

Higher genus Gromov-Witten invariants as genus zero invariants of symmetric products

By KEVIN COSTELLO

Abstract

I prove a formula expressing the descendent genus g Gromov-Witten invariants of a projective variety X in terms of genus 0 invariants of its symmetric product stack $S^{g+1}(X)$. When X is a point, the latter are structure constants of the symmetric group, and we obtain a new way of calculating the Gromov-Witten invariants of a point.

1. Introduction

Let X be a smooth projective variety. The genus 0 Gromov-Witten invariants of X satisfy relations which imply that they can be completely encoded in the structure of a Frobenius manifold on the cohomology $H^*(X, \mathbb{C})$. In this paper I prove a formula which expresses the descendent genus g Gromov-Witten invariants of a smooth projective variety X in terms of the descendent genus 0 invariants of the symmetric product stack $S^{g+1}X$. The latter are encoded in a Frobenius manifold structure on the orbifold cohomology group $H_{\text{orb}}^*(S^{g+1}(X), \mathbb{C})$. This implies that the Gromov-Witten invariants of X at all genera are described by a sequence of Frobenius manifold structures on the homogeneous components of the Fock space

$$\mathcal{F} = \text{Sym}^*(H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} t\mathbb{C}[t]) = \bigoplus_{d \geq 0} H_{\text{orb}}^*(S^d(X), \mathbb{C}).$$

Standard properties of genus 0 invariants, such as associativity, when applied to the symmetric product stacks S^dX , yield implicit relations among higher-genus Gromov-Witten invariants of X .

When $X = *$ is a point, the symmetric product is the classifying stack BS_d of the symmetric group. The Frobenius manifold associated to the genus 0 invariants of BS_d is in fact a Frobenius *algebra*, which is the center of the group algebra of the symmetric group, $\mathbb{C}[S_d]^{S_d}$. Our result therefore gives a new way of expressing the integrals of tautological classes on $\overline{\mathcal{M}}_{g,n}$ in terms of structure constants of $\mathbb{C}[S_d]$.

More generally, the associativity constraints, together with some other simple properties, are sufficient to determine the small quantum cohomology of the symmetric product stack $S^d X$ in terms of the small quantum cohomology of X . The construction of Lehn-Sorger [26] (see also Fantechi-Göttsche [15]), which calculates the orbifold cohomology of $S^d X$ in terms of the ordinary cohomology of X , applies verbatim to calculate the small quantum cohomology of $S^d X$ in terms of that of X . In general, the large quantum cohomology of $S^d X$ is not determined by that of X .

Let me sketch the geometric relation between Gromov-Witten invariants of X and $S^d X$. Stacks of stable maps to the symmetric product stack $S^d X$ are identified with stacks of certain correspondences $\mathcal{C} \leftarrow \mathcal{C}' \rightarrow X$, where \mathcal{C} and \mathcal{C}' are twisted balanced nodal curves, and $\mathcal{C}' \rightarrow \mathcal{C}$ is étale of degree d . We introduce stacks $\overline{\mathcal{M}}_\eta(X)$, parametrizing such correspondences with certain markings on \mathcal{C}' and \mathcal{C} , where $g(\mathcal{C}) = 0$.¹ Here η is some label remembering the genera of \mathcal{C}' and \mathcal{C} , the stack structure at the marked points, the homology class of the map $\mathcal{C}' \rightarrow X$, and so forth. There is a finite group G acting without fixed points on $\overline{\mathcal{M}}_\eta(X)$, by reordering marked points of \mathcal{C}' , such that $\overline{\mathcal{M}}_\eta(X)/G$ is a stack of stable maps from genus 0 curves to $S^d X$. This implies that integrals on $\overline{\mathcal{M}}_\eta(X)$ are Gromov-Witten invariants of $S^d X$.

There is a map $p : \overline{\mathcal{M}}_\eta(X) \rightarrow \overline{\mathcal{M}}_{g,r,\beta}(X)$, for some g, r and $\beta \in H_2(X)$, defined by taking the coarse moduli space C' of \mathcal{C}' , with its natural map $C' \rightarrow X$, and forgetting some marked points. We show that p is finite of degree $k \in \mathbb{Q}^\times$, in the virtual sense. By this we mean

$$(1.0.1) \quad p_*[\overline{\mathcal{M}}_\eta(X)]_{\text{virt}} = k[\overline{\mathcal{M}}_{g,r}(X)]_{\text{virt}}$$

We then express the pull back $p^*\psi_i$ of the tautological ψ classes on $\overline{\mathcal{M}}_{g,r}(X)$, in terms of ψ classes and boundary divisors of $\overline{\mathcal{M}}_{v' \rightarrow v}(X)$. The boundary cycles of $\overline{\mathcal{M}}_\eta(X)$ are again products of similar stacks of étale correspondences. Further, there is a commutative diagram of evaluation maps

$$\begin{array}{ccc} \overline{\mathcal{M}}_\eta(X) & \longrightarrow & X^r \\ \downarrow & \nearrow & \\ \overline{\mathcal{M}}_{g,r}(X) & & \end{array}$$

This allows us to translate integrals on $\overline{\mathcal{M}}_{g,r}(X)$ of ψ classes, and cohomology classes pulled back from X^r , into sums of products of similar integrals on $\overline{\mathcal{M}}_{0,n}(S^m X)$ for varying m and n .

The most technically difficult part of this procedure is proving the push-forward formula (1.0.1). We do this by working in a “universal” setting, where all the moduli stacks are smooth; and deduce it for arbitrary X by base change,

¹We identify the genus of a twisted curve with that of its coarse moduli space.

in the virtual sense, by the stack $\overline{\mathcal{M}}_{g,\beta}(X)$ of curves in X with no markings. We need to introduce moduli stacks of curves with markings in a semigroup. Let A be a semigroup with indecomposable zero; for each $a \in A$ we define a moduli stack $\mathfrak{M}_{g,n,a}$ of all (possibly unstable) connected nodal curves of genus g , with n marked points, and certain A -valued marking on the irreducible components. These curves must satisfy some stability conditions; for example when $a = 0$, but not otherwise, $\mathfrak{M}_{g,n,0} = \overline{\mathcal{M}}_{g,n}$ is the usual Deligne-Mumford moduli stack of stable curves. In general, $\mathfrak{M}_{g,n,a}$ is smooth, proper, locally of finite type, but nonseparated. The advantage of these moduli stacks over the more familiar stacks $\mathfrak{M}_{g,n}$ of all nodal curves, is that there are (proper, separated) contraction maps $\mathfrak{M}_{g,n,a} \rightarrow \mathfrak{M}_{g,n-1,a}$, which identify $\mathfrak{M}_{g,n,a}$ with the universal curve over $\mathfrak{M}_{g,n-1,a}$. This is not the case for $\mathfrak{M}_{g,n}$.

Let $C(X)$ be the cone of effective 1-cycles on X modulo numerical equivalence. For each $\beta \in C(X)$ we have the associated smooth moduli stacks $\mathfrak{M}_{g,n,\beta}$, and the stacks of stable maps $\overline{\mathcal{M}}_{g,n,\beta}(X)$. Now,

$$\overline{\mathcal{M}}_{g,n,\beta}(X) = \mathfrak{M}_{g,n,\beta} \times_{\mathfrak{M}_{g,\beta}} \overline{\mathcal{M}}_{g,\beta}(X).$$

More generally, for any connected modular graph γ with labellings in $C(X)$, so that γ defines a stratum of $\overline{\mathcal{M}}_{g,n,\beta}(X)$,

$$\overline{\mathcal{M}}_{\gamma}(X) = \mathfrak{M}_{\gamma} \times_{\mathfrak{M}_{g,\beta}} \overline{\mathcal{M}}_{g,\beta}(X).$$

Further, these fibre products are compatible with virtual fundamental classes. That is, the system of stacks of stable maps to X , together with their natural morphisms and virtual classes, is pulled back, via the map $\overline{\mathcal{M}}_{g,\beta}(X) \rightarrow \mathfrak{M}_{g,\beta}$, from the stacks $\mathfrak{M}_{g,n,a}$ with their natural morphisms.

We can extend this observation to stacks of étale correspondences to X :

$$\overline{\mathcal{M}}_{\eta}(X) = \mathfrak{M}_{\eta} \times_{\mathfrak{M}_{g,\beta}} \overline{\mathcal{M}}_{g,\beta}(X)$$

where \mathfrak{M}_{η} is some stack of étale maps of curves $\mathcal{C}' \rightarrow \mathcal{C}$. This fibre product is also compatible with virtual fundamental classes.

Since all of these fibre products are in the virtual sense, they behave quite like flat base changes for the purposes of intersection theory. We show that to prove the map $\overline{\mathcal{M}}_{\eta}(X) \rightarrow \overline{\mathcal{M}}_{g,r,\beta}(X)$ is finite in the virtual sense as in formula (1.0.1), it is sufficient to show that $\mathfrak{M}_{\eta} \rightarrow \mathfrak{M}_{g,r,\beta}$ is actually finite. With the correct choices of η , this is not difficult.

1.1. *Relation to previous work. Intersection numbers on moduli stacks of curves.* The Gromov-Witten theory of a point has been known since Kontsevich's proof [21] of Witten's conjecture [33]. There are two parts to Kontsevich's proof. Firstly, he reduces the geometric problem to a combinatorial problem, using a topological cell decomposition of the moduli stack of curves to derive formulae for integrals of tautological classes. Then he derives

a matrix integral formula for these expressions, and uses this to prove Witten's conjecture.

The results of this paper, applied to a point, give a new way to do the first part of this procedure; that is we find a combinatorial expression for integrals of tautological classes on the moduli stack. The techniques are purely algebro-geometric, and thus have a very different flavour from Kontsevich's topological model.

More recently, another proof of the Kontsevich-Witten theorem has appeared. A combinatorial expression for intersection numbers on the moduli stack of curves in terms of Hurwitz numbers was announced by Ekedahl, Lando, Shapiro and Vainshtein in [13] and proved in [14]. Another proof of this formula was obtained by Graber and Vakil [18], building on a special case proved by Fantechi and Pandharipande [16]. This result was used by Okounkov and Pandharipande [29] to give another proof of the Kontsevich-Witten theorem.

The geometric part of this proof relies on spaces of ramified covers of \mathbb{P}^1 to relate intersection numbers on $\overline{\mathcal{M}}_{g,n}$ to Hurwitz numbers. Spaces of covers of genus 0 curves also play a central role in this work. However, the compactifications we use are different, as are the methods for obtaining formulae for integrals on $\overline{\mathcal{M}}_{g,n}$. For example, in [18], Graber and Vakil calculate certain Gromov-Witten invariants of \mathbb{P}^1 in two different ways: firstly, using virtual localization, and secondly, by using a "branching map" to configuration spaces of points on \mathbb{P}^1 . Equating these yields the desired formula. On the other hand, the techniques used here can be viewed, in the case of a point, as firstly constructing a correspondence $\overline{\mathcal{M}}_{0,n} \leftarrow \overline{\mathcal{M}}_\eta \rightarrow \overline{\mathcal{M}}_{g,m}$, which is finite over both $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{g,m}$, and then calculating the pullbacks of tautological classes from $\overline{\mathcal{M}}_{g,m}$. The expressions we end up with are different from those obtained by the authors cited above.

The results presented here work for arbitrary target space, and not just a point; it does not seem to be clear how to generalize the results of [13], [14], [16], [18], [21] to arbitrary target X .

Orbifold Gromov-Witten theory. Gromov-Witten invariants for orbifolds were defined by Chen and Ruan [7], [8], [9] in symplectic topology and Abramovich, Graber, Vistoli [3], [2] in algebraic geometry. Chen and Ruan's work introduced the orbifold cohomology groups H_{orb}^* , as the ordinary cohomology of the space of twisted sectors of an orbifold. In orbifold Gromov-Witten theory the orbifold cohomology group H_{orb}^* plays the same role as the ordinary cohomology group plays in standard Gromov-Witten theory. In particular, orbifold quantum cohomology ($g = 0$ orbifold Gromov-Witten theory) gives H_{orb}^* the structure of a Frobenius manifold.

In this paper the algebraic approach to orbifold Gromov-Witten theory developed by Abramovich, Graber and Vistoli [2] is used. Their construction is

based on the foundational work of Abramovich and Vistoli [3], who introduced stable maps to Deligne-Mumford stacks, and proved that there are reasonable stacks parametrizing these stable maps. Stacks of stable maps to the classifying stack BG of a finite group G play an important role in this work. These stacks were studied by Abramovich, Corti and Vistoli [1], where they are related to more classical spaces of admissible covers.

The Gromov-Witten theory of the classifying stack BG of a finite group G was studied by Jarvis and Kimura [20]. In a recent preprint, Jarvis, Kaufmann and Kimura [19] study the algebraic structure defined by G -equivariant quantum cohomology for a finite group G . The reader should refer to these works for more details on the structure of the genus 0 Gromov-Witten invariants of BS_n and of $S^n X$. Note, however, that the notation for tautological classes, etc., used here, differs from their notation by constants.

1.2. *Plan of the paper.* We define some of the basic moduli stacks we need in Section 2. These are certain stacks of nodal curves with markings in a semigroup; we show they are smooth Artin algebraic stacks and describe certain maps between them, as well as tautological classes. Section 3 is devoted to setting up various categories of labelled graphs, together with functors which associate to a graph a certain moduli stack of curves. In Section 4 we calculate how tautological classes and cycles pull back under morphisms of moduli stacks, coming from morphisms of graphs. These pull backs are expressed as sums over graphs, weighted by tautological classes.

Section 5 contains the main technical theorem, which says roughly that a map of finite degree remains of finite degree in the virtual sense, after a virtual base change. We use Behrend-Fantechi's virtual fundamental class technology, and this result follows from an analysis of their "relative intrinsic normal cone stacks". In Section 6, we construct, for each g, r, β , a label η for a moduli stack of étale covers \mathfrak{M}_η , with a finite map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_{g,r,\beta}$.

In Section 7, we base change by $\overline{\mathcal{M}}_{g,\beta}(X)$, to get stacks of stable maps and étale correspondences to X . In Section 8, we put these results together to give a formula for all descendent Gromov-Witten invariants of X in terms of genus 0 invariants of symmetric products $S^d X$. Finally, in Section 9, I illustrate the general result by calculating some low-genus Gromov-Witten invariants of a point.

1.3. *Future work.* The results presented here provide implicit constraints on the Gromov-Witten invariants of an arbitrary variety, coming from associativity properties of quantum cohomology of symmetric products. It would be interesting to see what relation these constraints have with the conjectural Virasoro constraints, first proposed by Eguchi, Hori and Xiong [12]. A first step in this direction would be to use the results of this paper to give a new proof of Witten's conjecture. I imagine this is far from easy; in the two proofs of

the Kontsevich-Witten theorem of which I am aware, even once the geometric work has been done, significant insight is required to prove the theorem.

In another direction, I think that one can prove a reconstruction theorem, analogous to the first reconstruction theorem of Kontsevich and Manin [22], which would imply that for certain Fano manifolds X , the quantum cohomology of $S^d X$ is determined by the quantum cohomology of X . This would imply that all Gromov-Witten invariants of X are determined by the genus 0 invariants. It's not clear in what generality one can make such a statement: we need $K_X \ll 0$, which implies many genus 0 invariants of $S^d X$ vanish for dimension reasons.

Note that such a statement has a close relationship with certain corollaries of the Virasoro conjecture. Dubrovin and Zhang [10], [11] have shown that the Virasoro conjecture implies that when X has semisimple quantum cohomology, all higher genus Gromov-Witten invariants of X are determined by the genus 0 invariants. Conjecturally, many Fano manifolds have semisimple quantum cohomology.

I hope to return to these points in a future paper.

1.4. *Acknowledgements.* I am very grateful to my Ph.D. supervisor Ian Grojnowski for his support, both mathematical and moral, during my time at Cambridge. I would like to thank Alessio Corti, Ezra Getzler, Tim Perutz, Richard Thomas, Burt Totaro, Hsian-Hua Tseng, and especially Constantin Teleman and Ravi Vakil for their interest in this work, and for very helpful conversations and correspondence. A version of this paper has been accepted as my Ph.D. thesis at Cambridge University. I have been financially supported by the EPSRC, the Cambridge European Trust, and the Cecil King Memorial Foundation.

1.5. *Notation.* We work always over a field k , algebraically closed and of characteristic zero. Stacks are in the sense of Laumon and Moret-Bailly [27]. In particular, a stack is not required to have an atlas, an algebraic stack must admit a smooth atlas, and a Deligne-Mumford stack must admit an étale atlas. I will sometimes use the phrase Artin stack as a synonym for algebraic stack.

Later we will define various categories of graphs. Here is a summary of some notation needed for these:

Υ^u	Labels (g, I, a) for smooth connected curves, of genus g with marked point set I and class $a \in A$ in the semigroup.
Υ^t	Labels (g, I, m, a) for smooth connected twisted curves, of genus g , with marked point set I , stack structure at the marked points given by $m : I \rightarrow \mathbb{Z}_{>0}$, and class $a \in A$ in the semigroup.

$\exp(\Upsilon^u)$	Labels for disconnected smooth marked curves, and disconnected smooth marked twisted curves, respectively.
$\exp(\Upsilon^t)$	Labels for disconnected smooth marked twisted curves, respectively.
Υ^c	Labels for étale covers $\mathcal{C}' \rightarrow \mathcal{C}$ of smooth twisted curves, with \mathcal{C} connected.
$s : \Upsilon^c \rightarrow \exp(\Upsilon^t)$	Source map, which associates to a label for $\mathcal{C}' \rightarrow \mathcal{C}$ the label for \mathcal{C}' .
$t : \Upsilon^c \rightarrow \Upsilon^t$	Target map, which associates to a label for $\mathcal{C}' \rightarrow \mathcal{C}$ the label for \mathcal{C} .
Γ^u	Graphs built from vertices Υ^u , which label nodal connected curves.
Γ^t	Graphs built from vertices Υ^t , which label twisted nodal curves.
Γ^c	A certain type of map of graphs in Γ^t , which labels étale covers $\mathcal{C}' \rightarrow \mathcal{C}$ of twisted nodal curves.
$r : \Upsilon^t \rightarrow \Upsilon^u$	Associates to a label for a twisted curve \mathcal{C} the label for its coarse moduli space C .
$r : \Gamma^t \rightarrow \Gamma^u$	
$s, t : \Gamma^c \rightarrow \Gamma^t$	Source and target maps, which associate to a label for an étale cover $\mathcal{C}' \rightarrow \mathcal{C}$ the labels for \mathcal{C}' , \mathcal{C} respectively.

2. Moduli stacks

Let $g \in \mathbb{Z}_{\geq 0}$ and let I be a finite set. Let $\mathfrak{M}_{g,I}$ be the stack of all nodal curves of genus g with I marked smooth points. $\mathfrak{M}_{g,I}$ is a smooth algebraic stack; it is nonseparated, and locally but not globally of finite type.

Let $g \in \mathbb{Z}_{\geq 0}$, let I be a finite set and let $m : I \rightarrow \mathbb{Z}_{>0}$ be a function. Let $\mathfrak{M}_{g,I,m}$ be the moduli stack of all twisted (balanced) curves of genus g , with marked points labelled by I and the degree of twisting at the marked points given by m . Explicitly, $\mathfrak{M}_{g,I,m}$ is the category of commutative diagrams

$$\begin{array}{ccccc}
 & & \mathcal{C} & \xrightarrow{\quad} & \mathcal{C} \\
 & & \uparrow & \searrow & \downarrow \\
 U \times I & \longrightarrow & U \times \coprod_{i \in I} B\mu_{m(i)} & \longrightarrow & U
 \end{array}$$

where:

- U is a scheme of finite type.
- \mathcal{C} is a proper separated flat DM stack over U , étale locally a nodal curve over U .

- The map $\mathcal{C} \rightarrow C$ exhibits C as the coarse moduli space of \mathcal{C} , and C is connected of genus g .
- $U \times \coprod B\mu_{m(i)} \hookrightarrow \mathcal{C}$ is an embedding of a disjoint union of trivial $\mu_{m(i)}$ -gerbes into \mathcal{C} , and $U \times I \rightarrow U \times \coprod B\mu_{m(i)}$ are sections of these gerbes.
- $\mathcal{C} \rightarrow C$ is an isomorphism away from the nodes and marked points of C .
- Étale locally near a node of \mathcal{C} , $\mathcal{C} \rightarrow U$ looks like

$$(\text{Spec } A[u, v]/(uv - t))/\mu_r \rightarrow \text{Spec } A$$

where $t \in A$, and the group of r -th roots of unity μ_r acts on $A[u, v]/(uv - t)$ by $u \rightarrow lu, v \rightarrow l^{-1}v$, where $l \in \mu_r$.

This definition is due to Abramovich and Vistoli; for more details see [3]. Note that we use trivialized gerbes, where they use possibly nontrivial gerbes. Our stack is simply the fiber product of all the universal gerbes lying over their version.

PROPOSITION 2.0.1. $\mathfrak{M}_{g,I,m}$ is a smooth stack.

By smooth I mean in the sense of the formal criterion for smoothness over the base $\text{Spec } k$. I expect that $\mathfrak{M}_{g,I,m}$ is algebraic, although I do not know a reference for this. Presumably, one could prove this using the techniques of Abramovich and Vistoli [3]. However, we do not really need any properties of $\mathfrak{M}_{g,I,m}$; for us it is essentially a placeholder.

There is a map $\mathfrak{M}_{g,I,m} \rightarrow \mathfrak{M}_{g,I}$ which associates to a twisted curve its coarse moduli space.

We need variants of these definitions, which depend on a semigroup. Let A be a commutative semigroup, with unit $0 \in A$, such that

- A has indecomposable zero: $a + a' = 0$ implies $a = a' = 0$.
- A has finite decomposition: for every $a \in A$, the set $\{(a_1, a_2) \in A \times A \mid a_1 + a_2 = a\}$ is finite.

For example, $A = 0$, or A is the cone $C(X)$ of curves in a projective variety X up to numerical equivalence, or $A = \{0, 1\}$ where $1 + 1 = 1$.

Fix any such A . Let (g, I, a) be a triple where $g \in \mathbb{Z}_{\geq 0}$, I is a finite set, and $a \in A$. We say (g, I, a) is stable, if either $a \neq 0$ or $a = 0$ and $2g - 2 + \#I > 0$. For any such triple (g, I, a) we define the stack $\mathfrak{M}_{g,I,a}$ over $\mathfrak{M}_{g,I}$. Roughly, $\mathfrak{M}_{g,I,a}$ parametrizes curves C with I marked smooth points, together with a labelling of each irreducible component of C by an element of A . The sum over irreducible components of the associated elements of A must be a , and a certain stability condition must be satisfied. The simplest formal definition is inductive.

- (1) If (g, I, a) is unstable, then $\mathfrak{M}_{g,I,a}$ is empty.
- (2) Suppose (g, I, a) is stable. Then an object of $\mathfrak{M}_{g,I,a}$ is
 - An object of $\mathfrak{M}_{g,I}$, that is, a flat family $C \rightarrow U$, of nodal curves over a scheme U , together with I smooth marked points $U \times I \rightarrow C$.
 - Let $C_{\text{gen}} \rightarrow U$ be the complement of the nodes and marked points of C . The additional data we require is a constructible function $f : C_{\text{gen}} \rightarrow A$. f must be locally constant on the geometric fibres of $C_{\text{gen}} \rightarrow U$.
- (3) If $U_0 \subset U$ is the open subscheme parametrizing nonsingular curves $C_0 \rightarrow U_0$, then $f : C_{0,\text{gen}} \rightarrow A$ must be constant with value a .
- (4) We require that f satisfies a gluing condition along the boundary of $\mathfrak{M}_{g,I}$. Precisely, suppose we have a decomposition $g = g' + g''$ and $I = I' \amalg I''$, a map $V \rightarrow U$, and a factorization of the map $V \rightarrow \mathfrak{M}_{g,I}$ into

$$V \rightarrow \mathfrak{M}_{g',I' \amalg \{s'\}} \times \mathfrak{M}_{g'',I'' \amalg \{s''\}} \rightarrow \mathfrak{M}_{g,I}$$

where the second map is obtained by gluing the marked points s', s'' . Let $C'_V \rightarrow V$ and $C''_V \rightarrow V$ be the associated families of curves. We require that the pulled-back constructible functions $f' : C'_{V,\text{gen}} \rightarrow A$ and $f'' : C''_{V,\text{gen}} \rightarrow A$ define a morphism

$$V \rightarrow \coprod_{a=a'+a''} \mathfrak{M}_{g',I' \amalg \{s'\},a'} \times \mathfrak{M}_{g'',I'' \amalg \{s''\},a''}$$

- (5) In a similar way, suppose we have a map $V \rightarrow U$, and a factorization of the map $V \rightarrow \mathfrak{M}_{g,I}$ into

$$V \rightarrow \mathfrak{M}_{g-1,I \amalg \{s,s'\}} \rightarrow \mathfrak{M}_{g,I}$$

Then, the family of genus $g - 1$ curves $C_V \rightarrow V$, together with the pulled-back constructible function $f : C_{V,\text{gen}} \rightarrow A$, must define a map

$$V \rightarrow \mathfrak{M}_{g-1,I \amalg \{s,s'\},a}$$

PROPOSITION 2.0.2. *The map $\mathfrak{M}_{g,I,a} \rightarrow \mathfrak{M}_{g,I}$ is étale, and relatively a scheme of finite type. Therefore $\mathfrak{M}_{g,I,a}$ is a smooth algebraic stack.*

Define $\mathfrak{M}_{g,I,m,a} = \mathfrak{M}_{g,I,m} \times_{\mathfrak{M}_{g,I}} \mathfrak{M}_{g,I,a}$. The stack $\mathfrak{M}_{g,I,m,a}$ is smooth.

2.1. *Contraction maps.* The main advantage of $\mathfrak{M}_{g,I,a}$ over $\mathfrak{M}_{g,I}$ is the existence of contraction maps $\pi_i : \mathfrak{M}_{g,I,a} \rightarrow \mathfrak{M}_{g,I \setminus i,a}$ for each $i \in I$. Given a curve $C \in \mathfrak{M}_{g,I,a}$ with marked points $P_j, j \in I$, $\pi_i(C)$ is obtained from C by removing P_i , and contracting the irreducible component of C containing P_i to

a point if it is unstable. Unstable components are components of genus 0 with marking $0 \in A$ and containing < 3 nodes or marked points, and components of genus 1 with no nodes or marked points. To construct the map π_i , we need

PROPOSITION 2.1.1. *There is an isomorphism $\mathfrak{C}_{g,I \setminus i,a} \cong \mathfrak{M}_{g,I,a}$ where $\mathfrak{C}_{g,I \setminus i,a}$ is the universal curve over $\mathfrak{M}_{g,I \setminus i,a}$.*

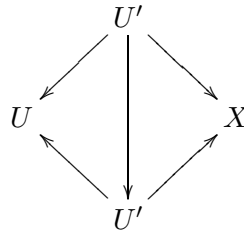
Proof. Just as in [24, Def. 2.3], there is a map $\mathfrak{C}_{g,n-1} \rightarrow \mathfrak{M}_{g,n}$. This lifts to a map $\mathfrak{C}_{g,n-1,a} \rightarrow \mathfrak{M}_{g,n,a}$, by labelling any irreducible component which is contracted in the map $\mathfrak{C}_{g,n-1,a} \rightarrow \mathfrak{M}_{g,n-1,a}$ by $0 \in A$. As in [4, Lemma 7], the formal criterion for étaleness shows that $\mathfrak{C}_{g,n-1,a} \rightarrow \mathfrak{M}_{g,n,a}$ is étale. To show it is an isomorphism, it is enough to show this on the level of k -points, which is easy. \square

2.2. *Maps to symmetric products.* Let X be a scheme. Let $S^d X = [X^d/S_d]$ be the symmetric product stack of X .

LEMMA 2.2.1. *The stack whose groupoid of U points, for U a scheme, has objects diagrams*

$$U \leftarrow U' \rightarrow X$$

where $U' \rightarrow U$ is proper, separated, surjective, and étale of degree d and has morphisms, isomorphisms $U' \rightarrow U'$ such that the diagram



commutes, is equivalent to the stack $S^d X$.

Proof. By definition, to give a map $U \rightarrow S^d X$ is to give a right, étale locally trivial, principal S_d -bundle $P \rightarrow U$, together with an S_d -equivariant map $P \rightarrow X^d$. Given such, let $U' = P \times^{S_d} \{1, \dots, d\}$. Then $U' \rightarrow U$ is étale of degree d . One can recover P from U' as the sheaf on the small étale site of U ,

$$P = \text{Iso}_{\text{et}}(U', U \times \{1, \dots, d\}).$$

Observe that P is an étale locally trivial principal S_d bundle. This is because the map $U' \rightarrow U$ is étale locally isomorphic to $U \times \{1 \dots d\}$ - proper, separated, surjective, étale maps of degree d are precisely the maps with this property. Then,

$$\begin{aligned}
 \text{Hom}(P, X^d)^{S_d} &= \text{Hom}(P \times \{1, \dots, d\}, X)^{S_d} \\
 &= \text{Hom}(P \times^{S_d} \{1, \dots, d\}, X) = \text{Hom}(U', X). \quad \square
 \end{aligned}$$

COROLLARY 2.2.2. *Let V be a DM stack. The 2-groupoid $\text{Hom}(V, S^d X)$ is equivalent to the 2-groupoid of diagrams $V \leftarrow V' \rightarrow X$, with $V' \rightarrow V$ proper, separated surjective, and étale of degree d . Further, the 2-groupoid $\text{HomRep}(V, S^d X)$ of representable maps $V \rightarrow S^d X$ is equivalent to the 2-groupoid of such diagrams $V \leftarrow V' \rightarrow X$, where the inertia groups of V act faithfully on the fibres of $V' \rightarrow V$.*

Proof. We prove the statement about representability. $V \rightarrow S^d X$ is representable if and only if the principal S_d -bundle, $P \rightarrow V$, is an algebraic space. This is equivalent to saying that the inertia groups of V act faithfully on the fibres of $V' \rightarrow V$. □

2.3. *Stacks of étale covers.* We need some notation to shorten the cumbersome g, I, m, a labels. Let $\Upsilon^t(A)$ be the groupoid of quadruples $\nu = (g(\nu), T(\nu), m, a(\nu))$ where $g(\nu) \in \mathbb{Z}_{\geq 0}$, $T(\nu)$ is a finite set, $m : T(\nu) \rightarrow \mathbb{Z}_{>0}$ is a function, and $a(\nu) \in A$. We impose the stability condition as before: if $a(\nu) = 0$, then $2g(\nu) - 2 + \#T(\nu) > 0$. The morphisms are isomorphisms preserving all the structure. Let $\Upsilon^u(A)$ be the groupoid of triples $v = (g(v), T(v), a(v))$ satisfying the stability condition. There is a map $r : \Upsilon^t(A) \rightarrow \Upsilon^u(A)$ sending $(g, I, m, a) \rightarrow (g, I, a)$. For $\nu \in \Upsilon^t$ we have the moduli stack \mathfrak{M}_ν ; similarly for $v \in \Upsilon^u$ we have \mathfrak{M}_v . There is a map $r : \mathfrak{M}_\nu \rightarrow \mathfrak{M}_{r(\nu)}$ which associates to a twisted curve its coarse moduli space.

We also want labels for moduli stacks of disconnected curves. We define a groupoid $\text{exp}(\Upsilon^t)$. An object $\alpha \in \text{exp}(\Upsilon^t)$, is a finite set $V(\alpha)$, and a map $V(\alpha) \rightarrow \text{Ob } \Upsilon^t$. A morphism $\alpha \rightarrow \alpha'$ in $\text{exp}(\Upsilon^t)$, is an isomorphism $\phi : V(\alpha) \cong V(\alpha')$ of finite sets, together with an isomorphism $v \cong \phi(v)$ of the associated element of Υ^t , for each $v \in V(\alpha)$. Define $\text{exp}(\Upsilon^u)$ in a similar way. Given $\alpha \in \text{exp}(\Upsilon^t)$, let

$$T(\alpha) = \coprod_{v \in V(\alpha)} T(v).$$

The groupoid $\text{exp}(\Upsilon^t)$ labels possibly disconnected nodal curves. Let

$$\mathfrak{M}_\alpha = \prod_{v \in V(\alpha)} \mathfrak{M}_v.$$

Next, we want to define labels for étale maps of twisted curves, $\mathcal{C}' \rightarrow \mathcal{C}$. \mathcal{C}' may be disconnected. A covering η , consists of

- (1) An element $s(\eta) \in \text{exp}(\Upsilon^t)$, the source, and an element $t(\eta) \in \Upsilon^t$, the target.
- (2) These must satisfy

$$\sum_{v \in V(s(\eta))} a(v) = a(t(\eta)) \in A.$$

(3) A map of finite sets, $p : T(s(\eta)) \rightarrow T(t(\eta))$.

(4) For each $t' \in T(s(\eta))$, we require that $m(t')$ divides $m(p(t'))$. Let $d(t') = m(p(t'))/m(t')$.

(5) We require that for each $t \in T(t(\eta))$,

$$m(t) = \text{lcm}\{d(t') \mid p(t') = t\}$$

where lcm stands for lowest common multiple.

(6) A function $d : V(s(\eta)) \rightarrow \mathbb{Z}_{\geq 1}$, the degree.

(7) For each $t \in T(t(\eta))$, and each $v \in V(s(\eta))$,

$$\sum_{\substack{t' \in T(v) \\ p(t')=t}} d(t') = d(v).$$

We define $d(\eta) = \sum_{v \in V(s(\eta))} d(v)$.

(8) The Riemann-Hurwitz formula holds: for each $v \in V(s(\eta))$,

$$2(g(s(\eta)_v) - 1) = 2d(v)(g(t(\eta)) - 1) + \sum_{t' \in T(v)} (d(t') - 1).$$

Let Υ^c be the groupoid of all coverings η , with the obvious isomorphisms. There are source and target functors,

$$\begin{aligned} s : \Upsilon^c &\rightarrow \exp(\Upsilon^t), \\ t : \Upsilon^c &\rightarrow \Upsilon^t. \end{aligned}$$

We will often write $\alpha \twoheadrightarrow \nu$ to mean a covering η with $s(\eta) = \alpha$ and $t(\eta) = \nu$.

Associated to a covering $\eta \in \Upsilon^c$, we define a stack \mathfrak{M}_η of étale covers $f : \mathcal{C}' \rightarrow \mathcal{C}$. \mathfrak{M}_η is the category whose objects are

- An object of $\mathfrak{M}_{t(\eta)}$, with associated family of twisted nodal curves $\mathcal{C} \rightarrow U$, sections $T(t(\eta)) \rightarrow \mathcal{C}$ and constructible function $f : C_{\text{gen}} \rightarrow U$, where C is the coarse moduli space of \mathcal{C} .
- An object of $\mathfrak{M}_{s(\eta)}$, with associated family of possibly disconnected twisted nodal curves $\mathcal{C}' \rightarrow U$, sections $T(s(\eta)) \rightarrow \mathcal{C}'$ and constructible function $f' : C'_{\text{gen}} \rightarrow U$, where C' is the coarse moduli space of \mathcal{C}' .
- An étale map $p : \mathcal{C}' \rightarrow \mathcal{C}$.

These must satisfy:

- The diagram

$$\begin{array}{ccc} \mathbb{T}(s(\eta)) \times U & \longrightarrow & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathbb{T}(t(\eta)) \times U & \longrightarrow & \mathcal{C} \end{array}$$

must be Cartesian over U . This implies, in particular, that the marked points of \mathcal{C}' are precisely those lying over marked points of \mathcal{C} .

- Let p_*f' be the constructible function on C_{gen} given by pushing forward f' ; we require that $p_*f' = f$.
- The map $\mathcal{C} \rightarrow BS_d$ associated to the étale map $\mathcal{C}' \rightarrow \mathcal{C}$ must be representable.

PROPOSITION 2.3.1. *The map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_{t(\eta)}$ is étale; therefore, \mathfrak{M}_η is smooth. Further, \mathfrak{M}_η is algebraic.*

Before proving this, we need a lemma.

LEMMA 2.3.2. *Let G be a finite group. The stack*

$$\mathfrak{M}_{g,I,m}(BG) \stackrel{\text{def}}{=} \text{HomRep}_{\mathfrak{M}_{g,I,m}}(\mathfrak{C}_{g,I,m}, BG \times \mathfrak{M}_{g,I,m})$$

of representable maps $\mathcal{C} \rightarrow BG$ from curves $\mathcal{C} \in \mathfrak{M}_{g,I,m}$ is algebraic.

Sketch of proof. A representable map from a twisted nodal curve \mathcal{C} to BG is the same as a principal G bundle $P \rightarrow \mathcal{C}$, whose total space is an ordinary nodal curve. This is the same, just as in [1, Th. 4.3.2], as a nodal curve P , with a G -action, such that the map $P \rightarrow P/G$ to the scheme quotient is generically a principal G -bundle; the G action must also have some compatibility at the nodes. We recover \mathcal{C} as $[P/G]$, the stack quotient.

The stack of nodal curves P is algebraic. Further, the stack of nodal curves with a G action is algebraic, because a G action on a curve P can be identified with its graph in $P \times P \times G$. It follows that $\mathfrak{M}_{g,I,m}(BG)$ is algebraic. \square

Proof of Proposition 2.3.1. The deformations of an étale cover $\mathcal{C}' \rightarrow \mathcal{C}$ are the same as deformations of the base, as in [1]; therefore the map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_{t(\eta)}$ is étale.

Consider the map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_{g(t(\eta)), \mathbb{T}(t(\eta)), m}(BS_d)$. This map is relatively a scheme; it follows that \mathfrak{M}_η is algebraic. \square

The fact that \mathfrak{M}_η is algebraic implies that its image in $\mathfrak{M}_{t(\eta)}$, which is an open substack, is also algebraic.

Let us look at coverings $\mathcal{C}' \rightarrow \mathcal{C} \in \mathfrak{M}_\eta$ locally. Given a tail $t \in \mathbb{T}(t(\eta))$, in an étale neighbourhood of the twisted marked point $t \rightarrow \mathcal{C}$, $\mathcal{C}' \rightarrow \mathcal{C}$ looks like

$$\left(\text{Spec} \left(\bigoplus_{t' \in p^{-1}(t)} \bigoplus_{i=1}^{d(t')} k[x_{t',i}] \right) \rightarrow \text{Spec} k[y] \right) / \mu_{m(t)}.$$

As before, $d(t') = m(t)/m(t')$. The algebra map which induces this map of schemes is of course $y \rightarrow \sum x_{t',i}$. The $\mu_{m(t)}$ action sends $y \rightarrow ly$, for $l \in \mu_{m(t)}$; and $x_{t',i} \rightarrow lx_{t',i+1 \bmod d(t')}$. This action is faithful, which is equivalent to representability of the associated map $\mathcal{C} \rightarrow BS_d$, because $m(t)$ is the lowest common multiple of $d(t')$ for $t' \in p^{-1}(t)$. The stabilizer of $x_{t',i}$ is $\mu_{m(t')} \subset \mu_{m(t)}$. There is a similar picture near the nodes, except $k[x]$ is replaced by $k[u, v]/uv$.

2.4. *Stacks of stable maps.* Let X be a smooth projective variety. We will let our semigroup A , be the cone of effective 1-cycles on X , up to numerical (or homological) equivalence. For each $v = (g, I, \beta) \in \Upsilon^u$, we have the stack $\overline{\mathcal{M}}_v(X)$ of Kontsevich stable maps in X . There is a map

$$\overline{\mathcal{M}}_v(X) \rightarrow \mathfrak{M}_v$$

which associates to a stable map $C \rightarrow X$ with marked points, the curve C , with its marked points, and the constructible function $C_{\text{gen}} \rightarrow C(X)$ given by taking the homology class of an irreducible component. In a similar way, for each $\alpha \in \exp(\Upsilon^u)$, labelling disconnected curves, we have a moduli stack $\overline{\mathcal{M}}_\alpha(X)$ with a map $\overline{\mathcal{M}}_\alpha(X) \rightarrow \mathfrak{M}_\alpha$.

LEMMA 2.4.1.

$$\overline{\mathcal{M}}_{g,n,\beta}(X) = \mathfrak{M}_{g,n,\beta} \times_{\mathfrak{M}_{g,n-1,\beta}} \overline{\mathcal{M}}_{g,n-1,\beta}(X).$$

Proof. It was shown in [6] that $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is the universal curve over $\overline{\mathcal{M}}_{g,n-1,\beta}(X)$, which is pulled back from $\mathfrak{M}_{g,n-1,\beta}$. But we have shown that $\mathfrak{M}_{g,n,\beta}$ is the universal curve over $\mathfrak{M}_{g,n-1,\beta}$. \square

More generally,

$$\overline{\mathcal{M}}_{g,n,\beta}(X) = \mathfrak{M}_{g,n,\beta} \times_{\mathfrak{M}_{g,\beta}} \overline{\mathcal{M}}_{g,\beta}(X)$$

so that all stacks of marked stable maps to X , arise by base change with the stack of unmarked stable maps $\overline{\mathcal{M}}_{g,\beta}(X)$.

Let V be a proper projective Deligne-Mumford stack. Abramovich and Vistoli [3] defined the stack of stable maps to V : this is the stack of representable maps $f : \mathcal{C} \rightarrow V$ from twisted nodal curves with marked points to V , such that $\text{Aut}(f)$ is finite. We are only interested in the case $V = S^d X$, the symmetric product stack of a smooth projective variety X . Take our semigroup to be $C(X)$ as above. For each $\nu = (g, I, m, \beta) \in \Upsilon^t$, let $\overline{\mathcal{M}}_\nu(S^d X)$ be the stack of stable maps from curves $\mathcal{C} \in \mathfrak{M}_{g,I,m}$, such that if $\mathcal{C} \leftarrow \mathcal{C}' \rightarrow X$ is the associated correspondence, then $\mathcal{C}' \rightarrow X$ has class $\beta \in C(X)$.

For each covering $\eta \in \Upsilon^c$, define

$$\overline{\mathcal{M}}_\eta(X) = \mathfrak{M}_\eta \times_{\mathfrak{M}_{s(\eta)}} \overline{\mathcal{M}}_{s(\eta)}.$$

Let $\text{Aut}(\eta | t(\eta))$ be the group of automorphisms of η which act trivially on $t(\eta)$.

LEMMA 2.4.2. *There is a natural isomorphism*

$$\coprod_{\eta, t(\eta)=v} \overline{\mathcal{M}}_\eta(X) / \text{Aut}(\eta | t(\eta)) \cong \overline{\mathcal{M}}_v(S^d X).$$

2.5. *Tautological line bundles.* Let $\nu \in \Upsilon^u$ or Υ^t . For each $t \in T(\nu)$, there is a section $\sigma_t : \mathfrak{M}_\nu \rightarrow \mathfrak{C}_\nu$ of the universal curve. Define $L_t = \Omega_{\sigma_t}^1$ to be the relative cotangent bundle which is the tautological line bundle. If $\mu \in \Upsilon^t$, so that $r(\mu) \in \Upsilon^u$, we have a map $\mathfrak{M}_\mu \rightarrow \mathfrak{M}_{r(\mu)}$. For each $t \in T(\mu) = T(r(\mu))$, we have $r^*L_t = L_t^{\otimes m(t)}$.

For $\eta \in \Upsilon^c$, for each $t \in T(s(\eta))$ (or $t \in T(t(\eta))$) there is a tautological line bundle L_t , pulled back from $\mathfrak{M}_{s(\eta)}$ (respectively $\mathfrak{M}_{t(\eta)}$). If $p(t') = t$ under the projection $T(s(\eta)) \rightarrow T(t(\eta))$, then $L_{t'} \cong L_t$.

Let $\psi_t = c_1(L_t)$ on any of the three types of moduli stacks.

2.6. *Automorphisms and deformations of twisted nodal curves.* Let $v \in \Upsilon^t$, let $\mathcal{C} \in \mathfrak{M}_v$ and let $C \in \mathfrak{M}_{r(v)}$ be the coarse moduli space of \mathcal{C} . We want to describe $\text{Aut}(\mathcal{C} | C)$, the group of automorphisms of \mathcal{C} which are trivial on the coarse moduli space C . This group splits as a product of contributions from each twisted node and twisted marked point of \mathcal{C} : a twisted node or marked point with inertia group μ_r contributes μ_r .

The fibre of $\mathfrak{M}_v \rightarrow \mathfrak{M}_{r(v)}$ over a curve $C \in \mathfrak{M}_{r(v)}$ can similarly be described as a product of local contributions from the nodes and marked points of C . For each tail $t \in T(v) = T(r(v))$, we have a factor of $B\mu_{m(t)}$. For each node of C , we have a factor of $\prod_{k \in \mathbb{Z}_{>0}} B\mu_k$. For more details, see [1].

For $v \in \Upsilon^t$, let $T_1(v) \subset T(v)$ be the set of tails with multiplicity $m(t) = 1$. For $\mathcal{C} \in \mathfrak{M}_v$ and $t \in T_1(v)$, the marked point $P_t \in \mathcal{C}$ is untwisted. The first-order deformations of \mathcal{C} are given by

$$\text{Ext}^1(\Omega_{\mathcal{C}}^1(\sum_{t \in T_1(v)} P_t), \mathcal{O}_{\mathcal{C}}).$$

The deformation theory is unobstructed. We can identify this space with

$$H^0 \left(\omega_{\mathcal{C}} \otimes \Omega_{\mathcal{C}}^1(\sum_{t \in T_1(v)} P_t) \right)^\vee$$

where $\omega_{\mathcal{C}}$ is the dualizing line bundle.

Let $C \in \mathfrak{M}_{r(v)}$ be the coarse moduli space of \mathcal{C} . We have a map $\pi : \mathcal{C} \rightarrow C$. In [3], it is shown that π_* is an exact functor, and so

$$H^0 \left(\omega_{\mathcal{C}} \otimes \Omega_{\mathcal{C}}^1 \left(\sum_{t \in T_1(v)} P_t \right) \right) = H^0 \left(\pi_* \left(\omega_{\mathcal{C}} \otimes \Omega_{\mathcal{C}}^1 \left(\sum_{t \in T_1(v)} P_t \right) \right) \right).$$

The space of first order deformations of C is similarly given by

$$H^0 \left(\omega_C \otimes \Omega_C^1 \left(\sum_{t \in T(r(v))} P_t \right) \right)^\vee.$$

Observe that we have a pole at all tails, not just those with multiplicity one. Clearly

$$\pi_* \left(\omega_{\mathcal{C}} \otimes \Omega_{\mathcal{C}}^1 \left(\sum_{t \in T_1(v)} P_t \right) \right) = \omega_C \otimes \Omega_C^1 \left(\sum_{t \in T(r(v))} P_t \right)$$

away from the nodes and marked points of C . In fact, this equality extends also to the marked points. At the nodes, however, this is no longer true. The map $\mathfrak{M}_v \rightarrow \mathfrak{M}_{r(v)}$ is ramified along the divisor of singular curves. The degree of ramification along the divisor in \mathfrak{M}_v corresponding to a node with inertia group μ_k is $k - 1$ (i.e. a function vanishing to degree 1 on the divisor in $\mathfrak{M}_{r(v)}$ vanishes to degree k along the divisor in \mathfrak{M}_v when it is pulled back). One can see this by looking at the local picture, as in [1, §3].

Let $\eta \in \Upsilon^c$ and let $\mathcal{C}' \rightarrow \mathcal{C} \in \mathfrak{M}_\eta$ be an étale cover of twisted balanced curves. Let $C' \rightarrow C$ be the corresponding ramified covering of the coarse moduli spaces of \mathcal{C}' and \mathcal{C} . We are interested in $\text{Aut}(\mathcal{C}' \rightarrow \mathcal{C} \mid C' \rightarrow C)$, the automorphisms which are trivial on the coarse moduli space. As before, this splits as a product with a contribution from each twisted node and twisted marking. Each $t \in T(t(\eta))$ – that is each marking of \mathcal{C} – contributes $\mu_{m(t)}$. However, in this case the contribution from the nodes is trivial. This follows from the fact that the map $\mathcal{C} \rightarrow BS_d$ is representable. The marked points of \mathcal{C}' do not contribute anything extra.

We have seen already what the deformations of $\mathcal{C}' \rightarrow \mathcal{C}$ are: they are the same as deformations of \mathcal{C} ; that is, the map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_{t(\eta)}$ is étale.

3. Graphs

We define categories of graphs, which label various flavours of nodal curve, as well as étale covers of twisted nodal curves. We introduce three categories: Γ^u , which contains labels for untwisted nodal curves; Γ^t , which has labels for twisted nodal curves; and Γ^c , which has labels for pairs of twisted nodal curves $\mathcal{C}', \mathcal{C}$, with an étale map $\mathcal{C}' \rightarrow \mathcal{C}$. These categories depend on a semi-group A , which we will usually not mention. The morphisms in these categories

correspond to degenerating curves, or dually to contracting graphs. There are functors, denoted \mathfrak{M} , from each of these categories to the category of stacks, which take a label to the moduli stack of all curves with that label; as well as functors

$$\Gamma^c \xrightarrow{s,t} \Gamma^t \xrightarrow{r} \Gamma^u$$

where s and t stand for source and target, and take the labels for a pair $\mathcal{C}' \rightarrow \mathcal{C}$ to the labels for \mathcal{C}' or \mathcal{C} respectively. Also, r takes the labels for \mathcal{C} to the labels for its coarse moduli space C . These functors get translated into morphisms of stacks after we apply the moduli stack functor \mathfrak{M} ; that is, there are maps of stacks $s : \mathfrak{M}_\sigma \rightarrow \mathfrak{M}_{s(\sigma)}$, and similarly for t and r .

Now we define these categories. Let Γ^t be the category whose objects are objects η of $\text{exp}(\Upsilon^t)$, together with an order-two isomorphism $\sigma : T(\eta) \cong T(\eta)$, commuting with the multiplicity map $m : T(\eta) \rightarrow \mathbb{Z}_{\geq 1}$. Define the objects of Γ^u in a similar fashion, using Υ^u instead of Υ^t and omitting references to m . There are forgetful maps $F : \Gamma^t \rightarrow \text{exp}(\Upsilon^t)$, and $F : \Gamma^u \rightarrow \text{exp}(\Upsilon^u)$. For $\gamma \in \Gamma^t$ (or $\gamma \in \Gamma^u$), the set of vertices of γ , written $V(\gamma)$, is the set of vertices $V(F(\gamma))$ of the underlying element of $\text{exp}(\Upsilon^t)$. The set of half-edges of γ is the set $H(\gamma) = T(F(\gamma))$. The set of edges of γ , $E(\gamma)$, is the set of free $\mathbb{Z}/2$ orbits on $H(\gamma)$. The set of tails of γ , $T(\gamma)$, is the set of σ -fixed points on $H(\gamma)$. We can fit the structure of a graph $\gamma \in \Gamma^t$ into the diagram

$$\begin{array}{ccc} \mathbb{Z}_{>0} & \xleftarrow{m} & H(\gamma) \cup^\sigma \\ & & \downarrow \\ \mathbb{Z}_{\geq 0} & \xleftarrow{g} & V(\gamma) \xrightarrow{a} A. \end{array}$$

Given $\gamma \in \Gamma^t$ or Γ^u , let $\mathfrak{M}_\gamma = \prod_{v \in V(\gamma)} \mathfrak{M}_v$. For every half-edge $h \in H(\gamma)$, there is a tautological line bundle L_h . For every edge $e \in E(\gamma)$ corresponding to the σ -orbit (h_1, h_2) let $L_e = L_{h_1} \otimes L_{h_2}$.

3.1. *Contractions.* Now we want to define the morphisms in the categories Γ^t and Γ^u , called *contractions*. In terms of the cell complex $C(\gamma)$ associated to a graph γ , a contraction $\gamma' \rightarrow \gamma$ is a surjective continuous map $C(\gamma') \rightarrow C(\gamma)$ which is an isomorphism away from the vertices of $C(\gamma)$, and possibly maps some edges of $C(\gamma')$ to vertices of $C(\gamma)$. It is better to describe contractions more formally. Let $\gamma, \gamma' \in \Gamma^t$. A contraction $\gamma' \rightarrow \gamma$ is a surjective map of sets $f : H(\gamma') \amalg V(\gamma') \rightarrow H(\gamma) \amalg V(\gamma)$, such that

- $V(\gamma') \subset f^{-1}(V(\gamma))$.

- The diagram

$$\begin{array}{ccc}
 \mathrm{H}(\gamma') \amalg \mathrm{V}(\gamma') & \xrightarrow{f} & \mathrm{H}(\gamma) \amalg \mathrm{V}(\gamma) \\
 \sigma \amalg 1 \downarrow & & \downarrow \sigma \amalg 1 \\
 \mathrm{H}(\gamma') \amalg \mathrm{V}(\gamma') & \xrightarrow{f} & \mathrm{H}(\gamma) \amalg \mathrm{V}(\gamma) \\
 \downarrow & & \downarrow \\
 \mathrm{V}(\gamma') & \xrightarrow{f} & \mathrm{V}(\gamma)
 \end{array}$$

commutes, where σ is the involution on $\mathrm{H}(\gamma')$ or $\mathrm{H}(\gamma)$, and $\mathrm{H}(\gamma) \amalg \mathrm{V}(\gamma) \rightarrow \mathrm{V}(\gamma)$ comes from the map $\mathrm{H}(\gamma) \rightarrow \mathrm{V}(\gamma)$ assigning to a half-edge the vertex it is attached to.

- f induces an isomorphism $\mathrm{H}(\gamma') \supset f^{-1}(\mathrm{H}(\gamma)) \cong \mathrm{H}(\gamma)$, commuting with the multiplicity functions.
- We require that f not contract any tails to vertices, so that $\mathrm{T}(\gamma') = \mathrm{H}(\gamma')^\sigma \subset f^{-1}(\mathrm{H}(\gamma))$. This implies that f induces an isomorphism $\mathrm{T}(\gamma') \cong \mathrm{T}(\gamma)$.
- For each $v \in \mathrm{V}(\gamma)$, we can define a graph $\gamma'_v \in \Gamma^t$, with vertices $f^{-1}(v) \cap \mathrm{V}(\gamma')$, edges $f^{-1}(v) \cap \mathrm{E}(\gamma')$, and tails $f^{-1}(\mathrm{H}(v)) \amalg (f^{-1}(v) \cap \mathrm{T}(\gamma'))$. We require that γ'_v be connected of genus $g(\gamma'_v) = g(v)$.

We define contractions of graphs $\gamma, \gamma' \in \Gamma^u$ in the same fashion, leaving out references to the multiplicity function.

Let $\mathrm{E}(f) \subset \mathrm{E}(\gamma')$ be the set of edges contracted by f . Given any subset $I \subset \mathrm{E}(\gamma')$ there is a unique contraction $f : \gamma' \rightarrow \gamma$ with $\mathrm{E}(f) = I$. One can think of $\mathrm{E}(f)$ as being the kernel of f .

Contractions correspond to degenerating curves by adding more nodes. If we have a contraction $f : \gamma' \rightarrow \gamma$, we can identify $\mathfrak{M}_{\gamma'} = \prod_{v \in \mathrm{V}(\gamma)} \mathfrak{M}_{\gamma'_v}$. There are maps $\mathfrak{M}_{\gamma'_v} \rightarrow \mathfrak{M}_v$, which induces a map

$$f_* : \mathfrak{M}_{\gamma'} \rightarrow \mathfrak{M}_\gamma.$$

The map f_* has cotangent complex on $\mathfrak{M}_{\gamma'}$,

$$f^* \Omega_{\mathfrak{M}_\gamma}^1 \rightarrow \Omega_{\mathfrak{M}_{\gamma'}}^1.$$

We can compute the cohomology of this complex in terms of tautological line bundles on $\mathfrak{M}(\gamma')$.

$$\begin{aligned}
 \mathrm{Ker} \left(f^* \Omega_{\mathfrak{M}_\gamma}^1 \rightarrow \Omega_{\mathfrak{M}_{\gamma'}}^1 \right) &= \bigoplus_{i \in \mathrm{E}(f)} L_i \\
 \mathrm{CoKer} \left(f^* \Omega_{\mathfrak{M}_\gamma}^1 \rightarrow \Omega_{\mathfrak{M}_{\gamma'}}^1 \right) &= 0.
 \end{aligned}$$

3.2. *Coverings.* We define labels for étale maps $\mathcal{C}' \rightarrow \mathcal{C}$ of twisted nodal curves. A covering $\gamma' \twoheadrightarrow \gamma$ is a map of sets $p : V(\gamma') \rightarrow V(\gamma)$, and for each $v \in V(\gamma)$, a covering $p^{-1}(v) \rightarrow v$. Here we consider $p^{-1}(v)$ as an object of $\exp(\Upsilon^t)$. We require that the associated map $H(\gamma') \rightarrow H(\gamma)$ is equivariant with respect to the involution, takes tails to tails, and edges to edges.

Given a covering $p : \gamma' \twoheadrightarrow \gamma$, for $v \in V(\gamma)$, let $\gamma'_v = p^{-1}(v) \in \exp(\Upsilon^t)$ be the vertices lying over v . We define a stack $\mathfrak{M}_{\gamma' \twoheadrightarrow \gamma}$ of coverings, by

$$\mathfrak{M}_{\gamma' \twoheadrightarrow \gamma} = \prod_{v \in V(\gamma)} \mathfrak{M}_{\gamma'_v \twoheadrightarrow v}.$$

As usual there are maps $\mathfrak{M}_{\gamma' \twoheadrightarrow \gamma} \rightarrow \mathfrak{M}_\gamma$, which are étale, and $\mathfrak{M}_{\gamma' \twoheadrightarrow \gamma} \rightarrow \mathfrak{M}_{\gamma'}$. If we also have a contraction $\gamma \rightarrow \eta$, there are a unique covering $\eta' \twoheadrightarrow \eta$ and a contraction $\gamma' \rightarrow \eta'$ such that the diagram

$$\begin{array}{ccc} \gamma' & \longrightarrow & \eta' \\ \downarrow & & \downarrow \\ \gamma & \longrightarrow & \eta \end{array}$$

commutes.

Let Γ^c be the category whose objects are coverings $\gamma' \twoheadrightarrow \gamma$ with $\gamma', \gamma \in \Gamma^t$ and whose morphisms are diagrams as above. There are source and target functors $s, t : \Gamma^c \rightarrow \Gamma^t$. There is a functor $\Gamma^c \rightarrow \text{stacks}$, sending a covering $\rho = s(\rho) \twoheadrightarrow t(\rho)$ to \mathfrak{M}_ρ , and an arrow $\rho \rightarrow \phi$ to the associated map of stacks $\mathfrak{M}_\rho \rightarrow \mathfrak{M}_\phi$.

Let $\rho, \phi \in \Gamma^c$ and suppose we have a morphism $f : \rho \rightarrow \phi$. There is an induced morphism $t(\rho) \rightarrow t(\phi)$: let $E(f) \subset E(t(\rho))$ be the set of edges contracted. There is a map $\mathfrak{M}_\rho \rightarrow \mathfrak{M}_\phi$, which are étale over its image. The relative cotangent bundle is $\Omega_{f_*}^1 = \bigoplus_{i \in E(f)} L_i$. The diagram

$$\begin{array}{ccc} \mathfrak{M}_\rho & \longrightarrow & \mathfrak{M}_\phi \\ \downarrow & & \downarrow \\ \mathfrak{M}_{t(\rho)} & \longrightarrow & \mathfrak{M}_{t(\phi)} \end{array}$$

commutes, and the vertical maps are étale.

4. Pull backs of tautological classes

We want to do intersection theory on our moduli stacks, later with virtual fundamental classes. We will work both with Artin algebraic stacks and with Deligne-Mumford stacks. For DM stacks, we will use the theory A_* of Vistoli [32], with \mathbb{Q} -coefficients. For Artin stacks, Kresch has defined in [25] intersection theory, as has Toen in [31]. However, all we ever need to do with

Artin stacks is intersect regularly embedded smooth divisors with regularly embedded smooth closed substacks, and also take first Chern classes of line bundles. No particularly sophisticated technology is required – what we need is basically the same as that used by Behrend in [4] to define Gromov-Witten classes.

We want to intersect cycles corresponding to graphs. If we have a map $f : \rho' \rightarrow \rho$ in one of our categories of graphs $\Gamma^c, \Gamma^t, \Gamma^u$, we have an associated map $f_* : \mathfrak{M}_{\rho'} \rightarrow \mathfrak{M}_{\rho}$. Let $\mathfrak{M}_f \hookrightarrow \mathfrak{M}_{\rho}$ be the closed substack of \mathfrak{M}_{ρ} which is supported on the image of f . Note that \mathfrak{M}_f is smooth, and the map $\mathfrak{M}_f \hookrightarrow \mathfrak{M}_{\rho}$ is a closed regular embedding. The map $\mathfrak{M}_{\rho'} \rightarrow \mathfrak{M}_f$ is étale, in general nonrepresentable, of degree $\|f\|$, where:

- (1) If $f : \rho' \rightarrow \rho$ is a morphism in Γ^u , then

$$\|f\| = \# \text{Aut}(\rho' \rightarrow \rho \mid \rho)$$

is the number of automorphisms of ρ' commuting with the map $\rho' \rightarrow \rho$.

- (2) If $f : \rho' \rightarrow \rho$ is a morphism in Γ^t , then

$$\|f\| = \frac{\# \text{Aut}(\rho' \rightarrow \rho \mid \rho)}{\prod_{i \in E(f)} m(i)}$$

where $E(f) \subset E(\rho')$ is the set of edges contracted.

- (3) If $f : \rho' \rightarrow \rho$ is a morphism in Γ^c , then

$$\|f\| = \frac{\# \text{Aut}(\rho' \rightarrow \rho \mid \rho)}{\prod_{i \in E(f)} m(i)^2}$$

where $E(f) \subset E(t(\rho'))$ is the set of edges contracted.

Let $e \in E(f)$. Observe that the line bundle L_e on $\mathfrak{M}_{\rho'}$ descends to a line bundle L_e on \mathfrak{M}_f . Let $\psi_e = c_1(L_e) \in A^1\mathfrak{M}_f$. The conormal bundle to the embedding $\mathfrak{M}_f \hookrightarrow \mathfrak{M}_{\rho}$ is $\bigoplus_{e \in E(f)} L_e$.

Let \mathcal{A} be one of the categories $\Gamma^c, \Gamma^t, \Gamma^u$. Suppose we have a diagram

$$\begin{array}{ccc} & & \rho'' \\ & & \downarrow g \\ \rho' & \xrightarrow{f} & \rho \end{array}$$

in \mathcal{A} . We want to calculate $f^*[\mathfrak{M}_g] \in A_*\mathfrak{M}_{\rho'}$. For simplicity we will assume that $\mathfrak{M}_g \hookrightarrow \mathfrak{M}_{\rho}$ is a divisor, or equivalently $\#E(g) = 1$. This the only case we will need.

LEMMA 4.0.1. *$f^*[\mathfrak{M}_g]$ can be expressed as a sum with a term for each map $h : \rho' \rightarrow \rho''$ such that $g \circ h = f$. Each such term is weighted by $-\psi_e$,*

where e is the unique element of $E(f) \setminus E(h)$. There is also a term $[\mathfrak{M}_h]$ for each isomorphism class of commutative diagrams

$$\begin{array}{ccc} \eta & \xrightarrow{k} & \rho'' \\ h \downarrow & & \downarrow g \\ \rho' & \xrightarrow{f} & \rho \end{array}$$

where $\#E(h) = 1$ and $E(h) \not\subset E(k)$.

The factor $-\psi_e$ is the first Chern class of the normal bundle. Now we calculate pull backs under the maps of stacks induced by the functor $r : \Gamma^t \rightarrow \Gamma^u$.

LEMMA 4.0.2. *Let $\gamma \in \Gamma^t$, and suppose $f : \alpha \rightarrow r(\gamma)$ is a morphism in Γ^u , with $\#E(f) = 1$. Let $r : \mathfrak{M}_\gamma \rightarrow \mathfrak{M}_{r(\gamma)}$ be the canonical map*

$$r^*[\mathfrak{M}_f] = \sum_{g:\gamma' \rightarrow \gamma} m(e)[\mathfrak{M}_g]$$

where the sum is over $g : \gamma' \rightarrow \gamma$ such that $r(g) = f$, and $e \in E(g)$ is the unique element.

The factors $m(e)$ come from the fact that the map $\mathfrak{M}_\gamma \rightarrow \mathfrak{M}_{r(\gamma)}$ has ramification along the boundary divisors of $\mathfrak{M}_{r(\gamma)}$.

Next we calculate pullbacks under s . Let $\rho \in \Gamma^c$, and let $f : \gamma \rightarrow s(\rho)$ be a morphism in Γ^t , where again $\#E(f) = 1$. There is a map $s : \mathfrak{M}_\rho \rightarrow \mathfrak{M}_{s(\rho)}$.

LEMMA 4.0.3. *$s^*[\mathfrak{M}_f] \in A_*\mathfrak{M}_\rho$ can be expressed as a sum over isomorphism classes of pairs of maps $g : \rho' \rightarrow \rho$ in Γ^c , and $h : s(\rho') \rightarrow \gamma$ in Γ^t , such that $f \circ h = s(g)$, and $\#E(g) = 1$:*

$$s^*[\mathfrak{M}_f] = \sum_{g,h} [\mathfrak{M}_g].$$

Observe that since $t : \mathfrak{M}_\eta \rightarrow \mathfrak{M}_{t(\eta)}$ is étale, it is easy to calculate pull backs under t .

4.1. *Pull backs of ψ -classes.* So far we have seen how to pull back divisors corresponding to graphs. Next, we want to pull back Chern classes of tautological line bundles.

Under a morphism $f : \gamma' \rightarrow \gamma$ in any of our categories $\Gamma^c, \Gamma^t, \Gamma^u$, for any half-edge $h \in H(\gamma)$ there is a unique half-edge $h' \in H(\gamma')$ with $f(h') = h$. Then, $f^*\psi_h = \psi_{h'}$.

For the functors $s, t : \Gamma^c \rightarrow \Gamma^t$, for any $\eta \in \Gamma^c$ and half-edge $h \in H(s(\eta))$, by definition $s^*\psi_h = \psi_h$; and similarly for t . Also, if $h_s \in H(s(\eta))$ lies above $h_t \in H(t(\eta))$, then $\psi_{h_s} = \psi_{h_t}$.

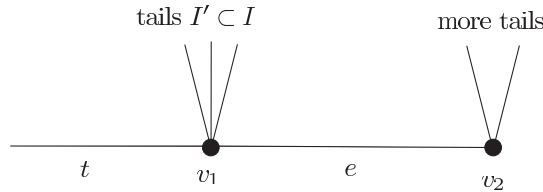
For the functor $r : \Gamma^t \rightarrow \Gamma^u$: if $\gamma \in \Gamma^t$, then $H(\gamma) = H(r(\gamma))$. If $h \in H(\gamma)$, then $r^*\psi_h = m(h)\psi_h$.

There is another case which is not so trivial. Let $\eta \in \Upsilon^c$. Let $I \subsetneq T(s(\eta))$ be such that after removing the tails in I , $s(\eta)$ remains stable. This always happens if $g(s(\eta)) > 0$ or if $a(s(\eta)) \in A$ is nonzero. Let $v \in \Upsilon^u$ be obtained from $r(s(\eta))$ by removing the tails in I . There is a map $\pi : \mathfrak{M}_{r(s(\eta))} \rightarrow \mathfrak{M}_v$. We are interested in calculating the pullbacks of tautological ψ classes in \mathfrak{M}_v , under the morphisms in the diagram

$$\mathfrak{M}_\eta \xrightarrow{s} \mathfrak{M}_{s(\eta)} \xrightarrow{r} \mathfrak{M}_{r(s(\eta))} \xrightarrow{\pi} \mathfrak{M}_v.$$

To do this we need to introduce yet more notation.

Let $t \in T(v)$ be any tail. Let $\gamma \in \Gamma^u$ with a contraction $\gamma \rightarrow r(s(\eta))$. Now, $I \subset T(\gamma)$ and $t \in T(\gamma) \setminus I$. For $e \in E(\gamma)$, let γ_e be the graph obtained by contracting all edges except e . We define $S(e, t, I) \in \{0, 1\}$ to be 1, if and only if t is in a vertex of γ_e that is contracted after forgetting the tails I . This happens if and only if γ_e looks like



where v_1, v_2 are the vertices of γ_e , the genus $g(v_1) = 0$, and the class $a(v_1) = 0 \in A$. Define $S(e, t, I) = 0$ otherwise. Observe that for each $e \in E(\gamma)$, there is at most one $t \in T(v) \leftrightarrow T(\gamma)$ such that $S(e, t, I) = 1$.

LEMMA 4.1.1. For each $t \in T(v)$,

$$\pi^*\psi_t = \psi_t - \sum_{f:\gamma \rightarrow r(s(\eta))} [\mathfrak{M}_f]S(e, t, I)$$

where the sum is over $f : \gamma \rightarrow r(s(\eta))$ with $\#E(\gamma) = \#E(f) = 1$, and $e \in E(\gamma)$ is the unique edge.

Proof. This is a rephrasing of a standard result. □

COROLLARY 4.1.2. For each $t \in T(v)$,

$$r^*\pi^*\psi_t = m(t)\psi_t - \sum_{f:\gamma \rightarrow s(\eta)} [\mathfrak{M}_f]m(e)S(e, t, I)$$

where the sum is over $f : \gamma \rightarrow s(\eta)$ with $\#E(\gamma) = \#E(f) = 1$, and $e \in E(\gamma)$ is the unique edge.

COROLLARY 4.1.3. For each $t \in T(v)$,

$$s^*r^*\pi^*\psi_t = m(t)\psi_t - \sum_{f:\gamma \rightarrow \eta} [\mathfrak{M}_f] \sum_{e \in E(s(\gamma))} m(e)S(e, t, I)$$

where the sum is over $f : \gamma \rightarrow \eta$ with $\#E(t(\gamma)) = \#E(f) = 1$.

For each tail $t \in T(v)$, let z_t be a formal parameter. For each map $\gamma \rightarrow \eta$ in Γ^c , and for each edge $e \in T(t(\eta))$ or tail $t \in T(t(\eta))$, define formal parameters w_e and w_t . We impose the relations between z and w parameters:

- For $t \in T(t(\eta))$, $w_t = \sum_{t' \in p^{-1}(t) \cap T(v)} m(t')z_{t'}$ where the sum is over tails of $T(v) \hookrightarrow T(s(\eta))$ lying over t .
- For $e \in E(t(\gamma))$, $w_e = \sum_{e' \in p^{-1}(e)} \sum_{t \in T(v)} m(e')S(e', t, I)z_t$.

PROPOSITION 4.1.4. With this notation,

(4.1.1)

$$s^*r^*\pi^*e^{\sum_{t \in T(v)} z_t \psi_t} = \sum_{f:\gamma \rightarrow \eta} i_* \left([\mathfrak{M}_f] e^{\sum_{t \in T(t(\gamma))} w_t \psi_t} \prod_{e \in E(t(\gamma))} \frac{1 - e^{w_e \psi_e}}{\psi_e} \right)$$

where the sum is over isomorphism classes of maps $f : \gamma \rightarrow \eta$, and $i : \mathfrak{M}_f \hookrightarrow \mathfrak{M}_\eta$ is the inclusion.

Proof. When all variables z, w are zero, both sides are evidently equal. Now apply the operator $\frac{d}{dz_t}$ to both sides; it suffices to show that $\frac{d}{dz_t}$ acts by intersection with $s^*r^*\pi^*\psi_t$ on the right-hand side. This follows from Corollary 4.1.3, and Lemma 4.0.1. \square

4.2. *Pullbacks from Deligne-Mumford space.* Let $v \in \Upsilon^u$, be such that $g(v) = 0$ and $\#T(v) \geq 3$. There is a map

$$\pi : \mathfrak{M}_v \rightarrow \overline{\mathcal{M}}_{0, T(v)}$$

where $\overline{\mathcal{M}}_{0, T(v)}$ is the usual Deligne-Mumford stack of genus 0 stable curves. This map is flat; this follows from the analogous result in [4]. Suppose $\#T(v) \geq 4$. For each distinct $i, j, k, l \in T(v)$, there is a map

$$\overline{\mathcal{M}}_{0, T(v)} \rightarrow \overline{\mathcal{M}}_{0, \{i, j, k, l\}}.$$

In the usual way, by pulling back two rationally equivalent divisors on $\overline{\mathcal{M}}_{0, \{i, j, k, l\}} \cong \mathbb{P}^1$, we get the associativity equation on \mathfrak{M}_v :

$$\sum_{f_{ij|jk}:\gamma \rightarrow v} [\mathfrak{m}_{f_{ij|kl}}] = \sum_{f_{ik|jl}:\gamma \rightarrow v} [\mathfrak{m}_{f_{ik|jl}}]$$

where the left-hand side, is the sum over graphs $\gamma \rightarrow v$, such that $\#E(\gamma) = 1$, and the tails i, j and k, l are on separate vertices of γ ; and similarly for the right-hand side.

For each vertex $\nu \in \Upsilon^t$, with $g(\nu) = 0$ and $\#\mathbf{T}(\nu) \geq 4$, pulling this relation back from the map $\mathfrak{M}_\nu \rightarrow \mathfrak{M}_{r(\nu)}$, we get the associativity relations on \mathfrak{M}_ν :

$$\sum_{f_{ij|jk}:\gamma \rightarrow v} [\mathfrak{M}_{f_{ij|kl}}]m(e) = \sum_{f_{ik|jl}:\gamma \rightarrow v} [\mathfrak{M}_{f_{ik|jl}}]m(e)$$

where, as before, the graphs $\gamma_{ij|kl}$ have one edge, e , and the tails i, j and k, l are in different vertices of γ .

For $v \in \Upsilon^u$ with $g(v) = 0$ and $\#\mathbf{T}(v) \geq 3$ as before, and for each distinct $i, j, k \in \mathbf{T}(v)$ consider the map $p : \overline{\mathcal{M}}_{0, \mathbf{T}(v)} \rightarrow \overline{\mathcal{M}}_{0, \{i, j, k\}}$. Because this is a point, $\psi_i = 0$ on $\overline{\mathcal{M}}_{0, \{i, j, k\}}$. It follows that, on \mathfrak{M}_v , we have the equation

$$\psi_i = \sum_{f_{i|kl}:\gamma \rightarrow v} [\mathfrak{M}_{f_{i|kl}}]$$

where the sum is over graphs $f_{i|kl} : \gamma \rightarrow v$, such that $\#\mathbf{E}(v) = 1$, and the tails i and $\{k, l\}$ are on different vertices.

Now let $\nu \in \Upsilon^t$, be such that $g(\eta) = 0$ and $\#\mathbf{T}(\eta) \geq 3$. Pulling back this relation from $\mathfrak{M}_{r(\nu)}$ to \mathfrak{M}_ν , gives us, for each $i, j, k \in \mathbf{T}(\nu)$,

$$m(i)\psi_i = \sum_{f_{i|kl}:\gamma \rightarrow v} [\mathfrak{M}_{f_{i|kl}}]m(e)$$

where the sum is over maps $f_{i,|kl} : \gamma \rightarrow v$ where $e \in \mathbf{E}(\gamma)$ is the unique edge, and the tails i and $\{k, l\}$ are on different vertices of γ .

Finally, observe that for each $\eta \in \Gamma^c$ with $t(\eta)$ having only one vertex, we can pull these relations back via the étale map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_{t(\eta)}$ in an obvious way.

5. Virtual fundamental classes

I will use the Behrend-Fantechi [5] construction of virtual fundamental classes.

Let F be a Deligne-Mumford stack, V an Artin stack, and suppose there is a map $F \rightarrow V$. A *perfect relative obstruction theory* [5] is a two-term complex of vector bundles $E = E^{-1} \rightarrow E^0$ on F , together with a map

$$E \rightarrow \mathbb{L}_{F/V}^*$$

in the derived category $D(F)$ to the relative cotangent complex $\mathbb{L}_{F/V}^*$, which is an isomorphism on H^0 and surjective on H^{-1} .

Associated to a perfect relative obstruction theory, Behrend and Fantechi in [5] associate a cone $C \hookrightarrow (E^{-1})^\vee$, and define the virtual fundamental class

$$[F, E] \in A_*F, \quad [F, E] = s^*[C]$$

where $s : F \hookrightarrow E$ is the zero section. Here s^* is the Gysin map, which is defined to be the inverse of the pull-back isomorphism $\pi^* : A_*F \rightarrow A_*E$. Behrend-Fantechi show that $[F, E]$ only depends on the quasi-isomorphism class of E .

Let me recall some of the details of their construction. The *relative intrinsic normal cone* of a map $F \rightarrow V$, $\mathfrak{C}_{F/V}$, is a cone stack over F , with the property that if locally we factor $F \rightarrow V$ into $F \xrightarrow{i} M \xrightarrow{p} V$, where $i : F \hookrightarrow M$ is a closed embedding and $p : M \rightarrow V$ is smooth, then, $\mathfrak{C}_{F/V}$ is the quotient stack

$$\mathfrak{C}_{F/V} = [C_{F/M}/i^*T_{M/V}].$$

Here $C_{F/M}$ is the usual normal cone, which is acted on by the additive group scheme $i^*T_{M/V}$.

Suppose we have a perfect relative obstruction theory $E \rightarrow \mathbb{L}_{F/V}^*$. Let $E_1 = E^{-1\vee}$ and $E_0 = E^{0\vee}$. One can show that there is a closed embedding $\mathfrak{C}_{F/V} \hookrightarrow [E_1/E_0]$, where $[E_1/E_0]$ is the quotient stack. Form the Cartesian diagram

$$\begin{array}{ccc} C & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ \mathfrak{C}_{F/V} & \longrightarrow & [E_1/E_0] \end{array}$$

where the vertical arrows are smooth, the horizontal arrows are closed embeddings, and $C \rightarrow F$ is a usual cone, in particular a scheme over F . We then define

$$[F, E] = s^*[C]$$

where $s : F \hookrightarrow E_1$ is the zero section.

Let X', X be pure dimensional schemes of the same dimension, with X irreducible, and let $f : X' \rightarrow X$ be a map between them. We say f is of degree d , if $f_*\mathcal{O}_{X'}$ is of rank d over the generic point of X . If X is not irreducible, we say f is of pure degree d if it is of degree d for every irreducible component of X . This property is local in the smooth topology of X . That is, if $U \rightarrow X$ is a surjective smooth map, and $U' = X' \times_X U$, then $U' \rightarrow U$ is of pure degree d if and only if $X' \rightarrow X$ is. Further, this property is local in the étale topology of X' , in the following sense. For each irreducible component X_i of X , pick an étale cover $\coprod_j U_{ij} \rightarrow X'_i$, where U_{ij} are connected and $U_{ij} \rightarrow X'_i$ is of degree e_{ij} . Then,

$$\deg(X'_i/X_i) = \sum_j \frac{\deg(U_{ij}/X_i)}{e_{ij}}.$$

Let V', V be Artin stacks of the same pure dimension, and let $V' \rightarrow V$ be a map of relative Deligne-Mumford type. This means that for every scheme $U \rightarrow V$, $U \times_V V' \rightarrow U$ is a Deligne-Mumford stack. We say $V' \rightarrow V$ is of pure degree d , if for some smooth surjective map $U \rightarrow V$ from a scheme, for each irreducible component U_i of U , for some étale atlas $\coprod_j U'_{ij} \rightarrow U'_i = V' \times_V U_i$, with $U'_{ij} \rightarrow U'_i$ of degree e_{ij} , $d = \sum_j \deg(U'_{ij}/U_i)/e_{ij}$.

This property does not depend on the choices of smooth and étale atlases, because for schemes it is local in the smooth and étale topologies, as above. In particular, if V', V are DM stacks, then this definition, using the smooth topology for V , agrees with the definition using the étale topology for V .

THEOREM 5.0.1. *Suppose there is a Cartesian diagram*

$$(5.0.1) \quad \begin{array}{ccc} F_2 & \xrightarrow{f} & F_1 \\ p_2 \downarrow & & \downarrow p_1 \\ V_2 & \xrightarrow{g} & V_1, \end{array}$$

such that:

- F_i are Deligne-Mumford stacks.
- V_i are Artin stacks of the same pure dimension.
- g is a morphism of relative Deligne-Mumford type, and of pure degree d for some $d \in \mathbb{Q}_{\geq 0}$.
- f is proper.
- $F_1 \rightarrow V_1$ has perfect relative obstruction theory E_1 , inducing a perfect relative obstruction theory $E_2 = f^*E_1$ on $F_2 \rightarrow V_2$.

Then

$$f_*[F_2, E_2] = d[F_1, E_1].$$

Proof. Let \mathfrak{C}_{F_i/V_i} be the relative intrinsic normal cone stack. First, we reduce the problem to proving that $\mathfrak{C}_{F_2/V_2} \rightarrow \mathfrak{C}_{F_1/V_1}$ is of pure degree d (observe that \mathfrak{C}_{F_i/V_i} are of the same pure dimension). When $E_1 = E_1^{-1} \rightarrow E_1^0$, let $E_{1,1} = E_1^{-1\vee}$ and $E_{1,0} = E_1^{0\vee}$. Recall that we have a closed embedding $\mathfrak{C}_{F_1/V_1} \hookrightarrow [E_{1,1}/E_{1,0}]$, where this is the stack quotient. Let $C_1 = \mathfrak{C}_{F_1/V_1} \times_{[E_{1,1}/E_{1,0}]} E_{1,1}$. Now, C_1 is an ordinary cone, so that $C_1 \rightarrow F_1$ is a scheme, and there is a closed embedding $C_1 \hookrightarrow E_{1,1}$. In a similar way define $C_2 \hookrightarrow E_{2,1}$. The map $C_1 \rightarrow \mathfrak{C}_{F_1/V_1}$ is smooth and surjective, and $C_2 = \mathfrak{C}_{F_2/V_2} \times_{\mathfrak{C}_{F_1/V_1}} C_1$. Now, $\mathfrak{C}_{F_2/V_2} \rightarrow \mathfrak{C}_{F_1/V_1}$, being of pure degree d , is equivalent to $C_2 \rightarrow C_1$ being of pure degree d . Form the diagram

$$\begin{array}{ccc} C_2 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ E_{2,1} & \xrightarrow{f'} & E_{1,1} \\ p_2 \downarrow & & \downarrow p_1 \\ F_2 & \xrightarrow{f} & F_1. \end{array}$$

By definition, $p_i^*[F_i, E_i] = [C_i]$ in $A_*E_{i,1}$. To show $f_*[F_2, E_2] = d[F_1, E_1]$ is equivalent to showing $f'_*[C_2] = d[C_1]$, for which it is enough to show that $C_2 \rightarrow C_1$ is of pure degree d .

To do this we work locally, and reduce to the case of schemes. We pick a local chart $U_1 \rightarrow V_1$, where U_1 is an irreducible scheme, and $U_1 \rightarrow V_1$ is smooth. Pick an étale map of degree n , $U_2 \rightarrow U_1 \times_{V_1} V_2$, where U_2 is a scheme, and an étale map from a scheme $X_1 \rightarrow F_1$. By possibly passing to smaller charts, we can pick a factorization of the map $X_1 \rightarrow U_1$ into $X_1 \hookrightarrow M_1 \rightarrow U_1$, where M_1 is a scheme, $X_1 \hookrightarrow M_1$ is a closed embedding, and $M_1 \rightarrow U_1$ is smooth. Without loss of generality, we can assume that U_2 , X_1 and M_1 are irreducible. We have a diagram

$$\begin{array}{ccc}
 X_2 & \xrightarrow{f_X} & X_1 \\
 i_2 \downarrow & & \downarrow i_1 \\
 M_2 & \xrightarrow{f_M} & M_1 \\
 p_2 \downarrow & & \downarrow p_1 \\
 U_2 & \xrightarrow{f_U} & U_1,
 \end{array}$$

where $U_2 \rightarrow U_1$ and $M_2 \rightarrow M_1$ are of degree dn . It is sufficient to show that $\mathfrak{C}_{X_2/U_2} \rightarrow \mathfrak{C}_{X_1/U_1}$ is of pure degree dn . Also, $\mathfrak{C}_{X_i/U_i} = [C_{X_i/M_i}/i_i^*T_{M_i/U_i}]$, where C_{X_i/M_i} is the usual normal cone. Since $f_M^*T_{M_1} = T_{M_2}$, it is sufficient to show that the map $C_{X_2/M_2} \rightarrow C_{X_1/M_1}$ is of pure degree dn , and we now have the case of closed embeddings of schemes.

Now, we prove it in this case using the flat deformation to the normal cone, as in [17], and the fact that the degree is constant in a flat family of maps. Let Z_i be the blowup of $M_i \times \mathbb{P}^1$ along $X_i \times \{\infty\}$. Let M'_i be the blowup of M_i along X_i . There is a commutative diagram

$$\begin{array}{ccc}
 Z_2 & \xrightarrow{g} & Z_1 \\
 & \searrow f_2 & \downarrow f_1 \\
 & & \mathbb{P}^1.
 \end{array}$$

The maps $f_i : Z_i \rightarrow \mathbb{P}^1$ are flat, and $f_i^{-1}(\infty) = \mathbb{P}(C_i \oplus 1) + M'_i$, as Cartier divisors, with multiplicity. Clearly $Z_2 \rightarrow Z_1$ is of pure degree dn , as is $M'_2 \rightarrow M'_1$. It follows that the map $\mathbb{P}(C_2 \oplus 1) \rightarrow \mathbb{P}(C_1 \oplus 1)$ is of pure degree dn . There are open embeddings $C_i \hookrightarrow \mathbb{P}(C_i \oplus 1)$, which implies $C_2 \rightarrow C_1$ is of pure degree dn as desired. \square

6. Finite degree theorem

Let $v \in \Upsilon^u$. We want to construct $\eta \in \Upsilon^c$, together with a set of tails $A \subset T(s(\eta))$, such that:

- $s(\eta)$ has just one vertex,
- v is obtained from $r(s(\eta))$ by removing the tails A ,
- $g(t(\eta)) = 0$,
- $\dim \mathfrak{M}_\eta = \dim \mathfrak{M}_v$ and the map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_v$ is of degree $d \in \mathbb{Q}_{>0}$.

The idea is quite simple: if $C \in \mathfrak{M}_v$ is a generic genus $g(v)$ curve with some marked points P_i , and $D = \sum \lambda_i P_i$ is a positive divisor of degree $g + 1$, there is a unique (up to isomorphism) map $f : C \rightarrow \mathbb{P}^1$ with $f^{-1}(\infty) = D$. If everything is generic this map is simply ramified. There is a unique étale map of twisted curves $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$, such that the map $\tilde{\mathcal{C}} \rightarrow BS_{g+1}$ is representable, which yields the ramified map $f : C \rightarrow \mathbb{P}^1$ after taking coarse moduli spaces. We let η be the combinatorial data which labels this étale map $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$, together with the marked points. There is a map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_v$, which is generically finite.

Let us construct η more formally. We assume that $g(v) > 0$ and $T(v) \neq \emptyset$. Pick a partition $T(v) = I \amalg J$ into subsets, and a multiplicity function $d : I \rightarrow \mathbb{Z}_{>0}$, with $\sum_{i \in I} d(i) = g + 1$. We define the covering $\eta = s(\eta) \rightarrow t(\eta)$, by

- $s(\eta), t(\eta)$ have just one vertex,
- $g(t(\eta)) = 0$ and $g(s(\eta)) = g$,
- $a(s(\eta)) = a(t(\eta)) = a(v) \in A$.
- For $k \in \mathbb{Z}_{>0}$ let $[k]$ be the finite set $\{1, \dots, k\}$. Then define,

$$T(t(\eta)) = J \amalg \{\infty\} \amalg [k]$$

where

$$k \stackrel{\text{def}}{=} \#I + 3g - 1.$$

- The degree of η is $g + 1$.
- The tails of $s(\eta)$ are

$$T(s(\eta)) = (J \times [g + 1]) \amalg I \amalg ([g] \times [k]).$$

- The map $T(s(\eta)) \rightarrow T(t(\eta))$ sends $I \rightarrow \infty$, and is the natural product map on the other factors,

$$J \times [g + 1] \rightarrow J, \quad [g] \times [k] \rightarrow [k].$$

- We define the multiplicity function m on $T(t(\eta))$ where $m(\infty)$ is the lowest common multiple of $d(i)$ for $i \in I$, $m(j) = 1$ for $j \in J$ and $m(r) = 2$ for $r \in [k]$.
- We define the multiplicity function on $s(\eta)$ where $m(i) = m(\infty)/d(i)$ for $i \in I$, $m(r, j) = 1$ for $(r, j) \in [g + 1] \times J$, and on $[g] \times [k]$ m is defined by

$$m(1, s) = 1, \quad m(r, s) = 2 \text{ if } r > 1.$$

This implies that for $(r, s) \in [g] \times [k]$,

$$d(1, s) = 2, \quad d(r, s) = 1 \text{ if } r > 1.$$

The Riemann-Hurwitz formula becomes

$$2g - 2 = -2(g + 1) + g + 1 - \#I + k$$

which is true by our choice of k . The formulae for the dimensions of \mathfrak{M}_η and \mathfrak{M}_v are

$$\dim \mathfrak{M}_\eta = k + \#J - 2, \quad \dim \mathfrak{M}_v = 3g - 3 + \#I + \#J$$

which are equal.

The map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_v$, comes from forgetting the tails

$$(J \times [g]) \coprod ([g] \times [k]) \hookrightarrow (J \times [g + 1]) \coprod ([g] \times [k]) \coprod I = T(s(\eta)).$$

LEMMA 6.0.1. *The map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_v$ is of degree*

$$\frac{k!(g!)^{\#J}((g - 1)!)^k}{2^k m(\infty)}.$$

Proof. Let $C \in \mathfrak{M}_v$ be generic, and define a divisor $D = \sum_{i \in I} d(i)i \subset C$. Now, $\deg D = g + 1$ and $D > 0$. For a generic curve with generic marked points C , I claim that there is a unique up to isomorphism map $f : C \rightarrow \mathbb{P}^1$ with $f^{-1}(\infty) = D$, and further f is simply ramified.

Let D' be a divisor on C with $0 \leq D' < D$. Let $I' \subset I$ be the set of points which occur with nonzero multiplicity in D' . Firstly, we would like to show that the locus of smooth curves C which admit a map $f : C \rightarrow \mathbb{P}^1$ with $f^{-1}(\infty) = D'$ is of positive codimension in \mathfrak{M}_v . Any such curve $C \in \mathfrak{M}_v$, with its marked points, is determined up to finite ambiguity by the branch points in \mathbb{P}^1 of the map $f : C \rightarrow \mathbb{P}^1$, up to $\mathbb{C} \times \mathbb{C}^*$ -action, and by the marked points $J \coprod (I \setminus I') \subset C$. It follows from the Riemann-Hurwitz formula that f has strictly less than $3g - 1 + \#I'$ branch points. The $\mathbb{C} \times \mathbb{C}^*$ -action on \mathbb{P}^1 reduces the dimension of the space of possible branch points by 2; so we find that the moduli space of smooth curves C with marked points, which admit such a map $f : C \rightarrow \mathbb{P}^1$, is of dimension strictly less than $3g - 3 + \#J + \#I$. That is, it is of positive codimension in \mathfrak{M}_v .

This implies that for a generic curve C , for all $0 \leq D' < D$, $\dim H^0(C, \mathcal{O}_C(D')) = 1$. Riemann-Roch tells us that $\dim H^0(C, \mathcal{O}_C(D)) \geq 2$. Take D' to be of degree g . We must have $H^1(C, \mathcal{O}_C(D')) = 0$, which implies $H^1(C, \mathcal{O}_C(D)) = 0$ and $\dim H^0(C, \mathcal{O}_C(D)) = 2$. This last fact implies that there is precisely one map $f : C \rightarrow \mathbb{P}^1$, up to isomorphism, such that $f^{-1}(\infty) = D$. I claim that if C is generic, this map is simply ramified. One can see this by observing, using the Riemann-Hurwitz formula as before, that the locus of smooth curves $C \in \mathfrak{M}_v$ which admit a map $f : C \rightarrow \mathbb{P}^1$ with $f^{-1}(\infty) = D$ and nonsimple ramification is of positive codimension.

The degree of our map $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_v$ can now be calculated from different ways of ordering tails of η , and automorphisms of twisted curves in \mathfrak{M}_η over their coarse moduli space. □

7. Stable curves in X

Let X be a smooth projective variety. Let $C(X)$ be the cone of effective one-cycles in X modulo numerical equivalence. $C(X)$ is a semigroup with indecomposable zero and finite decomposition. We define our categories of graphs and vertices using $C(X)$.

For every $\gamma \in \Gamma^u$, we have [4] the moduli stack $\overline{\mathcal{M}}_\gamma(X)$ of stable maps to X of type γ where $\overline{\mathcal{M}}_\gamma(X)$ is a separated, proper Deligne-Mumford stack of finite type. There is a map $\overline{\mathcal{M}}_\gamma(X) \rightarrow \mathfrak{M}_\gamma$, with a perfect relative obstruction theory $(R\pi_* f^* TX)^\vee$, where $\pi : \mathcal{C} \rightarrow \overline{\mathcal{M}}_\gamma(X)$ is the universal curve and $f : \mathcal{C} \rightarrow X$ the universal map. The target \mathfrak{M}_γ for this perfect relative obstruction theory is slightly different from the version used in Behrend’s construction [4], because of the labellings by elements of the semigroup A . The virtual fundamental classes, however, are the same. This follows from the fact that the map $\mathfrak{M}_{g,n,a} \rightarrow \mathfrak{M}_{g,n}$ is étale.

Suppose we have a map $\gamma' \rightarrow \gamma$ in Γ^u . Then there is a fibre square,

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\gamma'}(X) & \longrightarrow & \overline{\mathcal{M}}_\gamma(X) \\ p' \downarrow & & \downarrow p \\ \mathfrak{M}_{\gamma'} & \longrightarrow & \mathfrak{M}_\gamma. \end{array}$$

Further, the perfect relative obstruction theory of p' is pulled back from that of p .

If γ' is obtained from γ by adding on some tails, then we have a fibre square exactly as above, and again the perfect relative obstruction theory of $p' : \overline{\mathcal{M}}_{\gamma'}(X) \rightarrow \mathfrak{M}_{\gamma'}$ is pulled back from that of $p : \overline{\mathcal{M}}_\gamma(X) \rightarrow \mathfrak{M}_\gamma$.

If γ' is obtained by cutting an edge of γ , then $\mathfrak{M}_{\gamma'} = \mathfrak{M}_{\gamma}$ and we have a fibre square,

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\gamma}(X) & \longrightarrow & \overline{\mathcal{M}}_{\gamma'} \\ \downarrow & & \downarrow \\ X \times \mathfrak{M}_{\gamma} & \xrightarrow{\Delta} & X \times X \times \mathfrak{M}_{\gamma'} \end{array}$$

The perfect relative obstruction theories of $\overline{\mathcal{M}}_{\gamma'}(X)$ and $\overline{\mathcal{M}}_{\gamma}(X)$ over $\mathfrak{M}_{\gamma'} = \mathfrak{M}_{\gamma}$ are compatible with this Cartesian diagram, in the sense of [5, §7].

Let $\eta \in \Gamma^c$. Define $\overline{\mathcal{M}}_{\eta}(X)$ by the Cartesian square

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\eta}(X) & \xrightarrow{(ros)_X} & \overline{\mathcal{M}}_{r(s(\eta))} \\ \downarrow & & \downarrow \\ \mathfrak{M}_{\eta} & \xrightarrow{ros} & \mathfrak{M}_{r(s(\eta))} \end{array}$$

This is the stack of diagrams $\mathcal{C} \leftarrow \mathcal{C}' \rightarrow X$, where $\mathcal{C}' \rightarrow \mathcal{C}$ is a map from \mathfrak{M}_{η} and, if \mathcal{C}' is the coarse moduli space of \mathcal{C}' , the map $\mathcal{C}' \rightarrow X$ is a stable map from $\overline{\mathcal{M}}_{r(s(\eta))}(X)$. Give $\overline{\mathcal{M}}_{\eta}(X) \rightarrow \mathfrak{M}_{\eta}$ the perfect relative obstruction theory pulled back from that for $\overline{\mathcal{M}}_{r(s(\eta))}(X) \rightarrow \mathfrak{M}_{r(s(\eta))}$.

If $\eta' \rightarrow \eta$ is a map in Γ^c , then we have a fibre square

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\eta'}(X) & \longrightarrow & \overline{\mathcal{M}}_{\eta}(X) \\ p' \downarrow & & \downarrow p \\ \mathfrak{M}_{\eta'} & \longrightarrow & \mathfrak{M}_{\eta} \end{array}$$

and the perfect relative obstruction theory for p' is pulled back from that for p .

Let $\eta \in \Gamma^c$ and let $e \in E(t(\eta))$. Let $I \subset E(s(\eta))$ be the set of edges lying over e . Let $\eta' \in \Gamma^c$ be obtained from η by cutting the edges e, I . Then, $\mathfrak{M}_{\eta'} = \mathfrak{M}_{\eta}$, and we have a Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\eta} & \longrightarrow & \overline{\mathcal{M}}_{\eta'} \\ \downarrow & & \downarrow \\ \mathfrak{M}_{\eta} \times X^I & \xrightarrow{\Delta} & \mathfrak{M}_{\eta'} \times (X^2)^I \end{array}$$

which is compatible with perfect relative obstruction theories over $\mathfrak{M}_{\eta} = \mathfrak{M}_{\eta'}$.

Observe that since $\mathfrak{M}_{\eta} \rightarrow \mathfrak{M}_{t(\eta)}$ is étale, $\overline{\mathcal{M}}_{\eta}(X)$ has a perfect relative obstruction theory over $\mathfrak{M}_{t(\eta)}$ also.

7.1. *Stable maps to symmetric products.* We have described stacks of stable maps to a smooth projective variety X . We also have stacks of stable maps to a smooth DM stack V , as defined by Abramovich and Vistoli in [3]. We are interested in these when $V = S^d X$. We need something to play the

role of the cone of curves in X , that is, to hold homology classes of curves. We simply use again $C(X)$, the semigroup of effective one cycles on X up to numerical equivalence. For $\gamma \in \Gamma^t(C(X)) = \Gamma^t$, let $\overline{\mathcal{M}}_\gamma(S^d X)$ be the stack of stable representable maps from curves in \mathfrak{M}_γ to $S^d X$, in a way compatible with the $C(X)$ -markings on the curve.

LEMMA 7.1.1. *For a covering $\gamma' \twoheadrightarrow \gamma$, let $\text{Aut}(\gamma' | \gamma)$ be the group of automorphisms γ' , commuting with the covering $\gamma' \twoheadrightarrow \gamma$. Then,*

$$\overline{\mathcal{M}}_\gamma(S^d X) = \coprod_{\gamma' \twoheadrightarrow \gamma} \overline{\mathcal{M}}_{\gamma' \twoheadrightarrow \gamma}(X) / \text{Aut}(\gamma' | \gamma)$$

where the union is over all $\gamma' \twoheadrightarrow \gamma$ of degree d . Further, this identification is compatible with perfect relative obstruction theories over \mathfrak{M}_γ .

Proof. The isomorphism at the level of stacks follows from Section 2.2. We need to prove compatibility of perfect relative obstruction theories. Suppose we have a representable stable map $f : \mathcal{C} \rightarrow S^d X$, corresponding to a diagram $\mathcal{C} \xleftarrow{p'} \mathcal{C}' \xrightarrow{f'} X$, and equivalently to a principal S_d bundle $p : P \rightarrow \mathcal{C}$, where P is an algebraic space, and an S_d -equivariant map $g : P \rightarrow X^d$. The perfect relative obstruction theory for stable maps to $S^d X$, is given by $H^*(\mathcal{C}, f^* T S^d X)$. But,

$$f^* T S^d X = p_*^{S_d} g^* T X^d = p'_* f'^* T X.$$

Observe p'_* and $p_*^{S_d}$ are exact. Let $g' : \mathcal{C}' \rightarrow X$ be the map from the coarse moduli space of \mathcal{C}' to X , and let $m : \mathcal{C}' \rightarrow \mathcal{C}'$ be the canonical map. Observe that m_* is exact, and $f' = g' \circ m : \mathcal{C}' \rightarrow X$. Now,

$$\begin{aligned} H^*(\mathcal{C}, f^* T S^d X) &= H^*(\mathcal{C}', f'^* T X) \\ &= H^*(\mathcal{C}', m^* g'^* T X) \\ &= H^*(\mathcal{C}', g'^* T X) \end{aligned}$$

as desired. □

There are evaluation maps $\overline{\mathcal{M}}_{\gamma' \twoheadrightarrow \gamma}(X) \rightarrow X^{\Gamma(\gamma')}$. These come from the evaluation maps

$$\overline{\mathcal{M}}_\gamma(S^d X) \rightarrow \text{twisted sectors of } S^d X$$

which are used to define quantum cohomology of $S^d X$. The stack of twisted sectors \tilde{V} of a DM stack V is the stack of cyclic gerbes in V [2], [7],[31],

$$\tilde{V} = \coprod_{k \geq 1} \text{HomRep}(B\mu_k, V).$$

We can identify $\widetilde{S^d X}$ with a disjoint union $\coprod_{\sigma \in S_{d^*}} ((X^d)^\sigma) / C(\sigma)$, where the disjoint union is over conjugacy classes in S_d , σ is a representative of each

conjugacy class, $(X^d)^\sigma$ is the σ -fixed points and $C(\sigma)$ is the centralizer of σ . There is a commutative diagram of evaluation maps

$$\begin{CD} \overline{\mathcal{M}}_{\gamma' \rightarrow \gamma}(X) @>ev>> X^{T(\gamma')} \\ @VVV @VVV \\ \overline{\mathcal{M}}_\gamma(S^d X) @>ev>> (\widetilde{S^d X})^{T(\gamma)}. \end{CD}$$

Further, the tautological ψ -classes on $\overline{\mathcal{M}}_\gamma(S^d X)$ are pulled back to ψ classes on $\overline{\mathcal{M}}_{\gamma' \rightarrow \gamma}(X)$. Thus one can identify integrals of the form

$$\int_{[\overline{\mathcal{M}}_{\gamma' \rightarrow \gamma}]_{\text{virt}}} \prod_{t \in T(\gamma)} \psi_t^{k_t} \prod_{t' \in T(\gamma')} ev_{t'}^* h_{t'}$$

where $h_{t'} \in H^*(X)$, with Gromov-Witten invariants of $S^d X$.

8. From genus g invariants of X to genus 0 invariants of $S^d X$

Let $v \in \Upsilon^u$, so that v labels a stack of stable maps to X . Assume $g(v) > 0$ and $\#T(v) > 0$. We will use the notation of Section 6. There we constructed $\eta \in \Upsilon^c$, such that v is obtained by removing some tails of $r(s(\eta))$, and $g(t(\eta)) = 0$. The associated map

$$\mathfrak{M}_\eta \rightarrow \mathfrak{M}_v$$

was shown to be of degree

$$n \stackrel{\text{def}}{=} \frac{k!(g!)^{\#J}((g-1)!)^k}{2^k m(\infty)}.$$

Form the fibre square

$$\begin{CD} \overline{\mathcal{M}}_\eta(X) @>q>> \overline{\mathcal{M}}_v(X) \\ @VVV @VVV \\ \mathfrak{M}_\eta @>>> \mathfrak{M}_v. \end{CD}$$

LEMMA 8.0.2. *The map $q : \overline{\mathcal{M}}_\eta(X) \rightarrow \overline{\mathcal{M}}_v(X)$, is of degree n , in the virtual sense,*

$$q_*[\overline{\mathcal{M}}_\eta(X)]_{\text{virt}} = n[\overline{\mathcal{M}}_v(X)]_{\text{virt}}.$$

Proof. We apply Theorem 5.0.1. Observe that $\mathfrak{M}_\eta \rightarrow \mathfrak{M}_v$ is relatively of Deligne-Mumford type and generically finite of degree n , \mathfrak{M}_η and \mathfrak{M}_v are algebraic stacks, and that $\overline{\mathcal{M}}_\eta(X) \rightarrow \overline{\mathcal{M}}_v(X)$ is proper. □

We have seen in Section 4 how to express the pulled-back tautological classes $p^*\psi_t$, and their products, for $t \in T(v)$, in terms of tautological classes pushed forward under contractions $\rho \rightarrow \eta$ in Γ^c . Let us combine this result with the previous one to calculate Gromov-Witten invariants of X in terms of integrals over $\overline{\mathcal{M}}_\rho(X)$, which are genus 0 invariants of $S^d X$.

We have a commutative diagram

$$(8.0.1) \quad \begin{array}{ccc} X^{T(\eta)} & \xrightarrow{\pi} & X^{T(v)} \\ ev_\eta \uparrow & & \uparrow ev_v \\ \overline{\mathcal{M}}_\eta(X) & \xrightarrow{q} & \overline{\mathcal{M}}_v(X) \\ c_\eta \downarrow & & \downarrow c_v \\ \mathfrak{M}_\eta & \xrightarrow{p} & \mathfrak{M}_v. \end{array}$$

The integrals we want to calculate are

$$\int_{[\overline{\mathcal{M}}_v(X)]_{\text{virt}}} c_v^* e^{\sum_{t \in T(v)} z_t \psi_t} ev_v^* \alpha$$

where $\alpha \in H^* X^{T(v)}$.

Let us recall some of the notation of Section 4.1. For each map $f : \rho \rightarrow \eta$ in Γ^c , we defined

$$\|f\| = \frac{\# \text{Aut}(\rho \rightarrow \eta \mid \eta)}{\prod_{e \in E(t(\rho))} m(e)^2}$$

where $\text{Aut}(\rho \rightarrow \eta \mid \eta)$ is the group of automorphisms of ρ commuting with the contraction $\rho \rightarrow \eta$, or equivalently (in this special case) fixing all tails.

Let $I \subset T(s(\eta))$ be the set of tails we forget to obtain v . For each $f : \rho \rightarrow \eta$, each edge $e \in E(s(\rho))$, and each tail $t \in v$, we defined $S(e, t, I) \in \{0, 1\}$ as follows. Let $s(\rho)_e$ be obtained from contracting all edges other than e . If the vertex of $s(\rho)_e$ containing t becomes unstable after forgetting the tails I , we set $S(e, t, I) = 1$; otherwise $S(e, t, I) = 0$.

For each tail $t \in T(v)$, define a formal variable z_t , and for each edge $e \in E(t(\rho))$, define a variable w_e , with the relations

$$w_e = \sum_{e' \in p^{-1}(e) \subset E(s(\rho))} m(e') S(e', t, I) z_t.$$

Using this notation, we have

THEOREM 8.0.3 (Main theorem).

$$(8.0.2) \quad \int_{[\overline{\mathcal{M}}_v(X)]_{\text{virt}}} c_v^* e^{\sum_{t \in \mathbb{T}(v)} z_t \psi_t} ev_v^* \alpha$$

$$= \sum_{f: \rho \rightarrow \eta} \frac{1}{n \|f\|} \int_{[\overline{\mathcal{M}}_\rho(X)]_{\text{virt}}} c_\rho^* \left(e^{\sum_{t \in \mathbb{T}(t(\rho))} w_t \psi_t} \prod_{e \in \mathbb{E}(t(\rho))} \frac{1 - e^{w_e \psi_e}}{\psi_e} \right) ev_\rho^* \pi^* \alpha.$$

The left-hand side is the general form for descendent genus g invariants of X . The right-hand side is an expression in the genus 0 invariants of the symmetric product stack $S^{g+1}X$.

Proof. Firstly, the projection formula shows that

$$\int_{[\overline{\mathcal{M}}_v(X)]_{\text{virt}}} c_v^* e^{\sum_{t \in \mathbb{T}(v)} z_t \psi_t} ev_v^* \alpha = \frac{1}{n} \int_{[\overline{\mathcal{M}}_\eta(X)]_{\text{virt}}} q^* c_v^* e^{\sum_{t \in \mathbb{T}(v)} z_t \psi_t} q^* ev_v^* \alpha.$$

The formula 4.1.1 shows that

$$q^* c_v^* e^{\sum_{t \in \mathbb{T}(v)} z_t \psi_t} = c_\eta^* \left(\sum_{f: \rho \rightarrow \eta} e^{\sum_{t \in \mathbb{T}(t(\rho))} w_t \psi_t} \prod_{e \in \mathbb{E}(t(\rho))} \frac{1 - e^{w_e \psi_e}}{\psi_e} c_\eta^* [\mathfrak{M}_f] \right).$$

Now, for each term in the sum, $\mathfrak{M}_\rho \rightarrow \mathfrak{M}_f$ is étale of degree $\|f\|$. The standard compatibility of virtual fundamental classes shows that

$$c_\eta^* [\mathfrak{M}_f] = \frac{1}{\|f\|} f_* [\overline{\mathcal{M}}_\rho(X)]$$

where $f_* : \overline{\mathcal{M}}_\rho(X) \rightarrow \overline{\mathcal{M}}_\eta(X)$ is the canonical map. This implies the result. \square

9. Examples

9.1. *Generalities.* I will calculate some examples in the case where X is a point. We will work with the semigroup $A = 0$. For each $\eta \in \Upsilon^c$, we have the associated moduli stack $\overline{\mathcal{M}}_\eta$, with a map

$$p : \overline{\mathcal{M}}_\eta \rightarrow \overline{\mathcal{M}}_{g(t(\eta)), \mathbb{T}(t(\eta))}.$$

The stack $\overline{\mathcal{M}}_{g(t(\eta)), \mathbb{T}(t(\eta))}$ is the usual Deligne-Mumford stack of stable curves. It follows from the results of [1] that $\overline{\mathcal{M}}_\eta$ is closely related to the normalization of a stack of admissible covers.

In this section, we will always assume $g(t(\eta)) = 0$, and that $\#\mathbb{T}(t(\eta)) = n$. Further, we pick an ordering on the set $\mathbb{T}(t(\eta))$, and so obtain an isomorphism

$$\mathbb{T}(t(\eta)) \cong [n] = \{1, \dots, n\}.$$

The map p is now a map $\overline{\mathcal{M}}_\eta \rightarrow \overline{\mathcal{M}}_{0,n}$; we want to calculate its degree. For each vertex $v \in V(s(\eta))$, and tail $t' \in T(s(\eta))$, recall we have numbers $d(v), d(t') \in \mathbb{Z}_{\geq 1}$. Let $i \in [n] \cong T(t(\eta))$. Then, the set

$$\{d(t') \mid t' \in p^{-1}(i) \subset T(s(\eta)), t' \text{ is attached to } v\}$$

defines a partition of $d(v)$, and so a conjugacy class $C_{i,v} \subset S_{d(v)}$. Let

$$\chi(v) = \left\{ \sigma_i \in S_{d(v)} \text{ for } i = 1 \dots n \mid \sigma_i \in C_{i,v}, \prod_{i=1}^n \sigma_i = 1, \right. \\ \left. \#([d(v)] / \langle \sigma_1, \dots, \sigma_n \rangle) = 1 \right\}.$$

The last condition means that the group $\langle \sigma_1, \dots, \sigma_n \rangle$ acts transitively on the set $[d(v)] = \{1, \dots, d(v)\}$.

Let $\text{Aut}(\eta \mid t(\eta), V(s(\eta)))$ be the group of automorphisms of η acting trivially on $t(\eta)$ and the set of vertices $V(s(\eta))$.

PROPOSITION 9.1.1. *The map $\overline{\mathcal{M}}_\eta \rightarrow \overline{\mathcal{M}}_{0,n}$ is of degree*

$$\frac{\# \text{Aut}(\eta \mid t(\eta), V(s(\eta)))}{\prod_{i=1}^n m(i)} \prod_{v \in V(s(\eta))} \frac{\chi(v)}{d(v)!}.$$

Recall that for each $i \in [n]$,

$$p^* \psi_i = m(i) \psi_i.$$

Let $k_i \in \mathbb{Z}_{\geq 0}$, for $i = 1 \dots n$. We have

$$(9.1.1) \quad \int_{\overline{\mathcal{M}}_\eta} \prod_{i=1}^n \psi_i^{k_i} = \frac{\# \text{Aut}(\eta \mid t(\eta), V(s(\eta)))}{\prod_{i=1}^n m(i)^{k_i+1}} \prod_{v \in V(s(\eta))} \frac{\chi(v)}{d(v)!} \int_{\overline{\mathcal{M}}_{0,n}} \prod_{i=1}^n \psi_i^{k_i}$$

which allows us, at least in principle, to calculate the integrals on the left-hand side.

9.2. *Calculations.* The first example we will compute is

Example 9.2.1.

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 1/24.$$

We define η by

- $s(\eta)$ has just one vertex; $g(s(\eta)) = 1$ and $g(t(\eta)) = 0$.
- The tails are

$$\begin{aligned} T(s(\eta)) &= \{X_1, X_2, X_3, X_4\}, & d(X_i) &= 2, & m(X_i) &= 1, \\ T(t(\eta)) &= \{x_1, x_2, x_3, x_4\}, & m(x_i) &= 2. \end{aligned}$$

- The map $T(s(\eta)) \rightarrow T(t(\eta))$ sends $X_i \mapsto x_i$.

Forgetting the marked points X_2, X_3, X_4 gives us a map

$$\pi : \overline{\mathcal{M}}_\eta \rightarrow \overline{\mathcal{M}}_{1,1}.$$

Now we apply the main theorem. Note that the only graph $f : \rho \rightarrow \eta$ that arises with nonzero coefficient on the right-hand side of (8.0.2) is $\rho = \eta$. Therefore

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 3^{-1} \cdot 2^3 \int_{\overline{\mathcal{M}}_\eta} \psi_1.$$

Next, we apply formula (9.1.1). In this case, $\chi(v) = 1$ for the unique vertex $v \in V(s(\eta))$, and $\# \text{Aut}(\eta | t(\eta), V(s(\eta))) = 1$, so we find that

$$\int_{\overline{\mathcal{M}}_\eta} \psi_1 = 2^{-6} \int_{\overline{\mathcal{M}}_{0,4}} \psi_1 = 2^{-6}.$$

Combining these formulae yields

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 1/24.$$

Next we calculate

Example 9.2.2.

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = 1/1152 = 2^{-7}3^{-2}.$$

One can see this is correct by applying the Kontsevich-Witten theorem. We define $\eta \in \Upsilon^c$ in this case by

- $s(\eta)$ has just one vertex, with $g(s(\eta)) = 2$ and $g(t(\eta)) = 0$.
- We define the sets of tails by

$$\begin{aligned} \mathbb{T}(s(\eta)) &= \{X_1, X_2, Y_2, X_3, Y_3, \dots, X_7, Y_7\}, \\ \mathbb{T}(t(\eta)) &= \{x_1, x_2, x_3, \dots, x_7\} \end{aligned}$$

with the map $\mathbb{T}(s(\eta)) \rightarrow \mathbb{T}(t(\eta))$, sending $X_i \mapsto x_i$ and $Y_j \mapsto x_j$.

- The multiplicities of these tails are defined by

$$\begin{aligned} d(X_1) &= 3, & d(X_i) &= 2 \text{ if } i \geq 2, \\ m(X_i) &= 1 \text{ for all } i, \\ d(Y_i) &= 1, & m(Y_i) &= 2, \\ m(x_1) &= 3, & m(x_i) &= 2 \text{ if } i \geq 2. \end{aligned}$$

Forgetting the marked points X_i, Y_i for all $i \geq 2$, gives a map

$$\pi : \overline{\mathcal{M}}_\eta \rightarrow \overline{\mathcal{M}}_{2,1}.$$

Now we apply the main formula (8.0.2). In this case, we find that there are nontrivial graphs $f : \rho \rightarrow \eta$ appearing on the right-hand side. For each $i = 2 \dots 7$ define a graph $\rho_i \in \Gamma^c$ with a map $f_i : \rho_i \rightarrow \eta$, as follows.

- $s(\rho_i)$ has 3 vertices, V_1, V_2, W_2 , and $t(\rho_i)$ has two vertices v_1, v_2 . The map $V(s(\rho_i)) \rightarrow V(t(\rho_i))$ sends $V_i \rightarrow v_i$ and $W_2 \rightarrow v_2$.
- The genera of the vertices are given by

$$g(v_i) = 0, \quad g(V_1) = g(W_2) = 0, \quad g(V_2) = 2.$$

- There are an edge e joining v_1 and v_2 , an edge E joining V_1 and V_2 , and an edge F joining V_1 and W_2 . The multiplicities of these edges are given by

$$\begin{aligned} m(e) = 2, & & m(E) = 1, & & m(F) = 2, \\ & & d(E) = 2, & & d(F) = 1. \end{aligned}$$

- The sets of tails of the vertices are given by

$$\begin{aligned} T(V_1) &= \{X_1, X_i, Y_i\}, & T(V_2) &= \{X_j \mid j \neq 1, i\}, & T(W_2) &= \{Y_j \mid j \neq i\}, \\ T(v_1) &= \{x_1, x_i\}, & T(v_2) &= \{x_i \mid i \neq 1, i\}. \end{aligned}$$

Observe that after contracting the edge F , X_1 is on a genus 0 vertex that becomes unstable after removing the other marked points. This implies $f_i : \rho_i \rightarrow \eta$ occurs with nonzero coefficient in the expansion in the main formula (8.0.2). In fact, ρ_i are the only nontrivial graphs which occur. We have $\frac{1}{\|f_i\|} = 4$. Note also that on $\overline{\mathcal{M}}_{\rho_i}$, $\psi_1 = 0$ because the marked point x_1 is on a genus 0 curve with two marked points and one edge. Applying the main formula, we see

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = \frac{2^6 \cdot 3}{6!} \left(\int_{\overline{\mathcal{M}}_\eta} \psi_1^4 - 4 \sum_{i=2}^7 \int_{\overline{\mathcal{M}}_{\rho_i}} \psi_e^3 \right).$$

Next, we apply formula (9.1.1) to calculate $\int_{\overline{\mathcal{M}}_\eta} \psi_1^4$. One can calculate easily that

$$\begin{aligned} \#\{\sigma_1, \sigma_2, \dots, \sigma_7 \in S_3 \mid \sigma_1 \text{ is a 3-cycle,} \\ \sigma_i \text{ are transpositions for } i \geq 2, \prod \sigma_j = 1\} = 3^5 \cdot 2. \end{aligned}$$

Further, $\#\text{Aut}(\eta \mid t(\eta), V(s(\eta))) = 1$, so that

$$\int_{\overline{\mathcal{M}}_\eta} \psi_1^4 = 2^{-6} \cdot 3^{-1} \int_{\overline{\mathcal{M}}_{0,7}} \psi_1^4 = 2^{-6} \cdot 3^{-1}.$$

Now we calculate $\int_{\overline{\mathcal{M}}_{\rho_i}} \psi_e^3$. As in subsection 3.2, $\overline{\mathcal{M}}_{\rho_i}$ splits as a product of contributions from the vertices of $t(\rho_i)$. Let $p : V(s(\rho_i)) \rightarrow V(t(\rho_i))$ be the natural map. For each $v \in V(t(\rho_i))$, let $p^{-1}(v) \rightarrow v \in \Upsilon^c$ be the natural

covering, whose tails consist of tails and germs of edges in ρ_i at the vertices $v, p^{-1}(v)$. We have

$$\overline{\mathcal{M}}_{\rho_i} = \prod_{v \in V(t(\rho_i))} \overline{\mathcal{M}}_{p^{-1}(v) \rightarrow v}.$$

This implies that integrals split in a similar way. There are two vertices v_1, v_2 on $t(\rho_i)$. We have $p^{-1}(v_1) = V_1$ and $p^{-1}(v_2) = \{V_2, W_2\}$. Further,

$$\dim \overline{\mathcal{M}}_{p^{-1}(v_1) \rightarrow v_1} = 0.$$

Denote by e the germ of the edge at v_2 , which we consider as a tail. We have

$$\int_{\overline{\mathcal{M}}_{\rho_i}} \psi_e^3 = \left(\int_{\overline{\mathcal{M}}_{p^{-1}(v_1) \rightarrow v_1}} 1 \right) \times \left(\int_{\overline{\mathcal{M}}_{p^{-1}(v_2) \rightarrow v_2}} \psi_e^3 \right).$$

An easy application of formula (9.1.1) now shows that

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{p^{-1}(v_1) \rightarrow v_1}} 1 &= 3^{-1} \cdot 2^{-2} \\ \int_{\overline{\mathcal{M}}_{p^{-1}(v_2) \rightarrow v_2}} \psi_e^3 &= 2^{-10}. \end{aligned}$$

Finally, we see that

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{2,1}} \psi^4 &= \frac{2^6 \cdot 3}{6!} (2^{-6} \cdot 3^{-1} - 24 \cdot 2^{-12} \cdot 3^{-1}) \\ &= 2^{-7} \cdot 3^{-2} \end{aligned}$$

as desired.

IMPERIAL COLLEGE, LONDON, UNITED KINGDOM
 Current address: NORTHWESTERN UNIVERSITY, EVANSTON, IL
 E-mail address: costello@math.northwestern.edu

REFERENCES

- [1] D. ABRAMOVICH, A. CORTI, and A. VISTOLI, Twisted bundles and admissible covers, *Comm. Algebra* **31** (2003), 3547–3618.
- [2] D. ABRAMOVICH, T. GRABER, and A. VISTOLI, Algebraic orbifold quantum products, in *Orbifolds in Mathematics and Physics* (Madison, WI, 2001), *Contemp. Math.* **310**, 1–24, A. M. S., Providence, RI, 2002.
- [3] D. ABRAMOVICH and A. VISTOLI, Compactifying the space of stable maps, *J. Amer. Math. Soc.* **15** (2002), 27–75.
- [4] K. BEHREND, Gromov-Witten invariants in algebraic geometry, *Invent. Math.* **127** (1997), 601–617.
- [5] K. BEHREND and B. FANTECHI, The intrinsic normal cone, *Invent. Math.* **128** (1997), 45–88.

- [6] K. BEHREND and YU. MANIN, Stacks of stable maps and Gromov-Witten invariants, *Duke Math. J.* **85** (1996), 1–60.
- [7] W. CHEN and Y. RUAN, Orbifold quantum cohomology; math.AG/0005198.
- [8] ———, A new cohomology theory for orbifold, *Comm. Math. Phys.* **248** (2004), 1–31.
- [9] ———, Orbifold Gromov-Witten theory, in *Orbifolds in Mathematics and Physics* (Madison, WI, 2001), *Contemp. Math.* **310**, 25–85, A. M. S., Providence, RI, 2002.
- [10] B. DUBROVIN and Y. ZHANG, Frobenius manifolds and Virasoro constraints, *Selecta Math.* **5** (1999), 423–466.
- [11] ———, Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants; math.DG/0108160.
- [12] T. EGUCHI, K. HORI, and C.-S. XIONG, Quantum cohomology and Virasoro algebra, *Phys. Lett. B* **402** (1997), 71–80.
- [13] T. EKEDAHL, S. LANDO, M. SHAPIRO, and A. VAINSHTEIN, On Hurwitz numbers and Hodge integrals, *C. R. Acad. Sci. Paris Sér. Math.* **328** (1999), 1175–1180.
- [14] ———, Hurwitz numbers and intersections on moduli spaces of curves, *Invent. Math.* **146** (2001), 297–327.
- [15] B. FANTECHI and L. GÖTTSCHE, Orbifold cohomology for global quotients, *Duke Math. J.* **117** (2003), 197–227.
- [16] B. FANTECHI and R. PANDHARIPANDE, Stable maps and branch divisors, *Compositio Math.* **130** (2002), 345–364.
- [17] W. FULTON, *Intersection Theory*, Springer-Verlag, New York, 1984.
- [18] T. GRABER and R. VAKIL, Hodge integrals and Hurwitz numbers via virtual localization, *Compositio Math.* **135** (2003), 25–36.
- [19] T. J. JARVIS, R. KAUFMANN, and T. KIMURA, Pointed admissible G -covers and G -equivariant cohomological field theories; math.AG/0302316.
- [20] T. J. JARVIS and T. KIMURA, Orbifold quantum cohomology of the classifying space of a finite group, in *Orbifolds in Mathematics and Physics* (Madison, WI, 2001), *Contemp. Math.* **31**, 123–134, A. M. S., Providence, RI, 2002.
- [21] M. KONTSEVICH, Intersection theory on the moduli space of curves and the matrix Airy function, *Comm. Math. Phys.* **147** (1992), 1–23.
- [22] M. KONTSEVICH and YU. MANIN, Gromov-Witten classes, quantum cohomology and enumerative geometry, *Comm. Math. Phys.* **164** (1994), 525–562.
- [23] ———, Relations between the correlators of topological sigma-model coupled to gravity, *Comm. Math. Phys.* **196** (1998), 385–398.
- [24] F. F. KNUDSEN, On the projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$, *Math. Scand.* **52** (1983), 161–199.
- [25] A. KRESCH, Cycle groups for Artin stacks, *Invent. Math.* **138** (1999), 495–536.
- [26] M. LEHN and C. SORGER, The cup product of the Hilbert scheme for K3 surfaces, *Invent. Math.* **152** (2003), 305–329.
- [27] G. LAUMON and L. MORET-BAILLY, *Champs Algébriques*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **39**, Springer-Verlag, New York, 2000.
- [28] D. MUMFORD, Towards an enumerative geometry of the moduli space of curves, *Arithmetic and Geometry*, Vol. II, *Progr. Math.* **36**, 271–328, Birkhauser Boston, 1983.
- [29] A. OKOUNKOV and R. PANDHARIPANDE, Gromov-Witten theory, Hurwitz numbers and matrix models, I; math.AG/0101147.

- [30] ———, Gromov-Witten theory, Hurwitz theory, and completed cycles, *Ann. of Math.* **163** (2006), 517–560.
- [31] B. TOEN, Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford, *K-Theory* **18** (1999), 33–76.
- [32] A. VISTOLI, Intersection theory on algebraic stacks and their moduli spaces, *Invent. Math.* **97** (1989), 613–670.
- [33] E. WITTEN, 2-dimensional gravity and intersection theory on moduli space, in *Surveys in Differential Geometry* (Cambridge MA, 1990), 243–310.

(Received October 7, 2003)