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Higher-order tangents and Fefferman's paper on Whitney's extension problem

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Abstract

Whitney [W2] proved that a function defined on a closed subset of $\mathbb{R}$ is the restriction of a $\mathcal{C}^m$ function if the limiting values of all m^{th} divided differences form a continuous function. We show that Fefferman's solution of Whitney's problem for $\mathbb{R}^n$ [F, Th. 1] is equivalent to a variant of our conjecture in [BMP2] giving a criterion for $\mathcal{C}^m$ extension in terms of iterated limits of finite differences.

1. Introduction

Whitney [W2] proved that a function defined on a closed subset of $\mathbb{R}$ is the restriction of a $\mathcal{C}^m$ function if the limiting values of all m^{th} divided differences (with supports converging to points) form a continuous function. In [BMP2], we conjectured that a real-valued function φ defined on a closed subset E of $\mathbb{R}^n$ is the restriction of a $\mathcal{C}^m$ function provided that φ extends to a function on a "paratangent bundle" defined using iterated limits of finite-difference operators. The main purpose of this note is to verify the conjectures of [BMP2] with the paratangent bundle there replaced by a natural variant; we prove that these assertions (Theorems 1.7, 1.8 below) are equivalent to Fefferman's solution of Whitney's problem [F, Th. 1]. The "Glaeser refinements" that Fefferman iterates to give his criterion for φ to be $\mathcal{C}^m$ are dual to "Glaeser operations" in the sense of [BMP2]. (See Theorem 3.2.)

We will use the notation of [F] wherever possible. Let $\mathcal{P}$ denote the vector space of real m^{th} degree polynomials on $\mathbb{R}^n$, and let $\mathcal{P}^*$ denote the dual of $\mathcal{P}$. If $F \in \mathcal{C}^m(\mathbb{R}^n)$, let $T_y^m F$ denote the Taylor polynomial of F at y; i.e., $T_y^m F(x) = \sum_{\alpha \le m} \partial^\alpha F(y)(x-y)^\alpha/\alpha!$.

Using Theorem 1.8, we show (Theorem 1.9) that our geometric paratangent bundle coincides with the following analogue of the Zariski tangent

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bundle from algebraic geometry: Consider the ideal $\mathcal{I}^m(E) \subset \mathcal{C}^m(\mathbb{R}^n)$ of $\mathcal{C}^m$ functions vanishing on E. We define the $\mathcal{C}^m$ *Zariski paratangent bundle* $\mathcal{T}^m(E)$ as

$$\mathcal{T}^m(E) \ = \ \{(y,\xi) \in E \times \mathcal{P}^* : \ \xi(T_y^m F) = 0, \ F \in \mathcal{I}^m(E)\} \tag{1.1}$$

[BMP2, §2].

Given $F \in \mathcal{C}^m(\mathbb{R}^n)$, define $D^m F : E \times \mathcal{P}^* \to \mathbb{R}$ by $D^m F(y,\xi) = \xi(T_y^m F)$. Clearly, if $y \in E$, then $D^m F(y,\xi)$ depends only on $\varphi := F|E$ precisely when $(y,\xi) \in \mathcal{T}^m(E)$. Denote $D^m F|\mathcal{T}^m(E)$ by $\nabla^m \varphi : \mathcal{T}^m(E) \to \mathbb{R}$. If $y \in E$ and $\xi = \delta_y \in \mathcal{P}^*$ is the *delta function* $\delta_y(P) := P(y)$, $P \in \mathcal{P}$, then

$$\nabla^m \varphi(y, \lambda\delta_y) = \lambda\varphi(y), \quad \lambda \in \mathbb{R} \ . \tag{1.2}$$

Our criterion for $\mathcal{C}^m$ extension is based on the question: Does (1.2) determine, by means of appropriate limits, the value $\nabla^m\varphi(y,\xi)$, for all $\xi \in \mathcal{T}^m(E)(y)$?

1.1. *Glaeser operation.* Let E denote a closed subset of $\mathbb{R}^n$.

Definition 1.1. Let V be a finite-dimensional vector space. A *linear* (resp., *affine*) *subbundle* of $E \times V$ means a subset Γ of $E \times V$ such that, for all $y \in E$, the *fibre* $\Gamma(y) := \{v \in V : \ (y,v) \in E\}$ is a linear (resp., affine) subspace of V.

Definition 1.2. Let $S = \{y_1, \dots, y_k\}$ be a finite subset of $\mathbb{R}^n$. The space $W^m(S)$ of *Whitney* $\mathcal{C}^m$ *functions* is the space of sections of $S \times \mathcal{P}$. Then $W^m(S)$ is a finite-dimensional vector space. We write elements P of $W^m(S)$ as $P = (P_1, \dots, P_k)$, where each P_j belongs to the fibre of $W^m(S)$ over y_j (and $P_i = P_j$ if $y_i = y_j$). The *Whitney* $\mathcal{C}^m$ *norm* is defined as

$$\|P\|_{W^m(S)} = \max\left\{ \max_{\substack{1\le j\le k\\ |\alpha|\le m}} |\partial^\alpha P_j(y_j)|, \ \max_{\substack{y_i\neq y_j\\ |\alpha|\le m}} \frac{|\partial^\alpha (P_i - P_j)(y_j)|}{|y_i - y_j|^{m-|\alpha|}} \right\} . \tag{1.3}$$

There is a projection ("restriction mapping") $\mathcal{C}^m(\mathbb{R}^n) \ni F \mapsto P \in W^m(S)$ given by $P = (P_1, \dots, P_k)$, where each P_j is the Taylor polynomial $T_{y_j}^m F$.

For each $j = 1, \dots, k$, let $\mathcal{P}^* \ni \xi \mapsto \xi_{y_j} \in W^m(S)^*$ denote the dual to the projection $W^m(S) \ni P \mapsto P_j \in \mathcal{P}$; i.e., $\xi_{y_j}(P) = \xi(P_j)$, where $P = (P_1, \dots, P_k)$.

Given a Banach space B, with norm $\|\cdot\|_B$, we write $\|\cdot\|_{B^*}$ for the dual norm on B^*.

Definition 1.3. We fix a positive integer k. Given a linear subbundle T of $E \times \mathcal{P}^*$, we define a new linear subbundle $g(T)$ of $E \times \mathcal{P}^*$: The fibre $g(T)(y_0)$, where $y_0 \in E$, is defined as the linear span of all elements $\xi \in \mathcal{P}^*$ that are obtained in the following way: There is a sequence of subsets $S_i = \{y_{i1}, \dots, y_{ik}\} \subset E$, $i = 1, 2, \dots$ and there are elements $\xi_{ij} \in T(y_{ij})$, for $i = 1, 2, \dots$ and $j = 1, \dots, k$, such that:

(1) Each sequence $\{y_{ij}\} = \{y_{1j}, y_{2j}, \dots\}$ converges to y_0;

(2) $\Big\| \sum_{j=1}^{k} \xi_{ij,y_{ij}} \Big\|_{W^m(S_i)^*} \le c$, where c is a constant;

(3) $\xi = \lim_{i\to\infty} \sum_{j=1}^{k} \xi_{ij}$ in $\mathcal{P}^*$.

Then $T \mapsto g(T)$ is a *Glaeser operation* in the sense of [BMP2, Def. 3.2]; i.e., $\overline{T} \subset g(T)$ and g is *local* (i.e., if T_1, T_2 are linear subbundles of $E \times \mathcal{P}^*$ and $T_1(y) = T_2(y)$ for all y in an open subset U of E, then $g(T_1)(y) = g(T_2)(y)$, $y \in U$).

Definition 1.4. Let $f : T \to \mathbb{R}$ denote a function which is linear on the fibres of T. Let $y_0 \in E$. Suppose there exists a linear function $\tilde{f}(y_0) : g(T)(y_0) \to \mathbb{R}$ such that

$$\tilde{f}(y_0)(\xi) = \lim_{i\to\infty} \sum_{j=1}^{k} f(y_{ij})(\xi_{ij})$$

whenever $\xi = \lim_{i\to\infty} \sum_{j=1}^{k} \xi_{ij}$ in $\mathcal{P}^*$, where $\xi_{ij} \in T(y_{ij})$ and $S_i = \{y_{i1}, \dots, y_{ik}\} \subset E$ satisfy (1) and (2) of Definition 1.3. Then we write $\tilde{f}(y_0) = g(f)(y_0)$. Clearly, $g(f)(y_0)$ is unique if it exists. If $g(f)(y)$ exists for all $y \in E$, then we call $g(f) : g(T) \to \mathbb{R}$ the *Glaeser extension* of f.

Remark 1.5. In [BMP2], we defined a different Glaeser operation $\rho(T)$ by replacing condition (2) in Definition 1.3 by the condition

(2′) $|y_{ij} - y_{i1}|^{m-|\alpha|} |\xi_{ij,\alpha}(y_{ij})| \le c$, for all i, $2 \le j \le k$, $|\alpha| \le m$, where c is a constant and $\xi_\alpha(y)$ denotes $\xi((x-y)^\alpha/\alpha!)$, $\xi \in \mathcal{P}^*$.

Morevover, for every $f : T \to \mathbb{R}$ linear on the fibres, we defined a Glaeser extension $\rho(f) : \rho(T) \to \mathbb{R}$ as above, using the Glaeser operation ρ instead of g. Then $\rho(T) \subset g(T)$, by [BMP2, Lemma 4.7] and Lemma 2.1 below, and if $g(f)$ exists, then $\rho(f) = g(f)|\rho(T)$.

1.2. *Higher-order tangent bundle.* We fix a positive integer k. We define a *higher-order tangent bundle* (or *paratangent bundle*) $T_k^m(E) \subset E \times \mathcal{P}^*$ as follows: We begin with the bundle of lines $T_0 \subset E \times \mathcal{P}^*$ defined by

$$T_0 = \{(y, \lambda\delta_y) : \; y \in E, \; \lambda \in \mathbb{R}\} \; .$$

We then define a sequence of linear subbundles of $E \times \mathcal{P}^*$,

$$T_0 \subset T_1 \subset \cdots \; ,$$

by $T_l = g(T_{l-1})$, $l = 1, 2, \dots$.

Let $r = \dim \mathcal{P}$. By Glaeser's lemma [BMP2, Lemma 3.3] (cf. [F, Lemma 2.2]):

(1) $T_l = T_{2r}$, for all $l \geq 2r$;

(2) T_{2r} is a closed linear subbundle $T_k^m(E)$ of $E \times \mathcal{P}^*$;

(3) $\dim T_k^m(E)(y)$ is upper-semicontinuous on E.

Now consider $\varphi : E \to \mathbb{R}$. We define $f_0 : T_0 \to \mathbb{R}$ by

$$f_0(y, \lambda\delta_y) = \lambda\varphi(y) .$$

Clearly, f_0 is linear on the fibres of T_0. We inductively define $f_l : T_l \to \mathbb{R}$ by $f_l = g(f_{l-1})$, $l = 1, 2, \dots$, provided that the Glaeser extension $g(f_{l-1})$ exists. If f_l exists for all l, then we denote f_{2r} by $\nabla_k^m \varphi$ and we say that $\nabla_k^m \varphi : T_k^m(E) \to \mathbb{R}$ is the *Glaeser extension* of φ.

Remark 1.6. The Zariski paratangent bundle $\mathcal{T}^m(E)$ (1.1) has fibre $\mathcal{T}^m(E)(y) = (T_y^m \mathcal{I}^m(E))^\perp$, $y \in E$. For any k and $y \in E$, $T_k^m(E)(y) \subset (T_y^m \mathcal{I}^m(E))^\perp$ (by (1) $\Leftrightarrow$ (3) in Lemma 2.1). Therefore, $T_k^m(E) \subset \mathcal{T}^m(E)$. If φ is the restriction of a $\mathcal{C}^m$ function, then φ extends to $\nabla_k^m \varphi : T_k^m(E) \to \mathbb{R}$, and the latter coincides with the resriction of $\nabla^m \varphi : \mathcal{T}^m(E) \to \mathbb{R}$ defined above (by Lemma 2.1, (1) $\Leftrightarrow$ (3)).

1.3. *Main theorems.* For the following results, we use the positive integer $k^\#$ depending only on m and n given by Fefferman [F], and we write $T^m(E) = T_{k^\#}^m(E)$, $\nabla^m \varphi = \nabla_{k^\#}^m \varphi$.

THEOREM 1.7. *Let $\varphi : E \to \mathbb{R}$. Then φ is the restriction of a $\mathcal{C}^m$ function if and only if φ extends to*

$$\nabla^m \varphi : \; T^m(E) \to \mathbb{R} .$$

Moreover, if $F \in \mathcal{C}^m(\mathbb{R}^n)$ and $F|E = \varphi$, then, for all $y \in E$ and $\xi \in T^m(E)(y)$, $\nabla^m \varphi(y)(\xi) = \xi(T_y^m F)$.

THEOREM 1.8. *Let $\varphi : E \to \mathbb{R}$. Suppose that φ extends to $\nabla^m \varphi : T^m(E) \to \mathbb{R}$. If $y_0 \in E$ and $(\nabla^m \varphi)(y_0) = 0$, then there exists $F \in \mathcal{C}^m(\mathbb{R}^n)$ such that $F|E = \varphi$ and $T_{y_0}^m F = 0$.*

THEOREM 1.9. *$T^m(E) = \mathcal{T}^m(E)$.*

Theorems 1.7 and 1.8 answer Questions 1 and 2 in [F] using iterated limits of divided differences: It follows from Theorem 3.2 below that Theorems 1.7 and 1.8 are equivalent (by duality) to [F, Th. 1 (A) and (B)] (respectively). It is easy to see that Theorem 1.8 implies 1.9 and, conversely, if we assume Theorem 1.7, Theorem 1.9 implies 1.8.

Let $\tau_k^m(E)$ denote the paratangent bundle defined as in §1.2 above, using the Glaeser operation ρ of Remark 1.5 in place of g. (The term "paratangent bundle" comes from Glaeser's use of "paratingent" in [G].) Then

$$\tau_k^m(E) \subset T_k^m(E) \subset \mathcal{T}^m(E) \ .$$

In [BMP2], we asked whether Theorems 1.7-1.9 hold using $\tau_k^m(E)$ (for suitable k) in place of $T^m(E)$. These questions remain open in general.[1] (The assertions hold if $\tau_k^m(E) = E \times \mathcal{P}^*$ [BMP2, Proof of Th. 4.20]; e.g., with $k = 2$ if E has dense interior [*loc. cit.*]. Already $\rho(T_0) = E \times \mathcal{P}^*$ with $k = \dim \mathcal{P}$ for certain fractal sets E [I].)

Glaeser solved Whitney's problem for $\mathcal{C}^1$ functions [G] (cf. [Br]). His theorem is equivalent to the assertion that $\varphi : E \to \mathbb{R}$ is the restriction of a $\mathcal{C}^1$ function if and only if φ extends to $\nabla^1\varphi : \tau_2^1(E) \to \mathbb{R}$. In particular, [F, Th. 1] in the $\mathcal{C}^1$ case, with $k^{\#}(1, n) = 2$, follows from Glaeser's theorem.

Suppose that E is a compact subanalytic subset of $\mathbb{R}^n$. [BMP2, Th. 1.2] shows that there exists $q = q_E(m)$ such that $\varphi : E \to \mathbb{R}$ is the restriction of a $\mathcal{C}^m$ function provided that φ extends to $\nabla^q\varphi : \tau_2^q(E) \to \mathbb{R}$. (Note that $\tau_2^q(E)$ here is denoted $\tau_1^q(E)$ in [BMP2]. The loss of differentiability comes from our use of the composition theorem of [BMP1].) It follows that, if $\nabla^m\varphi : \tau_2^m(E) \to \mathbb{R}$ exists for all m, then φ extends to a $\mathcal{C}^\infty$ function, *provided that* E is "semicoherent"; moreover, in this case, there is a continuous linear $\mathcal{C}^\infty$ extension operator. Let $\mathcal{T}^{m,q}(E)$, $q \geq m$, denote the subbundle of $E \times \mathcal{P}^*$ with fibre $\mathcal{T}^{m,q}(E)(y) = (T_y^m \mathcal{I}^q(E))^\perp$, $y \in E$. Semicoherence is equivalent to the condition that, for all m, the increasing sequence of subbundles $\mathcal{T}^{m,q}(E)$ of $E \times \mathcal{P}^*$, $q \geq m$, stabilizes. (See [BM], [BMP2] for these and related matters.)

2. Whitney norm for a finite set

Let L denote a closed cube in $\mathbb{R}^n$. Let $y \in L$. Any element of $\mathcal{C}^m(L)^*$ with support $\{y\}$ has the form $F \mapsto \xi(T_y^m F)$, $F \in \mathcal{C}^m(L)$, where $\xi \in \mathcal{P}^*$.

Consider $S = \{y_1, \ldots, y_k\} \subset L$. The restriction mapping $\mathcal{C}^m(L) \to W^m(S)$ induces $W^m(S)^* \hookrightarrow \mathcal{C}^m(L)^*$. By Whitney's classical extension theorem [W1], there is an extension operator $\mathcal{E} : W^m(S) \to \mathcal{C}^m(L)$ such that

$$c_1 \|f\|_{W^m(S)} \ \leq \ \|\mathcal{E}(f)\|_{\mathcal{C}^m(L)} \ \leq \ c_2 \|f\|_{W^m(S)} \ ,$$

for all $f \in W^m(S)$, where the constants c_1, c_2 depend only on m, n and $\operatorname{diam} L$ (cf. [M, Complement 3.5]). Therefore, for all $\xi \in W^m(S)^*$,

$$\frac{1}{c_2}\|\xi\|_{W^m(S)^*} \ \leq \ \|\xi\|_{\mathcal{C}^m(L)^*} \ \leq \ \frac{1}{c_1}\|\xi\|_{W^m(S)^*} \ . \tag{2.1}$$

[1](Added in proof.) Counterexamples are given in E. Bierstone, C. Fefferman, P. D. Milman and W. Pawłucki, Examples concerning Whitney's $\mathcal{C}^m$ extension problem (to appear). An improved bound $k^{\#} = 2^{\dim \mathcal{P}}$ is given in E. Bierstone and P. D. Milman, $\mathcal{C}^m$ norms on finite sets and $\mathcal{C}^m$ extension criteria (*Duke Math. J.*, to appear).

Given $y_0 \in L$, we introduce another norm $\|\cdot\|^{y_0}_{W^m(S)}$ on $W^m(S)$ (for which the dual norm will be denoted $\|\cdot\|^{y_0}_{W^m(S)^*}$): If $y_0 \in S$, then we define $\|\cdot\|^{y_0}_{W^m(S)}$ as $\|\cdot\|_{W^m(S)}$. If $y_0 \notin S$, set $\tilde{S} = S \cup y_0$ and consider $W^m(S) \ni P \mapsto \tilde{P} \in W^m(\tilde{S})$, where $\tilde{P} = (0, P_1, \dots, P_k)$ if $P = (P_1, \dots, P_k)$. In this case, we define

$$\|P\|^{y_0}_{W^m(S)} \;:=\; \|\tilde{P}\|_{W^m(\tilde{S})} \,.$$

Let $P = (P_1, \dots, P_k) \in W^m(S)$ and let $P_0 \in \mathcal{P}$. Let (P_0) denote $(P_0, \dots, P_0) \in W^m(S)$. Assuming that $P_0 = P_j$ if $y_0 = y_j$ for some $j = 1, \dots, k$, we see from (1.3) that

$$\|(P_0) - P\|^{y_0}_{W^m(S)} \;\approx\; \max_{\substack{0 \le i,j \le k \\ y_i \ne y_j \\ |\alpha| \le m}} \frac{|\partial^\alpha (P_i - P_j)(y_j)|}{|y_i - y_j|^{m-|\alpha|}} \tag{2.2}$$

(where $\approx$ means that each side is majorized by the other times a constant depending only on m, n and $\operatorname{diam} L$); so we will regard (2.2) as an equality (cf. Fefferman's definition of "Glaeser refinement", (5) in [F, Intro.]).

Lemma 2.1. *Let $y_0 \in L$. For each $i = 1, 2, \dots,$ consider $S_i = \{y_{i1}, \dots, y_{ik}\} \subset L$, where $\lim_{i\to\infty} y_{ij} = y_0$, $j = 1, \dots, k$. Let $\xi_{ij} \in \mathcal{P}^*$, for $i = 1, 2, \dots$ and $j = 1, \dots, k$. Then the following conditions are equivalent*:

(1) $\lim_{i\to\infty} \sum_{j=1}^k \xi_{ij, y_{ij}}(F)$ *exists, for all* $F \in \mathcal{C}^m(L)$.

(2) (a) $\lim \sum \xi_{ij}$ *exists in* $\mathcal{P}^*$; (b) $\|\sum \xi_{ij,y_{ij}}\|_{\mathcal{C}^m(L)^*}$ *is bounded (independently of* i).

(3) (a) $\lim \sum \xi_{ij}$ *exists in* $\mathcal{P}^*$; (b) $\|\sum \xi_{ij,y_{ij}}\|_{W^m(S_i)^*}$ *is bounded.*

(4) (a) $\lim \sum \xi_{ij}$ *exists in* $\mathcal{P}^*$; (b) $\|\sum \xi_{ij,y_{ij}}\|^{y_0}_{W^m(S_i)^*}$ *is bounded.*

Proof. (1) is equivalent to (2), by the uniform boundedness principle. Let $\mathcal{C}^m(L, y_0)$ denote $\{F \in \mathcal{C}^m(L) :\; T^m_{y_0} F = 0\}$, and consider also

(5) (a) $\lim \sum \xi_{ij}$ exists in $\mathcal{P}^*$; (b) $\|\sum \xi_{ij,y_{ij}}\|^{y_0}_{\mathcal{C}^m(L,y_0)^*}$ is bounded.

Write $\xi_i = \sum_{j=1}^k \xi_{ij,y_{ij}}$, $i = 1, 2, \dots$. It is easy to see that (5) implies $\{\xi_i(F)\}$ converges to 0 for all $F \in \mathcal{C}^m(L, y_0)$, and that $\{\xi_i\}$ converges on elements of $\mathcal{C}^m(L)$ if and only if $\lim \sum \xi_{ij}$ exists in $\mathcal{P}^*$ and $\{\xi_i\}$ converges on elements of $\mathcal{C}^m(L, y_0)$. Therefore, (5) implies (1). By (2.1), (2)(b) is equivalent to (3)(b) and (2)(b) implies (4)(b). It is clear that (4)(b) implies (5)(b). □

3. Glaeser operation and Fefferman's Glaeser refinement

Let E denote a closed subset of $\mathbb{R}^n$. There is a one-to-one correspondence between linear subbundles I of $E\times\mathcal{P}$ and linear subbundles T of $E\times\mathcal{P}^*$ given by $T=I^\perp$ (i.e., by $T(y)=I(y)^\perp$, the orthogonal complement of $I(y)$ in the dual to $\mathcal{P}$, for all $y\in E$), so that $T^\perp=I^{\perp\perp}=I$.

Let H denote an affine subbundle of $E\times\mathcal{P}$. (By convention, we allow the empty subset of $\mathcal{P}$ as an affine subspace.) If $H(y)\neq\emptyset$, for all $y\in E$, then there is a unique linear subbundle $I=I_H$ of $E\times\mathcal{P}$ such that, for any $y\in E$ and $P\in H(y)$, $H(y)=I(y)+P$. Then H induces a function $f=f_H: T=I^\perp\to\mathbb{R}$, such that f is linear on the fibres of T and, for all $y\in E$ and $P\in H(y)$,

$$f(y)(\xi)=\xi(P)\ ,\quad \xi\in T(y)\ .$$

($f(y)$ denotes the restriction of f to the fibre $T(y)$.)

Conversely, given a linear subbundle T of $E\times\mathcal{P}^*$ and a function $f:T\to\mathbb{R}$ linear on the fibres, there is a uniquely determined affine subbundle H of $E\times\mathcal{P}$ such that $H(y)\neq\emptyset$ (for all $y\in E$), $T=I_H^\perp$ and $f=f_H$.

Remark 3.1. If $y\in E$ and $P\in\mathcal{P}$, then $P\in H(y)$ if and only if $\xi(P)=f(y)(\xi)$ for all $\xi\in T(y)$.

Fix a positive integer k. Let H denote an affine subbundle of $E\times\mathcal{P}$. Given $y\in E$, Fefferman defines an affine subspace $\tilde{H}(y)$ of $H(y)$ that he calls the "Glaeser refinement" ((5) in [F, Intro.]). We will write $\tilde{H}(y)=G(H)(y)$. Then $G(H)\subset H$ is an affine subbundle of $E\times\mathcal{P}$.

Theorem 3.2. *Fix a positive integer k. Let T denote a linear subbundle of $E\times\mathcal{P}^*$ and let $f:T\to\mathbb{R}$ denote a function that is linear on the fibres of T. Let $I=T^\perp\subset E\times\mathcal{P}$, and let H denote the affine subbundle of $E\times\mathcal{P}^*$ such that $I=I_H$ and $f=f_H$. Let $y_0\in E$. Then*:

(1) $g(T)(y_0)=G(I)(y_0)^\perp$.

(2) *The Glaeser extension $g(f)(y_0): g(T)(y_0)\to\mathbb{R}$ exists if and only if $G(H)(y_0)\neq\emptyset$.*

(3) *If $P_0\in G(H)(y_0)$, then $G(H)(y_0)=G(I)(y_0)+P_0$.*

Proof. (1)(a) $g(T)(y_0)\subset G(I)(y_0)^\perp$: Let $\xi\in g(T)(y_0)$ and let $P_0\in G(I)(y_0)\subset I(y_0)$. It is enough to show that $\xi(P_0)=0$ when $\xi=\lim_{i\to\infty}\sum_{j=1}^k\xi_{ij}$, where $\xi_{ij}\in T(y_{ij})$ ($i=1,2,\ldots$ and $j=1,\ldots,k$) and $S_i=\{y_{i1},\ldots,y_{ik}\}$ satisfy the conditions of Definition 1.3. We will show there is a constant c such that $|\xi(P_0)|\le c\varepsilon$, for any $\varepsilon>0$.

Let $B(y_0,\delta)$ denote the ball with centre y_0 and radius δ. Take $\delta=\delta(\varepsilon)$ satisfying the conditions in Fefferman's definition of $G(I)(y_0)$ [F, Intro., (5)].

There exists $i(\delta)$ such that, if $i \geq i(\delta)$, then $S_i \subset B(y_0, \delta)$, so that there exists $P_{ij} \in I(y_{ij})$, $j = 1, \dots, k$, as in Fefferman's definition. In particular, $\xi_{ij}(P_{ij}) = 0$, $j = 1, \dots, k$, $i \geq i(\delta)$.

Consider $(P_0) - P_i \in W^m(S_i)$, $i \geq i(\delta)$, where (P_0) denotes $(P_0, \dots, P_0)$ and $P_i = (P_{i1}, \dots, P_{ik})$. Then $\|(P_0) - P_i\|^{y_0}_{W^m(S_i)} \leq \varepsilon$, by (2.2). Now,

$$\xi(P_0) \;=\; \lim_{i\to\infty} \sum_{j=1}^{k} \xi_{ij}(P_0) \;=\; \lim_{i\to\infty} \sum_{j=1}^{k} \xi_{ij}(P_0 - P_{ij}) \;.$$

For all $i \geq i(\delta)$,

$$\Big| \sum_{j=1}^{k} \xi_{ij}(P_0 - P_{ij}) \Big| \;\leq\; \Big\| \sum_{j=1}^{k} \xi_{ij,y_{ij}} \Big\|^{y_0}_{W^m(S_i)^*} \cdot \Big\| (P_0) - P_i \Big\|^{y_0}_{W^m(S_i)} \;\leq\; c\varepsilon \;,$$

where c is independent of ε, by Lemma 2.1.

(1)(b) $g(T)(y_0)^\perp \subset G(I)(y_0)$: Let $P_0 \in g(T)(y_0)^\perp \subset I(y_0)$. Suppose that $P_0 \notin G(I)(y_0)$. Then (according to [F, Intro., (5)]) there exists $\varepsilon > 0$ such that, for all $\delta > 0$, there exists $S_\delta = \{y_{\delta 1}, \dots, y_{\delta k}\} \subset B(y_0, \delta) \cap E$ such that, for all $P_{\delta j} \in I(y_{\delta j})$, $j = 1, \dots, k$, it is not true that

$$|\partial^\alpha (P_{\delta i} - P_{\delta j})(y_{\delta j})| \;\leq\; \varepsilon |y_{\delta i} - y_{\delta j}|^{m-|\alpha|} \;,$$

for $|\alpha| \leq m$, $0 \leq i, j \leq k$ (where $y_{\delta 0} = y_0$ and $P_{\delta 0} = P_0$).

Let $\delta > 0$. There exists $l = l(\delta)$, $1 \leq l \leq k$, such that $y_0 \neq y_{\delta j}$ for precisely l of the elements $y_{\delta 1}, \dots, y_{\delta k}$ (which we can take to be $y_{\delta 1}, \dots, y_{\delta l}$. Let $R_\delta = \{y_{\delta 1}, \dots, y_{\delta l}\}$. Let $I(R_\delta)$ denote the linear subspace of $W^m(R_\delta)$ consisting of all $P = (P_1, \dots, P_l)$ such that $P_j \in I(y_{\delta j})$, $j = 1, \dots, l$. Take $P'_\delta = (P_{\delta 1}, \dots, P_{\delta l}) \in I(R_\delta)$ closest to $(P_0)' = (P_0, \dots, P_0)$ (l times) in the norm $\|\cdot\|^{y_0}_{W^m(R_\delta)}$.

There exist $\xi_{\delta j} \in T(y_{\delta j}) = I(y_{\delta j})^\perp$, $j = 1, \dots, l$, such that, if $\xi_\delta := \sum_{j=1}^{l} \xi_{\delta j, y_{\delta j}} \in W^m(R_\delta)^*$, then $\|\xi_\delta\|^{y_0}_{W^m(R_\delta)^*} = 1$ and

$$\big|\xi_\delta((P_0)')\big| \;=\; \big\|[(P_0)']\big\|^{y_0}_{W^m(R_\delta)/I(R_\delta)} \;=\; \big\|(P_0)' - P'_\delta\big\|^{y_0}_{W^m(R_\delta)} \;,$$

where $[(P_0)']$ denotes the class of $(P_0)'$ in the quotient space $W^m(R_\delta)/I(R_\delta)$, and the norm in the middle is the quotient norm from $\|\cdot\|^{y_0}_{W^m(R_\delta)}$.

Let $P_\delta = (P_{\delta 1}, \dots, P_{\delta k})$, where $P_{\delta j} = P_0$ if $j > l$, and let $(P_0) = (P_0, \dots, P_0)$ (k times). Then

$$\big\|(P_0)' - P'_\delta\big\|^{y_0}_{W^m(R_\delta)} \;=\; \big\|(P_0) - P_\delta\big\|^{y_0}_{W^m(S_\delta)} \;>\; \varepsilon. \tag{3.1}$$

There exists $\delta = \delta_i$, $i = 1, 2, \dots$, such that $\lim_{i\to\infty} \delta_i = 0$ and $\{\xi_{\delta_i}\}$ converges in $\mathcal{P}^*$. By passing to a subsequence if necessary, we can assume there exists l,

$1 \le l \le k$, such that $l(\delta_i) = l$, for all i. Now $\xi := \lim_{i\to\infty} \xi_{\delta_i} \in g(T)(y_0)$, by the definition of $g(T)(y_0)$ (and Lemma 2.1). Since $P_0 \in g(T)(y_0)^\perp$, then

$$0 \;=\; |\xi(P_0)| \;=\; \lim_{i\to\infty} \Big| \sum_{j=1}^{l} \xi_{\delta_i,j}(P_0) \Big| \;\ge\; \varepsilon \;;$$

a contradiction.

(2)(a) "If": Let $P_0 \in G(H)(y_0)$. Since $G(H)(y_0) \subset H(y_0)$, $f(y_0)(\xi) = \xi(P_0)$, for all $\xi \in T(y_0) = I(y_0)^\perp$. Now, $\xi \mapsto \xi(P_0)$ defines a linear function $g(T)(y_0) \to \mathbb{R}$. It is enough to show that, if $\xi = \lim_{i\to\infty} \sum_{j=1}^k \xi_{ij}$ as in (1)(a) above, then

$$\xi(P_0) \;=\; \lim_{i\to\infty} \sum_{j=1}^{k} f(y_{ij})(\xi_{ij}) \;.$$

The proof is essentially the same as in (1)(a). Let $\varepsilon > 0$. Take $\delta = \delta(\varepsilon)$ satisfying the conditions in the definition of $G(H)(y_0)$. There exists $i(\delta)$ as in (1)(a). So if $i \ge i(\delta)$, then there exists $P_{ij} \in H(y_{ij})$, $j = 1, \ldots, k$, as in Fefferman's definition; in particular, $\xi_{ij}(P_{ij}) = f(y_{ij})(\xi_{ij})$.

Then, for all $i \ge i(\delta)$, $\|(P_0) - P_i\|^{y_0}_{W^m(S_i)} \le \varepsilon$, so that

$$\Big| \Big(\sum_{j=1}^{k} \xi_{ij,y_{ij}} \Big) \left((P_0) - P_i \right) \Big| \;\le\; c\varepsilon \;,$$

where c is a constant, by Lemma 2.1. Therefore,

$$\xi(P_0) = \lim_{i\to\infty} \left(\sum_{j=1}^{k} \xi_{ij,y_{ij}} \right) ((P_0)),$$

and

$$\Big| \xi(P_0) - \lim_{i\to\infty} \sum_{j=1}^{k} f(y_{ij})(\xi_{ij}) \Big| \;\le\; c\varepsilon \;.$$

(2)(b) "Only if": Suppose there exists a Glaeser extension $g(f)(y_0)$. We first note that there exists $P_0 \in H(y_0)$ such that $\xi(P_0) = g(f)(y_0)(\xi)$, for all $\xi \in g(T)(y_0) = G(I)^\perp(y_0)$: Extend $g(f)(y_0)$ to a linear function λ on $\mathcal{P}^*$, and choose $P_0 \in \mathcal{P}$ such that $\lambda(\xi) = \xi(P_0)$ for all $\xi \in \mathcal{P}^*$. Since $\lambda | T(y_0) = f(y_0)$, we have $\xi(P_0) = f(y_0)(\xi)$, $\xi \in I(y_0)^\perp$. Therefore, $P_0 \in H(y_0)$ (by Remark 3.1).

Suppose that $P_0 \notin G(H)(y_0)$. Then there exists $\varepsilon > 0$, exactly as in (1)(b) above, except that here $P_{\delta j} \in H(y_{\delta j})$.

Let $\delta > 0$. Take $l = l(\delta)$ and R_δ as in (1)(b). Let $H(R_\delta)$ denote the affine subspace of $W^m(R_\delta)$ consisting of all $P = (P_1, \ldots, P_l)$ such that $P_j \in H(y_{\delta j})$,

$j = 1, \ldots, l$. We follow (1)(b): Take $P'_\delta \in H(R_\delta)$ closest to $(P_0)'$ in the norm $\|\cdot\|^{y_0}_{W^m(R_\delta)}$. There exist $\xi_{\delta j} \in T(y_{\delta j}) = I(y_{\delta j})^\perp$ such that $\|\xi_\delta\|^{y_0}_{W^m(R_\delta)^*} = 1$ and

$$\left|\xi_\delta((P_0)' - P'_\delta)\right| \;=\; \left\|[(P_0)' - P'_\delta]\right\|^{y_0}_{W^m(R_\delta)/I(R_\delta)} \,.$$

Let P_δ and (P_0) be as in (1)(b). Then (3.1) holds.

Choose δ_i as before. Then $\xi = \lim_{i\to\infty} \xi_{\delta_i} \in g(T)(y_0)$, and $\lim_{i\to\infty} \xi_{\delta_i}(P'_{\delta_i}) = g(f)(y_0)(\xi)$, by the definition of $g(f)(y_0)$. Therefore,

$$\varepsilon \;\le\; \lim_{i\to\infty} \left\|(P_0)' - P'_{\delta_i}\right\|^{y_0}_{W^m(R_\delta)^*} \;=\; \left|\lim_{i\to\infty} \xi_{\delta_i}((P_0)' - P'_{\delta_i})\right| \;=\; 0\;;$$

a contradiction.

(3) Let $P_0 \in G(H)(y_0)$. Then $G(H)(y_0) = G(I)(y_0) + P_0$ if and only if the following assertion holds: Let $P \in \mathcal{P}$. Then $P \in G(H)(y_0)$ if and only if $P - P_0 \in G(I)(y_0)$. This assertion follows from the definitions of $G(H)$ and $G(I)$. □

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