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Isometric actions of simple Lie groups on pseudoRiemannian manifolds

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Abstract

Let M be a connected compact pseudoRiemannian manifold acted upon topologically transitively and isometrically by a connected noncompact simple Lie group G. If m_0, n_0 are the dimensions of the maximal lightlike subspaces tangent to M and G, respectively, where G carries any bi-invariant metric, then we have $n_0 \leq m_0$. We study G-actions that satisfy the condition $n_0 = m_0$. With no rank restrictions on G, we prove that M has a finite covering $\widehat{M}$ to which the G-action lifts so that $\widehat{M}$ is G-equivariantly diffeomorphic to an action on a double coset $K\backslash L/\Gamma$, as considered in Zimmer's program, with G normal in L (Theorem A). If G has finite center and $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then we prove that we can choose $\widehat{M}$ for which L is semisimple and Γ is an irreducible lattice (Theorem B). We also prove that our condition $n_0 = m_0$ completely characterizes, up to a finite covering, such double coset G-actions (Theorem C). This describes a large family of double coset G-actions and provides a partial positive answer to the conjecture proposed in Zimmer's program.

1. Introduction

In this work, G will denote a connected noncompact simple Lie group and M a connected smooth manifold, which is assumed to be compact unless otherwise stated. Moreover, we will assume that G acts smoothly, faithfully and preserving a finite measure on M. We will assume that these conditions are satisfied unless stated otherwise. There are several known examples of such actions that also preserve some geometric structure and all of them are essentially of an algebraic nature (see [Zim3] and [FK]). Some of such examples are constructed from homomorphisms $G \hookrightarrow L$ into Lie groups L that admit a (cocompact) lattice Γ. For such setup, the G-action is then the one by left translations on $K\backslash L/\Gamma$, where K is some compact subgroup of $C_L(G)$. Moreover, if L is semisimple and Γ is irreducible, then the G-action is ergodic. This family of examples is a fundamental part in the questions involved in

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studying and classifying G-actions. In his program to study such actions, Robert Zimmer has proposed the problem of determining to what extent a general G-action on M as above is (or at least can be obtained from) an algebraic action, which includes the examples $K\backslash L/\Gamma$ as above (see [Zim3]).

Our goal is to make a contribution to Zimmer's program within the context of pseudoRiemannian geometry. Hence, from now on, we consider M furnished with a smooth pseudoRiemannian metric and assume that G acts by isometries of the metric. Note that G also preserves the pseudoRiemannian volume on M, which is finite since M is compact.

One of the first things we want to emphasize is the fact that G itself can be naturally considered as a pseudoRiemannian manifold. In fact, G admits bi-invariant pseudoRiemannian metrics and all of them can be described in terms of the Killing form (see [Her1] and [BN]). So it is natural to inquire about the relationship of the pseudoRiemannian invariants of both G and M. The simplest one to consider is the signature, which from now on we will denote with (m_1, m_2) and (n_1, n_2) for M and G, respectively, where we have chosen some bi-invariant pseudoRiemannian metric on G. Our notation is such that the first number corresponds to the dimension of the maximal timelike tangent subspaces and the second number to the dimension of the maximal spacelike tangent subspaces. We will also denote $m_0 = \min(m_1, m_2)$ and $n_0 = \min(n_1, n_2)$, which are the dimensions of maximal lightlike tangent subspaces for M and G, respectively. We observe that the signature (n_1, n_2) depends on the choice of the metric on G. However, as it was remarked by Gromov in [Gro], if (n_1, n_2) corresponds to the metric given by the Killing form, then any other bi-invariant pseudoRiemannian metric on G has signature given by either (n_1, n_2) or (n_2, n_1). In particular, n_0 does not depend on the choice of the bi-invariant metric on G, so it only depends on G itself. For these numbers, it is easy to check the following inequality. A proof is given later on in Lemma 3.2.

LEMMA 1.1. *For G and M as before, we have $n_0 \leq m_0$.*

The goal of this paper is to obtain a complete description, in algebraic terms, of the manifolds M and the G-actions that occur when the equality $n_0 = m_0$ is satisfied. We will prove the following result. We refer to [Zim6] for the definition of engagement.

THEOREM A. *Let G be a connected noncompact simple Lie group. If G acts faithfully and topologically transitively on a compact manifold M preserving a pseudoRiemannian metric such that $n_0 = m_0$, then the G-action on M is ergodic and engaging, and there exist:*

(1) *a finite covering $\widehat{M} \to M$,*

(2) *a connected Lie group L that contains G as a factor,*

(3) *a cocompact discrete subgroup Γ of L and a compact subgroup K of $C_L(G)$,*

for which the G-action on M lifts to $\widehat{M}$ so that $\widehat{M}$ is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$. Furthermore, there is an ergodic and engaging G-invariant finite smooth measure on L/Γ.

In other words, if the (pseudoRiemannian) geometries of G and M are closely related, in the sense of satisfying $n_0 = m_0$, then, up to a finite covering, the G-action is given by the algebraic examples considered in Zimmer's program. This result does not require any conditions on the center or real rank of G.

On the other hand, it is of great interest to determine the structure of the Lie group L that appears in Theorem A. For example, one might expect to able to prove that L is semisimple and Γ is an irreducible lattice. By imposing some restrictions on the group G, in the following result we prove that such conclusions can be obtained. In this work we adopt the definition of irreducible lattice found in [Mor], which applies for connected semisimple Lie groups with finite center, even if such groups admit compact factors. We also recall that a semisimple Lie group L is called isotypic if its Lie algebra $\mathfrak{l}$ satisfies $\mathfrak{l} \otimes \mathbb{C} = \mathfrak{d} \oplus \cdots \oplus \mathfrak{d}$ for some complex simple Lie algebra $\mathfrak{d}$.

THEOREM B. *Let G be a connected noncompact simple Lie group with finite center and $\operatorname{rank}_{\mathbb{R}}(G) \geq 2$. If G acts faithfully and topologically transitively on a compact manifold M preserving a pseudoRiemannian metric such that $n_0 = m_0$, then there exist*:

(1) *a finite covering $\widehat{M} \to M$*,

(2) *a connected isotypic semisimple Lie group L with finite center that contains G as a factor*,

(3) *a cocompact irreducible lattice Γ of L and a compact subgroup K of $C_L(G)$*,

for which the G-action on M lifts to $\widehat{M}$ so that $\widehat{M}$ is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$. Hence, up to fibrations with compact fibers, M is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$ and L/Γ.

To better understand these results, one can look at the geometric features of the known algebraic actions of simple Lie groups. This is important for two reasons. To verify that there actually exist examples of topologically transitive actions that satisfy our condition $n_0 = m_0$, and to understand to what extent Theorems A and B describe such examples.

First recall that every semisimple Lie group with finite center admits cocompact lattices. However, not every such group admits an *irreducible* cocompact lattice, which is a condition generally needed to provide ergodic actions. In the work of [Joh] one can find a complete characterization of the semisimple groups with finite center and without compact factors that admit irreducible lattices. Also, in [Mor], one can find conditions for the existence of irreducible lattices on semisimple Lie groups with finite center that may admit compact factors. Based on the results in [Joh] and [Mor] we state the following proposition that provides a variety of examples of ergodic pseudoRiemannian metric preserving actions for which $n_0 = m_0$. Its proof is an easy consequence of [Joh] and [Mor], and the remarks that follow the statement.

PROPOSITION 1.2. *Suppose that G has finite center and* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. *Let L be a semisimple Lie group with finite center that contains G as a normal subgroup. If L is isotypic, then L admits a cocompact irreducible lattice. Hence, for any choices of a cocompact irreducible lattice Γ in L and a compact subgroup K of $C_L(G)$, G acts ergodically, and hence topologically transitively, on $K\backslash L/\Gamma$ preserving a pseudoRiemannian metric for which $n_0 = m_0$.*

For the existence of the metric, we observe that there is an isogeny between L and $G \times H$ for some connected semisimple group H. On a product $G \times H$, we have $K \subset HZ(G)$ and we can build the metric from the Killing form of $\mathfrak{g}$ and a Riemannian metric on H which is K-invariant on the left and H-invariant on the right. For general L a similar idea can be applied.

Hence, Proposition 1.2 ensures that topological transitivity and the condition $n_0 = m_0$, assumed by Theorems A and B, are satisfied by a large and important family of examples, those built out of isotypic semisimple Lie groups containing G as a normal subgroup.

A natural problem is to determine to what extent topological transitivity and the condition $n_0 = m_0$ characterize the examples given in Proposition 1.2. We obtain such a characterization in the following result.

THEOREM C. *Let G be a connected noncompact simple Lie group with finite center and* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. *Assume that G acts faithfully on a compact manifold X. Then the following conditions are equivalent.*

(1) *There is a finite covering $\widehat{X} \to X$ for which the G-action on X lifts to a topologically transitive G-action on $\widehat{X}$ that preserves a pseudoRiemannian metric satisfying $n_0 = m_0$.*

(2) *There is a connected isotypic semisimple Lie group L with finite center that contains G as a factor, a cocompact irreducible lattice Γ of L and a compact subgroup K of $C_L(G)$ such that $K\backslash L/\Gamma$ is a finite covering of X with G-equivariant covering map.*

In words, up to finite covering maps, for topologically transitive G-actions on compact manifolds, to preserve a pseudoRiemannian metric with $n_0 = m_0$ is a condition that characterizes those algebraic actions considered in Zimmer's program corresponding to the double cosets $K\backslash L/\Gamma$ described in (2).

In the theorems stated above we are assuming the pseudoRiemannian manifold acted upon by G to be compact. However, it is possible to extend our arguments to finite volume manifolds if we consider complete pseudoRiemannian structures. In Section 8 we present the corresponding generalizations of Theorems A, B, and C that can be thus obtained.

With the results discussed so far, we completely describe (up to finite coverings) the isometric actions of noncompact simple Lie groups that satisfy our geometric condition $n_0 = m_0$. Moreover, we have actually shown that the collection of manifolds defined by such condition is (up to finite coverings) a very specific and important family of the examples considered in Zimmer's program: those given by groups containing G as a normal subgroup.

Given the previous remarks, we can say that we have fully described and classified a distinguished family of G-actions. Nevertheless, it is still of interest to conclude (from our classification) results that allow us to better understand the topological and geometric restrictions satisfied by the family of G-actions under consideration. This also allows us to make a comparison with results obtained in other works (see, for example, [FK], [LZ2], [SpZi], [Zim8] and [Zim3]). With this respect, in the theorems below, and under our standing condition $n_0 = m_0$, we find improvements and/or variations of important results concerning volume preserving G-actions. Based on this, we propose the problem of extending such theorems to volume preserving G-actions more general than those considered here.

In the remaining of this section, we will assume that G is a connected noncompact simple Lie group acting smoothly, faithfully and topologically transitively on a manifold M and preserving a pseudoRiemannian metric such that $n_0 = m_0$. We also assume that either M is compact or its metric is complete with finite volume. The results stated below basically follow from Theorems A, B and C (and their extensions to finite volume complete manifolds); the corresponding proofs can be found in Section 8.

The next result is similar in spirit to Theorem A in [SpZi], but requires no rank restriction on G.

THEOREM 1.3. *If the G-action is not transitive, then M has a finite covering space M_1 that admits a Riemannian metric whose universal covering splits isometrically. In particular, for such metric, M_1 has some zeros for its sectional curvature.*

Observe that any algebraic G-action of the form $K\backslash L/\Gamma$, as in Zimmer's program, is easily seen to satisfy the conclusion of Theorem 1.3 by just requiring L to have at least two noncompact factors. Hence, one may propose the problem of finding a condition, either geometric or dynamical, that characterizes the conclusion of Theorem 1.3 or an analogous property.

The following result can be considered as an improved version of Gromov's representation theorem. In this case we require a rank restriction.

THEOREM 1.4. *Suppose G has finite center and* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. *Then there exist a finite index subgroup Λ of $\pi_1(M)$ and a linear representation $\rho : \Lambda \to \mathrm{Gl}(p, \mathbb{R})$ such that the Zariski closure $\overline{\rho(\Lambda)}^Z$ is a semisimple Lie group with finite center in which $\rho(\Lambda)$ is a lattice and that contains a closed subgroup locally isomorphic to G. Moreover, if M is not compact, then $\overline{\rho(\Lambda)}^Z$ has no compact factors.*

Again, we observe that all algebraic G-actions in Zimmer's program, i.e. of the form $K\backslash L/\Gamma$ described before, are easily seen to satisfy the conclusions of Theorem 1.4. Actually, our proof depends on the fact that our condition $n_0 = m_0$ ensures that such a double coset appears. Still we may propose the problem of finding other conditions that can be used to prove this more general Gromov's representation theorem. Such a result, in a more general case, would provide a natural semisimple Lie group in which to embed G to prove that a given G-action is of the type considered in Zimmer's program.

Zimmer has proved in [Zim8] that when $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ any analytic engaging G-action on a manifold X preserving a unimodular rigid geometric structure has a fully entropic virtual arithmetic quotient (see [LZ1], [LZ2] and [Zim8] for the definitions and precise statements). The following result, with our standing assumption $n_0 = m_0$, has a much stronger conclusion than that of the main result in [Zim8]. Note that a sufficiently strong generalization of the next theorem for general finite volume preserving actions would mean a complete solution to Zimmer's program for finite measure preserving G-actions, even at the level of the smooth category.

THEOREM 1.5. *Suppose G and M satisfy the hypotheses of either Theorem* B *or Theorem* B′ *(see* §8*). Then the G-action on M has finite entropy. Moreover, there is a manifold $\widehat{M}$ acted upon by G and G-equivariant finite covering maps $\widehat{M} \to A(M)$ and $\widehat{M} \to M$, where $A(M)$ is some realization of the maximal virtual arithmetic quotient of M.*

The organization of the article is as follows. The proof of Theorem A relies on studying the pseudoRiemannian geometry of G and M. In that sense, the fundamental tools for the proof of Theorem A are developed in Sections 3 and 4. In Section 5 the proof of Theorem A is completed based on the results proved up to that point and a study of a transverse Riemannian structure associated to the G-orbits. The proofs of Theorems B and C (§§6 and 7) are based on Theorem A, but also rely on the results of [StZi] and [Zim5]. In Section 8 we show how to extend Theorems A, B and C to finite volume manifolds if we assume completeness of the pseudoRiemannian structure involved. Section 8 also contains the complete proofs of Theorems 1.3, 1.4 and 1.5.

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2. Some preliminaries on homogeneous spaces

We will need the following easy to prove result.

LEMMA 2.1. *Let H be a Lie group acting smoothly and transitively on a connected manifold X. If for some $x_0 \in X$ the isotropy group H_{x_0} has finitely many components, then H has finitely many components as well.*

Proof. Let $H_{x_0} = K_0 \cup \cdots \cup K_r$ be the component decomposition of H_{x_0}. Choose an element $k_i \in K_i$, for every $i = 0, \dots, r$.

For any given $h \in H$, let $\widehat{h} \in H_0$ be such that $h(x_0) = \widehat{h}(x_0)$ (see [Hel]). Hence, $\widehat{h}^{-1}h \in H_{x_0}$, so there exists i_0 such that $\widehat{h}^{-1}h \in K_{i_0}$. If γ is a continuous path from k_{i_0} to $\widehat{h}^{-1}h$, then $\widehat{h}\gamma$ is a continuous path from $\widehat{h}k_{i_0}$ to h. This shows that $H = H_0k_0 \cup \dots H_0k_r$. □

As an immediate consequence we obtain the following.

COROLLARY 2.2. *If X is a connected homogeneous Riemannian manifold, then the group of isometries* $\mathrm{Iso}(X)$ *has finitely many components. Moreover, the same property holds for any closed subgroup of* $\mathrm{Iso}(X)$ *that acts transitively on X.*

The following result is a well known easy consequence of Singer's Theorem (see [Sin]). Nevertheless, we state it here for reference and briefly explain its proof, from the results of [Sin], for the sake of completeness.

THEOREM 2.3 (Singer). *Let X be a smooth simply connected complete Riemannian manifold. If the pseudogroup of local isometries has a dense orbit, then X is a homogeneous Riemannian manifold.*

Proof. By the main theorem in [Sin], we need to show that X is infinitesimally homogeneous as considered in [Sin]. The latter is defined by the existence of an isometry $A : T_xX \to T_yX$, for any two given points $x, y \in X$, so that A transforms the curvature and its covariant derivatives (up to a fixed order) at x into those at y. Under our assumptions, this condition is satisfied only on a dense subset S of X. However, for an arbitrary $y \in X$, we can choose $x \in S$, a sequence $(x_n)_n \subset S$ that converges to y and a sequence of maps $A_n : T_xX \to T_{x_n}X$ that satisfy the infinitesimal homogeneity condition.

By introducing local coordinates at x and y, we can consider that (for n large enough) the sequence $(A_n)_n$ lies in a compact group and thus has a subsequence that converges to some map $A : T_xX \to T_yX$. By the continuity of the identities that define infinitesimal homogeneity in [Sin], it is easy to show that A satisfies such identities. This proves infinitesimal homogeneity of X, and so X is homogeneous. $\square$

3. Isometric splitting of a covering of M

We start by describing some geometric properties of the G-orbits on M when the condition $n_0 = m_0$ is satisfied.

PROPOSITION 3.1. *Suppose G acts topologically transitively on M preserving its pseudoRiemannian metric and satisfying $n_0 = m_0$. Then G acts everywhere locally freely with nondegenerate orbits. Moreover, the metric induced by M on the G-orbits is given by a bi-invariant pseudoRiemannian metric on G that does not depend on the G-orbit.*

Proof. Everywhere local freeness follows from topological transitivity by the results in [Sz].

Observe that the condition for G-orbits to be nondegenerate is an open condition, i.e. there exist a G-invariant open subset U of M so that the G-orbit of every point in U is nondegenerate.

On the other hand, given local freeness, it is well known that for $T\mathcal{O}$ the tangent bundle to the G-orbits, the following map is a G-equivariant smooth trivialization of $T\mathcal{O}$:

$$\begin{aligned} \varphi : M \times \mathfrak{g} &\to T\mathcal{O} \\ (x, X) &\mapsto X^*_x \end{aligned}$$

where X^* is the vector field on M whose one parameter group of diffeomorphisms is $\exp(tX)$, and the G-action on $M \times \mathfrak{g}$ is given by $g(x,X) = (gx, \mathrm{Ad}(g)(X))$. Then, by restricting the metric on M to $T\mathcal{O}$ and using the above trivialization, we obtain the smooth map:

$$\begin{aligned} \psi : M &\to \mathfrak{g}^* \otimes \mathfrak{g}^* \\ x &\mapsto B_x \end{aligned}$$

where $B_x(X,Y) = h_x(X^*_x, Y^*_x)$, for h the metric on M. This map is clearly G-equivariant. Hence, since the G-action is tame on $\mathfrak{g}^* \otimes \mathfrak{g}^*$, such map is essentially constant on the support of almost every ergodic component of M. Hence, if S is the support of one such ergodic component of M, then there is an $\mathrm{Ad}(G)$-invariant bilinear form B_S on $\mathfrak{g}$ so that, by the previous discussion, the metric on $T\mathcal{O}|_S \cong S \times \mathfrak{g}$ induced by M is almost everywhere given by B_S

on each fiber. Also, the $\mathrm{Ad}(G)$-invariance of B_S implies that its kernel is an ideal of $\mathfrak{g}$. If such kernel is $\mathfrak{g}$, then $T\mathcal{O}|_S$ is lightlike which implies $\dim \mathfrak{g} \leq m_0$. But this contradicts the condition $n_0 = m_0$ since $n_0 < \dim \mathfrak{g}$. Hence, being $\mathfrak{g}$ simple, it follows that B_S is nondegenerate, and so almost every G-orbit contained in S is nondegenerate. Since this holds for almost every ergodic component, it follows that almost every G-orbit in M is nondegenerate. In particular, the set U defined above is conull and so nonempty.

Moreover, the above shows that the image under ψ of a conull, and hence dense, subset of M lies in the set of $\mathrm{Ad}(G)$-invariant elements of $\mathfrak{g}^* \otimes \mathfrak{g}^*$. Since the latter set is closed, it follows that $\psi(M)$ lies in it. In particular, on every G-orbit the metric induced from that of M is given by an $\mathrm{Ad}(G)$-invariant symmetric bilinear form on $\mathfrak{g}$.

By topological transitivity, there is a G-orbit $\mathcal{O}_0$ which is dense and so it must intersect U. Since U is G-invariant it follows that $\mathcal{O}_0$ is contained in U. Let B_0 be the nondegenerate bilinear form on $\mathfrak{g}$ so that under the map ψ the metric of M restricted to $\mathcal{O}_0$ is given by B_0. Hence $\psi(\mathcal{O}_0) = B_0$ and so the density of $\mathcal{O}_0$ together with the continuity of ψ imply that ψ is the constant map given by B_0. We conclude that all G-orbits are nondegenerate as well as the last claim in the statement. □

The arguments in Proposition 3.1 allows us to prove the following result which is a generalization of Lemma 1.1.

LEMMA 3.2. *Let G be a connected noncompact simple Lie group acting by isometries on a finite volume pseudoRiemannian manifold X. Denote with (n_1, n_2) and (m_1, m_2) the signatures of G and X, respectively, where G carries a bi-invariant pseudoRiemannian metric. If we denote $n_0 = \min(n_1, n_2)$ and $m_0 = \min(m_1, m_2)$, then $n_0 \leq m_0$.*

Proof. With this setup we have local freeness on an open subset U of X by the results in [Zim4]. As in the proof of Proposition 3.1, we consider the map:

$$\begin{aligned} U &\to \mathfrak{g}^* \otimes \mathfrak{g}^* \\ x &\mapsto B_x \end{aligned}$$

which, from the arguments in such proof, is constant on the ergodic components in U for the G-action. On any such ergodic component, the metric along the G-orbits comes from an $\mathrm{Ad}(G)$-invariant bilinear form B_0 on $\mathfrak{g}$. As before, the kernel of B_0 is an ideal. If the kernel is all of $\mathfrak{g}$, then $B_0 = 0$ and the G-orbits are lightlike which implies that $n_0 < \dim \mathfrak{g} \leq m_0$. If the kernel is trivial, then B_0 is nondegenerate and the G-orbits are nondegenerate submanifolds of X. But this implies $n_0 \leq m_0$ as well, since n_0 does not depend on the bi-invariant metric on G. □

In the rest of this work we will denote with $T\mathcal{O}$ the tangent bundle to the orbits. From Proposition 3.1 it follows that $TM = T\mathcal{O} \oplus T\mathcal{O}^\perp$, when the G-action is topologically transitive and $n_0 = m_0$.

We will need the following result which provides large local isotropy groups. Its proof relies heavily on the arguments in [CQ] (see also [Gro]). Similar results appear in [Zim7] and [Fe], but in such works analyticity and compactness of the manifold acted upon is assumed.

Proposition 3.3. *Let G be a connected noncompact simple Lie group and X a smooth finite volume pseudoRiemannian manifold. Suppose that G acts smoothly on X by isometries. Then there is a dense subset S of X so that, for every $x \in S$, there exist an open neighborhood U_x of x and a Lie algebra $\mathfrak{g}(x)$ of Killing vector fields defined on U_x satisfying*:

(1) $Z_x = 0$, *for every $Z \in \mathfrak{g}(x)$*,

(2) *the local one-parameter subgroups of $\mathfrak{g}(x)$ preserve the G-orbits*,

(3) $\mathfrak{g}(x)$ *and $\mathfrak{g}$ are isomorphic Lie algebras*, *and*

(4) *for the isomorphism in* (3), *the canonical vector space isomorphism $T_xGx \cong \mathfrak{g}$ is also an isomorphism of $\mathfrak{g}$-modules.*

Proof. Without using analyticity, the arguments in Lemma 9.1 in [CQ] provide a conull set S_0 so that for every $x \in S_0$ one has a Lie algebra of infinitesimal Killing vector fields of order k that satisfy the above conclusions up to order k. Moreover, this is achieved for every k sufficiently large. Further on, in Theorem 9.2 in [CQ], such infinitesimal vector fields are extended to local ones by using analyticity. This extension ultimately depends on Proposition 6.6 in [CQ]. The latter result is based on the arguments in Nomizu [Nom].

In [Nom] a notion of regular point for X is introduced, which satisfy two key properties. The set of regular points is an open dense subset U of X and at regular points every infinitesimal Killing field of large enough order can be extended locally. The first property is found in [Nom] and the second one is proved in [CQ], both just using smoothness.

From these remarks we find that the set $S = U \cap S_0$ satisfies the conclusions without the need to assume analyticity, as one does in the statements of [CQ]. Also, S is obviously dense since U is open dense, S_0 is conull and the measure considered (the pseudoRiemannian volume) is smooth. Finally, we observe that even though the results in [Nom] are stated for Riemannian metrics, those that we use here extend with the same proof to pseudoRiemannian manifolds. A remark of this sort was already made in [CQ]. □

We now prove integrability of the normal bundle to the orbits.

PROPOSITION 3.4. *Suppose G acts topologically transitively on M preserving its pseudoRiemannian metric and satisfying $n_0 = m_0$. Then $T\mathcal{O}^\perp$ is integrable.*

Proof. Let $\omega : TM \to \mathfrak{g}$ be the $\mathfrak{g}$-valued 1-form on M given by $TM = T\mathcal{O} \oplus T\mathcal{O}^\perp \to T\mathcal{O} \cong M \times \mathfrak{g} \to \mathfrak{g}$, where the two arrows are the natural projections. Define the curvature of ω by the 2-form $\Omega = d\omega|_{T\mathcal{O}^\perp \wedge T\mathcal{O}^\perp}$. As remarked in [Gro] (see also [Her2]) it is easy to prove that $T\mathcal{O}^\perp$ is integrable if and only if $\Omega = 0$.

Choose S and $\mathfrak{g}(x)$ as in Proposition 3.3. Hence, the local one-parameter subgroups of $\mathfrak{g}(x)$ preserve $T\mathcal{O}_x^\perp$ for every $x \in S$. From this, and the isomorphism $\mathfrak{g}(x) \cong \mathfrak{g}$ described in the proof of Proposition 3.3, it is easy to show that the linear map $\Omega_x : T\mathcal{O}_x^\perp \wedge T\mathcal{O}_x^\perp \to \mathfrak{g}$ is a $\mathfrak{g}$-module homomorphism, for every $x \in S$. This fact is contained in the proof of Proposition 3.9 in [Her2].

On the other hand, Proposition 3.1 and the condition $n_0 = m_0$ imply that $T\mathcal{O}^\perp$ is either Riemannian or antiRiemannian. Since the elements of $\mathfrak{g}(x)$ are Killing fields, it follows that $\mathfrak{g}(x)$ can be linearly represented on $T\mathcal{O}_x^\perp \wedge T\mathcal{O}_x^\perp$ so that the elements of $\mathfrak{g}(x)$ define derivations of a definite inner product. This provides a homomorphism of $\mathfrak{g}(x)$ into the Lie algebra of an orthogonal group. Since $\mathfrak{g}(x)$ is simple noncompact, such homomorphism is trivial and it follows that the $\mathfrak{g}(x)$-module $T\mathcal{O}_x^\perp \wedge T\mathcal{O}_x^\perp$ is trivial. Hence no $\mathfrak{g}(x)$-irreducible subspace of $T\mathcal{O}_x^\perp \wedge T\mathcal{O}_x^\perp$ can be isomorphic to $\mathfrak{g}$ as a $\mathfrak{g}$-module. By Schur's Lemma we conclude that $\Omega_x = 0$, for every $x \in S$. Hence, $\Omega = 0$ on all of M and $T\mathcal{O}^\perp$ is integrable. □

The following result is fundamental to obtain an isometric splitting.

LEMMA 3.5. *With the conditions of Proposition* 3.4, *the leaves of the foliation defined by $T\mathcal{O}^\perp$ are complete for the metric induced by M.*

Proof. As observed in the proof of Proposition 3.4, the bundle $T\mathcal{O}^\perp$ is either Riemannian or antiRiemannian. Hence, the foliation by G-orbits on M carries a Riemannian or antiRiemannian structure obtained from $T\mathcal{O}^\perp$. By the basic properties of Riemannian foliations, the compactness of M implies that geodesic completeness is satisfied for geodesics orthogonal to the G-orbits (see [Mol]). This clearly implies the completeness for leaves of the foliation given by $T\mathcal{O}^\perp$. □

The next proposition provides a first description of the properties of M. It is similar in spirit to the main results in [Her2].

PROPOSITION 3.6. *Suppose G acts topologically transitively on M preserving its pseudoRiemannian metric and satisfying $n_0 = m_0$. Choose a leaf N of the foliation defined by $T\mathcal{O}^\perp$. Fix on G the bi-invariant pseudoRiemannian metric that induces on the G-orbits the metric inherited by M and* *consider N as a pseudoRiemannian manifold with the metric inherited by M as well. Then the map $G \times N \to M$, obtained by restricting the G-action to N, is a G-equivariant isometric covering map. In particular, this induces a G-equivariant isometric covering map $G \times \widetilde{N} \to M$, where $\widetilde{N}$ is the universal covering space of N.*

Proof. By our choices of metrics, the G-invariance of the metric on M and the previous results, it is easy to conclude that the map $G \times N \to M$ as above is a local isometry. On the other hand, the Levi-Civita connection on G is bi-invariant and, by the problems in Chapter II of [Hel], its geodesics are translates of one-parameter subgroups. In particular, G is complete. Hence, by Lemma 3.5, $G \times N$ is a complete pseudoRiemannian manifold. Then, Corollary 29 in page 202 in [O'N] implies that the restricted action map $G \times N \to M$ is an isometric covering map. The rest of the claims follow easily from this fact. □

As an immediate consequence we obtain the following result. The proof uses Proposition 4.5. We note that Section 4 is actually independent from this section and the rest of this work.

COROLLARY 3.7. *Let G, M and N be as in the hypotheses of Proposition 3.6. Then there is a discrete subgroup Γ_0 of $\mathrm{Iso}(G \times \widetilde{N})$ of deck transformations for $G \times \widetilde{N} \to M$ such that $(G \times \widetilde{N})/\Gamma_0 \to M$ is a G-equivariant finite covering.*

Our next goal is to prove that, by passing to a finite covering, Γ_0 can be replaced by a discrete subgroup of a group that contains G as a subgroup as well. In order to do that, we will study the isometry group of G with some bi-invariant pseudoRiemannian metric.

4. Geometry of bi-invariant metrics on G

Let G be a connected noncompact simple Lie group G as before. We will investigate some useful properties about the geometry of a bi-invariant pseudoRiemannian metric on G. Note that any such metric is analytic and by [Her1] can be described in terms of the Killing form (see also [BN]); however, we will not use such fact. In this section, we fix an arbitrary bi-invariant pseudoRiemannian metric on G and denote with $\mathrm{Iso}(G)$ the corresponding group of isometries. Also we denote $L(G)R(G) = \{L_g \circ R_h | g, h \in G\}$, the group generated by left and right translations, which is clearly a connected subgroup of $\mathrm{Iso}(G)$.

We will use some basic properties of pseudoRiemannian symmetric spaces, which are known to be a natural generalization of Riemannian symmetric spaces. For the definitions and basic properties of the objects involved we will refer to [CP]. Moreover, we will use in our proofs some of the results found in this reference.

From [CP], we recall that, in a pseudoRiemannian symmetric space X, a transvection is an isometry of the form $s_x \circ s_y$, where s_x is the involutive isometry that has x as an isolated fixed point. The group T generated by transvections is called the transvection group of X. This group is (clearly) invariant under the conjugation by s_o, any fixed involutive isometry. With this setup, the pseudoRiemannian symmetric triple associated to X is given by $(Lie(T), \sigma, B)$, where B is a suitable bilinear form on $Lie(T)$ and σ is the differential at $e \in T$ of the conjugation by some involution s_o. We refer to [CP] for a more precise description of this object. Here, we need to show the following features of the geometry of G associated with these notions.

PROPOSITION 4.1. *G is a pseudoRiemannian symmetric space whose associated pseudoRiemannian symmetric triple can be chosen to be of the form $(\mathfrak{g} \times \mathfrak{g}, \sigma, B)$, where $\sigma(X, Y) = (Y, X)$.*

Proof. Since the differential of the inversion map $g \mapsto g^{-1}$ at any point can be written as the composition of the differentials of a left and a right translations (see [Hel]), it follows that the bi-invariant metric on G is also invariant under the inversion map. Hence, for every $x \in G$, the map s_x defined by $s_x(g) = xg^{-1}x$ is an isometry of G and it is easily seen to be involutive with x as an isolated fixed point. Hence, G is pseudoRiemannian symmetric.

Let T be the transvection group of G. One can easily check that $s_x \circ s_y = L_{xy^{-1}} \circ R_{y^{-1}x}$, and so T is a subgroup of $L(G)R(G)$. On the other hand, since G is simple and connected we have $[G, G] = G$. Hence, for every $z \in G$, there exist $x, y \in G$ such that $z = [x, y]$. From this it is easy to prove that $L_z = s_e \circ s_{yx} \circ s_x \circ s_{y^{-1}} \in T$. This with a similar formula for right translations show that $T = L(G)R(G)$. Furthermore, if we define a $G \times G$-action on G by $(g, h)x = gxh^{-1}$, then the map $(g, h) \mapsto L_g \circ R_{h^{-1}}$ defines a local isomorphism $G \times G \to L(G)R(G) = T$, which implies $Lie(T) \cong \mathfrak{g} \times \mathfrak{g}$ as Lie algebras. Finally, a straightforward computation proves that using conjugation by s_e, the map σ on $\mathfrak{g} \times \mathfrak{g}$ has the required expression. □

As a consequence we obtain the following result. We recall that a connected pseudoRiemannian manifold is called weakly irreducible if the tangent space at some (and hence at every) point has no nonsingular proper subspaces invariant by the holonomy group at the point.

PROPOSITION 4.2. *For any bi-invariant pseudoRiemannian metric on G, the universal covering space $\widetilde{G}$ is weakly irreducible.*

Proof. Consider the representation ρ of the Lie algebra $\mathfrak{f} = \{(X,X)|X \in \mathfrak{g}\}$ (isomorphic to $\mathfrak{g}$) in the space $\mathfrak{p} = \{(Y,-Y)|Y \in \mathfrak{g}\}$ given by the expression:

$$\rho(X,X)(Y,-Y) = ([X,Y], -[X,Y]).$$

This clearly turns $\mathfrak{p}$ into an $\mathfrak{f}$-module isomorphic to the $\mathfrak{g}$-module given by the adjoint representation of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{p}$ is then an irreducible $\mathfrak{f}$-module. Then the conclusion follows from the description of the pseudo-Riemannian symmetric triple associated to G in our Proposition 4.1 and from Proposition 4.4 in page 18 in [CP]. $\square$

With the previous result at hand we obtain the next statement.

PROPOSITION 4.3. *Let N be a connected complete Riemannian* (*or antiRiemannian*) *manifold. Then any isometry of the pseudoRiemannian product $G \times N$ preserves the factors, in other words,* $\mathrm{Iso}(G \times N) = \mathrm{Iso}(G) \times \mathrm{Iso}(N)$.

Proof. Let $\widetilde{G} \times \widetilde{N}$ be the universal covering of $G \times N$. Let $\widetilde{N} = N_0 \times \cdots \times N_k$ be the de Rham decomposition of $\widetilde{N}$ as Riemannian (or antiRiemannian) manifold. By the de Rham-Wu decomposition theorem for pseudoRiemannian manifolds (see [Wu] and [CP]) and by Proposition 4.2, it follows that $\widetilde{G} \times \widetilde{N}$ has a de Rham decomposition and it is given by $\widetilde{G} \times N_0 \times \dots N_k$. Furthermore, it is known that such decomposition is unique up to order. In particular, every isometry of $\widetilde{G} \times N_0 \times \dots N_k$ permutes the factors, but since each N_i is Riemannian (or antiRiemannian) and $\widetilde{G}$ is not, then every isometry of $\widetilde{G} \times \widetilde{N}$ preserves these two factors.

Now let $f \in \mathrm{Iso}(G \times N)$ and lift it to an isometry $\widetilde{f}$ of $\widetilde{G} \times \widetilde{N}$. By the previous arguments, $\widetilde{f}$ preserves the product, i.e. if we write $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2)$, then $\widetilde{f}_1$ does not depend on $\widetilde{N}$ and $\widetilde{f}_2$ does not depend on $\widetilde{G}$. From this, it is easy to see that there exist isometries $f_1 \in \mathrm{Iso}(G)$ and $f_2 \in \mathrm{Iso}(N)$ such that $f = (f_1, f_2)$, thus showing the result. $\square$

We now proceed to obtain a fairly precise description of $\mathrm{Iso}(G)$. First we prove the following result.

LEMMA 4.4. *Denote by $\mathrm{Iso}(G)_e$ the isotropy subgroup at $e \in G$. Then the homomorphism*:

$$\begin{aligned} \varphi : \mathrm{Iso}(G)_e &\to \mathrm{Gl}(\mathfrak{g}) \\ h &\mapsto dh_e \end{aligned}$$

is an isomorphism onto a closed subgroup of $\mathrm{Gl}(\mathfrak{g})$.

Proof. By the arguments from Lemma 11.2 in page 62 in [Hel] the map is injective. Now let $L^{(1)}(G)$ be the linear frame bundle of G endowed with the parallelism given by the Levi-Civita connection on G. Consider the standard fiber of $L^{(1)}(G)$ given by $\mathrm{Gl}(\mathfrak{g})$. Then the proof of Theorem 3.2 in page 15 in [Ko] shows that the map:

$$\begin{aligned}\psi : \mathrm{Iso}(G) &\to L^{(1)}(G)\\ h &\mapsto dh_e\end{aligned}$$

realizes $\mathrm{Iso}(G)$ as a closed submanifold that defines the topology for which $\mathrm{Iso}(G)$ turns out to be a transformation Lie group. Then the result follows by observing that the image of φ is just the intersection of the image of ψ with the fiber of $L^{(1)}(G)$ at e, which is $\mathrm{Gl}(\mathfrak{g})$. $\square$

The next result turns out to be fundamental in our proof of Theorem A.

PROPOSITION 4.5. $\mathrm{Iso}(G)$ *has finitely many components and* $\mathrm{Iso}(G)_0 = L(G)R(G)$. *Also,* $(\mathrm{Iso}(G)_e)_0$ *is isomorphic to* $\mathrm{Ad}(G)$ *with respect to the map* φ *from Lemma* 4.4.

Proof. By Problem A.6.ii in page 148 in [Hel] and since $\mathrm{Iso}(G)_e$ preserves the (unique) bi-invariant connection we have that $\varphi(\mathrm{Iso}(G)_e) \subset \mathrm{Aut}(\mathfrak{g})$, where $\mathrm{Aut}(\mathfrak{g})$ is the group of Lie algebra automorphisms of $\mathfrak{g}$. Also, we clearly have $\mathrm{Ad}(G) \subset \varphi(\mathrm{Iso}(G)_e)$, because $L_g \circ R_{g^{-1}}$ is an isometry. Since $\mathfrak{g}$ is simple we know that $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})_0$, from which we conclude $\varphi(\mathrm{Iso}(G)_e)_0 = \mathrm{Ad}(G)$, and the last claim follows.

Since $\mathrm{Aut}(\mathfrak{g})$ is algebraic, it has finitely many components. Hence, the previous inclusions imply that $\mathrm{Iso}(G)_e$ has finitely many components and, by Lemma 2.1, the group $\mathrm{Iso}(G)$ satisfies such property as well.

On the other hand, G being homogeneous under $\mathrm{Iso}(G)$ it is also homogeneous under $\mathrm{Iso}(G)_0$ (see [Hel]). Hence, we observe that G has the following two expressions as a homogeneous pseudoRiemannian manifold:

$$\begin{aligned}G &= \mathrm{Iso}(G)_0/(\mathrm{Iso}(G)_0 \cap \mathrm{Iso}(G)_e)\\ &= L(G)R(G)/G_\Delta,\end{aligned}$$

where $G_\Delta = \{L_g \circ R_{g^{-1}} | g \in G\}$. This together with the following easy to prove identities:

$$\dim(\mathrm{Iso}(G)_0 \cap \mathrm{Iso}(G)_e) = \dim((\mathrm{Iso}(G)_e)_0) = \dim(\mathrm{Ad}(G)) = \dim(G_\Delta),$$

show that the inclusion $L(G)R(G) \subset \mathrm{Iso}(G)_0$ is a homomorphic inclusion of connected Lie groups of the same dimension. This implies $\mathrm{Iso}(G)_0 = L(G)R(G)$. $\square$

By Propositions 4.3, 4.5 and Corollary 2.2 we obtain the following.

Proposition 4.6. *Let N be a connected homogeneous Riemannian manifold. Then* $\mathrm{Iso}(G \times N)$ *has finitely many connected components and* $\mathrm{Iso}(G \times N)_0 = \mathrm{Iso}(G)_0 \times \mathrm{Iso}(N)_0 = L(G)R(G) \times \mathrm{Iso}(N)_0$.

5. Proof of Theorem A

Throughout this section, we assume that G and M satisfy the hypotheses of Theorem A. Let N be a leaf of the foliation defined by $T\mathcal{O}^\perp$, as described in Section 3, and $\widetilde{N}$ its universal covering space. We will prove that $\widetilde{N}$ is homogeneous. First observe that, by hypothesis, there is a dense G-orbit. By Proposition 3.6, we have $GN = M$, and so we can assume that for some $x_0 \in N$, the G-orbit of x_0 is dense in M.

Let y be any given point in N and V an open neighborhood of y in N with the leaf topology. Since the complementary distributions $T\mathcal{O}$ and $T\mathcal{O}^\perp$ (as defined in §3) are integrable, we can find a connected neighborhood U of y in M and a diffeomorphism $f = (f_1, f_2) : U \to \mathbb{R}^k \times \mathbb{R}^l$ that maps the pair of foliations for the distributions $T\mathcal{O}|_U$ and $T\mathcal{O}^\perp|_U$ diffeomorphically onto the foliations of $\mathbb{R}^{k+l}$ defined by the factors $\mathbb{R}^k$ and $\mathbb{R}^l$ (see Example 2.10 in page 12 in [Ko]). By shrinking either U or V, if necessary, we can assume that V is a plaque for the foliated chart that f defines for the foliation given by $T\mathcal{O}^\perp$. Then V is a transversal for the foliation by G-orbits that intersects every plaque (for the foliation by G-orbits) in U. Let x be a point in the G-orbit of x_0 lying in U, and P the plaque of the foliation by G-orbits that contains x. If we choose $z \in P \cap V$, since the plaque P is contained in the G-orbit of x_0 (because $x \in P$), then there exist $g \in G$ such that $z = gx_0$. But the G-action preserves $T\mathcal{O}^\perp$, and so the restriction of g to N defines an isometry of N that maps x_0 into z.

The previous arguments show that the isometry group of N has a dense orbit in N. By lifting isometries of N to $\widetilde{N}$ and using deck transformations, it is easy to prove that the isometry group of $\widetilde{N}$ has a dense orbit as well. Since $\widetilde{N}$ is either Riemannian or antiRiemannian (see the proof of Proposition 3.4), we conclude from Lemma 3.5 and Theorem 2.3 that $\widetilde{N}$ is a homogeneous pseudoRiemannian manifold. Let H be the identity component of the group of isometries of $\widetilde{N}$. Then there is a compact subgroup K of H such that $\widetilde{N} = K\backslash H$.

Let Γ_0 be a discrete subgroup $\mathrm{Iso}(G \times \widetilde{N})$ as given by Corollary 3.7. By Proposition 4.6, the group $\mathrm{Iso}(G \times \widetilde{N})$ has finitely many components and its identity component is $L(G)R(G) \times \mathrm{Iso}(N)_0 = L(G)R(G) \times H$. If we define $\Gamma = \Gamma_0 \cap (L(G)R(G) \times H)$, then Γ is a normal finite index subgroup of Γ_0, which provides a normal finite covering $\widehat{M} = (G \times \widetilde{N})/\Gamma = (G \times K\backslash H)/\Gamma$ of M. Now, let $\gamma \in \Gamma$ be given. Then, by the lifting property, we can find a diffeomorphism $\widetilde{\gamma}$ such that the following diagram commutes.

$$\begin{tikzcd}
\widetilde{G} \times \widetilde{N} \arrow[r, "\widetilde{\gamma}"] \arrow[d] & \widetilde{G} \times \widetilde{N} \arrow[d] \\
G \times \widetilde{N} \arrow[r, "\gamma"] \arrow[dr] & G \times \widetilde{N} \arrow[d] \\
& \widehat{M}
\end{tikzcd}$$

where the vertical arrows are given by the universal covering map of $G\times\widetilde{N}$ and the arrows into $\widehat{M}$ are given by the covering map $G \times \widetilde{N} \to \widehat{M}$. In particular, $\widetilde{\gamma}$ is a covering transformation for the universal covering map $\widetilde{G} \times \widetilde{N} \to \widehat{M}$, and so it is also a covering transformation for the universal covering map $\widetilde{G} \times \widetilde{N} \to M$. By Proposition 3.6, the G-action on M lifts to a G-action on $G \times \widetilde{N}$ and to a $\widetilde{G}$-action on $\widetilde{G} \times \widetilde{N}$ both by the expression $g(g_1, x) = (gg_1, x)$. On the other hand, the latter lifted action commutes with the action of $\pi_1(M)$ and in particular it commutes with $\widetilde{\gamma}$. Then we observe that, in the square of the commutative diagram above, the top horizontal arrow and the vertical arrows are $\widetilde{G}$-equivariant and so the map γ is G-equivariant. On the other hand, we can write $\gamma = (L_{g_1} \circ R_{g_2}, h)$, since it lies in $L(G)R(G) \times H$. Then, the G-equivariance of γ and the above expression for the G-action on $G \times \widetilde{N}$ yields:

$$(g_1gg_2, xh) = (g(e, x))(L_{g_1} \circ R_{g_2}, h) = g((e, x)(L_{g_1} \circ R_{g_2}, h)) = (gg_1g_2, xh)$$

for every $g \in G$ and $x \in \widetilde{N}$, which implies $g_1 \in Z(G)$. Hence, $L_{g_1} = R_{g_1}$ and then $\gamma \in R(G) \times H$.

The previous arguments show that Γ is a discrete subgroup of $R(G)\times H \cong G \times H$ acting on the right on $G \times \widetilde{N} = G \times K\backslash H$ and commuting with the (natural) G-action on the latter space. In particular, Γ is a discrete cocompact subgroup of $G \times H$. Since the Γ-action is on the right and the G-action is on the left, they both commute and the G-action lifts to $\widehat{M}$.

If we now take $L = G \times H$, $\widehat{M}$, Γ and K as above, then the conditions required in (1), (2) and (3) are satisfied, as well as the fact that $\widehat{M}$ is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$. Hence, to complete the proof of Theorem A, it remains to show that the G-actions on M and on L/Γ are ergodic and engaging. We will prove this by showing that we can replace H, Γ and $\widehat{M}$ by suitable choices so that the properties proved so far still hold and the required ergodicity and engagement conditions are now satisfied. To achieve this, we will study the properties of the Riemannian (or antiRiemannian) foliation by G-orbits on M.

We will use the following result about Riemannian and Lie foliations.

Lemma 5.1. *Let X be a compact manifold with a smooth foliation $\mathcal{F}$ carrying either a transverse Lie structure or a transverse Riemannian structure.*

Then the foliation $\mathcal{F}$ has a dense leaf if and only if it is ergodic. Furthermore, if $\mathcal{F}$ has a dense leaf, then the induced foliation on any connected finite covering of X has a dense leaf as well and so it is ergodic.

Proof. In both cases, the foliation carries a transverse holonomy invariant finite volume and so ergodicity is easily seen to imply the existence of a dense leaf.

For the converse, let us first assume that $\mathcal{F}$ carries a transverse Lie structure. Let F be a connected Lie group that models the transverse Lie structure, so that there exist a developing map and a holonomy representation given by:

$$D : \widehat{X} \to F$$
$$\rho : \pi_1(X) \to F$$

We refer to [Mol], [Ton] and [Zim5] for the definition of such objects.

Then it is easily seen that the existence of a dense leaf is equivalent to the density of $\rho(\pi_1(X))$ in F. By Lemma 2.2.13 in [Zim2] such density is equivalent to the ergodicity of the $\rho(\pi_1(X))$-action on F by left translations. Finally, the latter ergodicity is equivalent to the ergodicity of the foliation $\mathcal{F}$ (see for example [FHM] or [Zim5]).

We still consider X endowed with a transverse Lie structure and now we suppose that X has a dense leaf. Let $Y \to X$ be a connected finite covering map. If we choose a dense leaf $\mathcal{O}$ in X, then, for the induced foliation on Y, the inverse image of $\mathcal{O}$ in Y is a finite union of leaves $\mathcal{O}_1, \dots, \mathcal{O}_r$. We clearly have that $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r$ is dense in Y, i.e. we can write $Y = \overline{\mathcal{O}}_1 \cup \cdots \cup \overline{\mathcal{O}}_r$. On the other hand, since Y is compact, the transverse Lie structure on Y is complete and by the structure theorems for transversely parallelizable foliations (see [Mol]) it follows that the closures of leaves define a partition of Y. Since Y is connected, from the previous expression of Y it follows that $Y = \overline{\mathcal{O}}_i$, for some i, and so Y has a dense leaf. Moreover, from what we have proved so far it follows that the foliation in Y is ergodic.

We now consider the case where X carries a transverse Riemannian structure. Then, by the results in [Mol], there is a foliated fiber bundle $\pi : P \to X$ where P is compact and carries a transverse Lie structure. Moreover, if X has a dense leaf it is easy to see from the results in [Mol] (see page 153 therein) that we can choose a foliated reduction to assume that P has a dense leaf. In particular, from the above, the foliation on P is ergodic. Hence, if the foliation $\mathcal{F}$ in X is not ergodic, then there exist a leaf saturated measurable subset A of X which is neither null nor conull. From this, it is easily seen that $\pi^{-1}(A)$ is a measurable leaf saturated subset of P which is neither null nor conull. This contradiction proves that $\mathcal{F}$ is ergodic.

Finally, the last claim for transverse Riemannian structures is proved with the same argument used for Lie foliations using the corresponding properties for Riemannian foliations. □

As an immediate consequence of Lemma 5.1, the G-action on M is ergodic and engaging.

We now state the following result. Its proof follows easily from the definitions, and the (easy) fact that there is an isomorphism

$$\pi_1((G \times H)/\Gamma)/\pi_1(G \times H) \cong \Gamma$$

(see [Hu] or [KN]).

LEMMA 5.2. *Let G, H and Γ be given as above. Let $D : G \times H \to H$ be the natural projection. Define $\rho : \pi_1((G \times H)/\Gamma) \to H$ as the composition of the maps*:

$$\pi_1((G \times H)/\Gamma) \to \pi_1((G \times H)/\Gamma)/\pi_1(G \times H) \cong \Gamma \to H$$

where the last arrow is the restriction of D to Γ. Then D is a developing map with holonomy representation ρ for the Lie foliation on $(G \times H)/\Gamma$ given by the local factor H.

We observe that G preserves the (finite) pseudoRiemannian volume on M and so it preserves a finite smooth measure on $K\backslash L/\Gamma$. Since K is compact, it is easy to check that there is a G-invariant finite smooth measure on L/Γ. Moreover, from the proof of Lemma 5.1, it is easy to see that the G-action on L/Γ is ergodic (with respect to the latter G-invariant measure) if and only if $\mathrm{Im}(\rho)$ is dense in H. If such density holds with our current choices of groups, then by Lemma 5.1 the G-action on L/Γ is ergodic and engaging.

On the other hand, if $\mathrm{Im}(\rho)$ is not dense in H, we will show that we can replace H by a smaller connected closed subgroup H_1 and Γ by a normal finite index subgroup Γ_1 for which the conclusions of Theorem A proved up to this point still hold and such that the corresponding holonomy representation has dense image in H_1.

LEMMA 5.3. *Let P_T be the orthogonal transverse frame bundle of the Riemannian* (*or antiRiemannian*) *foliation on $\widehat{M}$. Then the G-action lifts to P_T and there is a G-equivariant map $\varphi : (G \times H)/\Gamma \to P_T$ for which the diagram*:

$$\begin{tikzcd}
(G \times H)/\Gamma \arrow[rr, "\varphi"] \arrow[dr] & & P_T \arrow[dl] \\
& \widehat{M} &
\end{tikzcd}$$

is commutative and realizes $(G \times H)/\Gamma$ as an embedded K-reduction of P_T. Here we recall that $\widetilde{N} = K\backslash H$.

Proof. Since $\widehat{M} = (G \times \widetilde{N})/\Gamma$, it is a simple matter to prove that the transverse frame bundle over $\widehat{M}$ associated to the foliation by G-orbits is given by $(G \times L^{(1)}(\widetilde{N}))/\Gamma$, where $L^{(1)}(\widetilde{N})$ is the linear frame bundle of $\widetilde{N}$. Note that in our notation the group of diffeomorphisms on both N and $\widetilde{N}$ has been chosen to act on the right and so $L^{(1)}(\widetilde{N})$, and its reductions, have left actions for their structure groups.

From the previous description, the G-action on $\widehat{M}$ lifts to the transverse frame bundle of the foliation on $\widehat{M}$ by G-orbits. Hence, since G preserves the Riemannian (or antiRiemannian) structure on the foliation, then the G-action preserves P_T. The latter action is thus the lift of the G-action on $\widehat{M}$.

Let $o = Ke \in K\backslash H$ and consider $\mathrm{Gl}(T_o\widetilde{N})$ as the structure group of $L^{(1)}(\widetilde{N})$. Define the map:

$$\begin{aligned} \widetilde{\varphi} : G \times H &\to G \times L^{(1)}(\widetilde{N}) \\ (g, h) &\mapsto (g, dh_o) \end{aligned}$$

Such map is clearly equivariant with respect to both G and Γ. On the other hand, the restriction of $\widetilde{\varphi}$ to $\{e\} \times K$ clearly realizes K as a closed subgroup of $O(T_o\widetilde{N})$. Also, since H acts by isometries on $\widetilde{N}$, it induces a G-equivariant map $\varphi : (G \times H)/\Gamma \to P_T$.

On the other hand, the natural projection $(G\times H)/\Gamma \to (G\times\widetilde{N})/\Gamma = \widehat{M}$ is clearly surjective, because $\widetilde{N} = K\backslash H$, and its fibers are K-orbits diffeomorphic to K since H acts effectively on $\widetilde{N}$. It follows that, with respect to φ, the manifold $(G \times H)/\Gamma$ is a K-reduction of P_T. □

Now let H, Γ and ρ be as before, where the latter is the holonomy map from Lemma 5.2. Recall that we are assuming that $\overline{\mathrm{Im}(\rho)}$ is a proper subgroup of H. First we observe that by the definition of ρ, the group Γ is a subgroup of $G \times \overline{\mathrm{Im}(\rho)}$. Hence, we clearly have that $(G \times \overline{\mathrm{Im}(\rho)})/\Gamma$ is a closed G-invariant subset of $(G \times H)/\Gamma$. In particular, $(G \times \overline{\mathrm{Im}(\rho)})/\Gamma$ is a union of closures of G-orbits. By the structure of the closures of leaves of a Riemannian foliation (see Theorem 5.1 in page 155 in [Mol]), we know that the closure of any G-orbit in P_T is mapped onto the closure of some G-orbit in $\widehat{M}$. Since we proved that the G-action on M is engaging, we also know that in $\widehat{M}$ every G-orbit is dense, in other words, $\widehat{M}$ is a single leaf closure. Hence, any G-invariant closed subset in $(G\times H)/\Gamma$ maps onto $\widehat{M}$ under the projection $(G\times H)/\Gamma \to (G\times K\backslash H)/\Gamma = \widehat{M}$. Since $(G \times \overline{\mathrm{Im}(\rho)})/\Gamma$ is G-invariant and closed, it is mapped onto $\widehat{M}$ under the latter natural projection.

LEMMA 5.4. *With the above setup,* $\overline{\mathrm{Im}(\rho)}$ *is a closed subgroup of* $\mathrm{Iso}(\widetilde{N})$ *that acts transitively on* $\widetilde{N}$. *In particular, if* H_1 *is the identity component of* $\overline{\mathrm{Im}(\rho)}$, *then* H_1 *acts transitively on* $\widetilde{N}$.

Proof. Let $o = Ke \in K\backslash H = \widetilde{N}$. Choose an arbitrary $y \in \widetilde{N}$. Then by the previous discussion, there exist $(g, h) \in G\times\overline{\mathrm{Im}(\rho)}$ and $(\gamma_1, \gamma_2) \in \Gamma \subset G\times\overline{\mathrm{Im}(\rho)}$

such that:

$$(e, y) = (g, Kh)(\gamma_1, \gamma_2) = (g\gamma_1, Kh\gamma_2) = (g\gamma_1, oh\gamma_2)$$

and so $y = oh\gamma_2$, where $h\gamma_2 \in \overline{\mathrm{Im}(\rho)}$. Hence, $\overline{\mathrm{Im}(\rho)}$ acts transitively on $\widetilde{N}$ and, by connectedness of $\widetilde{N}$, so does H_1. $\square$

By Lemma 5.4 and Corollary 2.2, it follows that H_1 is a normal finite index subgroup of $\overline{\mathrm{Im}(\rho)}$. Hence, the group $\Gamma_1 = (G \times H_1) \cap \Gamma$ is a normal finite index subgroup of Γ. Let K_1 be the stabilizer in H_1 of the point $o = Ke \in K\backslash H = \widetilde{N}$. If we define the manifold $\widehat{M_1} = (G \times K_1\backslash H_1)/\Gamma_1$, then, from the previous constructions, the natural projection map:

$$(G \times K_1\backslash H_1)/\Gamma_1 \to (G \times K\backslash H)/\Gamma = (G \times \widetilde{N})/\Gamma = \widehat{M}$$

defines a finite normal covering of $\widehat{M}$ and so $\widehat{M_1}$ is a finite covering of M.

We claim that if we choose $\widehat{M_1}$ for (1), $L = G \times H_1$ for (2) and $\Gamma_1 \subset L$ and $K_1 \subset C_L(G)$ for (3), then the conclusions of Theorem A are satisfied. In fact, the only point that remains to be proved is that the G-action on L/Γ_1 is ergodic and engaging.

To show this last requirement, we first observe that Lemma 5.2 still holds with our new choices. In other words, the developing map $D_1 : G \times H_1 \to H_1$ and the holonomy representation $\rho_1 : \pi_1((G \times H_1)/\Gamma_1) \to H_1$ of the transverse Lie structure for the foliation by G-orbits on $(G \times H_1)/\Gamma_1$ are the restrictions to $G \times H_1$ and $\pi_1((G \times H_1)/\Gamma_1)$, respectively, of the maps D and ρ as they are defined in Lemma 5.2.

On the other hand, from the proof of Lemma 5.1, to conclude ergodicity and engagement of the G-action on L/Γ_1 it is enough to show that $\overline{\mathrm{Im}(\rho_1)} = H_1$. From the expressions of ρ and ρ_1 in Lemma 5.2, if we denote with $pr : G \times H \to H$ the natural projection, then $\mathrm{Im}(\rho) = \mathrm{pr}(\Gamma)$ and $\mathrm{Im}(\rho_1) = \mathrm{pr}(\Gamma_1)$. So we want to prove that $\overline{\mathrm{pr}(\Gamma_1)} = H_1$.

Let $h \in H_1$ be given. Since $H_1 \subset \overline{\mathrm{Im}(\rho)} = \overline{\mathrm{pr}(\Gamma)}$, there exist a sequence $(\gamma_n)_n \subset \Gamma$ such that $(\mathrm{pr}(\gamma_n))_n$ converges to h. But H_1 is open in $\overline{\mathrm{Im}(\rho)}$, because it is its identity component. Hence, after dropping finitely many terms from the sequence, we can assume that $(\mathrm{pr}(\gamma_n))_n \subset H_1$. This implies that $(\gamma_n)_n \subset G \times H_1$ and so $(\gamma_n)_n \subset (G \times H_1) \cap \Gamma = \Gamma_1$. Hence, we obtain $\overline{\mathrm{Im}(\rho_1)} = H_1$, thus completing the proof of Theorem A.

6. Proof of Theorem B

To prove Theorem B, we assume that G and M satisfy its hypotheses. Hence, we can choose $\widehat{M}$, $L = G \times H$, Γ and K as in the conclusions of Theorem A. First observe that we can assume that $H \neq e$, since otherwise the conclusions of Theorem B are trivially satisfied. In particular, we will assume that the G-action on M is not transitive.

We state the following result. We refer to [Zim1] for the notion of compatible transverse measure.

Lemma 6.1. *Every nontransitive locally free ergodic action that preserves a compatible finite smooth measure transverse to the orbits is properly ergodic.*

Proof. The orbits define a foliation each of whose leaves intersect a foliated chart in a countable number of plaques. Hence, on a foliated chart every leaf intersects a transversal on a countable and thus null set, since such transversals have positive dimension. In particular, every orbit is null and so the action is properly ergodic. □

By Lemma 6.1 the G-action on M is properly ergodic. Then, the hypotheses of the main result in [StZi] are satisfied and such result implies that the G-action on M is essentially free. Now we conclude from this the following.

Lemma 6.2. *The G-action on $\widehat{M}$ is essentially free and the G-action on L/Γ is free.*

Proof. Since the G-action on $\widehat{M}$ is obtained as the lift of the G-action on M with respect to the covering map $\widehat{M} \to M$, it follows that every G-orbit in $\widehat{M}$ is a covering of a G-orbit in M. On the other hand, since the G-action on $\widehat{M}$ is locally free, every G-orbit on $\widehat{M}$ is a quotient of G by a discrete subgroup. If we consider such setup for a G-orbit $\mathcal{O}$ in M with trivial stabilizers and denote with $\widehat{\mathcal{O}}$ a G-orbit in $\widehat{M}$ that lifts $\mathcal{O}$ we obtain a commutative diagram of covering maps as follows:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "f"] & \widehat{\mathcal{O}} \arrow[d] \\ & \mathcal{O} \end{tikzcd}$$

where f is a diffeomorphism.

Such diagram induces a corresponding commutative diagram of fundamental groups given by:

$$\begin{tikzcd} \pi_1(G) \arrow[r] \arrow[dr, "f_*"] & \pi_1(\widehat{\mathcal{O}}) \arrow[d] \\ & \pi_1(\mathcal{O}) \end{tikzcd}$$

which can hold only if $\pi_1(\widehat{\mathcal{O}}) = \pi_1(\mathcal{O})$, since f_* is an isomorphism and the vertical arrow is an inclusion. Then the covering $\widehat{\mathcal{O}} \to \mathcal{O}$ is trivial, which implies that the arrow $G \to \widehat{\mathcal{O}}$ above is a diffeomorphism and so the G-orbit $\widehat{\mathcal{O}}$ has trivial stabilizers.

Hence we have proved that every G-orbit in M with trivial stabilizers lifts to $\widehat{M}$ to G-orbits with trivial stabilizers, and so the G-action on $\widehat{M}$ is essentially free.

A similar argument proves that the G-action on L/Γ is essentially free. For this we use the fact that, as found in the proof of Theorem A, L/Γ is a G-invariant reduction of the orthonormal frame bundle transverse to the foliation by G-orbits, so that the G-orbits in L/Γ are coverings of their projections onto $\widehat{M}$, on which we just proved essential freeness. Then we observe that for any given $l_1, l_2 \in L$, the stabilizers in G of the Γ-classes of such points satisfy $G_{l_2\Gamma} = l_2 l_1^{-1} G_{l_1\Gamma} l_1 l_2^{-1}$. Hence essential freeness of the G-action on L/Γ implies freeness for such action. $\square$

Let K_0 be a maximal compact subgroup of G. By Lemma 6.2 the K_0-action on L/Γ is free and so $K_0\backslash L/\Gamma$ is a compact connected manifold. Moreover, the G-orbits in L/Γ induce a foliation in $K_0\backslash L/\Gamma$ that carries a leafwise Riemannian metric so that each leaf is isometric to the simply connected Riemannian symmetric space $K_0\backslash G$. The simply connectedness of each leaf in $K_0\backslash L/\Gamma$ follows from Lemma 6.2. Also, by Theorem A, the G-action on L/Γ is topologically transitive. From this, it is easy to check that the foliation in $K_0\backslash L/\Gamma$, by Riemannian symmetric spaces, has a dense leaf. Finally, since $L = G \times H$, the foliation on $K_0\backslash L/\Gamma$ carries a transverse Lie structure modelled on H.

The previous discussion shows that the hypotheses in Theorem A in [Zim5] are satisfied and such result implies that H is semisimple, and so $L = G \times H$ is semisimple as well.

In what follows, for every connected semisimple Lie group F we denote with F^{is} the minimal connected closed normal subgroup of F such that F/F^{is} is compact. Then we clearly have $G \subset L^{is}$.

On the other hand, by the proof of Theorem A, Γ projects densely into H by the natural projection $L = G \times H \to H$, which implies that $G\Gamma$ is dense in L. Hence, $L^{is}\Gamma$ is dense in L as well. By Corollary 5.17 in [Rag] it follows that $\Gamma Z(L)$ is discrete (see also Lemma 6.1 in page 329 in [Mar]). Hence, $\Gamma Z(H)$ is discrete as well and, since it contains Γ, it is also a lattice. This clearly implies that Γ has finite index in $\Gamma Z(H)$. But it is easy to prove that there is a bijection:

$$(\Gamma Z(H))/\Gamma \cong Z(H)/(\Gamma \cap Z(H))$$

and so $\Gamma \cap Z(H)$ has finite index in $Z(H)$.

Now denote $Z = \Gamma \cap Z(H)$, and observe that there is an equivariant diffeomorphism:

$$\frac{L/Z}{\Gamma/Z} \cong L/\Gamma$$

as well as isomorphisms:

$$KZ/Z \cong K/(K \cap Z) \cong K$$

where the latter follows from $K \cap \Gamma = e$, which is a consequence of the freeness of the K-action on L/Γ.

Hence, if we replace $L = G \times H$, H, Γ and K with $L/Z = G \times H/Z$, H/Z, Γ/Z and KZ/Z, then the corresponding $\widehat{M}$ for the new choices is given by the double coset $(KZ/Z)\backslash(L/Z)/(\Gamma/Z)$, which is easily seen to be G-equivariantly diffeomorphic to our original choice $K\backslash L/\Gamma$. Moreover, it is also clear that all the properties proved so far still hold. Also, since we modded out by Z, which has finite index in $Z(H)$, it follows that for our new choices the group L has finite center.

To complete to proof of Theorem B we will show that, for the above choices, Γ is irreducible and L is isotypic. To achieve this, we state the following general result.

Lemma 6.3. *Let F be a connected semisimple Lie group with finite center and Λ a lattice in F such that $F^{is}\Lambda$ is dense in F. Then the following conditions are equivalent.*

(a) *For every closed connected noncompact normal subgroup N of F, the group $N\Lambda$ is dense in F.*

(b) *For every closed connected noncompact normal proper subgroups F', F'' of F that satisfy $F = F'F''$ with $F' \cap F''$ finite, the group $(\Lambda \cap F')(\Lambda \cap F'')$ has infinite index in Λ.*

Proof. This result is essentially contained in Chapter V of [Rag] although it is not explicitly stated. The proof is obtained from the results in [Rag] as follows.

When $F = F^{is}$, the equivalence is part of Corollary 5.21 in [Rag] and the definition of irreducible lattice in this reference. The proof of such corollary ultimately depends on Theorem 5.5 in [Rag].

We then observe that our condition on Λ implies that it satisfies property (SS) (see [Rag] for the definition) and so the hypotheses of Theorem 5.26 in [Rag] are satisfied. In our case, it is easy to see that, if we apply the arguments that prove Corollary 5.21 in [Rag] using Theorem 5.26 in [Rag] instead of Theorem 5.5 in [Rag], we conclude the equivalence between (a) and (b). □

The previous lemma shows that the definition of irreducible lattice in a connected semisimple Lie group with finite center (that may admit compact factors) as found in [Mor], which is given by condition (a), is equivalent to condition (b). We now use this to prove that our lattice Γ is irreducible in L, with irreducibility as defined in [Mor] to be able to apply results therein.

First recall that both $G\Gamma$ and $L^{is}\Gamma$ are dense in L. By Lemma 6.3, if Γ is not irreducible, then there exist closed connected noncompact normal proper subgroups L', L'' of L that satisfy $L = L'L''$ with $L' \cap L''$ finite and such that the group $(\Gamma \cap L')(\Gamma \cap L'')$ has finite index in Γ. It follows easily that $G(\Gamma\cap L')(\Gamma\cap L'')$ is dense in L as well. Since G is a simple factor of L, it is either contained in L' or in L''. If $G \subset L'$, then we have $G(\Gamma\cap L')(\Gamma\cap L'') \subset L'(\Gamma\cap L'')$, where the latter is a closed proper subgroup of L, which yields a contradiction. An analogous contradiction is obtained if $G \subset L''$ and so Γ is irreducible.

Finally, since L admits an irreducible lattice and is not isogenous to either $\mathrm{SO}(1,n)\times C$ or $\mathrm{SU}(1,n)\times C$, for any nontrivial connected compact group C, it follows from Problem 6.8 in [Mor] that L is isotypic. This concludes the proof of Theorem B.

7. Proof of Theorem C

The implication (2)$\Rightarrow$(1) is trivial by considering a pseudoRiemannian metric on $K\backslash L/\Gamma$ as in Proposition 1.2.

Conversely, if we assume (1), then we can apply Theorem B to conclude that $\widehat{X}$ has a finite covering of the form $K\backslash L/\Gamma$ to which the G-action lifts. But then $K\backslash L/\Gamma$ is a finite covering of X and (2) holds.

8. Further results and consequences

A careful examination of the proofs in the previous sections show that our assumption of compactness for M in Theorem A is used only to ensure completeness of the Riemannian or antiRiemannian structure transverse to the G-orbits. Hence, the same proof allows to obtain the following result.

THEOREM A$'$. *Let G be a connected noncompact simple Lie group. If G acts faithfully and topologically transitively on a* (*not necessarily compact*) *manifold M preserving a finite volume complete pseudoRiemannian metric such that $n_0 = m_0$, then the G-action on M is ergodic and engaging, and there exist*:

(1) *a finite covering $\widehat{M} \to M$,*

(2) *a connected Lie group L that contains G as a factor,*

(3) *a discrete subgroup Γ of L and a compact subgroup K of $C_L(G)$,*

for which the G-action on M lifts to $\widehat{M}$ so that $\widehat{M}$ is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$. Furthermore, there is an ergodic and engaging G-invariant finite smooth measure on L/Γ.

On the other hand, the proof of our Theorem B makes use of Theorem A in [Zim5]. The latter requires compactness only to ensure the existence of a G-invariant finite smooth measure on a suitable manifold (see Proposition 2.5 in [Zim5]). Besides that, no further use of the compactness condition is made in [Zim5]. For our setup, to have a corresponding version of Theorem A in [Zim5] for finite volume manifolds, we need to ensure that the G-action on L/Γ has an invariant finite smooth measure, which is a conclusion in our Theorem A$'$. The proof of Theorem B also applies Molino's work on Riemannian and Lie foliations, which are usually stated for compact manifolds. However, if we assume completeness of the manifold M, then the orthonormal transverse frame bundle that appears in our arguments is transversely complete and Molino's work still applies. Hence, the same arguments used to prove Theorem B allows to obtain the following result.

Theorem B$'$. *Let G be a connected noncompact simple Lie group with finite center and* $\operatorname{rank}_{\mathbb{R}}(G) \geq 2$. *If G acts faithfully and topologically transitively on a noncompact manifold M preserving a finite volume complete pseudo-Riemannian metric such that $n_0 = m_0$, then there exist*:

(1) *a finite covering $\widehat{M} \to M$,*

(2) *a connected isotypic semisimple Lie group L without compact factors and with finite center that contains G as a factor,*

(3) *an irreducible lattice Γ of L and a compact subgroup K of $C_L(G)$,*

for which the G-action on M lifts to $\widehat{M}$ so that $\widehat{M}$ is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$. Hence, up to fibrations with compact fibers, M is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$ and L/Γ.

The additional property for L in (2), its lack of compact factors, follows from the following facts. Being irreducible, the lattice Γ is arithmetic in L since the latter has finite center and is not isogenous to either $\mathrm{SO}(1,n) \times C$ or $\mathrm{SU}(1,n) \times C$ for a compact group C. On the other hand, arithmetic lattices which are not cocompact can only occur in semisimple Lie groups without compact factors. Such claims follow from Theorem 6.21 and Corollary 6.55 in [Mor].

From the above we obtain the following result.

Theorem C$'$. *Let G be a connected noncompact simple Lie group with finite center and* $\operatorname{rank}_{\mathbb{R}}(G) \geq 2$. *Assume that G acts faithfully on a noncompact manifold X. Then the following conditions are equivalent.*

(1) *There is a finite covering $\widehat{X} \to X$ for which the G-action on X lifts to a topologically transitive G-action on $\widehat{X}$ that preserves a finite volume complete pseudoRiemannian metric such that $n_0 = m_0$.*

(2) *There is a connected isotypic semisimple Lie group L without compact factors and with finite center that contains G as a factor, an irreducible lattice Γ of L and a compact subgroup K of $C_L(G)$ such that $K\backslash L/\Gamma$ is a finite covering of X with G-equivariant covering map.*

We end this section with detailed proofs of Theorems 1.3, 1.4 and 1.5.

Proof of Theorem 1.3. We present the arguments for compact M, but the proof is similar for M complete with finite volume. By Corollary 3.7, M has a finite covering of the form $(G \times \widetilde{N})/\Gamma_0$. Furthermore, from the proof of Theorem A it follows that the group Γ_0 has a finite index subgroup Γ_1 contained in $R(G) \times \mathrm{Iso}(\widetilde{N})$. In particular, $M_1 = (G \times \widetilde{N})/\Gamma_1$ is a finite covering space of M. Hence it is enough to define a metric on M_1 from one on $G \times \widetilde{N}$ which is the direct product of a right invariant metric on G and the metric on $\widetilde{N}$. □

Proof of Theorem 1.4. Let $\widehat{M}$ and Γ be as in the conclusions of Theorems B or B′. By (2) and (3) in Theorems B and B′, Γ admits a linear representation with the required properties. Then the result is a consequence of the following facts: 1) $\pi_1(\widehat{M})$ has finite index in $\pi_1(M)$, and 2) there is a surjective homomorphism $\pi_1(\widehat{M}) \to \Gamma$. □

Proof of Theorem 1.5. Choose $\widehat{M} = K\backslash L/\Gamma$ a finite covering of M as provided by Theorems B or B′. By general results, it follows that the G-action on M has finite entropy even if M is not compact (see [LZ1] and [Zim2]).

On the other hand, if we project L, K and Γ using the adjoint representation, then it is easy to obtain an arithmetic G-space (as defined in [LZ1]), say $K'\backslash L'/\Gamma'$, which admits a G-equivariant covering of $\widehat{M}$. Then the result is a consequence of the following two facts: 1) for our setup, $A(K'\backslash L'/\Gamma') = K'\backslash L'/\Gamma'$, 2) if $Y_1 \to Y_2$ is a G-equivariant covering map with both G-actions ergodic and with finite entropy, then $A(Y_1) = A(Y_2)$. Both facts follow easily from the results and definitions in [LZ1]. □

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