\dots,a_n)}X$ along the singular points determine $B_{(a_1,\dots,a_n)}X$ in a formal or analytic neighborhood of E (cf. [HR64, Lem. 9] or [Mor82, 3.33]). Thus every deformation of $B_{(a_1,\dots,a_n)}X$ is trivial in an analytic neighborhood of the exceptional divisor. We conclude:

Claim 39. If $a_1 = a_2 = a_3 = 1$ then every deformation of $B_{(a_1,\dots,a_n)}\mathbb{P}^n$ is obtained by changing the local coordinate system that defines the weighted blow up. □

It is easy to express nontrivial deformations of the weighted blow ups $B_{(1,d^{n-1})}\mathbb{P}^n$, but rigidity probably holds assuming only that $a_1 = a_2 = 1$.

40 (Resolution of quotient singularities). It is not easy to get resolutions for an arbitrary cyclic quotient singularity, but for the singularities $\mathbb{A}^{t+1}/\frac{1}{d}(1,(-1)^t)$ there is a rather simple resolution.

The key observation is that if we have the weighted blow up $B_{(d-1,1^t)}\mathbb{A}^{t+1}$ then the cyclic goup action $\frac{1}{d}(1,(-1)^t)$ on $\mathbb{A}^{t+1}$ lifts to an action on $B_{(d-1,1^t)}\mathbb{A}^{t+1}$ which acts trivially on the exceptional divisor.

Indeed, let us start with

$$\mathbb{A}^{t+1}(x_0,\dots,x_n)/\tfrac{1}{d}(1,(-1)^t).$$

The key chart $U_0 \subset B_{(d-1,1^t)}\mathbb{A}^{t+1}(x_0,\dots,x_n)$ is

$$U_0 \cong \mathbb{A}^{t+1}(y_0,\dots,y_n)/\tfrac{1}{d-1}(1,(-1)^t), \quad \text{where } x_0 = y_0^{d-1} \text{ and } x_i = y_i y_0.$$

Thus we see that the $\frac{1}{d}(1,(-1)^t)$ action on $\mathbb{A}^{t+1}(x_0,\dots,x_n)$ lifts to $\mathbb{A}^{t+1}(y_0,\dots,y_n)$ as $\frac{1}{d}(-1,0^t)$. Since

$$\mathbb{A}^{t+1}(y_0,\dots,y_n)/\tfrac{1}{d}(-1,0^t) \cong \mathbb{A}^{t+1}(y_0^d,y_1,\dots,y_n),$$

we conclude that the quotient of U_0 is

$$U_0/\mu_d \cong \mathbb{A}^{t+1}(y_0^d,y_2,\dots,y_n)/\tfrac{1}{d-1}(d,(-1)^t) = \mathbb{A}^{t+1}/\tfrac{1}{d-1}(1,(-1)^t).$$

Thus we get a partial resolution

$$g_1 : X_1 \to X_0 \cong \mathbb{A}^{t+1}/\tfrac{1}{d}(1,(-1)^t)$$

whose exceptional divisor is $E_1 \cong \mathbb{P}(d-1,1^t)$ and X_1 has a unique singular point of the form $\mathbb{A}^{t+1}/\frac{1}{d-1}(1,(-1)^t)$. Moreover, the discrepancy (cf. [KM98, §2.3]) of E_1 is $a(E_1) = \frac{t-1}{d}$.

Working by induction, we thus obtain a tower

$$X_{d-1} \stackrel{g_{d-1}}{\to} \cdots \stackrel{g_2}{\to} X_1 \stackrel{g_1}{\to} X_0 \cong \mathbb{A}^{t+1}/\tfrac{1}{d}(1,(-1)^t)$$

where X_{d-1} is smooth and $g_i : X_i \to X_{i-1}$ contracts a single exceptional divisor $E_i \cong \mathbb{P}(d-i, 1^t)$ to a point. Moreover, we also obtain that E_1 has minimal discrepancy among all exceptional divisors over the singular point of $\mathbb{A}^{t+1}/\frac{1}{d}(1,(-1)^t)$. (Indeed, any exceptional divisor other than E_1 is also exceptional over X_1; thus it either lies over the unique singular point and by induction has discrepancy at least $\frac{t-1}{d-1} > \frac{t-1}{d}$, or lies generically over the smooth locus of E_1 and has discrepancy at least $1+\frac{t-1}{d}$.) As in [Kol99, Prop. 6.5] this implies that $g_1 : X_1 \to X_0$ is unique. Indeed, given any other projective birational morphism $g_1' : X_1' \to X_0$ with a single exceptional divisor E_1' of discrepancy $\frac{t-1}{d}$, the induced birational map $h : X_1 \dashrightarrow X_1'$ is a local isomorphism near the generic point of E_1 since E_1 is the unique exceptional divisor of discrepancy $\frac{t-1}{d}$. Thus h is an isomorphism by (12) since the g_1-ample divisor $-E_1$ is transformed into the g_1'-ample divisor $-E_1'$. This implies that in the setting of (28) the local resolution process automatically globalizes.

(Note that this is a much stronger uniqueness than in the case of $B_{(a_1,\dots,a_n)}\mathbb{A}^n \to \mathbb{A}^n$, where we have uniqueness only up to a local coordinate change in $\mathbb{A}^n$. For $B_{(a_1,\dots,a_n)}\mathbb{A}^n \to \mathbb{A}^n$ the discrepancy is $\sum a_i$. This is not minimal unless all the $a_i = 1$, and the usual blow up is indeed unique. It is a general rule that for minimal discrepancy divisors we can expect stronger uniqueness results.)

As in (38) we also obtain that every deformation of X_i is trivial for $t \geq 3$.

Putting all of these together, we conclude:

Claim 41. If $t \geq 3$ then every deformation of the above constructed canonical resolution of $B_{(d^s,1^t)}\mathbb{P}^{s+t}$ is obtained by changing the local coordinate system that defines the weighted blow up. □

7. Open problems

The above examples show that moduli spaces of smooth polarized varieties can be complicated. My guess is that in fact they have a universality property with respect to subspaces.

CONJECTURE 42. *Let G be a linear algebraic group acting properly on a quasi-projective scheme W. Then there are*

(1) *a projective space* $\mathbb{P}$,

(2) *an open subset* $U \subset \operatorname{Hilb}(\mathbb{P})$ *parametrizing smooth varieties such that* $\operatorname{Aut}(\mathbb{P})$ *acts properly on* U,

(3) *a homomorphism* $G \to \operatorname{Aut}(\mathbb{P})$, *and*

(4) *a G-equivariant closed embedding $W \to U$,*

such that the corresponding morphism $W/G \to U/\operatorname{Aut}(\mathbb{P})$ is a closed embedding.

This naturally leads to the following question, which is quite interesting in its own right.

Question 43. Which algebraic spaces can be written as geometric quotients of quasi-projective schemes?

The paper [Tot04] contains a detailed review and a necessary and sufficient condition in terms of resolutions by locally free sheaves. Nonetheless, the answer is not known even for normal schemes or smooth algebraic spaces.

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