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Orbit equivalence rigidity and bounded cohomology

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Abstract

We establish new results and introduce new methods in the theory of measurable orbit equivalence, using bounded cohomology of group representations. Our rigidity statements hold for a wide (uncountable) class of groups arising from negative curvature geometry. Amongst our applications are (a) measurable Mostow-type rigidity theorems for products of negatively curved groups; (b) prime factorization results for measure equivalence; (c) superrigidity for orbit equivalence; (d) the first examples of continua of type II_1 equivalence relations with trivial outer automorphism group that are mutually not stably isomorphic.

Contents

1. Introduction

In this paper, a companion to [MS2], we continue our attempts to widen the scope of rigidity theory, using new techniques made available by the bounded cohomology methods recently developed by Burger and Monod [BM2], [M]. In the present paper, we focus our attention on rigidity of measurable orbit equivalence, an area which has seen remarkable achievements by R. Zimmer during the 80's, and in the last few years has flourished again with the striking work of A. Furman [F1], [F2], [F3] and D. Gaboriau [Ga1], [Ga2], [Ga3]. Our main purpose is to establish new rigidity phenomena, some reminiscent of those known in the case of higher rank lattices, for a large (uncountable) class of groups arising geometrically in the general framework of "negative curvature":

Examples 1.1. Consider the collection of all countable groups Γ which admit either: (i) A nonelementary simplicial action on some simplicial tree, proper on the set of edges; or (ii) A nonelementary proper isometric action on some proper CAT(-1) space; or (iii) A nonelementary proper isometric action on some Gromov-hyperbolic graph of bounded valency.

Non-Abelian free groups are outstanding examples of groups in this class; indeed, the main rigidity results below are already interesting in that case. Notice that since *any nontrivial free product of two countable groups* is in the list above (unless they are finite of order 2), this class is uncountable; it also contains the uncountable class of nonelementary subgroups of Gromov-hyperbolic groups. In particular, this collection of groups includes the fundamental group of any closed manifold of negative sectional curvature.

The Examples 1.1 are given as a matter of convenience to make this introduction more concrete; it is in fact only a certain cohomological property of these groups which plays a role in our approach. Indeed, we introduce the following:

Notation 1.2. Denote by $\mathcal{C}_{\mathrm{reg}}$ *the class of countable groups* Γ with $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(\Gamma)) \neq 0$.

This definition refers to the bounded cohomology of Γ with coefficients in the regular representation; see Sections 3 and 7 for the relevant background. When stating our results in Section 2 in full generality, we use a possibly larger class $\mathcal{C}$. For the time being, however, suffice it to indicate that indeed $\mathcal{C}_{\mathrm{reg}}$ is strongly related to the geometric notion of negative curvature, as the following indicates:

THEOREM 1.3. *All the groups of Examples* 1.1 *belong to* $\mathcal{C}_{\mathrm{reg}}$.

This statement can be seen as a cohomological property of negative curvature and relies on the results of [MS2] complemented with [MMS]. However, we shall offer in Section 7.2 a short independent proof that many examples, including free groups, belong to the class $\mathcal{C}_{\mathrm{reg}}$.

Before recalling the notion of measurable orbit equivalence, let us fix the following convention: For a discrete group Γ we say that a standard measure space (X, μ) is a *probability* Γ*-space* if $\mu(X) = 1$ and Γ acts measurably on X, preserving μ. In this paper, all such actions are assumed *essentially free*; i.e., the stabiliser of almost every point is trivial.

Definition 1.4. Let Γ and Λ be countable groups and (X, μ), (Y, ν) be probability Γ- and Λ-spaces respectively. A measurable isomorphism $F : X \to Y$ is said to be an *Orbit Equivalence* of the actions if for a.e. $x \in X$: $F(\Gamma x) = \Lambda F(x)$, i.e., if F takes almost every Γ-orbit bijectively onto a Λ-orbit.

In that case, the two actions are called *Orbit Equivalent* (OE), and we say that a (possibly different) isomorphism $\widetilde{F} : X \to Y$ induces this orbit equivalence if $\widetilde{F}(\Gamma x) = F(\Gamma x)$ for a.e. $x \in X$.

The starting point of orbit equivalence rigidity theory lies in the remarkable lack-of-rigidity phenomenon established by Ornstein-Weiss [OW] (generalised by Connes-Feldman-Weiss [CFW]), following H. Dye [Dy], for the class of amenable groups: *Any two ergodic probability measure-preserving actions of countable amenable groups are* OE. (Shortly we shall mention another different motivation for OE rigidity theory, related to geometric group theory.) To put our main results in a better perspective, we observe first that this absence of rigidity can be extended also to some nonamenable groups (see Theorem 2.27):

Any given probability measure-preserving action of a countable free group is orbit equivalent to actions of uncountably many different groups.

Of course, a similar lack of rigidity follows for product actions of products of free groups. The main point of several of our results is this: For such product groups, a surprisingly rigid behaviour occurs if we rule out product actions by the following ergodicity property.

Definition 1.5. Let $\Gamma = \Gamma_1 \times \Gamma_2$ be a product of countable groups. A Γ-space (X, μ) is called *irreducible* if both Γ_i act ergodically on X.

For clarity of the exposition we shall formulate here some of our main results for two factors only, and in partial generality; Section 2.2 contains the general statements.

Observe that irreducibility depends on the given product structure on Γ, rather than on Γ alone. Among the many natural examples of irreducible actions, we mention here those we shall make explicit use of: Bernoulli actions (see below), products of unbounded real linear groups acting on homogeneous spaces (see Section 2.5 below), and left-right multiplication actions of products of groups which are both embedded densely in one compact group (see the proof of Theorem 1.14 below).

THEOREM 1.6 (OE Strong Rigidity – Products). *Let Γ_1, Γ_2 be torsion-free groups in $\mathcal{C}_{\mathrm{reg}}$, $\Gamma = \Gamma_1 \times \Gamma_2$, and let (X, μ) be an irreducible probability Γ-space. Let (Y, ν) be any other probability Γ-space* (*not necessarily irreducible*). *If the Γ-actions on X and Y are* OE, *then they are isomorphic with respect to an automorphism of Γ. More precisely*, *there is $f \in \mathrm{Aut}(\Gamma)$ such that the orbit equivalence is induced by a Borel isomorphism $F : X \to Y$ with $F(\gamma x) = f(\gamma)F(x)$ for all $\gamma \in \Gamma$ and a.e. x.*

Notice that composing an action with a group automorphism yields an orbit equivalent action, but in general one which is not isomorphic. Unlike the case of higher rank lattices, for some groups covered by the theorem (such as products of free groups), there is an abundance of such automorphisms which should be "detected". As observed in Section 2.2 below, Theorem 1.6 is not valid in general if the groups are not in the class $\mathcal{C}_{\mathrm{reg}}$.

Using Theorem 1.6 we are able to produce the first examples of finitely generated groups outside the distinguished family of higher rank lattices in semi-simple Lie groups, possessing infinitely many nonorbit equivalent actions (see also the "exotic" infinitely generated groups in [BG]). In fact we show more:

THEOREM 1.7 (Many groups with many actions). *There exists a continuum $2^{\aleph_0}$ of finitely generated torsion-free groups, each admitting a continuum of measure-preserving free actions on standard probability spaces, such that no two actions in this whole collection are orbit equivalent.*

Although we are able to include products of (non-Abelian) free groups in this family, it is still an open problem to produce infinitely many mutually nonorbit equivalent actions of one free group.

(Added in proof: D. Gaboriau and S. Popa have since obtained a continuum of non-OE actions of a free group [GP], while G. Hjorth established that all infinite Kazhdan groups share this property [Hj].)

To proceed one step further, we recall the following notion:

Definition 1.8. A measure-preserving action of a group Λ on a measure space (Y, ν) is called *mildly mixing* if there are no nontrivial recurrent sets, i.e., if for any measurable $A \subseteq X$ and any sequence $\lambda_i \to \infty$ in Λ, one has $\nu(\lambda_i A \triangle A) \to 0$ only when A is null or co-null.

Here is now a *superrigidity*-type result:

THEOREM 1.9 (OE superrigidity for products – torsion free case). *Let $\Gamma = \Gamma_1 \times \Gamma_2$ and (X, μ) be as in Theorem* 1.6. *Let Λ be* any torsion-free *countable group and let (Y, ν) be any mildly mixing probability Λ-space.*

If the Γ- and Λ-actions are OE *then Λ is isomorphic to Γ, and the actions on X, Y are isomorphic (with respect to an isomorphism $\Gamma \cong \Lambda$).*

Actually we prove a more general statement, dropping the torsion-freeness assumption on Λ, thereby allowing "commensurable situations". We state here the following result, which is generalised further in Section 2:

THEOREM 1.10 (OE superrigidity – product). *Let $\Gamma = \Gamma_1 \times \Gamma_2$ and (X, μ) be as in Theorem* 1.6. *Let Λ be* any *countable group and let (Y, ν) be any mildly mixing probability Λ-space. If the Γ- and Λ-actions are* OE *then both the groups Γ and Λ, as well as the actions, are commensurable. More precisely*:

(i) *There exist a finite index subgroup* $\Gamma_0 < \Gamma$, *whose projections to both factors* Γ_i *are onto*, *a finite normal subgroup* $N \lhd \Lambda$ *with* $|N| = [\Gamma : \Gamma_0]$, *and a short exact sequence*

$$1 \to N \to \Lambda \to \Gamma_0 \to 1$$

such that:

(ii) *The* Γ-*action induced from the* $\Lambda/N \cong \Gamma_0$-*action on* $(N\backslash Y, \nu)$ *is* isomorphic *to its action on* (X, μ) (*with respect to an automorphism of* Γ).

In particular, *if either the* Γ-*action on* X *is aperiodic* (*i.e.*, *remains ergodic under any finite index subgroup*), *or* Λ *is torsion-free*, *then* Λ *is isomorphic to* Γ *and the actions on* X, Y *are isomorphic* (*with respect to an isomorphism* $\Gamma \cong \Lambda$).

This theorem is optimal in the sense that any Λ satisfying (i) above admits an action which is OE to an irreducible action of Γ. A crucial ingredient in the proof of this theorem is a remarkable idea of A. Furman from [F1] in the framework of simple Lie groups, which we adapt here for our purposes. In Example 2.22 below we show by means of a counter-example why the mild mixing condition is natural in our context, and how Theorem 1.10 may fail for actions which are weakly mixing, and "close to being" mildly mixing. Of course, the simplest examples of mildly mixing actions are (strongly) mixing actions, and those exist for any group, as in the following standard construction: For a countable group Γ and any probability distribution μ (different from Dirac) on the interval $[0,1]$, call the natural shift Γ-action on the product space $([0,1]^\Gamma, \mu^\Gamma)$ a *Bernoulli* Γ-action. Any such action can easily be seen to be mixing, and this takes care at the same time of irreducibility and aperiodicity. We therefore have:

COROLLARY 1.11. *Let* $\Gamma = \Gamma_1 \times \Gamma_2$ *where each* Γ_i *is a torsion-free group in* $\mathcal{C}_{\mathrm{reg}}$. *If a Bernoulli* Γ-*action is orbit equivalent to a Bernoulli* Λ-*action for some* arbitrary *group* Λ, *then* Γ *and* Λ *are isomorphic and with respect to some isomorphism* $\Gamma \cong \Lambda$ *the actions are isomorphic by a Borel isomorphism which induces the given orbit equivalence.* □

As shown by the result of Ornstein and Weiss cited above, amenable groups share a sharp lack of rigidity in the measurable orbit equivalence theory. Our next two results are analogous to two of the theorems above, only that here we replace the setting of products by one involving amenable radicals. We show a similar rigid behaviour modulo the intrinsic lack of rigidity caused by the presence of such radicals.

THEOREM 1.12 (OE Strong Rigidity – Radicals). *Let* Γ *be a group and* $M \lhd \Gamma$ *a normal amenable subgroup such that the quotient* $\bar{\Gamma} = \Gamma/M$ *is torsion-*

free and in $\mathcal{C}_{\text{reg}}$. *Let* (X,μ), (Y,ν) *be probability* Γ*-spaces on which* M *acts ergodically. If the two* Γ*-actions are* OE *then there is a Borel isomorphism* $F : X \to Y$ *such that for all* $\gamma \in \Gamma$ *and a.e.* $x \in X$: $F(\gamma M x) = f(\bar{\gamma}) M F(x)$, *where* $\bar{\gamma} = \gamma M$ *and* f *is some automorphism of* $\bar{\Gamma}$.

Here is the superrigidity-type version:

THEOREM 1.13 (OE Superrigidity – Radicals). *Let* Γ *and* (X,μ) *be as in Theorem* 1.12. *Let* Λ *be* any countable group *and let* (Y,ν) *be any mildly mixing probability* Λ*-space. If the* Γ*- and* Λ*-actions are* OE *then there exists an infinite normal amenable subgroup* $N \lhd \Lambda$ *such that* Λ/N *is isomorphic to* Γ/M. *Moreover*, *there is an isomorphism* $f : \Gamma/M \to \Lambda/N$ *such that the* OE *is induced by a Borel isomorphism* $F : X \to Y$ *satisfying* $F(\gamma M x) = f(\bar{\gamma}) N F(x)$.

In a different direction, we can apply Theorem 1.6 to study countable ergodic relations of type II_1. We first recall some terminology (see also [F2], [F3]).

Let Γ be a countable group and (X,μ) be an ergodic probability Γ-space. Let $\mathcal{R} = \mathcal{R}_{\Gamma,X} \subseteq X \times X$ denote the (type II_1) equivalence relation on X defined by that action, i.e. $(x,y) \in \mathcal{R}$ if and only if $\Gamma x = \Gamma y$. Two such relations are isomorphic if and only if the two actions are OE. Further, the group of automorphisms $\mathrm{Aut}(\mathcal{R})$ of the relation $\mathcal{R}$ is the group of measure-preserving isomorphisms $F : X \to X$ such that $F(\Gamma x) = \Gamma F(x)$ for a.e. $x \in X$. Moreover, one defines the inner and outer automorphism groups by

$$\mathrm{Inn}(\mathcal{R}) = \big\{ F \in \mathrm{Aut}(\mathcal{R}) \ : \ F(x) \in \Gamma x \ \ \mu\text{–a.e.} \big\}, \qquad \mathrm{Out}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\mathrm{Inn}(\mathcal{R}).$$

While $\mathrm{Inn}(\mathcal{R})$ (the so-called *full group*) is always very large (e.g. it acts essentially transitively on the collection of all measurable subsets of a given measure), it is of interest to find relations – or group actions – for which $\mathrm{Out}(\mathcal{R})$ is small, or even trivial. The first construction of some $\mathcal{R}_{\Gamma,X}$ with trivial outer automorphism group is due to S. Gefter [Ge1], [Ge2]. Recently A. Furman [F3] has produced more examples within a comprehensive study of the problem in the setting of higher rank lattices (these are used, along with Zimmer's cocycle superrigidity, by both authors). Furman constructs a continuum of mutually nonisomorphic type II_1 relations with trivial outer automorphism group which are all *weakly isomorphic* (see (i) in Definition 2.1 below), being obtained by restricting one fixed relation $\mathcal{R}_{\Gamma,X}$ to subsets of different measure. We show the following:

THEOREM 1.14 (Many Relations with Trivial Out). *There exists a continuum of mutually non weakly isomorphic relations of type* II_1 *with trivial outer automorphism group.*

As mentioned earlier, the study of orbit equivalence can be motivated also from an entirely different point of view, being a measurable counterpart to geometric (or quasi-isometric) equivalence of groups. This analogy, as well as the following notion, were suggested by M. Gromov [Gr, 0.5.E]:

Definition 1.15. Two countable groups Γ, Λ are called *Measure Equivalent* (ME) if there is a standard (infinite-) measure space (Σ, m) with commuting measure-preserving Γ- and Λ-actions, such that each one of the actions admits a finite measure fundamental domain. (In particular, both actions are free, even though not necessarily the product action – see also Remark 2.14 below.) The space (Σ, m) endowed with these actions is called an ME *coupling* of Γ and Λ.

The analogy with geometric group theory can be seen as follows: Replacement of Σ in in Definition 1.15 by a locally compact space on which Γ and Λ act properly, continuously and co-compactly, in a commuting way, results in a notion strictly equivalent to Γ being quasi-isometric to Λ, see [Gr, 0.2.C].

On the other side, ME relates back to OE because of the following fact, observed by Zimmer and Furman (see Section 2.1 below): For two discrete groups Γ and Λ, admitting some OE actions is equivalent to having an ME coupling where the two groups have the same co-volume. (The case of arbitrary co-volumes corresponds to *weak orbit equivalence* which we actually cover in all of our results, but preferred not to discuss in the introduction – see Section 2 below.) Thus, results concerning orbit and measure equivalence can be transformed one to the other (a fact we shall take advantage of, following Furman's approach), and may both come under the title "measurable group theory" – a counterpart to geometric group theory.

THEOREM 1.16 (ME Rigidity – Factors). *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and $\Lambda = \Lambda_1 \times \cdots \times \Lambda_{n'}$ be products of torsion-free countable groups. Assume that all the Γ_i's are in $\mathcal{C}_{\mathrm{reg}}$. If Γ is* ME *to Λ, then $n \geq n'$, and if equality holds then, after permutation of the indices, Γ_i is* ME *to Λ_i for all i.*

This may be viewed as a far reaching extension of the phenomena established by R. Zimmer [Z1] and S. Adams [A1] to the effect that the orbit relation generated by "negatively curved" groups is not a product relation. Illustrating the analogy with geometric group theory, we point out that the arguments of Eskin-Farb [EF1], [EF2] or Kleiner-Leeb [KL] can be used to show that if two products of nonelementary hyperbolic groups are quasi-isometric, then so are the factors (after permuting indices).

For amenable radicals we have the following analogue:

THEOREM 1.17 (ME Rigidity – Quotients by Radicals). *Let Γ, Λ be countable groups and let $M \lhd \Gamma, N \lhd \Lambda$ be amenable normal subgroups such that*

$\bar{\Gamma} = \Gamma/M$ and $\bar{\Lambda} = \Lambda/N$ are in $\mathcal{C}_{\rm reg}$ and are torsion-free. If Γ is ME *to Λ, then $\bar{\Gamma}$ is* ME *to $\bar{\Lambda}$.*

As mentioned earlier, our new approach to orbit equivalence rigidity uses notably the new approach to bounded cohomology recently developed by Burger-Monod [BM2], [M]. The latter provides both results as well as "working tools" which turn out to be very effective in the setting of measurable orbit equivalence. Aiming the paper at the broader audience interested in orbit equivalence rigidity, we shall assume here no prior familiarity with bounded cohomology, and present in Section 3 below a friendly and brief introduction to this theory, including the main results that we need from Burger-Monod's work. Suffice it to say at this point that we define (second) bounded cohomology similarly to usual (second) group cohomology, but using bounded cochains. As a by-product of our proofs, we get some new cohomological invariants of measure equivalence, and consequently some additional "softer" rigidity results, as in the following (see Corollary 7.6):

THEOREM 1.18. *The vanishing of the second bounded cohomology with coefficients in the regular representation is an* ME *invariant.*

COROLLARY 1.19. *A countable group containing an infinite normal amenable subgroup is not* ME *to any group in $\mathcal{C}_{\rm reg}$.*

It follows for instance that such a group cannot be ME to any (nonelementary) Gromov-hyperbolic group; the latter statement was established for the particular case of infinite *center* by S. Adams [A2].

Related results. In the framework of reducibility of Borel relations, G. Hjorth and A. Kechris [HK] established rigidity results for certain types of products in independent work carried out at about the same time.

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Added in Proof: Since the acceptance of this paper for publication, many new results in the emerging measurable group theory appeared, particularly with the ground-breaking work of S. Popa. We refer the reader to the accounts [Po] [Sh2] for further details and references.

2. Discussion and applications of the main results

2.1. *Weak orbit equivalence and measure equivalence.* In this subsection, we recall some basic facts about the relation between orbit and measure equivalence, which will enable us to reformulate a number of our main results in the stronger form in which they will be proved. The material of this subsection follows [F2, §§2–3] wherein the reader can find more details and proofs. As a matter of notation, we shall use only *left* actions and cocycles.

We recall our standing convention that (X, μ) is called a *probability* Γ*-space* if it is a standard probability space with an essentially free measurable Γ-action preserving μ. Thus all corresponding measurable equivalence relations will be of type II_1.

Definition 2.1 (Weak Orbit Equivalence). Let Γ and Λ be countable groups and (X, μ), (Y, ν) be probability Γ- and Λ-spaces respectively. The two actions are said to be *weakly orbit equivalent* (WOE) or stably orbit equivalent, if either one of the following two equivalent conditions holds:

(i) The two equivalence relations induced by the Γ- and Λ-actions are *weakly isomorphic*, i.e., there exist nonnull measurable subsets $A \subseteq X$, $B \subseteq Y$ on which the restrictions of the relations are isomorphic. More precisely, for some A, B as above, a measurable isomorphism $F : A \to B$ and all $x_1, x_2 \in A$, one has $\Gamma x_1 \cap A = \Gamma x_2 \cap A$ if and only if $\Lambda F(x_1) \cap B = \Lambda F(x_2) \cap B$.

(ii) There exist measurable maps $p : X \to Y$, $q : Y \to X$ such that:

1. $p_*\mu \prec \nu$, $q_*\nu \prec \mu$ (where $\prec$ denotes absolute continuity of measures).
2. $p(\Gamma x) \subseteq \Lambda p(x)$ and $q(\Lambda y) \subseteq \Gamma q(y)$ for a.e. $x \in X$, $y \in Y$.
3. $q \circ p(x) \in \Gamma x$ and $p \circ q(y) \in \Lambda y$ for a.e. $x \in X$, $y \in Y$.

Orbit equivalence as defined in the introduction is of course a special case of WOE with A, B of full measure in (i) or with p, q inverse measurable isomorphisms in (ii). As we shall see, WOE is a useful notion even if one is interested in OE only.

Definition 2.2 (Compression Constant). With assumptions and notation as in Definition 2.1, one defines the *compression constant*

$$C(X, Y) = \nu(B)/\mu(A),$$

where A, B are as in point (i) of Definition 2.1. The compression constant depends on the given WOE but not on the choice of A, B.

PROPOSITION 2.3. *With notation as above, assume that the* Γ- *and* Λ-*actions on* X, Y *are ergodic. Then* $C(X, Y) = 1$ *if and only if the actions are* OE. $\square$

Definition 2.4 (WOE Cocycles). Retain the notation of point (ii) of Definition 2.1. Due to essential freeness, one can define measurable cocycles $\alpha : \Gamma \times X \to \Lambda$ and $\beta : \Lambda \times Y \to \Gamma$ by the a.e. requirements $\alpha(\gamma, x)p(x) = p(\gamma x)$ and $\beta(\lambda, y)q(y) = q(\lambda y)$. Recall that the cocycle identity reads here $\alpha(\gamma\gamma', x) = \alpha(\gamma, \gamma' x)\alpha(\gamma', x)$.

When two actions are WOE – or even OE – the maps which send orbits into orbits are of course far from being unique. Supposing for simplicity that the actions on X, Y are OE, one can perturb an orbit equivalence $F : X \to Y$ by any measurable assignment $\varphi : X \to \Lambda$, thereby defining $\widetilde{F}(x) = \varphi(x)F(x)$, which induces the same OE. It is easy to see that any isomorphism $\widetilde{F}$ inducing the same OE is actually obtained in this way, and that this yields a cohomologous (or *equivalent*) cocycle $\widetilde{\alpha} \sim \alpha$. For later reference we record the following elementary result.

LEMMA 2.5. *With the above notation, suppose that the* Γ- *and* Λ-*actions are* OE *and that the associated cocycle* $\alpha : \Gamma \times X \to \Lambda$ *is equivalent to a cocycle* $\widetilde{\alpha}$ *which does not depend on* $x \in X$. *Then the essential value map* $f : \Gamma \to \Lambda$ *determined by* $\widetilde{\alpha}$ *is a group isomorphism and the* OE *is induced by an isomorphism* $\widetilde{F} : X \to Y$ *which intertwines the actions relatively to* f (*i.e. the actions are isomorphic with respect to* f). $\square$

We finally observe that even if one perturbs an OE map $F : X \to Y$ to obtain $\widetilde{F}$ as above, the latter will in general *not* be a bijection and hence *a priori* not describe an OE. However it will induce a WOE, and the WOE context is stable under this operation; hence this setting is more natural and convenient to work with. The viewpoint of measure equivalence, which we now turn to, enables us to remove completely the arbitrary choice of the map F inducing the (weak) orbit equivalence.

Recall from the introduction (Definition 1.15) the definition of an ME coupling (Σ, m) between two countable groups Γ, Λ. We shall say that the ME coupling Σ is *ergodic* if the $\Gamma \times \Lambda$-action on Σ is ergodic; this is equivalent to the ergodicity of Γ on $\Lambda\backslash\Sigma$, or to the ergodicity of Λ on $\Gamma\backslash\Sigma$.

Recall that the Γ-action on Σ admits by definition a measurable fundamental domain $Y \subseteq \Sigma$ with $0 < m(Y) < \infty$. Likewise, let X be such a fundamental domain for Λ. We shall always endow $\Gamma\backslash\Sigma$ with the measure *restricted* from m via the identification $\Gamma\backslash\Sigma \cong Y$, and likewise for $\Lambda\backslash\Sigma \cong X$.

In order to distinguish from the original Γ-action on Σ, we denote by $\gamma \cdot x$ the measurable measure-preserving Γ-action on X obtained by $\Lambda \backslash \Sigma \cong X$ from the commutativity of the Γ- and Λ-actions. Likewise, we have also a "dot" Λ-action $\lambda \cdot y$ on Y.

Definition 2.6 (Retractions, ME Cocycles). Let $\chi : \Sigma \to \Gamma$ be the measurable Γ-equivariant map defined by: $\chi(x)^{-1}x \in Y$ for all $x \in \Sigma$. Then we call χ the *retraction* associated to Y. Likewise, there is a Λ-equivariant retraction $\kappa : \Sigma \to \Lambda$ associated to X. We obtain thus cocycles $\alpha : \Gamma \times X \to \Lambda$ and $\beta : \Lambda \times Y \to \Gamma$ (with respect to the "dot" actions) by setting $\alpha(\gamma, x) = \kappa(\gamma x)^{-1}$ and $\beta(\lambda, y) = \chi(\lambda y)^{-1}$.

Thus we have for all $x \in X$ and $\gamma \in \Gamma$ the formula

$$\gamma \cdot x = \alpha(\gamma, x)\gamma x$$

and likewise for $\lambda \cdot y$. Observe also that one can define maps $p_\chi : X \to Y$ and $q_\kappa : Y \to X$ by $p_\chi(x) = \chi(x)^{-1}x$ and $q_\kappa(y) = \kappa^{-1}(y)y$.

Example 2.7 (Trivial Coupling). Let (Σ, m) be an ME coupling of Γ with Λ and assume that both actions on Σ are simply transitive (with m purely atomic). Then the choice of any base point $x \in \Sigma$ defines an isomorphism $f : \Gamma \to \Lambda$ by taking for $f(\gamma)$ the only $\lambda \in \Lambda$ such that $\lambda\gamma x = x$. We call Σ a *trivial* coupling and denote it by $\mathbf{T}_f$.

Observe that another choice of x gives a conjugated isomorphism. Observe also that upon identifying Λ with Σ as the orbit of x, the action becomes $(\gamma, \lambda)\,\eta = \lambda\eta f(\gamma)^{-1}$ for $\eta \in \Lambda$.

A less trivial (but still very straightforward) source of examples is the following:

Example 2.8 (Lattices). Let G be a locally compact, second countable group and Γ, Λ two lattices in G. The existence of lattices implies that any Haar measure m is left and right invariant; therefore, we obtain an ME coupling $\Sigma = (G, m)$ of Γ with Λ by considering the $\Gamma \times \Lambda$-action given by $(\gamma, \lambda)g = \gamma g \lambda^{-1}$. A very special case occurs when $G = \Gamma$ and Λ is a finite index subgroup of Γ.

Given an ME coupling (Σ, M) of Γ with Λ, we shall need the following concept which may seem pedantic at first sight, but will turn out to be extremely useful: Since Σ is technically a $\Gamma \times \Lambda$-space, we may define the *opposite coupling* $\check{\Sigma}$ of Λ with Γ to be the $\Lambda \times \Gamma$-space obtained via the canonical isomorphism $\Lambda \times \Gamma \cong \Gamma \times \Lambda$. As this will be particularly relevant in situations where $\Lambda = \Gamma$, we will (though rarely!) have to distinguish the Γ-actions on Σ

by writing $(\gamma, x) \mapsto A^1_\gamma x$ and $A^2_\gamma x$ respectively (then $\check{\Sigma}$ is obtained by switching A^1 and A^2).

Definition 2.9 (Coupling Composition). Assume we are furthermore given an ME coupling (Ω, n) of Λ with a third (countable) group Δ. Define the *composed coupling* $\Sigma \times_\Lambda \Omega$ to be the quotient space of $\Sigma \times \Omega$ by the product Λ-action. By commutativity, this is still a $\Gamma \times \Delta$-space, and we turn it into an ME coupling of Γ with Δ by endowing it with the measure obtained by restricting $m \otimes n$ to an (infinite measure) fundamental domain for Λ in $\Sigma \times \Omega$.

Definition 2.10 (Coupling Index). Given an ME coupling (Σ, m) of Γ with Λ, define its *coupling index* to be the following positive number:

$$[\Gamma : \Lambda]_\Sigma = \frac{m(\Lambda \backslash \Sigma)}{m(\Gamma \backslash \Sigma)}.$$

The notation reflects the fact that in the particular case where Λ is a finite index subgroup of Γ (Example 2.8) we recover indeed the index $[\Gamma : \Lambda] = \left|\Gamma/\Lambda\right|$. More generally, the coupling index corresponds to the ratio of co-volumes if Γ, Λ are lattices in one given locally compact group. It is straightforward to verify the formulae

$$(1) \qquad [\Gamma : \Lambda]_\Sigma = 1/[\Lambda : \Gamma]_{\check{\Sigma}}, \qquad [\Gamma : \Delta]_{\Sigma \times_\Lambda \Omega} = [\Gamma : \Lambda]_\Sigma \cdot [\Lambda : \Delta]_\Omega.$$

We need one more

Example 2.11 (Standard Coupling). Let Γ be a countable group and (X, μ) a probability Γ-space. We define an ME coupling of Γ with itself as follows: Endow $\Sigma = X \times \Gamma$ with the product measure and define the Γ-actions A^1, A^2 by $A^1_\gamma(x, \gamma_0) = (\gamma x, \gamma\gamma_0)$ and $A^2_\gamma(x, \gamma_0) = (x, \gamma_0\gamma^{-1})$; we call this the *standard* coupling associated to X. The two resulting Γ-actions on $A^1(\Gamma)\backslash\Sigma$ and $A^2(\Gamma)\backslash\Sigma$ are both isomorphic to the Γ-action on X. The subset $X \times \{e\} \subseteq \Sigma$ is a common fundamental domain for both actions on Σ, the associated cocycles are the identity isomorphism and there is a natural quotient map $\Sigma \to \mathbf{T}_{\mathrm{Id}}$ to the trivial coupling whose fibres can be identified with X.

Conversely, it is easy to verify that every ME coupling satisfying the properties listed above is measurably isomorphic to a standard coupling Σ as above.

We now state the fundamental observation concerning the relation between ME and WOE. The following is proved by A. Furman [F2] (who gives credit also to M. Gromov and R. Zimmer).

THEOREM 2.12 (ME-WOE). *Let* Γ, Λ *be countable groups and* (X_0, μ), (Y_0, ν) *be probability* Γ- *and* Λ-*spaces respectively. To any* WOE *given with* p, q *as in Definition* 2.1 *point* (ii) *corresponds an* ME *coupling* Σ *of* Γ *with* Λ, *together with a choice of* Γ- *and* Λ-*fundamental domains* Y, X *resp., such that*:

(i) *Modulo renormalisation of measures, one has isomorphisms of* Γ*-spaces* $X_0 \cong \Lambda\backslash\Sigma \cong X$ *and of* Λ*-spaces* $Y_0 \cong \Gamma\backslash\Sigma \cong Y$.

(ii) *Under these identifications,* $p_\chi = p$ *and* $q_\kappa = q$. *Moreover, the* WOE *cocycles* α, β *of Definition* 2.4 *coincide with the* ME *cocycles of Definition* 2.6.

(iii) $C(X_0, Y_0) = [\Gamma : \Lambda]_\Sigma$.

Moreover, in the ergodic case, $[\Gamma : \Lambda]_\Sigma = 1$ *if and only if the* WOE *is* (*induced by*) *an* OE, *and then one can choose in* Σ *a common* Γ*- and* Λ*-fundamental domain.*

Conversely, the above procedure produces WOE *probability* Γ*-respectively* Λ*-spaces out of any* ME *coupling* Σ *and the above three properties hold.* (*Yet, in contrast to our standing assumption these spaces need not be essentially free – see Remark* 2.14 *below.*)

On the proof. See 3.2 and 3.3 in Furman [F2] where ergodicity is assumed. However one can reduce to this case by [F1, 2.2]. □

We can now see what *commensurability* for actions should be.

Example 2.13 (Stability Properties). Here are two constructions that appear naturally and will be useful in the sequel; they are in a sense mutually dual.

(i). Consider a countable group Γ and a probability Γ-space (X, μ). Let $N \lhd \Gamma$ be a finite normal subgroup and set $\Lambda = \Gamma/N$. Consider the quotient $N\backslash X$ (with quotient measure) as a probability Λ-space. Then the Γ-action on X is WOE to the Λ-action on $N\backslash X$ since one can take for Definition 2.1 (ii) $p : X \to N\backslash X$ to be the quotient map and $q : N\backslash X \to X$ any measurable cross-section. Alternatively, one meets the other condition of that definition by taking $A = q(N\backslash X)$ for q as before and $B = N\backslash X$ (thus the compression constant is $C(X, N\backslash X) = |N|$). The ME coupling associated to this WOE is the following: First let Σ be the standard coupling of Γ with itself associated to X as in Example 2.11, and then consider the coupling $N\backslash\Sigma = A^2(N)\backslash\Sigma$ obtained from Σ by dividing out, say, the second N-action. Of course we have $[\Gamma : \Lambda]_{N\backslash\Sigma} = |N| = C(X, N\backslash X)$.

(ii). This time we consider a finite index subgroup Λ of a countable group Γ and a probability Λ-space (Y, ν). We write $Y\!\uparrow_\Lambda^\Gamma$ for the Γ-space which is the *suspension* (or *induction*) of the Λ-action on Y; this space is obtained (after dividing the measure by $[\Gamma : \Lambda]$) by considering the quotient of $\Gamma \times Y$ by the Λ-action $\lambda(\gamma, y) = (\lambda\gamma, \lambda y)$ endowed with the Γ-action descending from $\gamma_1(\gamma_2, y) = (\gamma_2\gamma_1^{-1}, y)$. Then the Γ-action on $Y\!\uparrow_\Lambda^\Gamma$ is WOE to the Λ-action on Y. Indeed, the first equivalent characterisation in Definition 2.1 is met by setting $B = Y$ and letting $A \subseteq Y{\uparrow}_\Lambda^\Gamma$ be the image of $\{e\} \times Y$. Alternatively, for the second characterisation, let $p : Y{\uparrow}_\Lambda^\Gamma \to Y$ be the natural quotient map and q be the section obtained by $y \mapsto (e, y)$. In particular, we have the compression constant $C(Y{\uparrow}_\Lambda^\Gamma, Y) = [\Gamma : \Lambda]$. To describe the ME coupling associated to this WOE, one considers again the standard coupling Σ (of Λ this time) associated to Y; then, either one takes the suspension of, say, the first Λ-action on Σ, or – equivalently – one composes the coupling Σ with the coupling arising from the inclusion $\Lambda < \Gamma$ as in the end of Example 2.8.

We conclude the example by remarking that if we have a probability Γ-space (X, μ) and a finite index subgroup $\Lambda < \Gamma$, then in general the restricted Λ-action on X will *not* be WOE to the original Γ-action. This can be seen for instance as follows: Suppose $\Gamma = \Gamma_1 \times \Gamma_2$, where the Γ_i's are torsion-free and in $\mathcal{C}_{\mathrm{reg}}$ (e.g. non-Abelian free groups). Let $\Lambda < \Gamma$ be a finite index subgroup not isomorphic to Γ. Now if X is mildly mixing Γ-space, then the Λ-action on X cannot be WOE to the Γ-action, since that would contradict the generalisation of Theorem 1.9 given below as Theorem 2.17 (ii). Notice however, that this stands in contrast to, but does not contradicts the fact that any ME coupling of Γ with some other countable group Δ also forms an ME coupling of the finite index subgroup $\Lambda < \Gamma$ with Δ.

Remark 2.14. There is some lack of symmetry in the relation between ME and WOE, because the WOE actions on probability spaces obtained as quotients of an ME coupling can be far from being free (consider e.g. Example 2.8 with Abelian G, or the trivial coupling for which the quotients reduce to a point). As far as proofs are concerned, this is not a difficulty for us, as we establish all our proofs in the setting of ME couplings and then deduce the WOE or OE statements, thereby using only the WOE $\longrightarrow$ ME direction. However, since the opposite direction will be useful to us when constructing some examples, we observe that the technicality arising in the inverse construction can easily be circumvented. This is achieved by composing a given ME coupling Σ (with potentially nonfree Γ- or Λ-quotients) with a standard self-coupling of Γ (Example 2.11) associated to any free probability Γ-space X. The relevant properties of Σ will be preserved in the composed coupling; ergodicity properties, such as irreducibility, are preserved if one chooses X to be "sufficiently ergodic" (e.g. mixing) Γ-space.

2.2. *Reformulation and discussion of the main results.* The relation between OE and ME, as discussed in the preceding subsection, enables us to reformulate our results in terms of the latter notion. In doing so we shall also generalise the main results to the framework of weak orbit equivalence.

As mentioned in the introduction, we will consider a family of groups more general than the class $\mathcal{C}_{\mathrm{reg}}$. The property relevant to our approach is described by the following:

Definition 2.15 (Class $\mathcal{C}$). Denote by $\mathcal{C}$ the class of groups admitting a mixing unitary representation π on a separable Hilbert space, such that $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi) \neq 0$.

Recall that a unitary representation is called *mixing* if all its matrix coefficients vanish at infinity; the outstanding example, and the one we shall actually use, is the regular representation. It follows from Theorem 1.3 that the Examples 1.1 introduced for the sake of concreteness are all contained in $\mathcal{C}$ (see Section 7 which has more on $\mathcal{C}$).

We next extend Definition 1.5 above in order to cover products of any number of groups:

Definition 2.16. Let $\Gamma_1, \dots, \Gamma_n$ be groups and set $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. A Γ-space (X, μ) is called *irreducible* if for every $1 \leq j \leq n$ the subproduct $\Gamma'_j = \prod_{i \neq j} \Gamma_i$ acts ergodically on X.

Notice that this definition forces $n > 1$ (unless X is trivial).

We begin reformulating our main results by considering Theorem 1.6. We shall in fact prove the following more general version of it:

THEOREM 2.17. *Let $\Gamma_1, \dots, \Gamma_n$ be torsion-free groups in $\mathcal{C}$ and (Σ, m) be an* ME *coupling of $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ with a product $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ of any torsion-free countable groups such that the Λ-action on $\Gamma\backslash\Sigma$ is irreducible. Assume that either*

(i) $[\Lambda : \Gamma]_\Sigma \geq 1$, *or*

(ii) *the Γ-action on $\Lambda\backslash\Sigma$ is irreducible.*

Then, upon permuting indices, there are isomorphisms $f_i : \Lambda_i \xrightarrow{\cong} \Gamma_i$ such that identifying Γ with Λ through $f = \prod f_i : \Lambda \cong \Gamma$, the coupling Σ is a standard coupling. Equivalently, by reference to Example 2.11 *for the notion of standard couplings, $[\Lambda : \Gamma]_\Sigma = 1$ and there is a common fundamental domain $Y \subseteq \Sigma$ for both actions such that $\lambda Y = f(\lambda) Y$ for all $\lambda \in \Lambda$.*

Thus, at the level of (W)OE, Theorem 2.17 implies that, under the assumptions corresponding to the above, *any* WOE *of the actions is in fact an* OE *induced by an isomorphism of the actions with respect to an isomorphism of the groups.*

Therefore, Theorem 1.6 follows from Theorem 2.17 in the particular case $\Lambda_i = \Gamma_i$, $n = 2$, $\mathcal{C}_{\mathrm{reg}}$ instead of $\mathcal{C}$ and essentially free quotients.

We give now an illustration of the necessity of the assumptions in Theorem 2.17:

Example 2.18 (Coupling Index Condition). Let F_n denote the free group on n generators. Realise F_3 and F_5 as index-two subgroups of F_2 and F_3

respectively, and view $F_2 \times F_5 < F_2 \times F_3$ and $F_3 \times F_3 < F_2 \times F_3$ as index-two subgroups. Thus $F_2 \times F_3$ is an ME coupling of $\Gamma = F_2 \times F_5$ with $\Lambda = F_3 \times F_3$ and the coupling index is one (observe that there is indeed a common Γ- and Λ-fundamental domain $\{(e,e),(x,y)\}$ in $F_2 \times F_3$ given by representatives x, y of the nontrivial cosets in F_2, F_3). Thus we see that the coupling index condition alone is not sufficient to derive the conclusion of Theorem 2.17. On the other hand, if we replace F_5 by F_3 then $F_2 \times F_3$ becomes a coupling of $F_2 \times F_3$ with $\Lambda = F_3 \times F_3$ for which the latter acts irreducibly (indeed, it acts on a point). However this time the inequality for the coupling constant is not satisfied, which accounts for the failure of the conclusion of the theorem. (Recall from Remark 2.14 that one can also build OE and WOE counter-examples out of the ME examples given here upon making the quotient actions essentially free by composition with, say, the standard coupling associated to a Bernoulli shift.)

Here is now an example showing how the statement breaks down for groups not in $\mathcal{C}$.

Example 2.19 (Class $\mathcal{C}$ Condition, I). Let G be a connected noncompact simple Lie group with trivial center and consider four copies of G labeled G_i, $1 \leq i \leq 4$. For each pair $1 \leq i \neq j \leq 4$ let Γ_{ij} be an irreducible lattice in the product $G_i \times G_j$. Of course, one may choose none, some, or all Γ_{ij} to be nonisomorphic as abstract groups. Now $\Gamma = \Gamma_{12} \times \Gamma_{34}$ as well as $\widetilde{\Gamma} = \Gamma_{13} \times \Gamma_{24}$ can both be realised naturally as lattices in $\prod_{1 \leq i \leq 4} G_i$, thus producing an ME coupling of Γ with $\widetilde{\Gamma}$. Using Howe-Moore's theorem, it is easy to check that this coupling is irreducible – namely each Γ_{ij} acts ergodically on the quotient of G^4 by the"other" product. Moreover, the conclusion of the theorem fails even if we take all Γ_{ij} isomorphic, as this ME coupling is not a standard one (Example 2.11). This also shows that a nontrivial assumption on the groups is needed in Theorem 1.6 from the introduction.

Next, we reformulate and generalise Theorem 1.10 (and thus Theorem 1.9) in the ME setting and discuss the assumptions made there. For the sake of clarity, we separate the statements for the groups (2.20) and for the actions (2.20*):

THEOREM 2.20. *Let $\Gamma_1, \ldots, \Gamma_n$ be torsion-free groups in $\mathcal{C}$ and let Λ be any countable group admitting an* ME *coupling (Σ, m) to $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$.*

If the Γ-action on $\Lambda \backslash \Sigma$ is irreducible and the Λ-action on $\Gamma \backslash \Sigma$ is mildly mixing, then Λ fits in an extension

$$1 \longrightarrow N \longrightarrow \Lambda \xrightarrow{\pi} \Gamma' \longrightarrow 1 \tag{2}$$

where N is finite and $\Gamma' < \Gamma$ is a finite index subgroup whose projections to each Γ_i are onto. Moreover,

$$[\Gamma : \Gamma'] \ = \ |N| \cdot [\Gamma : \Lambda]_\Sigma. \tag{3}$$

Let us call an exact sequence (2) with N finite and Γ' of finite index in Γ a *virtual isomorphism* of the groups Λ, Γ. We have seen above (Example 2.13) that in this setting the natural generalisation of isomorphic actions is a sort of commensurability of actions; we show in the proof of Theorem 2.20 that any WOE of actions as in the setting of that theorem are in fact a virtual isomorphism; more precisely:

THEOREM 2.20*. *Let $\Gamma_1, \ldots, \Gamma_n$ be torsion-free groups in $\mathcal{C}$ with an irreducible essentially free $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$-action on a probability space Y.*

If this action is WOE *to a mildly mixing, essentially free, action of any countable group Λ on a probability space X, then there is a virtual isomorphism as in* (2) *and the corresponding Γ-action on $(N\backslash X){\uparrow}_{\Gamma'}^{\Gamma}$ is isomorphic to Y.*

In the OE case we can deduce a stronger statement upon assuming aperiodicity of Γ:

COROLLARY 2.21. *Let $\Gamma_1, \ldots, \Gamma_n$ be torsion-free groups in $\mathcal{C}$, and let Y be an aperiodic irreducible essentially free $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$-space.*

If this action is OE *to a mildly mixing, essentially free action of any countable group Λ on a probability space X, then there exists an isomorphism of Λ and Γ with respect to which the actions on X, Y are isomorphic.*

(Observe that aperiodicity and irreducibility both hold if e.g. the Γ-action is mildly mixing.)

In the light of the discussion of Section 2.1, the above result imply indeed Theorems 1.9 and 1.10 stated in the introduction. In fact, we see that if we assume only that the actions in Theorem 1.10 are WOE, we still obtain both conclusions (i) and (ii), with the modified formula $[\Gamma : \Gamma'] = |N| \cdot C(X, Y)$.

Note that the OE assumption is equivalent to $C(X, Y) = 1$; that N is trivial as soon as Λ is torsion-free; and that on the other hand aperiodicity forces $[\Gamma : \Gamma'] = 1$ because in that case the action cannot be a suspension of an action of a proper finite index subgroup. This accounts for Theorem 1.9 and Corollary 2.21.

Example 2.22 (Mild Mixing Condition). In order to put the mild mixing assumption in a better perspective, consider the following situation. Let G be a connected, rank one, simple Lie group with trivial center (e.g. $\mathrm{PSL}_2(\mathbf{R})$) and choose two lattices $\Gamma_1, \Gamma_2 < G$ (in particular, the Γ_i's are in $\mathcal{C}_{\mathrm{reg}}$). Let $\Lambda < G \times G$ be an irreducible lattice. Then, as in Example 2.8, $G \times G$ is an ME coupling of $\Gamma = \Gamma_1 \times \Gamma_2$ with Λ, or equivalently, the Γ-action on G^2/Λ is WOE to the Λ-action on G^2/Γ (the essential freeness of these actions can be deduced from the center freeness of G). One can arrange to have the same co-volumes, so that then the actions are in fact OE. Furthermore, the irreducibility of the lattice Λ ensures that the Γ-action is irreducible. Thus, we have here a situation where the conclusion of Theorem 2.20 (and Theorem 1.9) fails because the Λ-action on G^2/Γ is not mildly mixing, even though it does have very strong ergodicity properties: It is weakly mixing, and moreover one can find Λ for which every nontrivial element acts ergodically (or weakly mixing). In fact, one can detect precisely how the mild mixing property in Definition 1.8 fails: By Howe-Moore's theorem, it can be shown that the only nontrivial recurrent sets are of the form $A \times G/\Gamma_2$ or $G/\Gamma_1 \times B$, and the associated recurrent sequences (λ_n) of Λ must satisfy $\mathrm{pr}_1(\lambda_n) \to e_1$ or $\mathrm{pr}_2(\lambda_n) \to e_2$, respectively, where pr_i is the i^{th} quotient map $G \times G \to G$ and e_i the trivial element in the i^{th} factor of the two.

Analogous to the case of products, we restate and generalise the rigidity results for groups with amenable radicals through the notion of measure equivalence:

THEOREM 2.23. *Let Γ, Λ be countable groups and $M \lhd \Gamma, N \lhd \Lambda$ amenable normal subgroups such that $\overline{\Gamma} = \Gamma/M$ and $\overline{\Lambda} = \Lambda/N$ are in $\mathcal{C}$ and torsion-free. Let (Σ, m) be an* ME *coupling of Γ with Λ.*

If N is ergodic on $\Gamma\backslash\Sigma$ and M on $\Lambda\backslash\Sigma$, then there is an isomorphism $f : \overline{\Gamma} \xrightarrow{\cong} \overline{\Lambda}$. Moreover, Σ admits a $\Gamma \times \Lambda$-equivariant factor $\Phi : \Sigma \to \mathbf{T}_f$, where the latter is the trivial coupling of $\overline{\Gamma}$ with $\overline{\Lambda}$ inducing f.

The superrigidity-type statement goes as follows:

THEOREM 2.24. *Let Γ be a countable group with an amenable normal subgroup $M \lhd \Gamma$ such that $\overline{\Gamma} = \Gamma/M$ is in $\mathcal{C}$ and torsion-free, and let Λ be any countable group with an ME coupling (Σ, m) to Γ. If the M-action on $\Lambda\backslash\Sigma$ is ergodic and the Λ-action on $\Gamma\backslash\Sigma$ is mildly mixing, then there is an amenable normal subgroup $N \lhd \Lambda$ such that $\overline{\Lambda} = \Lambda/N$ is isomorphic to $\overline{\Gamma}$.*

To verify that these results indeed imply Theorems 1.12 and 1.13 stated in the introduction, one appeals again to Theorem 2.12 above.

We now discuss some situations related to Theorem 1.16.

Example 2.25 (Class $\mathcal{C}$ Condition, II). Let G be any discrete group with Kazhdan's property (T) and H be a group without property (T). Set $\Gamma_1 = G \times G$, $\Gamma_2 = H \times H$, $\Lambda_1 = G \times H$ and $\Lambda_2 = H \times G$. Then $\Gamma = \Gamma_1 \times \Gamma_2$ is ME (indeed isomorphic) to $\Lambda = \Lambda_1 \times \Lambda_2$; however, Γ_1 is not ME to any Λ_i since property (T) is an ME invariant [F1, 1.4]. Thus, some nontrivial assumption on the groups in Theorem 1.16 is necessary. In fact, we do not have any natural candidate for a more general class of groups than $\mathcal{C}$ for which a similar result should hold.

The fact that groups in $\mathcal{C}$ cannot have infinite direct factors (see Section 7) is illustrated in a patent way in the above example. Indeed, we may arrange for both G and H to be in $\mathcal{C}$ or even in $\mathcal{C}_{\rm reg}$: Take for instance for G a lattice in $\mathrm{Sp}(n,1)$ with $n \geq 2$ and for H a free group on two generators. Then, as above, the conclusion of Theorem 1.16 fails for the products $\Gamma = \Gamma_1 \times \Gamma_2$, $\Lambda = \Lambda_1 \times \Lambda_2$, but of course after further splitting of the factors one can shuffle the groups to get the (trivial) self-couplings of G and of H respectively, in accordance with the theorem.

As another example, consider the construction described in Example 2.18, namely the ME coupling of $\Gamma = F_2 \times F_5$ and $\Lambda = F_3 \times F_3$ with coupling index one – so that these groups admit actions (which can be made free) that are indeed OE and not just WOE. By the recent result of D. Gaboriau [Ga3], the ℓ^2-Betti numbers are OE invariants, so that neither the couple F_2 and F_3, nor the couple F_5 and F_3, admit OE actions. Thus, even by assuming that two products $\Gamma = \Gamma_1 \times \Gamma_2$ and $\Lambda = \Lambda_1 \times \Lambda_2$ admit OE actions, one cannot arrive at a stronger conclusion in Theorem 1.16. The reader is invited to examine the proof of Theorem 1.16 in this very simple and concrete example to see how the equality of co-volumes can be lost in passing from the original ME coupling to couplings of the individual factors.

Remark 2.26. Suppose that a product $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ of groups in $\mathcal{C}_{\rm reg}$ is ME to a torsion-free product $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ of any (countable) groups Λ_i. Gaboriau's results [Ga2], [Ga3] imply that the ℓ^2-Betti numbers $\beta^i_{(2)}$ of Γ are proportional to those of Λ. But now Theorem 1.16 tells us that (after permutation of indices) we can also apply this to each pair, giving of course more restrictions on the possible values of the ℓ^2-Betti numbers of the factors.

For instance, suppose we have a group Γ_1 in $\mathcal{C}_{\rm reg}$ with the nonzero ℓ^2-Betti numbers $\beta^2_{(2)} = 2$, $\beta^3_{(2)} = 3$, $\beta^4_{(2)} = 1$ and set $\Gamma_2 = \Gamma_1$. Choose now any torsion-free countable groups Λ_i such that:

	$\beta^1_{(2)}$	$\beta^2_{(2)}$	$\beta^3_{(2)}$	$\beta^4_{(2)}$	$\beta^{\geq 5}_{(2)}$
Λ_1	0	1	2	1	0
Λ_2	0	4	4	1	0

Then $\Gamma = \Gamma_1 \times \Gamma_2$ has the same ℓ^2-Betti numbers as $\Lambda = \Lambda_1 \times \Lambda_2$, so that Gaboriau's result does not exclude an ME coupling of these two groups. However, such a coupling is impossible in view of Theorem 1.16 since then we would have individual couplings, and *that* would now contradict Gaboriau's proportionality.

Finally, we make some concluding remarks on the irreducibility property in Theorem 2.17. Suppose that we have an ME coupling Σ between two groups $\Gamma = \Gamma_1 \times \Gamma_2$ and $\Lambda_1 \times \Lambda_2$, where all four factors are torsion-free and in $\mathcal{C}_{\rm reg}$

but the Γ_i are not isomorphic to the Λ_j. (For instance, the Γ_i are countable non-Abelian free groups and the Λ_j surface groups of genus ≥ 2.) Then Theorem 1.16 tells us that (upon permuting indices) Γ_1 is ME to Λ_1 and Γ_2 to Λ_2; but on the other hand, the coupling Σ cannot be irreducible for both Γ and Λ because of Theorem 2.17 (ii). Can one deduce in certain situations that Σ is actually a *product coupling*? Likewise, if $[\Gamma : \Lambda]_\Sigma = 1$, the coupling cannot even be irreducible for one side in view of Theorem 2.17 (i); so, again, must it be a product coupling?

2.3. *Groups with many actions.* We begin by proving an observation made in the introduction:

THEOREM 2.27. *Let* Γ *be a* (*countable*) *free group. Then any given probability* Γ*-space* (X, μ) *is* OE *to actions of uncountably many nonisomorphic groups.*

Proof. We may assume that Γ has rank at least two in view of the Ornstein-Weiss result [OW] for amenable groups. In fact, for simplicity of notation only we shall take Γ of rank two. What we shall actually show is that for *every* pair of countable amenable groups A, B the Γ-action on X is OE to an action of $\Lambda = A * B$ on the same space X. Let u, v be free generators of Γ; to avoid technical issues, assume that both u and v are ergodic transformations of X (it is not difficult to remove this assumption, keeping the same strategy of proof). Consider the infinite (cyclic) amenable groups $\langle u \rangle$ and $\langle v \rangle$ and note that by the result of Ornstein-Weiss, there exist measure-preserving, essentially free actions of A and B on (X, μ), each of which has a.e. the same orbits as $\langle u \rangle$ and $\langle v \rangle$ respectively. These actions of $\langle u \rangle$ and $\langle v \rangle$ define by universality an action of their free product Λ, which has the same orbits as Γ. The whole point of the argument is to show that this action is essentially free. Indeed, otherwise there is a nontrivial element $a_1 b_1 \cdots a_n b_n$ of $A * B$ which fixes pointwise a measurable set $Y \subseteq X$ with $\mu(Y) > 0$. Now for a.e. $y \in Y$ there are integers $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ such that $u^{p_1} v^{q_1} \cdots u^{p_n} v^{q_n} y = a_1 b_1 \cdots a_n b_n y = y$. Since there are countably many n-tuples (p_i, q_i), this contradicts the essential freeness of the Γ-action. $\square$

Remark 2.28. More generally, it seems that whenever G, H, A, B are countable groups such that G and A admit OE actions, and likewise for H and B, then $G * H$ admits an action OE to an action of $A * B$ (and in particular $G * H$ is ME to $A * B$). This should follow from a similar idea, realising the OE for G and A on a common space X and the OE for H and B on a space Y, only that now one has to choose an isomorphism of standard probability spaces $X \cong Y$ such that the resulting actions of the free products are essentially free – e.g. by applying the Baire category theorem to the Polish space of such isomorphisms. Observe that this line of reasoning does not pass to WOE (this is not possible in general, as the example of finite groups of different order shows). The situation is reminiscent of the known difference between bi-Lipschitz and quasi-isometric equivalence for free products of finitely generated groups.

The above result stands in strong contrast to our Theorem 1.7 from the introduction; let us turn to the latter.

Proof of Theorem 1.7. The idea is to apply our Theorem 1.6 in order to get many actions of a given group that are mutually not OE; but actually, we shall rather use the stronger statement of Theorem 2.17 in order to be able to vary the groups as well. That way, we shall construct a family of actions as claimed in Theorem 1.7, but furthermore *no two of them will even be*WOE.

Let $\mathcal{F}$ be the continuum of isomorphism classes of all groups $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_i = A * B$ range over all free products of any two torsion-free countable groups. In view of Theorem 2.17, all we have to do is to find for each such Γ in $\mathcal{F}$ a continuum of *nonisomorphic* irreducible probability Γ-spaces. In order to produce the latter, we use the well known Gaussian measure construction that associates to any continuous unitary representation π of a locally compact, second countable group Γ, a measure-preserving Γ-action. As explained in [Z2] (see 5.2.13 and p. 111), one obtains a continuum of nonisomorphic Γ-actions once Γ has a continuum of nonequivalent irreducible unitary representations π, and furthermore, for any closed subgroup $H < \Gamma$, the following holds: If $\pi|_H$ is weakly mixing, then H acts ergodically on the measure space constructed in this manner.

On the other hand, it is a well known fact that any discrete infinite group Γ admits a continuum of irreducible unitary representations π that are weakly contained in $L^2(\Gamma)$ (this follows from Corollaire 1 in J. Dixmier [Dx], a remark for which we thank Bachir Bekka). But then, for any nonamenable closed subgroup $H < G$, the restriction $\pi|_H$ must be weakly mixing, since otherwise we would have (using $\prec$ to denote weak containment):

$$\mathbb{1}_H \subseteq (\pi \otimes \overline{\pi})|_H \prec (L^2(G) \otimes L^2(G))|_H \cong \bigoplus_{n=1}^{\infty} L^2(H),$$

contradicting nonamenability of H in view of the (generalised) Hulanicki criterion. Applying this discussion to $H = \Gamma_i < \Gamma$, one constructs a continuum of irreducible nonisomorphic probability Γ-spaces, thereby finishing the proof. □

2.4. *Outer automorphisms of certain type* II_1 *relations.* The goal of this subsection is to present the

Proof of Theorem 1.14. We shall use the following notation of A. Furman [F3]: If (X, μ) is any probability Γ-space for a countable group Γ, let

$$\mathrm{Aut}^*(X,\Gamma)=\big\{F\in\mathrm{Aut}(\mathcal{R}_{\Gamma,X})\ :\ \exists\, f\in\mathrm{Aut}(\Gamma)\ \forall\,\gamma\in\Gamma:\quad F(\gamma x)=f(\gamma)F(x)\big\}$$

and write $A^*(X,\Gamma)$ for the image of $\mathrm{Aut}^*(X,\Gamma)$ in $\mathrm{Out}(\mathcal{R}_{\Gamma,X})$. With this notation one deduces immediately the following from Theorem 1.6:

Let $\Gamma=\Gamma_1\times\Gamma_2$ *be a torsion-free group with both* Γ_i *in* $\mathcal{C}_{\mathrm{reg}}$, *and let* (X,μ) *be an irreducible probability* Γ*-space. Then* $\mathrm{Out}(\mathcal{R}_{\Gamma,X})=A^*(X,\Gamma)$.

Now let K be a (second countable) compact group, and μ be its normalised Haar measure. Let us fix $K=\mathrm{SO}(n)$ with n odd, which enjoys the property of having both trivial center and no nontrivial outer automorphisms. Let Λ be a Kazhdan group which admits a dense embedding into K and such that every injective homomorphism $\Lambda\to\Lambda$ is an inner automorphism. We note that for every $n\geq 5$ one can indeed find such a group Λ which is a lattice in an appropriate higher rank simple Lie group; indeed, the dense embedding into K is provided by a standard Galois twist argument, while for the condition on injective homomorphisms $\Lambda\to\Lambda$ we refer to [Pr]. Let F_p and F_q be non-Abelian free groups with $p\neq q$ and consider the free products $\Gamma_1=\Lambda*F_p$ and $\Gamma_2=\Lambda*F_q$. Suppose for the time being that we are given injective homomorphisms of F_p,F_q into K such that the induced maps $\Gamma_i\to K$ are still injective; then we can view K as a probability $\Gamma=\Gamma_1\times\Gamma_2$-space by letting Γ_1 and Γ_2 act by right and left multiplication respectively (it is easily verified that essential freeness here is satisfied once every open subgroup of K is center free).

Proposition 2.29. *The group* $\mathrm{Out}(\mathcal{R}_{\Gamma,K})$ *is trivial.*

Proof. By the above reformulation of Theorem 1.6, it is enough to show that $A^*(K,\Gamma)$ is trivial. Since we chose F_p and F_q nonisomorphic, every element of $\mathrm{Aut}^*(K,\Gamma)$ induces a (perhaps twisted) isomorphism of both Γ_1- and Γ_2-actions individually. We shall see that $\mathrm{Aut}^*(K,\Gamma_1)\cap\mathrm{Aut}^*(K,\Gamma_2)$ is trivial (even though each of these two groups is large).

A direct argument of A. Furman [F3, 7.2] enables one to describe $\mathrm{Aut}^*(K,\Gamma_1)$ as

$$\mathrm{Aut}^*(K,\Gamma_1)=\big\{a_{\sigma,t}(k)=t\sigma(k)\ :\ \sigma\in\mathrm{Aut}(K),\,t\in K,\,\sigma(\Gamma_1)=t^{-1}\Gamma_1 t\big\}.$$

Now, since $\mathrm{Out}(K)$ is trivial, we can write $\sigma(k)=c^{-1}kc$ for some $c\in K$; hence $a_{\sigma,t}(k)=tc^{-1}kc$ with the condition $c^{-1}\Gamma_1 c=t^{-1}\Gamma_1 t$; i.e. $(ct^{-1})^{-1}\Gamma_1(ct^{-1})=\Gamma_1$. Recall now that $\Gamma_1=\Lambda*F_p$. We claim that up to a conjugation in Γ_1 every $f\in\mathrm{Aut}(\Gamma_1)$ is trivial on Λ. Indeed, since any action of the Kazhdan group Λ on the Bass-Serre tree associated with the free product $\Lambda*F_p$ has a fixed vertex, it follows that $f(\Lambda)$ is contained in a conjugate of Λ or of F_p, the latter being of course impossible. Hence after conjugation every

$f \in \mathrm{Aut}(\Gamma_1)$ satisfies $f(\Lambda) \subseteq \Lambda$, and by the choice of Λ we deduce that after further conjugation f is trivial on Λ, proving the claim.

If we apply this and the claim above to the automorphism f given by conjugation by ct^{-1}, recalling that by density of Λ in K every continuous automorphism of the latter which is trivial on the former must be trivial, we find $tc^{-1} \in \Gamma_1$ and hence conclude

$$\mathrm{Aut}^*(K, \Gamma_1) = \big\{k \mapsto \gamma_1 k c_1 \ : \ \gamma_1 \in \Gamma_1,\, c_1 \in K\big\}.$$

The analogous argument for Γ_2 yields

$$\mathrm{Aut}^*(K, \Gamma_2) = \big\{k \mapsto c_2 k \gamma_2 \ : \ \gamma_2 \in \Gamma_2,\, c_2 \in K\big\}$$

since Γ_2 acts from the right. Thus, for

$$F \in \mathrm{Aut}^*(K, \Gamma) \subseteq \mathrm{Aut}^*(K, \Gamma_1) \cap \mathrm{Aut}^*(K, \Gamma_2)$$

we have $F(k) = c_2 k \gamma_2 = \gamma_1 k c_1$ and therefore $\gamma_1^{-1} c_2 k = k c_1 \gamma_2^{-1}$. Taking $k = e$ (or rather k sufficiently close to e since these equalities hold only almost everywhere), we deduce $\gamma_1^{-1} c_2 = c_1 \gamma_2^{-1}$. Since K has trivial center this forces $\gamma_1^{-1} c_2 = c_1 \gamma_2^{-1} = e$, i.e. $c_2 = \gamma_1$ and $c_1 = \gamma_2$ so that $\mathrm{Aut}^*(K, \Gamma)$ (and more generally the above intersection) consists of maps $k \mapsto \gamma_1 k \gamma_2$ which are of course trivial in $A^*(K, \Gamma)$. This concludes the proof of Proposition 2.29. □

We return now to the proof of Theorem 1.14. Fix once and for all one dense embedding of Λ into K as above. We considered for the statement of Proposition 2.29 injective homomorphisms of F_p, F_q into K such that the induced maps $\Gamma_i \to K$ are still injective. However, such embeddings not only exist, but are generic with respect to the Haar measure [E]; in particular there is a continuum of nonconjugate such homomorphisms. Now Proposition 2.29 shows that each member of the corresponding family of Γ-actions determines a relation with trivial $\mathrm{Out}(\mathcal{R}_{\Gamma,K})$. By a similar argument, Theorem 1.6 implies that no two distinct actions in this family can be WOE, since they are nonconjugate; thus the relations are not weakly isomorphic, as required. □

2.5. *Some examples with linear groups.* Using Howe-Moore's theorem, one can easily deduce from our OE rigidity results applications to rigidity for linear groups acting on homogeneous spaces. We bring here two examples.

Example 2.30. Let $\overline{\Gamma}$ be a torsion-free group in $\mathcal{C}$ with an injective homomorphism $\rho : \overline{\Gamma} \to \mathrm{SL}_n(\mathbf{Z})$. Form the semi-direct product $\Gamma = \mathbf{Z}^n \rtimes_\rho \overline{\Gamma}$, where $\overline{\Gamma}$ acts linearly on $\mathbf{Z}^n$ via ρ. This realises Γ as a subgroup of $\mathrm{SL}_{n+1}(\mathbf{Z}) < G = \mathrm{SL}_{n+1}(\mathbf{R})$. Let now Δ, Σ be any two lattices in G.

If the translation Γ-action on G/Δ is WOE *to a Λ-translation action on G/Σ, where $\Lambda < G$ is any discrete subgroup, then there is an infinite normal* *amenable subgroup* $N \lhd \Lambda$ *such that* Λ/N *is isomorphic to* Γ. (*In particular, the Zariski closure of* Λ *is not semi-simple.*)

This statement is a straightforward application of Theorem 2.24 together with Howe-Moore's theorem.

In our last example we consider linear embeddings that are not necessarily discrete. In fact, the following is of interest precisely in the nondiscrete cases:

Example 2.31. Let $F = F_p \times F_q$ be a product of non-Abelian free groups (or of any torsion-free groups in $\mathcal{C}$). Let $i_1, i_2 : F \to G = \mathrm{SL}_n(\mathbf{R})$ be two embeddings such that the image of each free group under both embeddings is unbounded. For simplicity, assume n is odd so that G has trivial center. Let Δ, Σ be two lattices in G.

If the F*-translation action on* G/Δ *through* i_1 *is* WOE *to the* F*-translation action on* G/Σ *through* i_2, *then there is an automorphism* f *of* F *such that the embeddings* i_1 *and* i_2 *are topologically equivalent modulo* f.

More precisely, $i_2 \circ f \circ i_1^{-1}$ *extends to an isomorphism between the closures of* $i_1(F)$ *and* $i_2(F)$ *in* G. *In particular*, $\overline{i_1(F)}$ *is isomorphic to* $\overline{i_2(F)}$.

Proof. By Howe-Moore's theorem, our assumption on the embeddings ensures that both F-actions are irreducible. By Theorem 2.17, it follows that the actions are isomorphic with respect to an automorphism f; we may assume $f = \mathrm{Id}$ upon composing one of the embeddings with f. If (g_k) is a sequence of F such that $i_1(g_k)$ tends to $e \in G$, then by Howe-Moore $i_2(g_k)$ is bounded since our two actions are isomorphic. Thus $i_2(g_k)$ has a limit point $g \in G$; now, since G has trivial center, $g = e$ because otherwise g would not act trivially on G/Σ. By symmetry of that argument we deduce that $i_1(g_k) \to e$ if and only if $i_2(g_k) \to e$, as required. □

3. Background in bounded cohomology

The purpose of this section is to offer the most elementary possible account of the bounded cohomology tools that we shall need. In the setting of this paper, it is possible to derive most relevant statements from two fundamental principles (Theorems 3.2 and 3.3 below). Thus, we shall indicate some proofs for the reader's convenience. For a more detailed introduction, we refer to [BM2], [M].

We consider throughout the paper bounded cohomology for countable (discrete) groups Γ. The coefficients will be taken almost always in unitary Γ-representations on separable Hilbert spaces. However, for the purpose of induction of such modules (see Section 4), it will be essential to allow the following more general setting:

Definition 3.1. A *coefficient* Γ-module (π, E) is an isometric linear Γ-representation π on a Banach space E such that: (i) E is the dual of some separable Banach space, (ii) π consists of adjoint operators (in this duality).

The *bounded cohomology* of Γ with coefficient module (π, E) is defined to be the cohomology of the complex

$$0 \longrightarrow \ell^\infty(\Gamma, E)^\Gamma \longrightarrow \ell^\infty(\Gamma^2, E)^\Gamma \longrightarrow \ell^\infty(\Gamma^3, E)^\Gamma \longrightarrow \cdots \tag{4}$$

of bounded invariant functions and is denoted by $\mathrm{H}^\bullet_\mathrm{b}(\Gamma, E)$ or $\mathrm{H}^\bullet_\mathrm{b}(\Gamma, \pi)$. This complex is just the subcomplex of bounded functions in the standard (homogeneous) bar complex for the Eilenberg-MacLane cohomology; in other word, invariance is understood with respect to the regular representation

$$\big(\lambda_\pi(\gamma) f\big)(\gamma_0, \ldots, \gamma_n) = \pi(\gamma)\Big(f(\gamma^{-1}\gamma_0, \ldots, \gamma^{-1}\gamma_n)\Big)$$

and the maps in (4) are the usual (Alexander-Spanier) coboundary maps.

The usual cohomological methods do not apply to bounded cohomology, which has proved difficult to compute. It is therefore essential to have at least some replacement for the intractable complex (4). More useful complexes arise in connection with standard Borel Γ-spaces with a finite quasi-invariant measure and are such that the Γ-action is amenable in R. Zimmer's [Z2] sense. For short, we call such a space an *amenable* Γ*-space*.

Theorem 3.2 ([BM2], [M]). *Let E be a coefficient Γ-module and S an amenable Γ-space. Then the complex*

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(S, E)^\Gamma \longrightarrow L^\infty_{\mathrm{w}*}(S^2, E)^\Gamma \longrightarrow L^\infty_{\mathrm{w}*}(S^3, E)^\Gamma \longrightarrow \cdots \tag{5}$$

realises canonically $\mathrm{H}^\bullet_\mathrm{b}(\Gamma, E)$. The corresponding statement holds for the subcomplex of alternating cochains. □

We do not make the meaning of *canonically* more precise here, but its importance will be obvious in certain arguments below. In the above, $L^\infty_{\mathrm{w}*}$ denotes the space of essentially bounded weak-* measurable functions. Below, we will often deal with cases where E is separable, in which case weak-* and strong measurability coincide; hence the simpler notation L^∞.

The point of Theorem 3.2 is that there are indeed examples of amenable spaces with very strong ergodicity properties: The following result was established in [BM2], [M] for finitely (or compactly) generated groups; the general version was then provided by V. Kaimanovich [K].

Theorem 3.3. *For every countable group Γ, there is an amenable Γ-space S such that for every separable coefficient Γ-module E, the space $L^\infty(S^2, E)^\Gamma$ is reduced to constant functions. In particular, there is a canonical isomorphism*

$$\mathrm{H}^2_\mathrm{b}(\Gamma, E) \cong ZL^\infty_\mathrm{alt}(S, E)^\Gamma,$$

where ZL^∞_alt denotes the space of alternating cocycles. □

(We point out that the conditions on E are not merely technical, and that there are counter-examples if one drops either the separability assumption or the duality of E.)

A first immediate application of this fact is the following.

COROLLARY 3.4. *Let* Γ *be a countable group and* $(\pi_n, \mathcal{H}_n)_{n=1}^{\infty}$ *a family of unitary* Γ*-representations in separable Hilbert spaces* $\mathcal{H}_n$*. Then*

$$\mathrm{H}_{\mathrm{b}}^2\left(\Gamma, \bigoplus_{n=1}^{\infty}\mathcal{H}_n\right) = 0 \iff \forall\ n \geq 1 :\ \mathrm{H}_{\mathrm{b}}^2(\Gamma, \mathcal{H}_n) = 0.$$

An analogous statement holds for direct integrals of unitary representations. $\square$

Here is another immediate consequence taken from [BM2], [M]:

COROLLARY 3.5. *Let* Γ *be a countable group and* $\alpha : E \to F$ *an adjoint* Γ*-map of coefficient* Γ*-modules. If* F *is separable, then the induced map* $\alpha_* : \mathrm{H}_{\mathrm{b}}^2(\Gamma, E) \to \mathrm{H}_{\mathrm{b}}^2(\Gamma, F)$ *is injective.*

Proof. By Theorem 3.2 and functoriality of (5), the map α_* is realised at the level of $\mathrm{H}_{\mathrm{b}}^n(\Gamma, -)$ by the corresponding map

$$\alpha_n :\ L_{\mathrm{w}*,\mathrm{alt}}^{\infty}(S^{n+1}, E) \longrightarrow L_{\mathrm{alt}}^{\infty}(S^{n+1}, F)$$

for any amenable Γ-space S; observe that this map ranges in measurable functions because α is adjoint. On the other hand, α_n is injective at the level of *cocycles*, so that the cohomological statement with $n = 2$ follows from $L_{\mathrm{alt}}^{\infty}(S^2, F) = 0$ with S as in Theorem 3.3. $\square$

We can also derive readily the following special case of a general exact sequence [M, N° 12.0.2]:

COROLLARY 3.6. *Let* Γ *be a countable group,* $N \lhd \Gamma$ *a normal subgroup and* $Q = \Gamma/N$ *the quotient. If* E *is a separable coefficient* Q*-module, then the inflation map*

$$\inf :\ \mathrm{H}_{\mathrm{b}}^2(Q, E) \longrightarrow \mathrm{H}_{\mathrm{b}}^2(\Gamma, E)$$

is injective.

Proof. Let S be an amenable Γ-space as in Theorem 3.3 and let S' be the Mackey realisation of $L^{\infty}(S)^N$. Then S' is an amenable Q-space satisfying the condition of Theorem 3.3. Thus we have canonical embeddings

$$L_{\mathrm{alt}}^{\infty}((S')^{n+1}, E)^Q \longrightarrow L_{\mathrm{alt}}^{\infty}(S^{n+1}, E)^{\Gamma}$$

which induce the inflation $\mathrm{H}_{\mathrm{b}}^n(Q, E) \to \mathrm{H}_{\mathrm{b}}^n(\Gamma, E)$. Since $L_{\mathrm{alt}}^{\infty}((S')^2, E)^Q$ vanishes, we deduce that at the level of $\mathrm{H}_{\mathrm{b}}^2$ the inflation is still injective. $\square$

A key ingredient for our use of bounded cohomology in this paper is the following product formula whose proof relies also on Theorem 3.3.

THEOREM 3.7 ([BM2], [M]). *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of countable groups Γ_i and let (π, E) be a separable coefficient Γ-module. Then*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, E) \cong \bigoplus_{i=0}^{n} \mathrm{H}^2_{\mathrm{b}}\big(\Gamma_i, E^{\Gamma'_i}\big),$$

where $E^{\Gamma'_i}$ denotes the subspace of vectors fixed by $\Gamma'_i = \prod_{j \neq i} \Gamma_j$. □

We will mostly use the consequence that $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E) \neq 0$ implies $E^{\Gamma'_i} \neq 0$ for some i; we emphasize that the above formula does not follow formally like some Künneth formula, and indeed may fail when E is not separable (or not dual). Since [BM2], [M] deal with the general case of group extensions, we indicated the simpler proof of the product case in [MS1].

We end with a simple fact that is well known in this setting ([J], [Gr], [I], [N]) but can also be seen as an application of Theorem 3.2:

PROPOSITION 3.8. *Let $N \lhd \Gamma$ be an amenable normal subgroup of the countable group Γ and let E be a coefficient Γ-module. Then the inflation*

$$\mathrm{inf}: \ \mathrm{H}^n_{\mathrm{b}}(\Gamma/N, E^N) \longrightarrow \mathrm{H}^n_{\mathrm{b}}(\Gamma, E)$$

is an isomorphism. In particular, $E^N \neq 0$ if $\mathrm{H}^n_{\mathrm{b}}(\Gamma, E) \neq 0$.

Proof. Let S be Γ/N endowed with some probability measure of full support. Then it is both an amenable Γ- and Γ/N-space. The statement now follows from Theorem 3.2 by realising the inflation by the isomorphisms

$$L^\infty_{\mathrm{w}*}(S^{n+1}, E^N)^{(\Gamma/N)} = L^\infty_{\mathrm{w}*}(S^{n+1}, E^N)^{\Gamma} \cong L^\infty_{\mathrm{w}*}(S^{n+1}, E)^{\Gamma}. \qquad \square$$

Remark 3.9. A degenerate case of Proposition 3.8 occurs when Γ itself is amenable: one deduces then that $\mathrm{H}^n_{\mathrm{b}}(\Gamma, E)$ vanishes for all $n \geq 1$ and every coefficient Γ-module E.

More advanced tools from [BM2], [M] include a low degree exact sequence for group extensions (used below for Proposition 7.4).

4. Cohomological induction through couplings

Before considering cohomological induction, we start with some properties of the operation of inducing *representations.* This is well known in the OE setting but becomes more transparent for ME.

4.1. *Induced representations.* Let (Σ, m) be an ME coupling of two countable groups Λ, Γ and let (π, E) be a unitary Γ-representation in a separable Hilbert space E, or more generally a separable coefficient module (Definition 3.1).

There are two equivalent ways to define the *induced representation* $\Sigma\mathbf{I}_\Gamma^\Lambda\pi$. First, one can define the Banach space

$$\Sigma\mathbf{I}_\Gamma^\Lambda\pi = L^{[2]}(\Sigma, E)^\Gamma \tag{6}$$

of Γ-equivariant measurable maps $f : \Sigma \to E$, wherein the notation $L^{[2]}$ means that the Γ-invariant function $\|f\|_E$ is to be in $L^2(\Gamma\backslash\Sigma)$. The Λ-action in this model is simply given by translation on Σ. Equivalently, one can define

$$\Sigma\mathbf{I}_\Gamma^\Lambda\pi = L^2(\Gamma\backslash\Sigma, E)$$

and endow it with the twisted Λ-action defined a.e. by

$$(\lambda f)(\Gamma x) = \pi(\chi(x)^{-1}\chi(\lambda^{-1}x))f(\lambda^{-1}\Gamma x),$$

where χ is a retraction as in Definition 2.6. Although this viewpoint is useful too, one should remember that the isomorphism between the latter and the more natural former depends on the choice of χ. We also mention that upon identifying $\Gamma\backslash\Sigma$ with a fundamental domain Y, the action on $f \in L^2(Y, E)$ becomes the well-known twisted action

$$(\lambda f)(y) = \pi(\beta(\lambda^{-1}, y)^{-1})f(\lambda^{-1} \cdot y)$$

for the associated cocycle $\beta : \Lambda \times Y \to \Gamma$. This model is relevant when thinking of OE or WOE.

At any rate, $\Sigma\mathbf{I}_\Gamma^\Lambda\pi$ is a separable coefficient Λ-module, and a unitary representation if E was unitary. The definition (6) implies that the construction is natural and that one has the following *transitivity* property: If Σ' is an ME coupling of Λ with a further group Δ, then

$$\Sigma'\mathbf{I}_\Lambda^\Delta(\Sigma\mathbf{I}_\Gamma^\Lambda\pi) \cong (\Sigma' \times_\Lambda \Sigma)\mathbf{I}_\Gamma^\Delta\pi. \tag{7}$$

Here are some elementary properties of the induction operation:

LEMMA 4.1. *Let Λ, Γ be two countable groups with commuting, measure-preserving actions on a measure space (Σ, m). Suppose that the Γ-action admits a finite measure fundamental domain and that the Λ-action admits some fundamental domain. Then, for every unitary Γ-representation π, σ one has:*

(i) *If π is mixing for Γ then $\Sigma\mathbf{I}_\Gamma^\Lambda\pi$ is mixing for Λ.*

(ii) *If $\pi \prec \sigma$ then $\Sigma\mathbf{I}_\Gamma^\Lambda\pi \prec \Sigma\mathbf{I}_\Gamma^\Lambda\sigma$.*

(iii) *$\Sigma\mathbf{I}_\Gamma^\Lambda\ell^2(\Gamma) \cong L^2(\Sigma)$ as Λ-representations.*

Proof. This follows e.g. from Lemma 6.2 in [Sh1], where for (i) we use the fact that the Λ-representation on $L^2(\Sigma)$ is mixing since the Λ-action on Σ admits a fundamental domain. □

COROLLARY 4.2. *Let M, N be two countable groups with commuting, measure-preserving actions on a σ-finite measure space (Σ, m). Suppose that N has a fundamental domain in Σ and that M is amenable and has a finite measure, fundamental domain. Then N is amenable, too.*

Proof. Recall that by A. Hulanicki's criterion [Hu], a group Λ is amenable if and only if the regular representation $\ell^2(\Lambda)$ contains weakly the trivial representation $\mathbb{1}_\Lambda$. Thus by assumption $\mathbb{1}_M \prec \ell^2(M)$. The first and third points of Lemma 4.1 imply $\Sigma\mathbf{I}_M^N \mathbb{1}_M \prec L^2(\Sigma)$. On the other hand, $\Sigma\mathbf{I}_M^N \mathbb{1}_M = L^2(M\backslash\Sigma)$ contains $\mathbb{1}_N$ since $M\backslash\Sigma$ has finite measure. Since $L^2(\Sigma) \cong \bigoplus_{n=1}^\infty \ell^2(N)$ we conclude that $\mathbb{1}_N \prec \bigoplus_{n=1}^\infty \ell^2(N)$ and thus N is amenable by a generalised version of A. Hulanicki's criterion, see [Z2, 7.3.6]. $\square$

4.2. *Cohomological induction.* Given an ME coupling (Σ, m) of two countable groups Λ, Γ and separable coefficient Γ-module (π, E), there is a natural way to induce the bounded cohomology. More specifically, we define a map

$$\Sigma\mathbf{i}_\Gamma^\Lambda : \ \mathrm{H}_\mathrm{b}^n(\Gamma, \pi) \longrightarrow \mathrm{H}_\mathrm{b}^n(\Lambda, \Sigma\mathbf{I}_\Gamma^\Lambda \pi) \tag{8}$$

as follows. Let $\chi : \Sigma \to \Gamma$ be a retraction for the Γ-action on Σ. Realise the left hand side of (8) by the complex (4) whose n^{th} term is $\ell^\infty(\Gamma^{n+1}, E)^\Gamma$. Likewise, the n^{th} term for the right hand side is $\ell^\infty\big(\Lambda^{n+1}, L^{[2]}(\Sigma, E)^\Gamma\big)^\Lambda$. For every f in the former, define $\Sigma\mathbf{i}_\Gamma^\Lambda f$ in the latter by

$$\Sigma\mathbf{i}_\Gamma^\Lambda f(\lambda_0, \ldots, \lambda_n)(x) = f(\chi(\lambda_0^{-1}x), \ldots, \chi(\lambda_n^{-1}x)). \tag{9}$$

It is straightforward to verify the statement:

LEMMA 4.3. *This map $\Sigma\mathbf{i}_\Gamma^\Lambda$ is a well defined linear map ranging in $\ell^\infty\big(\Lambda^{n+1}, L^{[2]}(\Sigma, E)^\Gamma\big)^\Lambda$. Moreover, it is continuous and when n varies one obtains a morphism of complexes.* $\square$

Observe that the very fact that $\Sigma\mathbf{i}_\Gamma^\Lambda$ yields cochains with square-summable coefficients would *a priori* not be true if we tried to induce general (unbounded) cocycles.

The main result of this section is:

THEOREM 4.4 (Induction). *Let (Σ, m) be an* ME *coupling of countable groups Λ, Γ. Then the induction map*

$$\Sigma\mathbf{i}_\Gamma^\Lambda : \ \mathrm{H}_\mathrm{b}^2(\Gamma, \pi) \longrightarrow \mathrm{H}_\mathrm{b}^2(\Lambda, \Sigma\mathbf{I}_\Gamma^\Lambda \pi)$$

is injective for every separable coefficient Γ-module (π, E). Moreover, it does not depend of the choice of χ.

The tools provided by Theorems 3.2 and 3.3 – more specifically, Corollary 3.5 – enable us to deduce Theorem 4.4 from the following:

Proposition 4.5. *Let* (Σ, m) *be an* ME *coupling of countable groups* Λ, Γ, *let* $\chi : \Sigma \to \Gamma$ *be a retraction for the* Γ*-action on* Σ *and let* (π, E) *be a coefficient* Γ*-module. Then the map*

$$\begin{aligned} \chi^* :\ \ell^\infty\big(\Gamma^{n+1}, L^\infty_{\mathrm{w}*}(\Lambda\backslash\Sigma, E)\big)^\Gamma &\longrightarrow \ell^\infty\big(\Lambda^{n+1}, L^\infty_{\mathrm{w}*}(\Sigma, E)^\Gamma\big)^\Lambda \\ \chi^* f(\lambda_0, \ldots, \lambda_n)(s) \ &= \ f(\chi(\lambda_0^{-1}s), \ldots, \chi(\lambda_n^{-1}s))(s) \end{aligned}$$

induces an injection

$$\mathfrak{s}_\Sigma :\ \mathrm{H}^n_{\mathrm{b}}\big(\Gamma, L^\infty_{\mathrm{w}*}(\Lambda\backslash\Sigma, E)\big) \longrightarrow \mathrm{H}^n_{\mathrm{b}}\big(\Lambda, L^\infty_{\mathrm{w}*}(\Sigma, E)^\Gamma\big).$$

Moreover, $\mathfrak{s}_\Sigma$ *does not depend on the choice of* χ.

Proof that Proposition 4.5 *implies Theorem* 4.4. Let E be separable. In view of formula (9), the induction map $\Sigma\mathbf{i}_\Gamma^\Lambda$ factors as

$$\begin{aligned} (10)\quad \mathrm{H}^n_{\mathrm{b}}(\Gamma, E) \xrightarrow{\varepsilon_*} \mathrm{H}^n_{\mathrm{b}}\big(\Gamma, L^\infty(\Lambda\backslash\Sigma, E)\big) &\xrightarrow{\mathfrak{s}_\Sigma} \\ \mathrm{H}^n_{\mathrm{b}}\big(\Lambda, L^\infty(\Sigma, E)^\Gamma\big) &\xrightarrow{\iota_*} \mathrm{H}^n_{\mathrm{b}}\big(\Lambda, L^{[2]}(\Sigma, E)^\Gamma\big), \end{aligned}$$

where $\varepsilon : E \to L^\infty(\Sigma, E)^\Lambda$ is the inclusion of constant functions and ι is the inclusion of $L^\infty(\Sigma, E)^\Gamma$ into $L^{[2]}(\Sigma, E)^\Gamma = \Sigma\mathbf{I}_\Gamma^\Lambda\pi$. The map ε admits a Γ-equivariant right inverse given by integration over the finite measure space $\Lambda\backslash\Sigma$. Therefore, the first map in (10) is injective. The second map is injective and independent of χ by Proposition 4.5. Since ι is the dual of the inclusion of $L^{[2]}(\Sigma, E)^\Gamma$ into $L^{[1]}(\Sigma, E)^\Gamma$ and $L^{[2]}(\Sigma, E)^\Gamma$ is separable, we are in situation to apply Corollary 3.5 and ι_* is also injective for $n = 2$. This is the only point where we use the separability of E and $n = 2$. □

Thus we are left to prove Proposition 4.5. Given the importance of Theorem 4.4 for the paper, we shall present two proofs: first, we give a functorial proof that actually shows more. Then, for the convenience of a reader who would want to avoid the use of cohomological machinery, we outline an independent down-to-earth proof.

4.3. *Functorial proof.* We prove a more general "reciprocity" statement:

Proposition 4.6. *Let* Λ, Γ *be countable groups*, S *a standard measure space with measure class preserving* $\Lambda\times\Gamma$*-action and let* (π, E) *be a coefficient* $\Lambda\times\Gamma$*-module. If both the* Λ*- and* Γ*-actions on* S *are amenable*, *then there is a canonical isomorphism*

$$(11)\qquad \mathfrak{s} :\ \mathrm{H}^n_{\mathrm{b}}\big(\Gamma, L^\infty_{\mathrm{w}*}(S, E)^\Lambda\big) \cong \mathrm{H}^n_{\mathrm{b}}\big(\Lambda, L^\infty_{\mathrm{w}*}(S, E)^\Gamma\big)$$

for all $n \geq 0$. *Moreover*, *if the* Γ*-action on* S *admits a retraction* $\chi : S \to \Gamma$, *then the map*

$$\begin{aligned} \chi^* :\ \ell^\infty\big(\Gamma^{n+1}, L^\infty_{\mathrm{w}*}(S, E)^\Lambda\big)^\Gamma &\longrightarrow \ell^\infty\big(\Lambda^{n+1}, L^\infty_{\mathrm{w}*}(S, E)^\Gamma\big)^\Lambda \\ \chi^* f(\lambda_0, \ldots, \lambda_n)(s) \ &= \ f(\chi(\lambda_0^{-1}s), \ldots, \chi(\lambda_n^{-1}s))(s) \end{aligned}$$

induces $\mathfrak{s}$ *on cohomology.*

This implies indeed Proposition 4.5 since an action with fundamental domain is amenable (and the Λ-representation on E is taken to be trivial).

Proof of Proposition 4.6. The $\Lambda\times\Gamma$-action on $\Gamma^{n+1}\times S$ given by diagonal Γ-action on $\Gamma^{n+1}\times S$ and Λ-action on S is amenable because it is isomorphic to the $\Lambda\times\Gamma$-action given by diagonal Γ-action on Γ^{n+1} and Λ-action on S. Therefore the coefficient $\Lambda\times\Gamma$-module

$$\ell^\infty\big(\Gamma^{n+1}, L^\infty_{\mathrm{w}*}(S,E)\big) \cong L^\infty_{\mathrm{w}*}(\Gamma^{n+1}\times S, E)$$

is relatively injective; see [M, N° 5.7.1]. Likewise, $\ell^\infty\big(\Lambda^{n+1}, L^\infty_{\mathrm{w}*}(S,E)\big)$ is relatively injective for $\Lambda\times\Gamma$. Therefore, there is a $\Lambda\times\Gamma$-morphism of complexes

$$\ell^\infty\big(\Gamma^{\bullet+1}, L^\infty_{\mathrm{w}*}(S,E)\big) \to \ell^\infty\big(\Lambda^{\bullet+1}, L^\infty_{\mathrm{w}*}(S,E)\big)$$

and any two such morphisms are $\Lambda\times\Gamma$-homotopic. (This follows from [M, §7]; the complexes above have indeed a contracting homotopy by evaluation of the first variable.) In particular – and due to the symmetry between Λ and Γ – there is a canonical isomorphism between the cohomology of the associated nonaugmented complexes of $\Lambda\times\Gamma$-invariants. But those identify to the complexes with n^{th} term $\ell^\infty(\Gamma^{n+1}, L^\infty_{\mathrm{w}*}(S,E)^\Lambda)^\Gamma$, respectively $\ell^\infty(\Lambda^{n+1}, L^\infty_{\mathrm{w}*}(S,E)^\Gamma)^\Lambda$, which compute canonically both sides of (11), whence the first part of the proposition.

If now we have a retraction χ, then the formula for χ^* also yields an example of such a morphism of complexes, thus inducing the same map since all morphisms are $\Lambda\times\Gamma$-homotopic. □

Observe that this proof shows that both sides in (11) are canonically isomorphic to $\mathrm{H}^n_{\mathrm{b}}\big(\Gamma\times\Lambda, L^\infty_{\mathrm{w}*}(S,E)\big)$. (Note that this situation contrasts sharply with Theorem 3.7 and illustrates the necessity of the separability assumption in the latter, an assumption not satisfied by $L^\infty_{\mathrm{w}*}(S,E)$ above.)

4.4. *Another proof.* We briefly indicate here another way to deduce Proposition 4.5, starting with an elementary proof for a special case (Lemma 4.8) and then dealing with increasing levels of generality. First we state the following fact, skipping the tedious verification of the computation (recall that it follows anyway from the previous functorial proof).

LEMMA 4.7. *The map* $\mathfrak{s}_\Sigma$ *in Proposition* 4.5 *does not depend on the choice of* χ. □

LEMMA 4.8. *Proposition* 4.5 *holds if the* Λ*-action on* Σ *admits a fundamental domain contained in some fundamental domain for the* Γ*-action.*

Proof. In that case, we can choose retractions $\chi:\Sigma\to\Gamma$ and $\kappa:\Sigma\to\Lambda$ such that $\kappa^{-1}(e_\Lambda)\subseteq\chi^{-1}(e_\Gamma)$. If we define now κ^* by a formula analogous to that of Proposition 4.5, we have

$$\begin{aligned}\kappa^*\chi^* f(\gamma_0,\ldots,\gamma_n)(s) &= \chi^* f(\kappa(\gamma_0^{-1}s),\ldots,\kappa(\gamma_n^{-1}s))(s)\\ &= f(\chi(\kappa(\gamma_0^{-1}s)^{-1}s),\ldots,\chi(\kappa(\gamma_n^{-1}s)^{-1}s))(s).\end{aligned}$$

Since

$$\kappa(\gamma_i^{-1}s)^{-1}s = \gamma_i\kappa(\gamma_i^{-1}s)^{-1}\gamma_i^{-1}s \in \gamma_i\kappa^{-1}(e_\Lambda) \subseteq \gamma_i\chi^{-1}(e_\Gamma),$$

we deduce $\kappa^*\chi^* f = f$ so that χ^* has a left inverse as morphism of complexes. This implies the injectivity of $\mathfrak{s}_\Sigma$ in view of Lemma 4.7. □

Corollary 4.9. *Proposition* 4.5 *holds if the coupling is ergodic and* $[\Gamma:\Lambda]_\Sigma \le 1$.

Proof. In the ergodic situation, the full group of automorphisms of the $\Gamma\times\Lambda$-action on Σ acts transitively on sets of equal measure (up to null-sets). Therefore, we can find a Λ-fundamental domain contained in some fundamental domain for the Γ-action. Now we can apply Lemmas 4.8 and 4.7. □

Proposition 4.10. *Proposition* 4.5 *holds if* $[\Gamma:\Lambda]_\Sigma \le 1$ (*without ergodicity assumption*).

Proof. By Lemma 2.2 in [F1], any ME coupling (Σ, m) can be disintegrated into ergodic couplings (Σ_t, m_t) with $m = \int_T m_t\, d\eta(t)$ for some probability space (T,η). The set

$$T_+ = \big\{t\in T\ :\ [\Gamma:\Lambda]_{\Sigma_t} \le 1\big\}$$

has positive η-measure since $[\Gamma:\Lambda]_\Sigma \le 1$. Let Σ_+ be the coupling obtained by integrating m_t over T_+; using Corollary 3.4, one checks that the injectivity of $\Sigma_+\mathbf{i}_\Gamma^\Lambda$ follows from the injectivity of η-a.e. $\Sigma_t\mathbf{i}_\Gamma^\Lambda$, which is granted by Corollary 4.9. One checks similarly that the injectivity of $\Sigma\mathbf{i}_\Gamma^\Lambda$ follows from the injectivity of $\Sigma_+\mathbf{i}_\Gamma^\Lambda$. □

Using Lemma 4.7, one verifies:

Lemma 4.11. *Let* Σ *be an* ME *coupling of countable groups* Λ,Γ *and* Σ' *an* ME *coupling of countable groups* Δ,Λ. *Then, for every coefficient* Γ*-module* (π,E), $\mathfrak{s}_{\Sigma'}\mathfrak{s}_\Sigma = \mathfrak{s}_{\Sigma'\times_\Lambda\Sigma}$ *and hence* $\Sigma'\mathbf{i}_\Lambda^\Delta(\Sigma\mathbf{i}_\Gamma^\Lambda\pi) = (\Sigma'\times_\Lambda\Sigma)\mathbf{i}_\Gamma^\Delta\pi$. □

End of second proof of Proposition 4.5. Consider the composed ME coupling of Γ with itself $\Omega = \check{\Sigma}\times_\Lambda\Sigma$. In view of Lemma 4.11, it is enough to show that the map

$$\mathfrak{s}_\Omega :\ \mathrm{H}^n_{\mathrm{b}}\big(\Gamma, L^\infty_{\mathrm{w}*}(\Gamma\backslash\Omega, E)\big) \longrightarrow \mathrm{H}^n_{\mathrm{b}}\big(\Gamma, L^\infty_{\mathrm{w}*}(\Omega, E)^\Gamma\big)$$

defined as in Proposition 4.5 is injective. By (1)

$$[\Gamma:\Gamma]_\Omega = [\Gamma:\Lambda]_\Sigma\cdot[\Lambda:\Gamma]_{\check{\Sigma}} = 1,$$

so that we may conclude by applying Proposition 4.10 to Ω. □

5. Strong rigidity

5.1. *Strong rigidity for products.* Our first goal is to prove Theorem 1.16 from the introduction, for the more general class of groups $\mathcal{C}$ defined in 2.15. The use of bounded cohomology in the proof of Theorem 1.16 is detailed in the following result which we isolate for further reference:

PROPOSITION 5.1. *Let (Σ, m) be an* ME *coupling of $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ with $\Lambda = \Lambda_1 \times \cdots \times \Lambda_{n'}$, where Γ_i are torsion-free groups in $\mathcal{C}$ and Λ_j are any countable groups. Then there are a surjective map $t : \{1, \ldots, n\} \to \{1, \ldots, n'\}$ and a fundamental domain $Y \subseteq \Sigma$ for the Γ-action such that $\Lambda'_{t(i)} Y \subseteq \Gamma'_i Y$ for all i. Moreover, if $n = n'$, then $\Lambda_{t(i)} Y \subseteq \Gamma_i Y$ for all i and the groups Λ_j are also in $\mathcal{C}$.*

Proof. First we perform an induction on $1 \le k \le n$ to construct a map t and fundamental domains $Y_k \subseteq \Sigma$ for the Γ-action such that

$$\Lambda'_{t(i)} Y_k \;\subseteq\; \Gamma'_i Y_k \qquad \forall\ 1 \le i \le k. \tag{12}$$

Let π_1 be a mixing unitary representation of Γ_1 such that $\mathrm{H}^2_{\mathrm{b}}(\Gamma_1, \pi_1) \neq 0$. We also denote by π_1 the corresponding representation of Γ (factoring through Γ_1), and recall that by Corollary 3.6 we have $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi_1) \neq 0$. Therefore, applying Theorem 4.4, we deduce that $\mathrm{H}^2_{\mathrm{b}}(\Lambda, \Sigma\mathbf{I}_\Gamma^\Lambda \pi_1)$ is nonx-trivial. We apply now the product formula of Theorem 3.7 to find some $1 \le t(1) \le n'$ such that

$$\mathrm{H}^2_{\mathrm{b}}\left(\Lambda_{t(1)}, (\Sigma\mathbf{I}_\Gamma^\Lambda \pi_1)^{\Lambda'_{t(1)}}\right) \;\neq 0. \tag{13}$$

Let Y_0 be any fundamental domain for Γ, and $\beta : \Lambda \times Y_0 \to \Gamma$ be the associated cocycle. Take for $\Sigma\mathbf{I}_\Gamma^\Lambda \pi_1$ the model $L^2(Y_0, \mathcal{H}_{\pi_1})$ with β-twisted action. Since (13) forces $(\Sigma\mathbf{I}_\Gamma^\Lambda \pi_1)^{\Lambda'_{t(1)}} \neq 0$, this amounts to the existence of a nonzero measurable function $f : Y_0 \to \mathcal{H}_{\pi_1}$ such that $f(\lambda' \cdot x) = \pi_1(\beta(\lambda', x))f(x)$ for every $\lambda' \in \Lambda'_{t(1)}$ and almost every $x \in Y_0$. Since the Γ_1-representation on $\mathcal{H}_{\pi_1}$ is mixing, the Γ-representation is tame and thus the cocycle reduction lemma (Lemma 5.2.11 in [Z2]) can be applied to the restriction $\beta : \Lambda'_{t(1)} \times Y_0 \to \Gamma$. Mind that the $\Lambda'_{t(1)}$-action on Y_0 is not assumed ergodic, but since Γ_1 is torsion-free and $\pi_1(\Gamma_1)$ mixing, the only possible stabiliser in Γ of nonzero elements of $\mathcal{H}_{\pi_1}$ is Γ'_1. Thus cocycle reduction applied to every ergodic component yields a measurable $\varphi : Y_0 \to \Gamma$ such that

$$\beta'(\lambda, x) = \varphi(\lambda \cdot x)\beta(\lambda, x)\varphi(x)^{-1}$$

ranges in Γ'_1 whenever $\lambda \in \Lambda'_{t(1)}$. Observe that if we replace φ by its composition with the projection $\Gamma \to \Gamma_1 < \Gamma$ and take for β' the resulting cocycle $\beta' : \Lambda \times Y_0 \to \Gamma$, we have still $\beta'(\Lambda'_{t(1)} \times Y_0) \subseteq \Gamma'_1$ almost everywhere; we make this change. We consider the new fundamental domain

$$Y_1 = \left\{\varphi(x)x \;:\; x \in Y_0\right\}$$

for the Γ-action on Σ. For every $\lambda' \in \Lambda'_{t(1)}$ and almost every $x \in Y_0$

$$\begin{aligned}\lambda'\varphi(x)x &= \varphi(x)\lambda' x = \varphi(x)\beta(\lambda', x)^{-1}\lambda' \cdot x \\ &= \big(\varphi(\lambda' \cdot x)\beta(\lambda', x)\varphi(x)^{-1}\big)^{-1}\varphi(\lambda' \cdot x)\lambda' \cdot x \ \in \ \Gamma'_1\varphi(\lambda' \cdot x)\lambda' \cdot x \subseteq \Gamma'_1 Y_1.\end{aligned}$$

This shows

$$\Lambda'_{t(1)} Y_1 \ \subseteq \ \Gamma'_1 Y_1. \tag{14}$$

Let now $k \geq 2$ and assume that Y_{k-1} and $t : \{1, \ldots, k-1\} \to \{1, \ldots, n'\}$ such that (12) holds for $k-1$. We take now a mixing unitary representation π_k of Γ_k such that $\mathrm{H}^2_{\mathrm{b}}(\Gamma_k, \pi_k) \neq 0$. Arguing as above, there is $1 \leq t(k) \leq n'$ with

$$\mathrm{H}^2_{\mathrm{b}}\Big(\Lambda_{t(k)}, (\Sigma \mathbf{I}_\Gamma^\Lambda \pi_k)^{\Lambda'_{t(k)}}\Big) \ \neq 0 \tag{15}$$

and thus $(\Sigma \mathbf{I}_\Gamma^\Lambda \pi_k)^{\Lambda'_{t(k)}} \neq 0$. We perform again a reduction of cocycle and obtain as above a map $\psi : Y_{k-1} \to \Gamma_k$ such that

$$Y_k = \big\{\psi(x)x \ : \ x \in Y_{k-1}\big\}$$

is a fundamental domain for the Γ-action on Σ satisfying

$$\Lambda'_{t(k)} Y_k \ \subseteq \ \Gamma'_k Y_k. \tag{16}$$

We claim that Y_k still satisfies

$$\Lambda'_{t(i)} Y_k \ \subseteq \ \Gamma'_i Y_k \qquad \forall\ 1 \leq i \leq k-1. \tag{17}$$

Indeed, for all $\lambda' \in \Lambda'_{t(i)}$ and almost every $x \in Y_{k-1}$ we have $\lambda'\psi(x)x = \psi(x)\lambda' x$ which by (14) is $\psi(x)\gamma' y$ for some $\gamma' \in \Gamma'_i$ and $y \in Y_{k-1}$. Thus

$$\lambda'\psi(x)x = \big(\psi(x)\gamma'\psi(y)^{-1}\big)\psi(y)y \ \in \ \Gamma'_i Y_k,$$

as claimed.

Thus (16) and (17) complete the induction step to prove (12). We let now $Y = Y_k$, so that for the first claim of Proposition 5.1, it remains only to show that t is surjective onto $\{1, \ldots, n'\}$. Suppose for a contradiction that there is $1 \leq j \leq n'$ not in the image of t. Then Λ_j is contained in $\bigcap_{i=1}^n \Lambda'_{t(i)}$, so that in view of (12) we have

$$\Lambda_j Y \subseteq \bigcap_{i=1}^n \Gamma'_i Y = Y,$$

contradicting the properness of the Λ-action, since Λ_j is nonx-trivial.

Under the additional assumption $n = n'$, the map t must be bijective. Then, using (12), we have for all $1 \leq i \leq n$ the inclusion

$$\Lambda_{t(i)} Y = \bigcap_{j \neq t(i)} \Lambda'_j Y \subseteq \bigcap_{\ell \neq i} \Gamma'_\ell Y = \Gamma_i Y, \tag{18}$$

as claimed. It remains only to show that $\Lambda_{t(i)}$ is in $\mathcal{C}$ for all i. Let π_i be the mixing Γ_i-representation that gave (15) for $k = i$ in the inductive argument above; in particular (15) implies

$$\mathrm{H}^2_{\mathrm{b}}\left(\Lambda_{t(i)}, \Sigma\mathbf{I}_\Gamma^\Lambda \pi_i\right) \neq 0. \tag{19}$$

The inclusion (18) shows that the set $\Sigma_i = \Gamma_i Y \subseteq \Sigma$ is $\Lambda_{t(i)} \times \Gamma_i$-invariant. The $\Lambda_{t(i)}$-action on Σ_i admits some fundamental domain because it is a subspace of Σ, and the Γ_i-action admits the finite measure fundamental domain Y. Therefore the $\Lambda_{t(i)}$-representation $\tau = \Sigma_i \mathbf{I}_{\Gamma_i}^{\Lambda_{t(i)}} \pi_i$ is mixing by point (i) in Lemma 4.1. On the other hand, since Σ_i is a fundamental domain for the Γ'_i-action on Σ, and Γ'_i acts trivially on $\mathcal{H}_{\pi_i}$, we have

$$\mathcal{H}_\tau = L^{[2]}(\Sigma_i, \mathcal{H}_{\pi_i})^{\Gamma_i} \cong L^{[2]}(\Sigma, \mathcal{H}_{\pi_i})^{\Gamma}.$$

Therefore, the $\Lambda_{t(i)}$-representation τ is isomorphic to the restriction to $\Lambda_{t(i)}$ of $\Sigma\mathbf{I}_\Gamma^\Lambda \pi_i$. Thus (19) shows indeed that $\Lambda_{t(i)}$ is in $\mathcal{C}$. □

Proof of Theorem 1.16 *for the class* $\mathcal{C}$ *instead of* $\mathcal{C}_{\mathrm{reg}}$. Whenever two groups Γ, Λ are measure equivalent, one can assume by disintegration that there is an *ergodic* ME coupling (Σ, m) of Γ with Λ; see [F1, 2.2]. First we apply Proposition 5.1 to obtain a bijection t and a Γ-fundamental domain Y with $\Lambda_{t(i)} Y \subseteq \Gamma_i Y$ for all i. Since by the proposition all Λ_j are in $\mathcal{C}$ and $n = n'$, the assumptions of the theorem (with class $\mathcal{C}$) now also hold if we reverse Γ and Λ. Therefore, a second application of Proposition 5.1 gives us a bijection s and a fundamental domain X for the Λ-action on Σ such that $\Gamma_{s(j)} X \subseteq \Lambda_j X$ for all j. Since all groups commute, the properties of our Γ-fundamental domain Y are not altered if we translate it by an element of Γ; therefore, we may and do assume that the intersection $A = X \cap Y$ has positive measure. We claim that the bijections s and t are inverse to each other.

Indeed, pick $1 \leq j \leq n$, write $i = s(j)$ and let us show that $t(i) = j$. Write A_Γ for the image of A in $\Gamma \backslash \Sigma$ and apply Poincaré recurrence to the $\Lambda_{t(i)}$-action on $\Gamma\backslash\Sigma$. This yields a nontrivial element $\lambda \in \Lambda_{t(i)}$ such that $\lambda A_\Gamma \cap A_\Gamma$ has positive measure. Therefore, there is $\gamma \in \Gamma$ such that $B = \lambda A \cap \gamma A$ has positive measure. Since $A \subseteq Y$, the inclusion $\Lambda_{t(i)} Y \subseteq \Gamma_i Y$ implies $B \subseteq \gamma Y \cap \Gamma_i Y$ and hence $\gamma \in \Gamma_i$. But now $\gamma A \subseteq \Gamma_i X = \Gamma_{s(j)} X \subseteq \Lambda_j X$ so that $B \subseteq \Lambda_j X \cap \lambda X$. It follows that $\lambda \in \Lambda_j$, and since λ is nontrivial we conclude $t(i) = j$, proving the claim.

In conclusion, we may permute the indices in such a way that

$$\Lambda_i Y \subseteq \Gamma_i Y \quad \text{and} \quad \Gamma_i X \subseteq \Lambda_i X \qquad \forall\, 1 \leq i \leq n. \tag{20}$$

Choose now some i and let $\overline{\Sigma}$ be the space of ergodic components of the $\Gamma'_i \times \Lambda'_i$-action on Σ. We shall show that for an appropriate measure ν the space $(\overline{\Sigma}, \nu)$ yields an ME coupling of Γ_i with Λ_i for the natural $\Gamma_i \times \Lambda_i$-action inherited from Σ. Note that we cannot take for ν the projection of m, since this could e.g. give infinite measure to every point in $\overline{\Sigma}$. Instead, we identify $\overline{\Sigma}$ with the space of ergodic components of the Λ'_i-action on $\Gamma'_i\backslash\Sigma$, the latter being measurably identified with $\Gamma_i Y$. We let ν be the projection of $m|_{\Gamma_i Y}$ under the corresponding map $\Gamma_i Y \to \overline{\Sigma}$.

We claim that Γ_i has a ν-finite fundamental domain in $\overline{\Sigma}$. Indeed, let $\overline{Y}$ be the image of Y in $\overline{\Sigma}$; one has $\Gamma_i\overline{Y} = \overline{\Sigma}$ and $\nu(\overline{Y})$ is finite because in view of (20) the pre-image of $\overline{Y}$ in $\Gamma'_i\backslash\Sigma$ is just Y under the identification $\Gamma'_i\backslash\Sigma \cong \Gamma_i Y$. Therefore, to conclude that $\overline{Y}$ is a finite measure fundamental domain for Γ_i in $(\overline{\Sigma}, \nu)$, it remains to show that $\gamma_i \in \Gamma_i$ must be trivial whenever $\gamma_i\overline{Y} \cap \overline{Y}$ is nonnull for ν. This is indeed the case, since then $\gamma_i\Gamma'_i\Lambda'_i Y \cap \Gamma'_i\Lambda'_i Y$ has positive measure in Σ; by (20), this set is $\gamma_1\Gamma'_i Y \cap \Gamma'_i Y$ so that $\gamma_i \in \Gamma'_i$ and hence γ_i is trivial.

Arguing in a symmetric manner, we consider the *a priori* different measure ν' on $\overline{\Sigma}$ obtained by projecting $m|_{\Lambda_i X}$ under $\Lambda_i X \to \overline{\Sigma}$ and deduce that Λ_i has a ν'-finite fundamental domain $\overline{X}$ in $\overline{\Sigma}$. In order to complete the proof that $(\overline{\Sigma}, \nu)$ yields an ME coupling of Γ_i with Λ_i, it is enough to show that ν' is a scalar multiple of ν. The measures ν, ν' are absolutely continuous with respect to each other since they are projected from $\Gamma_i Y$ and $\Lambda_i X$ respectively. Since we started by reducing to an ergodic ME coupling (Σ, m), the measures ν, ν' are both ergodic for the $\Gamma_i \times \Lambda_i$-action, so due to absolute continuity they are a scalar multiple one of the other. This completes the proof of Theorem 1.16. □

Proof of Theorem 2.17. Proposition 5.1 shows that the groups Λ_j are also in $\mathcal{C}$. Therefore, under the assumption (ii) the situation is symmetric and we may switch Γ with Λ if necessary so that in either case the assumption (i) holds; in other words, we may assume

$$m(\Gamma\backslash\Sigma) \geq m(\Lambda\backslash\Sigma). \tag{21}$$

Moreover, Proposition 5.1 gives us a fundamental domain $Y \subseteq \Sigma$ for Γ such that after possibly permuting indices

$$\Lambda_i Y \subseteq \Gamma_i Y \qquad (1 \leq i \leq n). \tag{22}$$

Fix some i. By the above, the subset $C = \Gamma'_i Y \subseteq \Sigma$ is preserved by $\Lambda'_i \times \Gamma'_i$. Since $C \cong \Gamma_i\backslash\Sigma$, the ergodicity of Λ'_i on $\Gamma\backslash\Sigma$ implies that C is an ergodic component of the $\Lambda'_i \times \Gamma'_i$-action on Σ. We know from the proof of Theorem 1.16 that the space $\overline{\Sigma}$ of ergodic components is an ME coupling of Λ_i with Γ_i for the measure projected from $m|_{\Gamma_i Y}$ (indeed, the ergodicity assumption of that proof is implied by the irreducibility of Λ on $\Gamma\backslash\Sigma$). Now C corresponds to an atom in $\overline{\Sigma}$ and thus the Γ'_i-invariant partition $\Sigma = \bigsqcup_{\gamma_i \in \Gamma_i} \gamma_i C$ shows that Γ_i acts simply transitively on $\overline{\Sigma}$.

Since C is an ergodic component and all actions commute, $\lambda_i C \cap C$ can only be C or $\varnothing$ (up to null-sets) when $\lambda_i \in \Lambda_i$. The first case implies in particular $\lambda_i Y \subseteq C = \Gamma'_i Y$. But (22) gives $\lambda_i Y \subseteq \Gamma_i Y$ and thus $\lambda_i Y \subseteq Y$. Since Λ_i is torsion-free and acts properly on Σ, we conclude that λ_i is trivial. We have shown that Λ_i acts freely on $\overline{\Sigma}$, and also that the sets $\lambda_i Y$ are mutually disjoint.

A dual argument shows that all $\lambda_j Y$ are mutually disjoint when λ_j ranges over Λ_j and $1 \leq j \leq n$. We claim that this is also the case for all λY with $\lambda \in \Lambda$. Indeed, assume that $\lambda Y \cap Y$ is nonnull and let us show that the component λ_k of λ in Λ_k vanishes for any k. Write $\lambda = \lambda_k \lambda'$ with $\lambda' \in \Lambda'_k$. Then the set $\lambda' Y \cap \lambda_k^{-1} Y$ is nonnull; however, the inclusions (22) give

$$\lambda' Y \cap \lambda_k^{-1} Y \ \subseteq\ \Gamma'_k Y \cap \Gamma_k Y \ =\ Y$$

and therefore $\lambda_k Y \cap Y$ has positive measure. By what we already know, this makes λ_k trivial as claimed.

The claim just proved forces Y to be a fundamental domain for the Λ-action on Σ because of the inequality (21). In particular, we have $[\Lambda : \Gamma]_\Sigma = 1$ and

$$\Lambda_j Y \ =\ \Gamma_j Y \qquad (1 \leq j \leq n). \tag{23}$$

Thus, returning now to fix some i, the arguments used to show that Γ_i acts simply transitively on $\overline{\Sigma}$ can be reversed to show that Λ_i does so too. Hence the coupling $\overline{\Sigma}$ is trivial (see Example 2.7) and yields an isomorphism $f_i : \Lambda_i \to \Gamma_i$. Observe that $\overline{\Sigma}$ can be viewed as the space of ergodic components of the Λ'_i-action induced on $\Gamma_i Y$ via $\Gamma_i Y \cong \Gamma'_i \backslash \Sigma$, and therefore by (23) it is the space of ergodic components of the corresponding action on $\Lambda_i Y$. These components are precisely the Λ_i-translates of Y by irreducibility; we choose for f_i the conjugate obtained by fixing as base point in $\overline{\Sigma}$ the component corresponding to Y in the above identification. We apply the whole argument to each index i and observe that with our choice of f_i we have indeed

$$\Pi_{i=1}^n \big(\lambda_i f_i(\lambda_i)\big) Y \ =\ Y$$

for all $\lambda_i \in \Lambda_i$. $\square$

5.2. *Strong rigidity with radicals.* The main goal of this subsection is to prove Theorem 2.23, which will actually be simpler than in the product case (unlike the case of the superrigidity theorem). We begin by proving a more general version of Theorem 1.17:

Proof of Theorem 1.17 *for the class* $\mathcal{C}$ *instead of* $\mathcal{C}_{\mathrm{reg}}$. Let (Σ, m) be an ME coupling of Γ with Λ; as mentioned before, we may assume it is ergodic [F1, 2.2]. The argument is similar to a part of the proof of Theorem 1.16, and we shall prove that the space $(\overline{\Sigma}, \nu)$ or ergodic components of the $M \times N$-action on Σ is an ME coupling of $\overline{\Gamma}$ with $\overline{\Lambda}$ for an appropriate measure ν. Let m_N be the measure m restricted to $C = N\backslash\Sigma$ (e.g. via a Λ-fundamental domain in Σ and a section of Λ/N) and take for ν the projection of m_N under the map $C \to \overline{\Sigma}$. As in Theorem 1.16, the ergodicity implies that one obtains the same measure up to a scalar by proceeding with $M\backslash\Sigma$ instead. Thus, due to the symmetry of the situation, it is enough to show that $\overline{\Lambda}$ has a ν-finite measure fundamental domain in $\overline{\Sigma}$.

Given a mixing unitary representation π of $\overline{\Lambda}$ with nonvanishing H^2_b, we use Corollary 3.6 and Theorem 4.4 exactly as in the proof of Proposition 5.1 to deduce that $\mathrm{H}^2_\mathrm{b}(\Gamma, \Sigma\mathbf{I}_\Lambda^\Gamma\pi)$ is nontrivial. By Proposition 3.8, the space $(\Sigma\mathbf{I}_\Lambda^\Gamma\pi)^M$ is nontrivial. Given a fundamental domain $X' \subseteq \Sigma$ for Λ and the associated cocycle $\alpha : \Gamma \times X' \to \Lambda$, we obtain a nonzero measurable function $f : X' \to \mathcal{H}_\pi$ such that $f(\gamma \cdot x) = \pi(\alpha(\gamma, x))f(x)$ for every $\gamma \in M$ and almost every $x \in X'$. Since π is tame as Λ-representation, we may apply cocycle reduction for the restriction $\alpha : M \times X' \to \Lambda$ in the nonergodic setting. We assumed $\overline{\Lambda}$ torsion-free and π mixing, and so the only possible stabiliser in Λ is N. Therefore, the reduction provides us with a measurable map $\varphi : X' \to \Lambda$ such that $\varphi(\gamma \cdot x)\alpha(\gamma, x)\varphi(x)^{-1}$ is in N for every $\gamma \in M$ and almost every $x \in X'$. Now the new Λ-fundamental domain $X = \{\varphi(x)x : x \in X'\}$ satisfies

$$MX \subseteq NX \tag{24}$$

because for almost every $x \in X'$ and all $\gamma \in M$ we have

$$\begin{aligned}\gamma\varphi(x)x = \varphi(x)\gamma x &= \varphi(x)\alpha(\gamma, x)^{-1}\gamma \cdot x \\ &= \big(\varphi(\gamma_2 \cdot x)\alpha(\gamma_2, x)\varphi(x)^{-1}\big)^{-1}(\varphi(\gamma \cdot x)\gamma \cdot x)\end{aligned}$$

which is in NX.

We turn back now to $\overline{\Sigma}$. Write $\overline{X}$ for the image of X in $\overline{\Sigma}$. We claim that for every $\lambda \in \Lambda$ the sets $\lambda\overline{X}$ and $\overline{X}$ are ν-essentially disjoint unless $\lambda \in N$. Indeed, otherwise the $N \times M$-invariant subset $NM\lambda X \cap NMX$ of Σ would have positive m-measure. Then (24) implies $m(N\lambda X \cap NX) > 0$ and thus $\lambda \in N$, as claimed. On the other hand, the Λ-translates of $\overline{X}$ cover $\overline{\Sigma}$.

It remains to see that $\nu(\overline{X})$ is finite. View C as a $\Gamma \times \overline{\Lambda}$-space and let X_N be the image of X in C; we have $m_N(X_N) = m(X)$ since X is a Λ-fundamental domain. Consider now $\overline{\Sigma}$ as the space of ergodic components of M in C. By (24) M preserves X_N and so $\nu(\overline{X})$ is finite. $\square$

Proof of Theorem 2.23. Let $\overline{\Sigma}$ and X be as in the proof of Theorem 1.17. The subset $B = NX \subseteq \Sigma$ is $M \times N$-invariant by (24). Since M is ergodic on $\Lambda\backslash\Sigma$, B is an ergodic component for the $M \times N$-action on Σ. It follows now from the proof of Theorem 1.17 that $\overline{\Lambda}$ acts simply transitively on $\mathrm{T} = \overline{\Sigma}$. By symmetry, we see that $\overline{\Gamma}$ acts also simply transitively on T and the statement follows. $\square$

6. Superrigidity

6.1. *Preliminaries.*

LEMMA 6.1. *Let (Z,ζ) be an ME coupling of countable groups Γ,Λ. Let $E\subseteq Z$ be a fundamental domain for Γ and $\vartheta:\Lambda\times E\to\Gamma$ the associated cocycle. If ϑ is equivalent to a cocycle ranging ζ-essentially in a subgroup $\Gamma_0<\Gamma$, then Γ_0 has finite index in Γ.*

Proof. Let $\chi: Z\to\Gamma$ be the Γ-equivariant retraction associated to F and assume that there is $\varphi: F\to\Gamma$ such that the cocycle ϑ' defined by

$$\vartheta'(\lambda,x)=\varphi(\lambda\cdot x)\vartheta(\lambda,x)\varphi(x)^{-1}$$

ranges in Γ_0. Define $\widetilde{\varphi}: Z\to\Gamma/\Gamma_0$ by

$$\widetilde{\varphi}(x)=\chi(x)\varphi(\chi(x)^{-1}x)^{-1}\Gamma_0;$$

this is Γ-equivariant. For $x\in F$, $\lambda\in\Lambda$ and $\gamma\in\Gamma$ one has

$$\begin{aligned}\widetilde{\varphi}(\lambda\gamma x)&=\gamma\chi(\lambda x)\varphi(\chi(\lambda x)^{-1}\lambda x)^{-1}\Gamma_0=\gamma\chi(\lambda x)\varphi(\lambda\cdot x)^{-1}\Gamma_0\\&=\gamma\chi(\lambda x)\varphi(\lambda\cdot x)^{-1}\vartheta'(\lambda,x)^{-1}\Gamma_0=\gamma\chi(\lambda x)\vartheta(\lambda,x)\varphi(x)^{-1}\Gamma_0.\end{aligned}$$

Since $x\in F$, we have $\vartheta(\lambda,x)=\chi(\lambda x)^{-1}$ and the above reduces to $\gamma\varphi(x)\Gamma_0=\widetilde{\varphi}(\gamma x)$. Therefore we obtain a Γ-equivariant map $\Lambda\backslash Z\to\Gamma/\Gamma_0$. Projecting the Γ-invariant finite measure to Γ/Γ_0, we deduce that the latter is finite. □

LEMMA 6.2 (Furman's homomorphism). *Let Λ be a countable group acting on a measure space (Σ,m), preserving the measure class. Let $F:\Sigma\times\Sigma\to\Gamma$ be a measurable map to a countable group Γ and assume that for all $\lambda\in\Lambda$*

(i) $F(x,y)=F(\lambda x,\lambda y)$ *for m^2-almost all $(x,y)\in\Sigma^2$.*

(ii) $F(\lambda x,y)F(x,y)^{-1}=F(\lambda x,z)F(x,z)^{-1}$ *for m^3-almost all $(x,y,z)\in\Sigma^3$.*

(iii) $F(x,y)=F(y,x)^{-1}$ *for m^2-almost all $(x,y)\in\Sigma^2$.*

(iv) $F(x,y)F(y,z)=F(x,z)$ *for m^3-almost all $(x,y,z)\in\Sigma^3$.*

Then for m-almost every $x\in\Sigma$ the map

$$\rho_x:\Lambda\longrightarrow\Gamma,\quad \rho_x(\lambda)=F(\lambda^{-1}x,y)F(x,y)^{-1}$$

is a homomorphism, and for m^2-almost all $(x,y)\in\Sigma^2$ the homomorphisms ρ_x and ρ_y are conjugated:

$$\rho_y(\lambda)=F(x,y)^{-1}\rho_x(\lambda)F(x,y).\qquad(\forall\,\lambda\in\Lambda)$$

Proof. This follows from Lemma 5.2 in [F1] upon modification of notation. □

Definition 6.3. A group is said to be ICC (for *infinite conjugacy classes*) if the conjugacy class of every nontrivial element is infinite.

We shall see (Proposition 7.11) that this property is automatically satisfied for torsion-free groups in $\mathcal{C}$. The following is a straightforward verification:

LEMMA 6.4. *If* Γ *is an* ICC *group, then the Dirac mass* δ_e *at the neutral element is the only probability measure on* Γ *which is invariant for the* Γ*-action by conjugation.* ☐

The following lemma isolates the use of the mild mixing assumption in the results where it appears. We recall that we have introduced in Section 2.1 the notation A^1, A^2 in order to distinguish between two commuting Γ-actions.

LEMMA 6.5. *Let* (Σ, m) *be an ME coupling of two countable groups* Γ, Λ, *let* $\Delta < \Gamma$ *be a subgroup and let* $\Omega = \Sigma \times_\Lambda \Lambda \times_\Lambda \check{\Sigma}$ *be the composed self-*Γ*-coupling* (*see Definition* 2.9 *and the paragraph preceding it*). *If the* Λ*-action on* $\Gamma\backslash\Sigma$ *is mildly mixing and the* Δ*-action on* $\Lambda\backslash\Sigma$ *is ergodic*, *then the* $A^i(\Delta)$*-action on* $A^j(\Gamma)\backslash\Omega$ *is ergodic for all* $i \neq j$.

Proof. By [SW] (a reference for which we thank Eli Glasner), the assumption on the Λ-action on $\Gamma\backslash\Sigma$ is equivalent to the following: for any standard Borel Λ-space (S, ν) with a nonatomic invariant ergodic σ-finite measure ν, the diagonal Λ-action on $\Gamma\backslash\Sigma \times S$ is ergodic. Assume now that we have a real-valued $A^i(\Delta)$-invariant measurable function F on $A^j(\Gamma)\backslash\Omega$. In view of $\Omega \cong \Sigma \times_\Lambda \check{\Sigma}$, the function F corresponds to a measurable function F' on $\Sigma \times \check{\Sigma}$ that is Λ-invariant for the diagonal action and $A^i(\Delta) \times A^j(\Gamma)$-invariant. View F' as a Λ-invariant function on $\Gamma\backslash\Sigma \times \Delta\backslash\Sigma$; the ergodicity of the Δ-action on $\Lambda\backslash\Sigma$ implies that $S = \Delta\backslash\Sigma$ is an ergodic Λ-space (notice that $\Delta\backslash\Sigma$ is not atomic since otherwise Σ and then $\Gamma\backslash\Sigma$ would be, contradicting the mild mixing assumption). Therefore, the above criterion for mild mixing shows that F' is constant. ☐

6.2. *Proof of Theorems* 2.20 *and* 2.20*. Let (Ω, ω) be the composed ME coupling of Γ with itself defined by $\Omega = \Sigma \times_\Lambda \Lambda \times_\Lambda \check{\Sigma}$. The class of $(x, \lambda, y) \in \Sigma \times \Lambda \times \check{\Sigma}$ in Ω will be denoted by $[x, \lambda, y]$.

By Lemma 6.5 applied to $\Delta = \Gamma_j$ for each $1 \leq j \leq n$, the coupling Ω of Γ with itself satisfies the assumptions of Theorem 2.17 point (ii). Thus there are an automorphism f of Γ and a factor map $\Phi : \Omega \to \mathrm{T}$ to a trivial coupling $\mathbf{T}$ such that for every $\gamma, \gamma' \in \Gamma$ and a.e. $x \in \Omega$ one has

$$\Phi(A^1_\gamma A^2_{\gamma'} x) = \gamma f(\gamma')\Phi(x).$$

We may identify Γ with T as in Example 2.7 so that we consider now Φ as a map $\Omega \to \Gamma$ satisfying

$$\Phi(A^1_\gamma A^2_{\gamma'} x) = \gamma \Phi(x) f(\gamma')^{-1}. \tag{25}$$

We prove now an analogue of A. Furman's crucial Lemma 5.5 in [F1] using a similar approach, but bypassing the seemingly nontrivial question of tameness of the conjugation action of a group on the space of probability measures on it. We point out that our approach also applies to Furman's setting and simplifies the argument therein.

LEMMA 6.6. *The map* $\Psi : \Sigma \times \Sigma \times \Sigma \to \Gamma$ *defined* m^3*-a.e. by*

$$\Psi(x, y, z) = \Phi([x, e, z])\Phi([y, e, z])^{-1} \tag{26}$$

does not depend on z *in the sense that* $\Psi(x, y, z_1) = \Psi(x, y, z_2)$ *holds for* m^4*-almost every* $(x, y, z_1, z_2) \in \Sigma^4$.

Proof. Define the map $\mathcal{T} : \Sigma^4 \to \Gamma$ by

$$\mathcal{T}(x, y, z_1, z_2) = \Psi(x, y, z_1)\Psi(x, y, z_2)^{-1}. \tag{27}$$

The above definitions imply that for m^4-a.e. $(x, y, z_1, z_2) \in \Sigma^4$ and every $\lambda \in \Lambda$, $\gamma \in \Gamma$ one has

$$\begin{aligned} \mathcal{T}(x, y, z_1, z_2) &= \mathcal{T}(\lambda x, \lambda y, \lambda z_1, \lambda z_2) \\ &= \mathcal{T}(x, \gamma y, z_1, z_2) \\ &= \mathcal{T}(x, y, \gamma z_1, z_2) \\ &= \mathcal{T}(x, y, z_1, \gamma z_2). \end{aligned}$$

Consider the $\Gamma \times \Lambda$-space $Z = \Sigma \times (\Gamma\backslash\Sigma)^3$, where Γ acts on the first factor and Λ by diagonal (fourtuple) action. This is again a coupling and $\mathcal{T}$ induces a map $\mathcal{T}_0 : \Lambda\backslash Z \to \Gamma$. Substituting (25) in (26) and then in (27), one gets

$$\mathcal{T}(\gamma x, y, z_1, z_2) = \gamma \mathcal{T}(x, y, z_1, z_2)\gamma^{-1},$$

so that $\mathcal{T}_0$ is Γ-equivariant for the conjugating action on Γ. If we project now the Γ-invariant measure of $\Lambda\backslash Z$ through $\mathcal{T}_0$, we get a conjugating invariant probability measure on Γ. Since the product of two ICC groups is again ICC, the (independent) Proposition 7.11 ensures that Γ is ICC, and so Lemma 6.4 shows that $\mathcal{T}_0$ is essentially constant with value $e \in \Gamma$, proving Lemma 6.6. □

We may now define $F : \Sigma\times\Sigma \to \Gamma$ as in Lemma 6.2 by $F(x, y) = \Psi(x, y, z)$. All properties listed in Lemma 6.2 are readily verified, and we obtain thus a family of conjugated homomorphisms $\rho_x : \Lambda \to \Gamma$. In particular, we obtain a well-defined subgroup $N \lhd \Lambda$ as the kernel of almost all ρ_x. For later reference, we record that the definition of ρ_x boils down to

$$\rho_x(\lambda) = \Phi([x, \lambda, z])\Phi([x, e, z])^{-1} \quad \forall\, \lambda \in \Lambda,\ m^2\text{-a.e. } (x, z) \in \Sigma^2. \tag{28}$$

Let $D \subseteq \Sigma$ be a fundamental domain for the Λ-action on Σ and consider the measure space

$$\widetilde{\Omega} = D \times \Lambda \times D \ \subseteq\ \Sigma \times \Lambda \times \Sigma.$$

This inclusion yields an isomorphism of measure spaces $\widetilde{\Omega} \cong \Omega$ and through this identification we let Γ act on $\widetilde{\Omega}$ by the second Γ-action A^2 on Ω. We also endow $\widetilde{\Omega}$ with the Λ-action coming from left multiplication on itself, so that we obtain on $\widetilde{\Omega}$ a $\Gamma \times \Lambda$-structure given explicitly by

$$(\gamma, \lambda)\,(x, \lambda_1, y) = (x, \lambda\lambda_1\alpha(\gamma, y)^{-1}, \gamma \cdot y), \tag{29}$$

where $\alpha : \Gamma \times D \to \Lambda$ is the cocycle corresponding to D; we recall also that the Γ-action on D is given by $\gamma \cdot y = \alpha(\gamma, y)\gamma y$. If we denote by $\widetilde{\Phi}$ the map $\widetilde{\Omega} \to \Gamma$ induced by Φ under the identification $\widetilde{\Omega} \cong \Omega$, then $E = \widetilde{\Phi}^{-1}(e) \subseteq \widetilde{\Omega}$ is a finite measure fundamental domain for the Γ-action since $\widetilde{\Phi} : \widetilde{\Omega} \to \Gamma$ is Γ-equivariant (with respect to right multiplication on Γ through the automorphism f). Thus $\widetilde{\Omega}$ is an ME coupling of Γ with Λ because the latter has an obvious finite measure fundamental domain. Equation (28) gives

$$\rho_x(\lambda) = \widetilde{\Phi}(x, \lambda, z)\widetilde{\Phi}(x, e, z)^{-1} \quad \forall\, \lambda \in \Lambda,\ m^2\text{-a.e. } (x, z) \in D^2, \tag{30}$$

but it shows further that for $\lambda_0 \in \Lambda$

$$\lambda_0 \in N \iff \begin{array}{l} \forall\ \lambda_1, \lambda_2 \in \Lambda,\ \text{for a.e. } (x, y) \in D^2\text{:} \\ \widetilde{\Phi}(x, \lambda_1\lambda_0\lambda_2, y) = \widetilde{\Phi}(x, \lambda_1\lambda_2, y). \end{array} \tag{31}$$

This characterisation shows in particular (for $\lambda_1 = e$) that N preserves E, and thus, by properness of the Λ-action, the group N is finite. This is the only point where we use the fact that $\widetilde{\Phi}$ has finite measure fibres.

By (30), we have for a.e. $t = (x, \lambda_1, y)$ in $\widetilde{\Omega}$ and every $\lambda \in \Lambda$,

$$\begin{aligned} \rho_x(\lambda) &= \rho_x(\lambda\lambda_1)\rho_x(\lambda_1)^{-1} \\ &= \widetilde{\Phi}(x, \lambda\lambda_1, z)\widetilde{\Phi}(x, e, z)^{-1}\Big(\widetilde{\Phi}(x, \lambda_1, z)\widetilde{\Phi}(x, e, z)^{-1}\Big)^{-1} \\ &= \widetilde{\Phi}(\lambda t)\widetilde{\Phi}(t)^{-1}. \end{aligned} \tag{32}$$

This is just $\widetilde{\Phi}(\lambda t)$ whenever $t \in E$. On the other hand, if $\vartheta : \Lambda \times E \to \Gamma$ is the cocycle associated to E, we have

$$e = \widetilde{\Phi}(\vartheta(\lambda, t)\lambda t) = \widetilde{\Phi}(\lambda t) f(\vartheta(\lambda, t))^{-1}.$$

We deduce

$$\rho_x(\lambda) = f(\vartheta(\lambda, t)) \qquad \forall\, \lambda \in \Lambda,\ \text{a.e. } t = (x, \lambda_1, y) \in E. \tag{33}$$

Applying the Fubini-Lebesgue theorem to the conclusion of Lemma 6.2, we have some $x_0 \in D$ such that $\rho = \rho_{x_0} : \Lambda \to \Gamma$ is a homomorphism with kernel N and such that for a.e. $x \in D$

$$\rho_x(\lambda) = F(x_0, x)^{-1}\rho(\lambda)F(x_0, x) \qquad \forall\, \lambda \in \Lambda. \tag{34}$$

We now proceed to show that $\rho(\Lambda)$ has finite index in Γ. Define $\varphi : E \to \Gamma$ by

$$\varphi(t) = f^{-1}\big(F(x_0, x)\big) \quad \text{for } t = (x, \lambda_1, y) \in E.$$

Observe that $\varphi(\lambda \cdot t) = \varphi(t)$ for $\lambda \in \Lambda$ because by (29)

$$\lambda \cdot t = \lambda\vartheta(\lambda, t)t = \Big(x, \lambda\lambda_1\alpha(\vartheta(\lambda, t), y)^{-1}, \vartheta(\lambda, t) \cdot y\Big).$$

Consider the cohomologous cocycle

$$\vartheta'(\lambda, t) = \varphi(\lambda \cdot t)\vartheta(\lambda, t)\varphi(t)^{-1}.$$

Applying successively (33) and (34), we have a.e.

$$\begin{aligned} f(\vartheta'(\lambda, t)) \quad &= F(x_0, x)f(\vartheta(\lambda, t))F(x_0, x)^{-1} \\ &= F(x_0, x)\rho_x(\lambda)F(x_0, x)^{-1} = \rho(\lambda). \end{aligned}$$

In other words, ϑ' ranges essentially in $f^{-1}(\rho(\Lambda))$. Applying Lemma 6.1, we deduce that $f^{-1}(\rho(\Lambda))$, and thus also $\rho(\Lambda)$, have finite index in Γ. Summing up, we obtain a virtual isomorphism of Λ and Γ:

$$1 \longrightarrow N \longrightarrow \Lambda \xrightarrow{\rho} \rho(\Lambda) \longrightarrow 1. \tag{35}$$

We proceed now to construct the virtual isomorphism (2) with the additional properties stated in Theorem 2.20 by considering again the ME coupling (Σ, m) of Γ, Λ but now in view of (35). Endow $N\backslash\Sigma$ with the restricted measure m_N (alternatively, we could work with the quotient measure $|N| \cdot m_N$); we consider this as a $\rho(\Lambda) \times \Gamma$-space via ρ. Since Γ is torsion-free, the N-action on $\Gamma\backslash\Sigma$ is essentially free and thus the Γ-action on $N\backslash\Sigma$ still admits a fundamental domain (of measure $m(\Gamma\backslash\Sigma)/|N|$). Therefore, $(N\backslash\Sigma, m_N)$ is an ME coupling of $\rho(\Lambda)$ with Γ, with coupling index satisfying

$$[\Gamma : \rho(\Lambda)]_{N\backslash\Sigma} = |N| \cdot [\Gamma : \Lambda]_{\Sigma}. \tag{36}$$

We write now Γ_i^* for the image of the projection of $\rho(\Lambda)$ to Γ_i and consider the finite index subgroup $\Gamma^* < \Gamma$ containing $\rho(\Lambda)$ defined by $\Gamma^* = \Gamma_1^* \times \cdots \times \Gamma_n^*$. Let (Σ^*, m^*) be the $\Gamma^* \times \Gamma$-space obtained by suspension from $(N\backslash\Sigma, m_N)$; that is, we have

$$\Sigma^* = \rho(\Gamma)\backslash\Big(N\backslash\Sigma \times \Gamma^*\Big),$$

wherein $\rho(\Lambda)$ acts diagonally, while the Γ^*-action is given by (inverted) right multiplication on the second factor and Γ acts on the first factor. Then Σ^* yields an ME coupling of Γ^* with Γ and we have

$$[\Gamma : \Gamma^*]_{\Sigma^*} \cdot [\Gamma^* : \rho(\Lambda)] = [\Gamma : \rho(\Lambda)]_{N\backslash\Sigma}. \tag{37}$$

The Γ-action on $\Gamma^*\backslash\Sigma^*$ is still irreducible by construction. On the other hand, we claim that the Γ^*-action on $\Gamma\backslash\Sigma^*$ is irreducible. Indeed, the ergodicity of the Γ_i^*-action on $\Gamma\backslash\Sigma^*$ is equivalent to the ergodicity of $\rho(\Lambda) \cap \Gamma_i$ on $\Gamma\backslash(N\backslash\Sigma)$, which in turn is equivalent to the ergodicity of $\rho^{-1}(\Gamma_i)$ on $\Gamma\backslash\Sigma$. The latter follows from mild mixing since $\rho^{-1}(\Gamma_i)$ is infinite, proving the claim.

At this point we are in position to apply Theorem 2.17 point (ii) to Σ^* and obtain isomorphisms $h_i : \Gamma_i^* \to \Gamma_i$ after possibly permuting factors. If we set $h = \Pi_{i=1}^n h_i$, $\pi = h \circ \rho$ and $\Gamma' = \pi(\Lambda)$, then we have a virtual isomorphism (2) such that the projections of Γ' to each Γ_i are onto. Moreover, Theorem 2.17 yields $[\Gamma : \Gamma^*]_{\Sigma^*} = 1$, which in view of (36) and (37) implies indeed the formula (3) since $[\Gamma : \Gamma'] = [\Gamma^* : \rho(\Lambda)]$. This completes the proof of Theorem 2.20. Theorem 2.20* then follows from the properties of the fundamental domain in Σ^* granted by Theorem 2.17 in view of the construction of Σ^*.

6.3. *Proof of Theorem* 2.24. We will start by constructing a homomorphism ρ of Λ, but in contrast to the proof of Theorem 2.20, ρ will range in $\overline{\Gamma}$ instead of Γ, and we shall show that its kernel is amenable rather than finite. Moreover, in order to show that it has finite index image, we shall need the following variation of Lemma 6.1 (which will be complemented by Lemma 6.9 below):

Lemma 6.7. *Let (Z, ζ) be an* ME *coupling of countable groups Γ, Λ and let $M \lhd \Gamma$ be a normal subgroup. Suppose that there is a measurable M-invariant set $E \subseteq Z$ such that $Z = \bigsqcup_{\gamma M \in \Gamma/M} \gamma E$. Pick a measurable map $\sigma : Z \to \Gamma$ such that $x \in \sigma(x)E$ for all $x \in Z$, so that in particular the induced map $\bar{\sigma} : Z \to \overline{\Gamma} = \Gamma/M$ is Γ-equivariant and determined by E. If $\bar{\sigma}|_{\Lambda E}$ ranges ζ-essentially in a subgroup $\overline{\Gamma}_0 < \overline{\Gamma}$, then $\overline{\Gamma}_0$ has finite index in $\overline{\Gamma}$.*

Proof. Denote by Γ_0 the pre-image of $\overline{\Gamma}_0$ in Γ, so that σ ranges essentially in Γ_0. It is enough to show that Γ_0 has finite index in Γ. Let $F \subseteq Z$ be a fundamental domain for the Γ-action and define $F' = \{\sigma(x)^{-1}x : x \in F\}$. One checks that F' is also a fundamental domain. Moreover, $F' \subseteq E$. Indeed, for $x \in F$ and $y = \sigma(x)^{-1}x$ we have $\bar{\sigma}(y) = \bar{\sigma}(x)^{-1}\bar{\sigma}(x)$ so that $\sigma(y) \in M$; now $y \in \sigma(y)E$ implies $y \in E$.

Let now χ, χ' be the retractions associated to F, F'. One computes from the definition of F' that $\chi'(x) = \chi(x)\sigma(\chi(x)^{-1}x)$ for a.e. $x \in Z$. Thus $\chi'(x)M = \bar{\sigma}(x)$ and in particular $\chi'(\lambda x)M \in \overline{\Gamma}_0$ for ζ-a.e. $x \in E$ and all $\lambda \in \Lambda$. Therefore the cocycle associated to F' ranges essentially in Γ_0. An application of Lemma 6.1 completes the proof. $\square$

Proof of Theorem 2.24. We write $\gamma \mapsto \bar{\gamma} = \gamma M$ for the natural map. We consider again the composed self-coupling $\Omega = \Sigma \times_\Lambda \Lambda \times_\Lambda \check{\Sigma}$ of Γ and apply Lemma 6.5 to $\Delta = M$. This time we may apply Theorem 2.23 and deduce an automorphism f of $\overline{\Gamma}$ and a factor map $\Phi : \Omega \to \mathrm{T}$ to a trivial self-coupling $\mathrm{T} \cong \overline{\Gamma}$ of $\overline{\Gamma}$, and we have

$$\Phi(A^1_\gamma A^2_{\gamma'} x) = \bar{\gamma}\Phi(x) f(\bar{\gamma}')^{-1}.$$

Since $\overline{\Gamma}$ is ICC by Proposition 7.11, we can argue exactly as for Theorem 2.20 and deduce that the map $\Psi : \Sigma \times \Sigma \times \Sigma \to \overline{\Gamma}$ defined as in Lemma 6.6 does not depend on the last variable. Taking the corresponding definitions for F, $\rho_x : \Lambda \to \overline{\Gamma}$, the $\Gamma \times \Lambda$-space $\widetilde{\Omega} = D \times \Lambda \times D$ and $\widetilde{\Phi}$, we need not change the arguments to apply Furman's Lemma 6.2 and get (30) and (31).

An important difference, though, is that $E = \widetilde{\Phi}^{-1}(e)$ is in general not a fundamental domain and may have infinite measure. However, M preserves E and we have

$$\widetilde{\Omega} = \bigsqcup_{\gamma M \in \Gamma/M} \gamma E. \tag{38}$$

LEMMA 6.8. *The group N is amenable.*

Proof. Equation (31) shows that N preserves E, which is thus a $M \times N$-space. Both actions are proper, measure-preserving and have some fundamental domain since they are the restriction of the actions on $\widetilde{\Omega}$. Therefore we can conclude by Corollary 4.2, provided M has a *finite measure* fundamental domain in E. Take a fundamental domain $F' \subseteq E$ for the Γ-action on $\widetilde{\Omega}$ as constructed in the proof of Lemma 6.7. It remains to show that the M-translates of F' cover E. Since $\Gamma F' = \widetilde{\Omega}$, this follows from the fact that the union (38) is disjoint. □

We fix a map $\sigma : \widetilde{\Omega} \to \Gamma$ as in Lemma 6.7 and we find, as after (32), that $\rho_x(\lambda) = \widetilde{\Phi}(\lambda t)$ whenever $t = (x, \lambda_1, y) \in E$. We have

$$e = \widetilde{\Phi}(\sigma(\lambda t)^{-1}\lambda t) = \widetilde{\Phi}(\lambda t) f(\bar{\sigma}(\lambda t))$$

whence again

$$\rho_x(\lambda) = f(\bar{\sigma}(\lambda t))^{-1} \qquad \forall\, \lambda \in \Lambda,\ \text{a.e. } t = (x, \lambda_1, y) \in E. \tag{39}$$

Fix $x_0 \in D$ with (34) and let $\rho = \rho_{x_0}$, $\overline{\Gamma}_0 = \rho(\Lambda)$ so that with (34) and (39) we get for a.e. $t \in E$

$$\rho(\lambda)^{-1} = F(x_0, x) f(\bar{\sigma}(\lambda t)) F(x_0, x)^{-1}. \tag{40}$$

Pick an arbitrary section $\tau : \overline{\Gamma} \to \Gamma$ of the projection and define $\psi : \widetilde{\Omega} \to \Gamma$ by $\psi(x, \lambda_1, y) = \tau f^{-1} F(x_0, x)$. Note that ψ is $\Gamma \times \Lambda$-invariant by (29). Define further $\sigma' : \widetilde{\Omega} \to \Gamma$ by $\sigma'(t) = \psi(t)\sigma(t)\psi(t)^{-1}$ and $E' = \{\psi(t)t : t \in E\}$. Now (40) implies

$$\overline{\sigma'}(\lambda t) \in f^{-1}(\overline{\Gamma}_0) \qquad \forall\, \lambda \in \Lambda,\ \text{a.e. } t \in E. \tag{41}$$

Define $\sigma'' : \widetilde{\Omega} \to \Gamma$ by $\sigma''(s) = \sigma'(\psi(s)^{-1}s)$. We replace cocycle equivalence by:

LEMMA 6.9. *E' and σ'' satisfy the assumptions of Lemma 6.7 for $Z = \widetilde{\Omega}$ and $f^{-1}(\overline{\Gamma}_0)$ instead of $\overline{\Gamma}_0$.*

Proof. First we check $ME' = E'$. Let $m \in M$, $t \in E$, $s = \psi(t)t \in E'$ and $m' = \psi(t)^{-1}m\psi(t) \in M$. The M-invariance of ψ gives

$$ms = m\psi(t)t = \psi(t)m't = \psi(m't)m't$$

which is in E' since $ME = E$. One checks in a similar way that $\widetilde{\Omega} = \bigsqcup_{\gamma M \in \Gamma/M} \gamma E'$. We check now $s \in \sigma''(s)E'$ for all $s \in \widetilde{\Omega}$. Setting $t = \psi(s)^{-1}s$ we have $s = \psi(t)t$ since $\psi(t) = \psi(s)$. Write

$$t = \sigma(t)\sigma(t)^{-1}t = \psi(t)^{-1}\sigma'(t)\psi(t)\sigma(t)^{-1}t,$$

so that

$$s = \sigma'(t)\psi(t)\sigma(t)^{-1}t = \sigma'(t)\psi(\sigma(t)^{-1}t)\sigma(t)^{-1}t.$$

By the choice of σ we have $\sigma(t)^{-1}t \in E$ so the above shows that s is in $\sigma'(t)E' = \sigma''(s)E'$ as claimed. Finally, for $t \in E$ and $s = \psi(t)t \in E'$ we have indeed

$$\overline{\sigma''}(\lambda s) = \overline{\sigma'}(\lambda\psi(s)^{-1}s)\overline{\sigma'}(\lambda t)$$

since $t = \psi(s)^{-1}s$ and thus $\overline{\sigma''}|_{\Lambda E'}$ ranges essentially in $f^{-1}(\overline{\Gamma}_0)$ by (41). □

In conclusion, Lemma 6.7 forces $f^{-1}(\overline{\Gamma}_0)$ and hence also $\overline{\Gamma}_0$ to have finite index in $\overline{\Gamma}$. Writing $\overline{\Lambda} = \Lambda/N \cong \overline{\Gamma}_0$, we observe that this group is torsion-free. It is also in $\mathcal{C}$ since this property passes to finite index subgroups (Lemma 7.5 below). Therefore, in order to complete the proof of Theorem 2.24 by an application of Theorem 2.23, it remains only to check that N is ergodic on $\Gamma\backslash\Sigma$. Since the Λ-action on $\Gamma\backslash\Sigma$ is mildly mixing, this is guaranteed if N is infinite. On the other hand, we claim that if N were finite then M would be so too and therefore the ergodicity assumption for the M-action on $\Lambda\backslash\Sigma$ would make Σ purely atomic, in which case Theorem 2.24 holds trivially.

Finally, to verify the claim above, observe that with N finite we deduce that Λ is in $\mathcal{C}$ (Lemma 7.3 below), so that by Corollary 7.6 below we deduce that Γ is in $\mathcal{C}$ as well. Since M is amenable and normal, by Proposition 3.8 it must fix a nonzero vector in any π with $H^2_b(\Gamma, \pi) \neq 0$. In our case such π which is mixing exists, hence M is finite. □

7. Groups in the class $\mathcal{C}$ and ME invariants

In this section, which is independent of the rest of the paper, we collect some information about the class $\mathcal{C}$ of groups to which our results apply.

Recall that we defined the class $\mathcal{C}$ as the collection of all countable groups Γ admitting a mixing unitary representation π such that $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi)$ is nonzero, and that $\mathcal{C}_{\mathrm{reg}}$ is the subclass of those for which one can take π to be the regular representation on $\ell^2(\Gamma)$. We do not know whether or not the inclusion $\mathcal{C}_{\mathrm{reg}} \subseteq \mathcal{C}$

is strict; it so happens that all our proofs that certain geometrically defined classes of groups belong to $\mathcal{C}$ actually show that they belong to $\mathcal{C}_{\mathrm{reg}}$.

We begin by justifying that the classes of groups listed in Examples 1.1 do indeed belong to the class $\mathcal{C}_{\mathrm{reg}}$. This is established in the companion paper [MS2] using also [MMS]; we refer to [MS2] for the geometric background and context. However, let us emphasize that we shall present at the end of this section an alternative argument to show that many groups in the list of Examples 1.1 are in $\mathcal{C}$, without appealing to [MS2] (or [MMS])

Proof of Theorem 1.3. Let Γ be a group as in Examples 1.1. In case (i), Corollary 7.8 of [MS2] states that Γ belongs to $\mathcal{C}_{\mathrm{reg}}$ (note that we may assume the tree is countable since Γ is). In case (ii), Corollary 7.6 of [MS2] ensures $\Gamma \in \mathcal{C}_{\mathrm{reg}}$. Case (iii) is Theorem 3 in [MMS]. $\square$

Via basic Bass-Serre theory, it follows that any nonelementary free product of groups is in $\mathcal{C}_{\mathrm{reg}}$, and that this is more generally so if one amalgamates over a finite subgroup (we recall here that an amalgamated product $A *_C B$ is *nonelementary* if $A \neq C$ and $[B : C] > 2$, or vice-versa). We have observed in 7.10, 7.11 of [MS2] that even if C is infinite, $A *_C B$ is in $\mathcal{C}_{\mathrm{reg}}$ as soon as C is malnormal (or almost malnormal) in one factor, and that there is in fact a sequence of weaker and weaker acylindricality conditions generalising this fact. Already the case of free products has the following immediate consequence:

COROLLARY 7.1. (i) *Every countable group embeds into a group in* $\mathcal{C}_{\mathrm{reg}}$.

(ii) *There is a continuum* $2^{\aleph_0}$ *of nonisomorphic finitely generated torsion-free groups in* $\mathcal{C}_{\mathrm{reg}}$. $\square$

We also point out that the third part of Examples 1.1 applies to the Cayley graph of Gromov-hyperbolic groups:

COROLLARY 7.2. *Every nonelementary Gromov-hyperbolic group is in* $\mathcal{C}_{\mathrm{reg}}$. *This holds more generally for nonelementary subgroups of Gromov-hyperbolic groups.* $\square$

7.1. *Stability properties and* ME *invariants.* We begin with a simple observation:

LEMMA 7.3. *Let* Γ *be a group and* $F \triangleleft \Gamma$ *a finite normal subgroup. If* Γ/F *is in* $\mathcal{C}_{\mathrm{reg}}$ *or* $\mathcal{C}$, *then so is* Γ.

Proof. If Γ/F is in $\mathcal{C}_{\mathrm{reg}}$, then $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(\Gamma/F)) \neq 0$ because an averaging argument gives $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(\Gamma/F)) \cong \mathrm{H}^2_{\mathrm{b}}(\Gamma/F, \ell^2(\Gamma/F))$. Since the Γ-representation $\ell^2(\Gamma/F)$ is contained in $\ell^2(\Gamma)$, we see that Γ is in $\mathcal{C}_{\mathrm{reg}}$. If Γ/F is in $\mathcal{C}$ and π is a mixing Γ/F-representation with $\mathrm{H}^2_{\mathrm{b}}(\Gamma/F, \pi) \neq 0$, then as before $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi) \neq 0$. But π is still mixing as Γ-representation, and so Γ is in $\mathcal{C}$. $\square$

Proposition 7.4. *Let Γ be a group in $\mathcal{C}$ (respectively $\mathcal{C}_{\rm reg}$) and $N \lhd \Gamma$ an infinite normal subgroup. Then N is in $\mathcal{C}$ (respectively $\mathcal{C}_{\rm reg}$).*

The normality assumption is necessary as can be seen readily by taking a group which is not in $\mathcal{C}$ (such as a product of two infinite groups) and applying Corollary 7.1. Observe that on the other hand if Γ is a group as in Examples 1.1, then *any* subgroup that acts still *nonelementarily* on the corresponding space is again in the list of Examples 1.1.

Proof of Proposition 7.4. Let π be a mixing unitary Γ-representation with nonvanishing $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi)$. Since N is infinite, it cannot fix any nonzero vector for π. Therefore the second exact sequence of Theorem 12.4.2 in [M] (wherein one must read Δ for N) shows that the restriction

$$\mathrm{res}: \ \mathrm{H}^2_{\mathrm{cb}}(\Gamma, \pi) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(N, \pi)$$

is injective and thus $\mathrm{H}^2_{\mathrm{cb}}(N, \pi) \neq 0$, so that N is in $\mathcal{C}$. In the case $\pi = \ell^2(\Gamma)$, we observe that $\pi|_N$ is a multiple of $\ell^2(N)$ so that we are done by Corollary 3.4. ☐

Recall that a subgroup Λ of a group Γ is called *co-amenable* (or one says that the coset Γ-space Γ/Λ is *amenable in Eymard's sense*) if there is a Γ-invariant mean on $\ell^\infty(\Gamma/\Lambda)$. This is, for example, the case if Λ has finite index in Γ, or if Λ is normal in Γ and the quotient is an amenable group.

Lemma 7.5. *Let Γ be a group in $\mathcal{C}$ (respectively $\mathcal{C}_{\rm reg}$) and $\Lambda < \Gamma$ a co-amenable subgroup. Then Λ is also in $\mathcal{C}$ (respectively $\mathcal{C}_{\rm reg}$).*

Proof. Let π be a mixing Γ-representation with $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi) \neq 0$. By [M, N° 8.6.2], the restriction map $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi) \to \mathrm{H}^2_{\mathrm{b}}(\Lambda, \pi)$ is injective, so that Λ is in $\mathcal{C}$ because π is also Λ-mixing. The case of $\mathcal{C}_{\rm reg}$ is handled with Corollary 3.4 as in Proposition 7.4. ☐

As implicitly observed in Section 5, Theorem 4.4 has the following consequence.

Corollary 7.6. *Let Λ, Γ be* ME *countable groups. Assume that Γ is in $\mathcal{C}$ (respectively $\mathcal{C}_{\rm reg}$). Then Λ is also in $\mathcal{C}$ (respectively $\mathcal{C}_{\rm reg}$).*

In other words, being in the classes $\mathcal{C}$ or $\mathcal{C}_{\rm reg}$ are ME invariants; in particular this proves Theorem 1.18 from the introduction.

Proof of the corollary. Let (Σ, m) be an ME coupling of Λ with Γ and let π be a mixing Γ-representation with $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \pi) \neq 0$. By Theorem 4.4, the space $\mathrm{H}^2_{\mathrm{b}}(\Lambda, \Sigma\mathbf{I}_\Gamma^\Lambda\pi)$ is nonzero. By Lemma 4.1 point (i), the Λ-representation $\Sigma\mathbf{I}_\Gamma^\Lambda\pi$ is mixing, and so Λ is in $\mathcal{C}$. In the case $\pi = \ell^2(\Gamma)$, Lemma 4.1 point (iii) gives $\Sigma\mathbf{I}_\Gamma^\Lambda\ell^2(\Gamma) \cong L^2(\Sigma)$. The latter is a multiple of $\ell^2(\Lambda)$ since there is a Λ-fundamental domain in Σ, so that Corollary 3.4 implies that $\mathrm{H}^2_\mathrm{b}(\Lambda, \ell^2(\Lambda))$ is nonzero. $\square$

We can in fact refine our cohomological ME-invariants to distinguish between some groups which are not in these two classes.

Definition 7.7. Denote by $w\mathcal{C}_\mathrm{reg}$ the class of countable groups Γ admitting a unitary representation π which is *weakly contained* in $\ell^2(\Gamma)$ and such that $\mathrm{H}^2_\mathrm{b}(\Gamma, \pi) \neq 0$.

Obviously, we have $\mathcal{C}_\mathrm{reg} \subseteq w\mathcal{C}_\mathrm{reg}$ and as we shall see, this inclusion is strict. All the stability properties established above for the classes $\mathcal{C}$ and $\mathcal{C}_\mathrm{reg}$ remain valid, with similar proofs, also for this class. Natural examples of groups in $w\mathcal{C}_\mathrm{reg}$ are provided by the following:

PROPOSITION 7.8. (i) *Suppose that $N \lhd \Gamma$ is a normal amenable subgroup. If Γ/N is in $\mathcal{C}_\mathrm{reg}$, then Γ is in $w\mathcal{C}_\mathrm{reg}$.*

(ii) *If a countable group Γ splits nontrivially as a free product over an amalgamated amenable subgroup then Γ is in $w\mathcal{C}_\mathrm{reg}$ (unless the amalgamated group has index ≤ 2 in both factors).*

Proof. (i) Because N is amenable we have $\mathbb{1} \prec \ell^2(N)$, and inducing both sides to Γ gives $\ell^2(\Gamma/N) \prec \ell^2(\Gamma)$. Thus it is enough to show $\mathrm{H}^2_\mathrm{b}(\Gamma, \pi) \neq 0$ for $\pi = \ell^2(\Gamma/N)$. Since we have $\mathrm{H}^2_\mathrm{b}(\Gamma/N, \pi) \neq 0$ by assumption, the result follows from Corollary 3.6.

(ii) It is enough to find an amenable subgroup $N < \Gamma$ with $\mathrm{H}^2_\mathrm{b}(\Gamma, \ell^2(\Gamma/N))$ $\neq 0$ since in the above argument the normality of N in Γ was not used for $\ell^2(\Gamma/N) \prec \ell^2(\Gamma)$. Here, all stabilisers for the Γ-action on the set E of edges of the Bass-Serre tree associated with the amalgamated decomposition are amenable; therefore, the action on $E \times E$ also has amenable stabilisers. Now our Corollary 7.8 in [MS2] completes the proof. $\square$

An argument similar to the one given in the proof of Corollary 7.6, by (ii) in Lemma 4.1, shows:

COROLLARY 7.9. *Being in the class $w\mathcal{C}_\mathrm{reg}$ is an* ME *invariant.* $\square$

Now that we have large families of groups in the various classes defined above, we add a few observations to complete the picture:

PROPOSITION 7.10. *Let Γ be a countable group.*

(i) *If Γ is amenable then Γ is not in $\mathcal{C}$ or $\mathcal{C}_\mathrm{reg}$, nor in $w\mathcal{C}_\mathrm{reg}$.*

(ii) *If Γ contains an infinite normal amenable (e.g. central) subgroup N then Γ is not in $\mathcal{C}$ or $\mathcal{C}_\mathrm{reg}$; but if Γ/N is in $\mathcal{C}_\mathrm{reg}$, then Γ is in $w\mathcal{C}_\mathrm{reg}$.*

(iii) *If* Γ *is the product of two infinite groups then* Γ *is not in* $\mathcal{C}$ *or* $\mathcal{C}_{\mathrm{reg}}$, *and if both groups are nonamenable then* Γ *is also not in* $w\mathcal{C}_{\mathrm{reg}}$ (*though* Γ *may be in* $w\mathcal{C}_{\mathrm{reg}}$ *if one of the factors is amenable as in* (ii) *above*).

(iv) *If* Γ *has infinitely many ends*, *i.e.* (*by Stallings*) *if it is a nontrivial free product over a finite amalgamated subgroup*, *then* Γ *is in* $\mathcal{C}$ *and* $\mathcal{C}_{\mathrm{reg}}$. *If* Γ *is a nontrivial free product over an amenable subgroup then it is in* $w\mathcal{C}_{\mathrm{reg}}$.

(v) Γ *is not in any of these classes if it is a lattice in a higher rank simple Lie group*, *or in a higher rank simple algebraic group over a local field.*

(vi) Γ *is not in any of these classes if it is an irreducible lattice in a product of nonamenable compactly generated locally compact groups.*

Proof. For (i), see Remark 3.9. The first part of (ii) follows from Proposition 3.8 since an infinite subgroup cannot fix a nonzero vector in a mixing representation; the second part was noted in Proposition 7.8. For point (iii), assume that $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\pi)\neq 0$ for $\Gamma=\Gamma_1\times\Gamma_2$. Theorem 3.7 implies $\mathcal{H}_\pi^{\Gamma_i}\neq 0$ for some $i\in\{1,2\}$. This forces Γ_i to be finite if π is mixing and to be amenable if $\pi\prec\ell^2(\Gamma)$. Point (iv) has been addressed above, see the above discussion after the proof of Theorem 1.3 and Proposition 7.8. One gets (v) by applying Theorem 1.4 from [MS2]. Theorem 16 in [BM2] implies (vi). ∎

At this point, we can complete the

Proof of Corollary 1.19. The result follows from the juxtaposition of Corollary 7.6 and point (ii) of Proposition 7.10. ∎

Next, recall that a group is said to be ICC if the conjugacy class of every nontrivial element is infinite. We used the following fact each time we needed Lemma 6.4.

PROPOSITION 7.11. *Any countable torsion-free group in* $\mathcal{C}$ *is* ICC.

Proof. Let Γ be as in the statement and suppose for a contradiction that it contains a nontrivial element γ_0 with finite conjugacy class. The centraliser Γ_0 of γ_0 has finite index in Γ so that it is also in $\mathcal{C}$. The subgroup C_0 generated by γ_0 is normal in Γ_0 and amenable, so by Proposition 3.8 the space $\mathcal{H}_\pi^{C_0}$ is nonzero for every Γ_0-representation such that $\mathrm{H}^2_{\mathrm{b}}(\Gamma_0,\pi)\neq 0$. However this can never happen for π mixing unless C_0 is finite, a contradiction since Γ is torsion-free and γ_0 is nontrivial. ∎

7.2. *A shortcut to* $\mathcal{C}$. Finally, we describe a short alternative approach to establish that certain groups (including free groups) are in the class $\mathcal{C}$, thereby avoiding completely the dependence on the companion paper [MS2]. The more restricted family we cover here is still rich enough to provide many interesting examples to which our foregoing results apply, as well as to establish Theorem 1.7 (see below) and Theorem 1.14 (with some modification of the proof).

PROPOSITION 7.12. *Every lattice in a simple* (*connected*, *center-free*) *Lie group of rank one is in* $\mathcal{C}$.

Before proving the proposition we notice that it yields yet another proof (in the finitely generated case) that non-Abelian free groups are in $\mathcal{C}$. Using this fact, we can give an alternative argument (independent of [MS2]) to show that *there is a continuum of nonisomorphic finitely generated torsion-free groups in* $\mathcal{C}$ (Corollary 7.1):

Indeed, if A, B are any two finitely generated amenable groups, then the argument given in the proof of Theorem 2.27 above shows that the free product $A * B$ is ME to a free group on two generators. Thus $A * B$ is in $\mathcal{C}$ by Corollary 7.6. In other words, it remains only to justify that there is a continuum of nonisomorphic finitely generated torsion-free amenable groups. This is true even for *soluble* groups; P. Hall proves in [Ha] that there are uncountably many (in fact, a continuum of) nonisomorphic groups G on two generators with $[G'', G] = 1$. A close examination of his proof (Theorem 6 and pages 433–435 in [Ha]) shows that G can be chosen torsion-free.

Proof of Proposition 7.12. Let G be a rank-one Lie group. Since all lattices in G are ME to each other, it is enough by Corollary 7.6 to prove the statement for *some* lattice $\Gamma < G$. We are therefore free to choose Γ cocompact in G, which implies that Γ is hyperbolic in the sense of Gromov and therefore the bounded cohomology with *trivial* coefficients $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is infinite-dimensional [EpF].

We now consider the quasi-regular G-representation $\pi = L^2(G/\Gamma)$ which splits as $\mathbb{1} \oplus \pi_0$, where π_0 denotes the kernel of the orthogonal projection $p : \pi \to \mathbb{1}$ to the constants. Observe that the induced Γ-representation $G\mathbf{I}_\Gamma^\Gamma \mathbb{1}$ as defined in Section 4.1 through the ME self-coupling of Γ given by the right and left Γ-actions on G is the restriction $\pi|_\Gamma$ of π to Γ. In fact, in the present case every definition of cohomological induction $G\mathbf{i}_\Gamma^\Gamma : \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma) \to \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma, G\mathbf{I}_\Gamma^\Gamma \mathbb{1})$ through this self-coupling (Section 4.2) coincides with the map

$$\mathrm{H}^\bullet_{\mathrm{b}}(\Gamma) \xrightarrow{\mathbf{i}_\Gamma^G} \mathrm{H}^\bullet_{\mathrm{cb}}(G, \pi) \xrightarrow{\mathrm{res}} \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma, \pi|_\Gamma)$$

wherein $\mathbf{i}_\Gamma^G$ is the induction to continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}$ as defined in [BM2], [M] and res is the restriction map. Thus, we have a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{H}_{\mathrm{b}}^{\bullet}(\Gamma, \pi|_{\Gamma}) \arrow[r, "p_*"] & \mathrm{H}_{\mathrm{b}}^{\bullet}(\Gamma) \\
\mathrm{H}_{\mathrm{b}}^{\bullet}(\Gamma) \arrow[ur, "G\mathbf{i}_{\Gamma}^{\Gamma}"] \arrow[dr, "\mathbf{i}_{\Gamma}^{G}"'] & & \\
 & \mathrm{H}_{\mathrm{cb}}^{\bullet}(G, \pi) \arrow[r, "p_*"] & \mathrm{H}_{\mathrm{cb}}^{\bullet}(G) \arrow[uu, "\mathrm{res}"']
\end{tikzcd}$$

However, in degree two we have the following additional information: (i) the space $\mathrm{H}_{\mathrm{cb}}^2(G)$ has dimension at most one (Lemma 6.1 in [BM1]; compare [BM2]); (ii) $G\mathbf{i}_{\Gamma}^{\Gamma}$ is injective (Theorem 4.4). It follows that $\mathrm{H}_{\mathrm{b}}^2(\Gamma, \pi_0|_{\Gamma})$ is infinite-dimensional (bounded cohomology is additive in the coefficients [M, 8.2.10]). This finishes the proof because π_0 is a mixing representation by the Howe-Moore theorem. $\square$

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