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The $\bar{\partial}_b$-complex on decoupled boundaries in $\mathbb{C}^n$

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1. Introduction

The purpose of this paper is to prove optimal estimates for solutions of the Kohn-Laplacian for certain classes of model domains in several complex variables. This will be achieved by applying a type of singular integral operator whose novel features (related to product theory and flag kernels) differ essentially from the more standard Calderón-Zygmund operators that have been used in these problems hitherto.

1.1. *Background.* We consider the Kohn-Laplacian on q-forms, $\Box_b^{(q)} = \Box_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b$, defined on the boundary $M = \partial\Omega$ of a smooth bounded pseudo-convex domain $\Omega \subset \mathbb{C}^n$. Our objective is the study of the (relative) inverse operator $\mathcal{K}$ and the corresponding Szegö projection $\mathcal{S}$ (when it exists), which satisfy $\Box_b \mathcal{K} = \mathcal{K}\,\Box_b = I - \mathcal{S}$. By definition $\mathcal{S}$ is the orthogonal projection on the L^2 null-space of $\Box_b$.

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In formulating the questions of regularity pertaining to the above, it is useful to recall Fefferman's hierarchy [Fef95] of the levels of understanding of the problem, which we rephrase as follows:

(1) Proof of C^∞ regularity.

(2) Derivation of optimal L^p, Hölder, and Sobolev-space estimates of solutions.

(3) Analysis of singularities of the distribution kernels of the operators $\mathcal{K}$ and S and derivation of the estimates in (2) from a corresponding theory of singular integrals.

Now as far as the C^∞ regularity is concerned, this has been resolved in the general situation where an appropriate "finite-type" condition holds (at least for the closely connected $\bar{\partial}$-Neumann problem) by the work of Kohn [Koh72], [Koh79], Catlin [Cat83], [Cat87], and D'Angelo [D'A82]. However, the more refined results of (2) and (3) have been obtained only in a more restrictive setting. This was carried out in a series of developments beginning with the work of Folland and Stein ([FS74]) in the strongly pseudo-convex case, and in later works of, among others, Christ ([Chr91b], [Chr88]), Fefferman and Kohn ([FK88]), Kohn ([Koh85]), McNeal ([McN89]), Nagel, Rosay, Stein, and Wainger ([NRSW89]), and Rothschild and Stein ([RS76]). This culminated in the work of Koenig ([Koe02]) on finite type domains whose Levi-form has comparable eigenvalues.

At the base of these results is a version of the Calderón-Zygmund theory for the following class of singular integrals: One considers operators $\mathcal{T}$ of the form $\mathcal{T}(f)(x) = \int T(x,y)\, f(y)dy$ whose kernels $T(x,y)$ are distributions that are smooth away from the diagonal, that satisfy the characteristic size estimates $|T(x,y)| \lesssim d(x,y)^a\, V(x,y)^{-1}$, and that satisfy corresponding differential inequalities and cancellation properties. Here $d(x,y)$ is the control metric determined by the vector fields which are the real and imaginary parts of the tangential Cauchy-Riemann operators, and $V(x,y)$ denotes the volume of the ball centered at x of radius $d(x,y)$. It can be shown that the relative fundamental solution $\mathcal{K}$ and the Szegö projection $\mathcal{S}$ are of this type, with $a = 2$ for $\mathcal{K}$, and $a = 0$ for $\mathcal{S}$. As a result, one obtains for these operators maximal sub-elliptic estimates in L^p, *etc.*

Unfortunately, while highly satisfactory, the above framework with a natural metric controlling all estimates cannot carry over in general. In fact, in more general circumstances there seem to arise a number of inequivalent metrics that control different aspects of the problem. This appears to be connected with earlier observations of Derridj [Der78] and Rothschild [Rot80] that maximal sub-ellipticity can hold only if the eigenvalues of the Levi-form are comparable. It is the purpose of this paper to make progress in the resolution of problems (2) and (3) in an illustrative model case - that of decoupled domains.

1.2. *A special case.* To get a better grasp of these problems and the results we obtain, we take a closer look at the special case of a decoupled domain where $\Omega = \{z \in \mathbb{C}^3\colon \Im\mathrm{m}[z_3] > |z_1|^n + |z_2|^m\}$, with n, m even integers. Then $M = \partial\Omega$ can be identified with $\{(z,t) \in \mathbb{C}^2 \times \mathbb{R},\ z = (z_1, z_2)\}$, and

$$\bar Z_1 = \frac{\partial}{\partial \bar z_1} - i\,\frac{n}{2}|z_1|^{n-2}\,z_1\frac{\partial}{\partial t}, \qquad \bar Z_2 = \frac{\partial}{\partial \bar z_2} - i\,\frac{m}{2}|z_2|^{m-2}z_2\frac{\partial}{\partial t}$$

form a basis for the tangential Cauchy-Riemann vector fields. The eigenvalues λ_1, λ_2 of the Levi-form at a point (z_1, z_2, t) are essentially $|z_1|^{n-2}$ and $|z_2|^{m-2}$, and are not comparable. With $\bar Z_j = \frac{1}{2}(X_j + iY_j)$, we can consider d_Σ, the control metric defined by X_1, Y_1, X_2, Y_2.

However, the above domain is also convex, so that there is another natural metric, which reflects the "flatness" of the boundary in different complex directions, the "Szegö metric" d_S; (see McNeal [McN94b], [McN94a], and Bruna, Nagel and Wainger [BNW88] for a real analogue). In our special case, if $n \le m$, when we measure the distance of the point $p = (z_1, z_2, t)$ from the origin 0 we have:

$$d_\Sigma(0,p) \approx |z_1| \quad + |z_2| \quad + |t|^{1/m};$$
$$d_S(0,p) \approx |z_1|^m + |z_2|^n + |t|.$$

Note that $d_S(0,p)^{1/m} \approx |z_1| + |z_2|^{n/m} + |t|^{1/m}$, and this is not equivalent to $d_\Sigma(0.p)$ if $n \ne m$. Thus these metrics, or powers of these metrics are in general not equivalent.

Now d_Σ controls the inverse of the sub-Laplacian $\mathcal{L} = -\frac{1}{2}\sum_{i=1}^2 (Z_i\bar Z_i + \bar Z_i Z_i)$, while d_S controls the Szegö kernel (the orthogonal projection on the null-space of the operator $-\sum_{i=1}^2 Z_i \bar Z_i$), and some mixture of d_Σ and d_S arises in the fundamental solution of the operator $\Box_b = -(Z_1\bar Z_1 + \bar Z_2 Z_2) = \Box_b^1 + \Box_b^2$, which is essentially the Kohn-Laplacian acting on 1-forms.

With this we can state a part of our main result obtained below, formulated in this special case, as follows:

THEOREM. *There is an operator $\mathcal{K}$ so that, when it is applied to smooth functions with compact support, there is the identity $\mathcal{K}\,\Box_b = \Box_b\,\mathcal{K} = I$. Moreover*

(a) *The four operators $Z_1\,\bar Z_1\,\mathcal{K} = \Box_b^1\,\mathcal{K}$, $\bar Z_2\,Z_2\,\mathcal{K} = \Box_b^2\,\mathcal{K}$, $\bar Z_1\,\bar Z_1\,\mathcal{K}$, and $Z_2\,Z_2\,\mathcal{K}$ are bounded on $L^p(M)$ for $1 < p < \infty$.*

(b) *Let B_1, B_2 be bounded functions on M, and suppose there are constants C_1, C_2 so that*

$$\lambda_1(z_1)\,B_1(z_1,z_2,t) \le C_1\,\lambda_2(z_2);$$
$$\lambda_2(z_2)\,B_2(z_1,z_2,t) \le C_2\,\lambda_2(z_1).$$

Then the two operators $B_1\,\bar{Z}_1\,Z_1\,\mathcal{K} = B_1\,\overline{\square}_b^1\,\mathcal{K}$ *and* $B_2\,Z_2\,\bar{Z}_2\,\mathcal{K} = B_2\,\overline{\square}_b^2\,\mathcal{K}$ *are bounded on* $L^p(M)$ *for* $1 < p < \infty$*. Here* $\lambda_1(z) = |z|^{m-2}$ *and* $\lambda_2(z) = |z|^{n-2}$ *are the eigenvalues of the Levi form.*

(c) *Let* B_1, B_2 *be bounded functions on* M, *and suppose there are constants* C_1, C_2 *so that*

$$B_1(z_1, z_2, t) \leq C_1\,\lambda_2(z_2);$$
$$B_2(z_1, z_2, t) \leq C_2\,\lambda_2(z_1).$$

Then the two operators $B_1\,Z_1\,Z_1\,\mathcal{K}$ *and* $B_2\,\bar{Z}_2\,\bar{Z}_2\,\mathcal{K}$ *are bounded on* $L^p(M)$ *for* $1 < p < \infty$.

(d) $\mathcal{K}$ *maps* $L^\infty(M)$ *to the isotropic Hölder space* $\Lambda^\alpha(M)$, *where*

$$\alpha = \min\left\{\frac{2}{n}, \frac{2}{m}\right\}.$$

The conclusion (b) is part of the optimal substitute for maximal subellipticity that holds in this case.

1.3. *Methods used.* To describe the methods used we continue with the case considered above. We begin by considering separately the component domains

$$M_1 = \left\{(z_1, w_1) \in \mathbb{C}^2 \,\middle|\, \Im[w_1] = |z_1|^n\right\} \simeq \{(z_1, t_1) \in \mathbb{C} \times \mathbb{R}\}, \quad \text{and}$$
$$M_2 = \left\{(z_2, w_2) \in \mathbb{C}^2 \,\middle|\, \Im[w_2] = |z_2|^m\right\} \simeq \{(z_2, t_2) \in \mathbb{C} \times \mathbb{R}\}.$$

We denote by $\widetilde{M}$ the Cartesian product $M_1 \times M_2$ and we let π be the projection of $\widetilde{M}$ to M given by $\pi : (z_1, t_1) \times (z_2, t_2) \to (z_1, z_2, t_1 + t_2)$.

The idea is to deduce the results about regularity of $\square_b$ on M from corresponding results on $\widetilde{M}$. Moreover, passing to the product allows one to consider various combinations of the separate metrics on each factor of $\widetilde{M}$, which in effect account for the different metrics on M. Our analysis proceeds as follows.

(1) *Analysis on each* M_j: Here the key point is the use of the nonhypoelliptic "heat" semi-group $e^{-s\square_j}$ on M_j where $\square_1 = Z_1\bar{Z}_1$, $\square_2 = \bar{Z}_2 Z_2$. (The needed estimates for this semi-group were obtained in [NS01a].) For later purposes one observes that if

$$\mathcal{K}_j = \int_0^\infty (e^{-s\square_j} - S_j)\,ds,$$

where S_j is the orthogonal projection on the null-space of $\square^j$, then $\mathcal{K}_j\square^j = \square^j\mathcal{K}_j = I - S_j$.

(2) *Results on the product* $\widetilde{M} = M_1 \times M_2$: In finding a (relative) inverse for $\Box_1 + \Box_2$ on $\widetilde{M}$ one considers

$$\widetilde{\mathcal{K}} = \int_0^\infty (e^{-s(\Box_1+\Box_2)} - S_1 \otimes S_2) ds$$

and also a substitute version

$$\widetilde{\mathcal{N}} = \int_0^\infty (e^{-s\Box_1} - S_1) \otimes (e^{-s\Box_2} - S_2) ds.$$

Now $\widetilde{\mathcal{N}}$ is more tractable than $\widetilde{\mathcal{K}}$ since any second order derivative in Z_j and $\bar{Z}_j$ of $\widetilde{\mathcal{N}}$ turns out to be a product-type singular integral on $M_1 \times M_2$. For such singular integrals an L^p theory has been worked out in [NS04]. However, $\widetilde{\mathcal{K}}$ is the desired relative inverse, since $(\Box_1 + \Box_2)\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}(\Box_1 + \Box_2) = I - S_1 \otimes S_2$; its properties can ultimately be deduced from those of $\widetilde{N}$ because of the identity

$$\widetilde{\mathcal{K}} = \widetilde{\mathcal{N}} + \mathcal{K}_1 \otimes S_2 + S_1 \otimes \mathcal{K}_2 .$$

(3) *Descent to* M: The operators above on $\widetilde{M} = M_1 \times M_2$ are translation-invariant in the t_1 and t_2 variables. Each appropriate operator T of this kind can be transferred by the projection π: $\widetilde{M} \longrightarrow M$ to an operator $T^{\#}$ on M, via the identity

$$T^{\#}(f) = J(T(f \circ \pi))$$

where $J(F)(z_1, w_1, z_2, w_2, t) = \int_{-\infty}^{\infty} F(z_1, w_1, t-s, z_2, w_2, s)\, ds$. This is then applied to $\widetilde{\mathcal{K}}$ to obtain $\mathcal{K} = (\widetilde{\mathcal{K}})^{\#}$, the inverse of $Z_1\bar{Z}_1 + \bar{Z}_2 Z_2$ on M.

There is however a fundamental issue that arises at this point. Operators like $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{N}}$ are not pseudo-local, because as product-like operators their kernels have singularities on the products of the diagonals of the M_i, and not just on the diagonal of $\widetilde{M}$. As a result the projections of such operators on M are thus in general again not pseudo-local. Why then is the operator $\mathcal{K}$ pseudo-local? Connected with this is the question of obtaining the appropriate differential inequalities satisfied by the kernel of $\mathcal{K}$ away from diagonal.

The resolution of these problems is connected with the key idea of "borrowing", which allows one to pass from smoothness inherent in the t_1 (and z_1) variable to the t_2 (and z_2) variable, and vice-versa. This technique is used in several places below where it takes a number of different forms. A particularly transparent example is the identity

$$\left(\frac{\partial}{\partial t_1} S_1 \otimes K_2\right)^{\#} = \left(S_1 \otimes \frac{\partial}{\partial t_2} K_2\right)^{\#}$$

which is used in obtaining conclusion (b) of the theorem above.

1.4. *Previous work.* Besides the results mentioned earlier which deal with the situation of comparable eigenvalues of the Levi-form, several other situations have been previously studied. The case of a decoupled domain in $\mathbb{C}^3$ with exactly one degenerate eigenvalue was dealt with in the paper of Machedon [Mac88] , where he also finds certain estimates for the fundamental solution which involve several metrics. In addition, Fefferman, Kohn, and Machedon [FKM90] have obtained results on Hölder regularity for $\Box_b$ on boundaries of diagonalizable domains (which is a larger class of domains than we consider). In contrast, here we obtain sharp L^p and Hölder estimates, and relevant differential inequalities for the solving operators and Szegö projections.

The general idea of "lifting" to a product (or "simpler" situation) is old, having already appeared in different forms in the study of the sub-Laplacian [RS76], and in [Mac88]. More recently it was used in [MRS95] to study certain operators on the Heisenberg group, and for $\Box_b$ on quadratic CR manifolds of higher-codimension in [NRS01]. The operators arising in [MRS95], related to the boundary operator of the $\bar{\partial}$-Neumann problem for the ball, which occurred in [PS86], already implicitly display the feature of the conflicting metrics which we have discussed above. There the kernels of the relevant operators arise as products of components that are homogeneous in different senses: the isotropic homogeneity reflecting the Euclidean metric, and the automorphic homogeneity of the Heisenberg group, reflecting the control metric.

1.5. *Organization of the paper.* Section 2 contains a review of background material and statements of the main results of the paper. The needed aspects of the geometry and analysis of each of the factors M_i and on their Cartesian product are set down in Section 3. Section 4 studies the various versions of the relative fundamental solutions of $\Box_b$ on $\widetilde{M}$. This leads to L^p results on M via transference, as is shown in Section 5. Section 6 deals with the various metrics on M and the resulting differential inequalities of the kernels are obtained in Section 7. In Section 8, we prove the Hölder regularity of the solutions, and in Section 9 we give examples to show that our regularity results are optimal.

2. Definitions and statement of results

2.1. *Definitions.* A domain $\Omega \subset \mathbb{C}^{n+1}$ and its boundary M are said to be *decoupled* if there are sub-harmonic, nonharmonic polynomials P_j such that

$$\Omega = \Big\{(z_1, \dots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} \,\Big|\, \Im\mathrm{m}[z_{n+1}] > \sum_{j=1}^{n} P_j(z_j)\Big\};$$

$$M = \Big\{(z_1, \dots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} \,\Big|\, \Im\mathrm{m}[z_{n+1}] = \sum_{j=1}^{n} P_j(z_j)\Big\}. \tag{2.1.1}$$

We call the integer $m_j = 2 + \text{degree}(\triangle P_j)$ the "degree" of P_j. (The actual degree of P_j may be larger, but the addition of a harmonic polynomial to P_j does not affect our analysis, and can be eliminated by a change of variables.) We identify M with $\mathbb{C}^n\times\mathbb{R}$ so that the point $\big(z_1,\dots,z_n,t+i\big(\sum_j P_j(z_j)\big)\big)\in M$ corresponds to the point $(z_1,\dots,z_n,t)\in\mathbb{C}^n\times\mathbb{R}$. M has real dimension $2n+1$. When integrating on M, we take the measure to be Lebesgue measure on $\mathbb{C}^n\times\mathbb{R}$.

In addition to the boundary of a decoupled domain as in (2.1.1), we also consider Cartesian products of boundaries of domains in $\mathbb{C}^2$. For $1\le j\le n$, let

$$\begin{aligned}\Omega_j &= \Big\{(z_j,w_j)\in\mathbb{C}^2\,\Big|\,\Im\mathrm{m}[w_j] > P_j(z_j)\Big\};\\ M_j &= \Big\{(z_j,w_j)\in\mathbb{C}^2\,\Big|\,\Im\mathrm{m}[w_j] = P_j(z_j)\Big\}.\end{aligned}\tag{2.1.2}$$

As before, we identify M_j with $\mathbb{C}\times\mathbb{R}$ so that the point $\big(z_j,t+iP_j(z_j)\big)$ corresponds to the point (z_j,t). When integrating on M_j we use Lebesgue measure on $\mathbb{C}\times\mathbb{R}$. The Cartesian product of these boundaries is

$$\widetilde{M} = M_1\times\cdots\times M_n\subset\mathbb{C}^{2n}.\tag{2.1.3}$$

Then $\widetilde{M}$ is the Shilov boundary of the product domain $\Omega_1\times\cdots\times\Omega_n$. It has real dimension $3n$ and real codimension n. We can identify $\widetilde{M}$ with $\mathbb{C}^n\times\mathbb{R}^n$ so that the point $p=\big(z_1,t_1+iP_1(z_1),\dots,z_n,t_n+iP_n(z_n)\big)\in\widetilde{M}$ corresponds to the point $(z_1,\dots,z_n,t_1,\dots,t_n)=(z,t)\in\mathbb{C}^n\times\mathbb{R}^n$. When integrating on $\widetilde{M}$, we take the measure to be Lebesgue measure on $\mathbb{C}^n\times\mathbb{R}^n$.

Let $\pi:\mathbb{C}^{2n}\to\mathbb{C}^{n+1}$ be the linear holomorphic mapping

$$\pi(z_1,\dots,z_n,w_1,\dots,w_n) = (z_1,\dots,z_n,w_1+\cdots+w_n).$$

This induces a mapping from $\widetilde{M}$ to M. In terms of the coordinates given by $\mathbb{C}^n\times\mathbb{R}^n$ and $\mathbb{C}^n\times\mathbb{R}$, we have

$$\pi(z_1,\dots,z_n,t_1,\dots,t_n) = (z_1,\dots,z_n,t_1+\cdots+t_n).\tag{2.1.4}$$

The mapping π allows us to transfer functions from $\widetilde{M}$ to M. If $\varphi\in\mathcal{C}_0^\infty(\mathbb{C}^n\times\mathbb{R}^n)$, we define a function $\varphi^\#\in\mathcal{C}_0^\infty(\mathbb{C}^n\times\mathbb{R})$ by setting

$$\begin{aligned}\varphi^\#(z,t) &= \int_{\mathbb{R}^{n-1}}\varphi\Big(z,r_1,\dots,r_{n-1},t-\sum_{j=1}^{n-1}r_j\Big)\,dr_1\cdots dr_{n-1}\\ &\equiv \int_{r\in\Sigma(t)}\varphi(z,r)\,d\tilde r\end{aligned}\tag{2.1.5}$$

where $\Sigma(t)=\{(r_1,\dots,r_n)\in\mathbb{R}^n\,|\,r_1+\cdots+r_n=t\}$ and $d\tilde r$ is $(n-1)$-dimensional Lebesgue measure on $\Sigma(t)$.

2.2. *The $\bar\partial_b$-complex and the $\Box_b$ operator on M and $\widetilde{M}$.* Let M be the boundary of a decoupled domain as in (2.1.1). Using coordinates $(z_1, \dots, z_n, t) \in \mathbb{C}^n \times \mathbb{R}$, bases for the Cauchy-Riemann operators of type $(1,0)$ and $(0,1)$ are given by the operators $\{Z_j,\, 1 \le j \le n\}$ and by $\{\bar Z_j,\, 1 \le j \le n\}$ where

$$(2.2.6)\qquad \begin{aligned} Z_j &= \frac{\partial}{\partial z_j} + i\,\frac{\partial P_j}{\partial z_j}(z_j)\,\frac{\partial}{\partial t} = X_j - iX_{n+j},\\ \bar Z_j &= \frac{\partial}{\partial \bar z_j} - i\,\frac{\partial P_j}{\partial \bar z_j}(z_j)\,\frac{\partial}{\partial t} = X_j + iX_{n+j}, \end{aligned}$$

where $\{X_1, \dots, X_{2n}\}$ are real vector fields.

Remark. For future reference, note that the operators Z_j, $\bar Z_j$, and their sums and products commute with translations in the variable t. The same will also be true of the inverses or relative inverses we construct for such operators. Hence the corresponding distribution kernels $K\big((z,t),(w,s)\big)$ will be of the form $K(z,w,t-s)$.

We recall the formalism of the $\bar\partial_b$-complex on M. If f is a function, then

$$\bar\partial_b[f] = \sum_{j=1}^{n} \bar Z_j[f]\, d\bar z_j.$$

Let ϑ_q denote the set of strictly increasing q-tuples of integers between 1 and n. Let $J = \{j_1, \dots, j_q\} \in \vartheta_q$, and let $d\bar z_J$ denote the $(0,q)$-form $d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q}$. Then $\{d\bar z_J\}_{J\in\vartheta_q}$ is a basis for the space of $(0,q)$ forms and

$$\bar\partial_b\Big[\sum_{J\in\vartheta_q} f\, d\bar z_J\Big] = \sum_{J\in\vartheta_q} \bar\partial_b[f] \,\wedge\, d\bar z_J.$$

One checks that $\bar\partial_b^2 = 0$.

Let $\bar\partial_b^*$ denote the formal adjoint of $\bar\partial_b$ so that $\bar\partial_b^*$ maps $(0,q+1)$-forms to $(0,q)$-forms. Thus for compactly supported $(0,q)$ and $(0,q+1)$ forms φ and ψ,

$$\big\langle \bar\partial_b[\varphi], \psi\big\rangle_{q+1} = \big\langle \varphi, \bar\partial_b^*[\psi]\big\rangle_q,$$

where $\big\langle\,\cdot\,,\,\cdot\,\big\rangle_q$ is the L^2-inner product on $(0,q)$-forms defined so that the forms $\{d\bar z_J\}_{J\in\vartheta_q}$ are orthonormal.

The Kohn-Laplacian

$$(2.2.7)\qquad \Box_b = \bar\partial_b\,\bar\partial_b^* + \bar\partial_b^*\,\bar\partial_b$$

is a second order system of partial differential operators which maps $(0,q)$-forms to $(0,q)$-forms. For the decoupled boundary M, $\Box_b$ acts as follows. For $1 \le j \le n$, let

$$(2.2.8)\qquad \begin{aligned} \Box_j^{(+)} &= -\bar Z_j\, Z_j;\\ \Box_j^{(-)} &= -Z_j\, \bar Z_j. \end{aligned}$$

For $J \in \vartheta_q$ and $1 \leq k \leq n$ set

$$J(k) = \begin{cases} (+) & \text{if } k \in J, \\ (-) & \text{if } k \notin J. \end{cases}$$

The operator $\Box_b$ acts diagonally, and is given by

$$\Box_b \left(\sum_{J \in \vartheta_q} \varphi_J \, d\bar{z}_J \right) = \sum_{J \in \vartheta_q} \Box_J(\varphi_J) \, d\bar{z}_J \tag{2.2.9}$$

where

$$\Box_J = \sum_{k=1}^{n} \Box_k^{J(k)}. \tag{2.2.10}$$

Thus, the study of the $\bar\partial_b$ complex on M on $(0,q)$-forms is reduced to the study of the $\binom{n}{q}$ operators $\Box_J$ for $J \in \vartheta_q$.

We can also consider the $\bar\partial_b$-complex on the product submanifold $\widetilde{M}$. Instead of the vector fields (2.2.6), we set

$$Z_j = \frac{\partial}{\partial z_j} + i \, \frac{\partial P_j}{\partial z_j}(z_j) \, \frac{\partial}{\partial t_j};$$
$$\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - i \, \frac{\partial P_j}{\partial \bar{z}_j}(z_j) \, \frac{\partial}{\partial t_j}.$$

The $\bar\partial_b$ complex on $\widetilde{M}$ is defined in the exactly the same way as on M. If we then define operators $\Box_j^{\pm}$ as before, the operator $\Box_b = \bar\partial_b \, \bar\partial_b^* + \bar\partial_b^* \, \bar\partial_b$ has exactly the same form as in equations (2.2.9) and (2.2.10).

The mapping $\pi : \widetilde{M} \to M$ given in equation (2.1.4) induces a mapping from functions on M to functions on $\widetilde{M}$, and hence induces a mapping $d\pi$ from tangent vectors on $\widetilde{M}$ to tangent vectors on M. In particular, if $T_j = \frac{\partial}{\partial t_j}$ on $\widetilde{M}$ and $T = \frac{\partial}{\partial t}$ on M, then $d\pi(T_j) = T$ for $1 \leq j \leq n$. This justifies our use of the same notation, i.e. Z_j and $\bar{Z}_j$ for vectors fields and $\Box_b$ for the Kohn Laplacian, on both $\widetilde{M}$ and M. The adjoint mapping $d\pi^*$ which carries differential forms on M to differential forms on $\widetilde{M}$ commutes with the mappings $\bar\partial_b$.

We have also considered a mapping $\varphi \to \varphi^{\#}$ in (2.1.5) which carries functions on $\widetilde{M}$ to functions on M. The following is then clear.

Proposition 2.2.1. *Let $\varphi \in \mathcal{C}_0^\infty(\widetilde{M})$ so that $\varphi^{\#} \in \mathcal{C}_0^\infty(M)$. Then $T[\varphi^{\#}] = (T_j[\varphi])^{\#}$ and so in particular $(T_j[\varphi])^{\#} = (T_k[\varphi])^{\#}$ for any $1 \leq j,k \leq n$.*

2.3. *Outline of the argument.* We now expand the discussion in Section 1.3 and describe the main ideas involved in the construction of relative fundamental solutions for the operators $\{\Box_J\}$. Let W_j denote either Z_j or $\bar{Z}_j$ where W_j is then a first order differential operator on M which depends only on the variables z_j and t, and which commutes with translation in t. Let $\Box_j = W_j^*\, W_j = -\overline{W}_j\, W_j$ be the corresponding nonnegative, self-adjoint second order differential operator on $L^2(M)$. Let $\mathcal{S}_j$ be the orthogonal projection of $L^2(M)$ onto the null space of $\Box_j$. Note that this space is the same as the null space of W_j. Let $\{e^{-s\Box_j}\}$ be the semi-group of contractions on $L^2(M)$ with infinitesimal generator $\Box_j$. Then $\Box = \sum_{j=1}^n \Box_j$ is one of the operators $\Box_J$.

Let M_j be the boundary in $\mathbb{C}^2$ given in equation (2.1.2). Then $\Box_j$ and $e^{-s\Box_j}$ also act on $L^2(M_j)$. From the theory of domains of finite type in $\mathbb{C}^2$, it is known that the projection $\mathcal{S}_j$ of $L^2(M_j)$ onto the null space of $\Box_j$ and the heat kernel $\mathcal{H}_j$ for the semi-group $\{e^{-s\Box_j}\}$ are given by operators

$$\begin{aligned}\mathcal{S}_j[f](z_j,t) &= \int_{\mathbb{C}\times\mathbb{R}} f(w_j,r)S_j(z_j,w_j,t-r)\,dw_j\,dr,\\ \mathcal{H}_j[f](s,z_j,t) &= \int_{\mathbb{C}\times\mathbb{R}} f(w_j,r)H_j(s,z_j,w_j,t-r)\,dw_j\,dr,\end{aligned}$$

where $S(z,w,t)$ and $H_j(s,z,w,t)$ are distributions on $\mathbb{C}\times\mathbb{C}\times\mathbb{R}$ and on $(0,\infty)\times\mathbb{C}\times\mathbb{C}\times\mathbb{R}$. We can think of the projection onto the null space of $\Box_j$ and the heat kernel $e^{-s\Box_j}$ as operators either on M_j or on the decoupled boundary M. In other words, when thinking of these operators acting on M we can also write

$$\begin{aligned}\mathcal{S}_j[f](z_1,\dots,z_n,t) &= \int_{\mathbb{C}\times\mathbb{R}} f(z_1,\dots,w_j,\dots,z_n,r)\,S_j(z_j,w_j,t-r)\,dw_j\,dr;\\ \mathcal{H}_j[f](z_1,\dots,z_n,t) &= \int_{\mathbb{C}\times\mathbb{R}} f(z_1,\dots,w_j,\dots,z_n,r)\,H_j(s,z_j,w_j,t-r)\,dw_j\,dr.\end{aligned}$$

We remark that the distribution kernels S_j and H_j are the limits of certain distributions S_j^ε and H_j^ε which are given by integration against infinitely differentiable functions with compact support. All the estimates we shall make on S_j and H_j hold for the approximating kernels, and the estimates are uniform in ε.

Now define an operator $\mathcal{S}$ on $L^2(M)$ by setting

$$\mathcal{S} = \prod_{j=1}^n \mathcal{S}_j.$$

This is the orthogonal projection onto the intersection of the null spaces of the operators $\{\Box_1,\dots,\Box_n\}$, which is the same as the projection onto the null space of the operator $\Box = \Box_1+\cdots+\Box_n$. Also note that since the operators $\{\Box_j\}$ commute, the operator $\mathcal{H} = \exp\big[-s\sum_{j=1}^n \Box_j\big]$ is just a product:

$$\mathcal{H} = e^{-s\,\left[\sum_{j=1}^n \Box_j\right]} = \prod_{j=1}^n e^{-s\,\Box_j} = \prod_{j=1}^n \mathcal{H}_j.$$

It thus follows that the operators $\mathcal{S}$ and $\mathcal{H}$ are then given on M by integral operators

$$\mathcal{S}[f](z,t) = \int_{\mathbb{C}^n\times\mathbb{R}} f(w,r)\, S(z,w,t-r)\, dw\, dr$$
$$e^{-s\Box}[f](z,t) = \int_{\mathbb{C}^n\times\mathbb{R}} f(w,r)\, H(s,z,w,t-r)\, dw\, dr$$

where the kernels $S(z,w,t)$ and $H(s,z,w,t)$ are given by the convolutions

$$S(z,w,t) = \int_{\Sigma(t)} \prod_{j=1}^{n} S_j(z_j,w_j,r_j)\, d\tilde{r};$$
$$H(s,z,w,t) = \int_{\Sigma(t)} \prod_{j=1}^{n} H_j(s,z_j,w_j,r_j)\, d\tilde{r}.$$

Here $\Sigma(t)$ and $d\tilde{r}$ are defined in equation (2.1.5).

Now by the spectral theorem, the operator $e^{-s\left[\sum_{j=1}^n \Box_j\right]}$ converges strongly to $\mathcal{S}$ as $s\to\infty$, and hence it will be easy to check that

$$\mathcal{K} = \int_0^\infty \left[e^{-s\left[\sum_{j=1}^n \Box_j\right]} - \mathcal{S}\right] ds$$

is a relative fundamental solution for $\Box = \sum_j \Box_j$ in the sense that

$$\mathcal{K}\,\Box = \Box\,\mathcal{K} = I - \mathcal{S}.$$

Also, if we set

$$\mathcal{N} = \int_0^\infty \prod_{j=1}^{n} \left(e^{-s\Box_j} - \mathcal{S}_j\right) ds,$$

then $\mathcal{N}$ is also a relative fundamental solution for $\Box$ in the sense that

$$\mathcal{N}\,\Box = \Box\,\mathcal{N} = \prod_{j=1}^{n}(I - \mathcal{S}_j).$$

Since the distribution kernels $\{S_j(z_j,w_j,t)\}$ and $\{H_j(s,z_j,w_j,t)\}$ are known, we have the following explicit formulas for the distribution kernels K and N for relative fundamental solutions for $\Box$ on M:

$$K(z,w,t) = \int_0^\infty \int_{\Sigma(t)} \Big[\prod_{j=1}^{n} H_j(s,z_j,w_j,r_j) - \prod_{j=1}^{n} S_j(z_j,w_j,r_j)\Big]\, d\tilde{r}\, ds,$$
$$N(z,w,t) = \int_0^\infty \int_{\Sigma(t)} \prod_{j=1}^{n} \Big(H_j(s,z_j,w_j,r_j) - S_j(z_j,w_j,r_j)\Big)\, d\tilde{r}\, ds.$$

There is one further idea in analyzing these last integrals. We write

$$\widetilde{K}(z,w,r) = \int_0^\infty \Big[\prod_{j=1}^n H_j(s,z_j,w_j,r_j) - \prod_{j=1}^n S_j(z_j,w_j,r_j)\Big]\,ds,$$
$$\widetilde{N}(z,w,r) = \int_0^\infty \prod_{j=1}^n \Big(H_j(s,z_j,w_j,r_j) - S_j(z_j,w_j,r_j)\Big)\,ds.$$

These are the kernels of operators which are relative fundamental solutions for the operator $\sum_{j=1}^n \Box_j$ acting not on M, but on the Cartesian product $\widetilde{M} = M_1 \times \cdots \times M_n$. Then, at least formally, we have

$$K(z,w,t) = \int_{\Sigma(t)} \widetilde{K}(z,w,r)\,d\tilde{r} = (\widetilde{K})^\#(z,w,t)$$
$$N(z,w,t) = \int_{\Sigma(t)} \widetilde{N}(z,w,r)\,d\tilde{r} = (\widetilde{N})^\#(z,w,t).$$

We shall first analyze these relative fundamental solutions $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{N}}$ on the product, and then integrate over $\Sigma(t)$ to obtain the relative fundamental solution on M. By doing this, we can take advantage of the product structure of the operators in the integrand, and in fact use product theory to establish L^p regularity. Transference methods show that integration over $\Sigma(t)$ then gives the same L^p regularity for $\mathcal{K}$.

2.4. *Statement of results.* We summarize our main results about relative inverses to the operators $\Box_J$ on the decoupled manifold M. The Kohn-Laplacian operator $\Box_b$ has an infinite dimensional null space in $L^2(M)$ when acting on $(0,0)$-forms or on $(0,n)$-forms, but has no null space in $L^2(M)_{(0,r)}$ when acting on $(0,r)$-forms for $1 \le r \le n-1$. Let $\mathcal{S}_0$ denote the orthogonal projection of $L^2(M)$ onto the null space of $\Box_b$ acting on functions, and let $\mathcal{S}_n$ denote the orthogonal projection of $L^2(M)_{(0,n)}$ onto the null space of $\Box_b$ acting on $(0,n)$-forms. We show that each $\Box_J$ has an inverse modulo the relevant projection.

Theorem 2.4.1. *For each of the 2^n possible operators $\{\Box_J\}$, we construct a distribution K_J on $M \times M$ so that if $\mathcal{K}_J$ denotes the linear operator*[1]

$$\mathcal{K}_J[\varphi](p) = \int_M \varphi(q)\,K_J(p,q)\,dq,$$

[1]By abuse of notation, this means that $\mathcal{K}_J[\varphi]$ is the distribution on M such that $\langle \mathcal{K}_J[\varphi],\psi\rangle = \langle K_j, \varphi\otimes\psi\rangle$ for $\varphi,\psi \in \mathcal{C}_0^\infty(M)$.

then

$$\mathcal{K}_J \,\Box_J = \Box_J \,\mathcal{K}_J = \begin{cases} I - S_0 & \text{if } \Box_J \text{ acts on functions;} \\ I & \text{if } \Box_J \text{ acts on a } (0,r)\text{-form with } 1 \le r \le n-1; \\ I - S_n & \text{if } \Box_J \text{ acts on } (0,n)\text{-forms.} \end{cases}$$

We study the regularity of the operators $\mathcal{K}_J$ in both L^p and Hölder spaces on M. Since $\mathcal{K}_J$ inverts a second order operator (modulo bounded projections), one expects that $\mathcal{K}_J$ should behave like an operator smoothing of order two. However as we have already pointed out in Section 1.1, Derridj [Der78] showed that maximal hypoelliptic estimates are possible only if the eigenvalues of the Levi form degenerate at the same rate. In particular, for decoupled domains in $\mathbb{C}^{n+1}$ with $n > 1$, the operator $\Box_b$ on M fails to be maximally subelliptic near $p \in M$ whenever $\triangle P_j(p) = 0$ for some j. Thus $Q(Z, \bar Z)\,\mathcal{K}_J$ cannot be a bounded operator on $L^2(M)$ for an arbitrary quadratic combination of "good" derivatives $\{Z_1, \dots, Z_n, \bar Z_1, \dots, \bar Z_n\}$.

THEOREM 2.4.2. *Write $\Box_J = W_1 \bar W_1 + \cdots + W_n \bar W_n$ where each W_j is one of the two operators $\{Z_j, \bar Z_j\}$, and $\bar W_j$ is the other. Let $\mathcal{K}_J$ be the distribution constructed in Theorem* 2.4.1. *Then*:

(1) *When $1 \le k, l \le n$ with $l \ne k$, the operators $W_k \bar W_k \mathcal{K}_J = -\Box_k \mathcal{K}_J$, $\bar W_k \bar W_k \mathcal{K}_J$, and $\bar W_l \bar W_k \mathcal{K}_J$ extend to bounded linear operators on $L^p(M)$ for $1 < p < \infty$.*

(2) *If B_k is a bounded smooth function on M and if there are constants $C_{k,l}$ so that for all $p = (z_1, \dots, z_n, t) \in M$*

$$|B_k(p)| \,\triangle P_k(z_k) \le C_{k,l} \,\triangle P_l(z_l)$$

for $1 \le k \le n$ and all l, then the operator $B_k \bar W_k W_k \mathcal{K}_J = -B \,\overline{\Box}_k \mathcal{K}_J$ extends to a bounded linear operator on $L^p(M)$ for $1 < p < \infty$.

(3) *If B_k is a bounded smooth function on M and if there are constants C_k so that for all $p = (z_1, \dots, z_n, t) \in M$*

$$|B_k(p)| \le C_k \inf_{l \ne k} \triangle P_l(z_l)$$

then the operator $B_k W_k W_k \mathcal{K}_J$ extends to a bounded linear operator on $L^p(M)$ for $1 < p < \infty$.

Note that this theorem does not make any assertion about operators of the form $W_l \bar W_k \mathcal{K}_J$. Theorems 2.4.1 and 2.4.2 will be proved in Section 5.3 and Section 5.4 below. That the estimates are optimal is shown in Section 9.

Precise Lipschitz regularity results require the introduction of various metrics on the space M. This is done below in Section 8. At this stage, however, we can state the following global result involving the standard isotropic Lipschitz spaces.

THEOREM 2.4.3. *Let $m = \max\{m_j\}$ be the largest of the degrees of the polynomials P_j. (This is the "type" of the boundary M.) Assume $m > 2$, and suppose that f is a function bounded and supported on a ball of radius one in M. Then for all J there is a constant C so that if $h \in \mathbb{C}^n \times \mathbb{R} \cong M$, then*

$$|\mathcal{K}_J[f](p+h) - \mathcal{K}_J[f](p)| \leq C\,|h|^{\frac{2}{m}}.$$

There is also a corresponding result when $m = 2$ (the strongly pseudo-convex case). These results are presented in Section 8.

We shall show that the distributions K_J are singular only on the diagonal of $M \times M$, and we obtain estimates on the size of these distributions and their derivatives away from these singularities. These estimates involve two different pseudo-metrics on M. The first, which we call d_Σ, is the control metric associated to the collection of vector fields which are the real and imaginary parts of the vector fields $\{\bar{Z}_1, \dots, \bar{Z}_n\}$. The second pseudo-metric, which we call d_S, describes the singularities of the singular integral which gives the Szegö projections S_0 and S_n. The corresponding balls are denoted by $B_\Sigma(p,\delta)$ and $B_S(p,\delta)$. These metrics are described in Section 6. The estimates of K_J also involve functions $\mu_j(p,\delta)$ which are defined in Section 3.1, equation (3.1.12) below.

THEOREM 2.4.4. *The distribution $K_J(p,q)$ is given by integration against a $\mathcal{C}^\infty$ function away from the diagonal $\{p = q \in M\}$, where there are the following estimates. Let $\partial_j^{\alpha_j}$ be a derivative of order $|\alpha_j|$ made up of the vector fields Z_j and $\bar{Z}_j$ in which each acts in either the variables p_j or q_j. Then for all $\alpha = (\alpha_1, \dots, \alpha_n)$ there is a constant $C_\alpha = C$ so that*

$$\begin{aligned}
&\Big|\Big[\prod_{j=1}^{n} \partial_j^{\alpha_j}\Big] K_J(p,q)\Big| \\
&\qquad \leq C \frac{\Big[\sum_{j=1}^n \mu_j(p, d_S(p,q))\Big]^2}{\Big|B_S(p, d_S(p,q))\Big|} \log\left[2 + \frac{\Big[\sum_{j=1}^n \mu_j(p, d_S(p,q))\Big]}{d_\Sigma(p,q)}\right] \\
&\qquad\qquad \prod_{j=1}^{n} \left[\mu_j(p, d_S(p,q))^{-1} + d_\Sigma(p,q)^{-1}\right]^{|\alpha_j|}.
\end{aligned}$$

3. Geometry and analysis on M_j and on $M_1 \times \cdots \times M_n$

In this section we summarize some geometric and analytic results which we require later in the paper. For $1 \leq j \leq n$, let M_j be the hypersurface given in equation (2.1.2). Let $\widetilde{M}$ be the Cartesian product as in (2.1.3). Subsections

3.1 through 3.3 deal with the study of the model hypersurfaces M_j in $\mathbb{C}^2$. Subsections 3.4 and 3.5 deal with geometry and analysis on $\widetilde{M}$.

3.1. *The control metric on M_j.* Recall that we write the complex vector field $\bar{Z}_j = X_j + iX_{n+j}$ where $\{X_j, X_{n+j}\}$ are real vector fields on M_j. Define a metric d_j on M_j as follows. If $p, q \in M_j$ and $\delta > 0$, let $AC(p, q, \delta)$ denote the set of absolutely continuous mappings $\gamma : [0, 1] \to M_j$ such that $\gamma(0) = p$ and $\gamma(1) = q$, and such that for almost all $t \in [0, 1]$ we have $\gamma'(t) = \alpha_j(t)\, X_j(\gamma(t)) + \alpha_{n+j}(t)\, X_{n+j}(\gamma(t))$ with $|\alpha_j(t)|^2 + |\alpha_{n+j}(t)|^2 < \delta^2$. Then we define

$$d_j(p, q) = \inf \left\{ \delta > 0 \,\middle|\, AC(p, q, \delta) \neq \emptyset \right\}.$$

The corresponding nonisotropic ball is

$$B_j(p, \delta) = \left\{ q \in M_j \,\middle|\, d_j(p, q) < \delta \right\},$$

and $\left|B_j(p, \delta)\right|$ denotes its volume. Set

$$V_j(p, q) = \left|B_j\big(p, d_j(p, q)\big)\right|.$$

The volume of the ball $B(p, \delta)$ is essentially a polynomial in δ with coefficients that depend on p. Let $T = \frac{\partial}{\partial t}$ so that at each point of M_j the tangent space is spanned by the vectors $\{X_j, X_{n+j}, T\}$. Write the commutator

$$[X_j, X_{n+j}] = \lambda_j\, T + a_j\, X_j + a_{n+j}\, X_{n+j} \tag{3.1.11}$$

where $\lambda_j, a_j, a_{n+j} \in \mathcal{C}^\infty(M_j)$. If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a k-tuple with each α_j equal to j or $n + j$, let $|\alpha| = k$ and let $X^\alpha = X_{\alpha_1} \cdots X_{\alpha_k}$ denote the corresponding k^{th} order differential operator. For $k \geq 2$ set

$$\Lambda_j^k(p) = \sum_{|\alpha| \leq k-2} |X^\alpha \lambda_j(p)|,$$

where λ_j is as defined in (3.1.11), and set

$$\Lambda_j(p, \delta) = \sum_{k=2}^{m_j} \Lambda_j^k(p)\, |\delta|^k.$$

We now have

Proposition 3.1.1. *There are constants C_1, C_2 depending only on m_j so that for $p \in M_j$ and $\delta > 0$*

$$C_1\, \delta^2\, \Lambda_j(p, \delta) \leq \left|B_j(p, \delta)\right| \leq C_2\, \delta^2\, \Lambda_j(p, \delta).$$

Also, $V_j(p, q) \approx V_j(q, p) \approx d_j(p, q)^2\, \Lambda_j\left(p, d_j(p, q)\right)$ where $A \approx B$ means that the ratio A/B is bounded and bounded away from zero.

There is an alternate description of the balls $\{B_j(p,\delta)\}$ and metric d_j given in terms of explicit inequalities. For $z, w \in \mathbb{C}$ let

$$T_j(w,z) = 2\,\Im\mathrm{m}\Big[\sum_{k=1}^{m_j} \frac{\partial^k P_j}{\partial z^k}(w)\,\frac{(z-w)^k}{k!}\Big].$$

Then with $p = (w,s) \in M_j$, set

$$\widetilde{B}_j(p,\delta) = \Big\{(z,t) \in M_j \,\Big|\, |z-w| < \delta \text{ and } |t - s + T_j(w,z)| < \Lambda_j(w,\delta)\Big\}.$$

Note that for $\delta > 0$, $\delta \to \Lambda_j(p,\delta)$ is a monotone increasing function. Hence there is a unique inverse function $\mu_j(p,\delta)$ such that for $\delta \geq 0$ we have $\Lambda_j\big(p,\mu_j(p,\delta)\big) = \mu_j\big(p,\Lambda_j(p,\delta)\big) = \delta$. We have

$$\mu_j(p,\delta)^{-1} \approx \sum_{k=2}^{m_j} \Lambda_j^k(p)^{\frac{1}{k}}\,|\delta|^{-\frac{1}{k}}. \tag{3.1.12}$$

PROPOSITION 3.1.2. *There are constants C_1 and C_2 depending only on m_j so that for $p \in M_j$ and $\delta > 0$*

$$\widetilde{B}_j(p, C_1\,\delta) \subset B_j(p,\delta) \subset \widetilde{B}_j(p, C_2, \delta).$$

Moreover, if $(z,t), (w,s) \in M_j$,

$$d_j\big((z,t),(w,s)\big) \approx |z-w| + \mu_j\big(w, \big|t - s - T_j(w,z)\big|\big).$$

Proofs of Propositions 3.1.1 and 3.1.2 can be found in [NSW85].

We say that the manifold M_j is normalized at a point (z,t) if $\frac{\partial^k P_j}{\partial z^k}(z) = 0$ for $1 \leq k \leq m_j$. If $p = (a, s + iP_j(a)) \in M_j \subset \mathbb{C}^2$, one can always make a biholomorphic change of variables of $\mathbb{C}^2$ which moves p to the origin, and which carries the manifold M_j to a new manifold M_j^p of the same type which is normalized at the origin. (See Subsection 6.1 below for further details.) It follows from Proposition 3.1.2 that if the domain M_j is normalized at the origin, then the balls and distances have the particularly simple form:

$$\widetilde{B}_j((0,0),\delta) = \Big\{(z,t) \in \mathbb{C}\times\mathbb{R} \,\Big|\, |z| < \delta \text{ and } |t| < \Lambda_j(0,\delta)\Big\};$$
$$d_j\big((z,t),(0,0)\big) \approx |z| + \mu_j(0,t).$$

3.2. NIS *operators on M_j.* We briefly review the definition of the class of nonisotropic smoothing (NIS) operators on M_j. These were introduced in [NRSW89], and the definition below is taken from Koenig [Koe02]. Let

$$\mathcal{T}[f](p) = \int_M T(p,q)\,f(q)\,dq$$

where T is a distribution on $M_j \times M_j$.

Definition 3.2.1. $\mathcal{T}$ is an NIS operator smoothing of order k if the distribution T is given away from the diagonal of $M_j \times M_j$ by integration against a $\mathcal{C}^\infty$ function and:

(1) There exists $\beta < \infty$, and for $s \geq 0$ there exists $\alpha(s) < \infty$ such that if $\zeta, \zeta' \in \mathcal{C}_0^\infty(M_j)$ with $\zeta' \equiv 1$ on the support of ζ, then there exists a constant $C_{s,\zeta,\zeta'}$ so that for all $f \in \mathcal{C}_0^\infty(M_j)$

$$\big|\big|\zeta\, \mathcal{T}[f]\big|\big|_s \leq C_{s,\zeta,\zeta'} \left[\big|\big|\zeta' f\big|\big|_{\alpha(s)} + \big|\big|f\big|\big|_\beta\right].$$

(2) Let X_p^α and X_q^β be derivatives of order $|\alpha|$ and $|\beta|$ in the vector fields X_j and X_{n+j} acting on the variables p and q. There exist constants $C_{\alpha,\beta}$ so that for $p \neq q$

$$\big|X_p^\alpha\, X_q^\beta T(p,q)\big| \leq C_{\alpha,\beta}\, d_j(p,q)^{k-|\alpha|-|\beta|}\, V_j(p,q)^{-1}.$$

(3) For each integer ℓ there are an integer N_ℓ and a constant C_ℓ so that if φ is a $\mathcal{C}^\infty$ function supported on $B_j(p,\delta)$, then for all $\varepsilon > 0$ and all α with $|\alpha| = \ell$

$$\big|X^\alpha \mathcal{T}[\varphi](p)\big| \leq C_\ell\, \delta^{k-\ell} \sup_{q\in M} \sum_{|J|\leq N_\ell} \delta^{|J|}\big|X^J\varphi(q)\big|.$$

(4) The above conditions also hold for the adjoint operator $\mathcal{T}^*$ with distribution kernel $T(y,x)$.

The constants $C_{s,\zeta,\zeta'}$, $C_{\alpha,\beta}$ and C_ℓ are called the NIS constants of the operator T.

Definition 3.2.2. With regard to condition (3), if φ is a smooth function supported on $B_j(p,\delta)$, we say that φ is *a normalized bump function* if

$$\sup_{q\in M} \sum_{|J|\leq N_\ell} \delta^{|J|}\big|X^J\varphi(q)\big| \leq 1.$$

3.3. *The $\bar\partial_b$-complex on M_j.* We shall need the following basic results concerning the $\bar\partial_b$ complex on M_j. Let Z_j and $\bar Z_j$ be the tangential Cauchy-Riemann operators of type $(1,0)$ and $(0,1)$ on M_j as given in equation (2.2.6). Then $\bar\partial_b[f] = \bar Z_j[f]\, d\bar z$, and the formal adjoint is $\bar\partial_b^*[g\, d\bar z] = -Z_j[g]$. We have operators $\Box_j^{(\pm)}$ as in equation (2.2.8). The Kohn-Laplacian is then $\Box_j^{(-)}$ when acting on functions, and is $\Box_j^{(+)}$ when acting on $(0,1)$-forms. Each operator $\Box_j^{\pm}$ extends to a closed, densely defined, nonnegative self-adjoint operator on $L^2(M_j)$. Let $\mathcal{S}_j^{(\pm)}$ denote the orthogonal projection of $L^2(M_j)$ onto the null space of $\Box_j^{(\pm)}$. The operator $\mathcal{S}_j^{(\pm)}$ is induced by a distribution $S_j^{(\pm)}(p,q)$ on $M_j \times M_j$ which is given away from the diagonal by integration against a $\mathcal{C}^\infty$

function. Let $e^{-s\Box_j^{(\pm)}}$ denote the semi-group generated by $\Box_j^{(\pm)}$. For each $s > 0$, there is a distribution heat kernel $H_j^{(\pm)}(s,p,q)$ on $M_j \times M_j$ such that

$$e^{-s\Box_j^{(\pm)}}[f](p) = \int_{M_j} H_j^{(\pm)}(s,p,q)\, f(q)\, dq.$$

The analysis of $\Box_j^{(-)}$ and $\Box_j^{(+)}$ is similar, and from now on we shall omit the superscript. Thus $\Box_j$ will stand for either $\Box_j^{(-)}$ or $\Box_j^{(+)}$, $\mathcal{S}_j$ will denote the projection onto its null space, and $H_j(s,p,q)$ will denote the corresponding heat kernel.

THEOREM 3.3.1. *The operator $\mathcal{S}_j$ is an* NIS *operator on M_j smoothing of order zero. Moreover, there is an* NIS *operators $\mathcal{K}_j$ smoothing of order two such that*

$$\mathcal{K}_j\,\Box_j = \Box_j\,\mathcal{K}_j = I - \mathcal{S}_j.$$

The distribution kernel of the operator $\mathcal{K}_j$ is related to the corresponding heat kernel H_j and the projection S_j by the formula

$$K_j(p,q) = \int_0^\infty H_j(s,p,q) - S_j(p,q)\, ds.$$

Proofs can be found in [Chr88], [Chr91b], [Chr91a], [CNS92], [NRSW89], and [NS01a].

THEOREM 3.3.2. *Let $B_j\big(z_j, w_j, \zeta\big)$ be the Bergman kernel for the domain Ω_j. Then the Szegö kernel is given by*

$$S_j(z_j, w_j, t) = \int_0^\infty B_j(z_j, w_j, t + ir)\, dr.$$

If

$$S_j^\varepsilon(z_j, w_j, t) = \int_\varepsilon^\infty B_j(z_j, w_j, t + ir)\, dr$$

then $S_j^\varepsilon \to S_j$ as $\varepsilon \to 0$, and for $\varepsilon > 0$, the kernels S_j^ε are smooth bounded functions on $M_j \times M_j$.

Further discussion, and the relevant estimates for the Bergman kernels, can be found in [NRSW89].

THEOREM 3.3.3. *There is a function $G_j \in \mathcal{C}^\infty\big((0,\infty) \times M_j \times M_j\big)$ so that*

$$H_j(s,p,q) = G_j(s,p,q) + S_j(p,q)$$

where $S_j(p,q)$ is the distribution kernel for the orthogonal projection operator $\mathcal{S}_j$. In particular, the distribution $H_j(s,p,q)$ is given by integration against a $\mathcal{C}^\infty$ function away from the diagonal. There is a constant $C_{\alpha,l}$ with the *following property. Let $\partial_{p,q}^{\alpha}$ denote a differentiation of total order $|\alpha|$ in Z_j and $\overline{Z}_j$, acting either on the p or q variables. Then for $p \neq q$*

$$\left|\partial_s^l\,\partial_{p,q}^{\alpha} G_j(s,p,q)\right| \le \begin{cases} C_{\alpha,l}\,d_j(p,q)^{-2l-|\alpha|}\,V_j(p,q)^{-1} & \text{if } s \le d_j(p,q)^2 \\ \\ C_{\alpha,l}\,s^{-l-\frac{1}{2}|\alpha|}\left|B_j(p,\sqrt{s})\right|^{-1} & \text{if } s \ge d_j(p,q)^2. \end{cases}$$

Moreover, for every nonnegative integer N there is a constant $C_{N,\alpha,l}$ so that

$$\left|\partial_s^l\,\partial_{p,q}^{\alpha} H_j(s,p,q)\right| \le C_{N,\alpha,l}\,\frac{d_{M_j}(p,q)^{-2l-|\alpha|}}{V_j(p,q)}\left[\frac{s^N}{s^N + d_j(p,q)^{2N}}\right].$$

Remark. It follows from the estimates on G_j and $|B_j(p,\sqrt{s})|$ that for $s \ge d_j(p_j,q_j)^2$,

$$\left|G_j(s,p_j,q_j)\right| \lesssim \left[\left|\lambda_j(p_j)\right| s^2 + s^{\frac{1}{2}m_j+1}\right]^{-1}. \tag{3.3.13}$$

In particular, each function $G_j(s,p_j,q_j)$ is integrable in s at infinity.

Theorem 3.3.4. *The operators*

$$H_{j,s}[f](p) = \int_{M_j} H_j(s,p,q)\,f(q)\,dq,$$
$$G_{j,s}[f](p) = \int_{M_j} G_j(s,p,q)\,f(q)\,dq$$

are NIS *operators smoothing of order zero, and the associated* NIS *constants are uniformly bounded for $s > 0$.*

Proofs of Theorems 3.3.3 and 3.3.4 can be found in [NS01a].

3.4. *Geometry on Cartesian products.* For $1 \le j \le n$, let M_j be a hypersurface in $\mathbb{C}^2$ as defined in equation (2.1.2), and let $\widetilde{M} = M_1 \times \cdots \times M_n$. Each of the nonisotropic distances d_j on M_j can be regarded as a function on $\widetilde{M}$ which depends only on the variables (z_j,t_j).

In addition, there is a nonisotropic metric d_Σ on $\widetilde{M}$ induced by all the real vector fields $\{X_1,\ldots,X_{2n}\}$. If $p,q \in M_j$ and $\delta > 0$, let $AC(p,q,\delta)$ denote the set of absolutely continuous mappings $\gamma : [0,1] \to \widetilde{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$, and such that for almost all $t \in [0,1]$ we have $\gamma'(t) = \sum_{j=1}^{2n} \alpha_j(t)\,X_j(\gamma(t))$ with $\sum_{j=1}^{2n} |\alpha_j(t)|^2 < \delta^2$. Then

$$d_\Sigma(p,q) = \inf\left\{\delta > 0 \,\middle|\, \mathrm{AC}(p,q,\delta) \neq \emptyset\right\}.$$

This metric is appropriate for describing the fundamental solution of the operator $\mathcal{L} = \sum_{j=1}^{2n} X_j^2$, and we refer to it as the *sum of squares* metric. (See [NSW85] for a discussion of the relationship between the metric and the operator $\mathcal{L}$.) The metric d_Σ can be explicitly described as follows. Let $p = (z_1, t_1, \dots, z_n, t_n) \in \widetilde{M}$. We can assume without loss of generality that each manifold M_j is normalized at the origin. We denote the origin of $\widetilde{M}$ by $\bar{0}$. Then

$$d_\Sigma(\bar{0}, p) \approx \sum_{j=1}^{n} [|z_j| + \mu_j(0, |t_j|)].$$

The ball centered at $\bar{0}$ of radius δ is, up to constants, given by

$$B_\Sigma(\bar{0}, \delta) = \Big\{(z,t) \in \widetilde{M} \,\Big|\, |z_j| < \delta \text{ and } |t_j| < \Lambda_j(0,\delta) \text{ for } 1 \le j \le n\Big\}.$$

We have

$$\big|B_\Sigma\,(\bar{0}, \delta)\,\big| \approx \delta^{2n} \prod_{j=1}^{n} \Lambda_j(0, \delta),$$

and

$$\big|B_\Sigma\big(\bar{0}, d_\Sigma(z,t)\big)\big| \approx \big[\sum_{j=1}^{n} |z_j| + \mu_j(0, |t_j|)\big]^{2n} \prod_{j=1}^{n} \Lambda_j\Big(0, \big[\sum_{j=1}^{n} |z_j| + \mu_j(0, |t_j|)\big]\Big).$$

3.5. *Product singular integral operators on $M_1 \times \cdots \times M_n$.* We introduce a class of operators on $\widetilde{M}$ (product NIS operators) which is the analogue of the standard product singular integrals in the Euclidean setting. (For a discussion of the Euclidean case, see for example [NRS01].) The definition will involve differential inequalities on the distribution kernel, and certain cancellation conditions expressed in terms of the action of the operator on normalized bump functions. These are generalizations of the single-factor NIS operators arising in Section 3.2. Because of the complicated cancellation conditions, it seems easiest to give the definition of product NIS operators on $\widetilde{M}$ by induction on the number n of factors. When $n = 1$, we are in the situation discussed in Section 3.2, and a product singular integral operator will just mean a standard NIS operator smoothing of order zero.

In general, product operators on $\widetilde{M}$ will be induced by distributions which are smooth functions away from the *product diagonal* given by

$$\widetilde{D} = \Big\{\big((p_1, \dots, p_n); (q_1, \dots q_n)\big) \in \widetilde{M} \times \widetilde{M} \,\Big|\, p_j = q_j \text{ for some } 1 \le j \le n\Big\}.$$

Definition 3.5.1. Assume that product singular integral operators have been defined on products where the number of factors is less than n. Let $\widetilde{M}$ be a product of n hypersurfaces M_j. Then $\widetilde{\mathcal{T}}$ is *a product singular integral operator on* $\widetilde{M}$ if $\widetilde{\mathcal{T}}$ is a continuous linear mapping from $\mathcal{C}_0^\infty(\widetilde{M})$ to $\mathcal{D}'(\widetilde{M})$ and if:

(1) The Schwartz kernel $\widetilde{T}(p,q)$ associated to $\widetilde{\mathcal{T}}$ is a distribution which is a $\mathcal{C}^\infty$ function on the set $(\widetilde{M} \times \widetilde{M} \setminus \widetilde{D})$. In particular, if $\varphi_j, \psi_j \in \mathcal{C}_0^\infty(M_j)$

have disjoint supports for $1 \le j \le n$,

$$\Big\langle \widetilde{T}(\varphi_1 \otimes \cdots \otimes \varphi_n),\, \psi_1 \otimes \cdots \otimes \psi_n \Big\rangle = \iint \widetilde{T}(p,q)\Big[\prod_{j=1}^{n} \varphi_1(q_j)\,\psi_1(p_j)\Big]\,dp\,dq.$$

(2) The function $\widetilde{T}$ satisfies the following differential inequalities on the set $(\widetilde{M} \times \widetilde{M} \setminus \widetilde{D})$. For any $(\alpha_1, \dots, \alpha_n)$ there is a constant $C = C_{\alpha_1,\dots,\alpha_n}$ with the following property. Suppose $\partial^{\alpha_j}_{p_j,q_j}$ is a differential operator of total order $|\alpha_j|$ made out of the operators Z_j and $\bar{Z}_j$ acting on either p_j or q_j. Then on $\widetilde{M} \times \widetilde{M} - \widetilde{D}$

$$\Big|\prod_{j=1}^{n} \partial^{\alpha_j}_{p_j,q_j}\ \widetilde{T}(p,q)\Big| \le C\,\prod_{j=1}^{n} d_j(p_j,q_j)^{-|\alpha_j|}\Big[\prod_{j=1}^{n} V_j(p_j,q_j)\Big]^{-1}.$$

(3) For each normalized bump function φ_n supported on a ball of radius δ_n in M_n and for each point $p_n \in M_n$ there is *a product singular integral operator* $\widetilde{T}^{\varphi_n,p_n}$ (of the $(n-1)$ factor type) defined on $M_1 \times \cdots \times M_{n-1}$ so that $p_n \to \widetilde{T}^{\varphi_n,p_n}$ is smooth and

$$\begin{aligned}&\big\langle \widetilde{T}(\varphi_1 \otimes \cdots \otimes \varphi_n), \psi_1 \otimes \cdots \otimes \psi_n \big\rangle \\ &\qquad = \int_{M_n} \big\langle \widetilde{T}^{\varphi_n,p_n}(\varphi_1 \otimes \cdots \otimes \varphi_{n-1}), (\psi_1 \otimes \cdots \otimes \psi_{n-1})\big\rangle\,\psi_n(p_n)\,dp_n.\end{aligned}$$

Moreover, the operator $\widetilde{T}^{\varphi_n,p_n}$ satisfies all the conditions for product singular integrals with $(n-1)$ factors, uniformly in φ_n and p_n, as do all operators $\delta_n^k\,\partial^k_{p_n}\widetilde{T}^{\varphi_n,p_n}$. Here $\partial^k_{p_n}$ is a differential operator of order k made out of Z_n and $\bar{Z}_n$.

(4) Condition (3) holds for any permutation of the indices $\{1, 2, \dots, n\}$.

(5) If for $1 \le j \le n$, φ_j is *a normalized bump function* (in the sense of Definition 3.2.2) supported on a ball $B_j(q_j, \delta_j) \subset M_j$, then

$$\Big|\prod_{j=1}^{n} \partial^{\alpha_j}_{p_j}\widetilde{T}(\varphi_1 \otimes \cdots \otimes \varphi_n)\Big| \lesssim \prod_{j=1}^{n} \delta_j^{-|\alpha_j|}$$

where the constants implied by the symbol $\lesssim$ are uniform.

(6) Properties (1) through (5) are also satisfied for all possible transposes of $\widetilde{T}$; i.e. those operators which arise by interchanging some collection of p_j and q_j.

Remark. If T_j is an NIS operator smoothing of order zero on M_j for $1 \le j \le n$, then the operator $T_1 \otimes \cdots \otimes T_n$ is a product singular integral operator on $\widetilde{M}$.

The main result on product singular integral operators, which is proved in [NS04], is the following.

THEOREM 3.5.2. *Suppose that $\widetilde{\mathcal{T}}$ is a product singular integral operator on $\widetilde{M}$ in the sense of Definition* 3.5.1. *Then $\widetilde{\mathcal{T}}$ is bounded on $L^p(\widetilde{M})$ for $1 < p < +\infty$.*

Remark. In addition to the notion of product singular integral operators defined in Section 3.5, there are also NIS operators smoothing of order zero with respect to the metric d_Σ. Both these families of operators might be called *singular integrals* on $\widetilde{M}$. In fact, NIS operators of order zero relative to the metric d_Σ are also product NIS operators. The corresponding result for Euclidean singular integrals is noted in [NRS01, Remark 2.1.6].

4. Relative fundamental solutions for $\Box_b$ on $M_1 \times \cdots \times M_n$

We now construct two relative fundamental solutions $\widetilde{\mathcal{K}}_J$ and $\widetilde{\mathcal{N}}_J$ for each of the 2^n Kohn-Laplacian operators $\Box_J$ on the product $\widetilde{M}$. Recall that for each increasing tuple J of integers between 1 and n, $\Box_J = \sum_{j=1}^n \Box_j^{J(j)}$. On $\widetilde{M}$ there is essentially no difference between these operators. Thus we shall fix J and abbreviate our notation so that we write $\Box_J = \Box_b = \sum_{j=1}^n \Box_j$ where each $\Box_j$ is either $Z_j \bar{Z}_j$ or $\bar{Z}_j Z_j$. As usual, H_j and S_j are the heat kernel and projection associated to the particular choice of $\Box_j$. We will write the corresponding relative fundamental solutions for $\Box_b$ as $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{N}}$.

4.1. *The relative fundamental solutions $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{N}}$.* Following the discussion in Section 2.3, we make the following definitions.

Definition 4.1.1. For $(p, q) \in \widetilde{M} \times \widetilde{M}$ set

$$\widetilde{K}(p,q) = \int_0^\infty \Big(\prod_{j=1}^n H_j(s, p_j, q_j) - \prod_{j=1}^n S_j(p_j, q_j) \Big)\, ds, \tag{4.1.14}$$

$$\widetilde{N}(p,q) = \int_0^\infty \prod_{j=1}^n \Big(H_j(s, p_j, q_j) - S_j(p_j, q_j) \Big)\, ds. \tag{4.1.15}$$

Also for any proper subset $A \subset \{1, \ldots, n\}$, set

$$\widetilde{N}_A(p,q) = \int_0^\infty \prod_{j \in A} \Big(H_j(s, p_j, q_j) - S_j(p_j, q_j) \Big)\, ds, \tag{4.1.16}$$

$$\widetilde{S}_A(p,q) = \bigotimes_{j \notin A} S_j(p_j, q_j). \tag{4.1.17}$$

If $A = \emptyset$ then $\widetilde{N}_A = 0$, and we set $S = \widetilde{S}_A = \prod_{j=1}^n S_j$. Note that $\widetilde{N}_A$ only involves variables (p_j, q_j) with $j \in A$, and $\widetilde{S}_A$ only involves variables (p_j, q_j) with $j \notin A$.

PROPOSITION 4.1.2. *The integrals* (4.1.14) *through* (4.1.16) *and* $\widetilde{S}_A$ *define distributions on* $\widetilde{M} \times \widetilde{M}$. *For* $\varepsilon > 0$ *and* $\emptyset \neq A \subset \{1, \dots, n\}$, *the integral*

$$\widetilde{N}_{A,\varepsilon}(p,q) = \int_\varepsilon^\infty \prod_{j\in A} \Big(H_j(s,p_j,q_j) - S_j(p_j,q_j)\Big)\, ds$$

converges absolutely to an infinitely differentiable function with bounded derivatives. As distributions, $\widetilde{N}_{A,\varepsilon} \to \widetilde{N}_A$ *as* $\varepsilon \to 0$.

Proof. We have

$$\widetilde{N}_{A,\varepsilon}(p,q) = \int_\varepsilon^\infty \prod_{j\in A} G_j(s,p_j,q_j)\, ds.$$

The estimates of Theorem 3.3.3 and equation (3.3.13) show that $\widetilde{N}_{A,\varepsilon}$ is smooth and bounded with bounded derivatives as long as $\varepsilon > 0$. Also, expanding $\prod_{j\in A}(H_j - S_j)$, we see that

$$\begin{aligned}\widetilde{N}_{A,\varepsilon}(p,q) = &\int_1^\infty \prod_{j\in A} G_j(s,p_j,q_j)\, ds + \int_\varepsilon^1 \prod_{j\in A} H_j(s,p_j,q_j)\, ds \\ &+(-1)^{|A\setminus B|} \sum_{B \subsetneq A} \Big[\int_\varepsilon^1 \prod_{j\in B} H_j(s,p_j,q_j)\, ds\Big] \bigotimes \prod_{j\in A\setminus B} S_j(p_j,q_j).\end{aligned}$$

It follows from Theorem 3.3.4 that $\int_\varepsilon^1 \prod_{j\in B} H_j(s,p_j,q_j)\, ds$ is a family of distributions which converges as $\varepsilon \to 0$. Hence the integral (4.1.16) converges to a distribution. This establishes the statements about the distributions $\widetilde{N}_{A,\varepsilon}$ and $\widetilde{N}_A$. Next $\widetilde{S}_A$ is a distribution since it is a tensor product of distributions. The statement that the integral in equation (4.1.14) defines a distribution follows[2] from the identity

$$\int_\varepsilon^\infty \Big(\prod_{j=1}^n H_j(s,p_j,q_j) - \prod_{j=1}^n S_j(p_j,q_j)\Big)\, ds \\ = \widetilde{N}_\varepsilon(p,q) + \sum_A \widetilde{N}_{A,\varepsilon}(p,q) \otimes \widetilde{\mathcal{S}}_A(p,q).$$

Since the right-hand side defines a distribution which has a limit as $\varepsilon \to 0$, the same is true of the left-hand side. This completes the proof. □

[2]This is a consequence of the Taylor expansion

$$\prod_{j=1}^n x_j = \prod_{j=1}^n y_j + \sum_{A \subsetneq \{1,\dots,n\}} \big[\prod_{j\in A} y_j \prod_{k\notin A} (x_k - y_k)\big].$$

LEMMA 4.1.3. *The distributions* $\widetilde{K}$, $\widetilde{N}$, *and* $\widetilde{N}_A$ *induce operators* $\widetilde{\mathcal{K}}$, $\widetilde{\mathcal{N}}$, *and* $\widetilde{\mathcal{N}}_A$ *on* $\widetilde{M}$ *which satisfy*

$$\widetilde{\mathcal{K}} = \widetilde{\mathcal{N}} + \sum_A \widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A \tag{4.1.18}$$

where the sum is take over all proper, nonempty subsets A *of* $\{1, \dots, n\}$*. Moreover,* $\widetilde{\mathcal{K}}$ *and* $\widetilde{\mathcal{N}}$ *are relative fundamental solutions of* $\Box_b$ *in the sense that*

$$\widetilde{\mathcal{K}}\,\Box_b = \Box_b\,\widetilde{\mathcal{K}} = I - \bigotimes_{j=1}^{n} \mathcal{S}_j; \tag{4.1.19}$$

$$\widetilde{\mathcal{N}}\,\Box_b = \Box_b\,\widetilde{\mathcal{N}} = \bigotimes_{j=1}^{n} (I - \mathcal{S}_j)\,. \tag{4.1.20}$$

Proof. The identity (4.1.18) is again just a consequence of the formula for the Taylor expansion for the function $\prod_{j=1}^{n} x_j$. To check identity (4.1.19), note that $\Box_j\,\mathcal{S}_j \equiv 0$, and $\Box_j\,H_j(s, p_j, q_j) = -\frac{\partial H_j}{\partial s}(s, p_j, q_j)$. Thus, working with distributions, we have

$$(\sum_{j=1}^{n} \Box_j)\,K(p,q) = -\int_0^\infty \frac{\partial(\prod_{j=1}^{n} H_j)}{\partial s}(s,p,q)\,ds = \prod_{j=1}^{n} \delta(p_j, q_j) - \prod_{j=1}^{n} S_j(p_j, q_j).$$

A similar argument gives identity (4.1.20). □

Remark. Note that $\widetilde{\mathcal{K}}$ inverts $\Box_b$ modulo the projection onto the intersection of the null spaces of the operators $\{\Box_j\}$ which is just the null space of $\Box_b$. On the other hand, $\widetilde{\mathcal{N}}$ inverts $\Box_b$ modulo the (much larger) subspace which is the direct sum of the null space of the operators $\{\Box_j\}$. Thus the operator $\widetilde{\mathcal{K}}$ is the *natural* relative fundamental solution which inverts the operator except on the smallest possible subspace. On the other hand, as we shall see, the operator $\widetilde{\mathcal{N}}$ has better regularity properties than the operator $\widetilde{\mathcal{K}}$. The identity (4.1.18) in Lemma 4.1.3 provides the link between the two.

4.2. *Analysis of the distributions* $\widetilde{N}$*.* In this section we show that we have global maximal hypoelliptic estimates for the relative fundamental solution $\widetilde{\mathcal{N}}$ by showing that any two good derivatives of $\widetilde{N}$ give a product singular integral in the sense of Definition 3.5.1. It then follows from Theorem 3.5.2 that such operators are bounded on $L^p(\widetilde{M})$ for $1 < p < \infty$.

THEOREM 4.2.1. *Let* $Q(Z, \bar{Z})$ *be any quadratic expression in the vector fields* $\{Z_1, \bar{Z}_1, \dots, Z_n, \bar{Z}_n\}$*. Then* $Q(Z, \bar{Z})\widetilde{\mathcal{N}}$ *is a product singular integral operator on* $\widetilde{M}$ *and consequently is bounded on* $L^p(\widetilde{M})$ *for all* $1 < p < +\infty$*. In particular, there is a constant* $C = C_{p,Q}$ *so that if* $\Box_b[u] = g$ *and if*

$\bigotimes_{j=1}^n (I - \mathcal{S}_j)[u] = u$, *then for* $1 < p < \infty$

$$\big|\big|Q(Z, \bar{Z})[u]\big|\big|_{L^p(\widetilde{M})} \leq C\,\big|\big|g\big|\big|_{L^p(\widetilde{M})}.$$

Before presenting the proof of Theorem 4.2.1, we make some remarks, and state two useful inequalities.

Remark 1. The condition that $\bigotimes_{j=1}^n (I - \mathcal{S}_j)[u] = u$ is equivalent to the statement that u is orthogonal to each of the null spaces $\big\{u \,\big|\, \Box_j[u] = 0\big\}$, $1 \leq j \leq n$.

Remark 2. The estimate in Theorem 4.2.1 follows from the boundedness of the operator $Q(Z, \bar{Z})\mathcal{N}$ and the identity (4.1.20).

Remark 3. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\chi(t) \equiv 1$ for $|t| \leq \frac{1}{2}$ and $\chi(t) \equiv 0$ for $|t| \geq 1$. Let ρ_j be a "regularized distance" on M_j, that is, a smooth function on $M_j \times M_j$ such that $\rho_j(p_j, q_j) \approx d_j(p_j, q_j)$ and $\big|X^\alpha \rho_j(p_j, q_j)\big| \lesssim d_j(p_j, q_j)^{1-|\alpha|}$, for any derivative X^α of order $|\alpha|$ in the vector fields X_j and X_{n+j} acting on either p_j or q_j. (The existence of such distances is established in [NS01b].) For each $R > 0$ define

$$\chi_R(p, q) = \prod_{j=1}^n \chi\left(\frac{\rho_j(p_j, q_j)}{R}\right).$$

Our proof of Theorem 4.2.1 will actually show that if we consider the kernel $\widetilde{N}_\varepsilon(p, q)\,\chi_R(p, q)$, then any two good derivatives composed with the corresponding operator yield a product NIS operator on $\widetilde{M}$, with constants independent of ε and $\mathbb{R}$. For $\varepsilon > 0$ and $R < +\infty$, the kernel $\widetilde{N}_\varepsilon(p, q)\,\chi_R(p, q)$ is bounded and has compact support. This observation will be important when we use transference results in Section 5 below to obtain information about operators on the decoupled boundary M.

The following two elementary estimates will be used frequently in what follows.

Proposition 4.2.2. (a) *If F is a positive, monotone decreasing function on $(0, \infty)$, and if there exists $\varepsilon > 0$ such that $F(2t) \leq 2^{-1-\varepsilon} F(t)$, then there is a constant C depending on ε so that for all $a > 0$,*

$$\int_a^\infty F(t)\,dt \leq C\,a\,F(a).$$

(b) *If F is a positive, monotone decreasing function on $(0, \infty)$, and if there exists $\varepsilon > 0$ such that $F(t/2) \leq 2^{-1+\varepsilon} F(t)$, then there is a constant C depending on ε so that for all $a > 0$,*

$$\int_0^a F(t)\,dt \leq C\,a\,F(a).$$

Now we turn to the proof of Theorem 4.2.1. For simplicity of exposition, we confine ourselves to the proof in the case that $n = 2$. The general situation only involves additional notational difficulty. Recall that in this two-dimensional case, the distributional kernel for $\widetilde{\mathcal{N}}$ is given by

$$\widetilde{N}(p_1, p_2, q_1, q_2) = \int_0^\infty \prod_{j=1}^{2} G_j(s, p_j, q_j)\, ds.$$

We only need to show that the operator $Q(Z, \bar{Z})\mathcal{N}$ is a product singular integral operator in the sense of Definition 3.5.1. In what follows, we establish the differential inequalities required for condition (2) in Lemma 4.2.3, the estimates required for condition (3) in Lemma 4.2.4, and the estimates for condition (5) in Lemma 4.2.5.

LEMMA 4.2.3. *On the set $(M_1 \times M_2) \times (M_1 \times M_2) - \widetilde{D}$, the distribution $\widetilde{N}$ is given by integration against an infinitely differentiable function. For $j = 1, 2$, let $\partial_j^{\alpha_j}$ be a derivative of order $|\alpha_j|$ in the vector fields Z_j or $\bar{Z}_j$ acting either on the variables p_j or q_j. If $p_1 \neq q_1$ and $p_2 \neq q_2$, the integral*

$$\partial_1^{\alpha_1}\, \partial_2^{\alpha_2}\, \widetilde{N}(p_1, q_1, p_2, q_2) = \int_0^\infty \partial_1^{\alpha_1} G_1(s, p_1, q_1)\, \partial_2^{\alpha_2} G_2(s, p_2, q_2)\, ds$$

converges absolutely, and there is a constant $C = C_{\alpha_1,\alpha_2}$ such that

$$\left|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\widetilde{N}(p_1, q_1, p_2, q_2)\right| \leq C \left[\frac{d_1(p_1, q_1)^{-|\alpha_1|}\, d_2(p_2, q_2)^{-|\alpha_2|}\, [\min\{d_1(p_1, q_1), d_2(p_2, q_2)\}]^2}{V_1\big(p_1, d_1(p_1, q_1)\big)\, V_2\big(p_2, d_2(p_2, q_2)\big)}\right]. \tag{4.2.21}$$

Proof. Fix $p_1 \neq q_1$ and $p_2 \neq q_2$, and assume without loss of generality that $0 < d_1(p_1, q_1) \leq d_2(p_2, q_2)$. Write

$$\begin{aligned}
&\int_0^\infty \partial_1^{\alpha_1} G_1(s, p_1, q_1)\, \partial_2^{\alpha_2} G_2(s, p_2, q_2)\, ds \\
&= \left[\int_0^{d_1(p_1,q_1)^2} + \int_{d_1(p_1,q_1)^2}^{d_2(p_2,q_2)^2} + \int_{d_2(p_2,q_2)^2}^{\infty}\right] [\partial_1^{\alpha_1} G_1(s, p_1, q_1)\, \partial_2^{\alpha_2} G_2(s, p_2, q_2)]\; ds \\
&= \quad I \quad + \quad II \quad + \quad III.
\end{aligned}$$

According to Theorem 3.3.3, we can estimate integral I by

$$\big|I\big| \leq C\, d_1(p_1, q_1)^2\, d_1(p_1, q_1)^{-|\alpha_1|} V_1(p_1, q_1)^{-1} d_2(p_2, q_2)^{-|\alpha_2|} V_2(p_2, q_2)^{-1}.$$

For integral II we use Theorem 3.3.3 and Proposition 4.2.2(a) to obtain

$$\begin{aligned}
\big|II\big| &\leq C\, d_2(p_2, q_2)^{-|\alpha_2|} V_2(p_2, q_2)^{-1} \int_{d_1(p_1,q_1)^2}^{\infty} s^{-|\alpha_1|/2}\, \big|B_1(p_1, \sqrt{s})\big|^{-1}\, ds \\
&\leq C\, d_1(p_1, q_1)^2\, d_1(p_1, q_1)^{-|\alpha_1|} V_1(p_1, q_1)^{-1} d_2(p_2, q_2)^{-|\alpha_2|} V_2(p_2, q_2)^{-1}
\end{aligned}$$

since $F(t) = t^{-1-|\alpha_1|/2}\Lambda_1\big(p_1,\sqrt{t}\big)\big)^{-1}$ satisfies $F(2t) \leq 2^{-2}\,F(t)$. Finally, for integral III, we again use Proposition 4.2.2(a) and obtain

$$\begin{aligned}
\big|III\big| &\leq C \int_{d_2(p_2,q_2)^2}^{\infty} s^{-(|\alpha_1|+|\alpha_2|)/2}\,\big|B_1(p_1,\sqrt{s})\big|^{-1}\,\big|B_2(p_2,\sqrt{s})\big|^{-1}\,ds \\
&\leq C\,d_2(p_2,q_2)^{2-(|\alpha_1|+|\alpha_2|)}\,|B_1(p_1,d_2(p_2,q_2))|^{-1}\,|B_2(p_2,d_2(p_2,q_2))|^{-1} \\
&\leq C\,d_2(p_2,q_2)^{-|\alpha_1|}\,\Lambda_1(p_1,d_2(p_2,q_2))^{-1}\,d_2(p_2,q_2)^{-|\alpha_2|}\,V_2(p_2,d_2(p_2,q_2))^{-1} \\
&\leq C\,d_1(p_1,q_1)^2\,d_1(p_1,q_1)^{-|\alpha_1|}V_1(p_1,q_1)^{-1}d_2(p_2,q_2)^{-|\alpha_2|}V_2(p_2,q_2)^{-1}.
\end{aligned}$$

Combining the estimates for I, II, and III completes the proof since $d_1(p_1,q_1) = \min\{d_1(p_1,q_1), d_2(p_2,q_2)\}$. □

We next study cancellation properties of the distribution $\widetilde{N}$ which are expressed by the action of the operator $\widetilde{\mathcal{N}}$ on normalized bump functions on one factor, say M_1.

LEMMA 4.2.4. *Let φ be a normalized bump function supported on $B_1(p_1,\delta) \subset M_1$. Let $\partial_j^{\alpha_j}$ be a derivative of order $|\alpha_j|$ in the vector fields Z_j or $\bar{Z}_j$ acting either on the variables p_j or q_j. There is a constant $C = C(\alpha_1,\alpha_2)$ so that if $p_2 \neq q_2$,*

$$\Big|\int_0^\infty \Big[\int_{M_1} \partial_1^{\alpha_1} G_1(s,p_1,q_1)\,\varphi(q_1)\,dq_1\Big]\,\partial_2^{\alpha_2} G_2(s,p_2,q_2)\,ds\,\Big| \leq C\,\delta^{-|\alpha_1|}\,\frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)}\,\min\{\delta^2, d_2(p_2,q_2)^2\}.$$

Proof. We have the following estimates:

$$\text{(a)}\qquad \Big|\int_{M_1} \partial_1^{\alpha_1} G_1(s,p_1,q_1)\,\varphi(q_1)\,dq_1\Big| \leq C \begin{cases} \delta^{-|\alpha_1|} & \text{if } s \leq \delta^2, \\ s^{-|\alpha_1|/2}\frac{V_1(p_1,\delta)}{V_1(p_1,\sqrt{s})} & \text{if } s \geq \delta^2. \end{cases}$$

$$\text{(b)}\qquad |\partial_2^{\alpha_2} G_2(s,p_2,q_2)| \leq C \begin{cases} \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)} & \text{if } s \leq d_2(p_2,q_2)^2, \\ \frac{s^{-|\alpha_2|/2}}{V_2(p_2,\sqrt{s})} & \text{if } s \geq d_2(p_2,q_2)^2. \end{cases}$$

(The first part of assertion (a) follows from Theorem 3.3.4. The second part of (a) as well as assertion (b) follow from Theorem 3.3.3.) Write

$$\begin{aligned}
&\int_0^\infty \left[\int_{M_1} \partial_1^{\alpha_1} G_1(s,p_1,q_1)\,\varphi(q_1)\,dq_1\right] \partial_2^{\alpha_2} G_2(s,p_2,q_2)\,ds\\
&\qquad = \left[\int_0^{\delta^2} + \int_{\delta^2}^\infty\right] \left[\int_{M_1} \partial_1^{\alpha_1} G_1(s,p_1,q_1)\,\varphi(q_1)\,dq_1\right] \partial_2^{\alpha_2} G_2(s,p_2,q_2)\,ds\\
&\qquad = A + B.
\end{aligned}$$

In integral A, $s \leq \delta^2$ so according to (a) we obtain

$$\big|A\big| \leq C\,\delta^{-|\alpha_1|} \int_0^{\delta^2} \big|\partial_2^{\alpha_2} G_2(s,p_2,q_2)\big|\,ds.$$

The analysis now depends on the relative sizes of $d_2(p_2,q_2)$ and δ.

Case 1: $d_2(p_2,q_2) \leq \delta$. In this case, using estimate (b) we get

$$\begin{aligned}
\big|A\big| &\leq C\,\delta^{-|\alpha_1|} \left[\int_0^{d_2(p_2,q_2)^2} \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)}\,ds + \int_{d_2(p_2,q_2)^2}^\infty \frac{|s|^{-|\alpha_2|/2}}{V_2(p_2,\sqrt{s})}\,ds\right]\\
&\leq C\,\delta^{-|\alpha_1|} \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)}\,d_2(p_2,q_2)^2\\
&= C\,\delta^{-|\alpha_1|}\,\frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)}\,\min\{\delta^2, d_2(p_2,q_2)^2\}.
\end{aligned}$$

Case 2: $d_2(p_2,q_2) \geq \delta$. In this case, the region of s integration only involves $s \leq d_2(p_2,q_2)^2$, and so using the first case of estimate (b), we obtain

$$\begin{aligned}
\big|A\big| &\leq C\,\delta^{-|\alpha_1|}\,\frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)}\,\delta^2\\
&= \delta^{-|\alpha|}\,\frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)}\,\min\{\delta^2, d_2(p_2,q_2)^2\}.
\end{aligned}$$

Next consider the integral B. Here the range of integration is $s \geq \delta^2$ and so by the first estimate in (a), we have

$$\big|B\big| \leq C \int_{\delta^2}^\infty s^{-|\alpha_1|/2} \frac{V_1(p_1,\delta)}{V_1(p_1,\sqrt{s})}\,\partial_2^{\alpha_2} G_2(s,p_2,q_2)\,ds.$$

Again, the analysis depends on the relative size of $d_2(p_2,q_2)$ and δ.

Case 1: $d_2(p_2,q_2) \leq \delta$. In this case we integrate where $s \geq d_2(p_2,q_2)^2$, and according to (b) we have

$$\begin{aligned}
\big|B\big| &\leq C \int_{\delta^2}^{\infty} s^{-(|\alpha_1|+|\alpha_2|)/2} \frac{V_1(p_1,\delta)}{V_1(p_1,\sqrt{s})\; V_2(p_2,\sqrt{s})}\, ds \\
&\leq C \int_{\delta^2}^{\infty} \frac{|s|^{-(|\alpha_1|+|\alpha_2|)/2}}{V_2(p_2,\sqrt{s})}\, ds \\
&\leq C\, \delta^{2-|\alpha_1|-|\alpha_2|}\, V_2(p_2,\delta)^{-1} \\
&\leq C\, \delta^{-|\alpha_1|-|\alpha_2|}\, \Lambda_2(p_2,\delta)^{-1} \\
&\leq C\, \delta^{-|\alpha_1|} \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)} \min\{\delta^2, d_2(p_2,q_2)^2\}.
\end{aligned}$$

Case 2: $d_2(p_2,q_2) \geq \delta$. In this case, we write

$$\begin{aligned}
B &= \left[\int_{\delta^2}^{d_2(p_2,q_2)^2} + \int_{d_2(p_2,q_2)^2}^{\infty}\right] \left(s^{-|\alpha_1|/2} \frac{V_1(p_1,\delta)}{V_1(p_1,\sqrt{s})}\, \partial_2^{\alpha_2} G_2(s,p_2,q_2)\right) ds \\
&= I + II.
\end{aligned}$$

In integral I, we integrate where $s \leq d_2(p_2,q_2)^2$ and so by the first estimate in (b) we have

$$\begin{aligned}
\big|I\big| &\leq C\, \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)} \int_{\delta^2}^{\infty} s^{-|\alpha_1|/2} \frac{V_1(p_1,\delta)}{V_1(p_1,\sqrt{s})}\, ds \\
&\leq C\, \delta^{-|\alpha_1|} \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)}\, \delta^2 \\
&= C\, \delta^{-|\alpha_1|} \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)} \min\{\delta^2, d_2(p_2,q_2)^2\}.
\end{aligned}$$

Finally, in integral II, we integrate where $s \geq d_2(p_2,q_2)^2$ and so by the second estimate in (b) we have

$$\begin{aligned}
\big|II\big| &\leq C \int_{d_2(p_2,q_2)^2}^{\infty} s^{-(|\alpha_1|+|\alpha_2|)/2} \frac{V_1(p_1,\delta)}{V_1(p_1,\sqrt{s})}\, V_2(p_2,\sqrt{s})^{-1}\, ds \\
&\leq C\, \frac{d_2(p_2,q_2)^{2-|\alpha_1|-|\alpha_2|}}{V_2(p_2,d_2(p_2,q_2))} \frac{\delta^2\, \Lambda_1(p_1,\delta)}{d_2(p_2,q_2)^2\, \Lambda_1\big(p_1,d_2(p_2,q_2)\big)} \\
&\leq C\, \frac{d_2(p_2,q_2)^{-|\alpha_1|-|\alpha_2|}}{V_2(p_2,d_2(p_2,q_2))}\, \delta^2 \\
&= C\, \delta^{-|\alpha_1|} \frac{d_2(p_2,q_2)^{-|\alpha_2|}}{V_2(p_2,q_2)} \min\{\delta^2, d_2(p_2,q_2)^2\}.
\end{aligned}$$

This completes the proof of Lemma 4.2.4. □

We finally need to study the action of the distribution $\widetilde{N}$ on pairs of bump functions.

Lemma 4.2.5. *Suppose φ_j is a normalized bump function supported on the ball $B_j(p_j,\delta_j)\subset M_j$. There is a constant C depending on α_j but independent of δ_j so that*

$$\left|\int_0^\infty\int_{M_1}\int_{M_2}\partial_1^\alpha G_1(s,p_1,q_1)\,\partial_2^\beta G_2(s,p_2,q_2)\,\varphi_1(q_1)\,\varphi_1(q_2)\,dq_1\,dq_2\,ds\right| \leq C\,\delta_1^{-|\alpha_1|}\,\delta_2^{-|\alpha_2|}\,\min\{\delta_1^2,\delta_2^2\}.$$

Proof. Suppose without loss of generality that $\delta_1\leq\delta_2$. Write

$$\int_0^\infty=\int_0^{\delta_1^2}+\int_{\delta_1^2}^{\delta_2^2}+\int_{\delta_2^2}^\infty=I+II+III.$$

We use the estimates

$$\left|\int_{M_j}\partial^{\alpha_j}G_j(s,p_j,q_j)\,\varphi_j(q_j)\,dq_j\right|\leq C\begin{cases}\delta^{-|\alpha_j|} & \text{if } s\leq\delta_j^2,\\ s^{-|\alpha_j|/2}\frac{V_j(p_j,\delta_j)}{V_j(p_j,\sqrt{s})} & \text{if } s\geq\delta_j^2.\end{cases}$$

For integral I, we have the estimate

$$\big|I\big|\lesssim\delta_1^{-|\alpha_1|}\,\delta_2^{-|\alpha_2|}\,\delta_1^2=\delta_1^{-|\alpha_1|}\,\delta_2^{-|\alpha_2|}\,\min\{\delta_1^2,\delta_2^2\}.$$

For integral II we have the estimate

$$\begin{aligned}\big|II\big|&\lesssim\delta_2^{-|\alpha_2|}\int_{\delta_1^2}^\infty s^{-|\alpha_1|/2}\,\frac{V_1(p_1,\delta_1)}{V_1(p_1,\sqrt{s})}\,ds\\&\lesssim\delta_1^{-|\alpha_1|}\,\delta_2^{-|\alpha_2|}\,\delta_1^2=\delta_1^{-|\alpha_1|}\,\delta_2^{-|\alpha_2|}\,\min\{\delta_1^2,\delta_2^2\}.\end{aligned}$$

Finally, for integral III we have the estimate

$$\begin{aligned}\big|III\big|&\lesssim\int_{\delta_2^2}^\infty s^{-(|\alpha_1|+|\alpha_2|)/2}\,\frac{V_1(p_1,\delta_1)}{V_j(p_1,\sqrt{s})}\,\frac{V_2(p_2,\delta_2)}{V_j(p_2,\sqrt{s})}\,ds\\&\lesssim\delta_2^{-|\alpha_1|+|\alpha_2|}\,\frac{\delta_1^2\,\Lambda_1(p_1,\delta_1)}{\delta_2^2\,\Lambda_j(p_1,\delta_2)}\delta_2^2\\&\lesssim\delta_1^{-|\alpha_1|}\,\delta_2^{-|\alpha_2|}\,\delta_1^2\\&=\delta_1^{-|\alpha_1|}\,\delta_2^{-|\alpha_2|}\,\min\{\delta_1^2,\delta_2^2\}.\end{aligned}$$

This completes the proof of Lemma 4.2.5, and consequently the proof of Theorem 4.2.1. □

4.3. *Analysis of the distribution $\widetilde{K}$.* Equation (4.1.18) gives a decomposition of the *operator* $\widetilde{\mathcal{K}}$. It will also be important to have a different kind of decomposition of the *kernel* $\widetilde{K}$ which is given by

$$\widetilde{K}(p,q) = \int_0^\infty \Big[\prod_{j=1}^n H_j(s,p_j,q_j) - \prod_{j=1}^n S_j(p_j,q_j)\Big]\,ds.$$

THEOREM 4.3.1. *There are distributions* $\widetilde{K}_0$, $\widetilde{K}_\infty$, *and for each nonempty proper subset* $A \subset \{1,\dots,n\}$ *a distribution* $\widetilde{K}_A$ *on* $\widetilde{M}\times\widetilde{M}$ *so that away from the product diagonal* $\widetilde{D}$ *on* $\widetilde{M}\times\widetilde{M}$,

$$\widetilde{K}(p,q) = \widetilde{K}_0(p,q) + \sum_A \Big(\prod_{j\notin A} S_j(p_j,q_j)\Big)\widetilde{K}_A(p,q) + \Big(\prod_{j=1}^n S_j(p_j,q_j)\Big)\widetilde{K}_\infty(p,q).$$

Moreover $\widetilde{K}_0$, $\widetilde{K}_A$, *and* $\widetilde{K}_\infty$ *are locally integrable functions, and the size of their derivatives can be estimated by the control metric* d_Σ. *Thus for all derivatives* X^α *of total order* $|\alpha|$ *in* $X_1,\dots,X_{2n}$ *acting on the variables* p_j *or* q_j,

$$\begin{aligned}
&\Big|X^\alpha\widetilde{K}_0(p,q)\Big| \le C_\alpha\,\frac{d_\Sigma(p,q)^{2-|\alpha|}}{\prod_{j=1}^n d_\Sigma(p,q)^2\Lambda_j(p_j,d_\Sigma(p,q))};\\
&\Big|X^\alpha\widetilde{K}_A(p,q)\Big| \le C_\alpha\,\frac{d_\Sigma(p,q)^{2-|\alpha|}}{\prod_{j\in A} d_\Sigma(p,q)^2\Lambda_j\big(p_j;d_\Sigma(p,q)\big)};\\
&\Big|X^\alpha\widetilde{K}_\infty(p,q)\Big| \le C_\alpha\,d_\Sigma(p,q)^{2-|\alpha|}.
\end{aligned} \tag{4.3.22}$$

To prove the estimates in Theorem 4.3.1, we decompose $H_j(s,x,y)$ into two parts. One part is supported close to the diagonal and is singular there. The other part is smooth everywhere.

Definition 4.3.2. Let ρ_j be *a regularized distance function on* M_j as defined in Remark 3 of Section 4.2. Let $\chi \in C^\infty(\mathbb{R})$ satisfy $\chi(t)\equiv 1$ if $t\le\frac{1}{2}$ and $\chi(t)\equiv 0$ if $t\ge 1$. Let

$$\chi_j(s,p_j,q_j) = \chi\left(\frac{\rho_j(p_j,q_j)^2}{s}\right)$$

and set

$$\begin{aligned}
\widetilde{S}_j(s,p_j,q_j) &= \chi_j(s,p_j,q_j)\,S_j(p_j,q_j);\\
\Phi_j(s,p_j,q_j) &= H_j(s,p_j,q_j) - \widetilde{S}_j(s,p_j,q_j)
\end{aligned}$$

PROPOSITION 4.3.3. $H_j(s,p_j,q_j) = \Phi_j(s,p_j,q_j) + \widetilde{S}_j(s,p_j,q_j)$. *Moreover*

$$\Phi_j(s,p_j,q_j) = \begin{cases} G_j(s,p_j,q_j) & \textit{when} \quad s\ge 2\,\rho_j(p_j,q_j)^2,\\ H_j(s,p_j,q_j) & \textit{when} \quad s\le \rho_j(p_j,q_j)^2.\end{cases}$$

$$\widetilde{S}_j(s,p_j,q_j) = \begin{cases} S_j(p_j,q_j) & \textit{when} \quad s\ge 2\rho_j(p_j,q_j)^2,\\ 0 & \textit{when} \quad s\le \rho_j(p_j,q_j)^2.\end{cases}$$

For every derivative X^α of order $|\alpha|$ in the vector fields X_j and X_{n+j} and every integer N there is a constant $C_{\alpha,N}$ so that

$$|X^\alpha \Phi_j(s,p_j,q_j)| \leq C_{\alpha,N} \begin{cases} s^{-\frac{1}{2}|\alpha|}\,|B_j(p_j,\sqrt{s})|^{-1} & \text{if } s \geq d_j(p_j,q_j)^2, \\ s^N\, d_j(p_j,q_j)^{-2-|\alpha|-2N}\,\Lambda_j(p_j,d_j(p_j,q_j))^{-1} & \text{if } s \leq d_j(p_j,q_j)^2. \end{cases}$$

Also,

$$|X^\alpha \chi_j(s,p_j,q_j)| \leq C_\alpha\, d_j(p_j,q_j)^{-|\alpha|}.$$

Moreover, as soon as α is different from zero, $X^\alpha\chi_j(s,p_j,q_j)$ is supported where $d_j(p_j,q_j)^2 \approx s$.

Proof. The estimates follow from the chain rule and the basic estimates in Theorem 3.3.3. □

The following result is an immediate consequence of the definitions:

PROPOSITION 4.3.4. *Suppose that $d_j(p_j,q_j) \leq d_\ell(p_\ell,q_\ell)$. For M large enough depending only on the type,*

$$\Lambda_j\big(p_j;d_j(p_j,q_j)\big)\left(\frac{d_j(p_j,q_j)}{d_\ell(p_\ell,q_\ell)}\right)^M \lesssim \Lambda_\ell(p_\ell,d_\ell(p_\ell,q_\ell)).$$

Proof of Theorem 4.3.1. The integrand in the integral defining $K(p,q)$ in equation (4.1.14) can be written

$$\begin{aligned}\prod_{j=1}^n H_j(s,p_j,q_j) - \prod_{j=1}^n S_j(p_j,q_j) &= \prod_{j=1}^n \Big(\Phi_j(s,p_j,q_j)+\widetilde{S}_j(s,p_j,q_j)\Big) - \prod_{j=1}^n S_j(p_j,q_j) \\ &= \prod_{j=1}^n \Phi_j(s,p_j,q_j) + \sum_A \left[\prod_{j\in A}\Phi_j(s,p_j,q_j)\prod_{j\notin A}\widetilde{S}_j(s,p_j,q_j)\right] \\ &\qquad + \left[\prod_{j=1}^n \widetilde{S}_j(s,p_j,q_j) - \prod_{j=1}^n S_j(p_j,q_j)\right]\end{aligned}$$

where the sum is taken over all nonempty, proper subsets $A \subset \{1,\dots,n\}$. Note that

$$\int_0^\infty \prod_{j\in A}\Phi_j(s,p_j,q_j)\prod_{j\notin A}\widetilde{S}_j(s,p_j,q_j)\,ds = \Big(\prod_{j\notin A}S_j(p_j,q_j)\Big)\int_0^\infty \prod_{j\in A}\Phi_j(s,p_j,q_j)\prod_{j\notin A}\chi\left(\frac{\rho_j(p_j,q_j)^2}{s}\right)ds$$

and that

$$\int_0^\infty \Big[\prod_{j=1}^n \widetilde{S}_j(s,p_j,q_j)-\prod_{j=1}^n S_j(p_j,q_j)\Big]\,ds = \Big(\prod_{j=1}^n S_j(p_j,q_j)\Big)\int_0^\infty \prod_{j=1}^n\left[1-\chi\left(\frac{\rho_j(p_j,q_j)^2}{s}\right)\right]ds.$$

Set

$$\begin{aligned}\widetilde{K}_0(p,q) &= \int_0^\infty \prod_{j=1}^m \Phi_j(s,p_j,q_j)\,ds,\\ \widetilde{K}_A(p,q) &= \int_0^\infty \prod_{j\in A}\Phi_j(s,p_j,q_j)\prod_{j\notin A}\chi\left(\frac{\rho_j(p_j,q_j)^2}{s}\right)ds,\\ \widetilde{K}_\infty(p,q) &= \int_0^\infty \prod_{j=1}^n\left[1-\chi\left(\frac{\rho_j(p_j,q_j)^2}{s}\right)\right]ds.\end{aligned}\tag{4.3.23}$$

To establish Theorem 4.3.1, we need to show that the integrals defined in equation (4.3.23) satisfy the estimates stated in equation (4.3.22).

4.3.1. *Estimates for* $\widetilde{K}_0$.

LEMMA 4.3.5.

$$\Big|\int_0^\infty \prod_{j=1}^n X^{\alpha_j}\Phi_j(s,p_j,q_j)\,ds\Big| \le C_\alpha \frac{d_\Sigma(p,q)^{2-|\alpha|}}{\prod_{j=1}^n\Big[d_\Sigma(p,q)^2\Lambda_j(p_j,d_\Sigma(p,q))\Big]}.$$

Proof. The case $n=2$ is entirely typical. We may suppose, without loss of generality, that

$$d_1(p_1,q_1)\le d_2(p_2,q_2),$$

so that $d_\Sigma(p,q)\approx d_2$. We split the integral into three parts; first where $0\le s\le d_1(p_1,q_1)^2$, next where $d_1(p_1,q_1)^2\le s\le d_2(p_2,q_2)^2$, and finally where $d_2(p_2,q_2)^2\le s<+\infty$. Using the estimates from Proposition 4.3.3 and also

Proposition 4.3.4, we have

$$\begin{aligned}
\Big|\int_0^{d_1(p_1,q_1)^2} & X_1^\alpha \Phi_1(s,p_1,q_1)\, X_2^\beta \Phi_2(s,p_2,q_2)\, ds\Big| \\
&\lesssim \frac{d_1(p_1,q_1)^{-2-|\alpha|-2M}\ d_2(p_2,q_2)^{-2-|\beta|-2N}}{\Lambda_1(p_1,d_1(p_1,q_1))\,\Lambda_2(p_2,d_2(p_2,q_2))} \int_0^{d_1(p_1,q_1)^2} s^{M+N}\, ds \\
&\lesssim \frac{d_1(p_1,q_1)^{-|\alpha|}\ d_2(p_2,q_2)^{-|\beta|}}{\Lambda_1(p_1,d_1(p_1,q_1))\ d_2(p_2,q_2)^2\ \Lambda_2(p_2,d_2(p_2,q_2))} \left(\frac{d_1(p_1,q_1)}{d_2(p_2,q_2)}\right)^{2M} \\
&\lesssim d_\Sigma(p,q)^{-2-|\alpha|-|\beta|}\,\Lambda_1(p_1,d_\Sigma(p,q))^{-1}\,\Lambda_2(p_2,d_\Sigma(p,q))^{-1}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\Big|\int_{d_1(p_1,q_1)^2}^{d_2(p_2,q_2)^2} & X_1^\alpha \Phi_1(s,p_1,q_1) X_2^\beta \Phi_2(s,p_2,q_2)\, ds\Big| \\
&\leq \frac{d_2(p_2,q_2)^{-2N-|\beta|}}{d_2(p_2,q_2)^2\,\Lambda_2(p_2;d_2(p_2,q_2))} \int_{d_1(p_1,q_1)^2}^{d_2(p_2,q_2)^2} s^{N-2-|\beta|}\,\Lambda_1\big(p_1;\sqrt{s}\big)^{-1} ds \\
&\leq \frac{d_2(p_2,q_2)^{-2-|\alpha|-2N-|\beta|}}{\Lambda_2(p_2;d_2(p_2,q_2))}\,\Lambda_1(p_1;d_2(p_2,q_2))^{-1}\,d_2(p_2,q_2)^{2N} \\
&\lesssim d_\Sigma(p,q)^{-2-|\alpha|-|\beta|}\,\Lambda_1(p_1;d_\Sigma(p,q))^{-1}\,\Lambda_2(p_2;d_\Sigma(p,q))^{-1}
\end{aligned}$$

since for N large enough, the function $s^{N-2-|\beta|}\,\Lambda_1\big(p_1;\sqrt{s}\big)^{-1}$ is monotone increasing, and hence we can estimate the integral by taking this function at the upper end point.

Finally

$$\begin{aligned}
\Big|\int_{d_2(p_2,q_2)^2}^{+\infty} & X_1^\alpha \Phi_1(s,p_1,y_1) X_2^\beta \Phi_2(s,p_2,y_2)\, ds\Big| \\
&\leq \int_{d_2(p_2,q_2)^2}^{+\infty} s^{-\frac{1}{2}(|\alpha|+|\beta|)-2}\Lambda_1(p_1;\sqrt{s})^{-1}\,\Lambda_2(p_2;\sqrt{s})^{-1}\, ds \\
&\lesssim d_\Sigma(p,q)^{-2-|\alpha|-|\beta|}\,\Lambda_1(p_1;d_\Sigma(p,q))^{-1}\,\Lambda_2(p_2;d_\Sigma(p,q))^{-1}. \qquad \square
\end{aligned}$$

4.3.2. *Estimates for* $\widetilde{K}_A$.

Lemma 4.3.6. *Let* $A \subset \{1,\dots,n\}$ *be a proper, nonempty subset. Then*

$$\begin{aligned}
\Big|\int_0^\infty \Big(\prod_{j\in A} X^{\alpha_j}\Phi_j(s,p_j,q_j)\Big)\Big(\prod_{j\notin A} X^{\beta_j}\chi_j(s,p_j,q_j)\Big)\, ds\Big| \\
\lesssim \frac{d_\Sigma(p,q)^{2-|\alpha|-|\beta|}}{\prod_{j\in A} d_\Sigma(p,q)^2\Lambda_j\big(p_j;d_\Sigma(p,q)\big)}.
\end{aligned} \tag{4.3.24}$$

Proof. Let $\delta_j = \rho_j(p_j, q_j)$. Then $d_\Sigma(p,q) \approx \max_j\{\delta_j\}$. Also set $\delta_A = \max_{j\notin A}\{\delta_j\}$. Then the support of the integrand in equation (4.3.24) is $s \geq \delta_A^2$.

Remark. If any of the integers $\{\beta_j\}$ is nonzero, the support of the integrand is between two multiples of the corresponding δ_j^2. Thus if some $\beta_j > 0$, all the $j \notin A$ for which $\beta_j > 0$ have the property that $\delta_j \approx \delta_A$.

Case 1: $\beta_j = 0$ *for every* $j \notin A$. There are now two subcases to consider, depending on the relative size of δ_A and $d_\Sigma(p,q)$. We always have $\delta_A \lesssim d_\Sigma(p,q)$.

Case 1a: $\delta_A \approx d_\Sigma(p,q)$. This is the case in which an index j for which δ_j is comparable to the maximum does not belong to the set A.

Now we must study

$$\Big| \int_{d_\Sigma(p,q)^2}^{\infty} \Big(\prod_{j\in A} X^{\alpha_j} \Phi_j(s, p_j, q_j)\Big)\Big(\prod_{j\notin A} \chi_j(s, p_j, q_j)\Big)\, ds \Big|. \tag{4.3.25}$$

The decay of the integrand at infinity allows us to estimate the integral by $d_\Sigma(p,q)^2$ times the value of the integrand at the lower endpoint. Since $|\chi_j(s,p_j,q_j)| \leq 1$, we obtain from Proposition 4.3.3 the estimate

$$d_\Sigma(p,q)^2 \Big(\prod_{j\in A} X^{\alpha_j} \Phi_j(d_\Sigma(p,q)^2, p_j, q_j)\Big) \leq \frac{d_\Sigma(p,q)^{2-\sum_{j\in A}|\alpha_j|}}{\prod_{j\in A} d_\Sigma(p,q)^2\, \Lambda_j\big(p_j, d_\Sigma(p,q)\big)}$$

since for every $j \in A$, $d_j(p_j,q_j) \leq d_\Sigma(p,q)$. This gives the desired estimate in this case.

Case 1b: $\delta_A \ll d_\Sigma(p,q)$. This is the case in which every index j for which δ_j is comparable to the maximum does belong to the set A.

This time we must study

$$\Big| \int_{\delta_A^2}^{\infty} \Big(\prod_{j\in A} X^{\alpha_j} \Phi_j(s, p_j, q_j)\Big)\Big(\prod_{j\notin A} \chi_j(s, p_j, q_j)\Big)\, ds \Big|. \tag{4.3.26}$$

We write

$$\int_{\delta_A^2}^{\infty} = \int_{\delta_A^2}^{d_\Sigma(p,q)^2} + \int_{d_\Sigma(p,q)^2}^{\infty}.$$

The second integral is handled in the same way as Case 1a. In dealing with the first integral, we would like to be able to take out the integrand at the *upper* endpoint rather than at the lower endpoint. However, note that there is an index $j \in A$ for which $\delta_j \approx d_\Sigma(p,q)$. For this j we use the estimate from Proposition 4.3.3:

$$\big| X_j^\alpha \Phi_j(s, p_j, q_j)\big| \leq C_{\alpha,N} s^N\, d_\Sigma(p,q)^{-2-|\alpha_j|-2N} \Lambda_j\big(p_j, d_\Sigma(p,q)\big)^{-1}.$$

If we take N large enough, we can make the entire integrand in equation (4.3.26) monotone increasing, and we get the desired estimate

$$\frac{d_\Sigma(p,q)^{2-\sum_{j\in A}|\alpha_j|}}{\prod_{j\in A} d_\Sigma(p,q)^2\,\Lambda_j\big(p_j, d_\Sigma(p,q)\big)}.$$

We now must consider what happens if some $\beta_j > 0$.

Case 2: *Some* $\beta_j > 0$. By the remark following Lemma 4.3.6, we are led to study

$$\prod_{j\notin A} \delta_A^{-|\beta_j|} \int_{\frac{1}{2}\delta_A^2}^{2\delta_A^2} \big|\prod_{j\in A} X^{\alpha_j}\Phi_j(s,p_j,q_j)\big|\,ds. \tag{4.3.27}$$

Case 2a: $\delta_A \approx d_\Sigma(p,q)$. As in Case 1a, we can estimate the integral by $d_\Sigma(p,q)^2$ times the value of the integrand at the lower endpoint. This gives us the estimate

$$\frac{d_\Sigma(p,q)^{2-\sum_{j\in A}|\alpha_j|-\sum_{j\notin A}|\beta_j|}}{\prod_{j\in A} d_\Sigma(p,q)^2\,\Lambda_j\big(p_j, d_\Sigma(p,q)\big)}$$

which is the desired result.

Case 2b: $\delta_A \ll d_\Sigma(p,q)$. Again, as in Case 1b, we know there exists $j \in A$ for which we have the estimate

$$\big|X_j^\alpha \Phi_j(s,p_j,q_j)\big| \le C_{\alpha,N}\, s^N\, d_\Sigma(p,q)^{-2-|\alpha_j|-2N}\Lambda_j\big(p_j, d_\Sigma(p,q)\big)^{-1}.$$

When we integrate $s^n\, d_\Sigma(p,q)^{-2N}$ between two multiples of δ_A^2, we get the factor

$$\delta_A^2\left(\frac{\delta_A^2}{d_\Sigma(p,q)^2}\right)^N \le d_\Sigma(p,q)^2\left(\frac{\delta_A^2}{d_\Sigma(p,q)^2}\right)^N.$$

If we take N large enough, by using Proposition 4.3.4, we can replace each factor $\delta_A^{-|\beta_j|}$ with $d_\Sigma(p,q)$, and again we get the required estimate. This completes the proof. □

4.3.3. *Estimates for* $\widetilde{K}_\infty$.

LEMMA 4.3.7.

$$\Big|\int_0^\infty \prod_{j=1}^n X^{\alpha_j}\,(1-\chi_j(s,p_j,q_j)]\;ds\Big| \lesssim d_\Sigma(p,q)^{2-\sum|\alpha_j|}.$$

Proof. The integrand is zero unless $s \le d_\Sigma(p,q)^2$, and each factor $\delta_j^{-|\alpha_j|}$ coming from a derivative is dominated by $d_\Sigma(p,q)^{-|\alpha_j|}$. This gives the desired estimate, and completes the proof. □

Thus the proof of Theorem 4.3.1 is complete. □

4.4. *Quadratic derivatives of* $\widetilde{K}$. If $Q(Z,\bar Z)$ is a quadratic expression in the vector fields $\{Z_1,\dots,Z_n,\bar Z_1,\dots,\bar Z_n\}$, it need not be the case that the operator $Q(Z,\bar Z)\,\widetilde{\mathcal{K}}$ is bounded on $L^2(\widetilde{M})$. In this section we obtain certain replacements for this loss of maximal hypoellipticity.

To describe the results, suppose that we are studying the relative fundamental solution operator $\widetilde{K}$ for the differential operator $\Box_b = \sum_{j=1}^n \Box_j$ where $\Box_j = W_j\,\overline{W}_j$, and each W_j is one of $\{Z_j,\overline{Z}_j\}$ so that $\overline{W}_j$ is the other. Let b be a bounded function on $\widetilde{M}$ (where we will write $b(p)=b(p_1,\dots,p_n)$). We obtain conditions on b that guarantee that the operators $b\,W_k\,W_l\,\widetilde{\mathcal{K}}$, $b\,W_k\,\overline{W}_l\,\widetilde{\mathcal{K}}$, $b\,\overline{W}_k\,W_l\,\widetilde{\mathcal{K}}$, and $b\,\overline{W}_k\,\overline{W}_l\,\widetilde{\mathcal{K}}$ are bounded on $L^p(\widetilde{M})$ for $1<p<\infty$. The size conditions that we need to impose on b will depend on which of these forms we consider, and are given in terms of the quantities $\left\{\lambda_j(p_j) = \frac{\partial^2 P_j}{\partial z_j \partial \bar z_j}(p_j)\right\}$ which are the eigenvalues of the Levi form on the decoupled boundary M.

THEOREM 4.4.1. *Let* $1\le k,l\le n$. *Then*

(1) *The operators* $W_k\,\overline{W}_k\,\widetilde{\mathcal{K}}$, $\overline{W}_k\,\overline{W}_k\,\widetilde{\mathcal{K}}$, *and* $\overline{W}_k\,\overline{W}_l\,\widetilde{\mathcal{K}}$ *are product singular integrals on* $\widetilde{M}$ *and hence are bounded on* $L^p(\widetilde{M})$ *for* $1<p<\infty$.

(2) *Let b be a bounded function on* $\widetilde{M}$ *and suppose there exists a constant C so that*
$$\lambda_k(p_k)\,|b(p_1,\dots,p_n)| \le C \inf_{l\neq k} \lambda_l(p_l).$$
Then
$$b\,\overline{W}_k\,W_k\,\widetilde{\mathcal{K}} = \sum_\alpha b_\alpha\,\mathcal{M}_\alpha + \sum_{l,A}\big(T_k - T_l\big) b_{l,A}\,\lambda_k\,\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A.$$
In the first sum on the right-hand side, each b_α is a bounded function on $\widetilde{M}$, *each $\mathcal{M}_\alpha$ is a product singular integral on* $\widetilde{M}$, $T_j = \frac{\partial}{\partial t_j}$, *and the sum is finite. In the second summation, each $b_{l,A}$ is a bounded function, and the sum is over all* $1\le l\le n$, *and all subsets* $A\subset\{1,\dots,n\}$ *with* $l\in A$ *and* $k\notin A$.

(3) *Let b be a bounded function on* $\widetilde{M}$ *which is independent of the variable* p_k, *and suppose there exists a constant C so that*
$$|b(p_1,\dots,p_n)| \le C \inf_{l\neq k} \lambda_l(p_l).$$
Then there are NIS *operators* $\{P_\alpha, P_{l,A}\}$ *of order zero acting only in the variable p_k so that*
$$b\,W_k\,W_k\,\widetilde{\mathcal{K}} = \sum_\alpha P_\alpha b_\alpha\,\mathcal{M}_\alpha + \sum_{l,A} P_{l,A}\,\big(T_k - T_l\big) b_{l,A}\,\lambda_k\,\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A.$$
Again, each b_α and $b_{l,A}$ is a bounded function on $\widetilde{M}$, *each $\mathcal{M}_\alpha$ is a product singular integral on* $\widetilde{M}$, $T_j = \frac{\partial}{\partial t_j}$, *and the second sum is over all* $1\le l\le n$, *and all subsets* $A\subset\{1,\dots,n\}$ *with* $l\in A$ *and* $k\notin A$.

Proof of (1). The key point is that $W_k\,\overline{W}_k\,\mathcal{S}_k = 0$, $\overline{W}_k\,\overline{W}_k\,\mathcal{S}_k = 0$, and $\overline{W}_k\,\overline{W}_l\,\mathcal{S}_k \otimes \mathcal{S}_l = 0$. The proof in all three cases is similar, so we only give the details for the first. According to equation (4.1.18) we have $\widetilde{K} = \widetilde{\mathcal{N}} + \sum_A \widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A$, and hence

$$W_k\,\overline{W}_k\,\widetilde{\mathcal{K}} = W_k\,\overline{W}_k\,\widetilde{\mathcal{N}} + \sum_A W_k\,\overline{W}_k\,(\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A).$$

But $W_k\,\overline{W}_k\,\widetilde{\mathcal{N}}$ is a product singular integral operator by Theorem 4.2.1. If $k \notin A$ then

$$W_k\,\overline{W}_k\,(\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A) = \widetilde{\mathcal{N}}_A \otimes W_k\,\overline{W}_k\,\widetilde{\mathcal{S}}_A = 0.$$

On the other hand, if $k \in A$, then

$$W_k\,\overline{W}_k\,(\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A) = W_k\,\overline{W}_k\,\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A.$$

But $W_k\,\overline{W}_k\,\widetilde{\mathcal{N}}_A$ is a product singular integral operator in the variables coming from the set A, as follows from Theorem 4.2.1 applied to fewer variables. Thus $W_k\,\overline{W}_k\,(\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A)$ (and hence $W_k\,\overline{W}_k\,\widetilde{\mathcal{K}}$) is a product singular integral operator. This completes the proof of (1).

Proof of (2). We now consider the operator $b\,\overline{W}_k\,W_k\,\widetilde{\mathcal{K}} = b\,\overline{\Box}_k\,\widetilde{\mathcal{K}}$. Arguing as in the proof of (1), we have

$$\begin{aligned} b\,\overline{\Box}_k\,\widetilde{\mathcal{K}} &= b\,\overline{\Box}_k\,\widetilde{\mathcal{N}} \;+ \sum_{A\subset\{1,\dots,n\}} b\,\overline{\Box}_k\big(\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A\big) \\ &= b\,\overline{\Box}_k\,\widetilde{\mathcal{N}} \;+ \sum_{k\in A\subset\{1,\dots,n\}} b\,\big(\overline{\Box}_k\,\widetilde{\mathcal{N}}_A\big) \otimes \widetilde{\mathcal{S}}_A + \sum_{k\notin A\subset\{1,\dots,n\}} b\,\widetilde{\mathcal{N}}_A \otimes \big(\overline{\Box}_k \widetilde{\mathcal{S}}_A\big). \end{aligned}$$

Now $\overline{\Box}_k\,\mathcal{N}$ is a product NIS operator by Theorem 4.2.1. Also $\overline{\Box}_k\widetilde{\mathcal{N}}_A$ is a product singular integral operator in the variables $k \in A$, and so $(\overline{\Box}_k\widetilde{\mathcal{N}}_A) \otimes \widetilde{\mathcal{S}}_A$ is a product singular integral operator on $\widetilde{M}$. Thus for any bounded function b on $\widetilde{M}$, the first two terms on the right in the last equation are bounded operators on $L^p(\widetilde{M})$ for $1 < p < \infty$.

The difficult terms are those involving $\widetilde{\mathcal{N}}_A \otimes \big(\overline{\Box}_k\widetilde{\mathcal{S}}_A\big)$ where $k \notin A$, since in terms of the product structure, $\big(\overline{\Box}_k\widetilde{\mathcal{S}}_A\big)$ is smoothing of order -2, while $\widetilde{\mathcal{N}}_A$ is smoothing of order $+2$. In general, such a product does not yield a product NIS operator.

In these bad terms, we transfer the two extra derivatives from the right-hand side of the tensor product to the left-hand side. Note that $\overline{\Box}_k = \Box_k \pm \lambda_k\,T_k$ where $T_k = \frac{\partial}{\partial t_k}$. Since $k \notin A$, we have $\Box_k\,\widetilde{\mathcal{S}}_A = 0$. Choose an index $l \in A$, so that in particular $l \neq k$. Then

$$\begin{aligned} b\,\widetilde{\mathcal{N}}_A \otimes \big(\overline{\Box}_k\widetilde{\mathcal{S}}_A\big) &= \pm b\,\lambda_k\,T_k\,\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A \\ &= \mp\Big(\frac{b\,\lambda_k}{\lambda_l}\Big)\Big[\lambda_l\,T_l\,\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A\Big] \pm b\,\lambda_k\big(T_k - T_l\big)\,\widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A. \end{aligned}$$

Now $\lambda_l\, T_l\, \widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A$ is a product singular integral operator on $\widetilde{M}$ since $\lambda_l\, T_l\, \widetilde{\mathcal{N}}_A$ is the commutator of two good derivatives applied to $\widetilde{\mathcal{N}}_A$. Thus

$$\Big(\frac{b\,\lambda_k}{\lambda_l}\Big)\Big[\lambda_l\, T_l\, \widetilde{\mathcal{N}}_A \otimes \widetilde{\mathcal{S}}_A\Big]$$

is a bounded operator on $L^p(\widetilde{M})$ provided that there is a constant C so that

$$\lambda_k(p_k)\,|b(p_1, \dots, p_n)| \leq C\lambda_l(p_l).$$

Finally, note that the operator $b\,\lambda_k(T_k - T_l) - (T_k - T_l)b\,\lambda_k$ is a bounded function. This completes the proof of (2).

Proof of (3). Recall that $\bar{Z}_j = X_j + iX_{n+j}$ where $\{X_j, X_{n+j}\}$ are real vector fields on the manifold M_j. It follows that $\Box_k + \overline{\Box}_k = Z_j\,\bar{Z}_j + \bar{Z}_j\, Z_j = 2(X_k^2 + X_{n+k}^2)$, and it is known that this operator can be inverted with an NIS operator on M_k, smoothing of order 2. It then follows that there are NIS operators P_1 and P_2 on M_k, smoothing of order zero, such that

$$W_k\, W_k = P_1\Box_k + P_2\overline{\Box}_k.$$

We can also regard P_1 and P_2 as operators on $\widetilde{M}$ which act only in the variable p_k. Thus if B is a bounded function on $\widetilde{M}$ which is independent of the variable p_k, the operator which is multiplication by B and the operator P_j commute. It follows that we have

$$B\,W_k\,W_k\,\widetilde{\mathcal{K}} = P_1\,B\,(\Box_k\,\widetilde{\mathcal{K}}) + P_2\,B\,(\overline{\Box}_k\,\widetilde{\mathcal{K}}).$$

The proof of part (3) now follows immediately from parts (1) and (2).

5. Transference from $M_1 \times \cdots \times M_n$ to M and L^p regularity of $\mathcal{K}$

In order to pass from results about operators on the product $\widetilde{M} = M_1 \times \cdots \times M_n$ to results about operators on the decoupled boundary M, we use the mapping $\pi : \widetilde{M} \to M$ given by $\pi(z_1, \dots, z_n, t_1, \dots, t_n) = (z_1, \dots, z_n, t_1 + \cdots + t_n)$. We have already observed that $d\pi$ maps the $\bar\partial_b$-complex on $\widetilde{M}$ to the $\bar\partial_b$-complex on M. As discussed in Section 2.3, we can also use π to transfer the relative fundamental solutions for $\Box_b$ on $\widetilde{M}$ to relative fundamental solutions for $\Box_b$ on M. In Section 5.1 below, we use standard transference techniques to show that the resulting operators have the same L^p norm on M as the original operators have on $\widetilde{M}$. In Section 5.2 we study what this transference does to products of projections on $\widetilde{M}$. In Section 5.3 we show the existence of relative fundamental solutions $\mathcal{K}$ and $\mathcal{N}$, and in Section 5.4 we obtain L^p-regularity results for the relative fundamental solutions $\mathcal{N}$ and $\mathcal{K}$ for $\Box_b$ on the decoupled boundary M.

5.1. *A general transference result.* Let $\widetilde{T}$ be a measurable function on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}^n$ with compact support. Suppose that

$$\sup_{z\in\mathbb{C}^n} \iint_{\mathbb{C}^n\times\mathbb{R}^n} |\widetilde{T}(z,w,t)|\,dw\,dt = C_1 < +\infty;$$
$$\sup_{w\in\mathbb{C}^n} \iint_{\mathbb{C}^n\times\mathbb{R}^n} |\widetilde{T}(z,w,t)|\,dz\,dt = C_2 < +\infty.$$

Define an operator $\widetilde{\mathcal{T}}$ acting on functions on $\mathbb{C}^n \times \mathbb{R}^n$ by setting

$$\widetilde{\mathcal{T}}[F](z,t) = \iint_{\mathbb{C}^n\times\mathbb{R}^n} \widetilde{T}(z,w,t-s)\,F(w,s)\,dw\,ds.$$

$\widetilde{\mathcal{T}}$ is then a bounded operator on $L^p(\mathbb{C}^n \times \mathbb{R}^n)$ with the bound at most $C_1^{\frac{1}{p}}\,C_2^{\frac{1}{p'}}$ where $\frac{1}{p}+\frac{1}{p'}=1$.

Next, if $(z,w,t) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$, set

$$\begin{aligned} T(z,w,t) &= \int_{\Sigma(t)} \widetilde{T}(z,w,s)\,d\widetilde{s} \\ &= \int \widetilde{T}(z,w,t-s_2-\cdots-s_n,s_2,\dots,s_n)\,ds_2\cdots ds_n, \end{aligned}$$

where as before $\Sigma(t)$ is the affine hyperplane $\{s\in\mathbb{R}^n \mid \sum_{j=1}^n s_j = t\}$, and $d\tilde{s}$ is $(n-1)$ dimensional measure on $\Sigma(t)$. Given a measurable function f on $\mathbb{C}^n \times \mathbb{R}$, for $(z,t)\in\mathbb{C}^n\times\mathbb{R}$ define

$$\begin{aligned} \mathcal{T}[f](z,t) &\equiv \iint_{\mathbb{C}^n\times\mathbb{R}} T(z,w,t-s)\,f(w,s)\,dw\,ds \\ &= \iint_{\mathbb{C}^n\times\mathbb{R}^n} \widetilde{T}(z,w,s)\,f(w,t-s_1-\cdots-s_n)\,dw\,ds_1\cdots ds_n \\ &= \iint_{\mathbb{C}^n\times\mathbb{R}^n} \widetilde{T}(z,w,s)\,R_s[f](w,t)\,dw\,ds_1\cdots ds_n \end{aligned}$$

where for $s=(s_1,\dots,s_n)$, $R_s[f](z,t) = f(z,t-s_1-\cdots-s_n)$. Note that $\mathcal{T}[f]\circ\pi = \widetilde{\mathcal{T}}[f\circ\pi]$. It follows that if $f\in L^\infty(\mathbb{C}^n\times\mathbb{R})$, these integrals converge absolutely for all $(z,t)\in\mathbb{C}^n\times\mathbb{R}$. Moreover,

$$\sup_{z\in\mathbb{C}^n} \iint_{\mathbb{C}^n\times\mathbb{R}} |T(z,w,t)|\,dw\,dt = C_1 < +\infty;$$
$$\sup_{w\in\mathbb{C}^n} \iint_{\mathbb{C}^n\times\mathbb{R}} |T(z,w,t)|\,dz\,dt = C_2 < +\infty.$$

Thus $\mathcal{T}$ is bounded on $L^p(\mathbb{C}^n\times\mathbb{R})$ with norm at most $C_1^{\frac{1}{p}}\,C_2^{\frac{1}{p'}}$.

We now have the following transference result which shows that a better *a priori* L^p bound for $\widetilde{\mathcal{T}}$ gives the same bound for $\mathcal{T}$.

THEOREM 5.1.1. *Suppose there is a constant A_p such that*

$$\big|\big|\widetilde{\mathcal{T}}[F]\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R}^n)} \leq A_p\,\big|\big|F\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R}^n)}.$$

Then the operator $\mathcal{T}$ satisfies

$$\big|\big|\mathcal{T}[f]\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R})} \leq A_p\,\big|\big|f\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R})}.$$

Proof. We follow the argument in [CW77, Ch. 2]. Let $E \subset \mathbb{R}^n$ denote the (compact) projection onto $\mathbb{R}^n$ of the compact support of the function $\widetilde{T}$. Thus $\widetilde{T}(z,w,s) \neq 0$ implies $s \in E$. Choose $\varepsilon > 0$, and choose a (large) bounded open set $V \subset \mathbb{R}^n$ so that if

$$V + E = \Big\{x \in \mathbb{R}^n \,\Big|\, x = v + e \text{ with } v \in V \text{ and } e \in E\Big\},$$

then

$$\frac{|V+E|}{|V|} \leq 1+\varepsilon.$$

We have

$$R_y\big[\mathcal{T}[f]\big](z,t) = \iint\limits_{\mathbb{C}^n\times\mathbb{R}^n} \widetilde{T}(z,w,s-y)\,R_s[f](w,t)\,dw\,ds.$$

Let χ be the characteristic function of $V+E$. Since for any $y \in \mathbb{R}^n$,

$$\big|\big|R_y[f]\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R})} = \big|\big|f\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R})},$$

we can average over V and obtain

$$\begin{aligned}
||\mathcal{T}[f]||^p_{L^p(\mathbb{C}^n\times R)} &= \frac{1}{|V|}\int_V ||R_y\big[\mathcal{T}[f]\big]||^p_{L^P(\mathbb{C}^n\times\mathbb{R})}\,dy\\
&= \frac{1}{|V|}\int_V \Big[\iint\limits_{\mathbb{C}^n\times\mathbb{R}} \big|R_y\big[\mathcal{T}[f](z,t)\big]\big|^p\,dz\,dt\Big]\,dy\\
&= \frac{1}{|V|}\int_V \Big[\iint\limits_{\mathbb{C}^n\times\mathbb{R}} \big|\mathcal{T}[f](z,t-y_1-\cdots-y_n)\big|^p\,dz\,dt\Big]\,dy\\
&= \frac{1}{|V|}\int_V \Big[\iint\limits_{\mathbb{C}^n\times\mathbb{R}} \Big|\iint\limits_{\mathbb{C}^n\times\mathbb{R}^n} \widetilde{T}(z,w,s-y)\,R_s[f](w,t)\,dw\,ds\Big|^p\,dz\,dt\Big]\,dy\\
&\leq \frac{1}{|V|}\int_{\mathbb{R}} \Big[\iint\limits_{\mathbb{C}^n\times\mathbb{R}^n} \Big|\iint\limits_{\mathbb{C}^n\times\mathbb{R}^n} \widetilde{T}(z,w,s-y)\,R_s[f](w,t)\,\chi(s)\,dw\,ds\Big|^p dz\,dy\Big]\,dt\\
&= \frac{1}{|V|}\int_{\mathbb{R}} \Big[\iint\limits_{\mathbb{C}^n\times\mathbb{R}^n} \Big|\iint\limits_{\mathbb{C}^n\times\mathbb{R}^n} \widetilde{T}(z,w,y-s)\,R_{-s}[f](w,t)\,\chi(-s)\,dw\,ds\Big|^p dz\,dy\Big]\,dt\\
&= \frac{1}{|V|}\int_{\mathbb{R}} ||\widetilde{\mathcal{T}}[\widetilde{F}_t]||^p_{L^p(\mathbb{C}^n\times\mathbb{R}^n)}\,dt\\
&\leq A_p^p\,\frac{1}{|V|}\,\int_{\mathbb{R}} ||\widetilde{F}_t||^p_{L^p(\mathbb{C}^n\times\mathbb{R}^n)}\,dt
\end{aligned}$$

where

$$\widetilde{F}_t(w,s) = R_{-s}[f](w,t)\,\chi(-s) = f(w, t + s_1 + \cdots + s_n)\,\chi(-s).$$

But then

$$\begin{aligned}\int_{\mathbb{R}} \big|\big|\widetilde{F}_t\big|\big|_{L^p}^p\,dt &= \iiint\limits_{\mathbb{C}^n\times\mathbb{R}^{n+1}} |f(w, t+s_1+\cdots+s_n)\,\chi(-s)|^p\,dw\,ds\,dt\\ &= \big|V+E\big|\;\big|\big|f\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R})}^p.\end{aligned}$$

It follows that

$$\big|\big|\mathcal{T}[f]\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R})}^p \leq A_p^p\,(1+\varepsilon)\,\big|\big|f\big|\big|_{L^p(\mathbb{C}^n\times\mathbb{R})}^p,$$

which completes the proof. □

5.2. *Transference of products of projections.* Recall that on $\widetilde{M}$, the relative fundamental solutions $\widetilde{\mathcal{N}}_J$ satisfy

$$\widetilde{\mathcal{N}}_J\,\Box_J = \Box_J\,\widetilde{\mathcal{N}}_J = \prod_{j=1}^n \big(I - \mathcal{S}_J^{J(j)}\big) = I + \sum_{\emptyset\neq A\subset\{1,\dots,n\}} (-1)^{|A|}\,\widetilde{\mathcal{S}}_{J,A}$$

where $\widetilde{\mathcal{S}}_{J,A}$ is the operator with distribution kernel

$$\widetilde{S}_{J,A}(p,q) = \bigotimes_{j\in A} S_j^{J(j)}(p_j,q_j).$$

We want to understand what happens when this operator is transferred to M by the mapping π. If $|A| = r$ we obtain an operator $\mathcal{S}_{J,A}$ on M whose distribution kernel is given by

$$S_{J,A}(z_1,\dots,z_n,w_1,\dots,w_n,t) = \int_{\Sigma_r(t)} \prod_{j\in A} S_j^{J(j)}(z_j,w_j,r_j)\,d\tilde{r}, \tag{5.2.28}$$

where $\Sigma_r(t) = \big\{(r_1,\dots,r_q)\,\big|\,r_1+\cdots+r_q = t\big\}$, and $d\tilde{r}$ denotes the $(r-1)$-dimensional Lebesgue measure on $\Sigma_r(t)$. In other words, $S_{J,A}$ is the convolution in the t-variable of the r functions $\{S_j^{J(j)}\}_{j\in A}$. We can study such distributions by taking the partial Fourier transform in the t-variable, defined by

$$\hat{f}(z,w,\tau) = \mathcal{F}[f](z,w,\tau) = \int_{-\infty}^{+\infty} e^{-2\pi i t\tau} f(z,w,t)\,dt.$$

We have the following basic fact about the partial Fourier transforms of the individual distributions $S_j^{\pm}$.

LEMMA 5.2.1. *Let $\varphi : \mathbb{C}\to\mathbb{R}$ be subharmonic, and let $Z = \dfrac{\partial}{\partial z} + i\dfrac{\partial\varphi}{\partial z}\dfrac{\partial}{\partial t}$ and $\bar{Z} = \dfrac{\partial}{\partial\bar{z}} - i\dfrac{\partial\varphi}{\partial\bar{z}}\dfrac{\partial}{\partial t}$. Let $f\in L^2(\mathbb{C}\times\mathbb{R})$. If $Z[f] = 0$, the support* *of the partial Fourier transform $\hat{f}$ is contained in* $\Big\{(z,\tau)\in\mathbb{C}\times\mathbb{R}\,\Big|\,\tau\le 0\Big\}$. *If $\bar{Z}[f]=0$, the support of the partial Fourier transform $\hat{f}$ is contained in* $\Big\{(z,\tau)\in\mathbb{C}\times\mathbb{R}\,\Big|\,\tau\ge 0\Big\}$. *If $\mathcal{S}^{(+)}$ is the orthogonal projection onto the null space of Z, then the support of the partial Fourier transform of the distribution kernel $S^{(+)}$ is supported where $\tau\le 0$. If $\mathcal{S}^{(-)}$ is the orthogonal projection onto the null space of $\bar{Z}$, then the support of the partial Fourier transform of the distribution kernel $S^{(-)}$ is supported where $\tau\ge 0$.*

Proof. Since the proofs are identical, we only deal with the case of the operator Z. We have

$$\widehat{Z}[g](z,\tau)=e^{\tau\varphi(z)}\,\frac{\partial}{\partial z}\big[e^{-\tau\varphi}\,g\big](z,\tau).$$

Since $\mathcal{F}$ is an isometry, it follows that if $Z[f]=0$, then for almost every τ the function $z\to e^{-\tau\varphi(z)}\mathcal{F}[f](z,\tau)$ is an anti-holomorphic function h_τ. Again since $\mathcal{F}$ is an isometry, it follows that for almost every such τ,

$$\int_{\mathbb{C}}|h_\tau(z)||^2e^{\tau\varphi(z)}\,dm(z)<\infty.$$

However, if $\tau>0$, this implies that $h_\tau\equiv 0$. In fact, if $\tau>0$, then

$$\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[|h_\tau|^2\,e^{\tau\varphi}\right]=e^{\tau\varphi}\left[\left|\frac{\partial f}{\partial z}+\tau\,f\,\frac{\partial\varphi}{\partial z}\right|^2+\tau\,|h_\tau|^2\frac{\partial^2\varphi}{\partial z\,\partial\bar{z}}\right]\ge 0.$$

Hence the function $z\to|h_\tau(z)|^2e^{\tau\varphi(z)}$ is subharmonic, and so its value at any point is dominated by its average over a disk centered at the point of radius r. Letting $r\to\infty$ shows that $|h_\tau(z)|^2\,e^{\tau\varphi(z)}=0$. Thus we have shown that if f is in the null space of the operator Z in $L^2(\mathbb{C}\times\mathbb{R})$, then the support of $\mathcal{F}[f]$ is contained in $\big\{(z,\tau)\,\big|\,\tau\le 0\big\}$.

Next, since $\mathcal{S}[f](z,t)=\iint S(z,w,t-s)\,f(w,s)\,dw\,ds$, it follows that $\hat{\mathcal{S}}[f](z,\tau)=\int\hat{S}(z,w,\tau)\,\hat{f}(w,\tau)\,dw$. Since $\hat{S}[f](z,\tau)\equiv 0$ for all $f\in L^2(\mathbb{C}\times\mathbb{R})$ and all $\tau>0$, it follows that $\hat{S}(z,w,\tau)\equiv 0$ for $\tau>0$. This completes the proof. □

If we apply this result to the operators $\{\mathcal{S}_j^{J(j)}\}$ and use equation (5.2.28), we obtain

LEMMA 5.2.2. *The operators $\mathcal{S}_{J,A}=0$ on $M\times M$ unless $A\subset J$ or $A\cap J=\emptyset$.*

5.3. *Relative fundamental solutions for $\Box_J$ on M.* Recall that for each $J\in\vartheta_q$, we have an operator

$$\Box_J=\sum_{j=1}^{n}\Box_j^{J(j)}$$

which acts either on $\widetilde{M}$ or on M. We have constructed relative fundamental solutions for the operator on $\widetilde{M}$. We then construct relative fundamental solutions for $\Box_J$ on M by transferring operators $\widetilde{\mathcal{N}}_J$ and $\widetilde{\mathcal{K}}_J$ on $\widetilde{M}$ to operators $\mathcal{N}_J$ and $\mathcal{K}_J$ on M. This gives the first of the main results of this paper:

THEOREM 2.4.1. *For each of the* 2^n *possible operators* $\{\Box_J\}$, *there is a distribution* K_J *on* $M \times M$ *so that if* $\mathcal{K}_J$ *denotes the linear operator*

$$\mathcal{K}_J[\varphi](p) = \int_M \varphi(q)\, K_J(p,q)\, dq,$$

then

$$\mathcal{K}_J\,\Box_J = \Box_J\,\mathcal{K}_J = \begin{cases} I - S_0 & \text{if } \Box_J \text{ acts on functions;} \\ I & \text{if } \Box_J \text{ acts on a } (0,r)\text{-form with } 1 \le r \le n-1; \\ I - S_n & \text{if } \Box_J \text{ acts on } (0,n)\text{-forms.} \end{cases}$$

Proof. Formally the kernel of the operator $\mathcal{K}_J$ is given by

$$K_J(z,w,t) = \int_{\Sigma(t)} \widetilde{K}_J(z,w,r)\, d\tilde{r}.$$

However, we have observed that $\widetilde{K}_J$ can be approximated by functions which are smooth and have compact support in $(\underbrace{z,w},t)$. For such approximations, the integral converges absolutely. Since on $\widetilde{M}$

$$\widetilde{\mathcal{K}}_J\,\Box_J = \Box_J\,\widetilde{\mathcal{K}}_J = I - \prod_{j=1}^{n} S_j^{J(j)},$$

it follows from Lemma 5.2.2 that on M we have the desired equation for $\mathcal{K}_J$. This completes the proof. □

By a similar argument we have a companion result for the transfer of the operator $\widetilde{\mathcal{N}}$.

THEOREM 2.4.1A. *For each of the* 2^n *possible operators* $\{\Box_J\}$, *there is a distribution* N_J *on* $M \times M$ *so that if* $\mathcal{N}_J$ *denotes the linear operator*

$$\mathcal{N}_J[\varphi](p) = \int_M \varphi(q)\, N_J(p,q)\, dq,$$

then

$$\mathcal{N}_J\,\Box_J = \Box_J\,\mathcal{N}_J = I + \sum_{A \subset J} (-1)^{|A|} S_{J,A} + \sum_{A \cap J = \emptyset} (-1)^{|A|} S_{J,A}.$$

5.4. *L^p-regularity and replacements for maximal hypoellipticity.* We have the following regularity results for the operators $\mathcal{N}_J$ and $\mathcal{K}_J$.

THEOREM 5.4.1. *Let $Q = Q(Z)$ be any quadratic expression in the vector fields $\{Z_1, \bar{Z}_1, \dots, Z_n, \bar{Z}_n\}$ on M. Then the operator $Q(Z)\mathcal{N}_J$ is a bounded operator on $L^p(M)$ for $1 < p < \infty$.*

Proof. The statement about $Q(Z)\mathcal{N}$ follows from Theorem 4.2.1, the remark following it, and Theorem 5.1.1. □

As we have pointed out, the operator $\mathcal{K}$ does not satisfy all maximal hypoelliptic estimates, and consequently $Q(Z)\mathcal{K}$ for a general quadratic expression Q in the vector fields $\{Z_1, \bar{Z}_1, \dots, Z_n, \bar{Z}_n\}$ will not in general be bounded on $L^2(M)$. The second main result of this paper gives the appropriate substitute result. It is Theorem 2.4.2 stated in Section 2.

Proof. The proof of the theorem follows from 4.4.1 and Theorem 5.1.1, since operators on $\widetilde{M}$ involving $(T_k - T_l)$ map to the zero operator on M under transference. □

6. Pseudo-metrics on M

So far, we have obtained various L^p regularity results for the relative fundamental solutions to the Kohn-Laplacian on M. We are also interested in describing the nature of the singularities of the corresponding kernels. As pointed out in the introduction, we cannot expect that these kernels behave like standard fractional integration operators on a space of homogeneous type where there is one distinguished metric. The object of this section is to study two different metrics or pseudo-metrics on the hypersurface M given in equation (2.1.1) that are relevant to the analysis of the $\bar{\partial}_b$-complex. We will then be able to describe the singularities of our kernels in terms of these metrics.

6.1. *The sum of squares metric.* The vector fields $\{X_1, \dots, X_{2n}\}$ have the property that they and all their commutators span the tangent space at each point. Hence ([NSW85]) they define a natural nonisotropic metric on M which we write d_Σ. This metric has the property that balls of radius δ are essentially ellipsoids of radius δ in the directions of the vector fields $\{X_1, \dots, x_{2n}\}$, but are much smaller in the missing T direction.

Explicitly, if $p = (z, t)$ and $q = (w, s)$ are two points of M, then

$$d_\Sigma(p,q) \approx \sum_{j=1}^{n} |z_j - w_j| + \min_j \Big\{ \mu_j \Big(w_j, \Big| t - s + 2\,\Im\mathrm{m} \Big[\sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{1}{k!} \frac{X^k P_j}{X z_j^k}(w_j)(z_j - w_j) \Big] \Big| \Big) \Big\}.$$

The corresponding ball centered at (w, s) of radius δ is given by

$$B_\Sigma\big((w,s),\delta\big) \approx \Bigg\{(z,t) \in M \,\Bigg|\, |z_j - w_j| < \delta \qquad \text{and}$$
$$\Big|t - s + 2\,\Im\mathrm{m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m_j}\frac{1}{k!}\frac{\partial^k P_j}{\partial z_j^k}(w_j)(z_j - w_j)\Big]\Big| < \sum_{j=1}^{n}\Lambda_j(w_j,\delta)\Bigg\}.$$

The volume of this ball is

$$\Big|B_\Sigma\big((w,s),\delta\big)\Big| \approx \delta^{2n}\Big[\sum_{j=1}^{n}\Lambda_j(w_j,\delta)\Big]. \tag{6.1.29}$$

The appearance of the expression

$$\Big|t - s + 2\,\Im\mathrm{m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m_j}\frac{1}{k!}\frac{\partial^k P_j}{\partial z_j^k}(w_j)(z_j - w_j)\Big]\Big|$$

occasionally makes it difficult to work with this distance. However, it is always possible to make a biholomorphic change of variables so that the point (w, s) is moved to the origin, and all pure z and $\bar{z}$ derivatives of the defining polynomials $\{P_j\}$ vanish there.

Suppose that $w = (w_1, \dots, w_n, w_{n+1}) \in M$. Let $\Phi^w : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the biholomorphic mapping given by $\Phi^w(z_1, \dots, z_n, z_{n+1}) = (\zeta_1, \dots, \zeta_n, \zeta_{n+1})$ where

$$\begin{aligned}\zeta_j &= z_j - w_j, \qquad 1 \le j \le n;\\ \zeta_{n+1} &= z_{n+1} - w_{n+1} - 2i\sum_{j=1}^{n}\sum_{k=1}^{m_j}\frac{1}{k!}\frac{\partial^k P_j}{\partial z_j^k}(w_j)\,(z_j - w_j)^k.\end{aligned} \tag{6.1.30}$$

Also, set

$$P_j^w(\zeta_j) = \sum_{\substack{k\ge1\\ \ell\ge1}}\frac{1}{k!\,\ell!}\frac{\partial^{k+\ell}P_j}{\partial z_j^k\,\partial\bar{z}_k^\ell}(w_j)\,\zeta_j^k\,\bar{\zeta}_j^\ell. \tag{6.1.31}$$

Note that each P_j^w is again a subharmonic, nonharmonic polynomial of degree m_j, and that

$$\frac{\partial^k P_j^w}{\partial\zeta_j^k}(0) = \frac{\partial^k P_j^w}{\partial\bar{\zeta}_j^k}(0) = 0, \quad 1 \le k \le m_j. \tag{6.1.32}$$

The mapping Φ^w maps the point w to the origin of $\mathbb{C}^{n+1}$ and maps the hypersurface M to the hypersurface

$$M^w = \Big\{(z_1, \dots, z_n, z_{n+1}) \in \mathbb{C}^{n+1}\,\Big|\,\Im\mathrm{m}[z_{n+1}] = \sum_{j=1}^{n}P_j^w(z_j)\Big\}. \tag{6.1.33}$$

Write $w_{n+1} = s + i\sum_{j=1}^{n} P_j(w_j)$. Since we can identify M and M^w with $\mathbb{C}^n \times \mathbb{R}$, the mapping Φ^w induces a change of variables on $\mathbb{C}^n \times \mathbb{R}$ given by $\Phi^a(z_1, \ldots, z_n, t) = (\zeta_1, \ldots, \zeta_n, s)$ where

$$\begin{aligned} \zeta_j &= z_j - w_j, \qquad 1 \leq j \leq n; \\ s &= t - s + 2\,\Im\mathrm{m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m_j}\frac{1}{k!}\frac{\partial^k P_j}{\partial z_j^k}(w_j)(z_j - w_j)\Big]. \end{aligned} \tag{6.1.34}$$

In this case, we say that M is normalized at the origin, and we have the much simpler expressions

$$B_\Sigma(0,\delta) \approx \Big\{(z,t) \in M \,\Big|\, |z| < \delta \text{ and } |t| < \sum_{j=1}^{n}\Lambda_j(0,\delta)\Big\}.$$

The volume of this ball is given by

$$\big|B_\Sigma(0,\delta)\big| \approx \delta^{2n}\Big[\sum_{j=1}^{n}\Lambda_j(0,\delta)\Big].$$

The corresponding distance from (z,t) to the origin is

$$d_\Sigma(z,t) = \sum_{j=1}^{m}|z_j| + \min_j\Big\{\mu_j(0,|t|)\Big\}.$$

6.2. *The Szegö metric.* There is a second pseudo-metric d_S on M which also plays an important role in our analysis. In general it is not equivalent to the sum of squares metric. The ball of radius δ is essentially an ellipsoid of length δ in the T direction, and of length $\mu_j(p,\delta)$ in the z_j direction. Thus unlike the sum of square balls, these balls are not isotropic in the complex directions $z_1, \ldots, z_n$. It follows from the scaling arguments in [McN89] and [NRSW88] that the Szegö kernel on M behaves like a singular integral operator relative to the metric d_S, and so we call this the Szegö metric.

Definition 6.2.1. Let $p = (z_1, \ldots, z_n, t)$ and $q = (w_1, \ldots, w_n, s)$ be two points in M. Set

$$\begin{aligned} d_S(p,q) = &\sum_{j=1}^{n}\Lambda_j(w_j, |z_j - w_j|) \\ &+ \Big|t - s + 2\,\mathrm{Im}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m_j}\frac{1}{k!}\,\frac{\partial^k P_j}{\partial z^k}(w_j)\,(z_j - w_j)^k\Big]\Big|. \end{aligned}$$

In particular, if $q = (0,0)$ and if the domain is normalized at the origin, then

$$d_S(p,0) = \sum_{j=1}^{n}\Lambda(0,z_j) + |t|.$$

It is not hard to check that the function d_S has the properties of a pseudo-metric given in the following proposition.

Proposition 6.2.2. *The function d_S has the following properties*:

(1) *For all $p, q \in M$ we have $d_S(p,q) \geq 0$, and $d_S(p,q) = 0$ if and only if $p = q$.*

(2) *There exists a constant C so that for all $p, q \in M$,*
$$d_S(p,q) \leq C\, d_S(q,p).$$

(3) *There exists a constant C so that for all $p, q, r \in M$,*
$$d_S(p,r) \leq C\big[d_S(p,q) + d_S(q,r)\big].$$

The balls corresponding to this pseudo-metric are given by
$$B_S(p,\delta) = \Big\{ q \in M \,\Big|\, d_S(p,q) < \delta \Big\}.$$
The measure of these Szegö balls is then given by
$$\Big| B_S(p,\delta) \Big| \approx \delta \prod_{j=1}^{n} \mu_j(p,\delta). \tag{6.2.35}$$

6.3. *Comparison of d_Σ and d_S.* There is also a more intrinsic way of defining these balls and distances. Recall that $\bar{Z}_j = X_j + iX_{n+j}$ where the X_j are real vector fields. The ball B_Σ centered at q of radius δ is essentially the set of points p to which one can flow from q along piecewise smooth curves tangent to one of the vectors $\{X_1, \ldots, X_{2n}\}$ for a total time less than δ. The ball B_S centered at q of radius δ is essentially the set of points p to which one can flow from q along piecewise smooth curves which are tangent to one of the vectors $\{X_j, X_{n+j}\}$ for a total time less than $\mu_j(q;\delta)$, for $1 \leq j \leq n$.

From this description, or by direct calculation of the two distances, we obtain the following inclusion of balls and relationship between distances.

Lemma 6.3.1. *Let $0 < \delta$. Then*
$$B_\Sigma(q;\delta) \subset B_S\big(q;\, \max_j\{\Lambda_j(q;\delta)\}\big),$$
$$B_S(q;\delta) \subset B_\Sigma\big(q;\, \max_j\{\mu_j(q,\delta)\}\big).$$
Also
$$\min_j \Big\{\Lambda_j(q;\, d_\Sigma(p,q))\Big\} \leq d_S(p,q) \leq \max_j \Big\{\Lambda_j(q;\, d_\Sigma(p,q))\Big\},$$
$$\min_j \Big\{\mu_j(q;\, d_S(p,q))\Big\} \leq d_\Sigma(p,q) \leq \max_j \Big\{\mu_j(q;\, d_S(p,q))\Big\}.$$

Proof. Suppose that $p \in B_\Sigma(q;\delta)$. We can flow from q to p along a piecewise smooth curve tangent to one of the vectors $\{X_1, \dots, X_n\}$ for time less than or equal to δ. Hence we can flow from q to p along a curve which is tangent to one of X_j, X_{n+j} for a total time at most $\mu_j\big(q;\, \Lambda_j(q;\delta)\big) \le \mu_j\big(q;\, \max_k\{\Lambda_k(q;\delta)\}\big)$. It follows that $p \in B_S\big(q; \max_j\{\Lambda_j(q;\delta)\}\big)$. This proves the first inclusion of balls. The second is proved in the same way.

The inequalities follow from these inclusions. For example, suppose $d_\Sigma(p,q) = \delta$. Then $p \in B_\Sigma(q;\delta) \subset B_S\big(q, \max_j\{\Lambda_j(q;\delta)\}\big)$. Hence

$$d_S(p,q) \le \max_j\{\Lambda_j(q;\delta)\} = \max_j\{\Lambda_j\big(q;\, d_\Sigma[p,q]\big)\}.$$

This is the second part of the first inequality. Similarly, if $d_S(p,q) = \delta$, then $p \in B_S(q;\delta) \subset B_\Sigma\big(q; \max_j\{\mu_j(q;\delta)\}\big)$, and so $d_\Sigma(p,q) \le \max_j\{\mu_j(q;\delta)\} = \max_j\{\mu_j(q;\, d_S(p,q))\}$, which is the second part of the second inequality. The first half of each inequality follows in the same way. □

There is a relationship between the volumes of the balls $B_\Sigma(p,\delta)$ and $B_S(p,\delta)$. More generally, there is a relationship between the sizes of fractional integral operators with respect to these two metrics. Recall that if d is a metric, then a singular integral kernel $S(p,q)$ relative to d has size

$$|S(p,q)| \lesssim \big|B\big(p, d(p,q)\big)\big|^{-1}$$

and a fractional integral kernel $K(p,q)$ smoothing of order α has size

$$|K(p,q)| \lesssim d(p,q)^\alpha \,\big|B\big(p, d(p,q)\big)\big|^{-1}.$$

In the case of d_Σ and d_S we have

COROLLARY 6.3.2. *Suppose that $\alpha \ge 0$. Then*

$$\frac{\big(d_\Sigma(p,q)\big)^\alpha}{\big|B_\Sigma\big(q;\, d_\Sigma(p,q)\big)\big|} \lesssim \frac{\big(\max_j\{\mu_j\big(q;\, d_S(p,q)\big)\}\big)^\alpha}{\big|B_S\big(q;\, d_S(p,q)\big)\big|}.$$

Proof. We shall use the abbreviations $d_\Sigma(p,q) = d_\Sigma$ and $d_S(p,q) = d_S$. Using the volumes of the balls B_S and B_Σ given in equations (6.1.29) and (6.2.35), the stated inequality is equivalent to

$$d_\Sigma^{\alpha-2n}\Big[\sum_{j=1}^n \Lambda_j(p,d_\Sigma)\Big]^{-1} \lesssim d_S^{-1}\Big[\prod_{j=1}^n \mu_j(p,d_S)^2\Big]^{-1}\Big[\max_j\{\mu_j(p,d_S)\}\Big]^\alpha.$$

However, according to Lemma 6.3.1, we have $d_\Sigma \le \max_j\{\mu_j(p,d_S)\}$, $d_\Sigma^{-2n} \le \Big[\prod_{j=1}^n \mu_j(p,d_S)^2\Big]^{-1}$ and $d_S \le \sum_{j=1}^n \Lambda_j(p,d_\Sigma)$. This completes the proof. □

7. Differential inequalities for the relative fundamental solution K

We now show that the distribution kernel K_J for the relative fundamental solution $\mathcal{K}_J$ is singular only on the diagonal of $M \times M$, and we obtain estimates on the size of the kernel and its derivatives away from the diagonal.

7.1. *Statement of the main result.* Let d_S denote the Szegö metric on M and let d_Σ denote the sum of squares metric on M. One of the main results stated in Section 2 is Theorem 2.4.4. It gives the following estimate:

$$\Big|\Big[\prod_{j=1}^{n}\partial_j^{\alpha_j}\Big]\,K_J(p,q)\Big| \lesssim \frac{\Big[\sum_{j=1}^n \mu_j\big(p,d_S(p,q)\big)\Big]^2}{\Big|B_S\big(p,d_S(p,q)\big)\Big|}\;\log\left[2+\frac{\sum_{j=1}^n \mu_j\big(p,d_S(p,q)\big)}{d_\Sigma(p,q)}\right] \times \prod_{j=1}^{n}\Big[\mu_j\big(p,d_S(p,q)\big)^{-1}+d_\Sigma(p,q)^{-1}\Big]^{|\alpha_j|}.$$

Before beginning the proof of this, we make several remarks.

(1) At least formally, the distribution kernel $K(z,w,t)$ is given by

$$K(z,w,t) = \int_{\Sigma(t)} \widetilde{K}(z,w,r)\,d\tilde{r}. \tag{7.1.36}$$

The kernel $\widetilde{K}(z,w,r)$ has singularities whenever $z_j = w_j = r_j = 0$, and our formal integral (7.1.36) runs over these nonintegrable singularities. We deal with this difficulty as follows. We have observed that $\widetilde{K}(z,w,r)$ is the limit as $\varepsilon \to 0$ of kernels $\widetilde{K}_\varepsilon(z,w,r)$ which are, for $\varepsilon > 0$, smooth, bounded functions on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}^n$. We write $K(z,w,t) = \lim_{\varepsilon\to 0} K_\varepsilon(z,w,t)$ where

$$K_\varepsilon(z,w,t) = \int_{\Sigma(t)} \widetilde{K}_\varepsilon(z,w,r)\,d\tilde{r}$$

and where this integral now converges. Near points where the kernel $\widetilde{K}$ becomes singular, we integrate the corresponding $\widetilde{K}_\varepsilon$ by parts to obtain good estimates which are independent of ε. This will justify the formal calculations. In the discussion below, we will suppress the dependence on ε with the understanding that all estimates are uniform in ε.

(2) For simplicity of exposition, we shall deal only with the case $n = 2$. The estimates in this case are sufficiently complicated, and for larger values of n, the arguments require similar computations together with appropriate induction hypotheses.

(3) We shall further simplify the notation by making our computations of $K(z,w,t)$ at the point $w = 0$ and we shall assume that our domain is

normalized at the origin. As we have seen, there is no loss of generality in doing this, and various expressions involving distance functions become easier to write.

7.2. *Szegö kernels as derivatives.* In this section we establish the estimates used in integration by parts near points where the integrand in (7.1.36) has nonintegrable singularities. These singularities are caused by the presence of distribution kernel $S_j(z,w,t)$ of the Szegö projection $\mathcal{S}_j$ either onto the null space of $\bar Z_j$ or Z_j in $L^2(M_j)$:

$$\mathcal{S}_j[f](z,t) = \iint\limits_{\mathbb{C}\times\mathbb{R}} S_j(z,w,t-s)\, f(w,s)\, dw\, ds.$$

Recall that $S_j(z,w,t)$ is a distribution on $\mathbb{C}\times\mathbb{C}\times\mathbb{R}$ which is singular only on the set where $z=w$ and $t=0$. If ∂_j^α denotes a derivative of order $|\alpha|$ in the vector fields Z_j and $\bar Z_j$ acting either in the (z,t) or (w,t) variables, then we have the estimate

(7.2.37)
$$\left|\partial_j^\alpha\, \partial_t^k S_j(z,w,t-s)\right| \le C_{\alpha,k}\, d_j\big((z,t),(w,s)\big)^{-2-|\alpha|}\Lambda_j\big(z, d_j\big((z,t),(w,s)\big)\big)^{-1-k}.$$

In transferring from $\widetilde M$ to M, we shall be forced to integrate formally over the nonintegrable singularities that occur when $(z,t)=(w,s)$. What will save us is the fact that $\partial_j^\alpha\, S_j(z,w,t)$ is essentially a high derivative in t of a bounded function, and so we are able to integrate by parts. To see this, we recall that the Szegö kernel is given as an integral of the corresponding Bergman kernel. If ∂_j^α denotes a derivative of order α in the vector fields Z_j and $\bar Z_j$ acting either in the (z,t) or (w,s) variables, we have

$$\partial_j^\alpha S_j(z,w,t-s) = \int_0^\infty \partial_j^\alpha B_j(z,w,t-s+ir)\, dr.$$

Moreover, the Bergman kernel B_j satisfies the size estimates

(7.2.38)
$$\begin{aligned}&\left|\partial_j^\alpha\, \partial_t^k\, \partial_r^\ell B_j(z,w,t-s+ir)\right|\\ \le{}& C_{\alpha,k}\, \left[d_j\big((z,t),(w,s)\big)+\mu_j(z,r)\right]^{-2-|\alpha|}\left[\Lambda_j\big(z,d_j\big((z,t),(w,s)\big)\big)+r\right]^{-2-k-r}.\end{aligned}$$

(The relationship between the Szegö and Bergman projections and the estimates (7.2.37) and (7.2.38) for the distribution kernels can be found in [NRSW88].)

We shall use the following decomposition when we need to integrate by parts.

LEMMA 7.2.1. *Fix $\delta > 0$. Then for each $\alpha \geq 0$ and each integer $m > 2 + \frac{1}{2}\alpha$, there is a constant $C = C(\alpha, m)$, and there are functions $F_j^{(m)}(z,w,t)$ and $G_j^{(m)}(z,w,t)$ so that*

$$\partial_j^\alpha S_j(z,w,t) = F_j^{(m)}(z,w,t) + \partial_t^m G_j^{(m)}(z,w,t)$$

where

$$|F_j^{(m)}(z,w,t)| \leq C \, \frac{1}{[d_j((z,t),(w,0)) + \mu_j(z,\delta)]^{\alpha+2} \, [\Lambda_j(z, d_j(z,t),(w,0)) + \delta]}$$

$$|G_j^{(m)}(z,w,t)| \leq C \, \frac{\delta^m}{[d_j((z,t),(w,0)) + \mu_j(z,\delta)]^{\alpha+2} \, [\Lambda_j(z, d_j(z,t),(w,0)) + \delta]}.$$

Proof. Write

$$\partial_j^\alpha S_j(z,w,t) = \int_0^\delta \partial_j^\alpha B_j(z,w,t+is)\,ds + \int_\delta^\infty \partial_j^\alpha B_j(z,w,t+is)\,ds.$$

Now for any smooth function φ on the interval $[0,\delta]$, Taylor's theorem gives for any positive integer m

$$\int_0^\delta \varphi(s)\,ds = \sum_{k=0}^{m-1} \frac{(-1)^k}{(k+1)!}\,\delta^{k+1}\,\varphi^{(k)}(\delta) + \frac{(-1)^m}{m!}\int_0^\delta s^m\,\varphi^{(m)}(s)\,ds.$$

We use this with $\varphi(s) = \partial_j^\alpha\, B_j(z,w,t_i s)$, and get

$$\partial_j^\alpha S_j(z,w,t) = \sum_{k=0}^{m-1} \frac{(-1)^k\,\delta^{k+1}}{(k+1)!}\,\partial_s^k\,\partial_j^\alpha B_1(z,t+i\delta) + \int_\delta^\infty \partial_j^\alpha B_j(z,w,t+is)\,ds$$
$$+ \frac{(-1)^m}{m!}\int_0^\delta s^m\,\partial_s^m\,\partial_j^\alpha B_j(z,w,t+is)\,ds.$$

Set

$$F_j^{(m)}(z,t) = \sum_{j=0}^{m-1} \frac{(-1)^j\,\delta^{j+1}}{(j+1)!}\,\partial_s^j\,X^\alpha B_j(z,w,t+i\delta) + \int_\delta^\infty \partial_j^\alpha B_j(z,w,t+is)\,ds.$$

Because of the rate of decrease of $|\partial_j^\alpha B_j(z,t+is)|$ as $s \to \infty$, we have

$$\Big|\int_\delta^\infty \partial_j^\alpha B_j(z,w,t+is)\,ds\Big|$$
$$\lesssim \frac{1}{[d_j((z,t),(w,0)) + \mu_j(z,\delta)]^{\alpha+2} \, [\Lambda_j(z, d_j(z,t),(w,0)) + \delta]}.$$

Also, the estimates in (7.2.38) show that

$$\big|\partial_s^k\,\partial_j^\alpha B_j(z,w,t+i\delta)\big| \lesssim \frac{1}{[d_j((z,t),(w,0)) + \mu_j(z,\delta)]^{\alpha+2} \, [\Lambda_j(z, d_j(z,t),(w,0)) + \delta]}.$$

Thus we have established the correct estimate for $\big|F_j^{(m)}(z,t)\big|$.

On the other hand, since $B(z,w,\zeta)$ is holomorphic in ζ, we have

$$\frac{(-1)^m}{m!}\int_0^\delta s^m\,\partial_s^m\,\partial_j^\alpha B_j(z,w,t{+}is)\,ds = \partial_t^m\Big[\frac{(-i)^m}{m!}\int_0^d s^m\,\partial_j^\alpha B_j(z,w,t{+}is)\,ds\Big].$$

Set

$$G_j^{(m)}(z,t) = \frac{(-i)^m}{m!}\int_0^\delta s^m\,X^\alpha B_j(z,w,t+is)\,ds.$$

We have

$$\begin{aligned}
&\Big|\int_0^\delta s^m\,\partial_j^\alpha B_j(z,w,t+is)\,ds\Big| \\
&\qquad\le \int_0^\delta \frac{s^m}{[d_j((z,t),(w,0))+\mu_j(z,s)]^{\alpha+2}\,[\Lambda_j(z,d_j((z,t),(w,0))+s]^2}\,ds \\
&\qquad\le \int_0^{\mu_j(z,\delta)} \frac{\Lambda_j(z,s)^m\,\Lambda_j'(z,s)\,ds}{[d_j((z,t),(w,0))+s]^{\alpha+2}\;[\Lambda_j(z,d_j((z,t),(w,0))+\Lambda_j(z,s)]^2}.
\end{aligned}$$

Now $\Lambda_j(z,s)^{m-2}\,\Lambda_j'(z,s) = s^{2m-3}\varphi(s)$ where φ is increasing. Thus if $2m-3 > \alpha+1$, we can estimate this last integral by the length of the interval times the value of the integrand at the right-hand endpoint. Thus we get

$$\begin{aligned}
&\Big|\int_0^\delta s^m X^\alpha B_j(z,w,t+is)\,ds\Big| \\
&\qquad\lesssim \frac{\delta^m}{[d_j((z,t),(w,0))+\mu_j(z,\delta)]^{\alpha+2}\;[\Lambda_j(z,d_j((z,t),(w,0)))+\delta]}.
\end{aligned}$$

This gives us the required estimate for $G_j^{(m)}(z,t)$, and completes the proof of Lemma 7.2.1. □

We remark that if $\alpha = 0$ and $m = 2$, then the term corresponding to $G_2(z,w,t)$ is not bounded, but involves a logarithm. In fact, suppose $A, B > 0$. Then

$$\begin{aligned}
\int_0^B \frac{ds}{\mu_j(z,A+s)^2} &= \int_A^{A+B} \frac{ds}{\mu_j(z,s)^2} \\
&= \int_{\mu_j(z,A)}^{\mu_j(z,A+B)} \frac{\Lambda_j'(z,t)}{t^2}\,dt \\
&\lesssim \frac{(A+B)}{\mu_j(z,A+B)^2}\,\log\left[\frac{\mu_j(z,A+B)}{\mu_j(z,A)}\right].
\end{aligned}$$

7.3. *Proof of Theorem* 2.4.4. When $n = 2$, according to Theorem 4.3.1, we need to consider four integrals:

$$\begin{aligned} K(z_1, z_2, 0, 0, t) = &\int_{-\infty}^{+\infty} K_0(z_1, r, z_2, t-r)\, dr \\ &+ \int_{-\infty}^{+\infty} S_1(z_1, r)\, K_1(z_1, r, z_2, t-r)\, dr \\ &+ \int_{-\infty}^{+\infty} S_2(z_2, t-r)\, K_2(z_1, r, z_2, t-r)\, dr \\ &+ \int_{-\infty}^{+\infty} S_1(z_1, r)\, S_2(z_2, t-r)\, K_\infty(z_1, r, z_2, t-r)\, dr. \end{aligned}$$

The first of these is the easiest to analyze since the integral does not run across any singularities unless $z_1 = z_2 = t = 0$. The last of the four is the hardest since there are two possible places where we will need to integrate by parts: when $z_1 = r = 0$ and when $z_2 = t - r = 0$. We shall do the computations only in the first and last cases, since the other two integrals are then very similar.

7.3.1. *The estimate for* K_0. We shall use the abbreviations

$$d_\Sigma\big((z_1, r, z_2, t-r), (0,0,0,0)\big) = d_\Sigma(z_1, r, z_2, t-r) = d_\Sigma.$$

Recall that on $\widetilde{M}$,

$$d_\Sigma\big((z_1, r, z_2, t-r), (0,0,0,0)\big) \approx |z_1| + \mu_1(0, r) + |z_2| + \mu_2(0, t-r).$$

Also, recall that on M

$$d_\Sigma(z_1, z_2, t) \approx |z_1| + |z_2| + \min\big\{[\mu_1(0,t), \mu_2(0,t)\big\}.$$

Thus to estimate K_0 we have

$$\begin{aligned} X^\alpha\Big[\int_{-\infty}^{+\infty} K_0(z_1, r, z_2, t-r)\, dr\Big] \\ \lesssim \int_{-\infty}^{+\infty} d_\Sigma(z_1, r, z_2, t-r)^{-2-|\alpha|}\big[\Lambda_1(0, d_\Sigma) + \Lambda_2(0, d_\Sigma)\big]^{-2}\, dr. \end{aligned}$$

We split this integral into three parts: the first is where $|r| \leq \frac{1}{2}|t|$ in which case $|t - r| \approx |t|$; the second is where $|t - r| \leq \frac{1}{2}|t|$ in which case $|r| \approx |t|$, and finally the complement where $|t - r| \approx |r|$.

For the first integral we get the estimate

$$\begin{aligned} (|z_1| + |z_2| + \mu_2(t))^{-2-|\alpha|}\left[\Lambda_1(0, |z_1| + |z_2| + \mu_2(t)) + \Lambda_2(0, |z_1| + |z_2| + \mu_2(t))\right]^{-1} \\ \lesssim \frac{d_\Sigma(z_1, z_2, t)^{2-|\alpha|}}{\big|B_\Sigma\big(0, d_\Sigma(z_1, z_2, t)\big)\big|} \end{aligned}$$

which is an estimate of the correct sort. For the second integral, we interchange the roles of r and $t-r$, and obtain the same estimate. Finally for the third integral, we can make the estimate

$$\int_{|r|\geq\frac{1}{2}|t|} d_\Sigma(z_1,r,z_2,r)^{-2-|\alpha|}\left[\Lambda_1(0,d_\Sigma(z_1,r,z_2,r))+\Lambda_2(0,d_\Sigma(z_1,r,z_2,r))\right]^{-1}.$$

Because of the decay of the integral as $|r|\to\infty$, we can estimate this by $|t|$ times the value of the integrand at $r=|t|$. This again gives the correct sort of estimate, and completes the analysis of the integral involving K_0.

7.3.2. *The estimate for K_∞.* We now turn to the hardest estimate involving K_∞. Let X denote a derivative using either Z_1 or $\bar{Z}_1$, and let Y denote a derivative using either Z_2 or $\bar{Z}_2$. We want to estimate integrals of the form

$$\int_{-\infty}^{+\infty} X^{\alpha_1}S_1(z_1,r)\,Y^{\beta_1}S_2(z_2,t-r)\,X^{\alpha_2}\,Y^{\beta_2}K_\infty(z_1,r,z_2,t-r)\,dr \tag{7.3.39}$$

by

(7.3.40)
$$\frac{(\mu_1(d_S)+\mu_2(d_S))^2}{d_S\,\mu_1(d_S)^2\,\mu_2(d_S)^2}\,\log\left[2+\frac{d_S}{\min\left\{\Lambda_1(d_\Sigma),\,\Lambda_2(d_\Sigma)\right\}}\right] \cdot\left[d_\Sigma^{-1}+\mu_1(d_S)^{-1}\right]^{\alpha_1}\,\left[d_\Sigma^{-1}+\mu_2(d_s)^{-1}\right]^{\beta_1}\,d_\Sigma^{-(\alpha_2+\beta_2)}.$$

The integrand is dominated by

$$\begin{aligned}&[|z_1|+\mu_1(0,|r|)]^{-2-\alpha_1}\,[\Lambda_1(0,|z_1|)+|r|]^{-1}\,[|z_2|+\mu_2(0,|t-r|)]^{-2-\beta_1}\\&\quad\cdot[\Lambda_2(0,|z_2|)+|t-r|]^{-1}\,\left[|z_1|+\mu_1(0,|r|)+|z_2|+\mu_2(0,|t-r|)\right]^{2-(\alpha_2+\beta_2)}\end{aligned}$$

which in turn is dominated by a sum

$$\begin{aligned}&[|z_1|+\mu_1(0,|r|)]^{-\alpha_1}\,[\Lambda_1(0,|z_1|)+|r|]^{-1}\,[|z_2|+\mu_2(0,|t-r|)]^{-2-\beta_1}\\&\quad\cdot[\Lambda_2(0,|z_2|)+|t-r|]^{-1}\,\left[|z_1|+\mu_1(0,|r|)+|z_2|+\mu_2(0,|t-r|)\right]^{-(\alpha_2+\beta_2)}\\&\qquad+[|z_1|+\mu_1(0,|r|)]^{-2-\alpha_1}\,[\Lambda_1(0,|z_1|)+|r|]^{-1}\,[|z_2|+\mu_2(0,|t-r|)]^{-\beta_1}\\&\quad\cdot[\Lambda_2(0,|z_2|)+|t-r|]^{-1}\,\left[|z_1|+\mu_1(0,|r|)+|z_2|+\mu_2(0,|t-r|)\right]^{-(\alpha_2+\beta_2)}.\end{aligned}$$

In estimating the integral (7.3.39), the integrand can possibly become infinite when $r=0$ (if $|z_1|=0$), and when $r=t$ (if $|z_2|=0$).

We shall assume

$$\Lambda_1(0,|z_1|)\leq\Lambda_2(0,|z_2|). \tag{7.3.41}$$

Also, we have

$$d_S(z_1,z_2,t)\approx\Lambda_1(0,|z_1|)+\Lambda_2(0,|z_2|)+|t|, \tag{7.3.42}$$

and

$$d_\Sigma(z_1, z_2, t) \approx |z_1| + |z_2| + \min\big\{\mu_1(0, |t|),\ \mu_2(0, |t|)\big\}. \tag{7.3.43}$$

Case 1. Suppose $\Lambda_1(0, |z_1|) \le \Lambda_2(0, |z_2|) \le 2|t|$. In this case $d_S(z_1, z_2, t) \approx |t|$. First consider the integral over the region where $|r| \ge \frac{1}{2}|t|$ and $|t - r| \ge \frac{1}{2}|t|$. In this region, $|t - r| \approx |r|$, and we can dominate this part of the integral (7.3.39) by

$$\begin{aligned}
&\int_{|r|\ge\frac{1}{2}|t|} [|z_1| + \mu_1(r)]^{-2-\alpha_1}\ [\Lambda_1(z_1) + |r|]^{-1}\ [|z_2| + \mu_2(r)]^{-2-\beta_1} \\
&\qquad\qquad \cdot [\Lambda_2(z_2) + |r|]^{-1}\ [|z_1| + |z_2| + \mu_1(r) + \mu_2(r)]^{2-(\alpha_2+\beta_2)}\ dr \\
&\qquad \lesssim |t|\ [|z_1| + \mu_1(t)]^{-2-\alpha_1}\ [\Lambda_1(z_1) + |t|]^{-1}\ [|z_2| + \mu_2(t)]^{-2-\beta_1} \\
&\qquad\qquad \cdot [\Lambda_2(z_2) + |t|]^{-1}\ [|z_1| + |z_2| + \mu_1(t) + \mu_2(t)]^{2-(\alpha_2+\beta_2)} \\
&\qquad \lesssim \mu_1(d_S)^{-2-\alpha_1}\ \mu_2(d_S)^{-2-\beta_1}\ d_S^{-1}\ [\mu_1(d_S) + \mu_2(d_S)]^{2-(\alpha_2+\beta_2)}
\end{aligned}$$

since $|t| \approx d_S$. But since $\mu_1(d_S)+\mu_2(d_S) \ge d_\Sigma$, this part of the integral (7.3.39) is dominated by a constant times 7.3.40.

Next, we split the integral for $|r| \le \frac{1}{2}|t|$ into the region $A = \{\Lambda_1(0, \mu_2(0, |t|)) \le |r| \le \frac{1}{2}|t|\}$ and the region $B = \{|r| \le \min\big\{\Lambda_1(0, \mu_2(0, |t|)),\ \frac{1}{2}|t|\big\}\}$. In both of these regions we have $|t - r| \approx |t|$.

The integral of (7.3.39) over the region A is dominated by

$$\begin{aligned}
&\int_A [|z_1| + \mu_1(0, |r|)]^{-\alpha_1} [\Lambda_1(0, |z_1|) + |r|]^{-1} [|z_2| + \mu_2(0, |t|)]^{-2-\beta_1} \\
&\qquad \cdot [\Lambda_2(0, |z_2|) + |t|]^{-1}\ [|z_1| + \mu_1(0, |r|) + |z_2| + \mu_2(0, |t|)]^{-(\alpha_2+\beta_2)}\ dr \\
&\qquad + \int_A [|z_1| + \mu_1(0, |r|)]^{-2-\alpha_1} [\Lambda_1(0, |z_1|) + |r|]^{-1} [|z_2| + \mu_2(0, |t|)]^{-\beta_1} \\
&\qquad \cdot [\Lambda_2(0, |z_2|) + |t|]^{-1}\ [|z_1| + \mu_1(0, |r|) + |z_2| + \mu_2(0, |t|)]^{-(\alpha_2+\beta_2)}\ dr.
\end{aligned}$$

Note that $d_\Sigma(z_1, z_2, t) \lesssim |z_1| + \mu_2(0, |t|)$. Hence the first term in this sum is dominated by a constant times

$$d_\Sigma^{-\alpha_1} \mu_2\big(d_S\big)^{-2-\beta_1}\ d_S^{-1}\ d_\Sigma^{-(\alpha_2+\beta_2)}\ \log\left[2 + \frac{d_S}{\Lambda_1(d_\Sigma)}\right].$$

The second integral in this sum is dominated by

$$d_\Sigma^{-2-\alpha_1} \mu_2\big(d_S\big)^{-\beta_1}\ d_S^{-1}\ d_\Sigma^{-(\alpha_2+\beta_2)}.$$

since the integral converges at infinity. Thus since $\min\{\mu_1(d_S), \mu_2(d_S)\} \lesssim d_\Sigma$, both terms are dominated by 7.3.40.

In the integral of (7.3.39) over the region B, we integrate the term $X_1^\alpha\, S(z_1, r)$ by parts. This means replacing $X_1^\alpha\, S(z_1, r)$ by a sum $F_1^{(m)} + \partial_r^m\, G_1^{(m)}$. In estimating the term with $F_1^{(m)}$, we can still replace $|t - r|$ by $|t|$.

Let $r^* = \min\{\Lambda_1(\mu_2(t)), \frac{1}{2}|t|\}$. Then this term is dominated by a constant multiple of

$$\int\limits_{|r|\le r^*} [|z_1| + \mu_1(r^*)]^{-2-\alpha_1}\ [\Lambda_1(z_1) + |r^*|]^{-1}\, [|z_2| + \mu_t(t)]^{-2-\beta_1}\, [\Lambda_2(z_2) + |t|]^{-1}$$
$$\cdot [|z_1| + |z_2| + \mu_2(t)]^{-(\alpha_2+\beta_2)}\ [|z|_1 + |z_2| + \mu_1(t) + \mu_2(t)]^2\ dr.$$

If $r^* = \Lambda_1(\mu_2(t))$, we get the estimate

$$d_\Sigma^{-2-\alpha_1}\, \mu_2(d_S)^{-2-\beta_1}\, d_S^{-1}\, d_\Sigma^{-(\alpha_2+\beta_2)}.$$

If $r^* = \frac{1}{2}|t|$ we get the estimate

$$d_S^{-2-\alpha_1}\, \mu_2(d_S)^{-2-\beta_1}\, d_S^{-1}\, d_\Sigma^{-(\alpha_2+\beta_2)}.$$

Either case is still dominated by 7.3.40.

Now consider the actual integration of $G_1^{(m)}$ by parts. Each time we integrate, we introduce a factor of $(r^*)^{+1}$ from the integration and also introduce $\min\left\{(\Lambda_2(z_2) + |t|)^{-1},\ \Lambda_1(|z_1| + |z_2| + \mu_2(t))^{-1}\right\}$ from the differentiation. In either case, the ratio is bounded, and we are reduced to the same estimate as before.

Finally, we need to consider the integral of (7.3.39) over the region where $|t - r| \le \frac{1}{2}|t|$. But now we can interchange r and $t - r$, and the arguments are the same as those for the region $|r| \le \frac{1}{2}|t|$ with the roles of the subscripts 1 and 2 interchanged. We again get the estimate 7.3.40. This completes the analysis of Case 1.

Case 2. Suppose $2|t| \le \Lambda_2(0, |z_2|)$. In this case

$$d_S \approx \Lambda_2(z_2).$$

Also, $\min\{\mu_1(t),\ \mu_2(t)\} \le \mu_2(t) \le \mu_2(2t) \le |z_2|$. Thus

$$d_\Sigma(z_1, z_2, t) \approx |z_1| + |z_2|.$$

First consider the integral over the region where $|r| \ge \Lambda_2(z_2)$. Because of our hypothesis on $|t|$, it follows in this region that $|t - r| \approx |r|$, and so we need to estimate

$$\int\limits_{|r|\ge\Lambda_2(z_2)} [|z_1| + \mu_1(0, |r|)]^{-2-\alpha_1}\, [\Lambda_1(0, |z_1|) + |r|]^{-1}\, [|z_2| + \mu_2(0, |r|)]^{-2-\beta_1}$$
$$\cdot [\Lambda_2(0, |z_2|) + |r|]^{-1}\ [|z_1| + \mu_1(0, |r|) + |z_2| + \mu_2(0, |r|)]^{2-(\alpha_2+\beta_2)}\ dr.$$

We break up the term $[|z_1| + \mu_1(0, |r|) + |z_2| + \mu_2(0, |r|)]^2$ into two parts, and integrate the resulting integrals separately. Because of the rate of decay of the integrand, we can estimate the integrals by $\Lambda_2(z_2)$ times the value of the integrand at the left endpoint. We obtain the estimate

$$\mu_1(d_S)^{-2-\alpha_1}\,\mu_2(d_S)^{-\beta_2}\,d_S^{-1}\,d_\Sigma^{-(\alpha_2+\beta_2)}+\mu_1(d_S)^{-\alpha_1}\,\mu_2(d_S)^{-2-\beta_2}\,d_S^{-1}\,d_\Sigma^{-(\alpha_2+\beta_2)}.$$

This gives us the desired estimate 7.3.40.

Thus it remains to integrate over the region $|r|\le\Lambda_2(z_2)$ when $2|t|\le\Lambda_2(z_2)$ and hence $d_S=\Lambda_2(z_2)$ and $d_\Sigma=|z_1|+|z_2|$. There are four subcases to deal with.

Case 2a. Assume that $|z_1|\le|z_2|$ so that $d_\Sigma\approx|z_2|$ and assume that $\Lambda_2(z_2)\le\Lambda_1(z_2)$. In this case we integrate by parts, integrating $X_1^\alpha S_1(z,r)$, and differentiating the other parts of the integrand over the whole interval $|r|\le\Lambda_2(z_2)$. We must estimate

(7.3.44)

$$\int_0^{\Lambda_2(z_2)}\Big[F_1^{(m)}(z_1,r,d)+\partial_r^m[G_1^{(m)}(z_1,r,d)]\Big]\cdot Y^{\beta_1}S_2(z_2,t-r)X^{\alpha_2}Y^{\beta_2}K_\infty(z_1,r,z_2,t-r)\,dr.$$

We choose $d=\Lambda_2(z_2)=d_S$, and write

$$H(z_1,r,z_2,t-r)=Y^{\beta_1}S_2(z_2,t-r)X^{\alpha_2}Y^{\beta_2}K_\infty(z_1,r,z_2,t-r).$$

We can make the following estimates when $|r|\le\Lambda_2(z_2)$:

$$\begin{aligned}
\big|F_1^{(m)}(z_1,r,d)\big|&\le[|z_1|+\mu_1(d_S)]^{-2-\alpha_1}\,[\Lambda_1(z_1)+d_S]^{-1},\\
\big|G_1^{(m)}(z_1,r,t)\big|&\le d_S^m\,[|z_1|+\mu_1(d_S)]^{-2-\alpha_1}\,[\Lambda_1(z_1)+d_S]^{-1},\\
\big|H(z_1,r,z_2,t-r)\big|&\lesssim\Big[|z_2|^{-\beta_1}\,\Lambda_2(z_2)^{-1}\,[|z_1|+|z_2|]^{-(\alpha_2+\beta_2)}\\
&\qquad+|z_2|^{-2-\beta_1}\,\Lambda_2(z_2)^{-1}[|z_1|+|z_2|]^{-(\alpha_2+\beta_2)}\,[|z_1|+\mu_1(d_S)]^2\Big],\\
\big|\partial_r^m H(z_1,r,z_2,t-r)\big|&\lesssim\Big[|z_2|^{-\beta_1}\,\Lambda_2(z_2)^{-1}\,[|z_1|+|z_2|]^{-(\alpha_2+\beta_2)}\\
&\qquad+|z_2|^{-2-\beta_1}\,\Lambda_2(z_2)^{-1}[|z_1|+|z_2|]^{-(\alpha_2+\beta_2)}\,[|z_1|+\mu_1(d_S)]^2\Big]\\
&\qquad\times\max\{\Lambda_1(z_2)^{-1},\Lambda_2(z_2)^{-1}\}^m.
\end{aligned}$$

Now the term in the integral (7.3.44) that does not require integration by parts (the part involving $F_1^{(m)}$) can be estimated by

$$\mu_1(d_S)^{-2-\alpha_1}\,\mu_2(d_S)^{-\beta_1}\,d_S^{-1}\,d_\Sigma^{-(\alpha_2+\beta_2)}+\mu_1(d_S)^{-\alpha_1}\,\mu_2(d_S)^{-2-\beta_1}\,d_S^{-1}\,d_\Sigma^{-(\alpha_2+\beta_2)}$$

which is dominated by the estimate 7.3.40. After we integrate by parts, the term involving $G_1^{(m)}$ can be estimated in the same way since $\Lambda_2(z_2)\le\Lambda_1(z_2)$ and hence

$$\max\big\{\Lambda_1(z_2)^{-1},\,\Lambda_2(z_2)^{-1}\big\}=\Lambda_2(z_2)^{-1}=d_S^{-1}.$$

Case 2b. Assume that $|z_1| \le |z_2|$ so that $d_\Sigma \approx |z_2|$, and assume that $\Lambda_1(z_2) \le \Lambda_2(z_2) = d_S$. In this case, we integrate by parts only over the interval $|r| \le \Lambda_1(z_2)$, and just make elementary estimates on the part where $\Lambda_1(z_2) \le |r| \le \Lambda_2(z_2)$.

For the first integral, we must estimate

$$\int_0^{\Lambda_1(z_2)} \Big[F_1^{(m)}(z_1,r,d) + \partial_r^m[G_1^{(m)}(z_1,r,d)]\Big] \cdot Y^{\beta_1}S_2(z_2,t-r)X^{\alpha_2}Y^{\beta_2}K_\infty(z_1,r,z_2,t-r)\,dr. \tag{7.3.45}$$

We choose $d = \Lambda_1(z_2)$, and again write

$$H(z_1,r,z_2,t-r) = Y^{\beta_1}S_2(z_2,t-r)X^{\alpha_2}Y^{\beta_2}K_\infty(z_1,r,z_2,t-r).$$

We can make the following estimates when $|r| \le \Lambda_1(z_2)$:

$$\begin{aligned}
\big|F_1^{(m)}(z_1,r,d)\big| &\le [|z_1|+|z_2|]^{-2-\alpha_1}\,[\Lambda_1(z_1)+\Lambda_1(z_2)]^{-1}\\
&\approx d_\Sigma^{-2-\alpha_1}\,\Lambda_1(d_\Sigma)^{-1},\\
\big|G_1^{(m)}(z_1,r,t)\big| &\le (\Lambda_1(z_2))^m\,[|z_1|+|z_2|]^{-2-\alpha_1}\,[\Lambda_1(z_1)+\Lambda_1(z_2)]^{-1}\\
&\approx (\Lambda_1(z_2))^m\,d_\Sigma^{-2-\alpha_1}\,\Lambda_1(d_\Sigma)^{-1},\\
\big|H(z_1,r,z_2,t-r)\big| &\lesssim \Lambda_2(z_2)^{-1}\,[|z_1|+|z_2|]^{-(\beta_1+\alpha_2+\beta_2)}\\
&\approx d_S^{-1}\,d_\Sigma^{-(\beta_1+\alpha_2+\beta_2)},\\
\big|\partial_r^m H(z_1,r,z_2,t-r)\big| &\lesssim \Lambda_2(z_2)^{-1}\,[|z_1|+|z_2|]^{-(\beta_1+\alpha_2+\beta_2)}\\
&\qquad\cdot\max\{\Lambda_1(z_2)^{-1},\Lambda_2(z_2)^{-1}\}^m\\
&\approx d_S^{-1}\,d_\Sigma^{-(\beta_1+\alpha_2+\beta_2)}\,\Lambda_1(z_2)^{-m}.
\end{aligned}$$

The part of the integral in (7.3.45) is estimated by

$$d_S^{-1}\,d_\Sigma^{-(2+\alpha_1+\beta_1+\alpha_2+\beta_2)}$$

which is dominated by the estimate 7.3.40 since $\min\big\{\mu_1(d_S),\,\mu_2(d_S)\big\} \lesssim d_\Sigma$.

For the second integral, we must estimate

$$\int_{\Lambda_1(z_2)}^{d_S} d_\Sigma^{-\alpha_1}[\Lambda_1(z_1)+r]^{-1}\,[\Lambda_2(z_2)]^{-1}\,d_\Sigma^{-(2+\beta_1+\alpha_2+\beta_2)}\,dr + \int_{\Lambda_1(z_2)}^{d_S}[|z_1|+\mu_1(r)]^{-\alpha_1}[\Lambda_1(z_1)+r]^{-1}\,[\Lambda_2(z_2)]^{-1}\,d_\Sigma^{-(2+\beta_1+\alpha_2+\beta_2)}\,dr.$$

Both of these integrals are dominated by

$$d_S^{-1}\,d_\Sigma^{-(2+\alpha_1+\beta_1+\alpha_2+\beta_2)}\,\log\left[2+\frac{d_S}{\Lambda_1(d_\Sigma)}\right],$$

although the logarithm term only appears in the first integral. This again gives the estimate 7.3.40.

Cases 2c *and* 2d. Here we assume that $|z_2| \leq |z_1|$. This time we integrate the term $Y_1^{\beta} S_2(z_2, t-r)$, and everything goes as in Cases 2a and 2b with the roles of the subscripts 1 and 2 interchanged.

This completes the analysis of the integral involving K_∞, and consequently completes the proof of the estimates in Theorem 2.4.4 when $n = 2$.

8. Hölder regularity for $\mathcal{K}$

We now turn to the study of the smoothing properties of the relative fundamental solutions $\mathcal{K}_J$ on the scale of (isotropic) Hölder spaces, and the proof of Theorem 2.4.3.

8.1. *L^1 modulus of continuity of K.* We first observe that the presence of the logarithm term in the estimate for K does not affect the L^1 norm of K over small balls.

PROPOSITION 8.1.1. *There is a constant C such that for $\delta > 0$,*

$$\int_{d_S(p,q)<\delta} \big|K(p,q)\big| dq \leq C\Big[\sum_{j=1}^{n} \mu_j(p,\delta)\Big]^2.$$

Proof. Without loss of generality, we can assume that p is the origin, and that the domains M_j are normalized at the origin. We abbreviate $d_S\big((z,t),(0,0)\big)$ by $d_S(z,t)$, and the same for d_Σ. We write

$$\int_{d_S\left(z,t\right)<\delta} \big|K\big((z,t),(0,0)\big)\big|\, dz\, dt$$
$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \int_{\substack{d_S(z,t)\approx 2^{-j}\,\delta \\ \inf_j\{\Lambda_j(0,d_\Sigma(z,t))\}\approx 2^{-k} d_S(z,t)}} K\big((z,t),(0,0)\big)\, dz\, dt.$$

The size of the integrand of the $(j,k)^{\text{th}}$ integral is dominated by a constant times

$$k\,\frac{\sum_{j=1}^{n} \mu_j(0, 2^{-j}\,\delta)^2}{\big|B_S(0,2^{-j}\,\delta)\big|}$$

where the factor k comes from the logarithm term. On the other hand, the region of integration is given by

$$\sum_{j=1}^{n} \Lambda_j(0,|z_j|) + |t| \approx 2^{-j}\,\delta,$$
$$\inf_j \Big\{\Lambda_j\Big(0, \sum |z_j| + \min\{\mu_j(0,t)\}\Big)\Big\} \approx 2^{-k-j}\delta.$$

The volume of this is dominated by

$$2^{-\varepsilon k}\left|B_S(0, 2^{-j}\,\delta)\right|.$$

Hence the integral is dominated by

$$\sum_{j=0}^{\infty}\left[\sum_{\ell=1}^{n}\mu_\ell(0, 2^{-j}\delta)^2\sum_{k=0}^{\infty}k\,2^{-\varepsilon k}\right]\approx\left[\sum_{\ell=1}^{n}\mu_\ell(0,\delta)\right]^2. \qquad \square$$

Next, we have the following estimates on the L^1 modulus of continuity of the kernel K.

PROPOSITION 8.1.2. *Let $h \in \mathbb{C}^2 \times \mathbb{R}$ be a vector with Euclidean length $|h|$. Then*

$$\int_{B_S(p,1)}\left|K(p+h,q)-K(p,q)\right|dq \lesssim \sum_{j=1}^{n}\mu_j^{\#}(p,|h|)^2$$

where

$$\mu_j^{\#}(p,\delta)^2 = \delta\int_\delta^1 \mu_j(p,t)^2\,\frac{dt}{t^2} \geq \mu_j(p,\delta)^2.$$

Proof. We split the integral into two parts. The first is where $d_S(p,q) \leq 10|h|$, and here we simply estimate the L^1 norm of K as in Proposition 8.1.1. In the second integral, where $d_S(p,q) \geq 10|h|$, we use the estimate

$$\begin{aligned}\left|K(p+h,q)-K(p,q)\right| &\leq h\left|\partial_t K(p,q)\right|\\ &\leq h\,\frac{\sum_{j=1}^{n}\mu_j(p,d_S(p,q))^2}{d_S(p,q)\,V(p,q)}.\end{aligned}$$

A similar estimate then gives the desired result, and completes the proof. $\square$

If we use a second difference, we can improve on Proposition 8.1.2.

PROPOSITION 8.1.3. *Let $h \in \mathbb{C}^2 \times \mathbb{R}$. Then*

$$\int_{B_S(p,1)}\Big|K(p+h,q)+K(p-h,q)-2K(p,q)\Big|\,dq \lesssim \sum_{j=1}^{n}\mu_j(p,|h|)^2.$$

8.2. *Applications to smoothness.* As an immediate consequence of the above we have the following result:

THEOREM 8.2.1. *Suppose that f is a bounded function supported in $B_S(p_0, 10)$. Then for $p \in B_S(p_0, 1)$,*

$$\big|\mathcal{K}[f](p+h) - \mathcal{K}[f](p)\big| \leq C \left[\sum_{j=1}^{n} \mu_j^{\#}(p, |h|)^2\right],$$

$$\big|\mathcal{K}[f](p+h) + \mathcal{K}[f](p-h) - 2\mathcal{K}[f](p)\big| \leq C \left[\sum_{j=1}^{n} \mu_j(p, |h|)^2\right].$$

In particular, if $m = \max\{m_1, m_2\} > 2$ is the maximum type, then

$$\big|\mathcal{K}[f](p+h) - \mathcal{K}[f](p)\big| \leq C\,|h|^{\frac{2}{m}}.$$

In general, (when $m \geq 2$),

$$\big|\mathcal{K}[f](p+h) + \mathcal{K}[f](p-h) - 2\mathcal{K}[f](p)\big| \leq C\,|h|^{\frac{2}{m}}.$$

Indeed, $\mu_j(p,\delta)^2 \lesssim \delta^{\frac{2}{m_j}}$ by Definition 3.1.2 of μ_j. Next, since $(\mu_j^{\#})^2 = \delta \int_\delta^1 \mu_j(t)^2 \,\frac{dt}{t}$, we see that $(\mu_j^{\#}(\delta))^2 \lesssim \delta^{\frac{2}{m_j}}$, when $m_j > 2$. The desired conclusions are therefore established.

9. Examples

In this section we provide examples that show where our regularity results for $\Box_b$ are optimal. We study the same decoupled boundary as in Section 1.2, and let

$$M = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \,\Big|\, \Im\mathrm{m}[z_3] = |z_1|^n + |z_2|^m\right\}$$

where m and n are positive even integers. Assume that $m \geq n$. We consider the operator $\Box_b$ as it acts on $(0,1)$-forms of the form $f\,d\bar{z}_2$. Then

$$\Box_b[f\,d\bar{z}_2] = (\Box_1 + \Box_2)[f]\,d\bar{z}_2$$

where $\Box_1 = -Z_1\,\bar{Z}_1$ and $\Box_2 = -\bar{Z}_2\,Z_2$. Since we are not in degree zero or two, the operator $\Box_b$ has no null space in $L^2(M)$, and we have constructed an operator $\mathcal{K}$ so that $\mathcal{K}\,\Box_b = \Box_b\,\mathcal{K} = I$.

If we identify M with $\mathbb{C}^2 \times \mathbb{R}$ then as in Section 1.2 we have

$$\bar{Z}_1 = \frac{\partial}{\partial \bar{z}_1} - i\,\frac{n}{2}|z_1|^{n-2}\,z_1\frac{\partial}{\partial t}, \qquad \bar{Z}_2 = \frac{\partial}{\partial \bar{z}_2} - i\,\frac{m}{2}|z_2|^{m-2} z_2\frac{\partial}{\partial t}.$$

For $\gamma \in \mathbb{R}$ set

$$F_\gamma(z_1, z_2, t) = (t + i|z_1|^n + i|z_2|^m)^\gamma.$$

It is easy to check that

$$\begin{aligned}
&\Box_1[F_\gamma] = 0,\\
&\Box_2[F_\gamma] = -im(m-2)|z_2|^{m-4}z_2^2\,F_{\gamma-1},\\
&\overline{\Box}_1[F_\gamma] = -in(n-2)|z_1|^{n-4}z_1^2\,F_{\gamma-1},\\
&\overline{\Box}_2[F_\gamma] = 0,
\end{aligned}$$

and hence that

$$|\Box_b[F_\gamma]| \approx |z_2|^{m-2}\,|F_{\gamma-1}|.$$

We now show that the Hölder regularity established in Theorem 2.4.3 is optimal. Observe that $\Box_b[F_\gamma]$ is bounded near the origin on M if and only if $m-2+m(\gamma-1)\geq 0$, that is, if and only if $\gamma \geq 2/m$. However, if F_γ satisfies an isotropic Hölder condition of order α, then so does its restriction to the line $z_1 = z_2 = 0$. But $F_\gamma(0,0,t) = t^\gamma$ which does not satisfies a Hölder condition of order greater than γ. Thus on M, if a function g is bounded, the equation $\Box_b[u] = g$ has solutions which do not satisfy Hölder conditions of any order greater than $2/m = 2/\max\{m,n\}$.

Next, Theorem 2.4.2 (b) provides a replacement for estimates giving maximal hypoellipticity. Suppose that $\Box_b[u] \in L^p(M)$. It need not follow that $\overline{\Box}_1[u] \in L^p(M)$, but we show that if $|z_1|^{n-2}\,|B(z_1,z_2,t)| \lesssim |z_2|^{m-2}$, it does follow that $B\,\Box_1[u] \in L^p(M)$. Let $\chi \in \mathcal{C}_0^\infty(M)$ be with $\chi \equiv 1$ near the origin. Then

$$\begin{aligned}
B\,\overline{\Box}_1[\chi F_\gamma] &\approx B\,|F_{\gamma_1}|\,|z_1|^{n-2}\\
&= \Big(B\,\frac{|z_1|^{n-2}}{|z_2|^{m-2}}\Big)\Big[|z_2|^{n-2}\,|F_{\gamma-1}|\Big] \approx \Big(B\,\frac{|z_1|^{n-2}}{|z_2|^{m-2}}\Big)\,\big|\Box_b[\chi F_\gamma]\big|.
\end{aligned}$$

Since $F_\gamma \in L^p_{\mathrm{loc}}(M)$ if and only if $\gamma\,p > -\big(1+\dfrac{2}{m}+\dfrac{2}{n}\big)$, we see that our condition is essentially optimal.

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References

[BNW88] J. Bruna, A. Nagel, and S. Wainger, Convex hypersurfaces and Fourier transforms, *Ann. of Math.* **127** (1988), 333–365.

[Cat83] D. Catlin, Necessary conditions for subellipticity of the $\overline{\partial}$- Neumann problem, *Ann. of Math.* **117** (1983), 147–171.

[Cat87] ———, Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains, *Ann. of Math.* **126** (1987), 131–191.

[Chr88] M. Christ, Regularity properties of the $\overline{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension 3, *J. Amer. Math. Soc.* **1** (1988), 587–646.

[Chr91a] ———, On the $\bar{\partial}$ equation in weighted L^2 norms in $\mathbb{C}^1$, *J. Geom. Anal.* **1** (1991), 193–230.

[Chr91b] ———, On the $\bar{\partial}_b$ equation for three-dimensional CR manifolds, in *Several Complex Variables and Complex Geometry*, *Part* 3, *Proc. Symposia Pure Math.* **52**, 63–82, A. M. S., Providence, RI, 1991.

[CNS92] D.-C. Chang, A. Nagel, and E. M. Stein, Estimates for the $\bar{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$, *Acta Math.* **169** (1992), 153–228.

[CW77] R. R. Coifman and G. Weiss, *Transference Methods in Analysis*, CBMS *Reg. Conf. Series in Math.* **31**, A. M. S., Providence, RI, 1977.

[D'A82] J. P. D'Angelo, Real hypersurfaces, orders of contact, and applictions, *Ann. of Math.* **115** (1982), 615–637.

[Der78] M. Derridj, Regularité pour $\bar{\partial}$ dans quelques domaines faiblement pseudoconvexes, *J. Differential Geom.* **13** (1978), 559–576.

[Fef95] C. Fefferman, On Kohn's microlocalization of $\overline{\partial}$ problems, in *Modern Methods in Complex Analysis*, *Ann. of Math. Studies* **137**, 119–133, Princeton Univ. Press, Princeton, NJ, 1995.

[FK88] C. Fefferman and J. J. Kohn, Hölder estimates on domains of complex dimension two and on three dimensional CR manifolds, *Adv. in Math.* **69** (1988), 233–303.

[FKM90] C. Fefferman, J. J. Kohn, and M. Machedon. Hölder estimates on CR manifolds with a diagonalizable Levi form, *Adv. in Math.* **84** (1990), 1–90.

[FS74] G. B. Folland and E. M. Stein, Estimates for the $\bar{\partial}_b$-complex and analysis on the Heisenberg group, *Comm. Pure and Appl. Math.* **27** (1974), 429–522.

[Koe02] K. Koenig, On maximal Sobolev and Hölder estimates for the tangential Cauchy-Riemann operator and boundary Laplacian, *Amer. J. Math.* **124** (2002), 129–197.

[Koh72] J. J. Kohn, Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two, *J. Differential Geom.* **6** (1972), 523–542.

[Koh79] ———, Subellipticity of the $\overline{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions, *Acta Math.* **142** (1979), 79–122.

[Koh85] ———, Estimates for $\bar{\partial}_b$ on compact pseudoconvex CR manifolds, in *Proc. Symposia Pure Math.* **43**, 207–217, A. M. S., Providence, RI, 1985.

[Mac88] M. Machedon, Estimates for the parametrix of the Kohn Laplacian on certain domains, *Invent. Math.* **91** (1988), 339–364.

[McN89] J. D. McNeal, Boundary behavior of the Bergman kernel function in $\mathbb{C}^2$, *Duke Math. J.* **58** (1989), 499–512.

[McN94a] ———, The Bergman projection as a singular integral operator, *J. Geom. Anal.* **4** (1994), 91–104.

[McN94b] ———, Estimates on the Bergman kernels of convex domains, *Adv. Math.* **109** (1994), 108–139.

[MRS95] D. Müller, F. Ricci, and E. M. Stein, Marcinkiewicz multipliers and multiparameter structure on Heisenberg (-type) groups, I, *Invent. Math.* **119** (1995), 119–233.

[NRS01] A. Nagel, F. Ricci, and E. M. Stein, Singular integrals with flag kernels and analysis on quadratic CR manifolds, *J. Funct. Anal.* **181** (2001), 29–118.

[NRSW88] A. Nagel, J.-P. Rosay, E. M. Stein, and S. Wainger, Estimates for the Bergman and Szegö kernels in certain weakly pseudoconvex domains, *Bull. Amer. Math. Soc.* **18** (1988), 55–59.

[NRSW89] ———, Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$, *Ann. of Math.* **129** (1989), 113–149.

[NS01a] A. Nagel and E. M. Stein, The $\Box_b$-heat equation on pseudoconvex manifolds of finite type in $\mathbb{C}^2$, *Math. Z.* **238** (2001), 37–88.

[NS01b] ———, Differentiable control metrics and scaled bump functions, *J. Differential Geom.* **5** (2001), 465–492.

[NS04] ———, On the product theory of singular integrals, *Rev. Mat. Iberoamericana* **20** (2004), 531–561.

[NSW85] A. Nagel, E. M. Stein, and S. Wainger, Balls and metrics defined by vector fields I: Basic properties, *Acta Math.* **155** (1985), 103–147.

[PS86] D. H. Phong and E. M. Stein, Hilbert integrals, singular integrals, and Radon transforms. I, *Acta Math.* **157** (1986), 99–157.

[Rot80] L. P. Rothschild, Nonexistence of optimal L^2 estimates for the boundary Laplacian operator on certain weakly pseudoconvex domains, *Comm. Partial Differential Equations* **5** (1980), 897–912.

[RS76] L. P. Rothschild and E. M. Stein, Hypoelliptic differential operators and nilpotent groups, *Acta Math.* **137** (1976), 247–320.

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