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Deligne's integrality theorem in unequal characteristic and rational points over finite fields

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À Pierre Deligne, à l'occasion de son 60*-ième anniversaire, en témoignage de profonde admiration*

Abstract

If V is a smooth projective variety defined over a local field K with finite residue field, so that its étale cohomology over the algebraic closure $\bar{K}$ is supported in codimension 1, then the mod p reduction of a projective regular model carries a rational point. As a consequence, if the Chow group of 0-cycles of V over a large algebraically closed field is trivial, then the mod p reduction of a projective regular model carries a rational point.

1. Introduction

If Y is a smooth, geometrically irreducible, projective variety over a finite field k, we singled out in [10] a motivic condition forcing the existence of a rational point. Indeed, if the Chow group of 0-cycles of Y fulfills base change $CH_0(Y \times_k \overline{k(Y)}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, then the number of rational points of Y is congruent to 1 modulo $|k|$. In general it is hard to control the Chow group of 0-cycles, but if Y is rationally connected, for example if Y is a Fano variety, then the base change condition is fulfilled, and thus, rationally connected varieties over a finite field have a rational point. Recall the Leitfaden of the proof. By S. Bloch's decomposition of the diagonal acting on cohomology as a correspondence [2, Appendix to Lecture 1], the base change condition implies that étale cohomology $H^m(\bar{Y}, \mathbb{Q}_\ell)$ is supported in codimension ≥ 1 for all $m \geq 1$, that is that étale cohomology for $m \geq 1$ lives in coniveau 1. Here ℓ is a prime number not dividing $|k|$. On the other hand, by Deligne's integrality theorem [6, Cor. 5.5.3], the coniveau condition implies that the eigenvalues of the geometric Frobenius acting on $H^m(\bar{Y}, \mathbb{Q}_\ell)$ are divisible by $|k|$ as algebraic

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integers for $m \geq 1$; thus the Grothendieck-Lefschetz trace formula [16] allows us to conclude. Summarizing, we see that the cohomological condition which forces the existence of a rational point is the coniveau condition. The motivic condition is here to allow us to check geometrically in concrete examples the coniveau condition.

If Y is no longer smooth, then homological cycle classes no longer act on cohomology; thus the base change condition is no longer the right condition to force the existence of a rational point. Indeed, J. Kollár constructed an example of a rationally connected projective variety, but without any rational point. On the other hand, the classical theorem by Chevalley-Warning [4], [22], and its generalization by Ax-Katz [1], [19], asserting that the number of rational points of a closed subset Y of $\mathbb{P}^n$ defined by r equations of degree d_i, with $\sum_1^r d_i \leq n$, is congruent to 1 modulo $|k|$, suggests that when Y is smoothly deformable, the rational points of the smooth fibres singled out in [10] produce rational points on the singular fibres of the deformation. Indeed, N. Fakhruddin and C. S. Rajan showed that if $f : X \to S$ is a projective dominant morphism over a finite field, with X, S smooth connected, and if the base change condition is generically satisfied, that is if $CH_0(X \times_{k(S)} \overline{k(X)}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, then the number of rational points of a closed fibre is congruent to 1 modulo the cardinality of its field of definition [14, Th. 1.1]. The method is a refined version of the one explained above in the smooth case, that is when S is the spectrum of a finite field. However, it does not allow us to finish the proof if only the coniveau condition on the geometric general fibre is known. On the other hand, the previous discussion in the smooth case indicates that it should be sufficient to assume that the geometric general fibre fulfills the cohomological coniveau condition to force the singular fibres to acquire a rational point. According to Grothendieck's and Deligne's philosophy of motives, which links the level for the congruence of rational points over finite fields to the level for the Hodge type over the complex numbers, this is supported by the fact that if $f : X \to S$ is a projective dominant morphism over the field of complex numbers, with X, S smooth, S a connected curve, and if the Hodge type of some smooth closed fibre is at least 1, then so is the Hodge type of all closed fibres [12, Th. 1.1]).

We state now our theorem and several consequences. Let K be a local field, with ring of integers $R \subset K$ and finite residue field k. We choose a prime number ℓ not dividing $|k|$. If V is a variety defined over K, we denote by $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ its ℓ-adic cohomology. We say that $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ has coniveau 1 if each class in this group dies in $H^m(U \times_K \bar{K}, \mathbb{Q}_\ell)$ after restriction on some nonempty open $U \subset V$.

THEOREM 1.1. *Let V be an absolutely irreducible, smooth projective variety over K, with a regular projective model X over R. If étale cohomology $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ has coniveau 1 for all $m \geq 1$, then the number of rational points of the special fibre $Y = X \times_R k$ is congruent to 1 modulo $|k|$.*

Let $K_0 \subset K$ be a subfield of finite type over its prime field over which V is defined, i.e. $V = V_0 \times_{K_0} K$ for some variety V_0 defined over K_0, and let Ω be a field extension of $K_0(V_0)$. For example if K has unequal characteristic, we may take $\Omega = K$. Using the decomposition of the diagonal mentioned before, one obtains

COROLLARY 1.2. *Let V be an absolutely irreducible, smooth projective variety over K, with a regular projective model X over R. If the Chow group of 0-cycles fulfills base change $CH_0(V_0 \times_{K_0} \bar{\Omega}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, then the number of rational points of Y is congruent to 1 modulo $|k|$.*

(See [14, Question 4.1] for the corollary, where the regularity of X is not asked for.) In particular, our corollary applies for Fano varieties, and more generally, for rationally connected varieties V.

If the local field K has equal characteristic, this is a certain strengthening of [14, Th. 1.1]. Indeed, our basis $\mathrm{Spec}(R)$ has only Krull dimension 1, but our coniveau assumption is the one which was expected, as indicated above. If the local field K has unequal characteristic, we see directly Deligne's philosophy at work. To our knowledge, this is the first such example. In this case, the coniveau 1 condition for étale cohomology is equivalent to the coniveau 1 condition for de Rham cohomology $H^m_{\mathrm{DR}}(V \times_K \bar{K})$. By Deligne's mixed Hodge theory [7], it implies that the Hodge type of de Rham cohomology $H^m_{\mathrm{DR}}(V)$ is ≥ 1 for all $m \geq 1$, or equivalently that $H^m(V, \mathcal{O}_V) = 0$ for all $m \geq 1$. Conversely, Grothendieck's generalized conjecture predicts that those two conditions are equivalent; that is the Hodge type being ≥ 1 should imply that the coniveau is 1. Thus one expects that if V is a smooth projective variety over K, with $H^m(V, \mathcal{O}_V) = 0$ for all $m \geq 1$, then if X is a regular projective model of V, the number of rational points of $Y = X \times_R k$ is congruent to 1 modulo $|k|$. In particular this holds for surfaces.

THEOREM 1.3. *Let V be an absolutely irreducible, smooth projective surface defined over a finitely generated $\mathbb{Q}$-algebra L. If*

$$H^1(V, \mathcal{O}_V) = H^2(V, \mathcal{O}_V) = 0,$$

then for any prime place of L with p-adic completion K, for which $V \times_L K$ has a regular model X, the number of rational points of the mod p reduction $X \times_R k$, where $R \subset K$ is the ring of integers and k is the finite residue field, is congruent to 1 modulo $|k|$.

An example of such a surface is Mumford's fake $\mathbb{P}^2$ [20], a surface in characteristic 0 which has the topological invariants of $\mathbb{P}^2$, yet is of general type. We still do not know whether its Chow group of 0-cycles fulfills base change, as predicted by Bloch's conjecture. The surface is constructed by 2-adic uniformization, and the special fibre over $\mathbb{F}_2$, says Mumford quoting Lewis Carroll to express his "confusion", is a $\mathbb{P}^2$ blown up 7 times, crossing itself in 7 rational double curves, themselves crossing in 7 triple points ... Theorem 1.3 allows one to say (in a less entertaining way) that at other bad primes with a regular projective model, there are rational points as well.

We now describe our method. Our goal is to show that the eigenvalues of the geometric Frobenius $F \in \mathrm{Gal}(\bar{k}/k)$ acting on $H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell)$ are $|k|$-divisible algebraic integers for $m \geq 1$. Indeed, this will imply, by the Grothendieck-Lefschetz trace formula [16], that Y has modulo $|k|$ the same number of rational points as $\mathbb{P}^N_k$.

To this aim, we consider the specialization map $H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ which is the edge homomorphism in the vanishing cycle spectral sequence ([8, p. 214, (7)], [21, p. 23]). Let G be the Deligne-Weil group of the local field K. This is an extension of $\mathbb{Z}$, generated multiplicatively by the geometric Frobenius F of $\mathrm{Gal}(\bar{k}/k)$, by the inertia I. It acts on $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$, on $H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell)$ via its quotient $\mathbb{Z} \cdot F$, and the specialisation map is G-equivariant. On the other hand, denoting by K^u the maximal unramified extension of K in $\bar{K}$, that is $K^u = K^I$, the specialization map has a G-equivariant factorization

$$sp : H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell) \xrightarrow{sp_u} H^m(V \times_K K^u, \mathbb{Q}_\ell) \to H^m(V \times_K \bar{K}, \mathbb{Q}_\ell),$$

where on the first two terms, G acts via its quotient $\mathbb{Z} \cdot F$. We first show

THEOREM 1.4. *Let V be a smooth projective variety over a local field K with finite residue field k . If X is a regular projective model over R, then the eigenvalues of F on the kernel of the specialization map sp_u are $|k|$-divisible algebraic integers.*

Theorem 1.4 is a consequence of Deligne's integrality theorem loc. cit. and of Gabber's purity theorem [15, Th. 2.1.1].

This reduces the problem to showing $|k|$-divisibility of the eigenvalues of F on $\mathrm{Im}(sp_u) \subset H^m(V \times_K K^u, \mathbb{Q}_\ell)$. The latter group is an F-equivariant extension of the inertia invariants $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)^I$ by the first inertia cohomology group $H^1(I, H^{m-1}(V \times_K \bar{K}, \mathbb{Q}_\ell))$. By Grothendieck [17], as k is finite, I acts quasi-unipotently on $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$. As a consequence, modulo multiplication by roots of unity, the eigenvalues of a lifting $\Phi \in G$ of F acting on $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ depend only on F ([8, Lemme (1.7.4)]). In particular, if for one choice of Φ, there are algebraic integers, then they are algebraic integers for all choices. We denote by $N^1H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ the subgroup of $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ consisting of the classes which die in $H^m(U \times_K \bar{K}, \mathbb{Q}_\ell)$ after restriction on some nonempty open $U \subset V$. It is a G-submodule. Then Theorem 1.1 is a consequence of

Theorem 1.5. *Let V be a smooth irreducible projective variety defined over a local field K with finite residue field k. Let Φ be a lifting of the geometric Frobenius of k in the Deligne-Weil group of K. Then the eigenvalues of Φ*

i) *on $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ are algebraic integers for all m,*

ii) *on $N^1 H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ are $|k|$-divisible algebraic integers.*

Theorem 1.5 is a consequence of Deligne's integrality theorem loc. cit., of de Jong's alterations [5] and of Rapoport-Zink's weight spectral sequence [21].

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2. The kernel of the specialization map over the maximal unramified extension

Let V be a smooth projective variety over a local field K with projective model X over the ring of integers and special fibre $Y = X \times_R k$ over the residue field k which we assume throughout to be finite.

In the following, K^u is the maximal unramified extension of K, R^u its ring of integers, with residue field $\bar{k}$. The specialization map sp_u [8, p. 213 (6)], is constructed by applying base change $H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell) = H^m(X \times_R R^u, \mathbb{Q}_\ell)$ for X proper over R, followed by the restriction map $H^m(X \times_R R^u, \mathbb{Q}_\ell) \to H^m(V \times_K K^u, \mathbb{Q}_\ell)$. In particular, one has an exact sequence

$$(2.1) \ldots \to H^m_Y(X \times_R R^u, \mathbb{Q}_\ell) \to H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell) \xrightarrow{sp_u} H^m(V \times_K K^u, \mathbb{Q}_\ell) \\ \to H^{m+1}_Y(X \times_R R^u, \mathbb{Q}_\ell) \to \ldots \ .$$

Here in the notation: $H_Y(() \times_R R^u, ())$ means $H_{Y \times_R R^u}(() \times_R R^u, ())$ etc. The geometric Frobenius $F \in \mathrm{Gal}(\bar{k}/k)$ acts on all terms in (2.1) and the exact sequence is F-equivariant. Theorem 1.4 is then a consequence of

Theorem 2.1. *If X is a regular scheme defined over R, with special fibre $Y = X \times_R k$, then the eigenvalues of F acting on $H^m_Y(X \times_R R^u, \mathbb{Q}_\ell)$ are algebraic integers in $|k| \cdot \bar{\mathbb{Z}}$ for all m.*

Proof. We proceed as in [10, Lemma 2.1]. One has a finite stratification $\ldots \subset Y_i \subset Y_{i-1} \subset \ldots \subset Y_0 = Y$ by closed subsets defined over k such that $Y_{i-1} \setminus Y_i$ is smooth. It yields the F-equivariant localization sequence

$$(2.2)\quad \ldots \to H^m_{Y_i}(X \times_R R^u, \mathbb{Q}_\ell) \to H^m_{Y_{i-1}}(X \times_R R^u, \mathbb{Q}_\ell) \\ \to H^m_{(Y_{i-1}\setminus Y_i)}((X \setminus Y_i) \times_R R^u, \mathbb{Q}_\ell) \to \ldots .$$

Thus Theorem 2.1 is a consequence of

THEOREM 2.2. *If X is a regular scheme defined over R, and $Z \subset Y = X \times_R k$ is a smooth closed subvariety defined over k, then the eigenvalues of F acting on $H^m_Z(X \times_R R^u, \mathbb{Q}_\ell)$ lie in $|k| \cdot \bar{\mathbb{Z}}$ for all m.*

Proof. The scheme X defined over R being regular, its base change $X \times_R R^u$ by the unramified map Spec $R^u \to$ Spec R is regular as well. By Gabber's purity theorem [15, Th. 2.1.1], one has an F-equivariant isomorphism

$$(2.3)\qquad H^m(Z \times_k \bar{k}, \mathbb{Q}_\ell)(-c) \cong H^{m+2c}_Z(X \times_R R^u, \mathbb{Q}_\ell),$$

where c is the codimension of Z in X. Thus in particular, F acts on $H^{m+2c}_Z(X \times_R R^u, \mathbb{Q}_\ell)$ as it does on $H^m(Z \times_k \bar{k}, \mathbb{Q}_\ell)(-c)$. We are back to a problem over finite fields. Since $c \geq 1$, we only need to know that the eigenvalues of F on $H^m(Z \times_k \bar{k}, \mathbb{Q}_\ell)$ lie in $\bar{\mathbb{Z}}$. This is [6, Lemma 5.5.3 iii] (via duality as Z is smooth). □

This finishes the proof of Theorem 2.1. □

Remark 2.3. We observe that (2.1) together with Theorem 2.1 implies that if V is a smooth projective variety defined over a local field K, and V admits a regular model over R, then the eigenvalues of F on $H^m(V \times_K K^u, \mathbb{Q}_\ell)$ are algebraic integers, and they are $|k|$-divisible algebraic integers for some m if and only if the eigenvalues of F on $H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell)$ are $|k|$-divisible algebraic integers for the same m.

3. Eigenvalues of a lifting of Frobenius on étale cohomology of smooth projective varieties

Let V be a smooth projective variety over a local field K with projective model X over the ring of integers R and special fibre $Y = X \times_R k$ over the finite residue field k. Let Φ be a lifting of Frobenius in the Deligne-Weil group of K. The aim of this section is to prove Theorem 1.5.

Recall that X/R is said to be strictly semi-stable if Y is reduced and is a strict normal crossing divisor. In this case, X is necessarily regular as well. Recall from [5, (6.3)] that if $A \subset X, A = \sum_i A_i$ is a divisor, (X, A) is said to be a strictly semi-stable pair if X/R is strictly semi-stable, $A + Y$ is a normal crossing divisor, and all the strata $A_I/R, I = (i_1, \dots, i_s)$ of A are strictly semi-stable as well.

Proof of Theorem 1.5 i). Let V be as in Theorem 1.5 i). Let $K' \supset K$ be a finite extension, with residue field $k' \supset k$, and Deligne-Weil group $G' \subset G$. Let $\sigma : V' \to V$ be an alteration; that is, V' is smooth projective over K', σ is proper, dominant and generically finite. Then $\sigma^* : H^m(V \times_K \bar{K}, \mathbb{Q}_\ell) \to H^m(V' \times_{K'} \overline{K'}, \mathbb{Q}_\ell)$ is injective, and G'-equivariant. In particular, it is Φ'-equivariant for a lifting $\Phi' \in G'$ of $F^{[k':k]}$. Thus Theorem 1.5 for Φ' implies Theorem 1.5 for Φ. By de Jong's fundamental alteration theorem ([5, Th. 6.5]), we may find such K', V' with the property that V' has a strictly semi-stable model over the ring of integers of K'. Thus by the above, without loss of generality, we may assume that V defined over K has a strictly semi-stable model X over the ring of integers $R \subset K$. We denote by $Y = X \times_R k$ the closed fibre. It is a strict normal crossing divisor. We denote by $Y^{(i)}$ the disjoint union of the smooth strata of codimension i in X. Thus $Y^{(0)} = X$, $Y^{(1)}$ is the disjoint union of the components of Y etc. We apply now the existence of the weight spectral sequence [21, Satz 2.10] by Rapoport-Zink (see also [18, (3.6.11), (3.6.12)] for a résumé),

$$(3.1) \quad {}_W E_1^{-r,m+r} = \oplus_{q \geq 0, r+q \geq 0} H^{m-r-2q}(Y^{(r+1+2q)} \times_k \bar{k}, \mathbb{Q}_\ell)(-r-q) \\ \Rightarrow H^m(V \times_K \bar{K}, \mathbb{Q}_\ell) \ .$$

It is G-equivariant and converges in E_2 ([18, p. 41]). Thus the eigenvalues of Φ on the right-hand side are (some of) the eigenvalues of F on the left-hand side. We apply again Deligne's integrality theorem [6], loc. cit. to conclude the proof. $\square$

Proof of Theorem 1.5 ii). Let V be as in Theorem 1.5 ii). Since étale cohomology $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ is a finite dimensional $\mathbb{Q}_\ell$-vectorspace, there is a divisor A^0 defined over K with a G-equivariant surjection $H^m_{A^0}(V \times_K \bar{K}, \mathbb{Q}_\ell) \twoheadrightarrow N^1 H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$. Let $K' \supset K$ be a finite extension, let $\sigma : V' \to V$ be an alteration. Then

$$(3.2) \quad \sigma^*(\mathrm{Im}(H^m_{A^0}(V \times_K \bar{K}, \mathbb{Q}_\ell)) \\ \subset \mathrm{Im}(H^m_{\sigma^{-1}(A^0)}(V' \times_{K'} \overline{K'}, \mathbb{Q}_\ell)) \subset H^m(V' \times_{K'} \overline{K'}, \mathbb{Q}_\ell).$$

Since as in the proof of i), $\sigma^* : H^m(V \times_K \bar{K}, \mathbb{Q}_\ell) \to H^m(V \times_{K'} \overline{K'}, \mathbb{Q}_\ell)$ is G'-equivariant and injective, Theorem 1.5 ii) for Φ' implies Theorem 1.5 ii) for Φ. We use again de Jong's alteration theorem [5, Th. 6.5]. There is a finite extension $K' \supset K$, with an alteration $\sigma : V' \to V$ such that V' has a strict semi-stable model X' over R', the ring of integers of K', and is such that the Zariski closure A' of $\sigma^{-1}(A^0)$ in X' has the property that (X', A') is strictly semi-stable. Thus by the above, we may assume that (X, A) is a strictly semi-stable pair, where A is the Zariski closure of A^0 in X. If I is a sequence $(i_1, i_2, \ldots, i_a)$ of pairwise distinct indices, we denote by A_I the intersection $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_a}$. One has the G-equivariant Mayer-Vietoris spectral sequence

$$E_1^{-a+1,b} = \oplus_{|I|=a} H^b_{A_I}(V \times_K \bar{K}, \mathbb{Q}_\ell) \Rightarrow H_A^{1-a+b}(V \times_K \bar{K}, \mathbb{Q}_\ell) \tag{3.3}$$

together with the G-equivariant purity isomorphism (e.g. [15, Th. 2.1.1])

$$H^{b-2c_I}(A_I \times_R \bar{K}, \mathbb{Q}_\ell)(-c_I) \cong H^b_{A_I}(V \times_K \bar{K}, \mathbb{Q}_\ell), \tag{3.4}$$

where c_I is the codimension of A_I in X. Since $c_I \geq 1$, we conclude with Theorem 1.5 i). □

4. The proof of Theorem 1.1 and its consequences

Proof of Theorem 1.1. We denote by Φ a lifting of Frobenius in the Deligne-Weil group of K. By Remark 2.3, $|k|$-divisibility of the eigenvalues of F acting on $H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell)$ is equivalent to $|k|$-divisibility of the eigenvalues of Φ acting on $H^m(V \times_k K^u, \mathbb{Q}_\ell)$. On the other hand, one has the F-equivariant exact sequence [8, p. 213, (5)]

$$\begin{aligned} 0 \to H^{m-1}(V \times_K \bar{K}, \mathbb{Q}_\ell)_I(-1) \to H^m(V \times_K K^u, \mathbb{Q}_\ell) \\ \to H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)^I \to 0\ . \end{aligned} \tag{4.1}$$

Here $_I$ means the inertia coinvariants while ^{I} means the inertia invariants. The quotient map $H^{m-1}(V \times_K \bar{K}, \mathbb{Q}_\ell) \twoheadrightarrow H^{m-1}(V \times_K \bar{K}, \mathbb{Q}_\ell)_I$ is Φ-equivariant. Thus by Theorem 1.5 i), the eigenvalues of F acting on $H^{m-1}(V \times_K \bar{K}, \mathbb{Q}_\ell)_I$ are algebraic integers for all m; thus on $H^{m-1}(V \times_K \bar{K}, \mathbb{Q}_\ell)_I(-1)$ they are $|k|$-divisible algebraic integers for all m. The injection $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)^I \hookrightarrow H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)$ is Φ-equivariant; thus by the coniveau assumption of Theorem 1.1 and Theorem 1.5 ii), the eigenvalues of F acting on $H^m(V \times_K \bar{K}, \mathbb{Q}_\ell)^I$ are $|k|$-divisible algebraic integers for $m \geq 1$. Thus we conclude that the eigenvalues of F acting on $H^m(Y \times_k \bar{k}, \mathbb{Q}_\ell)$ are $|k|$-divisible algebraic integers for all $m \geq 1$. By the Grothendieck-Lefschetz trace formula [16] applied to Y, this shows that the number of rational points of Y is congruent to 1 modulo $|k|$. This finishes the proof of Theorem 1.1. □

Proof of Corollary 1.2. One applies Bloch's decomposition of the diagonal [2, Appendix to Lecture 1], as mentioned in the introduction and detailed in [10], in order to show that the base change condition on the Chow group of 0-cycles implies the coniveau condition of Theorem 1.1. Indeed, $CH_0(V_0 \times_{K_0} \bar{\Omega}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ implies the existence of a decomposition $N\Delta \equiv \xi \times V_0 + \Gamma$ in $CH_{\dim(V)}(V_0 \times_{K_0} V_0)$, where $N \geq 1, N \in \mathbb{N}$, ξ is a 0-cycle of V_0 defined over K_0, Γ is a $\dim(V)$-cycle lying in $V_0 \times_{K_0} A$, where A is a divisor in V_0. This decomposition yields *a fortiori* a decomposition in $CH_{\dim(V)}((V\times_K V)\times_K \bar{K})$. The correspondence with Γ has image in $\mathrm{Im}(H^m_A(V\times_K \bar{K},\mathbb{Q}_\ell))\subset H^m(V\times_K \bar{K},\mathbb{Q}_\ell)$, while the correspondence with $\xi\times V_0$ kills $H^m(V\times_K \bar{K},\mathbb{Q}_\ell)$ for $m\geq 1$ as it factors through the restriction to $H^m(\xi\times_{K_0}\bar{K},\mathbb{Q}_\ell)$. Thus

$$N^1H^m(V\times_k \bar{K},\mathbb{Q}_\ell)=H^m(V\times_K \bar{K},\mathbb{Q}_\ell).$$

We apply Theorem 1.1 to conclude the proof. □

Proof of Theorem 1.3. In order to apply Theorem 1.1, we just have to know that $H^1(V,\mathcal{O}_V)=0$ is equivalent to the vanishing of de Rham cohomology $H^1_{\mathrm{DR}}(V)$. Thus by the comparison theorem, this implies $H^1_{\text{ét}}(V\times_K \bar{K},\mathbb{Q}_\ell)=0$. Furthermore, $H^2(V,\mathcal{O}_V)=0$ is equivalent to $N^1H^2_{\mathrm{DR}}(V)=H^2_{\mathrm{DR}}(V)$; thus by the comparison theorem, $N^1H^2_{\text{ét}}(V\times_K \bar{K},\mathbb{Q}_\ell)=H^2_{\text{ét}}(V\times_K \bar{K},\mathbb{Q}_\ell)$. Thus we can apply Theorem 1.1. □

5. Some comments and remarks

5.1. Theorem 1.5 ii) is formulated for N^1 and not for the higher coniveau levels N^κ of étale cohomology. The appendix to this article fills in this gap: if V is smooth over a local field K with finite residue field k, then the eigenvalues of Φ on $N^\kappa H^m_{\mathrm{prim}}(V\times_K \bar{K},\mathbb{Q}_\ell)$ lie in $|k|^\kappa\cdot\bar{\mathbb{Z}}$. Here the subscript prim means one mods out by the powers of the class of the polarization coming from a projective embedding $Y\subset\mathbb{P}^N$. So for example, in the good reduction case, the N^κ condition on the smooth projective fibre V will imply that $|Y(k)|\equiv|\mathbb{P}^N(k)|$ modulo $|k|^\kappa$. In general, only a strong minimality condition on the model X could imply this conclusion, as blowing up a smooth point of Y keeps the same number of rational points only modulo $|k|$.

5.2. Kollár's example of a rationally connected surface (personal communication) over a finite field k, but without a rational point, is birational (over $\bar{k}$) to the product of a genus ≥ 2 curve with $\mathbb{P}^1$. In particular it is not a Fano variety. Here we define a projective variety Y over a field k to be Fano if it is geometrically irreducible, Gorenstein, and if the dualizing sheaf ω_Y is anti-ample. If the characteristic of k is 0, then one defines the ideal sheaf $\mathcal{I}=\pi_*\omega_{Y'/Y}$, where $\pi: Y'\to Y$ is a desingularization. This ideal does not depend on the choice of Y' (and is called in our days the multiplier ideal). The Kawamata-Viehweg vanishing theorem applied to $\pi^*\omega_Y^{-1}$ shows that $H^m(Y,\mathcal{I})=0$, for all m if $\mathcal{I}$ is not equal to $\mathcal{O}_Y$, otherwise for $m\geq 1$. In the cases where the support S of $\mathcal{I}$ is the empty set or where S equals the singular locus of X, this implies by [9, Prop. 1.2] that the Hodge type of $H^m_{\mathrm{DR}}(X,S)$ is ≥ 1 for all m if $\mathcal{I}$ is not equal to $\mathcal{O}_Y$, otherwise for $m\geq 1$. Using again Deligne's philosophy as mentioned in the introduction, one would expect that a suitable definition of S in positive characteristic for a Fano variety (note the definition above requires the existence of a desingularization) would lead to the prediction that over a finite field k, the number of rational points of Y is congruent to the number of rational points of S modulo $|k|$ if $S=\emptyset$ or S equals the singular locus of X.

5.3. The correct motivic condition for a projective variety defined Y over a finite field k, which implies that the number of rational points of Y is congruent to 1 modulo $|k|$, is worked out in [3]. Let us briefly recall it. If $Y \subset \mathbb{P}^N$ is any projective embedding defined over k, with complement U, and $\sigma : \mathbb{P} \to \mathbb{P}^N$ is an alteration of $\mathbb{P}^N$ which makes $\sigma^{-1}Y =: Z$ a normal crossing divisor, then there is a divisor $A \subset \mathbb{P}$ in good position with respect to all strata of Z so that rationally, the motivic class $\Gamma_\sigma \in H^{2N}((\mathbb{P}\times U)\times_k \bar{k}, (Z\times U)\times_k \bar{k}, N)$ dies on $\mathbb{P}\setminus A$

$$\begin{aligned}(5.1)\qquad 0 = {} & \Gamma_\sigma|_{\mathbb{P}\setminus A}\\ & \in H^{2N}(((\mathbb{P}\setminus A)\times U)\times_k \bar{k}, ((Z\setminus Z\cdot A)\times U)\times_k \bar{k}, N)\otimes_{\mathbb{Z}} \mathbb{Q}.\end{aligned}$$

Here Γ_σ is the graph of the alteration σ. And we know by the main theorem in [3] that hypersurfaces $Y \subset \mathbb{P}^N$ of degree $d \le N$ fulfill this motivic condition. Given Corollary 1.2, it is conceivable that the condition $CH_0(V_0 \times_{K_0} \bar{\Omega}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ directly implies our motivic condition (5.1). We could not compute this, as moving cycles over discrete valuation rings is not easy.

5.4. In light of [12, Th. 1.1], one would expect that if V is projective smooth over K with smooth model X over R, if K has unequal characteristic, and if $H^m(V, \mathcal{O}_V) = 0$ for all $m \ge 1$, then there is a way using rigid geometry which allows us to conclude that Y has a point. Grothendieck's generalized Hodge conjecture surely predicts that the coniveau 1 condition on étale cohomology of $V \times_K \bar{K}$ and the vanishing of $H^m(V, \mathcal{O}_V)$ for $m \ge 1$ are equivalent, but we do not know the answer so far. It would give some hope to link higher Hodge levels κ for $H^m_{\mathrm{DR}}(V)$ to higher levels κ for divisibility of Frobenius eigenvalues in étale cohomology and to higher levels κ for congruences for the number of points of Y. In particular, it would give a natural explanation of the main results in [11] and [13] where it is shown that for a closed subset $Y \subset \mathbb{P}^N$ defined over a finite field, the divisibility of the eigenvalues of F is controlled by the divisibility for the number of rational points of $\mathbb{P}^N \setminus Y$ as stated in the Ax-Katz theorem [1], [19].

5.5. We give a concrete nontrivial example of Theorem 1.1 due to X. Sun (personal communication). Moduli $M(C, r, L)$ of vector bundles of rank r and fixed determinant L of degree d with $(r, d) = 1$ on a smooth projective curve C over a field are known to be smooth projective Fano varieties, to which we can apply our Theorem [10] if the field is finite. If (C, L) is defined over the local field K, with model $(\mathcal{C}, \mathcal{L})$ over R and reduction (C_k, L_k) over k, then if C_k has a node and $d = 1$, $M(C, 2, L)$ has a model $\mathcal{M}(\mathcal{C}, 2, \mathcal{L})$ with closed fibre

$M(C_k, 2, L_k)$ such that the underlying reduced variety parametrizes torsion-free sheaves E of rank 2 which are endowed with a morphism $\Lambda^2 E \to L_k$ which is an isomorphism off the double point. By [10], there is a rational point $E_0 \in M(\tilde{C}_k, 2, \tilde{L}_k)(k)$, where $\pi : \tilde{C}_k \to C_k$ is the normalization, $\tilde{L}_k = \pi^*(L_k \otimes \mathfrak{m})$, and $\mathrm{Spec}(\mathcal{O}_{C_k}/\mathfrak{m})$ is the node. Then $E = \pi_* E_0 \in M(C_k, 2, L_k)(k)$ is the wanted rational point of the modulo p reduction.

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Appendix

By Pierre Deligne and Hélène Esnault

We generalize in this appendix Theorem 1.5 to nontrivial coefficients on varieties V which are neither smooth nor projective. We thank Alexander Beilinson, Luc Illusie and Takeshi Saito for very helpful discussions.

The notation is as in the article. Thus K is a local field with finite residue field k, $R \subset K$ is the ring of integers, Φ is a lifting of the geometric Frobenius in the Galois group of K. We consider ℓ-adic sheaves on schemes of finite type defined over K in the sense of [3, (1.1)]. One generalizes the definition [2, Déf. 5.1] of T-integral ℓ-adic sheaves on schemes of finite type defined over finite fields to ℓ-adic sheaves on schemes of finite type defined over local fields with finite residue field. Recall, to this aim, that if $\mathcal{C}$ is an ℓ-adic sheaf on a K-scheme V and v is a closed point of V, then the stalk $\mathcal{C}_{\bar{v}}$ of $\mathcal{C}$ at $\bar{v}$ is a $\mathrm{Gal}(\bar{K}/K_v)$-module, where $K_v \supset K$ is the residue field of v, with residue field $\kappa(v) \supset k$. On $\mathcal{C}_{\bar{v}}$ the inertia $I_v = \mathrm{Ker}(\mathrm{Gal}(\bar{K}/K_v) \to \mathrm{Gal}(\kappa(v)/k))$ acts quasi-unipotently ([4]). Consequently the eigenvalues of a lifting $\Phi_v \in \mathrm{Gal}(\bar{K}/K_v)$ of the geometric Frobenius $F_v \in \mathrm{Gal}(\overline{\kappa(v)}/\kappa(v))$ are, up to multiplication by roots of unity, well defined ([3, Lemma (1.7.4)]). Let $T \subset \mathbb{Z}$ be a set of prime numbers.

Definition 0.1. The ℓ-adic sheaf $\mathcal{C}$ is T-integral if the eigenvalues of Φ_v acting on $\mathcal{C}_{\bar{v}}$ are integral over $\mathbb{Z}[\frac{1}{t}, t \in T]$ for all closed points $v \in V$.

Theorem 0.2. *Let V be a scheme of finite type defined over K, and let $\mathcal{C}$ be a T-integral ℓ-adic sheaf on V. Then if $f : V \to W$ is a morphism to another K-scheme of finite type W defined over K, the ℓ-adic sheaves $R^i f_! \mathcal{C}$ are T-integral as well. More precisely, if $w \in W$ is a closed point, then the eigenvalues of both F_w and $|\kappa(w)|^{n-i} F_w$ acting on $(R^i f_! \mathcal{C})_{\bar{w}}$ are integral over $\mathbb{Z}[\frac{1}{t}, t \in T]$, with $n = \dim(f^{-1}(w))$.*

Proof. Let w be a closed point of W. By base change for $Rf_!$ and by $\{w\} \hookrightarrow W$, one is reduced to the case where W is the spectrum of a finite extension K' of K. If $\bar{V} := V \otimes_{K'} \bar{K}'$, for $\bar{K}'$ an algebraic closure of K', one has to check the integrality statements for the eigenvalues of a lifting of Frobenius on $H^i_c(\bar{V}, \mathcal{C})$.

Let us perform the same reductions as in [2, p. 24]. Note that in loc. cit. U can be shrunk so as to be affine, with the sheaf smooth on it (= a local system). This reduces us to the cases where V is of dimension zero or is an affine irreducible curve smooth over $W = \operatorname{Spec}(K')$ and $\mathcal{C}$ is a smooth sheaf.

The integrality statement to be proven is insensitive to a finite extension of scalars K''/K'. The 0-dimensional case reduces in this way to the trivial case where V is a sum of copies of $\operatorname{Spec}(K')$. In the affine curve case, H^0_c vanishes, while, as in [2, Lemma 5.2.1], there is a 0-dimensional $Z \subset V$ such that the natural (Φ-equivariant) map from $H^0(Z, \mathcal{C})(-1)$ to $H^2_c(\bar{V}, \mathcal{C})$ is surjective, leaving us only H^1_c to consider.

Let $\mathcal{C}_{\mathbb{Z}_\ell}$ be a smooth $\mathbb{Z}_\ell$-sheaf from which $\mathcal{C}$ is deduced by $\otimes\mathbb{Q}_\ell$, and let $\mathcal{C}_\ell$ be the reduction modulo ℓ of $\mathcal{C}_{\mathbb{Z}_\ell}$. For some r, it is locally (for the étale topology) isomorphic to $(\mathbb{Z}/\ell)^r$. Let $\pi\colon V' \to V$ be the étale covering of V representing the isomorphisms of $\mathcal{C}_\ell$ with $(\mathbb{Z}/\ell)^r$. It is a $\mathrm{GL}(r, \mathbb{Z}/\ell)$-torsor over V. As $H^*_c(\bar{V}, \mathcal{C})$ injects into $H^*_c(\overline{V'}, \pi^*\mathcal{C})$, renaming irreducible components of V' as V and K' as K, we may and shall assume that $W = \operatorname{Spec}(K)$ and that $\mathcal{C}_\ell$ is a constant sheaf.

Let V_1 be the projective and smooth completion of V, and $Z := V_1 \setminus V$. Extending scalars, we may and shall assume that Z consists of rational points and that V_1, marked with those points, has semi-stable reduction. It hence is the general fiber of X regular and proper over $\operatorname{Spec}(R)$, smooth over $\operatorname{Spec}(R)$ except for quadratic nondegenerate singular points, with Z defined by disjoint sections z_α through the smooth locus.

Let Y be the special fiber of X, and $\bar{Y}$ be $Y \times_k \bar{k}$, for $\bar{k}$ the residue field of the algebraic closure $\bar{K}$ of K.

$$\begin{tikzcd}
V \arrow[d, hook, "j"'] & \text{: complement of disjoint sections } z_\alpha & & & \\
V_1 \arrow[r, hook] \arrow[d] & X \arrow[d] & Y \arrow[l, hook'] \arrow[d] \\
\mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) & \mathrm{Spec}(k) \arrow[l, hook'] .
\end{tikzcd}$$

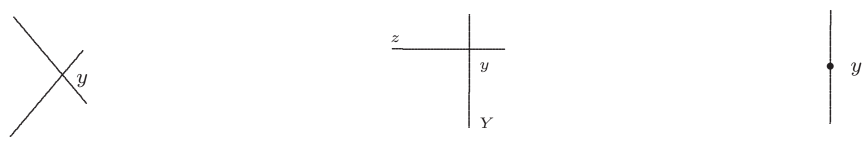
The cohomology with compact support $H^1_c(\bar{V}, \mathcal{C})$ is $H^1(\overline{V_1}, j_!\mathcal{C})$, and vanishing cycles theory relates this H^1 to the cohomology groups on $\bar{Y}$ of the nearby cycle sheaves $\psi^i(j_!\mathcal{C})$, which are ℓ-adic sheaves on $\bar{Y}$, with an action of $\mathrm{Gal}(\bar{K}/K)$ compatible with the action of $\mathrm{Gal}(\bar{K}/K)$ (through $\mathrm{Gal}(\bar{k}/k)$) on $\bar{Y}$. The choice of a lifting of Frobenius, i.e. of a lifting of $\mathrm{Gal}(\bar{k}/k)$ in $\mathrm{Gal}(\bar{K}/K)$, makes them come from ℓ-adic sheaves on Y, to which the integrality results of [2] apply.

Using the exact sequence

$$0 \to H^1(\bar{Y}, \psi^0(j_!\mathcal{C})) \to H^1(\overline{V_1}, j_!\mathcal{C}) \to H^0(\bar{Y}, \psi^1(j_!\mathcal{C}))$$

and [2] Théorème 5.2.2, we are reduced to check integrality of the sheaves $\psi^i(j_!\mathcal{C})$ $(i = 0, 1)$. It even suffices to check it at any k-point y of Y, provided we do so after any unramified finite extension of K.

Let $X_{(y)}$ be the henselization of X at y, and $Y_{(y)}$, $V_{1(y)}$ and $V_{(y)}$ be the inverse image of Y, V_1 or V in $X_{(y)}$. There are three cases:



(1) y singular on Y (2) y on a z_α (3) general case.

The restriction of ψ^i to y depends only on the restriction of $\mathcal{C}$ to $V_{(y)}$, and short exact sequences of sheaves give rise to long exact sequences of ψ.

Because $\mathcal{C}_\ell$ is a constant sheaf, $\mathcal{C}$ is tamely ramified along Y and the z_α. More precisely, it is given by a representation of the pro-ℓ fundamental group of $V_{(y)}$. It is easier to describe the group deduced from the profinite fundamental group by pro-ℓ completing only the kernel of its map to $\widehat{\mathbb{Z}} = \mathrm{Gal}(\bar{k}/k)$. By Abhyankhar's lemma, this group is an extension of $\widehat{\mathbb{Z}}$, generated by Frobenius, by $\mathbb{Z}_\ell(1)^2$ in case (1) or (2) or $\mathbb{Z}_\ell(1)$ in case (3). The representation is given by $r \times r$ matrices congruent to $1 \bmod \ell$. For $\ell \neq 2$, such a matrix, if quasi-unipotent, is unipotent. Indeed, it is the exponential of its logarithm and the eigenvalues of its logarithm are all zero. For $\ell = 2$, the same holds if the congruence is $\bmod 4$, hence if $\mathcal{C}_{\mathbb{Z}_2} \bmod 4$ is constant, a case to which one reduces by the same argument we used $\bmod 2$. By Grothendieck's argument [6, p. 515], the action of $\mathbb{Z}_\ell(1)$ or $\mathbb{Z}_\ell(1)^2$ is quasi-unipotent, hence unipotent, and we can filter $\mathcal{C}$ on $V_{(y)}$ by smooth sheaves such that the successive quotients Q extend to smooth sheaves on $X_{(y)}$. If Q extends to a smooth sheaf $\mathcal{L}$ on $X_{(y)}$, the corresponding ψ are known by Picard-Lefschetz theory: ψ^0 is $\mathcal{L}$ restricted to $Y_{(y)}$ in cases (1) and (3), and $\mathcal{L}$ outside of y extended by zero in case (2); ψ^1 is nonzero only in case (1), where it is $\mathcal{L}(-1)$ on $\{y\}$ extended by zero. By dévissage, this gives the required integrality. □

Corollary 0.3. *Let V be a scheme of finite type defined over K. Then the eigenvalues of Φ on $H^i(\bar{V}, \mathbb{Q}_\ell)$ are integral over $\mathbb{Z}$.*

Proof. We fix an integer $n > 2i$. By de Jong's theorem [1, Th. 6.5], there is a finite extension $K' \supset K$ and a simplicial truncated alteration $V_n \to \ldots \to V_0 \to V$ defined over K'. By the Mayer-Vietoris spectral sequence, $H^*(\bar{V}, \mathbb{Q}_\ell)$ is filtered by sub $\mathrm{Gal}(\bar{K}/K')$-modules, the graduation of which is a subquotient of $\oplus_0^n H^*(\bar{V}_j, \mathbb{Q}_\ell)$. Since integrality of eigenvalues can be computed on a finite extension of K, we may assume that V is smooth. If K has characteristic zero, there is a good compactification $j : V \hookrightarrow W$, with W smooth proper over K and $D = W \setminus V = \cup D_i$ a strict normal crossing divisor. Then the long exact sequence

$$\text{(0.1)} \qquad \ldots \to H^i_D(\bar{W}, \mathbb{Q}_\ell) \to H^i(\bar{W}, \mathbb{Q}_\ell) \to H^i(\bar{V}, \mathbb{Q}_\ell) \to \ldots$$

and Theorem 0.2 applied to the cohomology of W reduce to showing integrality for $H^i_D(\bar{W}, \mathbb{Q}_\ell)$. As in (3.3) of the main article, the Mayer-Vietoris spectral sequence

$$\text{(0.2)} \qquad E_1^{-a+1,b} = \oplus_{|I|=a} H^b_{D_I}(\bar{W}, \mathbb{Q}_\ell) \Rightarrow H^{1-a+b}_D(\bar{W}, \mathbb{Q}_\ell),$$

with $D_I = \cap_{i\in I} D_i$, reduces to the case where D is smooth projective of codimension ≥ 1. Then purity together with Theorem 0.2 allow us to conclude. If K has equal positive characteristic, we apply [1, Th. 6.5] again to find $\pi : V' \to V$ generically finite and $j : V' \hookrightarrow W$ a good compactification. As $\pi^* : H^i(\bar{V}, \mathbb{Q}_\ell) \hookrightarrow H^i(\overline{V'}, \mathbb{Q}_\ell)$ is injective, we conclude as above. □

Corollary 0.3 gives some flexibility as we do not assume that V is projective. In particular, one can apply the same argument as in the proof of Theorem 2.1 of the main article in order to show an improved version of Theorem 1.5, (ii) there:

COROLLARY 0.4. *Let V be a smooth scheme of finite type over K, and $A \subset V$ be a codimension κ subscheme. Then the eigenvalues of Φ on $H^i_A(\bar{V}, \mathbb{Q}_\ell)$ are divisible by $|k|^\kappa$ as algebraic integers.*

Proof. One has a stratification $\ldots \subset A_i \subset A_{i-i} \subset \ldots A_0 = A$ by closed subschemes defined over K with $A_{i-1} \setminus A_i$ smooth. The Φ-equivariant long exact sequence

$$\text{(0.3)} \qquad \ldots \to H^m_{A_i}(\bar{V}, \mathbb{Q}_\ell) \to H^m_{A_{i-1}}(\bar{V}, \mathbb{Q}_\ell) \to H^m_{(A_{i-1}\setminus A_i)}(\overline{V \setminus A_i}, \mathbb{Q}_\ell) \to \ldots$$

together with purity and Corollary 0.3 allow us to conclude by induction on the codimension. □

Remark 0.5. One has to pay attention to the fact that even if Theorem 0.2 generalizes Theorem 1.5 i) of the article to V not necessarily smooth, there is no such generalization of Theorem 1.5 ii) to the nonsmooth case, even on a finite field. Indeed, let V be a rational curve with one node. Then $H^1(\bar{V}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(0)$ as we see from the normalization sequence, yet $H^1_{\text{node}}(\bar{V}, \mathbb{Q}_\ell) = H^1(\bar{V}, \mathbb{Q}_\ell)$ as the localization map $H^1(\bar{V}, \mathbb{Q}_\ell) \to H^1(\overline{V \setminus \text{node}}, \mathbb{Q}_\ell)$ factor through

$$H^1(\overline{\text{normalization}}, \mathbb{Q}_\ell) = 0.$$

So we cannot improve the integrality statement to a divisibility statement in general. In order to force divisibility, one needs the divisor supporting the cohomology to be in good position with respect to the singularities.

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