

Analytic representation of functions and a new quasi-analyticity threshold

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Abstract

We characterize precisely the possible rate of decay of the anti-analytic half of a trigonometric series converging to zero almost everywhere.

1. Introduction

1.1. In 1916, D. E. Menshov constructed an example of a nontrivial trigonometric series on the circle \mathbb{T}

$$(1) \quad \sum_{n=-\infty}^{\infty} c(n)e^{int}$$

which converges to zero almost everywhere (a.e.). Such series are called null-series. This result was the origin of the modern theory of uniqueness in Fourier analysis, see [Z59], [B64], [KL87], [KS94].

Clearly for such a series $\sum |c(n)|^2 = \infty$. A less trivial observation is that a null series cannot be analytic, that is, involve positive frequencies only. Indeed, it would then follow by Abel's theorem that the corresponding analytic function

$$(2) \quad F(z) = \sum_{n \geq 0} c(n)z^n$$

has nontangential boundary values equal to zero a.e. on the circle $|z| = 1$. Privalov's uniqueness theorem (see below in §2.3) now shows that F is identically zero.

Definition. We say that a function f on the circle \mathbb{T} belongs to PLA (which stands for Pointwise Limit of Analytic series) if it admits a representation

$$(3) \quad f(t) = \sum_{n \geq 0} c(n)e^{int}$$

by an a.e. converging series.

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The discussion above shows that such a representation is unique. Further, for example, e^{-int} is not in PLA for any $n > 0$ since multiplying with z^n would lead to a contradiction to Privalov's theorem.

If f is an L^2 function with positive Fourier spectrum, or in other words, if it belongs to the Hardy space H^2 , then it is in PLA according to the Carleson convergence theorem. On the other hand, we proved in [KO03] that L^2 contains in addition PLA functions which are not in H^2 . The representation (3) for such functions is “nonclassical” in the sense that it is different from the Fourier expansion.

One should contrast this phenomenon against some results in the Riemannian theory (see [Z59, Chap. 11]) which say that whenever a representation by harmonics is unique then it is the Fourier one. Compare for examples the Cantor theorem to the du Bois-Reymond theorem. In an explicit form this principle was stated in [P23]: If a function $f \in L^1(\mathbb{T})$ has a *unique* pointwise decomposition (1) outside of some compact K then it is the Fourier expansion of f . Again, for analytic expansions (3) this is not true.

1.2. Taking a function f from the “nonclassic” part of $\text{PLA} \cap L^2$ and subtracting from the representation (3) the Fourier expansion of f , one gets a null-series with a small anti-analytic part in the sense that

$$\sum_{n < 0} |c(n)|^2 < \infty.$$

Note that there are many investigations of the possible size of the coefficients of a null-series. They show that the coefficients may be arbitrarily close to l^2 . See [I57], [A84], [P85], [K87]. In all known constructions the behavior of the amplitudes in the positive and the negative parts of the spectrum is the same. [KO03] shows that a substantial nonsymmetry may occur. How far may this nonsymmetry go? Is it possible for the anti-analytic amplitudes to decrease fast? Equivalently, may a function in $\text{PLA} \setminus H^2$ be smooth?

The method used in [KO03] is too coarse to approach this problem. However, we proved recently that smooth and even C^∞ functions do exist in $\text{PLA} \setminus H^2$. Precisely, in [KO04] we sketched the proof of the following:

THEOREM 1. *There exists a null-series (1) with amplitudes in negative spectrum ($n < 0$) satisfying the condition*

$$(4) \quad c(n) = O(|n|^{-k}), \quad k = 1, 2, \dots$$

Hence we are lead to the following question: *what is the maximal possible smoothness of a “nonclassic” PLA function?* In other words we want to characterize the possible rate of decreasing the amplitudes $|c(n)|$ of a null-series as $n \rightarrow -\infty$. This is the main problem considered here.

1.3. It should be mentioned that if one replaces convergence a.e. by convergence on a set of positive measure, then the characterization is given by the classic quasi-analyticity condition. Namely, the class of series (1) satisfying

$$(5) \quad c(n) = O(e^{-\rho(|n|)}) \quad \forall n < 0$$

for some $\rho(n)$ (with some regularity) is prohibited from containing a nontrivial series converging to zero on a set E of positive measure if and only if

$$(6) \quad \sum \frac{\rho(n)}{n^2} = \infty.$$

The part “only if” is well known: if this sum converges one may construct a function vanishing on an interval E whose Fourier coefficients satisfy (5), and for n positive as well (see e.g. [M35, Chap. 6]). The “if” part follows from a deep theorem of Beurling [Be89], extended by Borichev [Bo88]. See more details below in Section 2.3.

It turns out that in our situation the threshold is completely different. The following uniqueness theorem with a much weaker requirement on coefficients is true.

THEOREM 2. *Let ω be a function $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\omega(t)/t$ increase and*

$$(7) \quad \sum \frac{1}{\omega(n)} < \infty.$$

Then the condition:

$$(8) \quad c(n) = O(e^{-\omega(\log |n|)}), \quad n < 0$$

for a series (1) converging to zero a.e. implies that all $c(n)$ are zero.

It is remarkable that the condition is sharp. The following strengthened version of Theorem 1 is true:

THEOREM 3. *Let ω be a function $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, let $\omega(t)/t$ be concave and*

$$(9) \quad \sum \frac{1}{\omega(n)} = \infty.$$

Then there exists a null-series (1) such that (8) is fulfilled.

So the maximal possible smoothness of a “nonclassical” PLA function f is precisely characterized in terms of its Fourier transform by the condition

$$\widehat{f}(n) = O(e^{-\omega(\log |n|)}), \quad n \in \mathbb{Z}$$

where ω satisfies (9). As far as we are aware this condition has never appeared before as a smoothness threshold.

We mention that whereas the usual quasi-analyticity is placed near the “right end” in the scale of smoothness connecting C^∞ and analyticity, this

new quasi-analyticity threshold is located just in the opposite side, somewhere between $n^{-\log \log n}$ and $n^{-(\log \log n)^{1+\varepsilon}}$.

The main results of this paper were announced in our recent note [KO04].

2. Preliminaries

In this section we give standard notation, needed background and some additional comments.

2.1. We denote by \mathbb{T} the circle group $\mathbb{R}/2\pi\mathbb{Z}$. We denote by \mathbb{D} the disk in the complex plane $\{z : |z| < 1\}$ and $\partial\mathbb{D} = \{e^{it} : t \in \mathbb{T}\}$. For a function F (harmonic, analytic) on \mathbb{D} and a $\zeta \in \partial\mathbb{D}$ we shall denote the nontangential limit of F at ζ (if it exists) by $F(\zeta)$.

We denote by C and c constants, possibly different in different places. By $X \approx Y$ we mean $cX \leq Y \leq CX$. By $X \ll Y$ we mean $X = o(Y)$. Sometimes we will use notation such as $-O(\cdot)$. While this seems identical to just $O(\cdot)$ we use this notation to remind the reader that the relevant quantity is negative.

The notation $\lfloor x \rfloor$ will stand for the lower integral value of x . $\lceil x \rceil$ will stand for the upper integral value.

When x is a point and K some set in \mathbb{T} or \mathbb{D} , the notation $d(x, K)$ stands, as usual, for $\inf_{y \in K} d(x, y)$.

2.2. For a $z \in \mathbb{D}$ we shall denote the Poisson kernel at the point z by P_z and the conjugate Poisson kernel by Q_z . We denote by H the Hilbert kernel on \mathbb{T} . See e.g. [Z59]. If $f \in L^2(\mathbb{T})$ we shall denote by $F(z)$ the harmonic extension of f to the disk, i.e.

$$(10) \quad F(z) = \int_0^{2\pi} P_z(t) f(t) dt, \quad \forall z \in \mathbb{D}.$$

Similarly, the harmonic conjugate to F can be derived directly from f by

$$\tilde{F}(z) = \int_0^{2\pi} Q_z(t) f(t) dt, \quad \forall z \in \mathbb{D}.$$

It is well known that F and \tilde{F} have nontangential boundary values a.e. and that $F(e^{it}) = f(t)$ a.e. We shall denote $\tilde{f}(t) := \tilde{F}(e^{it})$. We remind the reader also that

$$\tilde{f}(x) = (f * H)(x) = \int_0^{2\pi} f(t) H(x - t) dt$$

where the integral is understood in the principal value sense.

For a function F on the disk, the notation $F^{(D)}$ denotes tangent differentiation, namely $F'(re^{i\theta}) := \frac{\partial F}{\partial \theta}$. The representations above admit differentiation.

For example,

$$F^{(D)}(z) = \int P_z^{(D)}(t) f(t) dt, \quad \forall z \in \mathbb{D}.$$

We shall use the following well known estimates for P , Q and their derivatives:

$$(11) \quad |P_z^{(D)}(t)| \leq \frac{C(D)}{|e^{it} - z|^{D+1}}, \quad |Q_z^{(D)}(t)| \leq \frac{C(D)}{|e^{it} - z|^{D+1}} \quad \forall D \geq 0;$$

for H we shall need the symmetry $H(t) = -H(-t)$ and

$$(12) \quad |H^{(D)}(t)| \leq \frac{(CD)^{CD}}{|e^{it} - 1|^{D+1}}.$$

2.3. Uniqueness theorems. In 1918 Privalov proved the following fundamental theorem:

Let F be an analytic function on \mathbb{D} such that $F(e^{it}) = 0$ on a set E of positive measure. Then F is identically zero.

See [P50], [K98]. The conclusion also holds under the condition

$$F(e^{it}) = \sum_{n=-\infty}^{-1} c(n)e^{int} \quad \text{on } E$$

with the $|c(n)|$ decreasing exponentially. When one goes further the picture gets more complicated. Examine the following result of Levinson and Cartwright [L40]:

Let F be an analytic function on \mathbb{D} with the growth condition

$$(13) \quad |F(z)| < \nu(1 - |z|) \int_0^1 \log \log \nu < \infty.$$

Assume that F can be continued analytically through an arc $E \subset \partial\mathbb{D}$ to an f in $\mathbb{C} \setminus \mathbb{D}$ which satisfies

$$f(z) = \sum_{n=-\infty}^{-1} c(n)z^n.$$

and the $c(n)$ satisfy the quasi-analyticity conditions (5), (6). Then F and f are identically zero.

It follows if a series (1) converges to zero on an interval and the “negative” coefficients decrease quasianalytically then it is trivial.

In 1961 Beurling extended the Levinson-Cartwright theorem from an arc to any set E with positive measure (see [Be89]):

Let $f \in L^2$ vanish on E and let its Fourier coefficients $c(n)$ satisfy (5), (6). Then f is identically zero.

Borichev [Bo88] proved that the L^2 condition in this theorem could be replaced by a very weak growth condition on the analytic part F in \mathbb{D} , similar in spirit to (13). Certainly this condition would be fulfilled if the series converged pointwise on E . Note again that the classic quasi-analyticity condition in all these results cannot be improved. Our proof of uniqueness uses the same general framework used in [Bo88], [BV89], [Bo89].

Other results about the uncertainty principle in analytic settings exist, namely connecting smallness of support with fast decrease of the Fourier coefficients. See for example [H78] for an analysis of support of measures with smooth Cauchy transform. The connection between the smoothness of the boundary value of a function F and the increase of F near the singular points of the boundary was investigated for F from the Nevanlinna class; see Shapiro [S66], Shamoyan [S95] and Bourhim, El-Fallah and Kellay [BEK04]. In particular, applying theorem A of [BEK04] to our case shows that one cannot construct a C^1 function in $\text{PLA} \setminus H^2$ by taking the boundary value of a Nevanlinna function. For comparison, our first example of a function from $\text{PLA} \setminus H^2$ (see [KO03]) is a boundary value of a Nevanlinna class function. That example is L^∞ and can be made continuous, but it cannot be made smooth in any reasonable sense without leaving the Nevanlinna class.

2.4. *The harmonic measure.* Let \mathcal{D} be a connected open set in \mathbb{C} such that $\partial\mathcal{D}$ is a finite collection of Jordan curves, and let $v \in \mathcal{D}$. Let B be Brownian motion (see [B95, I.2]) starting from v . Let T be the stopping time on the boundary of \mathcal{D} , i.e.

$$T := \inf\{t : B(t) \in \partial\mathcal{D}\}.$$

See [B95, Prop. I.2.7]. Then $B(T)$ is a random point on $\partial\mathcal{D}$, or in other words, the distribution of $B(T)$ is a measure on $\partial\mathcal{D}$ called the *harmonic measure* and denoted by $\Omega(v, \mathcal{D})$. The following result is due to Kakutani [K44].

Let f be a harmonic function in a domain \mathcal{D} and continuous up to the boundary. Let $v \in \mathcal{D}$. Then

$$(14) \quad f(v) = \int f(\theta) d\Omega(v, \mathcal{D})(\theta).$$

It follows that the definition of harmonic measure above is equivalent to the original definition of Nevanlinna which used solutions of Dirichlet's problem. We shall also need the following version of Kakutani's theorem:

Let f be a subharmonic function in a domain \mathcal{D} and upper semi-continuous up to $\partial\mathcal{D}$. Let $v \in \mathcal{D}$. Then

$$(15) \quad f(v) \leq \int f(\theta) d\Omega(v, \mathcal{D})(\theta).$$

See [B95, Propositions II.6.5 and II.6.7]. See also [*ibid*, Theorem II.1.15 and Proposition II.1.13].

3. Construction of smooth PLA functions

3.1. In this section we prove Theorem 3. We wish to restate it in a form which makes explicit the fact that the singular set is in fact compact:

THEOREM 3'. *Let ω be a function $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\omega(t)/t$ be concave and $\sum \frac{1}{\omega(n)} = \infty$. Then there exists a series (1) converging to zero outside a compact set K of measure zero such that (8) is fulfilled.*

The regularity condition that $\omega(t)/t$ be concave in Theorem 3' implies the very rough estimate $\omega(t) = e^{o(t)}$, which is what we will use. Actually, one may strengthen the theorem slightly by requiring only that $\omega(t)/t$ is increasing and $\omega(t) = e^{o(t)}$, and the result would still hold.

Without loss of generality it is enough to prove

$$(16) \quad c(n) = O(e^{-c\omega(\log |n|)}), \quad n < 0$$

for some $c > 0$, instead of (8). Also we may assume $\omega(t)/t$ increases to infinity (otherwise, just consider $\omega(t) = t \log(t+2)$ instead).

The c above, like all notation c and C , \ll , o and O in this section, is allowed to depend on ω . In general we will consider ω as given and fixed, and will not remind the reader that the various parameters depend on it.

A rough outline of the proof is as follows: we shall define a probabilistically-skewed thick Cantor set K and a random harmonic function G on the disk such that the boundary values of G on K are positive infinite, while the boundary values outside K are finite negative (except a countable set of points where they are infinite negative). Further, the function G is “not integrable” in the sense that $\int_0^{2\pi} |G(re^{i\theta})| d\theta \rightarrow \infty$ as $r \rightarrow 1$. The thickness of the set K would depend on ω . For example, if $\omega(t) = t \log t$ (which is enough for the construction of a nonclassic PLA $\cap C^\infty$ function, i.e. for the proof of Theorem 1) then K would have infinite $\delta \log \log 1/\delta$ -Hausdorff measure. Then we shall define $F = e^{G+i\tilde{G}}$ and f its boundary value (f is a nonclassic PLA function). We shall arrange for $G|_{\partial\mathbb{D}}$ to converge to $-\infty$ sufficiently fast near K , and it would follow that f is smooth. A bound for the growth of G to $+\infty$ near K would ensure that the Taylor coefficients of F go to zero with probability one. Finally the desired null-series would be defined by

$$(17) \quad c(n) := \hat{f}(n) - \begin{cases} \hat{F}(n) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

where \hat{f} is the Fourier transform of f while \hat{F} are the Taylor coefficients of F :

$$(18) \quad F(z) = \sum_{n=0}^{\infty} \hat{F}(n)z^n.$$

3.2. *Auxiliary sequences.* Let ω_2 satisfy that $\omega_2(t)/t$ is increasing, $\sum \frac{1}{\omega_2} = \infty$, and

$$(19) \quad \omega(t) \ll \omega_2(t) = \omega(t)t^{o(1)}$$

(note that $\omega_2(t) = e^{o(t)}$). Define

$$\Phi(n) := \exp\left(-\sum_{k=1}^n \frac{1}{\omega_2(k)}\right)$$

and in particular $\Phi(0) = 1$. Also, Φ decreases slowly (depending on ω_2), and the fact that $\frac{\omega_2(t)}{t}$ increases to ∞ gives

$$(20) \quad \Phi(n) = n^{-o(1)}.$$

Another regularity condition over Φ that will be used is the following:

LEMMA 1.

$$(21) \quad \sum_{k=1}^n \Phi(k) = O(n\Phi(n)).$$

Proof. Fix N such that $\omega_2(n)/n > 100$ for $n > N$. Then for all $n > 3N$,

$$\sum_{k=\lceil \frac{1}{3}n \rceil}^n \frac{1}{\omega_2(k)} \leq 0.03$$

and hence $\Phi(k) \leq 1.04\Phi(n)$ for all $k \in [\lceil \frac{1}{3}n \rceil, n]$. Inductively we get $\Phi(k) \leq 1.04^l\Phi(n)$ for any $k \in [\lfloor n3^{-l} \rfloor, \lfloor n3^{1-l} \rfloor] \cap \{N, \dots\}$. Hence we get

$$\begin{aligned} \sum_{k=1}^n \Phi(k) &= \sum_{l=1}^{\lfloor \log_3 n \rfloor + 1} \sum_{k=\lfloor n3^{-l} \rfloor + 1}^{\lfloor n3^{1-l} \rfloor} \Phi(k) \leq N + \sum_{l=1}^{\lfloor \log_3 n \rfloor + 1} \left\lfloor 2 \cdot 3^{-l} n \right\rfloor \Phi(n)(1.04)^l \\ &= O(n\Phi(n)). \end{aligned}$$

Notice that in the last equality we used the fact that $\Phi(n) \gg 1/n$ (20). □

Next, define

$$(22) \quad \sigma_n = 2\pi \cdot 2^{-n}\Phi(n), \quad \tau_n = \frac{1}{12}(\sigma_{n-1} - 2\sigma_n), \quad n \geq 0.$$

The purpose behind the definition of Φ is so that the following (which can be verified with a simple calculation) holds:

$$(23) \quad \frac{\tau_n}{\sigma_n} = \frac{1}{6\omega_2(n)} + O\left(\frac{1}{\omega_2^2(n)}\right).$$

From this and the regularity conditions $\omega_2(n) = e^{o(n)}$ and (20) we get a rough but important estimate for τ_n :

$$(24) \quad \tau_n = 2^{-n-o(n)}.$$

3.3. *The functions g_n .* Next we define some auxiliary functions. Let $a \in C^\infty([0, 1])$ be a nonnegative function satisfying

$$a|_{[0,1/3]} \equiv 0, \quad a|_{[1/2,1]} \equiv 1, \quad \max |a^{(D)}| \leq (CD)^{CD}.$$

Since the standard building block $e^{-1/x}$ satisfies the estimate for the growth of the derivatives above (even a very rough estimate can show this — say, use Lemma 7 below), and since such constraints are preserved by multiplication, there is no difficulty in constructing a .

Let l be defined by

$$l(t) = -t^{-1/3}a(1-t) - a(t).$$

Then l satisfies

$$(25) \quad \begin{aligned} l(t) &= -t^{-1/3}, & t \in]0, \frac{1}{3}] , \\ l(t) &= -1, & t \in [\frac{2}{3}, 1], \end{aligned}$$

and $l \leq -1$ on $]0, 1]$.

Using l , define functions on \mathbb{R} depending on a parameter $s \in [0, 1]$,

$$(26) \quad l^\pm(s; x) := \begin{cases} l(x) & 0 < x \leq 1 \\ -1 & 1 < x \leq 2 \pm s \\ l(3 \pm s - x) & 2 \pm s < x \leq 3 \pm s \end{cases}$$

and 0 otherwise. The estimate for the derivatives of a translates to

$$(27) \quad \left| (l^\pm)^{(D)}(s; x) \right| \leq \frac{(CD)^{CD}}{d(x, \{3 \pm s, 0\})^{D+1/3}}.$$

Let $s(n, k)$ be a collection of numbers between 0 and 1, for each $n \in \mathbb{N}$ and each $0 \leq k < 2^n$. Most of the proof will hold for any choice of $s(n, k)$, but in the last part we shall make them random, and prove that the constructed function will have the required properties for almost any choice of $s(n, k)$. Define now inductively intervals $I(n, k) = [a(n, k), a(n, k) + \sigma_n]$ (we call these

$I(n, k)$ “intervals of rank n ”) as follows: $I(0, 0) = [0, 2\pi]$ and for $n \geq 0$, $0 \leq k < 2^n$,

$$(28) \quad \begin{aligned} a(n + 1, 2k) &= a(n, k) + \tau_{n+1}(3 + s(n + 1, 2k)), \\ a(n + 1, 2k + 1) &= a(n, k) + \frac{1}{2}\sigma_n + \tau_{n+1}(3 + s(n + 1, 2k + 1)). \end{aligned}$$

In other words, at the n^{th} step, inside each interval of rank n (which has length σ_n), situate two disjoint intervals of rank $n + 1$ of lengths σ_{n+1} in random places (but not too near the boundary of $I(n, k)$ or its middle). Define

$$\begin{aligned} K &:= \bigcap_{n=1}^{\infty} K_n, & K_n &:= \bigcup_{k=0}^{2^n-1} I(n, k). \\ K^\circ &:= e^{iK}, & K_n^\circ &:= e^{iK_n}. \end{aligned}$$

Note that $\sum \frac{1}{\omega_2} = \infty$ shows that $\Phi(n) \rightarrow 0$ and hence K has zero measure.

We now define the most important auxiliary functions, $g_n \in L^2(\mathbb{T})$. We define them inductively, with $g_0 \equiv 0$. For one n and k , let I be the interval of rank $n - 1$ containing $I(n, k)$, and let I' be its half containing $I(n, k)$. Now, $I' \setminus I(n, k)$ is composed of two intervals, which we denote by J_1 (left) and J_2 (right). Define the function $g_n(t)$ on the set $I' \setminus I(n, k)$ by

$$(29) \quad \begin{aligned} g_n(t) &:= \omega(n)l^+(s(n, k); \varphi_1(t)), \quad t \in J_1, \quad \varphi_1 : J_1 \rightarrow [0, 3 + s(n, k)], \\ g_n(t) &:= \omega(n)l^-(s(n, k); \varphi_2(t)), \quad t \in J_2, \quad \varphi_2 : J_2 \rightarrow [0, 3 - s(n, k)], \end{aligned}$$

where the φ_i -s are linear, increasing and onto so that they are defined uniquely by their domain and range. As will become clear later, the $\omega(n)$ factor above is what determines the rate of decrease of the coefficients of the null series.

This defines g_n on $K_{n-1} \setminus K_n$. On $\mathbb{T} \setminus K_{n-1}$ we define $g_n \equiv g_{n-1}$. On K_n we define g_n to be a constant such that $\int_{\mathbb{T}} g_n = 0$. Note that g_n is negative on $\mathbb{T} \setminus K_n$ and positive on K_n . Also note that the definition of g_n shows that $\int_{I(n-1, k)} g_n^-$ is independent of $s(n - 1, k)$ — what you earn on the left you lose on the right. See Figure 1.

Extend $g_n(e^{it})$ to a harmonic function in the interior of the disk (remember that each g_n is in L^2), and denote the extension by G_n . Denote by \widetilde{G}_n the harmonic conjugate to G_n .

3.4. *The growth of the g_n .* We need to estimate the positive part of g_n . We have

$$(30) \quad \int_{I(n-1, k)} |g_n^-(x)| \stackrel{(*)}{\approx} \tau_n \omega(n) \stackrel{(**)}{\ll} \sigma_n$$

where $(*)$ comes from the definition of g_n (29) and $(**)$ comes from $\tau_n/\sigma_n \approx 1/\omega_2(n)$ (23) and $\omega \ll \omega_2$ (19). Summing (and using $\sigma_n = 2^{-n}\Phi(n)$) we get

$$(31) \quad \int_{\mathbb{T}} |g_n^-| = \sum_{l=0}^{n-1} \sum_{k=0}^{2^l-1} \int_{I(l, k)} |g_{l+1}^-(x)| \stackrel{(30)}{=} \sum_{l=1}^n o(\Phi(l)) \stackrel{(*)}{=} o(n\Phi(n)),$$

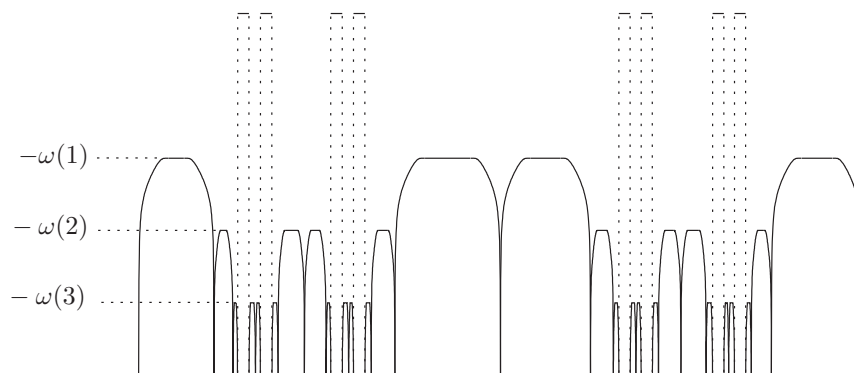


Figure 1: g_3 (not drawn to scale). Notice the random perturbations in the widths of the constant parts of g_3^- but the fixed width of the intervals in K_3 .

where (*) comes from Lemma 1 and $\Phi(n) \gg 1/n$ (20). Hence

$$(32) \quad \max g_n = o(n).$$

This crucial inequality is the one that guarantees in the end that our function F would satisfy $\widehat{F}(m) \rightarrow 0$. Comparing this to (29) we observe that even though K has zero measure, one can balance superlinear growth outside K (the $\omega(n)$ factor in (29)) with sublinear growth inside K .

We will also need a simple estimate from the other side. The same calculations, but using $\omega(n) = \omega_2(n) \cdot n^{-o(1)}$ (the second half of (19)) and $\Phi(n) = n^{-o(1)}$ (20) give

$$(33) \quad \int_{\mathbb{T}} g_n^- = -n^{1-o(1)}.$$

3.5. *The limit of the G_n .* First we want to show that the G_n 's converge to a harmonic function G on compact subsets of the disk, and to discuss the boundary behavior of G and \tilde{G} . For this purpose we need to examine the singularities of g_n . First, and more important is K . Clearly, $\lim_{n \rightarrow \infty} g_n(t) = +\infty$ while $\lim_{t' \rightarrow t, t' \notin K} \lim_{n \rightarrow \infty} g_n(t') = -\infty$ for every $t \in K$. Additionally we have a countable set of points where the g_n 's have $t^{-1/3}$ -type singularities, namely

$$Q := \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} \{a(n, k), a(n, k) + \frac{1}{2}\sigma_n, a(n, k) + \sigma_n\}.$$

Denote $K' := K \cup Q$, $(K')^\circ := e^{iK'}$. We will need the following calculation:

LEMMA 2. *For any $z \in \overline{\mathbb{D}} \setminus K_n^\circ$, and any $D \geq 0$,*

$$(34) \quad |G_{n+1}^{(D)}(z) - G_n^{(D)}(z)| \leq \frac{C(D)}{2^n d(z, K_n^\circ)^{D+1}}.$$

Proof. On the circle \mathbb{T} , $g_{n+1} - g_n$ is nonzero only on the intervals $I(n, k)$, and on each interval we have

$$(35) \quad \int_{I(n,k)} (g_{n+1}(x) - g_n(x)) \, dx = 0.$$

Further, the negative part of $g_{n+1} - g_n$ on $I(n, k)$, which is simply $g_{n+1}^- - \max g_n$ restricted to $I(n, k) \setminus (I(n + 1, 2k) \cup I(n + 1, 2k + 1))$ can be estimated using (30) and (32) to get

$$\begin{aligned} \int_{I(n,k)} |g_{n+1}(x) - g_n(x)| &= 2 \int (g_{n+1} - g_n)^- \approx \tau_{n+1}(\omega(n + 1) + o(n)) \\ &\stackrel{(*)}{\ll} 2^{-n} \Phi(n + 1) \ll 2^{-n} \end{aligned}$$

where in (*) we use that $n \ll \omega(n)$. Hence by (35),

$$(36) \quad \left| \int_t^u g_{n+1}(x) - g_n(x) \, dx \right| \leq C 2^{-n} \quad \forall t, u \in [0, 1], \forall n.$$

Write $G^{(D)}(z) = \int_{\mathbb{T}} g(t) P_z^{(D)}(t)$, where P_z is the Poisson kernel. We divide into two cases: if $1 - |z| > \frac{1}{2}d(z, K_n^\circ)$ then we have from (11) that

$$\int_{\mathbb{T}} |P_z^{(D+1)}| \leq \frac{C(D)}{(1 - |z|)^{D+1}} \leq \frac{C(D)}{d(z, K_n^\circ)^{D+1}}.$$

On the other hand, if $1 - |z| \leq \frac{1}{2}d(z, K_n^\circ)$ then $g_{n+1} - g_n$ is zero in an interval $J := [t - cd(z, K^\circ), t + cd(z, K^\circ)]$ for some c sufficiently small, where t is given by $e^{it} = z/|z|$, and

$$\int_{\mathbb{T} \setminus J} |P_z^{(D+1)}| \leq \frac{C(D)}{d(z, K_n^\circ)^{D+1}}.$$

In either case, a simple integration by parts gives (34).

Finally, on $\partial\mathbb{D}$ we have $G_{n+1}(e^{it}) - G_n(e^{it}) = g_{n+1}(t) - g_n(t) = 0$ for every $t \notin K_n$. □

A similar calculation with the conjugate Poisson kernel (and the Hilbert kernel on the boundary) shows

$$(37) \quad |\widetilde{G}_{n+1}^{(D)}(z) - \widetilde{G}_n^{(D)}(z)| \leq \frac{C(D)}{2^n d(z, K_n^\circ)^{D+1}}.$$

From (34) and (37) it is now clear that both G_n and \widetilde{G}_n converge uniformly on compact subsets of $\overline{\mathbb{D}} \setminus (K')^\circ$. Denote their respective limits by G and \widetilde{G} — clearly they are indeed harmonic conjugates which justifies the notation \widetilde{G} . Also we remind the reader the known fact that if g_n is C^D in some interval $I \subset \mathbb{T}$ then G_n is C^D in e^{iI} and in particular $G_n^{(D)}$ is continuous there. The following lemma is now clear:

LEMMA 3. (i) $G + i\tilde{G}$ is analytic in \mathbb{D} and continuous up to the boundary except at $(K')^\circ$.

(ii) If $t \in \mathbb{T} \setminus K_n$ then $G(e^{it}) = g_n(t) = g_n^-(t)$.

(iii) $(G_n + i\tilde{G}_n)^{(D)}$ converges to $(G + i\tilde{G})^{(D)}$ uniformly on compact subsets of $\overline{\mathbb{D}} \setminus (K')^\circ$.

3.6. *The function F .* We can now define a crucial element of the construction:

$$F = \exp(G + i\tilde{G}).$$

Clearly Lemma 3, (i) shows that F is an analytic function with almost everywhere defined boundary values. Denote by $f(t)$ the boundary value of F at e^{it} . Define similarly g and \tilde{g} and get that $f = e^{g+i\tilde{g}}$.

The reader should keep in mind that the relation between F and f is not similar to the one between an H^2 function and its boundary value (for example, between $G_n + i\tilde{G}_n$ and $g_n + i\tilde{g}_n$). In our case there is a singular distribution (supported on K') which is “lost” when taking the limit. The Fourier series of this singular distribution is exactly the null series we are trying to construct.

LEMMA 4. (i) F is not in $H^1(\mathbb{D})$.

(ii) $f \in L^\infty(\mathbb{T})$.

The first follows from Lemma 3, (ii) if we notice that the L^1 norms of g_n tend to ∞ according to (33) so that $\log |f| = g \notin L^1(\mathbb{T})$. The second is also a direct consequence of Lemma 3, (ii). These properties taken together show that the $c(n)$ (17) are nontrivial. The theorem now divides into the following two claims:

LEMMA 5. $f \in C^\infty(\mathbb{T})$. Further, f has the smoothness in the statement of the theorem:

$$(38) \quad \hat{f}(n) = O(e^{-c\omega(\log |n|)}), \quad n \in \mathbb{Z}.$$

LEMMA 6. With probability 1,

$$\hat{F}(n) = o(1).$$

Now we use the Riemann localization principle in a form due to Kahane-Salem [KS94, p. 54]:

If S is a distribution with $\hat{S}(n) = o(1)$ and I is an interval outside the support of S then $\sum \hat{S}(n)e^{int} = 0$ on I .

Lemma 3, (i) shows that the $c(n)$ defined by (17) represent a singular distribution supported on K' , and the last estimate shows that $c(n) = o(1)$.

Hence (1) is a nontrivial series convergent to 0 everywhere on $\mathbb{T} \setminus K'$. This, along with Lemma 5, proves the theorem.

The purpose of the next section is to prove Lemma 5.

3.7. *Smoothness.* The following two lemmas are self-contained; that is, their f -s, g -s, ω -s and K -s are not necessarily the same ones as those defined in the previous parts of the proof.

LEMMA 7. *Let $f = \exp(g)$. Then*

$$(39) \quad f^{(D)} = f \cdot \sum_{l_1 + \dots + l_i = D} a_{\vec{l}} \prod_{j=1}^i g^{(l_j)},$$

and $\sum_{\vec{l}} |a_{\vec{l}}| \leq D!$

This is a straightforward induction and we shall skip the proof.

LEMMA 8. *Let $\omega(t)$ satisfy that $\omega(t)/t$ is increasing to ∞ and $\omega(t) = e^{o(t)}$. Let K be some compact and let $g \in C^\infty(\mathbb{T} \setminus K)$ satisfy*

- (i) $\operatorname{Re} g(x) \leq -\omega(\log 1/d(x, K))$;
- (ii) $|g^{(D)}(x)| \leq \frac{(CD)^{CD}}{d(x, K)^{2D}}$ for every $D \geq 1$.

Let $f = e^g$ outside K , $f|_K \equiv 0$. Then $\widehat{f}(m) = O(e^{-c\omega(\log |m|)})$.

We remark that condition (ii) interfaces only with the regularity condition $\omega = e^{o(t)}$. The important point here is the interaction between condition (i) and the estimate for \widehat{f} .

Proof. Denote $d = d(x, K)$. Plugging the inequality for $g^{(D)}$ into (39) gives

$$|f^{(D)}(x)| \leq |f(x)| D! \frac{(CD)^{CD}}{d^{2D}} \leq |f(x)| \frac{(CD)^{CD}}{d^{2D}}.$$

In particular, $\omega(t)/t \rightarrow \infty$ shows that $|f(x)| \leq C e^{-\omega(\log 1/d)}$ decreases superpolynomially near K which shows that $f^{(D)}(x) = 0$ for all $x \in K$ and (inductively) for all D and hence $f \in C^\infty([0, 1])$. Further (assume $D > 1$),

$$(40) \quad |f^{(D)}(x)| \leq C \exp(-\omega(\log 1/d) + CD \log D + 2D \log 1/d).$$

For any m sufficiently large, choose now

$$D = \left\lceil 2 \frac{\omega(\frac{1}{4} \log |m|)}{\log |m|} \right\rceil.$$

Note that the condition $\omega(t) = e^{o(t)}$ shows that $D = |m|^{o(1)}$. To estimate the maximum of $f^{(D)}$ in (40), we notice that if $\log 1/d > \frac{1}{4} \log |m|$ then $\omega(\log 1/d) \geq 2D \log 1/d$ (here $\omega(t)/t$ is increasing); hence we may estimate roughly that

$$\max_d -\omega(\log 1/d) + 2D \log 1/d \leq \frac{1}{2}D \log |m|$$

and get

$$\|f^{(D)}\|_\infty \leq C \exp\left(\frac{1}{2}D \log |m| + CD \log D\right) \stackrel{(*)}{=} \exp\left(D \log |m| \left(\frac{1}{2} + o(1)\right)\right)$$

where $(*)$ comes from $D = |m|^{o(1)}$.

We now use the fact that $|\widehat{f}(m)| \leq |m|^{-D} \|f^{(D)}\|_\infty$ to get

$$\begin{aligned} |\widehat{f}(m)| &\leq C \exp\left(-\left(\frac{1}{2} - o(1)\right)D \log |m|\right) \\ &= C \exp\left(-\left(1 - o(1)\right)\omega\left(\frac{1}{4} \log |m|\right) + O(\log |m|)\right). \end{aligned}$$

Remembering that $\omega\left(\frac{1}{4} \log |m|\right) \leq \frac{1}{4}\omega(\log |m|)$ (again, because $\omega(t)/t$ is increasing) and that $\omega(t)/t \rightarrow \infty$ we see that the lemma is proved. \square

We remark that, in some sense, the lemma actually hides two applications of the Legendre transform, $(\mathfrak{L}h)(x) := \max_t h(t) - xt$. Roughly speaking, the norms of $f^{(D)}$ are the Legendre transform of the rate of decrease of g to $-\infty$ (condition (i) of the lemma) and $\widehat{f}(m)$ are the Legendre transform of $f^{(D)}$. Combining both facts allowed us not to use explicitly the notation \mathfrak{L} and to simplify somewhat.

Proof of Lemma 5. Our goal is to use Lemma 8 with the function $g + i\widetilde{g}$, the compact K' and the ω of the lemma being $c\omega$ for some $c > 0$ sufficiently small. The condition (i) on the size of the negative decrease of $\operatorname{Re}(g + i\widetilde{g}) = g$ is easiest to show. Let $x \in K_{n-1} \setminus K_n$. We divide into two cases: if $d(x, K') > e^{-n}$ then we may estimate

$$(41) \quad -\omega(\log 1/d(x, K')) \geq -\omega(n) \stackrel{(29)}{\geq} g_n^-(x) \stackrel{(*)}{=} g(x)$$

where $(*)$ comes from Lemma 3, (ii). If $d(x, K') \leq e^{-n}$ then $\tau_n = e^{-n(\log 2 + o(1))} \geq cd(x, K')^{0.7}$ (24) and

$$\begin{aligned} (42) \quad g(x) &\leq -c\omega(n) \left(\frac{d(x, K')}{\tau_n}\right)^{-1/3} \stackrel{(*)}{\leq} -cd(x, K')^{-0.1} \\ &= -c \exp\left(\frac{1}{10} \log 1/d(x, K')\right) \stackrel{(**)}{\ll} -\omega(\log 1/d(x, K')) \end{aligned}$$

where in $(*)$ we estimated trivially $\omega(n) \geq c$ and in $(**)$ we used the regularity condition $\omega(n) = e^{o(n)}$. Hence we get $g(x) \leq -c\omega(\log 1/d(x, K'))$ for all x , i.e. the condition (i) of Lemma 8.

To estimate $g^{(D)}$ outside K' , start from (27) and get for $x \in K_{n-1} \setminus K_n$ that

$$|g_n^{(D)}(x)| \leq \omega(n) \frac{(CD)^{CD}}{d(x, K')^{D+1/3}}.$$

Since $|I_{n,k}| \leq 2\pi \cdot 2^{-n}$ we get that for every $x \in I_{n,k}$

$$d(x, K')^{2/3} \leq C2^{-2n/3} \ll 1/\omega(n)$$

so that

$$(43) \quad |g^{(D)}(x)| = |g_n^{(D)}(x)| \leq \frac{(CD)^{CD}}{d(x, K')^{D+1}}.$$

Note that (43) holds for g_n and any $x \notin K_n$ (not necessarily in K_{n-1}).

For \tilde{g} we need to examine \tilde{g}_n and take $n \rightarrow \infty$ (remember Lemma 3, (iii)). Let $x \notin K_n \cup K'$, let $\rho = \frac{1}{2}d(x, K')$ and let $I = [-\rho, \rho]$. Now,

$$(44) \quad \tilde{g}_n(x) = \int_{\mathbb{T}} H(t)g_n(x-t) dt = \left(\int_I + \int_{\mathbb{T} \setminus I} \right) H(t)g_n(x-t) dt.$$

In general, the symmetry of H and $|H(t)| \approx \frac{1}{t}$ (12) allows to estimate for any h

$$(45) \quad \left| \int_I h(t)H(t) \right| \leq C|I| \max_I |h'|$$

which we use as follows: For the first integral, D differentiations under the integral sign (which can be justified easily using (45)) show that

$$(46) \quad \left| \left(\frac{d}{dx} \right)^D \int_I H(t)g_n(x-t) \right| = \left| \int_I H(t) (g_n)^{(D)}(x-t) \right| \stackrel{(45)}{\leq} C\rho \max_I |(g_n)^{(D+1)}| \leq \stackrel{(43)}{\leq} \frac{(CD)^{CD}}{\rho^{D+1}}.$$

For the second half of (44) we consider g_n and H as periodic functions and change variables to get $\int_{x+\rho}^{x+2\pi-\rho} H(x-t)g_n(t)$. This we differentiate D times under the integral sign and shift back, and we get

$$(47) \quad \left(\frac{d}{dx} \right)^D \int_{\mathbb{T} \setminus I} H(t)g_n(x-t) = \sum_{i=0}^{D-1} H^{(i)}(t) (g_n)^{(D-1-i)}(x-t) \Big|_{-\rho}^{\rho} + \int_{\mathbb{T} \setminus I} H^{(D)}(t)g_n(x-t) dt$$

where as usual $g|_a^b$ stands for $g(b) - g(a)$. Denote $a(s) = \int_{\rho}^s g_n(x-t) dt$, and remember that (36) gives that $|a| \leq C$. Hence when we integrate by parts the

integral on the right hand side of (47) gives

$$\begin{aligned} \left| \int_{\mathbb{T} \setminus I} H^{(D)}(t) g_n(x-t) dt \right| &\leq \left| H^{(D)}(-\rho) a(2\pi - \rho) \right| + \left| \int_{\rho}^{2\pi - \rho} H^{(D+1)}(t) a(t) dt \right| \\ &\stackrel{(36)}{\leq} C \left| H^{(D)}(-\rho) \right| + C \int_{\rho}^{2\pi - \rho} \left| H^{(D+1)}(t) \right| dt \\ &\stackrel{(12)}{\leq} \frac{(CD)^{CD}}{\rho^{D+1}}. \end{aligned}$$

Similarly we can use (43) and (12) to estimate the sum in (47) and this gives

$$(48) \quad \left| \left(\frac{d}{dx} \right)^D \int_{\mathbb{T} \setminus I} H(t) g_n(x-t) dt \right| \leq \frac{(CD)^{CD}}{\rho^{D+1}}.$$

Together with (46) and (44) this gives

$$(49) \quad \left| \tilde{g}_n^{(D)}(x) \right| \leq \frac{(CD)^{CD}}{\rho^{D+1}} \quad \forall x \notin K_n \cup K'.$$

Any $x \notin K'$ is also not in some K_m and hence (49) holds for any $n > m$ and hence it holds for \tilde{g} . With (43) and (42) we can use Lemma 8 and get (38) which proves Lemma 5. \square

3.8. *The Taylor coefficients of F .* In this section we prove Lemma 6, namely show that with probability 1, $\widehat{F}(m) \rightarrow 0$ as $m \rightarrow \infty$. First we define F_n to be the harmonic extension of $f_n = e^{g_n + i\tilde{g}_n}$ to \mathbb{D} (each f_n is bounded) and we want to find some n such that $\widehat{f}_n(m) = \widehat{F}_n(m)$ approximates $\widehat{F}(m)$. Summing (34) and (37) over n we get

$$\left| (G_n + i\tilde{G}_n)(z) - (G + i\tilde{G})(z) \right| \leq \frac{C}{(1 - |z|)2^n}.$$

Fix, therefore, $n = n(m) := \lceil C \log m \rceil$ for some C sufficiently large, and get, for every z with $|z| = 1 - \frac{1}{m}$ that $\left| (G_n + i\tilde{G}_n)(z) - (G + i\tilde{G})(z) \right| \leq 1/m$. Further, we have that

$$(50) \quad \max |F_n| \stackrel{(32)}{\leq} e^{o(n)} \leq m^{o(1)}$$

which means that, for $|z| = 1 - \frac{1}{m}$,

$$\left| F_n(z) - F(z) \right| \leq |F_n(z)| \left| 1 - \exp((G_n + i\tilde{G}_n)(z) - (G + i\tilde{G})(z)) \right| \leq C m^{-1/2}.$$

Finally we use

$$\widehat{F}(m) = \frac{1}{m!} F^{(m)}(0) = \int_{|z|=1-1/m} z^{-m-1} F(z) dz$$

so that

$$(51) \quad |\widehat{F}_n(m) - \widehat{F}(m)| = \left| \int_{|z|=1-1/m} z^{-m-1} (F_n(z) - F(z)) dz \right| \leq Cm^{-1/2}$$

and we see that it is enough to calculate $\widehat{f}_n(m)$.

3.9. *Probability.* At this point we use the fact that the $s(n, k)$ are random. Take them to be independent and uniformly distributed on $[0, 1]$. We shall perform a (probabilistic) estimate of $\widehat{f}_n(m)$ by moment methods. Unfortunately, it seems we need the fourth moment. We start with a lemma that contains the calculation we need without referring to analytic functions

For $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ we denote $i \subset j$ if $i = j$ or $i = 4$ and $j = 3$. The inverse will be denoted by $i \not\subset j$.

LEMMA 9. *Let I_i be 4 intervals and let $\tau, \alpha, \beta > 0$ be some numbers. Let h_1, h_2, h_3 be functions satisfying*

$$(52) \quad \int_{I_i} |h_j| \leq \alpha, \quad i \subset j,$$

$$(53) \quad |h_j(x)| = 1, |h'_j(x)| \leq \beta, |h''_j(x)| \leq \beta^2 \quad \forall x \in I_i + [-\tau, \tau], \quad i \not\subset j,$$

where “+” stands for regular set addition. Let t_1 and t_2 be two random variables, uniformly distributed on $[0, \tau]$, and let $t_3 = t_4 = 0$. Define

$$(54) \quad f(x) = f_{t_1, t_2}(x) := h_1(x - t_1)h_2(x - t_2)h_3(x).$$

Then

$$(55) \quad E := \left| \mathbb{E} \left(\prod_{i=1}^4 \int_{I_i+t_i} f(x_i) e^{-imx_i} dx_i \right) \right| \leq C \frac{\alpha^4}{m^2} \left(\max \beta, \frac{1}{\tau} \right)^2.$$

Proof. Denote $\beta' = \max \beta, \frac{1}{\tau}$. Define $S_i = \int_{I_i+t_i} f(x_i) e^{-imx_i} dx_i$. Translate S_1 and S_2 to get

$$S_1 = \int_{I_1} h_1(x_1)h_2(x_1 + t_1 - t_2)h_3(x_1 + t_1) e^{-im(x_1+t_1)} dx_1,$$

$$S_2 = \int_{I_2} h_1(x_2 + t_2 - t_1)h_2(x_2)h_3(x_2 + t_2) e^{-im(x_2+t_2)} dx_2.$$

Changing the order of integration we get

$$(56) \quad E = \left| \int_{I_1} h_1(x_1) e^{-imx_1} \cdots \int_{I_3} h_3(x_3) e^{-imx_3} \int_{I_4} h_3(x_4) e^{-imx_4} A dx_1 \cdots dx_4 \right|$$

where A , the central element, is defined by

$$A = \frac{1}{\tau^2} \int_0^\tau \int_0^\tau e^{-im(t_1+t_2)} A(t_1, t_2) dt_1 dt_2$$

and where

$$A(t_1, t_2) := h_2(x_1 + t_1 - t_2)h_3(x_1 + t_1)h_1(x_2 + t_2 - t_1)h_3(x_2 + t_2) \\ h_1(x_3 - t_1)h_2(x_3 - t_2)h_1(x_4 - t_1)h_2(x_4 - t_2).$$

The lemma will be proved once, we estimate A , which will be done by integrating by parts over t_1 and then over t_2 . We notice that A contains only expressions of the type $h_j(x)$ where $x \in I_i + [-\tau, \tau]$ and $i \not\subset j$. Therefore, using (53) we get

$$|A(t_1, t_2)| = 1, \quad \left| \frac{\partial A(t_1, t_2)}{\partial t_i} \right| \leq 5\beta, \quad \left| \frac{\partial^2 A(t_1, t_2)}{\partial t_1 \partial t_2} \right| \leq 25\beta^2.$$

Integrating by parts once, we get

$$\left| \int_0^\tau A(t_1, t_2) e^{-imt_1} dt_1 \right| \leq \frac{2}{|m|} + \frac{5\tau\beta}{|m|} \leq \frac{7\tau\beta'}{|m|}.$$

Further,

$$\left| \frac{\partial}{\partial t_2} \int_0^\tau A(t_1, t_2) e^{-imt_1} dt_1 \right| = \left| \int_0^\tau \frac{\partial A(t_1, t_2)}{\partial t_2} e^{-imt_1} dt_1 \right| \\ = \left| \frac{\partial A(t_1, t_2)}{\partial t_2} \frac{e^{-imt_1}}{-im} \Big|_0^\tau - \int_0^\tau \frac{\partial^2 A(t_1, t_2)}{\partial t_2 \partial t_1} \frac{e^{-imt_1}}{-im} dt_1 \right| \\ \leq \frac{10\beta}{|m|} + \frac{25\tau\beta^2}{|m|} \leq \frac{35\tau\beta'^2}{|m|}.$$

These two statements allow us to perform integration by parts over t_2 getting

$$|A| = \frac{1}{\tau^2} \left| \int_0^\tau \int_0^\tau A(t_1, t_2) e^{-im(t_1+t_2)} dt_1 dt_2 \right| \leq \frac{1}{\tau^2} \left(\frac{14\tau\beta'}{m^2} + \frac{35\tau^2\beta'^2}{m^2} \right) \leq \frac{49\beta'^2}{m^2}.$$

Plugging this into (56) and integrating using (52) we conclude the proof of the lemma. □

Continuing the proof of the theorem, for every $0 \leq k < 2^n$ denote

$$\mathcal{I}_k = \int_{I(n,k)} f_n(x) e^{-imx} dx.$$

We note that (32) shows that

$$(57) \quad |\mathcal{I}_k| \leq \int_{I(n,k)} |f_n(x)| \leq \sigma_n e^{o(n)} =: \gamma.$$

In other words, $\gamma = \gamma(n)$ is a bound for $|\mathcal{I}_k|$ independent of k satisfying

$$(58) \quad \gamma = \sigma_n e^{o(n)} = 2^{-n} m^{o(1)}$$

(for the last equality, remember that $\sigma_n = 2^{-n} \Phi(n)$, $\Phi(n) = n^{-o(1)}$ (20) and $n \approx \log m$).

LEMMA 10. *Let $0 \leq k_1, k_2, k_3, k_4 < 2^n$ and let $1 \leq r < n$, and assume that the $I(n, k_i)$ belong to at least three different intervals of rank r . Then*

$$\mathbb{E}(\mathcal{I}_{k_1}\mathcal{I}_{k_2}\mathcal{I}_{k_3}\mathcal{I}_{k_4}) \leq \gamma^4 \frac{C\omega(n)^2}{m^2\tau_r^3}.$$

Proof. Define q_1, \dots, q_4 using $I(n, k_i) \subset I(r, q_i)$. We may assume without loss of generality that the two q_i -s which may be equal are q_3 and q_4 . Let \mathcal{X} be the σ -field spanning all s -es *except* $s(r, q_1)$ and $s(r, q_2)$. We shall show

$$\mathbb{E}(\mathcal{I}_{k_1}\mathcal{I}_{k_2}\mathcal{I}_{k_3}\mathcal{I}_{k_4}|\mathcal{X}) \leq \gamma^4 \frac{C\omega(n)^2}{m^2\tau_r^3}$$

and then integrating over \mathcal{X} will give the result. We note that conditioning by \mathcal{X} is in effect fixing everything except the positions of $I(r, q_1)$ and $I(r, q_2)$ inside $I(r-1, \lfloor q_i/2 \rfloor)$. To be more precise, two copies of l also move with $I(r, q_i)$. Therefore define $J_j := I(r, q_j) + [-\tau_r, \tau_r]$ ($j = 1, 2$), which is the part of f_n that moves when $s(r, q_j)$ changes (there are zones where $f_n \equiv -\mu(r)$ which expand and contract on the sides of J_j) and denote $J_3 = \mathbb{T} \setminus (J_1 \cup J_2)$. Assume for a moment that $s(r, q_1) = s(r, q_2) = 0$ and define, using this assumption,

$$(59) \quad \begin{aligned} \eta_j &:= (g_n + \omega(r))|_{J_j}, \quad j = 1, 2, 3, & h_j &:= e^{\eta_j + i\tilde{\eta}_j}, \\ I_i &:= I(n, k_i), & i &= 1, 2, 3, 4. \end{aligned}$$

Under the assumption $s(r, q_1) = s(r, q_2) = 0$ we clearly have $f_n = h_1 h_2 h_3 e^{-\omega(r)}$ and when we remove this assumption, the only change is a translation of h_1 and h_2 . In other words, if we define $t_i = s(r, q_i)\tau_r$ then

$$f_n(x) = h_1(x - t_1)h_2(x - t_2)h_3(x)e^{-\omega(r)}.$$

Examining (54) we see that $|\mathbb{E}(\mathcal{I}_{k_1}\mathcal{I}_{k_2}\mathcal{I}_{k_3}\mathcal{I}_{k_4}|\mathcal{X})| = e^{-4\omega(r)}E$ where E is defined by (55); where the I_i of (55) are the same as those of (59); and where the τ of (55) is τ_r . To make (55) concrete we need to specify values for the α and β of (52) and (53) and prove that they hold. We define

$$\alpha = \gamma e^{\omega(r)}, \quad \beta = C \frac{\omega(n)}{\tau_r^{3/2}}.$$

Notice that β is obviously larger than $1/\tau_r$. With all these, Lemma 10 would follow from Lemma 9 once we show (52) and (53). (52) is clear from the definitions of α above, γ (57) and h_i (59), so we need only show (53).

Examining the definitions of η_j and $I(n, k)$ we see easily that $\eta_j(x) = 0$ for $x \in I_i + [-2\tau_r, 2\tau_r]$ whenever $i \not\subset j$ (we defined l^\pm (26) with a slightly larger “space” so that this fact would be true). This immediately shows $|h_i(x)| = 1$. Further, $h'_i = h_i(\eta'_i + i\tilde{\eta}'_i)$ gives $|h'_i| = |\tilde{\eta}'_i|$ and $h''_i = h_i((\eta'_i + i\tilde{\eta}'_i)^2 + \eta''_i + i\tilde{\eta}''_i)$ gives $|h''_i| \leq |\tilde{\eta}'_i|^2 + |\tilde{\eta}''_i|$. As in (47), the derivatives of $\tilde{\eta}_i$ have the representations

$$\tilde{\eta}'_i(x) = \int_{\mathbb{T}} \eta_i(x-t)H'(t) dt, \quad \tilde{\eta}''_i(x) = \int_{\mathbb{T}} \eta_i(x-t)H''(t) dt$$

where H is the Hilbert kernel. In general there are boundary terms (as in the calculation in (47)), but, as remarked, in our case η_i is zero in $[x - \tau_r, x + \tau_r]$ (when $x \in I_j + [-\tau_r, \tau_r]$) so these terms disappear. We may therefore estimate

$$(60) \quad \tilde{\eta}'_i(x) \leq \|\eta\|_2 \|H'|_{\mathbb{T} \setminus [-\tau_r, \tau_r]}\|_2, \quad \tilde{\eta}''_i(x) \leq \|\eta\|_2 \|H''|_{\mathbb{T} \setminus [-\tau_r, \tau_r]}\|_2.$$

The second terms are a straightforward calculation from (12) and we get

$$\|H'|_{[-\tau_r, \tau_r]^c}\|_2 \approx \tau_r^{-3/2}, \quad \|H''|_{[-\tau_r, \tau_r]^c}\|_2 \approx \tau_r^{-5/2}.$$

The first terms can be estimated easily: since the singularities in η (which originally came from the $x^{-1/3}$ factor in l) are in L^2 . Indeed, it is for this point that we defined l using $x^{-1/3}$. We easily get

$$(61) \quad \|\eta\|_2 \leq C\omega(n)$$

which gives us the estimate we need:

$$|h'_i| \leq C\tau_r^{-3/2}\omega(n), \quad |h''_i| \leq C\tau_r^{-3}\omega^2(n).$$

With this the conditions of Lemma 9 are fulfilled and we are done. □

Proof of Lemma 6. Define

$$X = X_m = \sum_{k=0}^{2^n-1} \int_{I(n,k)} f_n(x) e^{-imx} dx.$$

Now the difference $\widehat{f}_n(m) - X$ is exactly $f_n \widehat{\mathbf{1}_{\mathbb{T} \setminus K_n}}(m)$. The functions $f_n \mathbf{1}_{\mathbb{T} \setminus K_n}$ are uniformly C^1 so this difference converges to zero. To see this last claim, note that (43) and (49) show that $g'_n + i(\tilde{g}_n)'$ has a bound of $C/d(x, K_n)^2$ while (41) and (42) show that $f_n \leq Cd(x, K_n)^{10}$ (actually f_n converges to zero near K_n superpolynomially uniformly).

Therefore we want to bound X , and we shall estimate $\mathbb{E}X^4$. Let

$$E(k_1, k_2, k_3, k_4) := \mathbb{E} \prod \mathcal{I}_{k_i};$$

let $r(k_1, \dots, k_4)$ be the minimal r such that the $I(n, k_i)$ -s are contained in at least 3 distinct intervals of rank r . A simple calculation shows

$$\#\{(k_1, \dots, k_4) : r(k_1, \dots, k_4) = r\} \approx 2^{4n-2r}.$$

If τ_r is too small then the estimate of the lemma is useless and it would be better to estimate $|E(k_1, \dots, k_4)| \leq \gamma^4$. Let R be some number. Then for large r we have the estimate

$$(62) \quad E_1 := \sum_{r(k_1, \dots, k_4) \geq R} E(k_1, \dots, k_4) \leq C\gamma^4 2^{4n-2R} \stackrel{(58)}{=} m^{o(1)} 2^{-2R}.$$

For small r we use the lemma to get a better estimate. Examine one such k_1, \dots, k_4 and let $r = r(k_1, \dots, k_4)$. Lemma 10 gives

$$E(k_1, \dots, k_4) \leq \gamma^4 \frac{C\omega^2(n)}{m^2\tau_r^3} \stackrel{(*)}{=} \frac{\gamma^4}{m^{2-o(1)}\tau_r^3}$$

where $(*)$ comes from the regularity condition $\omega(n) = e^{o(n)}$ and $n \approx \log m$. Therefore

$$(63) \quad E_2 := \sum_{r(k_1, \dots, k_4) < R} E(k_1, \dots, k_4) \leq \gamma^4 2^{4n} m^{-2+o(1)} \sum_{r=1}^R 2^{-2r} \tau_r^{-3}$$

$$\stackrel{(24,58)}{=} m^{-2+o(1)} \sum_{r=1}^R 2^{r+o(r)} = m^{-2+o(1)} 2^{R+o(R)}.$$

Taking $R = \lfloor \frac{2}{3} \log_2 m \rfloor$ and summing (62) and (63) we get

$$(64) \quad \mathbb{E}X^4 \leq m^{-4/3+o(1)}.$$

This gives that

$$\mathbb{E} \sum_m X_m^4 \leq \sum_m \mathbb{E}X_m^4 < \infty.$$

In particular, with probability 1, $X_m \rightarrow 0$. As remarked above, this shows that $\widehat{f}_n(m) \rightarrow 0$ and hence $\widehat{F}(m) \rightarrow 0$ which concludes Lemma 6 and the theorem. \square

3.10. *Remarks.* 1. It is clear that if $f \in \text{PLA}$ then the associated analytic function F defined by (2), (3) has the estimate

$$(65) \quad F(z) = o\left(\frac{1}{1-|z|}\right)$$

simply because $\widehat{F}(n) \rightarrow 0$. It turns out that in some vague sense, this inequality is the “calculatory essence” of $\text{PLA} \setminus H^2$. In other words, if you have a singular distribution whose analytic part F satisfies (65) and its boundary value is in L^2 then you are already quite close to constructing a nonclassic PLA function. Note that (65) is enough to prove uniqueness (see Theorem 2' below) and the additional information $\widehat{F}(n) \rightarrow 0$ does not help. This ideology also stands behind the proof above. To understand why, let K be a nonprobabilistic Cantor set with the same thickness; namely at step n the total length of the 2^n intervals is $\Phi(n)$. Let G be a harmonic function constructed similarly; i.e. “hang” copies of $-x^{-1/3}$ suitably dilated and shifted from all intervals contiguous to K and define $F = e^{G+i\tilde{G}}$. Then a much simpler calculation shows that F satisfies (65), and even the stronger

$$(66) \quad F(z) = \left(\frac{1}{1-|z|}\right)^{o(1)}.$$

This implies

$$\sum_{k=2^n}^{2^{n+1}} |\widehat{F}(k)|^2 = o(2^n)$$

so that “in average” the coefficients tend to zero, with no need for probability in the construction. Thus the probabilistic skewing introduced above “smears” the spectrum of F uniformly and allows us to conclude the stronger $\widehat{F}(m) \rightarrow 0$.

Question. Is the nonprobabilistic construction (e.g. taking all $s(n, k)$ to be 0) in PLA?

The use of stochastic perturbations of the time domain to smooth singularities in the spectrum is not new. One may find examples of such techniques in [K85], notably the use of Brownian images in Chapter 17, and in [KO98]. A reader fluent in these techniques would probably assume it is possible to simplify the proof of Lemma 6 significantly along the following rough lines: find some event \mathcal{X} that would separate \mathcal{I}_{k_1} from the rest of the \mathcal{I}_{k_i} ’s making them independent, perhaps similar to the \mathcal{X} actually used. Now calculate $\mathbb{E}(\mathcal{I}_k | \mathcal{X})$ using one simple integration by parts. Next multiply $\mathbb{E}(\mathcal{I}_{k_i} | \mathcal{X})$ out, and integrate over \mathcal{X} . Unfortunately, it seems that no proof can be constructed this way. The problem is that, while g has a local structure and would be amenable to such a handling, \tilde{g} does not, and any change to one $s(n, k)$ affects the values of \tilde{g} globally.

2. The regularity condition $\omega(n) = e^{o(\log n)}$ can be relaxed somewhat, but it is not clear whether it can be removed altogether. For example, there is an inherent difficulty in generalizing Lemma 8 without this condition.

3. It is also of interest to ask how fast does $\widehat{F}(m) \rightarrow 0$, or in other words how much do we pay in $\{c(n)\}_{n>0}$ for the quick decrease of the $\{c(n)\}_{n<0}$. Using Chebyshev’s inequality with (64) it is easy to see that $|\widehat{F}(m)| \leq m^{-1/12+o(1)}$. This, however, can be improved significantly. Indeed, one may change the definition of l , (25) to have a singularity of type $x^{-\varepsilon}$ and then replace (60) with an $L^p - L^q$ estimate and get $\tau_r^{2+\varepsilon}$ in the formulation of Lemma 10, which would end up as $X_m \leq m^{-1/4+\varepsilon}$ almost surely. Further, it is possible to use higher moment estimates. To estimate the $2k^{\text{th}}$ moment, use a generalized version of Lemmas 9 and 10 for k moving intervals and k stationary ones to get an estimate of $m^{-k} \tau_r^{k+\varepsilon}$ and the final outcome would be $X_m \leq m^{-1/2+1/2k+\varepsilon}$ almost surely. Thus in effect we may construct a function F satisfying $\widehat{F} \in l^{2+\varepsilon}$ for all $\varepsilon > 0$, almost surely.

4. It is possible to characterize precisely the size of exceptional sets for the “nonclassic” part of $\text{PLA} \cap L^2$. Namely, denote by Λ the (generalized) Hausdorff measure generated by the function $t \mapsto t \log 1/t$. Then the following is true

THEOREM. (i) *There exists a function $f \in L^2 \setminus H^2$ which admits a decomposition (3) converging everywhere outside of some compact K of finite Λ -measure.*

(ii) *The result fails if one replaces the condition $\Lambda(K) < \infty$ with $\Lambda(K) = 0$ even if K is not required to be compact.*

Part (i) can be proved by the construction of the section, with a nonnegligible simplification since we do not watch for smoothness. Part (ii) follows easily from Phragmén-Lindelöf-like theorems for analytic functions of slow growth in \mathbb{D} . See [B92], [D77].

Note that in the symmetric settings, the corresponding exceptional sets (so called M -sets) could have dimension zero [B64], [KS94], [KL87] and moreover, may be “thin” with respect to any (generalized) Hausdorff measure [I68].

5. We also have some structural results about the set $\text{PLA} \cap C$. Namely:

THEOREM. *$\text{PLA} \cap C$ is the first Baire category and has zero Wiener measure.*

THEOREM. *PLA is dense in C (in sharp contrast to H^2).*

We intend to publish proofs of these three results elsewhere.¹

4. Uniqueness

4.1. The most natural settings for the statement of the uniqueness result is that of boundary behavior of analytic functions. Let us therefore restate Theorem 2 in a stronger form

THEOREM 2'. *Let F be an analytic function on \mathbb{D} satisfying*

$$(67) \quad F(z) = O\left(\frac{1}{(1-|z|)^M}\right) \quad \text{for some } M.$$

Assume that F has nontangential boundary limit almost everywhere and that

$$(68) \quad F(\zeta) = f(\zeta) := \sum_{n=-\infty}^{-1} c(n)\zeta^n \quad \text{a.e. on } \partial\mathbb{D}$$

and assume the $c(n)$ satisfy (8) with some $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\omega(t)/t$ increasing and $\sum \frac{1}{\omega(n)} < \infty$. Then F and f are identically zero.

To see that Theorem 2' generalizes Theorem 2 define $F(z)$ by (2) and note that (65) is stronger than (67). And as usual, Abel's theorem shows that F has nontangential boundary limit a.e.

In this section the notation C and c will be allowed to depend on the function F — here we consider F as given and fixed. By \mathbf{m} we denote the arc

¹The last two results are to appear at *Bull. London Math. Soc.*

length on the circle, normalized so that $\mathbf{m}\partial\mathbb{D} = 1$. For $\theta \in \partial\mathbb{D}$ we denote by $I(\theta, \varepsilon)$ an arc centered around θ with $\mathbf{m}I(\theta, \varepsilon) = \varepsilon$.

For a compact subset E of $\partial\mathbb{D}$ we shall define the *Privalov domain* over E , $\mathcal{P}(E)$, to be a subset of \mathbb{D} created by removing, for every arc I from the complement of E , a disk D_I orthogonal to \mathbb{D} at the end points of I . If I is larger than a half circle, remove $\mathbb{D} \setminus D_I$ instead of D_I so that in both cases you remove the component containing I (this definition is slightly different from the standard one). The following is well known:

LEMMA 11. *Let F be an analytic function on \mathbb{D} with almost everywhere nontangential boundary values, and let $\delta > 0$. Then there exists a compact set $E \subset \partial\mathbb{D}$ with $\mathbf{m}E > 1 - \delta$ such that F is continuous on $\mathcal{P}(E)$.*

4.2. The following lemma is simple but plays a crucial part in the proof.

LEMMA 12. *Let L be a function on some measure space with a probability measure μ with $A \leq L \leq B$. Assume for some $\varepsilon \in [0, 1]$,*

$$(69) \quad \int L d\mu = \varepsilon B + (1 - \varepsilon)A.$$

Let $D \in]A, B[$. Then

$$(70) \quad \int \max\{L, D\} d\mu \leq \varepsilon B + (1 - \varepsilon)D.$$

Proof. By considering $L - D$, we may assume without loss of generality that $D = 0$. Assume by contradiction that

$$\int L^+ > \varepsilon B.$$

This shows that the support of the positive part of L has measure $> \varepsilon$ so that the support of the negative part must have measure $< 1 - \varepsilon$ and therefore $\int L^- > A(1 - \varepsilon)$, a contradiction to (69). \square

LEMMA 13. *Let E be compact in $\partial\mathbb{D}$ and let $\varepsilon > 0$. Let $z \in \mathcal{P}(E)$ with $|z| > 1 - \varepsilon$ and define $\zeta := z/|z|$ and $l := (1 - |z|)/\varepsilon$. Assume that*

$$(71) \quad \mathbf{m}(I(\zeta, l) \setminus E) \leq \varepsilon^2 l.$$

Let \mathcal{D} be the component of $\mathcal{P}(E) \setminus (1 - l)\mathbb{D}$ containing z . Then the harmonic measure of E has the estimate

$$(72) \quad \Omega(z, \mathcal{D})(E) > 1 - C_1\varepsilon.$$

Here C_1 is an absolute constant. Similarly all constants in the proof of the lemma, both explicit and implicit in \approx notations are absolute.

Proof. Without loss of generality, assume $\varepsilon < \frac{1}{4}$. For any arc I which is a component of $\partial\mathbb{D} \setminus E$ let D_I be the Privalov disk as in the definition of a Privalov domain. Let B be a Brownian motion starting from z . Let T be the stopping time of B on $\partial\mathbb{D}$ and T_I be the stopping time of B on D_I .

We start with an estimate of $p := \Omega(z, \mathbb{D} \setminus D_I)(\partial D_I)$. We need to consider two cases:

- (i) If I is “far” from z we use $p \leq C(1 - |z|)/d(z, I)$.
- (ii) If I is “close” to z we use $p \leq C\mathbf{m}I/(1 - |z|)$.

Both follow from the conformal invariance of the harmonic measure [B95, Theorem V.1.2] which gives an explicit formula for $\Omega(z, \mathbb{D} \setminus D_I)$ and (i) and (ii) with a simple calculation.

Let J be $\mathbb{T} \setminus I(\zeta, l/2)$, and let T_J be the hitting time of J . Then (i) shows that $\Omega(z, \mathbb{D} \setminus D_J)(\partial D_J) \leq C\varepsilon$ and hence

$$\mathbb{P}(T_J < T) \leq C\varepsilon.$$

For any $I \subset J$ we have $D_I \subset D_J$ so that $T_I > T_J$. Now,

$$(73) \quad \mathbb{P}\left(\inf_{I \subset J} T_I < T\right) \leq C\varepsilon.$$

Next, if $I \subset I(\zeta, l)$ we use (ii) above and (71) to get

$$(74) \quad \mathbb{P}\left(\inf_{I \subset I(\zeta, l)} T_I < T\right) \leq \sum_{I \subset I(\zeta, l)} \mathbb{P}(T_I < T) \stackrel{(ii)}{\leq} C \sum_{I \subset I(\zeta, l)} \frac{\mathbf{m}I}{1 - |z|} \stackrel{(71)}{\leq} C\varepsilon.$$

The assumption $\varepsilon \leq \frac{1}{4}$ together with (71) shows that any arc I in the complement of E is either a subset of $I(\zeta, l)$ or of J . Hence the lemma is almost finished. We still have to deal with $\{|z| = 1 - l\}$, which is easy: define T^* to be the stopping time of B on the circle $\{|z| = 1 - l\}$. Let \mathcal{A} be the annulus $\{1 - l \leq |z| \leq 1\}$. The harmonic measure $\Omega(z, \mathcal{A})(\{|z| = 1 - l\})$ can be calculated explicitly from the fact that the solution h of Dirichlet’s problem on \mathcal{A} with boundary conditions 1 on $\{|z| = 1 - l\}$ and 0 on $\{|z| = 1\}$ has the explicit form

$$(75) \quad h(w) = \frac{\log |w|}{\log(1 - l)}.$$

Since $\Omega(z, \mathcal{A})(\{|z| = 1 - l\}) \stackrel{(14)}{=} h(z) \leq C\varepsilon$,

$$\mathbb{P}(T^* < T) \leq C\varepsilon$$

and this together with (73) and (74) gives

$$\mathbb{P}\left(\min\left\{\inf_I T_I, T^*\right\} < T\right) \leq C\varepsilon.$$

This is equivalent to (72) and the lemma is proved. □

4.3. *Proof of Theorem 2'.* Without loss of generality we may assume the log in (8) is to base 2. Also, we may assume without loss of generality that $\omega(n) \leq n^2$, since otherwise just take $\omega'(n) := \min\{\omega(n), n^2\}$. For $n \geq 0$ denote the Taylor coefficients of F by $c(n)$:

$$c(n) := \widehat{F}(n), \quad n \geq 0$$

(remember that for $n < 0$, $c(n)$ are defined by (68)).

Next, define for $k \in \mathbb{N}$,

$$f_k(z) := \sum_{n=-2^k}^{\infty} c(n)z^n, \quad z \in \mathbb{D}, \quad l_k(z) := \log |f_k(z)|, \quad r_k := 1 - 4^{-k}.$$

A simple calculation shows that

$$(76) \quad |f_k(z) - F(z)| \leq C \quad \forall |z| \geq 1 - 2^{-k}.$$

Hence, from (67) we get

$$(77) \quad F(z) = O\left(\frac{1}{(1-|z|)^M}\right) \Rightarrow f_k = O\left(\frac{1}{(1-|z|)^M}\right) \text{ uniformly in } k \\ \Rightarrow l_k(z) \leq C_2 k \quad \forall z, 1 - 2^{-k} \leq |z| \leq 1 - 8^{-k}.$$

Define $A_k := -\omega(k)/2$.

LEMMA 14. *For every $\delta > 0$ there exists a $K = K(\delta)$ such that*

$$\mathbf{m}\{\theta : l_K(\theta r_K) > A_K\} < \delta.$$

Proof. Use Lemma 11 to find a compact $S \subset \partial\mathbb{D}$ of measure $> 1 - \delta/2$ satisfying the fact that F is continuous on $\mathcal{P}(S)$. Let $S' \subset S$ be a compact set of measure $> 1 - \delta$ such that

$$(78) \quad \lim_{\eta \rightarrow 0} \frac{\mathbf{m}(S \cap I(\theta, \eta))}{\eta} = 1 \text{ uniformly in } \theta \in S'.$$

Our purpose is to use Lemma 13 with $\varepsilon = 1/4C_1$. Therefore, define η_0 by the condition

$$\frac{\mathbf{m}(S \cap I(\theta, \eta))}{\eta} \geq 1 - \varepsilon^2 \quad \forall \theta \in S', \forall \eta < \eta_0.$$

For any k and any $z = \theta r_k$, $\theta \in S'$, let $\mathcal{A} = \mathcal{A}(k, \theta)$ be the component of the set $\mathcal{P}(S) \setminus (1 - 4^{-k}/\varepsilon)\mathbb{D}$ containing z . Examine the harmonic measure Ω of \mathcal{A} . Lemma 13 gives, for all k sufficiently large such that $4^{-k} < \eta_0\varepsilon$,

$$(79) \quad \Omega(z, \mathcal{A})(S \cap \partial\mathcal{A}) \geq \frac{3}{4}.$$

Now, if k is sufficiently large so as to satisfy in addition that $4^{-k}/\varepsilon < 2^{-k}$ then we can use (76) and the continuity of F on $\mathcal{P}(S)$ to get

$$(80) \quad l_k(z) \leq C \quad \forall k, \forall \theta \in S', \forall z \in \partial\mathcal{A}(k, \theta).$$

(80) will be used on $\partial\mathcal{A} \setminus S$. On $\partial\mathcal{A} \cap S$ we write

(81)

$$|f_k(\theta)| \leq \sum_{n < -2^k} |c(n)| \leq C \sum_{n < -2^k} e^{-\omega(\log|n|)} \leq C \sum_{j=k}^{\infty} 2^j e^{-\omega(j)} \leq C e^{-(1-o(1))\omega(k)}$$

(in the last inequality we used the fact that $\omega(n)/n$ is increasing to infinity, due to (7)). Hence if k is sufficiently large we get

$$(82) \quad l_k(\theta) \leq -\frac{3}{4}\omega(k) + C \quad \forall \theta \in S.$$

Now l is the logarithm of an analytic function and is therefore subharmonic. This allows us to use (15) and from (79), (80) and (82) we get

$$l_k(z) \leq -\frac{9}{16}\omega(k) + C.$$

With K sufficiently large to satisfy all requirements so far, as well as $\omega(K)/16 > C$, the lemma is proved. \square

4.4. Continuing the proof of the theorem, let $\delta > 0$ be some arbitrary number, and let $z_0 \in \mathbb{D}$ satisfy $|z_0| = 1 - \sqrt{\delta}$. Let $P_k := \Omega(z_0, r_k\mathbb{D})$ (this is just the Poisson kernel with appropriate parameters). Naturally, we assume $z_0 \in r_k\mathbb{D}$, so that everything below holds for $k > \log_4 1/(1 - |z_0|)$. Our purpose is to show that the integrals

$$\int l_k(z) dP_k(z)$$

increase only in a precisely controlled manner. It turns out that this is difficult to do directly, and we need to “regularize” before doing so. Define therefore

$$\mathcal{I}_k := \int [l_k(z)]_{A_k} dP_k(z)$$

where $[f]_M := \max(f, M)$. The proof will revolve around a comparison of \mathcal{I}_{k+1} and \mathcal{I}_k . Since the calculation is long, we shall perform it in two stages, first moving the circle of integration inward but keeping the f_k and only then changing f_k .

LEMMA 15. *With the notation above,*

$$\int [l_{k+1}(z)]_{A_{k+1}} dP_k(z) \leq \mathcal{I}_{k+1} + Ck^2 2^{-k}.$$

Proof. We wish to compare the harmonic measure on $r_{k+1}\mathbb{D}$ to the one on the annulus $\mathcal{A} := \{1 - 2^{-k-1} \leq |z| \leq r_{k+1}\}$. The probability of a Brownian motion starting from z , $|z| = r_k$ to hit $\{|z| = 1 - 2^{-k-1}\}$ before $\{|z| = r_{k+1}\}$

is $\leq C2^{-k}$ (this follows from the explicit formula for the solution of Dirichlet's problem in an annulus (75)), and therefore

$$(83) \quad \|\Omega(z, \mathcal{A}) - \Omega(z, r_{k+1}\mathbb{D})\| \leq C2^{-k} \quad \forall |z| = r_k$$

where the norm is the usual norm in the space of measures on \mathbb{D} . We use this with the subharmonic function $\varphi(z) := [l_{k+1}(z)]_{A_{k+1}}$ and get

$$(84) \quad \begin{aligned} \varphi(z) &\stackrel{(15)}{\leq} \int \varphi d\Omega(z, \mathcal{A}) \stackrel{(83)}{\leq} \int \varphi d\Omega(z, r_{k+1}\mathbb{D}) + C2^{-k} \max_{\partial\mathcal{A}} |\varphi| \\ &\stackrel{(77)}{\leq} \int \varphi d\Omega(z, r_{k+1}\mathbb{D}) + C2^{-k}(k + |A_{k+1}|) \\ &\leq \int \varphi d\Omega(z, r_{k+1}\mathbb{D}) + C2^{-k}k^2. \end{aligned}$$

This we integrate to get

$$\begin{aligned} \int [l_{k+1}(z)]_{A_{k+1}} dP_k(z) &\stackrel{(84)}{\leq} \int \int [l_{k+1}(x)]_{A_{k+1}} d\Omega(z, r_{k+1}\mathbb{D})(x) dP_k(z) + C2^{-k}k^2 \\ &= \int [l_{k+1}(z)]_{A_{k+1}} dP_{k+1}(z) + C2^{-k}k^2. \end{aligned}$$

The last equality is the well known semigroup property of the Poisson kernel. In probabilistic terms, this is integration over conditional expectation. \square

LEMMA 16. *With the notation above,*

$$(85) \quad \mathcal{I}_k \leq \int [l_{k+1}(z)]_{A_k} dP_k(z) + Ce^{-c\omega(k)}.$$

Proof. The difference between f_k and f_{k+1} can be estimated as in (81) (for $|z| = r_k$) by

$$|f_k(z) - f_{k+1}(z)| \leq \sum_{n=-2^{k+1}}^{-2^k-1} |c(n)|r_k^n \leq C \sum e^{-\omega(\log_2 n)} \leq Ce^{-(1-o(1))\omega(k)}.$$

Therefore, if $|f_k(z)| \geq e^{-\omega(k)/2}$ then $|f_k| \leq |f_{k+1}|(1 + Ce^{-\omega(k)(1/2-o(1))})$ which gives

$$l_k(z) \leq l_{k+1}(z) + Ce^{-c\omega(k)} \quad \forall |z| = r_k, l_k(z) \geq A_k$$

or in other words

$$[l_k]_{A_k} \leq [l_{k+1}]_{A_k} + Ce^{-c\omega(k)}$$

which immediately gives (85) and proves the lemma. \square

The gap between the $[l_{k+1}]_{A_{k+1}}$ of Lemma 15 and the $[l_{k+1}]_{A_k}$ of Lemma 16 is bridged by Lemma 12. It will be convenient to reparametrize as follows: define $B_k = C_2k$ so that by (77) we would have

$$A_k \leq [l_k]_{A_k}(z) \leq B_k \quad \forall |z| = r_{k-1}.$$

Define ε'_k by the relation

$$\int [l_k(z)]_{A_k} dP_{k-1}(z) = \varepsilon'_k B_k + (1 - \varepsilon'_k) A_k.$$

Then Lemma 12 gives

$$\int [l_{k+1}(z)]_{A_k} dP_k(z) \leq \varepsilon'_{k+1} B_{k+1} + (1 - \varepsilon'_{k+1}) A_k.$$

Lemma 15 and then Lemma 16 now give

$$\begin{aligned} (86) \quad \varepsilon'_k B_k + (1 - \varepsilon'_k) A_k &= \int [l_k(z)]_{A_k} dP_{k-1}(z) \leq \mathcal{I}_k + Ck^2 2^{-k} \\ &\leq \varepsilon'_{k+1} B_{k+1} + (1 - \varepsilon'_{k+1}) A_k + Ck^2 2^{-k} + Ce^{-c\omega(k)} \end{aligned}$$

so that

$$(87) \quad \varepsilon'_k \leq \varepsilon'_{k+1} \left(1 + \frac{C_2}{B_k - A_k} \right) + \frac{Ck^2 2^{-k}}{B_k - A_k} \leq \varepsilon'_{k+1} \left(1 + \frac{C}{\omega(k)} \right) + Ck2^{-k}.$$

The same holds for the more natural quantity ε_k defined by

$$(88) \quad \int [l_k(z)]_{A_k} dP_k(z) = \varepsilon_k B_k + (1 - \varepsilon_k) A_k.$$

Indeed, (86) shows that

$$\varepsilon'_k - Ck2^{-k} \leq \varepsilon_k \leq \varepsilon'_{k+1} \left(1 + \frac{C}{\omega(k)} \right) + Ce^{-c\omega(k)}.$$

4.5. With this, the theorem is now easy. Lemma 14 combined with the fact that $|P_K(z)| \leq C\delta^{-1/2}$ for all $z \in \partial(r_K\mathbb{D})$ shows that for some K sufficiently large

$$\int [l_K(z)]_{A_K} dP_K(z) \leq CK\sqrt{\delta} - (1 - C\sqrt{\delta}) \frac{\omega(K)}{2}$$

so that $\varepsilon_K \leq C\sqrt{\delta}$. Applying (87) repeatedly, we get that for all applicable k , $\varepsilon_k \leq C\sqrt{\delta} + Ck2^{-k}$ (here is where we use the condition $\sum \frac{1}{\omega(k)} < \infty$). Let k_0 be the minimal applicable k , namely $\lceil \log_2 1/\delta \rceil + 1$. In particular

$$(89) \quad \varepsilon_{k_0} \leq C\sqrt{\delta}.$$

We can now estimate $l_{k_0}(z_0)$ as in Lemma 15: define $\mathcal{A} := \{1 - 2^{-k_0} \leq |z| \leq r_{k_0}\}$ and get

$$\begin{aligned} l_{k_0}(z_0) &\leq \int l_{k_0} d\Omega(z_0, \mathcal{A}) \stackrel{(83)}{\leq} \int l_{k_0} dP_{k_0} + C2^{-k_0} \max_{\partial\mathcal{A}} |l_{k_0}| \\ &\stackrel{(*)}{\leq} \varepsilon_{k_0} B_{k_0} + (1 - \varepsilon_{k_0}) A_{k_0} + Ck_0^2 2^{-k_0} \stackrel{(89)}{\leq} -c\omega(\log_2 1/\delta), \end{aligned}$$

where $(*)$ comes from the definition of ε_k , (88) for the left term and (77) and $\omega(k) \leq k^2$ for the right term. Therefore $|f_{k_0}(z_0)| \leq e^{-c\omega(\log_2 1/\delta)}$. Since this holds for all z_0 with $|z_0| = 1 - \sqrt{\delta}$ we may estimate $c(n)$ using Laurent's formula (we need to assume $n \geq -2^{k_0}$ so assume $n \geq -1/\delta$) and get

$$|c(n)| \leq (1 - \sqrt{\delta})^{-1-n} e^{-c\omega(\log_2 1/\delta)}.$$

Since this holds for all $\delta > 0$, we get that $c(n) = 0$ for all n , and the theorem is proved. \square

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