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Whitney's extension problem for C^m

By Charles Fefferman*

Dedicated to Julie

Abstract

Let f be a real-valued function on a compact set in $\mathbb{R}^n$, and let m be a positive integer. We show how to decide whether f extends to a $\mathbb{C}^m$ function on $\mathbb{R}^n$.

Introduction

Continuing from [F2], we answer the following question ("Whitney's extension problem"; see [hW2]).

Question 1. Let φ be a real-valued function defined on a compact subset E of $\mathbb{R}^n$. How can we tell whether there exists $F \in C^m(\mathbb{R}^n)$ with $F = \varphi$ on E?

Here, $m \geq 1$ is given, and $C^m(\mathbb{R}^n)$ denotes the space of real-valued functions on $\mathbb{R}^n$ whose derivatives through order m are continuous and bounded on $\mathbb{R}^n$. We fix $m, n \geq 1$ throughout this paper. We write $\mathcal{R}_x$ for the ring of m-jets of functions at $x \in \mathbb{R}^n$, and we write $J_x(F)$ for the m-jet of the function F at x. As a vector space, $\mathcal{R}_x$ is identified with $\mathcal{P}$, the vector space of real m^{th} degree polynomials on $\mathbb{R}^n$; and $J_x(F)$ is identified with the Taylor polynomial $\sum_{|\beta| \leq m} \frac{1}{\beta!} (\partial^\beta F(x)) \cdot (y - x)^\beta$.

We answer also the following refinement of Question 1.

Question 2. Let φ and E be as in Question 1. Fix $\tilde{x} \in E$ and $P \in \mathcal{R}_{\tilde{x}}$. How can we tell whether there exists $F \in C^m(\mathbb{R}^n)$ with $F = \varphi$ on E and $J_{\tilde{x}}(F) = P$?

In particular, we ask which m-jets at $\tilde{x}$ can arise as the jet of a C^m function vanishing on E. This is equivalent to determining the "Zariski paratangent space" from Bierstone-Milman-Pawłucki [BMP1].

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A variant of Question 1 replaces $C^m(\mathbb{R}^n)$ by $C^{m,\omega}(\mathbb{R}^n)$, the space of C^m functions whose m^{th} derivatives have a given modulus of continuity ω. This variant is well-understood, thanks to Brudnyi and Shvartsman [B], [BS1,2,3,4], [S1,2,3], and my own papers [F1,2,4]. (See also Zobin [Z1,2] for a related problem.) In particular, [F2], [F4] broaden the issue, by answering the following.

Question 3. Suppose we are given a modulus of continuity ω, an arbitrary subset $E \subset \mathbb{R}^n$, and functions $\varphi : E \to \mathbb{R}$, $\sigma : E \to [0, \infty)$. How can we tell whether there exist $F \in C^{m,\omega}(\mathbb{R}^n)$ and $M < \infty$ such that $|F(x) - \varphi(x)| \leq M \cdot \sigma(x)$ for all $x \in E$?

Specializing to $\sigma = 0$, we recover the analogue of Whitney's problem for $C^{m,\omega}$. A further generalization will play a crucial role in our solution of Questions 1 and 2. We will need to understand the following.

Question 4. Let ω be a modulus of continuity, and let E be an arbitrary subset of $\mathbb{R}^n$. Suppose that for each $x \in E$ we are given an m-jet $f(x) \in \mathcal{R}_x$ and a convex subset $\sigma(x) \subset \mathcal{R}_x$, symmetric about the origin. How can we tell whether there exist $F \in C^{m,\omega}(\mathbb{R}^n)$ and $M < \infty$ such that $J_x(F) - f(x) \in M \cdot \sigma(x)$ for all $x \in E$?

If the convex sets $\sigma(x)$ satisfy a condition which we call "Whitney convexity," then we can give a complete answer to Question 4, analogous to our earlier work [F2,4] on Question 3. This will be one of the main steps in our proof. Here, we announce our result on Question 4, and use it to answer Questions 1 and 2. A detailed proof of our result on Question 4 appears in [F3].

We discuss briefly the previous work on Whitney's problem. The history of this problem goes back to three papers of Whitney [hW1,2,3] in 1934, giving the classical Whitney extension theorem, and solving Question 1 in one dimension (i.e., for $n = 1$). G. Glaeser [G] solved Whitney's problem for $C^1(\mathbb{R}^n)$ using a geometrical object called the "iterated paratangent space." Glaeser's paper influenced all the later work on Whitney's problem.

Afterwards came the work of Brudnyi and Shvartsman mentioned above. They conjectured a solution to the analogue of Question 1 for $C^{m,\omega}(\mathbb{R}^n)$, and proved their conjecture in the case $m = 1$. Their work and that of N. Zobin contain numerous additional results and conjectures related to Question 1.

The next progress on Question 1 was the work of Bierstone-Milman-Pawłucki [BMP1]. They found an analogue of the iterated paratangent space relevant to $C^m(\mathbb{R}^n)$. They conjectured a geometrical solution to Questions 1 and 2 based on their construction, and they found supporting evidence for their conjecture. (A version of their conjecture holds for subanalytic sets E.) Our results on Questions 1 and 2 are equivalent to the main conjectures in [BMP1], with the paratangent space there replaced by a natural variant. This equivalence, and other related results, are proven in Bierstone-Milman-Pawłucki

[BMP2]. Regarding the conjectures of [BMP1] in their original form, we refer the reader to a forthcoming paper by Bierstone, Fefferman, Milman, and Pawłucki.

Our solution to Questions 1 and 2 is based on the idea of associating to each point $y \in E$ an affine subspace $H(y) \subset \mathcal{P}$, with the crucial property:

(1) If $F \in C^m(\mathbb{R}^n)$ and $F = \varphi$ on E, then $J_y(F) \in H(y)$ for all $y \in E$.

Here, we make the convention that the empty set is allowed as an affine subspace of $\mathcal{P}$. Clearly, if $H(y)$ is empty for some $y \in E$, then (1) shows that φ cannot be extended to a C^m function F.

If (1) holds for an affine subspace $H(y) \subseteq \mathcal{P}$, then we call $H(y)$ a "holding space" for φ.

Answering Questions 1 and 2 amounts to computing the smallest possible holding space for φ. To carry this out, we will start with a trivial holding space $H_0(y)$. We will then produce a sequence of affine subspaces:

(2) $H_0(y) \supseteq H_1(y) \supseteq H_2(y) \supseteq \cdots$, for all $y \in E$, with each $H_\ell(y)$ being a holding space for φ. Each H_ℓ arises from the previous space $H_{\ell-1}$ by an explicit construction that we call the "Glaeser refinement", to be explained below. At stage $L = 2\dim\mathcal{P} + 1$, the process stabilizes; we have

$$H_\ell(y) = H_L(y) \text{ for all } \ell \geq L\,. \tag{3}$$

The space $H_L(y)$ will turn out to be the smallest possible holding space for φ.

To start the above process, we just take

$$H_0(y) = \{P \in \mathcal{P} : P(y) = \varphi(y)\} \text{ for all } y \in E\,. \tag{4}$$

To define the Glaeser refinement, suppose that for each $y \in E$ we are given an affine subspace $H(y) \subseteq \mathcal{P}$. We fix a large integer constant $k^\#$ depending only on m and n. We write $B(y, \delta)$ for the ball in $\mathbb{R}^n$ with center y and radius δ. For each $y \in E$, we will define a new affine subspace $\widetilde{H}(y) \subseteq \mathcal{P}$.

Given $y_0 \in E$ and $P_0 \in \mathcal{P}$, we say that $P_0 \in \widetilde{H}(y_0)$ if and only if the following condition holds:

(5) Given $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $y_1, \ldots, y_{k^\#} \in E \cap B(y_0, \delta)$, there exist $P_1, \ldots, P_{k^\#} \in \mathcal{P}$, with $P_j \in H(y_j)$ for $j = 0, 1, \ldots, k^\#$ and $|\partial^\alpha(P_i - P_j)(y_j)| \leq \varepsilon |y_i - y_j|^{m-|\alpha|}$ for $|\alpha| \leq m$, $0 \leq i, j \leq k^\#$.

(Here and throughout, we adopt the convention that $|y_i - y_j|^{m-|\alpha|} = 0$ in the degenerate case $y_i = y_j$, $m = |\alpha|$.)

Evidently, $\widetilde{H}(y)$ is an affine subspace of $H(y)$ for each $y \in E$. We call $\widetilde{H}(y)$ the "Glaeser refinement" of $H(y)$.

Note that if $H(y)$ is a holding space for all $y \in E$, then so is $\widetilde{H}(y)$. This follows trivially from (5) and Taylor's theorem.

Thus, we have produced the spaces $H_0, H_1, H_2, \ldots$ in (2), by starting with (4) and repeatedly passing to the Glaser refinement (5). The crucial stabilization property (3) follows from an ingenious, simple lemma in [BMP1], which in turn was adapted from an ingenious, simple lemma in [G]. (We give a proof in Section 2.) In view of (3), the holding space $H_L(y)$ is its own Glaeser refinement. We call $H_L(y)$ the "stable holding space" for φ, and we denote it by $H_*(y)$.

Note that, if $H_\ell(y)$ is nonempty, then it has the form $f_\ell(y) + I_\ell(y)$, where $f_\ell(y) \in \mathcal{R}_y$ and $I_\ell(y)$ is an ideal in $\mathcal{R}_y$. Moreover, $I_\ell(y)$ is determined by y, E and ℓ, independently of φ. This follows by an easy induction on ℓ, using (4) and (5). (Again, see Section 2.)

In principle, the stable holding space $H_*(y)$ is computable from the function φ and the set E.

Our answer to Questions 1 and 2 is as follows.

THEOREM 1. *Let φ be a real-valued function defined on a compact subset $E \subset \mathbb{R}^n$. For $y \in E$, let $H_*(y)$ be the stable holding space for φ. Then*

(A) *φ extends to a C^m function on $\mathbb{R}^n$ if and only if $H_*(y)$ is nonempty for all $y \in E$. Moreover, suppose φ extends to a C^m function on $\mathbb{R}^n$. Then*

(B) *Given $y_0 \in E$ and $P_0 \in \mathcal{P}$, we have $P_0 \in H_*(y_0)$ if and only if there exists $F \in C^m(\mathbb{R}^n)$ with $F = \varphi$ on E and $J_{y_0}(F) = P_0$.*

It is easy to deduce Theorem 1 from the following result.

THEOREM 2. *Let $E \subset \mathbb{R}^n$ be compact. Suppose that, for each $y \in E$, we are given an affine subspace $H(y) \subseteq \mathcal{R}_y$ having the form $H(y) = f(y) + I(y)$, where $f(y) \in \mathcal{R}_y$ and $I(y)$ is an ideal in $\mathcal{R}_y$. Assume that $H(y)$ is its own Glaeser refinement, for each $y \in E$. Then there exists $F \in C^m(\mathbb{R}^n)$, with $J_y(F) \in H(y)$ for all $y \in E$.*

In fact, part (A) of Theorem 1 is immediate from Theorem 2 and the observation that φ cannot extend to a C^m function on $\mathbb{R}^n$ if $H_*(y)$ is empty for any y. (Note that $J_y(F) \in H_*(y)$ implies $J_y(F) \in H_0(y)$ by (2); hence $F(y) = \varphi(y)$ by (4).) Similarly, part (B) of Theorem 1 is immediate from the following corollary of Theorem 2.

COROLLARY. *Let $E, H(y)$ be as in Theorem 2. Given any $y_0 \in E$ and $P_0 \in H(y_0)$, there exists $F \in C^m(\mathbb{R}^n)$ with $J_y(F) \in H(y)$ for all $y \in E$, and $J_{y_0}(F) = P_0$.*

To prove the corollary, we define $\hat{H}(y_0) = \{P_0\}$ and $\hat{H}(y) = H(y)$ for $y \in E \setminus \{y_0\}$. The hypotheses of Theorem 2 hold for $\hat{H}$. The corollary follows at once by application of Theorem 2 to $\hat{H}$.

To prove Theorem 2, we formulate a more precise, quantitative result, in which we control the C^m-norm of F.

THEOREM 3. *There exist constants $k^{\#}$, C, depending only on m and n, for which the following holds*:

Let $E \subset \mathbb{R}^n$ be compact. Suppose that for each $x \in E$ we are given an m-jet $f(x) \in \mathcal{R}_x$ and an ideal $I(x)$ in $\mathcal{R}_x$. Assume that the following conditions are satisfied:

(I) *Given $x_0 \in E$, $P_0 \in f(x_0) + I(x_0)$, and $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_1, \dots, x_{k^{\#}} \in E \cap B(x_0, \delta)$, there exist polynomials $P_1, \dots, P_{k^{\#}} \in \mathcal{P}$, with $P_i \in f(x_i) + I(x_i)$ for $0 \le i \le k^{\#}$, and $|\partial^\alpha (P_i - P_j)(x_j)| \le \varepsilon |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le k^{\#}$.*

(II) *Given $x_1, \dots, x_{k^{\#}} \in E$, there exist polynomials $P_1, \dots, P_{k^{\#}} \in \mathcal{P}$, with $P_i \in f(x_i) + I(x_i)$ for $1 \le i \le k^{\#}$; $|\partial^\alpha P_i(x_i)| \le 1$ for $|\alpha| \le m$, $1 \le i \le k^{\#}$; and $|\partial^\alpha (P_i - P_j)(x_j)| \le |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $1 \le i, j \le k^{\#}$.*

Then there exists $F \in C^m(\mathbb{R}^n)$, with C^m-norm at most C, and with $J_x(F) \in f(x) + I(x)$ for all $x \in E$.

Theorem 3 easily implies Theorem 2 via the following lemma, proven in Section 2.

FINITENESS LEMMA. *Let E, $f(x)$, $I(x)$ be as in the hypotheses of Theorem 2. Then there exists a finite constant A such that the following holds*:

Given $x_1, \dots, x_{k^{\#}} \in E$, there exist polynomials $P_1, \dots, P_{k^{\#}} \in \mathcal{P}$, with $P_i \in f(x_i) + I(x_i)$ for $1 \le i \le k^{\#}$; $|\partial^\alpha P_i(x_i)| \le A$ for $|\alpha| \le m$, $1 \le i \le k^{\#}$; $|\partial^\alpha (P_i - P_j)(x_j)| \le A\, |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $1 \le i, j \le k^{\#}$.

The finiteness lemma is proven by contradiction, and gives no control over the constant A. Theorem 2 follows by applying Theorem 3, with $f(x)/A$ in place of $f(x)$, where A is as in the finiteness lemma. I know of no way to prove Theorem 2 without going through Theorem 3. Thus, the heart of the matter is Theorem 3. We set up a bit more notation, and discuss some ideas from the proof of Theorem 3.

Recall that $\mathcal{R}_x$ is the ring of m-jets of functions at x. Let $\bar{\mathcal{R}}_x$ be the ring of $(m-1)$-jets of functions at x, and let $\pi_x : \mathcal{R}_x \to \bar{\mathcal{R}}_x$ be the natural projection. For E, $f(x)$, $I(x)$ as in Theorem 3, we define the *signature* of a point $x \in E$ to be

$$\text{sig}\,(x) = (\dim I(x), \dim\left[\ker \pi_x \cap I(x)\right])\ , \tag{6}$$

where $I(x)$ and $\ker \pi_x \cap I(x)$ are regarded as vector spaces. For given integers k_1, k_2, the set

$$E(k_1, k_2) = \{x \in E : \text{sig}(x) = (k_1, k_2)\} \tag{7}$$

is called a *stratum.* Note that $0 \leq k_2 \leq k_1 \leq \dim\mathcal{P}$ for a nonempty stratum. Among all nonempty $E(k_1, k_2)$ we first take k_1 as small as possible, and then take k_2 as large as possible for the given k_1. With k_1, k_2 picked in this manner, the stratum $E(k_1, k_2)$ is called the "lowest stratum" and denoted by E_1. Thus, there is a lowest stratum whenever E is nonempty. Finally, the "number of strata" in E is simply the number of distinct (k_1, k_2) for which $E(k_1, k_2)$ is nonempty.

Our proof of Theorem 3 proceeds by induction on the number of strata. If the number of strata is zero, then E is empty, and Theorem 3 holds trivially, with $k^{\#} = 1$, $C = 1$, and $F \equiv 0$. For the induction step, let $\wedge \geq 1$ be a given integer, and suppose Theorem 3 holds whenever the number of strata is less than $\wedge$. We show that Theorem 3 holds also when the number of strata is equal to $\wedge$.

Thus, let $E, f(x), I(x)$ be as in the hypotheses of Theorem 3, with the number of strata equal to $\wedge$. Let E_1 be the lowest stratum. It is easy to see that E_1 is compact (Lemma 2.3 below). We partition $\mathbb{R}^n \smallsetminus E_1$ into Whitney cubes $\{Q_\nu\}$. Thus, each Q_ν satisfies:

(8) Q_ν^* is disjoint from E_1, and

(9) distance $(Q_\nu^*, E_1) < C$ diameter (Q_ν^*) if diameter $(Q_\nu) < 1$,

where Q_ν^* is a (closed) cube having the same center and three times the diameter of Q_ν. We write δ_ν for the diameter of Q_ν, and we introduce a "Whitney partition of unity" $\{\theta_\nu\}$, with

$$\sum_\nu \theta_\nu = 1 \text{ on } \mathbb{R}^n \smallsetminus E_1, \tag{10}$$

$$\text{supp } \theta_\nu \subset Q_\nu^*, \text{ and} \tag{11}$$

$$|\partial^\alpha \theta_\nu| \leq C\, \delta_\nu^{-|\alpha|} \text{ for } |\alpha| \leq m\ . \tag{12}$$

Our strategy is as follows.

Step 1. Find a function $\widetilde{F} \in C^m(\mathbb{R}^n)$, with

$$J_x(\widetilde{F}) \in f(x) + I(x) \text{ for all } x \in E_1\ . \tag{13}$$

Step 2. For each ν, apply the induction hypothesis (a rescaled form of Theorem 3 for fewer than $\wedge$ strata) with $E \cap Q_\nu^*$, $f(x) - J_x(\widetilde{F})$, $I(x)$ in place of E, $f(x), I(x)$. Note that $E \cap Q^*_\nu$ has fewer than $\wedge$ strata, thanks to (8). Thus, for each ν, we obtain a function $F_\nu \in C^m(\mathbb{R}^n)$, with

$$J_x(F_\nu) \in [f(x) - J_x(\widetilde{F})] + I(x) \text{ for all } x \in E \cap Q^*_\nu, \tag{14}$$

and with good control over the derivatives of F_ν up to order m.

Step 3. We define

$$F = \widetilde{F} + \sum_\nu \theta_\nu F_\nu \text{ on } \mathbb{R}^n .$$

Using (8)–(14) and our control on the derivatives of the F_ν, we conclude that $F \in C^m(\mathbb{R}^n)$, and that $J_x(F) \in f(x) + I(x)$ for all $x \in E$. We will also control the C^m-norm of F. This shows that Theorem 3 holds for $E, f(x)$, $I(x)$, completing the induction on $\wedge$ and establishing Theorem 3.

To obtain the desired control on the derivatives of the F_ν, we have to strengthen (13). For $x \in E$, $k^\# \geq 1$, $A > 0$, we will introduce a convex set $\Gamma_f(x, k^\#, A) \subset f(x) + I(x)$. In place of (13), we will need to make sure that $\widetilde{F}$ satisfies

$$J_x(\widetilde{F}) \in \Gamma_f(x, k^\#, A) \subset f(x) + I(x) \text{ for all } x \in E_1. \tag{15}$$

Once $\widetilde{F}$ satisfies (15), we can gain enough control over the derivatives of the F_ν to make our strategy work. However, to achieve (15), we must be able to produce a C^m function whose m-jet belongs to a given convex set at each point of E. This is how Question 4 above enters our solution of Whitney's extension problem.

As in [F2], the constant $k^\#$ in Theorems 1,2,3 can be bounded explicitly in terms of m and n, but new ideas will be needed to obtain the best possible $k^\#$.

It would be natural to try to extend our results to answer the following generalization of Questions 1 and 2.

Question 5. Let $E \subset \mathbb{R}^n$ be a compact set. Suppose that for each $x \in E$ we are given an m-jet $f(x) \in \mathcal{R}_x$ and a Whitney convex set $\sigma(x) \subset \mathcal{R}_x$. Assume there is a uniform Whitney constant for all the $\sigma(x)$. (See Section 1.) How can we tell whether there exist a function $F \in C^m(\mathbb{R}^n)$ and a finite constant M such that $J_x(F) \in f(x) + M\sigma(x)$ for all $x \in E$?

Let $C^m(E)$ denote the space of functions on E that extend to C^m functions on $\mathbb{R}^n$. In a forthcoming paper, we will show that there exists a bounded linear operator $T : C^m(E) \to C^m(\mathbb{R}^n)$ such that $T\varphi\big|_E = \varphi$ for $\varphi \in C^m(E)$. (See [BS1,3], [F1], [G], [hW2].)

It is a pleasure to acknowledge the great influence of Bierstone-Milman-Pawłucki [BMP1] on this paper, and to thank Bierstone and Milman for valuable discussions. It is a pleasure also to thank Gerree Pecht for TEX-ing my manuscript, expertly as always.

1. Whitney convexity

Recall that $\mathcal{R}_x$ denotes the ring of m-jets of functions at x. Suppose Ω is a subset of $\mathcal{R}_x$ and A is a positive real number. We will say that Ω is "Whitney convex (at x) with Whitney constant A" if the following conditions are satisfied:

(1) Ω is closed, convex, and symmetric about the origin. (That is, $P \in \Omega$ if and only if $-P \in \Omega$.)

(2) Let $P \in \Omega$, $Q \in \mathcal{R}_x$ and $\delta \in (0,1]$ be given.

Assume that

$$|\partial^\alpha P(x)| \le \delta^{m-|\alpha|} \text{ and } |\partial^\alpha Q(x)| \le \delta^{-|\alpha|}\,, \text{ for } |\alpha| \le m\,.$$

Then $P \cdot Q \in A\Omega$, where $P \cdot Q$ denotes the product of P and Q in $\mathcal{R}_x$.

The motivation for this definition goes back to the proof of the classical Whitney extension theorem. There, one studies sums of the form $F = \sum_\nu P_\nu \cdot \theta_\nu$ on $\mathbb{R}^n$, where the θ_ν form a partition of unity. In a small neighborhood of a given point x, there is a lengthscale $\delta \le 1$ for which the θ_ν satisfy $|\partial^\alpha \theta_\nu| \le \delta^{-|\alpha|}$ if $x \in \operatorname{supp} \theta_\nu$. If $\delta \ll 1$ then the derivatives of the θ_ν are large, yet F has bounded m^{th} derivatives provided we have $|\partial^\alpha(P_\mu - P_\nu)| \le \delta^{m-|\alpha|}$ on $\operatorname{supp}\theta_\mu \cap \operatorname{supp}\theta_\nu$. Thus, the estimates in (2) are natural in connection with Whitney's extension problem.

We will be studying $C^{m,\omega}(\mathbb{R}^n)$ for suitable ω. A function $\omega : [0,1] \to [0,\infty)$ is called a "regular modulus of continuity" if it satisfies the following conditions:

(3) $\omega(0) = \lim_{t\to 0+} \omega(t) = 0$ and $\omega(1) = 1$.

(4) $\omega(t)$ is increasing on $[0,1]$.

(5) $\omega(t)/t$ is decreasing on $(0,1]$.

In (4) and (5), we do not demand that ω be strictly increasing, or that $\omega(t)/t$ be strictly decreasing.

If ω is a regular modulus of continuity, then $C^{m,\omega}(\mathbb{R}^n)$ denotes the space of all C^m functions F on $\mathbb{R}^n$ for which the norm
$\| F \|_{C^{m,\omega}(\mathbb{R}^n)} =$

$$\max\left\{ \max_{|\beta|\le m} \sup_{x\in\mathbb{R}^n} |\partial^\beta F(x)|\,,\ \max_{|\beta|=m} \sup_{\substack{x,x'\in\mathbb{R}^n \\ 0<|x-x'|\le 1}} \frac{|\partial^\beta F(x) - \partial^\beta F(x')|}{\omega(|x-x'|)} \right\}$$

is finite.

By adapting the proof of the sharp Whitney theorem from [F2,4], we obtain the following result.

THE GENERALIZED SHARP WHITNEY THEOREM. *There exists a constant $k^{\#}_{\mathrm{GSW}}$, depending only on m and n, for which the following holds: Let ω be a regular modulus of continuity, and let $E \subset \mathbb{R}^n$ be an arbitrary subset. Suppose that for each $x \in E$ we are given an m-jet $f(x) \in \mathcal{R}_x$ and a subset $\sigma(x) \subset \mathcal{R}_x$.*

Assume that each $\sigma(x)$ is Whitney convex (at x), with a Whitney constant A_0 independent of x. Assume also that, given any subset $S \subset E$ with cardinality at most $k^{\#}_{\mathrm{GSW}}$, there exists a map $x \mapsto P^x$ from S into $\mathcal{P}$, with

(a) $P^x \in f(x) + \sigma(x)$ *for all* $x \in S$;

(b) $|\partial^\alpha P^x(x)| \le 1$ *for all* $x \in S$, $|\alpha| \le m$; *and*

(c) $|\partial^\alpha (P^x - P^y)(y)| \le \omega(|x-y|) \cdot |x-y|^{m-|\alpha|}$ *for all* $x, y \in S$, $|x-y| \le 1$, $|\alpha| \le m$.

Then there exists $F \in C^{m,\omega}(\mathbb{R}^n)$, with $\| F \|_{C^{m,\omega}(\mathbb{R}^n)} \le A_1$, and $J_x(F) \in f(x) + A_1 \cdot \sigma(x)$ for all $x \in E$. Here, A_1 depends only on m, n and the Whitney constant A_0.

This result is our answer to Question 4 from the introduction. The proof of the generalized sharp Whitney theorem appears in [F3]. It would be interesting to gain some understanding of Whitney convex sets.

2. Some elementary verifications

In this section, we sketch the proofs of some elementary assertions from the introduction.

LEMMA 2.1. *Let $H_0(y) \supseteq H_1(y) \supseteq \cdots$ be as in the introduction. If a given $H_\ell(y)$ is nonempty, then it can be written as $H_\ell(y) = f_\ell(y) + I_\ell(y)$, where $I_\ell(y)$ is an ideal in $\mathcal{R}_y$. Moreover, $I_\ell(y)$ is determined by ℓ, y, E, independently of φ.*

Sketch of proof. We can take $f_\ell(y)$ to be any element of $H_\ell(y)$. The $I_\ell(y)$ are defined by the following induction.

(1) $I_0(y) = \{P \in \mathcal{P} : P(y) = 0\}$.

(2) $P_0 \in I_{\ell+1}(y_0)$ if and only if the following holds:

Given $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $y_1, \ldots, y_{k^\#} \in E \cap B(y_0, \delta)$, there exist $P_1, \ldots, P_{k^\#} \in \mathcal{P}$, with $P_j \in I_\ell(y_j)$ for $j = 0, \ldots, k^{\#}$; and

$$|\partial^\alpha (P_i - P_j)(y_j)| \le \varepsilon |y_i - y_j|^{m-|\alpha|} \text{ for } |\alpha| \le m, 0 \le i, j \le k^{\#} .$$

The only assertion in the lemma that requires any proof is that $I_\ell(y)$ is an ideal in $\mathcal{R}_y$. To check that assertion, we use induction on ℓ. The case $\ell = 0$ is obvious. For the induction step, fix $\ell \geq 0$, and suppose each $I_\ell(y)$ is an ideal in $\mathcal{R}_y (y \in E)$. Suppose $P_0 \in I_{\ell+1}(y_0)$ and $Q \in \mathcal{P}$. Let $\widetilde{P}_0$ be the product of P_0 and Q in $\mathcal{R}_{y_0}$. We must check that $\widetilde{P}_0$ belongs to $I_{\ell+1}(y_0)$. This follows from (2), by using $\widetilde{P}_1, \ldots, \widetilde{P}_{k^\#}$ there, with $\widetilde{P}_j$ defined as the product of P_j with Q in $\mathcal{R}_{y_j}$. □

For the next lemma, we adopt the convention that the empty set has dimension $-\infty$ as an affine space.

LEMMA 2.2 (after Lemma 3.3 in [BMP1]). *Let* $H_0(y) \supseteq H_1(y) \supseteq \cdots$ *be as in the introduction, and let* $k \geq 0$, $x \in E$ *be given. If* $\dim H_{2k+1}(x) \geq \dim \mathcal{P} - \mathrm{k}$, *then* $H_\ell(x) = H_{2k+1}(x)$ *for all* $\ell \geq 2k+1$.

Proof. We use induction on k. For $k = 0$, the lemma asserts that

(3) if $H_1(x) = \mathcal{P}$, then $H_\ell(x) = \mathcal{P}$ for all $\ell \geq 1$.

From the definition of the H_ℓ in the introduction, one sees that

(4) $$\dim \mathrm{H}_{\ell+1}(\mathrm{x}) \leq \liminf_{\mathrm{y} \to \mathrm{x}} \mathrm{H}_\ell(\mathrm{y}).$$

Hence, if $H_1(x) = \mathcal{P}$, then $H_0(y) = \mathcal{P}$ for all y in a neighborhood of x. Consequently, $H_\ell(y) = \mathcal{P}$ in a neighborhood of x, for all $\ell \geq 1$, proving (3).

For the induction step, fix $k \geq 0$, and assume the lemma holds for that k. We must show that

(5) if $\dim H_{2k+3}(x) \geq \dim \mathcal{P} - k - 1$, then $H_\ell(x) = H_{2k+3}(x)$ for all $\ell \geq 2k+3$.

If $\dim H_{2k+1}(x) \geq \dim \mathcal{P} - k$, then (5) holds, since we are assuming Lemma 2.2 for k. Hence, in proving (5), we may assume that $\dim H_{2k+1}(x) \leq \dim \mathcal{P} - k - 1$. Thus,

(6) $$\dim H_{2k+1}(x) = \dim H_{2k+2}(x) = \dim H_{2k+3}(x) = \dim \mathcal{P} - k - 1.$$

Note that

(7) $\dim H_{2k+1}(y) \geq \dim \mathcal{P} - k - 1$ for all y near enough to x since otherwise (4) (with $\ell = 2k+1$) would contradict (6).

We claim that also

(8) $H_{2k+2}(y) = H_{2k+1}(y)$ for all y near enough to x.

In fact, suppose (8) fails; i.e., suppose that

(9) $$\dim H_{2k+2}(y) < \dim H_{2k+1}(y) \text{ for } y \text{ arbitrarily near } x.$$

Then, since we are assuming Lemma 2.2 for k, we must have $\dim H_{2k+1}(y) < \dim \mathcal{P} - k$ for all y as in (9), and therefore

(10) $$\dim H_{2k+2}(y) \leq \dim H_{2k+1}(y) - 1 \leq \dim \mathcal{P} - k - 2$$

for y arbitrarily close to x. From (4) and (10), we get $\dim H_{2k+3}(x) \leq \dim \mathcal{P} - k - 2$, contradicting (6). Thus, (8) cannot fail.

From (8) we see easily that $H_\ell(y) = H_{2k+1}(y)$ for all $\ell \geq 2k+1$, and all $y \in E$ close enough to x. In particular, $H_\ell(x) = H_{2k+3}(x)$ for all $\ell \geq 2k+3$. This completes the inductive step, and proves Lemma 2.2. □

In Lemma 2.2, we set $k = \dim \mathcal{P}$. Thus, for $L = 2\dim\mathcal{P} + 1$, we have $H_L(x) = H_{L+1}(x) = H_{L+2}(x) = \ldots$, provided $H_L(x)$ is nonempty. Of course, the same conclusion holds trivially when $H_L(x)$ is empty. This proves the assertions in the introduction, concerning the stabilization of the H_ℓ.

Next, we sketch the proof of the Finiteness Lemma from the introduction. We proceed by contradiction.

If the Finiteness Lemma fails, then, for each $\nu = 1, 2, 3, \ldots$ we can find $x_1^{(\nu)}, \ldots, x_{k^\#}^{(\nu)} \in E$, and a positive constant $A^{(\nu)}$, such that

(11) $$A^{(\nu)} \to \infty \text{ as } \nu \to \infty\ ,$$

and, for each ν,

(12) There do not exist polynomials $P_1, \ldots, P_{k^\#} \in \mathcal{P}$, with

(a) $P_j \in f(x_j^{(\nu)}) + I(x_j^{(\nu)})$ for $j = 1, \ldots, k^\#$;

(b) $|\partial^\alpha P_j(x_j^{(\nu)})| \leq A^{(\nu)}$ for $j = 1, \ldots, k^\#$ and $|\alpha| \leq m$; and

(c) $|\partial^\alpha (P_i - P_j)(x_j^{(\nu)})| \leq A^{(\nu)} |x_i^{(\nu)} - x_j^{(\nu)}|^{m-|\alpha|}$ for $|\alpha| \leq m$, $0 \leq i, j \leq k^\#$.

Recall that E is compact. Hence, by passing to a subsequence, we may arrange that, in addition to (11), (12), we have

(13) $$x_j^{(\nu)} \to x_j^{(\infty)} \in E \text{ as } \nu \to \infty, \text{ for each } j = 1, \ldots, k^\#\ .$$

The points $x_1^{(\infty)}, \ldots, x_{k^\#}^{(\infty)}$ need not be distinct.

Let $z_1, \ldots, z_\mu$ be an enumeration of the distinct elements of the set $\{x_1^{(\infty)}, \ldots, x_{k^\#}^{(\infty)}\}$. For each $\lambda (1 \leq \lambda \leq \mu)$, let $S(\lambda)$ be the set of all j for which $x_j^{(\infty)} = z_\lambda$. Thus, if ν is large enough, we have the following:

(14) $x_j^{(\nu)}$ is close to z_λ for all $j \in S(\lambda)$; and

(15) $|x_j^{(\nu)} - x_{j'}^{(\nu)}| > c > 0$ whenever $j \in S(\lambda)$ and $j' \in S(\lambda')$ with $\lambda \neq \lambda'$.

(In (15), we may take $c = \frac{1}{2} \min_{\lambda \neq \lambda'} |z_\lambda - z_{\lambda'}| > 0$.) Here, and for the rest of the proof of the Finiteness Lemma, we write c, C, C', etc. to denote constants independent of ν.

We now apply the hypothesis that $H(y) = f(y) + I(y)$ $(y \in E)$ is its own Glaeser refinement. We fix λ. In the definition of the Glaeser refinement, we take $y_0 = z_\lambda$, $P_0 = f(z_\lambda)$ and $\varepsilon = 1$; and, for ν large enough, we set $y_j = x_j^{(\nu)}$ for $j \in S(\lambda)$, $y_j = z_\lambda$ for $j \notin S(\lambda)$ $(1 \leq j \leq k^\#)$. Since $H(\cdot)$ is its own Glaeser refinement, we conclude from (14) that we can find $P_j^{(\nu)} \in f(x_j^{(\nu)}) + I(x_j^{(\nu)})$ ($j \in S(\lambda)$, ν large enough), with $|\partial^\alpha P_j^{(\nu)}(x_j^{(\nu)})| \leq |\partial^\alpha P_0(x_j^{(\nu)})| + 1$ $(|\alpha| \leq m)$ and $|\partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)})| \leq |x_i^{(\nu)} - x_j^{(\nu)}|^{m-|\alpha|}$ for $i, j \in S(\lambda)$, $|\alpha| \leq m$.

We carry this out for each $\lambda = 1, \ldots, \mu$. Thus, for large enough ν, we obtain polynomials $P_1^{(\nu)}, \ldots, P_{k^\#}^{(\nu)}$, with the following properties:

(16) $P_j^{(\nu)} \in f(x_j^{(\nu)}) + I(x_j^{(\nu)})$ for $j = 1, \ldots, k^\#$;

(17) $|\partial^\alpha P_j^{(\nu)}(x_j^{(\nu)})| \leq C$ for $|\alpha| \leq m$, $j = 1, \ldots, k^\#$;

(18) $|\partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)})| \leq |x_i^{(\nu)} - x_j^{(\nu)}|^{m-|\alpha|}$ for $|\alpha| \leq m$, $i, j \in S(\lambda)$, $1 \leq \lambda \leq \mu$.

Moreover, (15) and (17) show that

$$|\partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)})| \leq C' |x_i^{(\nu)} - x_j^{(\nu)}|^{m-|\alpha|}$$

for $|\alpha| \leq m$, $i \in S(\lambda)$, $j \in S(\lambda')$, $\lambda \neq \lambda'$.

Together with (18), this implies

(19) $|\partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)})| \leq C'' |x_i^{(\nu)} - x_j^{(\nu)}|^{m-|\alpha|}$ for $|\alpha| \leq m$, $1 \leq i, j \leq k^\#$.

Now let ν be large enough that (16), (17), (19) apply, and also large enough that $A^{(\nu)} > \max(C, C'')$, with C, C'' as in (17) and (19).

Then (16), (17), (19) together contradict (12). This contradiction completes the proof of the finiteness lemma. □

LEMMA 2.3. *Let E, f, I be as in the hypotheses of Theorem 3. Then the lowest stratum E_1 is compact.*

Proof. We keep the notation of the introduction. Let $x_0 \in E$, and suppose $\dim I(x_0) = d$. Let $P_0^{(0)}, \ldots, P_0^{(d)}$ be the vertices of a nondegenerate affine d-simplex in $f(x_0) + I(x_0)$. If we perturb the $P_0^{(j)}$ slightly in $\mathcal{P}$, then we obtain the verticies of a nondegenerate affine d-simplex in $\mathcal{P}$. Moreover, hypothesis (I) of Theorem 3 shows that, for any $x_1 \in E$ close enough to x_0, we may find $P_1^{(0)}, \dots, P_1^{(d)} \in f(x_1) + I(x_1)$ with $P_1^{(j)}$ close to $P_0^{(j)}$ in $\mathcal{P}$. Therefore, for any $x_1 \in E$ close enough to x_0, the affine space $f(x_1) + I(x_1)$ contains a nondegenerate affine d-simplex; hence $\dim I(x_1) \geq d$. It follows that $\{x \in E : \dim I(x) < d\}$ is a closed set, for any integer d. In particular, the set $\widetilde{E}$ of all $x \in E$ with $\dim I(x)$ equal to $k_1 = \min_{y \in E} \dim I(y)$ is closed.

Another application of hypothesis (I) of Theorem 3 shows that $x \mapsto I(x)$ is a continuous map from $\widetilde{E}$ to the Grassmannian of k_1-planes in $\mathcal{P}$.

Now let $k_2 = \max\limits_{y \in \widetilde{E}} \dim\,(\ker \pi_y \cap I(y))$. Then by definition,

$$E_1 = \{x \in \widetilde{E} : \dim\,(\ker \pi_x \cap I(x)) = k_2\}\,.$$

We will show that E_1 is closed. Suppose $x_\nu \in E_1$ for $\nu = 1, 2, \dots,$ and suppose $x_\nu \to x$ in $\mathbb{R}^n$. Then $x \in \widetilde{E}$, and $I(x_\nu) \to I(x)$ in the Grassmannian of k_1-planes in $\mathcal{P}$. Passing to a subsequence, we may assume that $\ker \pi_{x_\nu} \cap I(x_\nu)$ tends to a limit J in the Grassmannian of k_2-planes in $\mathcal{P}$.

We then have $J \subset I(x)$ and $\pi_x|_J = 0$. Hence, $\dim\,(\ker \pi_x \cap I(x)) \geq k_2$. By definition of k_2, it follows that $\dim\,(\ker \pi_x \cap I(x)) = k_2$, i.e., $x \in E_1$. Thus, as claimed, E_1 is closed. Since E_1 is also a subset of the compact set E, the proof of the lemma is complete. □

3. Further elementary results

In this section we collect a few standard facts and elementary results that will be used later. We begin with two lemmas about "clusters". We write $\#(S)$ for the cardinality of a set S.

Lemma 3.1. *Let $S \subset \mathbb{R}^n$, with $2 \leq \#(S) \leq k^\#$. Then we may partition S into subsets $S_1, S_2, \dots, S_M$, with the following properties*:

(a) $\#(S_i) < \#(S)$ *for each* i.

(b) *If $x \in S_i$ and $y \in S_j$ with $i \neq j$, then $|x - y| > c \cdot \operatorname{diam}(S)$ with c depending only on $k^\#$.*

Lemma 3.2. *Let $S \subset \mathbb{R}^n$, with $\#(S) \leq k^\#$, and let $\delta > 0$ be given. Then we can partition S into subsets $S_1, \dots, S_M$, with*

(a) $\operatorname{diam}\,(S_i) \leq \delta$ *for each i, and*

(b) $\operatorname{dist}\,(S_i, S_j) \geq c \cdot \delta$ *for $i \neq j$, where c depends only on $k^\#$.*

To prove Lemma 3.2, we note that there are at most $\binom{k^\#}{2}$ distances $|x - y|$ $(x, y \in S, x \neq y)$; hence, at least one of the intervals $I_\ell = (2^{-\ell}\delta, 2^{1-\ell}\delta]$ $\left(\ell = 1, 2, \dots, \binom{k^\#}{2} + 1\right)$ contains none of the distances between points of S.

Fix such an I_ℓ. If $x, y, z \in S$ with $|x-y|$, $|y-z| \leq 2^{-\ell}\delta$, then since $|x-z| \notin I_\ell$, we have $|x-z| \leq 2^{-\ell}\delta$. Hence, the relation $|x-y| \leq 2^{-\ell}\delta$ $(x, y \in S)$ is an equivalence relation. Taking $S_1, \ldots, S_M$ to be the equivalence classes for this equivalence relation, we easily confirm (a) and (b). This proves Lemma 3.2. □

To prove Lemma 3.1, we just apply Lemma 3.2 with $\delta = \frac{1}{2}$ diam (S). Since diam $(S_i) \leq \frac{1}{2}$ diam (S) for each i, we must have $\#(S_i) < \#(S)$. This proves Lemma 3.1. □

Next, we prove a linear algebra perturbation lemma.

LEMMA 3.3. *Suppose we are given an r-dimensional affine subspace $H \subseteq \mathbb{R}^N$, and the vertices $v_0, \ldots, v_r$ of a nondegenerate affine r-simplex in H. Then, for each $A > 0$, there exists $\varepsilon > 0$ for which the following holds*:

Let $H' \subseteq \mathbb{R}^N$ be another r-dimensional affine subspace of $\mathbb{R}^N$, and let $v_0', \cdots, v_r' \in H'$, with $|v_i' - v_i| \leq \varepsilon$ for each i. Let $v = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_r v_r$, with $\lambda_1 + \cdots + \lambda_r = 1$ and $|\lambda_i| \leq A$ for each i.

Suppose $v' \in H'$, with $|v' - v| \leq \varepsilon$. Then we may express v' in the form $v' = \lambda_0' v_0' + \lambda_1' v_1' + \cdots + \lambda_r' v_r'$, with $\lambda_0' + \cdots + \lambda_r' = 1$ and $|\lambda_i| \leq 2A$ for each i.

Proof. If ε is small enough, then, since $|v_i' - v_i| \leq \varepsilon$, the v_i' form the vertices of a nondegenerate affine r-simplex in H'. Since also H' is r-dimensional and $v' \in H'$,

$$v' = \lambda_0' v_0' + \cdots + \lambda_r' v_r', \text{ with } \lambda_0' + \cdots + \lambda_r' = 1. \tag{1}$$

It remains to show that $|\lambda_i'| \leq 2A$ for each i. Let $\xi_1, \ldots, \xi_r$ be an orthonormal basis for span $(v_1 - v_0, \ldots, v_r - v_0)$.

The $\lambda_0, \ldots, \lambda_r$ satisfy the system of linear equations

$$\lambda_0(v_0 \cdot \xi_i) + \lambda_1(v_1 \cdot \xi_i) + \cdots + \lambda_r(v_r \cdot \xi_i) = (v \cdot \xi_i) \ i = 1, \ldots, r, \tag{2}$$

$$\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1. \tag{3}$$

Since the v_i form the vertices of a nondegenerate r-simplex in an r-dimensional affine space H, the system of equations (2), (3) has nonzero determinant.

On the other hand, the $\lambda_0', \ldots, \lambda_r'$ satisfy

$$\lambda_0'(v_0' \cdot \xi_i) + \lambda_1'(v_1' \cdot \xi_i) + \cdots + \lambda_r'(v_r' \cdot \xi_i) = (v' \cdot \xi_i) \quad i = 1, \ldots, r, \tag{4}$$

$$\lambda_0' + \cdots + \lambda_r' = 1. \tag{5}$$

The matrix elements $v_j' \cdot \xi_i$ and right-hand sides $v' \cdot \xi_i$ in (4), (5) lie within ε of the corresponding matrix elements and right-hand sides of (2), (3). Consequently, if $|\lambda_i| \leq A$, then we can force the λ_i' to be arbitrarily close to the λ_i by taking ε small enough. In particular, if $|\lambda_i| \leq A$ for each i, and if ε is small enough, then $|\lambda_i'| \leq 2A$ for each i. The proof of Lemma 3.3 is complete. □

We recall two basic properties of convex sets in $\mathbb{R}^N$.

LEMMA 3.4 (Helly's theorem). *Let $(\mathcal{K}_\alpha)_{\alpha\in\mathcal{A}}$ be a family of compact convex subsets of $\mathbb{R}^N$. If any $N+1$ of the $\mathcal{K}_\alpha$ have nonempty intersection, then the whole family has nonempty intersection.*

LEMMA 3.5 (Lemma of Fritz John). *Let $\Omega \subset \mathbb{R}^N$ be compact, convex, and symmetric about the origin. Suppose also that Ω has nonempty interior. Then there exist vectors $v_1, \ldots, v_N \in \mathbb{R}^N$, such that*

$$\left\{\sum_1^N \lambda_i v_i : |\lambda_i| \le c \text{ for all } i\right\} \subseteq \Omega \subseteq \left\{\sum_1^N \lambda_i v_i : |\lambda_i| \le 1 \text{ for all } i\right\}$$

with $c > 0$ depending only on N.

For proofs of these results, see [rW].

Finally, for future reference, we give the standard Whitney extension theorem for finite sets.

LEMMA 3.6. *Let $S \subset \mathbb{R}^n$ be a finite set, and suppose that, for each $x \in S$, we are given an m-jet $P^x \in \mathcal{P}$. Assume that the P^x satisfy*

$$|\partial^\alpha P^x(x)| \le A \text{ for } |\alpha| \le m, \qquad x \in S;$$

and

$$|\partial^\alpha (P^x - P^y)(y)| \le A \cdot |x-y|^{m-|\alpha|} \text{ for } |\alpha| \le m, \qquad x, y \in S.$$

Then there exists $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \le C \cdot A$ and $J_x(F) = P^x$ for all $x \in S$.

Here, C depends only on m and n; and $\partial^\alpha P^x(x)$ denotes the α^{th} derivative of the polynomial P^x, evaluated at x.

See [M], [emS], [hW1] for a proof of Lemma 3.6.

4. Setup for the main induction

As explained in the introduction, we will prove Theorem 3 by induction on the number of strata. For the rest of the paper, we fix an integer $\wedge \ge 1$, and assume that Theorem 3 holds whenever the number of strata is less than $\wedge$. We write $k^\#_{\text{old}}$ to denote the constant called $k^\#$ in Theorem 3, for the case of fewer than $\wedge$ strata. Thus $k^\#_{\text{old}}$ is determined by m, n.

We must show that Theorem 3 holds for $\wedge$ strata. We let $k^\#$ be a large enough integer, determined by m and n, to be fixed later, and let E, $f(x)$, $I(x)$ be as in the hypotheses of Theorem 3 for our value of $k^\#$, assuming that the number of strata is equal to $\wedge$.

We fix $\wedge$, $k^{\#}$, E, $f(x)$, $I(x)$, and we keep the above assumptions, for the rest of this paper. From now on, we write c, C, C', etc., to denote constants depending only on m and n; and we call such constants "controlled."

5. The basic convex sets

Let E, f, I be as in Section 4. For $x_0 \in E$, $\bar{k} \geq 1$, $A > 0$, we define the set $\Gamma_f(x_0, \bar{k}, A)$ to consist of all $P_0 \in f(x_0) + I(x_0)$ for which the following holds:

(1) Given $x_1, \ldots, x_{\bar{k}} \in E$, there exist polynomials $P_1, \ldots, P_{\bar{k}} \in \mathcal{P}$, with

(a) $P_i \in f(x_i) + I(x_i)$ for $i = 0, 1, \ldots, \bar{k}$;

(b) $|\partial^\alpha P_i(x_i)| \leq A$ for $|\alpha| \leq m$, $0 \leq i \leq \bar{k}$; and

(c) $|\partial^\alpha (P_i - P_j)(x_j)| \leq A|x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \leq m$, $0 \leq i, j \leq \bar{k}$.

Also, for $x_0 \in E$, $\bar{k} \geq 1$, we define the set $\sigma(x_0, \bar{k})$ to consist of all $P_0 \in I(x_0)$ such that:

(2) Given $x_1, \ldots, x_{\bar{k}} \in E$, there exist polynomials $P_1, \ldots, P_{\bar{k}} \in \mathcal{P}$, with

(a) $P_i \in I(x_i)$ for $i = 0, 1, \ldots, \bar{k}$;

(b) $|\partial^\alpha P_i(x_i)| \leq 1$ for $|\alpha| \leq m$, $0 \leq i \leq \bar{k}$; and

(c) $|\partial^\alpha (P_i - P_j)(x_j)| \leq |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \leq m$, $0 \leq i, j \leq \bar{k}$.

Thus, $\Gamma_f(x_0, \bar{k}, A)$ and $\sigma(x_0, \bar{k})$ are compact, convex subsets of $\mathcal{P}$, and $\sigma(x_0, \bar{k})$ is symmetric about the origin. The set $\sigma(x_0, \bar{k})$ is determined by $x_0, \bar{k}, E, I(x)(x \in E)$; it is independent of the jets $f(x)(x \in E)$. The convex sets $\Gamma_f(x_0, \bar{k}, A)$ and $\sigma(x_0, \bar{k})$ will play a fundamental rôle in our proof of Theorem 3.

Recall that $\bar{\mathcal{R}}_x$ denotes the ring of $(m-1)$-jets of functions at x, and that $\pi_x : \mathcal{R}_x \to \bar{\mathcal{R}}_x$ denotes the natural projection. We identify $\bar{\mathcal{R}}_x$ with the vector space $\bar{\mathcal{P}}$ of $(m-1)^{\underline{\text{rst}}}$ degree polynomials on $\mathbb{R}^n$. We define

$$\bar{\Gamma}_f(x, \bar{k}, A) = \pi_x \Gamma_f(x, \bar{k}, A), \tag{3}$$

$$\bar{\sigma}(x, \bar{k}) = \pi_x \sigma(x, \bar{k}), \tag{4}$$

$$\bar{f}(x) = \pi_x f(x), \text{ and} \tag{5}$$

$$\bar{I}(x) = \pi_x I(x) \text{ for } x \in E\ . \tag{6}$$

Recall also that E_1 denotes the lowest stratum of E. Thus, E_1 is compact, and the quantities dim $I(x)$, dim (ker $\pi_x \cap I(x)$) are constant functions of x on I_1. We set

$$d = \dim I(x) \text{ for all } x \in E_1, \text{ and} \tag{7}$$

$$\bar{d} = \dim \bar{I}(x) \text{ for all } x \in E_1\ . \tag{8}$$

Note that if $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \leq C$ and $J_x(F) \in f(x) + I(x)$ for all $x \in E$, then obviously $J_{x_0}(F) \in \Gamma_f(x_0, k^\#, C')$. (To see this, just set $P_i = J_{x_i}(F)$, $i = 0, 1, \ldots, k^\#$ in definition (1).) This suggests that working to guarantee (0.15), as explained in the introduction, is a prudent idea.

LEMMA 5.1. *Suppose $A, A' > 0$, $\bar{k} \geq 1$, $x \in E$, and $P \in \Gamma_f(x, \bar{k}, A)$. Then*

$$P + A'\sigma(x, \bar{k}) \subseteq \Gamma_f(x, \bar{k}, A + A') \subseteq P + (2A + A')\sigma(x, \bar{k}).$$

The proof is immediate from definitions (1) and (2).

LEMMA 5.2. *Suppose $A > 0$, $x_0 \in E$, $P_0 \in \ker \pi_{x_0} \cap I(x_0)$. Assume that*

$$|\partial^\alpha P_0(x_0)| \leq A \ for \ |\alpha| \leq m\,.$$

Then $P_0 \in C\, A\sigma(x_0, \bar{k})$ for any $\bar{k} \geq 1$.

To prove Lemma 5.2, we just set $P_1 = P_2 = \cdots = P_{\bar{k}} = 0$ *in* (2).

LEMMA 5.3. *For any $x_0 \in E$ and $\bar{k} \leq k^\#$, the set $\sigma(x_0, \bar{k})$ is Whitney convex, with a controlled Whitney constant independent of x_0.*

Proof. We noted already that $\sigma(x_0, \bar{k})$ is compact, convex, and symmetric about the origin. Suppose we are given $P_0 \in \sigma(x_0, \bar{k})$, $Q \in \mathcal{R}_{x_0}$, and $0 < \delta \leq 1$, with

$$(9) \qquad |\partial^\alpha P_0(x_0)| \leq \delta^{m-|\alpha|} \text{ and } |\partial^\alpha Q(x_0)| \leq \delta^{-|\alpha|} \text{ for } |\alpha| \leq m\,.$$

We must show that the jet $P_0 \cdot Q$ belongs to $C\sigma(x_0, \bar{k})$, where the dot denotes multiplication in $\mathcal{R}_{x_0}$. Let $x_1, \ldots, x_{\bar{k}} \in E$ be given. Since $P_0 \in \sigma(x_0, \bar{k})$, there exist $P_1, \ldots, P_{\bar{k}} \in \mathcal{P}$ satisfying (2). Hence, by Whitney's extension theorem for finite sets, there exists

$$(10) \quad F \in C^m(\mathbb{R}^n), \text{ with } \| F \|_{C^m(\mathbb{R}^n)} \leq C \text{ and } J_{x_i}(F) = P_i \quad (0 \leq i \leq \bar{k})\,.$$

Also, (9) shows that we may find $\theta \in C^m(\mathbb{R}^n)$, with

$$(11) \qquad J_{x_0}(\theta) = Q, |\partial^\alpha \theta| \leq C\delta^{-|\alpha|} \text{ on } \mathbb{R}^n, \text{ and supp } \theta \subset B(x_0, \delta)\,.$$

By (9) and (10) we have $|\partial^\alpha F(x_0)| \leq \delta^{m-|\alpha|}$ for $|\alpha| \leq m$, and $|\partial^\alpha F| \leq C$ on $\mathbb{R}^n$ for $|\alpha| = m$. Consequently, $|\partial^\alpha F(x)| \leq C\delta^{m-|\alpha|}$ for $|\alpha| \leq m$, $x \in B(x_0, \delta)$. Together with (11), this shows that $|\partial^\alpha(\theta F)| \leq C\delta^{m-|\alpha|}$ on $B(x_0, \delta)$ for $|\alpha| \leq m$. In particular, $\| \theta F \|_{C^m(\mathbb{R}^n)} \leq C$, since supp $\theta \subset B(x_0, \delta)$.

Setting $\hat{P}_i = J_{x_i}(\theta F) = J_{x_i}(\theta) \cdot J_{x_i}(F) = J_{x_i}(\theta) \cdot P_i$ $(0 \leq i \leq \bar{k})$, with the dots denoting multiplication in $\mathcal{R}_{x_i}$, we have the following remarks.

(a) $\hat{P}_i \in I(x_i)$ for $i = 0, \ldots, \bar{k}$, since $P_i \in I(x_i)$ and $I(x_i) \subset \mathcal{R}_{x_i}$ is an ideal;

(b) $|\partial^\alpha \hat{P}_i(x_i)| \leq C$ for $|\alpha| \leq m$, $0 \leq i \leq \bar{k}$, since $\| \theta F \|_{C^m(\mathbb{R}^n)} \leq C$; and

(c) $|\partial^\alpha(\hat{P}_i - \hat{P}_j)(x_i)| \leq C|x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \leq m$, $0 \leq i, j \leq \bar{k}$, again because $\| \theta F \|_{C^m(\mathbb{R}^n)} \leq C$.

Since $\hat{P}_0 = J_{x_0}(\theta) \cdot P_0 = Q \cdot P_0$, remarks (a), (b), (c) above show that $cQ \cdot P_0$ belongs to $\sigma(x_0, \bar{k})$ for a small enough controlled constant c. Thus, $Q \cdot P_0 \in C\sigma(x_0, \bar{k})$. The proof of Lemma 5.3 is complete. □

The next lemma shows in particular that $\Gamma_f(x_0, \bar{k}, A)$ is nonempty for suitable $\bar{k}, A$. Let $D = \dim \mathcal{P}$.

LEMMA 5.4. *Suppose* $\bar{k} \cdot (\bar{k}D + 2) \leq k^\#$. *Then, given* $x_1, \ldots, x_{\bar{k}} \in E$, *there exist* $P_1, \ldots, P_{\bar{k}} \in \mathcal{P}$, *with*

(a) $P_i \in \Gamma_f(x_i, \bar{k}, 1) \subseteq f(x_i) + I(x_i)$ *for* $i = 1, \ldots, \bar{k}$;

(b) $|\partial^\alpha P_i(x_i)| \leq 1$ *for* $|\alpha| \leq m$, $i = 1, \ldots, \bar{k}$; *and*

(c) $|\partial^\alpha(P_i - P_j)(x_j)| \leq |x_i - x_j|^{m-|\alpha|}$ *for* $|\alpha| \leq m$, $0 \leq i, j \leq \bar{k}$.

Proof. Fix $x_1, \ldots, x_{\bar{k}} \in E$. Given a finite set $S \subset E$, define $S^+ = S \cup \{x_1, \ldots, x_{\bar{k}}\}$, and define $\mathcal{K}(S)$ to be the set of all $(P_1, \ldots, P_{\bar{k}}) \in \mathcal{P}^{\bar{k}}$ for which there exists a map $x \in S^+ \mapsto P^x \in f(x) + I(x)$, such that $P^{x_i} = P_i$ for $1 \leq i \leq \bar{k}$, $|\partial^\alpha P^x(x)| \leq 1$ for $|\alpha| \leq m$ and $x \in S^+$, and $|\partial^\alpha(P^x - P^y)(y)| \leq |x - y|^{m-|\alpha|}$ for $|\alpha| \leq m$, $x, y \in S^+$. Each $\mathcal{K}(S)$ is a compact, convex subset of $\mathcal{P}^{\bar{k}}$, which has dimension $\bar{k}D$.

We have $\mathcal{K}(S') \subseteq \mathcal{K}(S)$ for $S \subseteq S'$. Also, since E, f, I are assumed to satisfy hypothesis (II) of Theorem 3, we know that $\mathcal{K}(S)$ is nonempty whenever $\#(S^+) \leq k^\#$, hence, whenever $\#(S) \leq k^\# - \bar{k}$.

Therefore, if $S_1, \ldots, S_{\bar{k}D+1} \subseteq E$ with $\#(S_i) \leq \bar{k}$ for each i, then $\mathcal{K}(S_1) \cap \cdots \cap \mathcal{K}(S_{\bar{k}D+1})$ is nonempty, since it contains $\mathcal{K}(S_1 \cup \cdots \cup S_{\bar{k}D+1})$, and $\#(S_1 \cup \cdots \cup S_{\bar{k}D+1}) \leq \bar{k} \cdot (\bar{k}D + 1) \leq k^\# - \bar{k}$.

Helly's theorem now shows that there exists $(P_1, \ldots, P_{\bar{k}})$ belonging to $\mathcal{K}(S)$ for all $S \subseteq E$ with $\#(S) \leq \bar{k}$.

Taking S to be the empty set, we see that the P_i satisfy

$$|\partial^\alpha P_i(x_i)| \leq 1 \text{ for } |\alpha| \leq m, \quad i = 1, \ldots, \bar{k}; \text{ and} \tag{12}$$

$$|\partial^\alpha(P_i - P_j)(x_j)| \leq |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, \quad 1 \leq i, j \leq \bar{k}. \tag{13}$$

We will check that $P_i \in \Gamma_f(x_i, \bar{k}, 1)$ for each i. In fact, given $\tilde{x}_0, \ldots, \tilde{x}_{\bar{k}} \in E$ with $\tilde{x}_0 = x_i$, we take $S = \{\tilde{x}_0, \ldots, \tilde{x}_{\bar{k}}\}$. Since $(P_1, \ldots, P_{\bar{k}}) \in \mathcal{K}(S)$, there exist polynomials $\widetilde{P}_0, \ldots, \widetilde{P}_{\bar{k}} \in \mathcal{P}$, with $\widetilde{P}_0 = P_i$; $\widetilde{P}_j \in f(\tilde{x}_j) + I(\tilde{x}_j)$ for $j = 0, \ldots, \bar{k}$; $|\partial^\alpha \widetilde{P}_j(\tilde{x}_j)| \leq 1$ for $|\alpha| \leq m$, $j = 0, \ldots, \bar{k}$; and $|\partial^\alpha(\widetilde{P}_j - \widetilde{P}_\ell)(\tilde{x}_\ell)| \leq |\tilde{x}_j - \tilde{x}_\ell|^{m-|\alpha|}$ for $|\alpha| \leq m$, $0 \leq j, \ell \leq \bar{k}$. Thus,

(14) $P_i \in \Gamma_f(x_i, \bar{k}, 1)$, as claimed.

Our results (12), (13), (14) are the conclusions of Lemma 5.4. □

The goal of the next several lemmas is to show that, roughly speaking, if $P \in \Gamma_f(x, \bar{k}, C)$, and if x' is close to x and $P' \in f(x') + I(x')$ is close to P, then P' belongs to $\Gamma_f(x', \tilde{k}, C')$, with $\tilde{k}$ somewhat smaller than $\bar{k}$, and with C' somewhat larger than C. More precisely, the next several lemmas will be used to establish Lemma 5.10 below.

LEMMA 5.5. *If $d \neq 0$ (see (7)), then $\sigma(x_0, \bar{k})$ has nonempty interior in $I(x_0)$, for every $x_0 \in E$ and $\bar{k} \leq k^{\#}$.*

Proof. Since $\sigma(x_0, \bar{k}) \subseteq I(x_0)$ is convex and symmetric about the origin, it is enough to prove the following.

(15) Given $x_0 \in E$ and $P_0 \in I(x_0)$, there exists $\lambda > 0$ with $\lambda P_0 \in \sigma(x_0, \bar{k})$.

To show (15), we recall that E, f, I are assumed to satisfy the hypotheses of Theorem 3. We apply hypothesis (I) with $\varepsilon = 1$, to the jets $f(x_0)$, $f(x_0) + P_0 \in f(x_0) + I(x_0)$. Thus, there exists $\delta > 0$ for which the following holds.

Given $x_1, \ldots, x_{\bar{k}} \in E \cap B(x_0, \delta)$, there exist $P_0', P_1', \ldots, P_{\bar{k}}' \in \mathcal{P}$ and $P_0'', P_1'', \ldots, P_{\bar{k}}'' \in \mathcal{P}$, with

(16)
$$\begin{gathered} P_0' = f(x_0); \\ P_i' \in f(x_i) + I(x_i) \text{ for } i = 0, 1, \ldots, \bar{k}; \\ |\partial^\alpha (P_i' - P_j')(x_j)| \leq |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, \qquad 0 \leq i, j \leq \bar{k}; \end{gathered}$$

(17)
$$\begin{gathered} P_0'' = f(x_0) + P_0; \\ P_i'' \in f(x_i) + I(x_i) \text{ for } i = 0, 1, \ldots, \bar{k}; \\ |\partial^\alpha (P_i'' - P_j'')(x_j)| \leq |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, \qquad 0 \leq i, j \leq \bar{k}\,. \end{gathered}$$

Setting $P_i = P_i'' - P_i'$ for $i = 0, 1, \ldots, \bar{k}$ (which agrees with the given P_0 in (15) when $i = 0$, thanks to (16), (17)), we find that

(18) $$P_i \in I(x_i) \text{ for } i = 0, 1, \ldots, \bar{k}; \text{ and}$$

(19) $$|\partial^\alpha (P_i - P_j)(x_j)| \leq 2|x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, \qquad 0 \leq i, j \leq k^{\#}\,.$$

We may assume that $\delta < 1/2$, hence $|x_i - x_j| \leq 1$ in (19), and therefore

(20) $$|\partial^\alpha P_j(x_j)| \leq 2 + \max_{B(x_0, \delta)} |\partial^\alpha P_0| \text{ for } |\alpha| \leq m, \qquad 0 \leq j \leq k^{\#}\,.$$

From (19), (20) and Whitney's extension theorem for finite sets, we obtain $F \in C^m(\mathbb{R}^n)$, with

(21)
$$\begin{gathered} \| F \|_{C^m(\mathbb{R}^n)} \leq C \cdot \{2 + \max_{\substack{y \in B(x_0, \delta) \\ |\alpha| \leq m}} |\partial^\alpha P_0(y)|\} \equiv K \text{ and} \\ J_{x_i}(F) = P_i \text{ for } i = 0, 1 \ldots, \bar{k}\,. \end{gathered}$$

In particular,

(22) $J_{x_i}(F) \in I(x_i)$ for $i = 0, 1, \ldots, \bar{k}$ (by (18)); and

(23) $J_{x_0}(F) = P_0$.

We can achieve (21), (22), (23) for any $x_1, \ldots, x_{\bar{k}} \in E \cap B(x_0, \delta)$.

Now let $\theta \in C^m(\mathbb{R}^n)$ be a cutoff function, with

(24) $$J_{x_0}(\theta) = 1, \text{ supp } \theta \subset B(x_0, \delta), |\partial^\alpha \theta| \leq C\delta^{-|\alpha|} \text{ on } \mathbb{R}^n (|\alpha| \leq m).$$

Given any $x_1, \ldots, x_{\bar{k}} \in E$, we define $x'_1, \ldots, x'_{\bar{k}} \in E$ by setting $x'_i = x_i$ if $x_i \in B(x_0, \delta)$, $x'_i = x_0$ otherwise. Thus, all the x'_i belong to $E \cap B(x_0, \delta)$. Applying (21), (22), (23) with $x'_1 \cdots x'_{\bar{k}}$ in place of $x_1, \ldots, x_{\bar{k}}$, we obtain $F \in C^m(\mathbb{R}^n)$, with

(25) $$\| F \|_{C^m(\mathbb{R}^n)} \leq K, J_{x_0}(F) = P_0, J_{x_i}(F) \in I(x_i) \text{ if } x_i \in B(x_0, \delta).$$

From (24) and (25), we see that

(26) $$\| \theta F \|_{C^m(\mathbb{R}^n)} \leq C\, K \delta^{-m}, J_{x_0}(\theta F) = P_0, \text{ and}$$

(27) $$J_{x_i}(\theta F) \in I(x_i) \text{ for } i = 0, 1, \ldots, \bar{k}\,.$$

In fact, (27) follows from (25) for $x_i \in B(x_0, \delta)$, since $I(x_i)$ is an ideal. For $x_i \notin B(x_0, \delta)$, (27) follows from (24).

Setting $P_i = J_{x_i}(\theta F)$ for $i = 1, \ldots, \bar{k}$, we obtain the following result, for our given P_0: Given $x_1, \ldots, x_{\bar{k}} \in E$, there exist $P_1, \ldots, P_{\bar{k}} \in \mathcal{P}$, with $P_i \in I(x_i)$ for $i = 0, 1, \ldots, \bar{k}$;

$$|\partial^\alpha P_i(x_i)| \leq \left[C' K \delta^{-m}\right] \text{ for } |\alpha| \leq m,\, i = 0, 1, \ldots, \bar{k}\,; \text{ and}$$
$$|\partial^\alpha (P_i - P_j)(x_j)| \leq \left[C' K \delta^{-m}\right] |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m\,,\, 0 \leq i, j \leq \bar{k}\,.$$

This immediately implies (15), with $\lambda = [C' K \delta^{-m}]^{-1}$. The proof of Lemma 5.5 is complete. ☐

LEMMA 5.6. *Let $A > 0$, and suppose $1 + (D + 1) \cdot \tilde{k} \leq \bar{k}$. Let $x, x' \in E$, and let $P \in \Gamma_f(x, \bar{k}, A)$. Then there exists $P' \in \Gamma_f(x', \tilde{k}, A)$, with*

$$|\partial^\alpha (P - P')(x)|, |\partial^\alpha (P - P')(x')\,| \leq A|x - x'|^{m-|\alpha|} \text{ for } |\alpha| \leq m\,.$$

Proof. Given a finite set $S \subseteq E$, define $S^+ = \{x, x'\} \cup S$, and define $\mathcal{K}(S)$ as the set of all $P' \in \mathcal{P}$ for which there exists a map $y \mapsto P^y$ from S^+ to $\mathcal{P}$, with $P^x = P$; $P^{x'} = P'$, $P^y \in f(y) + I(y)$ for all $y \in S^+$;

$$|\partial^\alpha P^y(y)| \leq A \text{ for } |\alpha| \leq m \text{ and } y \in S^+; \text{ and}$$
$$|\partial^\alpha (P^y - P^z)(z)| \leq A|y - z|^{m-|\alpha|} \text{ for } |\alpha| \leq m, y, z \in S^+\,.$$

Each $\mathcal{K}(S)$ is a compact, convex subset of $\mathcal{P}$, which has dimension D. If $S \subseteq S'$ then $\mathcal{K}(S') \subseteq \mathcal{K}(S)$. If $S \subseteq E$ with $\#(S^+) \leq \bar{k} + 1$, then we see by using $P \in \Gamma_f(x, \bar{k}, A)$ that $\mathcal{K}(S)$ is nonempty.

If $S_1, \ldots, S_{D+1} \subseteq E$ with $\#(S_i) \le \tilde{k}$ for each i, then $S = S_1 \cup \cdots \cup S_{D+1}$ satisfies $\#(S^+) \le 2 + \#(S) \le 2 + (D+1)\tilde{k} \le \bar{k} + 1$. Hence, $\mathcal{K}(S)$ is nonempty, and $\mathcal{K}(S) \subseteq \mathcal{K}(S_i)$ for each i. Thus $\mathcal{K}(S_1) \cap \cdots \cap \mathcal{K}(S_{D+1})$ is nonempty.

Consequently, by Helly's theorem, there exists P' belonging to $\mathcal{K}(S)$ for every $S \subseteq E$ with $\#(S) \le \tilde{k}$. It follows easily that $P' \in \Gamma_f(x', \tilde{k}, A)$. Also, taking $S =$ empty set, we learn that

$$|\partial^\alpha (P - P')(x)|, |\partial^\alpha (P - P')(x')| \le A|x - x'|^{m-|\alpha|} \text{ for } |\alpha| \le m ,$$

since $P' \in \mathcal{K}(S)$. The proof of Lemma 5.6 is complete. □

For the next lemma, recall definitions (3)–(8).

Lemma 5.7. *Suppose $A > 0$ and $1 + (D+1)\tilde{k} \le \bar{k} \le k^\#$. Then, given $x \in E_1$, there exist $\varepsilon_0, \delta_0 > 0$ such that for any $\bar{Q} \in \bar{\Gamma}_f(x, \bar{k}, A)$, any $x' \in E_1 \cap B(x, \delta_0)$, and any $\bar{Q}' \in \bar{f}(x') + \bar{I}(x')$, if $|\partial^\alpha(\bar{Q}' - \bar{Q})(x)| \le \varepsilon_0$ for $|\alpha| \le m-1$, then $\bar{Q}' \in \bar{\Gamma}_f(x', \tilde{k}, A')$, with A' depending only on A, m, n.*

Proof. If $\bar{d} = 0$ then $\bar{f}(x') + \bar{I}(x')$ contains only the single point $\bar{f}(x')$, and Lemma 5.7 follows from Lemma 5.6. Suppose $\bar{d} \ne 0$. By Lemma 5.5 and Fritz John's Lemma, there exist $\bar{P}_1, \ldots, \bar{P}_{\bar{d}} \in \bar{I}(x)$ with the following properties.

(28) $$\bar{P}_i \in \bar{\sigma}(x, \bar{k}) \text{ for } i = 1, \ldots \bar{d},$$

(29) Any $\bar{P} \in \bar{\sigma}(x, \bar{k})$ may be written as $\bar{P} = \lambda_1 \bar{P}_1 + \cdots + \lambda_{\bar{d}} \bar{P}_{\bar{d}}$ with $|\lambda_i| \le C$ for $i = 1, \ldots, \bar{d}$.

In particular, $\bar{P}_1, \ldots, \bar{P}_{\bar{d}}$ are linearly independent. In this proof, we write $A_1, A_2, A_3, \cdots$ for constants determined by A, m, n. If $\bar{\Gamma}_f(x, \bar{k}, A)$ is empty, then Lemma 5.7 holds vacuously. Otherwise, fix

(30) $$\bar{Q}_0 \in \bar{\Gamma}_f(x, \bar{k}, A) \subseteq \bar{f}(x) + \bar{I}(x) ,$$

and define

(31) $$\bar{Q}_i = \bar{Q}_0 + \bar{P}_i \in \bar{f}(x) + \bar{I}(x) \text{ for } i = 1, \ldots, \bar{d} .$$

In view of (28), (30), (31) and Lemma 5.1, we have

(32) $$\bar{Q}_i \in \bar{\Gamma}_f(x, \bar{k}, A_1) \text{ for } i = 0, 1, \ldots, \bar{d} .$$

Also, from (30), (31) and the linear independence of $\bar{P}_1, \ldots, \bar{P}_{\bar{d}}$, we see that

(33) $\bar{Q}_0, \ldots, \bar{Q}_{\bar{d}}$ form the vertices of a nondegenerate affine $\bar{d}$-simplex in $\bar{f}(x) + \bar{I}(x)$.

Suppose $\bar{Q} \in \bar{\Gamma}_f(x, \bar{k}, A)$. Then (30) and Lemma 5.1 give $\bar{Q} - \bar{Q}_0 \in A_2 \bar{\sigma}(x, \bar{k})$; hence (29) shows that we may write $\bar{Q} - \bar{Q}_0 = \lambda_1 \bar{P}_1 + \cdots + \lambda_{\bar{d}} \bar{P}_{\bar{d}}$ with $|\lambda_i| \le A_3$ for $i = 1, \ldots, \bar{d}$. Thus, $\bar{Q} = \{1 - \lambda_1 - \cdots - \lambda_{\bar{d}}\}\bar{Q}_0 + \lambda_1 \bar{Q}_1 + \cdots + \lambda_{\bar{d}} \bar{Q}_{\bar{d}}$ and we have proven the following result:

(34) Any $\bar{Q} \in \bar{\Gamma}_f(x, \bar{k}, A)$ may be expressed in the form $\bar{Q} = \lambda_0 \bar{Q}_0 + \cdots + \lambda_{\bar{d}} \bar{Q}_{\bar{d}}$, with $\lambda_0 + \cdots + \lambda_{\bar{d}} = 1$, and $|\lambda_i| \leq A_4$ for $i = 0, 1, \ldots, \bar{d}$.

We now apply the linear algebra perturbation Lemma 3.3 to the affine subspaces $H = \bar{f}(x) + \bar{I}(x) \subseteq \bar{\mathcal{P}}$, $H' = \bar{f}(x') + \bar{I}(x') \subseteq \bar{\mathcal{P}}$, the vectors $\bar{Q}_0, \ldots, \bar{Q}_{\bar{d}} \in H$, and the constant A_4 in (34). Thus, we obtain $\varepsilon_0 > 0$ for which the following holds.

(35) Suppose $\bar{Q} = \lambda_0 \bar{Q}_0 + \cdots + \lambda_{\bar{d}} \bar{Q}_{\bar{d}}$ with $\lambda_0 + \cdots + \lambda_{\bar{d}} = 1$ and $|\lambda_i| \leq A_4$ (all i).

Suppose we are given $x' \in E_1$ and $\bar{Q}', \bar{Q}'_0, \ldots, \bar{Q}'_{\bar{d}} \in \bar{f}(x') + \bar{I}(x')$, with

(a) $|\partial^\alpha(\bar{Q}'_i - \bar{Q}_i)(x)| \leq \varepsilon_0$ for $|\alpha| \leq m-1$ and $0 \leq i \leq \bar{d}$, and

(b) $|\partial^\alpha(\bar{Q}' - \bar{Q})(x)| \leq \varepsilon_0$ for $|\alpha| \leq m-1$.

Then we may express $\bar{Q}'$ in the form

(c) $\bar{Q}' = \lambda'_0 \bar{Q}'_0 + \cdots + \lambda'_{\bar{d}} \bar{Q}'_{\bar{d}}$, with $\lambda'_0 + \cdots + \lambda'_{\bar{d}} = 1$ and $|\lambda'_i| \leq A_5$ (all i).

Next, we will show that there exists $\delta_0 > 0$ for which the following holds:

(36) Given any $x' \in E_1 \cap B(x, \delta_0)$, there exist

(α) $\bar{Q}'_0, \ldots, \bar{Q}'_{\bar{d}} \in \bar{\Gamma}_f(x', \tilde{k}, A_1) \subseteq \bar{f}(x') + \bar{I}(x')$, with

(β) $|\partial^\alpha(\bar{Q}'_i - \bar{Q}_i)(x)| \leq \varepsilon_0$ for $|\alpha| \leq m-1$ and $0 \leq i \leq \bar{d}$.

To see this, fix $i (0 \leq i \leq \bar{d})$. By (32) and (3), there exists $Q_i \in \Gamma_f(x, \bar{k}, A_1)$ with $\pi_x(Q_i) = \bar{Q}_i$. Now suppose $x' \in E_1 \cap B(x, \delta_0)$, for a small enough $\delta_0 > 0$ to be picked below. Lemma 5.6 gives us $Q'_i \in \Gamma_f(x', \tilde{k}, A_1)$, with

(37) $|\partial^\alpha(Q'_i - Q_i)(x)| \leq A_1 |x' - x|^{m-|\alpha|} \leq A_1 \delta_0^{m-|\alpha|} \leq A_1 \delta_0$ for $|\alpha| \leq m-1$, provided $\delta_0 \leq 1$.

We take $\bar{Q}'_i = \pi_{x'} Q'_i \in \bar{\Gamma}_f(x', \tilde{k}, A_1) \subseteq \bar{f}(x') + \bar{I}(x')$. (See (3), (5), (6).) Thus $\bar{Q}'_i$ satisfies (36) (α).

For $|\alpha| \leq m$, we have

$$\begin{aligned}
(38) \qquad \partial^\alpha Q'_i(x) &= \sum_{|\beta| \leq m-|\alpha|} \frac{1}{\beta!} \left(\partial^{\beta+\alpha} Q'_i(x')\right) \cdot (x - x')^\beta \\
&= \sum_{|\beta| \leq m-1-|\alpha|} \text{etc.} + \sum_{|\beta| = m-|\alpha|} \text{etc.} \\
&= \partial^\alpha \bar{Q}'_i(x) + \sum_{|\beta| = m-|\alpha|} \frac{1}{\beta!} \left(\partial^{\beta+\alpha} Q'_i(x')\right) \cdot (x - x')^\beta .
\end{aligned}$$

Also, since $Q'_i \in \Gamma_f(x', \tilde{k}, A_1)$, we have $|\partial^\alpha Q'_i(x')| \le A_1$ for $|\alpha| \le m$. (See (1)(b).) Hence, (38) implies that

(39) $$|\partial^\alpha \bar{Q}'_i(x) - \partial^\alpha Q'_i(x)| \le A_6\delta_0 \text{ for } |\alpha| \le m-1, \text{ provided } \delta_0 \le 1\ .$$

Since $\partial^\alpha \bar{Q}_i(x) = \partial^\alpha Q_i(x)$ for $|\alpha| \le m-1$, estimates (37) and (39) show that

(40) $$|\partial^\alpha(\bar{Q}'_i - \bar{Q}_i)(x)| \le A_7\delta_0 \text{ for } |\alpha| \le m-1, \text{ provided } \delta_0 \le 1\ .$$

We now pick $\delta_0 \le 1$ small enough that $A_7\delta_0 \le \varepsilon_0$. Thus, (40) holds, and it shows that $\bar{Q}'_i$ satisfies (36) (β). The proof of (36) is complete.

We fix $\varepsilon_0, \delta_0 > 0$ as in (35), (36). Now suppose $\bar{Q} \in \bar{\Gamma}_f(x, \bar{k}, A)$, $x' \in E_1 \cap B(x, \delta_0), \bar{Q}' \in \bar{f}(x') + \bar{I}(x')$, and assume that

(41) $$|\partial^\alpha(\bar{Q}' - \bar{Q})(x)| \le \varepsilon_0 \text{ for } |\alpha| \le m-1\ .$$

Then the hypotheses of (35) hold, thanks to (34) and (36). Applying (35), we may express $\bar{Q}'$ in the form $\bar{Q}' = \lambda'_0\bar{Q}'_0 + \cdots + \lambda'_{\bar{d}}\bar{Q}_{\bar{d}}$, with $\lambda'_0 + \cdots + \lambda'_{\bar{d}} = 1$, $|\lambda'_i| \le A_5$ (all i), and $\bar{Q}'_0, \ldots, \bar{Q}'_{\bar{d}} \in \bar{\Gamma}_f(x', \tilde{k}, A_1)$ as in (36)(α). Equivalently,

(42) $$\bar{Q}' = \bar{Q}'_0 + \sum_{i=1}^{\bar{d}} \lambda'_i(\bar{Q}'_i - \bar{Q}'_0)\ .$$

We have $\bar{Q}'_i - \bar{Q}'_0 \in 2A_1\bar{\sigma}(x', \tilde{k})$ by Lemma 5.1, hence

(43) $$\sum_{i=1}^{\bar{d}} \lambda'_i(\bar{Q}'_i - \bar{Q}'_0) \in A_8\bar{\sigma}(x', \tilde{k})\ .$$

From (42), (43), (36)(α), and another application of Lemma 5.1, we see that $\bar{Q}' \in \bar{\Gamma}_f(x', \tilde{k}, A_9)$. Thus, we have shown that, whenever $x' \in E_1 \cap B(x, \delta_0)$, $\bar{Q} \in \bar{\Gamma}_f(x, \bar{k}, A)$, $\bar{Q}' \in \bar{f}(x') + \bar{I}(x')$, with $|\partial^\alpha(\bar{Q}' - \bar{Q})(x)| \le \varepsilon_0$ for $|\alpha| \le m-1$, we have $\bar{Q}' \in \bar{\Gamma}_f(x', \tilde{k}, A_9)$. The proof of Lemma 5.7 is complete. □

Note that we had to restrict to $x, x' \in E_1$ in Lemma 5.7, because one of the crucial hypotheses in the linear algebra perturbation lemma was that the affine spaces H and H' have the same dimension.

LEMMA 5.8. *Suppose $A_1, A_2 > 0$ and $1 + (D+1)\cdot\tilde{k} \le \bar{k} \le k^{\#}$. Then, given $x \in E_1$, there exist $\varepsilon, \delta > 0$ such that, for any $Q \in \Gamma_f(x, \bar{k}, A_1)$, any $x' \in E_1 \cap B(x, \delta)$, and any $Q' \in f(x') + I(x')$, if*

(44) $$|\partial^\alpha(Q' - Q)(x)| \le \varepsilon \text{ for } |\alpha| \le m-1$$

and

(45) $$|\partial^\alpha Q'(x)| \le A_2 \text{ for } |\alpha| = m\ ,$$

then $Q' \in \Gamma_f(x', \tilde{k}, A')$, with A' determined by A_1, A_2, m, n.

Proof. In this proof, we write $A_3, A_4, A_5, \cdots$ to denote constants determined by A_1, A_2, m, n. Given $x \in E_1$, let ε_0, δ_0 be as in Lemma 5.7 with $A = A_1$. Let $\varepsilon, \delta > 0$ be small enough numbers, to be picked below, depending only on $A_1, A_2, m, n, \varepsilon_0, \delta_0$. Suppose $Q \in \Gamma_f(x, \bar{k}, A_1)$, $x' \in E_1 \cap B(x, \delta)$, $Q' \in f(x') + I(x')$, and assume (44) and (45). Since $Q \in \Gamma_f(x, \bar{k}, A_1)$, we have

$$|\partial^\alpha Q(x)| \leq A_1 \text{ for } |\alpha| \leq m. \qquad \text{(See (1)(b).)} \tag{46}$$

Hence, (44) and (45) show that

$$|\partial^\alpha Q'(x)| \leq A_3 \text{ for } |\alpha| \leq m\,. \tag{47}$$

We will take $\delta \leq 1$. Hence (47) implies

$$|\partial^\alpha Q'(x')| \leq A_4 \text{ for } |\alpha| \leq m\,, \tag{48}$$

since $x' \in B(x, \delta)$. Set $\bar{Q} = \pi_x Q$, $\bar{Q}' = \pi_{x'} Q'$. Thus,

(49) $\bar{Q} \in \bar{\Gamma}_f(x, \bar{k}, A_1)$, $x' \in E_1 \cap B(x, \delta_0)$, and $\bar{Q}' \in \bar{f}(x') + \bar{I}(x')$, provided we take $\delta \leq \delta_0$.

By expanding Q' about x', we see that

$$\partial^\alpha Q'(x) = \partial^\alpha \bar{Q}'(x) + \sum_{|\beta| = m - |\alpha|} \frac{1}{\beta!} \left(\partial^{\beta+\alpha} Q'(x')\right) \cdot (x - x')^\beta \text{ for } |\alpha| \leq m - 1\,.$$

Therefore, (48) implies that

$$|\partial^\alpha \bar{Q}'(x) - \partial^\alpha Q'(x)| \leq A_5 |x - x'|^{m - |\alpha|} \leq A_5 \delta^{m - |\alpha|} \leq A_5 \delta \text{ for } |\alpha| \leq m - 1. \tag{50}$$

Since also $\partial^\alpha \bar{Q}(x) = \partial^\alpha Q(x)$ for $|\alpha| \leq m - 1$, we learn from (44) and (50) that

$$|\partial^\alpha (\bar{Q}' - \bar{Q})(x)| \leq \varepsilon + A_5 \delta \text{ for } |\alpha| \leq m - 1\,. \tag{51}$$

We now pick $\varepsilon = \frac{1}{2}\varepsilon_0$ and $\delta = \min\{1, \delta_0, \varepsilon_0/(2A_5)\}$. Thus, the above arguments are valid for our ε, δ; and (51) gives

$$|\partial^\alpha (\bar{Q}' - \bar{Q})(x)| \leq \varepsilon_0 \text{ for } |\alpha| \leq m - 1\,. \tag{52}$$

In view of (49) and (52), we may apply Lemma 5.7, with $A = A_1$. Thus, we learn that $\bar{Q}' \in \bar{\Gamma}_f(x', \tilde{k}, A_6)$. That is,

$$\pi_{x'} Q' = \pi_{x'} \widetilde{Q} \text{ for some } \widetilde{Q} \in \Gamma_f(x', \tilde{k}, A_6) \subseteq f(x') + I(x')\,. \tag{53}$$

Fix $\widetilde{Q}$ as in (53). In particular, we have

$$|\partial^\alpha \widetilde{Q}(x')| \leq A_6 \text{ for } |\alpha| \leq m. \text{ (See (1)(b).)} \tag{54}$$

From (48), (53), (54), we see that

$$Q' - \widetilde{Q} \in \ker \pi_{x'} \cap I(x')\,, \text{ with } |\partial^\alpha (Q' - \widetilde{Q})(x')| \leq A_7 (|\alpha| \leq m)\,.$$

Together with Lemma 5.2, this shows that

$$Q' - \widetilde{Q} \in A_8 \sigma(x', \tilde{k}) \ . \tag{55}$$

We now have $Q' = \widetilde{Q} + (Q' - \widetilde{Q})$, with $\widetilde{Q} \in \Gamma_f(x', \tilde{k}, A_6)$ and $Q' - \widetilde{Q}$ satisfying (55). Applying Lemma 5.1, we conclude that $Q' \in \Gamma_f(x', \tilde{k}, A_9)$, completing the proof of Lemma 5.8. □

Lemma 5.9. *Suppose* $A_1, A_2 > 0$, $1+(D+1)\cdot\tilde{k} \le \bar{k}_2$, $1+(D+1)\cdot\bar{k}_2 \le \bar{k}_1$, $\bar{k}_1 \le k^{\#}$. *Let* $x_0 \in E_1$. *Then there exists* $\eta > 0$ *for which the following holds*:

Suppose $x', x'' \in E_1$, *with* $|x_0 - x'|$, $|x' - x''| < \eta$. *Let* $Q' \in \Gamma_f(x', \bar{k}_1, A_1)$ *and* $Q'' \in f(x'') + I(x'')$, *with*

$$|\partial^\alpha (Q'' - Q')(x')| \le A_2\, \eta^{m-|\alpha|} \text{ for } |\alpha| \le m \ .$$

Then $Q'' \in \Gamma_f(x'', \tilde{k}, A')$, *with* A' *determined by* A_1, A_2, m, n.

Proof. In this proof, we write $A_3, A_4, A_5, \cdots$ for constants determined by A_1, A_2, m, n. Suppose x_0, x', x'', Q', Q'' are as in the hypotheses of Lemma 5.9, with η a small enough positive number, independent of x', x'', Q', Q'', to be picked later. Since $Q' \in \Gamma_f(x', \bar{k}_1, A_1)$, Lemma 5.6 produces a polynomial

$$Q_0 \in \Gamma_f(x_0, \bar{k}_2, A_1) \ , \tag{56}$$

with

$$|\partial^\alpha (Q' - Q_0)(x_0)| \le A_1 |x_0 - x'|^{m-|\alpha|} \text{ for } |\alpha| \le m \ . \tag{57}$$

For $|\alpha| \le m$, we have also that

$$\begin{aligned} |\partial^\alpha (Q'' - Q')(x_0)| &= |\sum_{|\beta| \le m-|\alpha|} \frac{1}{\beta!}\, (\partial^{\beta+\alpha}(Q'' - Q')(x')) \cdot (x_0 - x')^\beta| \\ &\le \sum_{|\beta| \le m-|\alpha|} \frac{1}{\beta!}\, A_2\, \eta^{m-|\beta|-|\alpha|}\, |x_0 - x'|^{|\beta|} \\ &\le C\, A_2 \cdot \eta^{m-|\alpha|} \, . \end{aligned}$$

Together with (57), this yields

$$|\partial^\alpha (Q'' - Q_0)(x_0)| \le C\, A_3 \eta^{m-|\alpha|}$$

for $|\alpha| \le m$. In particular, we have

$$|\partial^\alpha (Q'' - Q_0)(x_0)| \le A_4 \eta \text{ for } |\alpha| \le m - 1 \ , \tag{58}$$

and

$$|\partial^\alpha (Q'' - Q_0)(x_0)| \le A_4 \text{ for } |\alpha| = m \ , \tag{59}$$

since we may take $\eta \le 1$. From (56), we see that $|\partial^\alpha Q_0(x_0)| \le A_1$ for $|\alpha| \le m$. (See (1)(b).) Hence, (59) shows that

$$|\partial^\alpha Q''(x_0)| \le A_5 \text{ for } |\alpha| = m \ . \tag{60}$$

We are ready to apply Lemma 5.8, which tells us the following. There exist $\varepsilon, \delta > 0$ determined by $A_1, A_5, \tilde{k}, \bar{k}_2, x_0$, such that:

(61) If $Q_0 \in \Gamma_f(x_0, \bar{k}_2, A_1)$, $x'' \in E_1 \cap B(x_0, \delta)$, $Q'' \in f(x'') + I(x'')$, $|\partial^\alpha(Q'' - Q_0)(x_0)| \le \varepsilon$ for $|\alpha| \le m-1$, and $|\partial^\alpha Q''(x_0)| \le A_5$ for $|\alpha| = m$, then $Q'' \in \Gamma_f(x'', \tilde{k}, A_6)$.

Note that, since $x'' \in E_1$ and $|x_0 - x'|$, $|x' - x''| < \eta$, we have

$$x'' \in B(x_0, 2\eta) \cap E_1 \ . \tag{62}$$

Recall that we assumed that

$$Q'' \in f(x'') + I(x'') \ . \tag{63}$$

If we now pick $\eta \le 1$ to satisfy $A_4 \eta < \varepsilon$ and $2\eta < \delta$, then the hypotheses of (61) hold, thanks to (56), (62), (63), (58), and (60). Hence, (61) shows that $Q'' \in \Gamma_f(x'', \tilde{k}, A_6)$. The proof of Lemma 5.9 is complete. □

Lemma 5.10. *Suppose* $A_1, A_2 > 0$, $1+(D+1)\cdot\bar{k}_3 \le \bar{k}_2$, $1+(D+1)\cdot\bar{k}_2 \le \bar{k}_1$, $\bar{k}_1 \le k^\#$. *Then there exists* $\eta > 0$ *for which the following holds*: *Suppose* $x', x'' \in E_1$, *with* $|x' - x''| < \eta$. *Let* $Q' \in \Gamma_f(x', \bar{k}_1, A_1)$ *and* $Q'' \in f(x'') + I(x'')$, *with* $|\partial^\alpha(Q'' - Q')(x')| \le A_2\, \eta^{m-|\alpha|}$ *for* $|\alpha| \le m$. *Then* $Q'' \in \Gamma_f(x'', \bar{k}_3, A')$ *with* A' *determined by* A_1, A_2, m, n.

Proof. We say that an open ball $B(y, \bar{\eta})$ with center $y \in E_1$ is "useful" if the following holds: Given $x' \in B(y, \bar{\eta}) \cap E_1$, $x'' \in B(x', \bar{\eta}) \cap E_1$, $Q' \in \Gamma_f(x', \bar{k}_1, A_1)$, and $Q'' \in f(x'') + I(x'')$, if $|\partial^\alpha(Q'' - Q')(x')| \le A_2\, \bar{\eta}^{m-|\alpha|}$ for $|\alpha| \le m$, then $Q'' \in \Gamma_f(x'', \bar{k}_3, A')$, with A' as in Lemma 5.9 (with $\bar{k}_3$ in place of $\tilde{k}$).

Lemma 5.9 shows that every point of E_1 is the center of a useful ball. Since E_1 is compact, it is therefore covered by finitely many useful balls $B(y_1, \eta_1), \ldots, B(y_N, \eta_N)$. We take $\eta = \min\{\eta_1, \ldots, \eta_N\}$.

Now suppose x', x'', Q', Q'' are as in the hypotheses of Lemma 5.10, for the η we just picked. Since the balls $B(y_\nu, \eta_\nu)$ cover E_1, we have $x' \in B(y_\nu, \eta_\nu) \cap E_1$ for some ν. For that ν, we have also $x'' \in B(x', \eta_\nu) \cap E_1$, since $|x' - x''| < \eta \le \eta_\nu$. In addition, $Q' \in \Gamma_f(x', \bar{k}_1, A_1)$, $Q'' \in f(x'') + I(x'')$, and $|\partial^\alpha(Q'' - Q')(x')| \le A_2\, \eta^{m-|\alpha|} \le A_2\, \eta_\nu^{m-|\alpha|}$ for $|\alpha| \le m$, by hypothesis of Lemma 5.10. Since $B(y_\nu, \eta_\nu)$ is useful, it follows that $Q'' \in \Gamma_f(x'', \bar{k}_3, A')$. The proof of Lemma 5.10 is complete. □

6. A modulus of continuity

Let E, f, I etc. be as in Section 4. We again write c, C, C', etc., to denote controlled constants. Our goal in this section is to produce a regular modulus of continuity ω^+, and a large enough integer constant $\bar{k}$, for which the following holds:

(1) Given $x_1, \ldots, x_{\bar{k}} \in E_1$, there exist $P_1, \ldots, P_{\bar{k}} \in \mathcal{P}$, with $P_i \in \Gamma_f(x_i, \bar{k}, C) \subseteq f(x_i) + I(x_i)$ for $i = 1, \ldots, \bar{k}$; $|\partial^\alpha P_i(x_i)| \le C$ for $|\alpha| \le m$, $i = 1, \ldots, \bar{k}$; and $|\partial^\alpha (P_i - P_j)(x_j)| \le C\omega^+(|x_i - x_j|) \cdot |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $|x_i - x_j| \le 1$, $1 \le i, j \le \bar{k}$.

(See Lemma 6.6 below.)

Here, $\Gamma_f(x_i, \bar{k}, C)$ is the convex set defined in Section 5. Once we have achieved (1), we can appeal to the Generalized Sharp Whitney theorem to construct the function $\tilde{F}$ described in the introduction.

The first few lemmas below tell us that, roughly speaking, the small number δ in hypothesis (I) of Theorem 3 may be picked independently of x_0 and P_0. As before, let $D = \dim \mathcal{P}$.

Lemma 6.1. *Suppose $1 + (D+1)\bar{k} \le k^{\#}$. Let $x \in E$, $P \in f(x) + I(x)$, $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that for every $x' \in E \cap B(x, \delta)$, there exists $P' \in f(x') + I(x')$, with*

(2) $$|\partial^\alpha (P - P')(x)| \le \varepsilon |x - x'|^{m-|\alpha|} \text{ for } |\alpha| \le m\ ,$$

and satisfying the following condition:

(3) *Given $x'_0, x'_1, \ldots, x'_{\bar{k}} \in E \cap B(x, \delta)$ with $x'_0 = x'$, there exist $P'_0, \ldots, P'_{\bar{k}} \in \mathcal{P}$, with $P'_0 = P'$, and with $P'_i \in f(x'_i) + I(x'_i)$ for $i = 0, 1, \ldots, \bar{k}$; and $|\partial^\alpha (P'_i - P'_j)(x'_j)| \le \varepsilon |x'_i - x'_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le \bar{k}$.*

Proof. Recall that E, f, I are assumed to satisfy the hypotheses of Theorem 3. Let $\delta > 0$ be as in hypothesis (I) (with x, P in place of x_0, P_0), and let $x' \in E \cap B(x, \delta)$ be given. If $x' = x$, then we may set $P' = P$, and conclusions (2), (3) hold, thanks to hypothesis (I). Suppose $x' \ne x$. For any finite set $S \subset E \cap B(x, \delta)$ containing x and x', let $\mathcal{K}(S)$ denote the set of all $P' \in f(x') + I(x')$ such that there exists a map $y \mapsto P^y$ from S to $\mathcal{P}$, with $P^x = P$, $P^{x'} = P'$, $P^y \in f(y) + I(y)$ for $y \in S$, and $|\partial^\alpha (P^y - P^z)(z)| \le \varepsilon |y - z|^{m-|\alpha|}$ for $|\alpha| \le m$, $y, z \in S$. Each $\mathcal{K}(S)$ is a compact, convex subset of $\mathcal{P}$, which has dimension D. Moreover, suppose we are given $S_1, S_2, \ldots, S_{D+1} \subset E \cap B(x, \delta)$, each containing x and x', with $\#(S_i) \le \bar{k} + 2$ for each i. Then $S = S_1 \cup \cdots \cup S_{D+1} \subset E \cap B(x, \delta)$, with $x, x' \in S$, and $\#(S) \le 2 + (D+1)\bar{k} \le 1 + k^{\#}$. Hence, hypothesis (I) shows that there exists a map $y \mapsto P^y$ defined on S, with $P^x = P$, $P^y \in f(y) + I(y)$ for all $y \in S$, and $|\partial^\alpha (P^y - P^z)(z)| \le \varepsilon |y - z|^{m-|\alpha|}$ for $|\alpha| \le m$, $y, z \in S$.

We can check trivially that $P^{x'}$ then belongs to $\mathcal{K}(S_i)$ for each i. Thus, $\mathcal{K}(S_1), \ldots, \mathcal{K}(S_{D+1})$ have nonempty intersection. Consequently, Helly's theorem shows that there exists $P' \in f(x') + I(x')$, belonging to $\mathcal{K}(S)$ whenever $S \subset E \cap B(x,\delta)$, $x, x' \in S$, $\#(S) \le \bar{k}+2$. One checks easily, from the definition of $\mathcal{K}(S)$, that P' satisfies properties (2) and (3). The proof of Lemma 6.1 is complete. □

LEMMA 6.2. *Suppose* $1 + (D+1)\bar{k} \le k^{\#}$. *Let* $x \in E_1$ *and* $\varepsilon > 0$ *be given. Then there exists* $\delta > 0$ *such that for any* $x_0, \ldots, x_{\bar{k}} \in E \cap B(x,\delta)$ *with* $x_0 \in E_1$, *and for any* $P_0 \in f(x_0) + I(x_0)$, *there exist* $P_1, \ldots, P_{\bar{k}} \in \mathcal{P}$, *with*

(4) $P_i \in f(x_i) + I(x_i)$ *for* $i = 0, 1, \ldots, \bar{k}$; *and*

(5) $|\partial^\alpha (P_i - P_j)(x_j)| \le \varepsilon |x_i - x_j|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \le m} |\partial^\beta P_0(x_0)|)$ *for* $|\alpha| \le m$, $0 \le i, j \le \bar{k}$.

Proof. If $d = 0$ (see (5.7)), then there is only one $P_0 \in f(x_0) + I(x_0)$, and therefore Lemma 6.2 follows from Lemma 6.1. Suppose $d \ne 0$. Given $y \in E$, we define a norm on $\mathcal{P}$ by taking $\| P \|_y^2 = \sum_{|\alpha| \le m} (\partial^\alpha P(y))^2$. We write $\langle P, Q \rangle_y$ for the corresponding inner product. Fix $x \in E_1$ and $\varepsilon > 0$. Let $Q_1, \ldots, Q_d$ be an orthonormal basis for $I(x)$ with respect to the norm $\| \cdot \|_x$. If $P \in I(x)$, then we may write

$$P = \lambda_1 Q_1 + \cdots + \lambda_d Q_d, \text{ with } |\lambda_i| \le C \max_{|\beta| \le m} |\partial^\beta P(x)| \qquad \text{(all } i).$$

Also, hypothesis (II) of Theorem 3 (which is assumed to hold for E, f, I) shows that there exists

$$\hat{P}_0 \in f(x) + I(x) \;, \tag{6}$$

with

$$|\partial^\alpha \hat{P}_0(x)| \le 1 \text{ for } |\alpha| \le m \;. \tag{7}$$

We set $\hat{P}_i = \hat{P}_0 + Q_i$ for $i = 1, \ldots, d$. Thus,

$$\hat{P}_i \in f(x) + I(x) \text{ for } i = 0, 1, \ldots, d \;. \tag{8}$$

With $\varepsilon' < \varepsilon$ to be picked below, we apply Lemma 6.1 to each $\hat{P}_i$. Thus, we obtain $\delta' > 0$ for which the following holds: Given $x' \in E_1 \cap B(x, \delta')$, there exist $\tilde{P}_i \in f(x') + I(x')$ $(0 \le i \le d)$ satisfying

(9) $|\partial^\alpha (\tilde{P}_i - \hat{P}_i)(x)| \le \varepsilon' |x' - x|^{m-|\alpha|}$ for $|\alpha| \le m$, $\quad 0 \le i \le d$; and

(10) Given $x_0, \ldots x_{\bar{k}} \in \cap B(x, \delta')$ with $x_0 = x'$, there exist $P_i^0, \ldots, P_i^{\bar{k}} \in \mathcal{P}$ $(0 \le i \le d)$, with $P_i^0 = \tilde{P}_i (0 \le i \le d)$; $P_i^j \in f(x_j) + I(x_j)$ $(0 \le i \le d, 0 \le j \le \bar{k})$; and $|\partial^\alpha (P_i^j - P_i^\ell)(x_\ell)| \le \varepsilon' |x_j - x_\ell|^{m-|\alpha|}$ $(|\alpha| \le m;\ 0 \le i \le d;\ 0 \le j, \ell \le \bar{k})$.

Suppose $x' \in E_1 \cap B(x,\delta)$ with $\delta < \delta'$ to be picked below. Then we may pick $\tilde{P}_i \in f(x') + I(x')(0 \le i \le d)$ satisfying (9) and (10). Note that, since $x, x' \in E_1$, we have $\dim I(x) = \dim I(x') = d$. Note also that

$$\langle (\hat{P}_i - \hat{P}_0), (\hat{P}_{i'} - \hat{P}_0)\rangle_x = \delta_{ii'} \text{ for } 1 \le i, i' \le d\ ,$$

by definition of the $\hat{P}_i$. (Here, $\delta_{ii'}$ denotes the Kronecker delta.) In view of (9), this implies that

$$\text{(11)} \qquad |\langle (\tilde{P}_i - \tilde{P}_0), (\tilde{P}_{i'} - \tilde{P}_0)\rangle_x - \delta_{ii'}| \le C\varepsilon' \text{ for } 1 \le i, i' \le d\ .$$

If δ is small enough, then (11) implies

$$\text{(12)} \qquad |\langle (\tilde{P}_i - \tilde{P}_0), (\tilde{P}_{i'} - \tilde{P}_0)\rangle_{x'} - \delta_{ii'}| \le C'\varepsilon' \text{ for } 1 \le i, i' \le d\ ,$$

since $x' \in B(x,\delta)$.

Note also that (7), (9) give $|\partial^\alpha \tilde{P}_0(x)| \le 1 + \varepsilon'(|\alpha| \le m)$, if $\delta \le 1$. Hence, if δ is small enough, we have

$$\text{(13)} \qquad |\partial^\alpha \tilde{P}_0(x')| \le 2 \text{ for } |\alpha| \le m\ .$$

Once ε' is determined, we fix $\delta < \delta'$ to be small enough that (12) and (13) hold. We have still not fixed ε'. We recall that $\tilde{P}_0, \dots, \tilde{P}_d \in f(x') + I(x')$, and that $\dim I(x') = d$. Hence, if ε' is small enough, then (12) shows that any $P \in I(x')$ may be expressed in the form

$$\text{(14)} \qquad P = \mu_1(\tilde{P}_1 - \tilde{P}_0) + \cdots + \mu_d(\tilde{P}_d - \tilde{P}_0) \text{ with } |\mu_i| \le C \max_{|\beta| \le m} |\partial^\beta P(x')|\ .$$

Together with (13), this implies the following result.

(15) Any $P' \in f(x') + I(x')$ may be expressed in the form $P' = \lambda_0 \tilde{P}_0 + \cdots + \lambda_d \tilde{P}_d$, with $\lambda_0 + \cdots + \lambda_d = 1$, and $|\lambda_i| \le C \cdot (1 + \max_{|\beta| \le m} |\partial^\beta P'(x')|)$ for $i = 0, \dots, d$. (To prove (15), we just apply (14) to $P' - \tilde{P}_0$.)

Now suppose we are given $P' \in f(x') + I(x')$, as well as $x_0, \dots, x_{\bar{k}} \in E \cap B(x,\delta)$ with $x_0 = x'$. We express P' in the form (15), and let P_i^j ($0 \le i \le d$, $0 \le j \le \bar{k}$) be as in (10). Now,

$$\text{(16)} \qquad P^j = \lambda_0 P_0^j + \cdots + \lambda_d P_d^j \in \mathcal{P} \text{ for } 0 \le j \le \bar{k}\ .$$

In particular,

$$P^0 = \lambda_0 P_0^0 + \cdots + \lambda_d P_d^0 = \lambda_0 \tilde{P}_0 + \cdots + \lambda_d \tilde{P}_d \text{ (see (10))} = P' \text{ (see (15))}.$$

Also, since $P_i^j \in f(x_j) + I(x_j)$ and $\lambda_0 + \cdots + \lambda_d = 1$, (16) gives $P^j \in f(x_j) + I(x_j)$ for $0 \le j \le \bar{k}$. Moreover, (10), (15), (16) show that

$$\begin{aligned} |\partial^\alpha(P^j - P^\ell)(x_\ell)| &\le \sum_{i=0}^{d} |\lambda_i| \cdot |\partial^\alpha(P_i^j - P_i^\ell)(x_\ell)| \\ &\le C \cdot (1 + \max_{|\beta| \le m} |\partial^\beta P'(x')|) \cdot \varepsilon' |x_j - x_\ell|^{m-|\alpha|} \end{aligned}$$

for $|\alpha| \le m$, $0 \le j, \ell \le \bar{k}$.

If $C\varepsilon' \le \varepsilon$, then

$$|\partial^\alpha(P^j - P^\ell)(x_\ell)| \le \varepsilon |x_j - x_\ell|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\,\partial^\beta P'(x')|) \ .$$

We now fix $\varepsilon' > 0$ small enough that the above arguments work. This in turn fixes δ' and δ. We have now proven the following result.

Let $\varepsilon > 0$ and $x \in E_1$. Then there exists $\delta > 0$ such that for any $x' \in E_1 \cap B(x,\delta)$, any $P' \in f(x') + I(x')$, and any $x_0, \dots, x_{\bar{k}} \in E \cap B(x,\delta)$ with $x_0 = x'$, there exist $P^0, \dots, P^{\bar{k}} \in \mathcal{P}$, with $P^0 = P'$; $P^j \in f(x_j) + I(x_j)$ for $0 \le j \le \bar{k}$; and

$$|\partial^\alpha(P^j - P^\ell)(x_\ell)| \le \varepsilon |x_j - x_\ell|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P'(x')|)$$

for $|\alpha| \le m$, $0 \le j, \ell \le \bar{k}$. This statement is obviously equivalent to Lemma 6.2. $\square$

LEMMA 6.3. *Suppose* $\bar{k} \ge 1$, $1 + (D+1)\cdot \bar{k} \le k^\#$. *Then, given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for any* $x_0 \in E_1$, *any* $P_0 \in f(x_0) + I(x_0)$, *and any* $x_1, \dots, x_{\bar{k}} \in E \cap B(x_0, \delta)$, *there exist* $P_1, \dots, P_{\bar{k}} \in \mathcal{P}$, *with*

(17) $$P_i \in f(x_i) + I(x_i) \text{ for } i = 0, 1, \dots, \bar{k}\ ; \text{ and}$$

(18) $$|\partial^\alpha(P_i - P_j)(x_j)| \le \varepsilon |x_i - x_j|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|)$$

for $|\alpha| \le m$, $0 \le i, j \le \bar{k}$.

Proof. Let us say that an open ball $B(y,\delta)$ is "useful" if, for any $x_0, \dots, x_{\bar{k}} \in E \cap B(y, 2\delta)$ with $x_0 \in E_1$, and for any $P_0 \in f(x_0) + I(x_0)$, there exist $P_1, \dots, P_{\bar{k}} \in \mathcal{P}$, satisfying (17) and (18). Lemma 6.2 shows that every point of E_1 is the center of a useful ball. Since E_1 is compact, it is covered by finitely many useful balls $B(y_\nu, \delta_\nu)$ $(\nu = 1, \dots, N)$.

We take $\delta = \min\{\delta_1, \dots, \delta_N\}$. Suppose we are given $x_0 \in E_1$, $P_0 \in f(x_0) + I(x_0)$, and $x_1, \dots, x_{\bar{k}} \in E \cap B(x_0, \delta)$. Then $x_0 \in B(y_\mu, \delta_\mu)$ for some μ, since the $B(y_\nu, \delta_\nu)$ cover E_1. Consequently, $x_0, x_1, \dots, x_{\bar{k}} \in B(y_\mu, 2\delta_\mu)$, as $\delta \le \delta_\mu$. Since $B(y_\mu, \delta_\mu)$ is useful, there exist $P_1, \dots, P_{\bar{k}} \in \mathcal{P}$ satisfying (17) and (18). Thus, Lemma 6.3 holds. $\square$

COROLLARY. *Suppose* $k^\# \ge D + 2$. *Then, given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, given any* $x_0, x_1 \in E_1$ *with* $|x_0 - x_1| < \delta$, *and given any* $P_0 \in f(x_0) + I(x_0)$, *there exists* $P_1 \in f(x_1) + I(x_1)$, *with*

$$|\partial^\alpha(P_1 - P_0)(x_i)| \le \varepsilon |x_0 - x_1|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \text{ for } |\alpha| \le m, i = 0, 1\ .$$

The corollary is an immediate consequence of the case $\bar{k} = 1$ of Lemma 6.3. Exploiting the above corollary, we can now prove the following result.

LEMMA 6.4. *Suppose $k^{\#} \geq D+2$. Then there exist a positive number $\delta_0 < 1$, and a regular modulus of continuity ω, for which the following holds: Given $x, x' \in E_1$ with $|x - x'| \leq \delta_0$, and given $P \in f(x) + I(x)$, there exists $P' \in f(x') + I(x')$, with*

$$|\partial^\alpha(P' - P)(x)| \leq \omega(|x - x'|) \cdot |x - x'|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P(x)|) \text{ for } |\alpha| \leq m .$$

Proof. Set $\varepsilon_\nu = 2^{-\nu}$ for $\nu = 0, 1, 2, \ldots$. By the corollary to Lemma 6.3, we may pick successively $\delta_0, \delta_1, \delta_2, \ldots$ with the following properties:

(19) $\delta_0 = 1$.

(20) $0 < \delta_{\nu+1} < \frac{1}{2}\delta_\nu$.

(21) If $\nu \geq 1$, then given $x, x' \in E_1$ with $|x - x'| \leq \delta_\nu$, and given $P \in f(x) + I(x)$, there exists $P' \in f(x') + I(x')$, with

$$|\partial^\alpha(P' - P)(x)| \leq \frac{1}{2}\varepsilon_\nu |x' - x|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P(x)|) \text{ for } |\alpha| \leq m .$$

Now define $\omega(t)$ on $[0,1]$ by setting

(22) $$\omega(0) = 0, \omega(\delta_\nu) = \varepsilon_\nu, \omega(t) \text{ linear on each } [\delta_{\nu+1}, \delta_\nu], \quad \nu \geq 0 .$$

It is routine to check that $\omega(t)$ is a regular modulus of continuity. (In particular, to see that $\omega(t)/t$ is decreasing, one checks that $\omega(t)/t = A_\nu + B_\nu/t$ on $[\delta_{\nu+1}, \delta_\nu]$, with $B_\nu > 0$ thanks to (20).)

Now suppose $x, x' \in E_1$, with $0 < |x - x'| \leq \delta_1$, and suppose $P \in f(x) + I(x)$. Pick $\nu \geq 1$ so that $\delta_{\nu+1} < |x - x'| \leq \delta_\nu$. Then, by (21), there exists $P' \in f(x') + I(x')$ with

(23) $$|\partial^\alpha(P' - P)(x)| \leq \tfrac{1}{2}\varepsilon_\nu |x' - x|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P(x)|) \text{ for } |\alpha| \leq m.$$

On the other hand, since $\delta_{\nu+1} < |x' - x|$, we have $\omega(|x' - x|) \geq \omega(\delta_{\nu+1}) = \varepsilon_{\nu+1} = \frac{1}{2}\varepsilon_\nu$. Therefore, (23) gives

(24) $$\begin{aligned} |\partial^\alpha(P' - P)(x)| \leq\, & \omega(|x' - x|) \cdot |x - x'|^{m-|\alpha|} \\ & \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P(x)|) \text{ for } |\alpha| \leq m . \end{aligned}$$

The above argument omits the case $x' = x$. However, in that trivial case, we can just put $P' = P \in f(x') + I(x')$. Thus, given $x, x' \in E_1$ with $|x - x'| \leq \delta_1$, and given $P \in f(x) + I(x)$, there exists $P' \in f(x') + I(x')$ satisfying (24). The proof of Lemma 6.4 is complete. □

Now we bring our clustering lemma (Lemma 3.1) into play.

LEMMA 6.5. *Suppose $k^{\#} \geq D+2$, and let ω, δ_0 be as in Lemma 6.4. Then, given any $\bar{k} \geq 1$, there exists a controlled constant $\hat{C}_{\bar{k}}$, for which the following holds: Let $x_0 \in S \subseteq E_1$, with* $\mathrm{diam}(S) \leq \delta_0$ *and* $\#(S) \leq \bar{k}$. *Then, given $P_0 \in f(x_0) + I(x_0)$, there exists a map $x \mapsto P^x$ from S to $\mathcal{P}$, with*

$$P^{x_0} = P_0 \; ; \tag{25}$$

$$P^x \in f(x) + I(x) \text{ for all } x \in S \; ; \tag{26}$$

$$(1 + \max_{|\beta| \leq m} |\partial^\beta P^x(x)|) \leq \hat{C}_{\bar{k}} \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P_0(x_0)|) \text{ for all } x \in S \; ; \tag{27}$$

and

$$\begin{aligned} |\partial^\alpha (P^x - P^y)(y)| \leq & \hat{C}_{\bar{k}} \cdot \omega(|x-y|) \\ & \cdot |x-y|^{m-|\alpha|} (1 + \max_{|\beta| \leq m} |\partial^\beta P_0(x_0)|) \text{ for } |\alpha| \leq m, x, y \in S \; . \end{aligned} \tag{28}$$

Proof. We use induction on $\bar{k}$. If $\bar{k} = 1$, then $S = \{x_0\}$, and we may just set $P^{x_0} = P_0$. Conditions (25)–(28) trivially hold, with $\hat{C}_1 = 1$.

Next, fix $\bar{k} \geq 2$, and suppose Lemma 6.5 holds, with a controlled constant $\hat{C}_{\bar{k}-1}$, whenever $\#(S) \leq \bar{k} - 1$. Let x_0, S, P_0 be as in the hypotheses of Lemma 6.5, with $\#(S) = \bar{k}$. Applying Lemma 3.1, we may partition S into $S_0, \ldots, S_M$, with

$$\#(S_\ell) \leq \bar{k} - 1 \text{ for each } \ell (0 \leq \ell \leq M) \; , \text{ and} \tag{29}$$

$$\mathrm{dist}(S_\ell, S_{\ell'}) > c_{\bar{k}} \cdot \mathrm{diam}(S) \text{ for } \ell \neq \ell' \; . \tag{30}$$

Without loss of generality, we may suppose that $x_0 \in S_0$, and that each S_ℓ is nonempty. For each $\ell = 1, \ldots, M$, fix an $x_\ell \in S_\ell$. Note that, for $1 \leq \ell \leq M$, we have $|x_\ell - x_0| \leq \mathrm{diam}(S) \leq \delta_0$. Hence, by Lemma 6.4, there exist polynomials $P_1, \ldots, P_M \in \mathcal{P}$, with

$$P_\ell \in f(x_\ell) + I(x_\ell) \text{ for } \ell = 1, \ldots, M \text{ (and of course also for } \ell = 0\text{), and} \tag{31}$$

$$|\partial^\alpha (P_\ell - P_0)(x_0)| \leq \omega(|x_\ell - x_0|) \cdot |x_\ell - x_0|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P_0(x_0)|) \tag{32}$$

for $|\alpha| \leq m$, $1 \leq \ell \leq M$. Set

$$\delta = \mathrm{diam}(S) \; . \tag{33}$$

From Lemma 6.4 we have

$$\delta \leq \delta_0 < 1 \; , \tag{34}$$

hence (32) yields

$$\begin{aligned} |\partial^\alpha (P_\ell - P_0)(x_0)| \leq & \omega(\delta) \cdot \delta^{m-|\alpha|} \\ & \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P_0(x_0)|), \text{ for } |\alpha| \leq m, 1 \leq \ell \leq M \; . \end{aligned} \tag{35}$$

This in turn implies that

$$|\partial^\alpha P_\ell(x_0)| \le C \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \text{ for } |\alpha| \le m \ , 1 \le \ell \le M \ .$$

Since $|x_\ell - x_0| \le \delta \le 1$ by (33) and (34), it follows that

$$(36) \quad (1 + \max_{|\beta|\le m} |\partial^\beta P_\ell(x_\ell)|) \le C' \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \text{ for } \ell = 1, \dots, M \ .$$

Now, for each $\ell (0 \le \ell \le M)$, we apply our induction hypothesis (Lemma 6.5 for $\#(S) \le \bar{k} - 1$), with x_ℓ, S_ℓ in place of x_0, S. Note that the induction hypothesis applies, thanks to (29). Thus on each S_ℓ, we obtain a map $x \mapsto P^x \in \mathcal{P}$, with

$$(37) \quad P^{x_\ell} = P_\ell \ ,$$

$$(38) \quad P^x \in f(x) + I(x) \text{ for } x \in S_\ell \ ,$$

$$(39) \quad (1 + \max_{|\beta|\le m} |\partial^\beta P^x(x)|) \le \hat{C}_{\bar{k}-1} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_\ell(x_\ell)|) \text{ for } x \in S_\ell \ ,$$

and

(40)
$$|\partial^\alpha (P^x - P^y)(y)| \le \hat{C}_{\bar{k}-1} \cdot \omega(|x-y|) \cdot |x-y|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_\ell(x_\ell)|)$$

for $|\alpha| \le m$, $x, y \in S_\ell$.

Since $S_0, S_1, \dots, S_M$ form a partition of S, the above maps $x \mapsto P^x$ may be combined into a single map $x \mapsto P^x$, defined on S. From (37) and (38), we have

$$(41) \quad P^{x_0} = P_0, \text{ and}$$

$$(42) \quad P^x \in f(x) + I(x) \text{ for all } x \in S \ .$$

From (36) and (39), we obtain the estimate

$$(43) \quad (1 + \max_{|\beta|\le m} |\partial^\beta P^x(x)|) \le C' \hat{C}_{\bar{k}-1} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \text{ for } x \in S \ .$$

Also, (36) and (40) show that

(44)
$$|\partial^\alpha (P^x - P^y)(y)| \le C' \hat{C}_{\bar{k}-1} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \cdot \omega(|x-y|) \cdot |x-y|^{m-|\alpha|}$$

whenever x and y belong to the same S_ℓ.

Suppose instead that $x \in S_\ell$ and $y \in S_{\ell'}$, with $\ell' \ne \ell$. From (36) and (40), we have

(45)
$$\begin{aligned} |\partial^\alpha (P^x - P_\ell)(x)| &\le C' \hat{C}_{\bar{k}-1} \cdot \omega(|x-x_\ell|) \cdot |x-x_\ell|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \\ &\le C' \hat{C}_{\bar{k}-1} \cdot \omega(\delta) \cdot \delta^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \end{aligned}$$

and

(46)
$$|\partial^\alpha(P^y - P_{\ell'})(y)| \le C'\hat{C}_{\bar{k}-1} \cdot \omega(|y - x_{\ell'}|) \cdot |y - x_{\ell'}|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|)$$
$$\le C'\hat{C}_{\bar{k}-1} \cdot \omega(\delta)\delta^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \text{ for } |\alpha| \le m\ .$$

Since $|x-y|, |x_0 - y| \le \delta$ by (33), estimates (45) and (35) (for ℓ and ℓ') imply

$$(47)\quad |\partial^\alpha(P^x - P_\ell)(y)| \le C'' \cdot \hat{C}_{\bar{k}-1} \cdot \omega(\delta) \cdot \delta^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \text{ and}$$

$$(48)\quad |\partial^\alpha(P_{\ell'} - P_\ell)(y)| \le C''\omega(\delta) \cdot \delta^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|),\ \text{for } |\alpha| \le m\ .$$

Summing (46), (47), (48), we find that

$$(49)\quad |\partial^\alpha(P^x - P^y)(y)| \le \left[C''' \cdot \hat{C}_{\bar{k}-1} + C'''\right]$$
$$\cdot\omega(\delta)\delta^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|) \text{ for } |\alpha| \le m\ .$$

Moreover, since $x \in S_\ell$ and $y \in S_{\ell'}$ with $\ell \ne \ell'$, (30) gives $|x-y| \ge c_{\bar{k}} \cdot \delta$. Since ω is a regular modulus of continuity, it follows that $\omega(|x-y|) \ge \omega(c_{\bar{k}}\cdot\delta) \ge c_{\bar{k}} \cdot \omega(\delta)$. Putting these remarks into (49), we conclude that

$$(50)\qquad |\partial^\alpha(P^x - P^y)(y)| \le \tilde{C}\left[\hat{C}_{\bar{k}-1} + 1\right] \cdot \omega(|x-y|)$$
$$\cdot|x-y|^{m-|\alpha|} \cdot (1 + \max_{|\beta|\le m} |\partial^\beta P_0(x_0)|)$$

for $|\alpha| \le m$, provided x and y do not both belong to the same S_ℓ.

In view of (41)–(44) and (50), we see that Lemma 6.5 holds for $\#(S) = \bar{k}$, with a suitable controlled constant $\hat{C}_{\bar{k}}$. This completes the induction step, and with it the proof of Lemma 6.5. □

LEMMA 6.6. *Suppose*

$$(51)\qquad (\bar{k}_1 D + 2) \cdot \bar{k}_1 \le k^\#, \quad 1 + (D+1) \cdot \bar{k}_2 \le \bar{k}_1, \quad 1 + (D+1) \cdot \bar{k}_3 \le \bar{k}_2\ .$$

Then there exists a regular modulus of continuity ω^+, for which the following holds. Given $S \subset E_1$ with $\#(S) \le \bar{k}_3$, there exists a map $x \mapsto P^x$ from S into $\mathcal{P}$, with

$$(52)\qquad P^x \in \Gamma_f(x, \bar{k}_3, C) \text{ for each } x \in S\ ;$$

$$(53)\qquad |\partial^\alpha P^x(x)| \le C \text{ for each } x \in S, \qquad |\alpha| \le m\ ; \text{ and}$$

$$(54)\qquad |\partial^\alpha(P^x - P^y)(y)| \le C\omega^+(|x-y|) \cdot |x-y|^{m-|\alpha|}$$

for $x, y \in S, |x-y| \le 1, |\alpha| \le m$.

Proof. Let ω, δ_0 be as in Lemma 6.4, let δ_1 be a small positive number to be picked later, and define

(55) $$\omega^+(t) = \omega(t)/\omega(\delta_1) \text{ if } 0 \le t \le \delta_1; \quad \omega^+(t) = 1 \text{ if } \delta_1 \le t \le 1 \;.$$

This makes sense for

(56) $$\delta_1 < 1 \;,$$

and one checks trivially that ω^+ is a regular modulus of continuity.

Suppose $S \subset E_1$, with $\#(S) \le \bar{k}_3$. By the clustering Lemma 3.2, we may partition S into subsets $S_1, \dots, S_L$, with

(57) $$\text{diam}(S_\ell) \le \delta_1 \text{ for } \ell = 1, \dots, L \;; \text{ and}$$

(58) $$\text{dist}(S_\ell, S_{\ell'}) > c\delta_1 \text{ for } \ell \ne \ell', 1 \le, \ell' \le L \;.$$

We may assume that each S_ℓ is nonempty. We pick some

(59) $$y_\ell \in S_\ell \text{ for each } \ell = 1, \dots, L \;,$$

and we define

(60) $$S_{\text{rep}} = \{y_1, \dots, y_L\} \subseteq S \subseteq E_1 \;.$$

From (60), we have $\#(S_{\text{rep}}) \le \#(S) \le \bar{k}_3 \le \bar{k}_1$ (see (51)), hence Lemma 5.4 gives us polynomials $P_1, \dots, P_L \in \mathcal{P}$ with the following properties.

(61) $$P_\ell \in \Gamma_f(y_\ell, \bar{k}_1, 1) \subseteq f(y_\ell) + I(y_\ell) \text{ for } 1 \le \ell \le L \;.$$

(62) $$|\partial^\alpha P_\ell(y_\ell)| \le 1 \text{ for } |\alpha| \le m, 1 \le \ell \le L \;.$$

(63) $$|\partial^\alpha (P_\ell - P_{\ell'})(y_{\ell'})| \le |y_\ell - y_{\ell'}|^{m-|\alpha|} \text{ for } |\alpha| \le m, 1 \le \ell, \ell' \le L \;.$$

For fixed ℓ, we have $y_\ell \in S_\ell \subseteq E_1$ with $\#(S_\ell) \le \bar{k}_3$ and $\text{diam}(S_\ell) \le \delta_1$. If we make sure that

(64) $$\delta_1 < \delta_0 \;,$$

then Lemma 6.5 applies, with $\bar{k}_3$ in place of $\bar{k}$.

Note that the constant called $\hat{C}_{\bar{k}}$ in Lemma 6.5 is controlled, since $\bar{k}_3 \le k^\#$, and $k^\#$ depends only on m and n. Hence, we obtain a map $x \mapsto P^x$, from S_ℓ into $\mathcal{P}$, with the following properties.

(65) $$P^{y_\ell} = P_\ell \;.$$

(66) $$P^x \in f(x) + I(x) \text{ for all } x \in S_\ell \;.$$

(67) $$|\partial^\alpha P^x(x)| \le C \cdot (1 + \max_{|\beta| \le m} |\partial^\beta P_\ell(y_\ell)|) \text{ for } x \in S_\ell, |\alpha| \le m \;.$$

(68) $$|\partial^\alpha (P^x - P^{x'})(x')| \le C\omega(|x - x'|) \cdot |x - x'|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \le m} |\partial^\beta P_\ell(y_\ell)|)$$

for $|\alpha| \le m$, $x, x' \in S_\ell$.

Putting (62) into (67) and (68), we find that

$$|\partial^\alpha P^x(x)| \le C_1 \text{ for } x \in S_\ell, |\alpha| \le m; \text{ and} \tag{69}$$

$$|\partial^\alpha (P^x - P^{x'})(x')| \le C_1 \omega(|x-x'|) \cdot |x-x'|^{m-|\alpha|} \text{ for } |\alpha| \le m, x, x' \in S_\ell \; . \tag{70}$$

Next, fix $\bar{x} \in S_\ell$. We prepare to apply Lemma 5.10, with $A_1 = 1, A_2 = 1$, $x' = y_\ell, x'' = \bar{x}$, $Q' = P_\ell, Q'' = P^{\bar{x}}$. We check that the hypotheses of that lemma hold here. In fact, (51) tells us that $\bar{k}_1, \bar{k}_2, \bar{k}_3$ are as in Lemma 5.10. Also, $y_\ell, \bar{x} \in S_\ell \subseteq S \subseteq E_1$, hence $|y_\ell - \bar{x}| \le \operatorname{diam}(S_\ell) \le \delta_1 < \eta$, provided we take

$$\delta_1 < \eta \; , \tag{71}$$

with η as in Lemma 5.10 for $A_1 = A_2 = 1$, and for our $\bar{k}_1, \bar{k}_2, \bar{k}_3$. Also, $P_\ell \in \Gamma_f(y_\ell, \bar{k}_1, 1)$ (see (61)), and $P^{\bar{x}} \in f(\bar{x}) + I(\bar{x})$ (see (66)). Finally, (70) and (65) show that

$$|\partial^\alpha (P^{\bar{x}} - P_\ell)(y_\ell)| \le C_1 \omega(\delta_1) \cdot |\bar{x} - y_\ell|^{m-|\alpha|} \le |\bar{x} - y_\ell|^{m-|\alpha|}$$

for $|\alpha| \le m$, provided δ_1 is so small that

$$C_1 \omega(\delta_1) \le 1 \; . \tag{72}$$

We now pick $\delta_1 > 0$ to satisfy (56), (64), (71), (72). Thus, as claimed, the hypotheses of Lemma 5.10 hold here. Applying that lemma, we learn that $P^{\bar{x}} \in \Gamma_f(\bar{x}, \bar{k}_3, C)$. Thus,

$$P^x \in \Gamma_f(x, \bar{k}_3, C) \text{ for all } x \in S_\ell \; . \tag{73}$$

We recall that S is partitioned into $S_1, \dots, S_L$, and that we have defined a map $x \mapsto P^x$ from each S_ℓ into $\mathcal{P}$. We may therefore combine these maps on the S_ℓ into a single map $x \mapsto P^x$ defined on all of S. We will check that this map satisfies the conclusions of Lemma 6.6. In fact, (73) shows that

$$P^x \in \Gamma_f(x, \bar{k}_3, C) \text{ for all } x \in S \; , \tag{74}$$

and (69) shows that

$$|\partial^\alpha P^x(x)| \le C \text{ for } |\alpha| \le m, x \in S \; . \tag{75}$$

To complete the proof of Lemma 6.6, it remains to prove (54). If x and y belong to the same S_ℓ, then we have $|x-y| \le \operatorname{diam}(S_\ell) \le \delta_1$; hence $\omega(|x-y|) \le \omega^+(|x-y|)$ (see (55) and (57)), and therefore (54) follows from (70).

On the other hand, suppose $x \in S_\ell, y \in S_{\ell'}$ with $\ell \ne \ell'$. Then (58) gives $|x-y| \ge c\delta_1$, and therefore $\omega^+(|x-y|) \ge \omega^+(c\delta_1) \ge c\omega^+(\delta_1) = c$, by virtue of (55) and the fact that ω^+ is a regular modulus of continuity. Thus, to prove Lemma 6.6, it is enough to show that

$$|\partial^\alpha (P^x - P^y)(y)| \le C|x-y|^{m-|\alpha|} \text{ for } |\alpha| \le m, x \in S_\ell, y \in S_{\ell'}, \ell \ne \ell' \; . \tag{76}$$

Fixing $x \in S_\ell, y \in S_{\ell'}, \ell \neq \ell'$, we have

(77) $$|x - y_\ell| \leq \delta_1, \quad |y - y_{\ell'}| \leq \delta_1, \quad |x - y| \geq c\delta_1 \ ,$$

thanks to (57), (58), (59). Also,

(78)
$$|\partial^\alpha (P^x - P_\ell)(x)|, |\partial^\alpha (P^y - P_{\ell'})(y)| \leq C\delta_1^{m-|\alpha|} \leq C'|x-y|^{m-|\alpha|} \text{ for } |\alpha| \leq m \ ,$$

by (70) and (77). These estimates imply

(79) $$|\partial^\alpha (P^x - P_\ell)(y)|, \quad |\partial^\alpha (P^y - P_{\ell'})(y)| \leq C''|x-y|^{m-|\alpha|} \text{ for } |\alpha| \leq m \ .$$

Since also $|y_\ell - y_{\ell'}| \leq C|x-y|$ by (77), we obtain from (63) the estimates

(80) $$|\partial^\alpha (P_\ell - P_{\ell'})(y_{\ell'})| \leq C|x-y|^{m-|\alpha|} \text{ for } |\alpha| \leq m \ .$$

We have $|y - y_{\ell'}| \leq C|x-y|$ by (77); hence (80) implies

(81) $$|\partial^\alpha (P_\ell - P_{\ell'})(y)| \leq C'|x-y|^{m-|\alpha|} \text{ for } |\alpha| \leq m \ .$$

The desired estimate (76) is immediate from (79) and (81), and the proof of Lemma 6.6 is complete. □

7. Picking the constant $k^\#$

From the Generalized Sharp Whitney theorem and the setup for the main induction, we recall the constants $k^\#_{\text{GSW}}$ and $k^\#_{\text{old}}$. (See Sections 1 and 4.) These constants have already been picked, and they depend only on m and n.

We now fix constants $\bar{k}_1, \bar{k}_2, \bar{k}_3, k^\#$, depending only on m and n, so that the following conditions are satisfied.

(1) $$\bar{k}_3 \geq k^\#_{\text{old}} + 5 \ .$$

(2) $$\bar{k}_3 \geq k^\#_{\text{GSW}} + 5 \ .$$

(3) $$\bar{k}_2 \geq 1 + (D+1) \cdot \bar{k}_3 \ .$$

(4) $$\bar{k}_1 \geq 1 + (D+1) \cdot \bar{k}_2 \ .$$

(5) $$k^\# \geq (\bar{k}_1 D + 2) \cdot \bar{k}_1 \ .$$

8. Constructing the auxiliary function

As before, we suppose E, f, I, etc. are as in Section 4; and we write c, C, C', etc. to denote controlled constants. Our goal in this section is to carry out Step 1 of the proof of Theorem 3, as explained in the introduction.

Comparing estimates (51) in Section 6 with our choice of $\bar{k}_1, \bar{k}_2, \bar{k}_3, k^\#$ in Section 7, we see that Lemma 6.6 applies to E, f, I. Let ω^+ be the regular modulus of continuity given by Lemma 6.6.

Thus, given $S \subset E_1$ with $\#(S) \leq \bar{k}_3$, there exists a map $x \mapsto P^x$ from S to $\mathcal{P}$, with

(1) $$P^x \in \Gamma_f(x, \bar{k}_3, C) \text{ for each } x \in S\ ;$$

(2) $$|\partial^\alpha P^x(x)| \leq C \text{ for } |\alpha| \leq m, x \in S; \text{ and}$$

(3) $$|\partial^\alpha (P^x - P^y)(y)| \leq C\omega^+(|x-y|) \cdot |x-y|^{m-|\alpha|} \text{ for } |\alpha| \leq m, x, y \in S, |x-y| \leq 1\ .$$

In particular, taking $S = \{x\}$ for $x \in E_1$, we obtain from (1) that $\Gamma_f(x, \bar{k}_3, C)$ is nonempty for every $x \in E_1$. Pick

(4) $$g(x) \in \Gamma_f(x, \bar{k}_3, C) \text{ for each } x \in E_1\ .$$

Then Lemma 5.1 shows that $\Gamma_f(x, \bar{k}_3, C) \subseteq g(x) + C'\sigma(x, \bar{k}_3)$ for $x \in E_1$. Hence, (1), (2), (3) imply the following.

(5) Given $S \subset E_1$ with $\#(S) \leq \bar{k}_3$, there exists a map $x \mapsto P^x$ from S into $\mathcal{P}$, with

(a) $P^x \in g(x) + C'\sigma(x, \bar{k}_3)$ for $x \in S$;

(b) $|\partial^\alpha P^x(x)| \leq C'$ for $|\alpha| \leq m, x \in S$;

(c) $|\partial^\alpha (P^x - P^y)(y)| \leq C'\omega^+(|x-y|) \cdot |x-y|^{m-|\alpha|}$ for $|\alpha| \leq m$, $x, y \in S$, $|x-y| \leq 1$.

Also, Lemma 5.3 tells us that

(6) For each $x \in E_1$, the set $\sigma(x, \bar{k}_3)$ is Whitney convex, with Whitney constant C''.

Recall from Section 7 that $\bar{k}_3 \geq k^{\#}_{\mathrm{GSW}}$. Hence, (5) and (6) show that the hypotheses of the Generalized Sharp Whitney theorem are satisfied, with our present $\omega^+, E_1, g(x)/C'$, $\sigma(x, \bar{k}_3)$, C'', in place of $\omega, E, f(x), \sigma(x), A_0$. Hence, the Generalized Sharp Whitney theorem produces a function $\tilde{F} \in C^{m,\omega^+}(\mathbb{R}^n)$, with

(7) $$\| \tilde{F} \|_{C^{m,\omega^+}(\mathbb{R}^n)} \leq C''', \text{ and}$$

(8) $$J_x(\tilde{F}) \in g(x) + C'''\sigma(x, \bar{k}_3) \text{ for all } x \in E_1\ .$$

In particular, (7) implies

(9) $$\| \tilde{F} \|_{C^m(\mathbb{R}^n)} \leq C'''\ ,$$

and (4), (8) and Lemma 5.1 yield

(10) $$J_x(\tilde{F}) \in \Gamma_f(x, \bar{k}_3, \tilde{C}) \text{ for all } x \in E_1\ .$$

Thus, we have proven the following result, completing Step 1 from the introduction.

LEMMA 8.1. *There exists* $\tilde{F} \in C^m(\mathbb{R}^n)$, *with* $\| \tilde{F} \|_{C^m(\mathbb{R}^n)} \leq C$, *and* $J_x(\tilde{F}) \in \Gamma_f(x, \bar{k}_3, C)$ *for all* $x \in E_1$.

9. Rescaling the induction hypothesis

Recall that we are assuming that Theorem 3 holds when the number of strata is less than $\wedge$. After an obvious rescaling, we obtain the following result.

LEMMA 9.1 (Rescaled Induction Hypothesis). *Let* $\tilde{\delta} > 0$, *and let* $E \subseteq \mathbb{R}^n$ *be compact. Suppose that for each* $x \in E$ *we are given an* m*-jet* $f(x) \in \mathcal{R}_x$ *and an ideal* $I(x) \subset \mathcal{R}_x$. *Assume that the following conditions are satisfied.*

(I) *Given* $x_0 \in E$, $P_0 \in f(x_0) + I(x_0)$, *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any* $x_1, \ldots, x_{k^\#_{\rm old}} \in E \cap B(x_0, \delta)$, *there exist polynomials* $P_1, \ldots, P_{k^\#_{\rm old}} \in \mathcal{P}$, *with* $P_i \in f(x_i) + I(x_i)$ *for* $0 \leq i \leq k^\#_{\rm old}$; *and*
$$|\partial^\alpha (P_i - P_j)(x_j)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k^\#_{\rm old}\ .$$

(II) *Given* $x_1, \ldots, x_{k^\#_{\rm old}} \in E$, *there exist polynomials* $P_1, \ldots, P_{k^\#_{\rm old}} \in \mathcal{P}$, *with* $P_i \in f(x_i) + I(x_i)$ *for* $1 \leq i \leq k^\#_{\rm old}$; $|\partial^\alpha P_i(x_i)| \leq \tilde{\delta}^{m-|\alpha|}$ *for* $|\alpha| \leq m$, $1 \leq i \leq k^\#_{\rm old}$; *and* $|\partial^\alpha (P_i - P_j)(x_j)| \leq |x_i - x_j|^{m-|\alpha|}$ *for* $|\alpha| \leq m$, $1 \leq i$, $j \leq k^\#_{\rm old}$.

Assume also that E *has fewer than* $\wedge$ *strata. Then there exists* $F \in C^m(\mathbb{R}^n)$, *with* $|\partial^\alpha F| \leq C \tilde{\delta}^{m-|\alpha|}$ *on* $\mathbb{R}^n$ *for* $|\alpha| \leq m$, *and* $J_x(F) \in f(x) + I(x)$ *for all* $x \in E$.

Lemma 9.1 will be used to carry out Step 2 of the plan described in the introduction.

10. The Whitney decomposition

In this section, we introduce the Whitney cubes mentioned in the introduction, and carry out Step 2 of the plan given in the introduction for proving Theorem 3.

We first partition $\mathbb{R}^n$ into a grid of cubes $\{Q^0_\nu\}$ of diameter 1. Next, we repeatedly subdivide the Q^0_ν into dyadic subcubes, in Calderón-Zygmund fashion. Once we have reached a given subcube Q of one of the Q^0_ν, we decide whether to retain Q or to subdivide it, according to Whitney's rule:

If $Q^* \cap E_1$ is empty, then we retain Q. Otherwise, we subdivide Q into 2^n congruent subcubes $Q_1, \ldots, Q_{2^n}$, and continue. Here, Q^* denotes a closed cube in $\mathbb{R}^n$, with the same center as Q, and with three times the diameter of Q. Recall that $E_1 \subseteq \mathbb{R}^n$ is compact. Thus $\mathbb{R}^n \smallsetminus E_1$ is partitioned into cubes $\{Q_\nu\}$, with the following properties, where we set

(1) $\delta_\nu = \mathrm{diam}(Q_\nu) \leq 1$:

(2) $\mathbb{R}^n \setminus E_1 = \bigcup_\nu Q_\nu$;

(3) $Q_\nu^* \cap E_1$ is empty;

(4) If $\delta_\nu < 1$, then there exists $x_0^{(\nu)} \in E_1$ with $\mathrm{dist}(x_0^{(\nu)}, Q_\nu) < C\delta_\nu$;

(5) If the closures of Q_μ and Q_ν have nonempty intersection, then $c\delta_\mu < \delta_\nu < C\delta_\mu$.

As in the proof of the standard Whitney extension theorem (see [M], [emS], [hW1]), these geometrical properties of the Q_ν allow us to construct a partition of unity $\{\theta_\nu\}$, with the following properties.

(6) $$1 = \sum_\nu \theta_\nu \text{ on } \mathbb{R}^n \setminus E_1 .$$

(7) $$\mathrm{supp}\, \theta_\nu \subset Q_\nu^* .$$

(8) $$|\partial^\alpha \theta_\nu| \leq C\delta_\nu^{-|\alpha|} \text{ on } \mathbb{R}^n, \text{ for } |\alpha| \leq m+1 .$$

(9) Any given point of $\mathbb{R}^n \setminus E_1$ has an open neighborhood that meets at most C of the supports of the θ_ν.

Let $\tilde{F} \in C^m(\mathbb{R}^n)$ be as in Lemma 8.1. Thus,

(10) $$\| \tilde{F} \|_{C^m(\mathbb{R}^n)} \leq C, \text{ and}$$

(11) $$J_x(\tilde{F}) \in \Gamma_f(x, \bar{k}_3, C) \subseteq f(x) + I(x) \text{ for all } x \in E_1 .$$

Thanks to (10), the function $\tilde{F}$ satisfies (12) and (13) below. (Recall that E is compact.)

(12) Given $\varepsilon > 0$, there exists $\delta > 0$ for which the following holds. Suppose $x_0 \in E$ and $x_1, \ldots, x_{\bar{k}_3} \in B(x_0, \delta)$. Set $\tilde{P}_i = J_{x_i}(\tilde{F})$ for $i = 0, 1, \ldots, \bar{k}_3$. Then

$$|\partial^\alpha(\tilde{P}_i - \tilde{P}_j)(x_j)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \bar{k}_3 .$$

(13) Suppose $x_0, \ldots, x_{\bar{k}_3} \in \mathbb{R}^n$. Set $\tilde{P}_i = J_{x_i}(\tilde{F})$ for $i = 0, 1, \ldots, \bar{k}_3$. Then

$$|\partial^\alpha(\tilde{P}_i - \tilde{P}_j)(x_j)| \leq C |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \bar{k}_3 .$$

From (10), (11) and Lemma 6.3, we have

(14) Given $\varepsilon > 0$, there exists $\delta > 0$ for which the following holds. Suppose $x_0 \in E_1$ and $x_1, \ldots, x_{\bar{k}_3} \in E \cap B(x_0, \delta)$. Then there exist $P_0, P_1, \ldots, P_{\bar{k}_3} \in \mathcal{P}$, with

(a) $P_0 = J_{x_0}(\tilde{F})$;

(b) $P_i \in f(x_i) + I(x_i)$ for $i = 0, 1, \ldots, \bar{k}_3$; and

(c) $|\partial^\alpha (P_i - P_j)(x_j)| \le \varepsilon |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le \bar{k}_3$.

Also, from (11) and the definition of $\Gamma_f(x, \bar{k}_3, C)$, we have

(15) Suppose $x_0 \in E_1$ and $x_1, \ldots, x_{\bar{k}_3} \in E$. Then there exist $P_0, P_1, \ldots, P_{\bar{k}_3} \in \mathcal{P}$, with

(a) $P_0 = J_{x_0}(\tilde{F})$;

(b) $P_i \in f(x_i) + I(x_i)$ for $i = 0, 1, \ldots, \bar{k}_3$;

(c) $|\partial^\alpha (P_i - P_j)(x_j)| \le C |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le \bar{k}_3$.

From (12) and (14), we deduce (16) below, by taking as our polynomials $P_i - \tilde{P}_i$ with P_i as in (14), and with $\tilde{P}_i$ as in (12).

(16) Given $\varepsilon > 0$, there exists $\delta > 0$ for which the following holds. Suppose $x_0 \in E_1$ and $x_1, \ldots, x_{\bar{k}_3} \in E \cap B(x_0, \delta)$. Then there exist $P_0, P_1, \ldots, P_{\bar{k}_3} \in \mathcal{P}$, with

(a) $P_0 = 0$;

(b) $P_i \in [f(x_i) - J_{x_i}(\tilde{F})] + I(x_i)$ for $i = 0, 1, \ldots, \bar{k}_3$; and

(c) $|\partial^\alpha (P_i - P_j)(x_j)| \le \varepsilon |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le \bar{k}_3$.

Similarly, from (13) and (15), we obtain

(17) Suppose $x_0 \in E_1$ and $x_1, \ldots, x_{\bar{k}_3} \in E$. Then there exist $P_0, P_1, \ldots, P_{\bar{k}_3} \in \mathcal{P}$, with

(a) $P_0 = 0$;

(b) $P_i \in [f(x_i) - J_{x_i}(\tilde{F})] + I(x_i)$ for $i = 0, 1, \ldots, \bar{k}_3$; and

(c) $|\partial^\alpha (P_i - P_j)(x_j)| \le C |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le \bar{k}_3$.

Now suppose Q_ν is one of our Whitney cubes, with diameter $\delta_\nu < 1$. Taking $x_0^{(\nu)}$ as in (4), and applying (17), we learn the following.

(18) Suppose $x_1, \ldots, x_{\bar{k}_3} \in E \cap Q_\nu^*$. Then there exist $P_1, \ldots, P_{\bar{k}_3} \in \mathcal{P}$, with

(a) $P_i \in [f(x_i) - J_{x_i}(\tilde{F})] + I(x_i)$ for $i = 1, \ldots, \bar{k}_3$;

(b) $|\partial^\alpha P_i(x_i)| \le C \delta_\nu^{m-|\alpha|}$ for $|\alpha| \le m$, $i = 1, \ldots, \bar{k}_3$; and

(c) $|\partial^\alpha(P_i - P_j)(x_j)| \le C|x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $1 \le i, j \le \bar{k}_3$.

Here, we take $x_0 = x_0^{(\nu)}$ in (17). Estimate (18)(b) follows from (17)(a) and (17)(c) with $i = 0$, by virtue of (4). Similarly, (16) and (4) imply:

(19) Given $\varepsilon > 0$, there exists $\delta > 0$ for which the following holds. Suppose $x_1, \dots, x_{\bar{k}_3} \in E \cap Q_\nu^*$, with $\delta_\nu < \delta$. Then there exist $P_1, \dots, P_{\bar{k}_3} \in \mathcal{P}$, with

(a) $P_i \in [f(x_i) - J_{x_i}(\tilde{F})] + I(x_i)$ for $i = 1, \dots, \bar{k}_3$;

(b) $|\partial^\alpha P_i(x_i)| \le \varepsilon \delta_\nu^{m-|\alpha|}$ for $|\alpha| \le m$, $i = 1, \dots, \bar{k}_3$; and

(c) $|\partial^\alpha(P_i - P_j)(x_i)| \le \varepsilon|x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le \bar{k}_3$.

From (18) and (19), it is easy to produce a function $A(t)$, mapping $(0,1]$ to the positive reals, for which the following results hold.

(20) $0 < A(t) \le C$ for all $t \in (0,1]$.

(21) $\lim_{t \to 0} A(t) = 0$.

(22) Suppose $\delta_\nu < 1$, and suppose $x_1, \dots, x_{\bar{k}_3} \in E \cap Q_\nu^*$. Then there exist $P_1, \dots, P_{\bar{k}_3} \in \mathcal{P}$, with

(a) $P_i \in [f(x_i) - J_{x_i}(\tilde{F})] + I(x_i)$ for $i = 1, \dots, \bar{k}_3$;

(b) $|\partial^\alpha P_i(x_i)| \le A(\delta_\nu) \cdot \delta_\nu^{m-|\alpha|}$ for $i = 1, \dots, \bar{k}_3$, $|\alpha| \le m$; and

(c) $|\partial^\alpha(P_i - P_j)(x_j)| \le A(\delta_\nu) \cdot |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $1 \le i, j \le \bar{k}_3$.

Moreover, because E, f, I are assumed to satisfy hypothesis (I) of Theorem 3, we obtain the following result, thanks to (12).

(23) Given $x_0 \in E$, $P_0 \in [f(x_0) - J_{x_0}(\tilde{F})] + I(x_0)$, and $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x_1, \dots, x_{\bar{k}_3} \in E \cap B(x_0, \delta)$ there exist $P_1, \dots, P_{\bar{k}_3} \in \mathcal{P}$, such that

(a) $P_i \in [f(x_i) - J_{x_i}(\tilde{F})] + I(x_i)$ for $i = 0, 1, \dots, \bar{k}_3$; and

(b) $|\partial^\alpha(P_i - P_j)(x_j)| \le \varepsilon|x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le \bar{k}_3$.

For any Whitney cube Q_ν with diameter $\delta_\nu < 1$, we may now apply the Rescaled Induction Hypothesis (Lemma 9.1), with

(24) $\delta_\nu, E \cap Q_\nu^*$, $[f(x) - J_x(\tilde{F})]/A(\delta_\nu)$, $I(x)$ in place of $\tilde{\delta}, E, f(x), I(x)$ in Lemma 9.1.

Note that the hypotheses of Lemma 9.1 hold for the data (24). In fact, hypotheses (I) and (II) of that lemma are immediate from (22) and (23), since $\bar{k}_3 \geq k^{\#}_{\text{old}}$. (See Section 7.) The number of strata for $I(x)$ on $E \cap Q^*_\nu$ is strictly less than $\wedge$, since the number of strata in E is precisely $\wedge$, and Q^*_ν does not intersect the lowest stratum E_1. (See (3).) Finally, $E \cap Q^*_\nu$ is compact, since we took Q^*_ν to be a closed cube. Thus, as claimed, the hypotheses of Lemma 9.1 hold for the data (24).

Applying Lemma 9.1, we now learn the following, for any Whitney cube Q_ν with diameter $\delta_\nu < 1$:

(25) There exists a function $F_\nu \in C^m(\mathbb{R}^n)$, with

(a) $|\partial^\alpha F_\nu| \leq CA(\delta_\nu) \cdot \delta_\nu^{m-|\alpha|}$ on $\mathbb{R}^n$, for $|\alpha| \leq m$; and

(b) $J_x(F_\nu) \in [f(x) - J_x(\tilde{F})] + I(x)$ for all $x \in E \cap Q^*_\nu$.

We can also show that in effect (25) holds when the Whitney cube Q_ν has diameter $\delta_\nu = 1$. In fact, we may simply apply our induction hypothesis (Theorem 3 with fewer than $\wedge$ strata), with $E \cap Q^*_\nu, f(x), I(x)$ in place of $E, f(x), I(x)$. One checks trivially that the hypotheses of Theorem 3 hold for $E \cap Q^*_\nu$, $f(x), I(x)$, since they are assumed to hold for $E, f(x), I(x)$. Again, $E \cap Q^*_\nu$ has fewer than $\wedge$ strata because Q^*_ν does not meet the lowest stratum E_1. Applying the inductive hypothesis, we obtain a function $\check{F}_\nu \in C^m(\mathbb{R}^n)$, with $|\partial^\alpha \check{F}_\nu| \leq C$ on $\mathbb{R}^n$, for $|\alpha| \leq m$; and $J_x(\check{F}_\nu) \in f(x) + I(x)$ for all $x \in E \cap Q^*_\nu$.

Setting $F_\nu = \check{F}_\nu - \tilde{F}$, and recalling (10), we see that $F_\nu \in C^m(\mathbb{R}^n)$, with

(26) $$|\partial^\alpha F_\nu| \leq C \text{ on } \mathbb{R}^n, \text{ for } |\alpha| \leq m; \text{ and}$$

(27) $$J_x(F_\nu) \in [f(x) - J_x(\tilde{F})] + I(x) \text{ for all } x \in E \cap Q^*_\nu\,.$$

Replacing $A(t)$ by $A^+(t) = A(t) + t$, we preserve (20), (21), (25). Moreover, the analogue of (25), with $A(t)$ replaced by $A^+(t)$, holds also for $\delta_\nu = 1$, thanks to (26), (27) and the obvious estimate $A^+(1) \geq 1$. Thus, we have proven the following result.

Lemma 10.1. *There exist functions* $F_\nu \in C^m(\mathbb{R}^n)$ *and* $A : (0,1] \longrightarrow (0,\infty)$, *for which the following hold*:

(a) $J_x(F_\nu) \in [f(x) - J_x(\tilde{F})] + I(x)$ *for all* $x \in E \cap Q^*_\nu$, *and for all* ν;

(b) $|\partial^\alpha F_\nu| \leq CA(\delta_\nu) \cdot \delta_\nu^{m-|\alpha|}$ *on* $\mathbb{R}^n$, *for* $|\alpha| \leq m$ *and for all* ν;

(c) $0 < A(t) \leq C$ *for all* $t \in (0,1]$; *and*

(d) $\lim_{t \to 0+} A(t) = 0$.

This completes Step 2 of the plan of the proof of Theorem 3, as outlined in the introduction.

11. Proof of the main result

In this section, we carry out Step 3 of the plan given in the introduction, and complete the proof of Theorem 3. Since we have already reduced Theorems 1 and 2 to Theorem 3, this will establish those results as well.

We let $E, f(x), I(x), E_1, \wedge$ be as in Section 4. We retain the Whitney cubes Q_ν and the cutoff functions θ_ν from Section 10. Finally, we let F_ν and $A(t)$ be as in Lemma 10.1.

For $\delta > 0$, we define

(1) $$F^{[\delta]}(x) = \sum_{\delta_\nu > \delta} \theta_\nu(x) F_\nu(x) \ .$$

From (10.3), (10.7), (10.9), we see that any $x \in E_1$ has an open neighborhood (depending on δ) that meets none of the supports of the θ_ν with $\delta_\nu > \delta$; while any $x \in \mathbb{R}^n \smallsetminus E_1$ has an open neighborhood that meets at most C of the supports of the θ_ν. Together with (10.8) and Lemma 10.1, this shows that each $F^{[\delta]}$ belongs to $C^m(\mathbb{R}^n)$, and that

(2) $$J_x(F^{[\delta]}) = 0 \text{ for all } x \in E_1, \text{ and}$$

(3) $$J_x(F^{[\delta]}) = \sum_{\substack{\delta_\nu > \delta \\ \operatorname{supp}\theta_\nu \ni x}} J_x(\theta_\nu) \cdot J_x(F_\nu) \text{ for all } x \in \mathbb{R}^n \smallsetminus E_1 \ .$$

On the right side of (3), there are only finitely many summands, and the dot denotes multiplication in $\mathcal{R}_x$.

Since $\operatorname{supp}\theta_\nu \subset Q_\nu^*$ and $I(x)$ is an ideal in $\mathcal{R}_x$ for $x \in E$, Lemma 10.1(a) shows that

$$J_x(\theta_\nu) \cdot J_x(F_\nu) \in J_x(\theta_\nu) \cdot [f(x) - J_x(\tilde{F})] + I(x) \text{ for } x \in E \cap \operatorname{supp}\theta_\nu \ .$$

Hence, (3) implies

(4) $$J_x(F^{[\delta]}) \in [\sum_{\substack{\delta_\nu > \delta \\ \operatorname{supp}\theta_\nu \ni x}} J_x(\theta_\nu)] \cdot [f(x) - J_x(\tilde{F})] + I(x) \text{ for } x \in E \smallsetminus E_1 \ .$$

Fix $x \in \mathbb{R}^n \smallsetminus E_1$. Then x belongs to only finitely many of supports of the θ_ν, say, $\operatorname{supp}\theta_{\nu_1}, \ldots, \operatorname{supp}\theta_{\nu_N}$. If $0 < \delta < \min\{\delta_{\nu_1}, \ldots, \delta_{\nu_N}\}$, then $\sum_{\substack{\delta_\nu > \delta \\ \operatorname{supp}\theta_\nu \ni x}} J_x(\theta_\nu) = \sum_{\operatorname{supp}\theta_\nu \ni x} J_x(\theta_\nu) = 1$, thanks to (10.6). Therefore, (4) shows that

(5) $$J_x(F^{[\delta]}) \in [f(x) - J_x(\tilde{F})] + I(x) \text{ for } x \in E \smallsetminus E_1, \delta < \tilde{\delta}(x) \ ,$$

where $\tilde{\delta}(x)$ is a small enough positive number depending on x.

Next, we estimate the C^m-norm of $F^{[\delta]}$. From Lemma 10.1(b),(c) and (10.8), we obtain

(6) $$|\partial^\alpha(\theta_\nu F_\nu)| \le CA(\delta_\nu) \cdot \delta_\nu^{m-|\alpha|} \le C'\delta_\nu^{m-|\alpha|} \le C' \text{ on } \mathbb{R}^n \text{ for } |\alpha| \le m \ .$$

Since also each $x \in \mathbb{R}^n$ belongs to at most C of the supports of the θ_ν, it follows from (1) and (6) that

$$\| F^{[\delta]} \|_{C^m(\mathbb{R}^n)} \leq C'' \text{ for all } \delta > 0 \ . \tag{7}$$

Similarly, if $0 < \delta_1 < \delta_2$, then we can estimate $F^{[\delta_1]} - F^{[\delta_2]}$. In fact, for $|\alpha| \leq m$ and $x \in \mathbb{R}^n$, (1) and (6) show that

$$\begin{aligned} |\partial^\alpha F^{[\delta_1]}(x) - \partial^\alpha F^{[\delta_2]}(x)| &= |\sum_{\substack{\delta_1 < \delta_\nu \leq \delta_2 \\ \mathrm{supp}\,\theta_\nu \ni x}} \partial^\alpha(\theta_\nu F_\nu)(x)| \\ &\leq \sum_{\substack{\delta_1 < \delta_\nu \leq \delta_2 \\ \mathrm{supp}\,\theta_\nu \ni x}} CA(\delta_\nu) \cdot \delta_\nu^{m-|\alpha|} \leq C' \cdot \sup\{A(\delta) : \delta \leq \delta_2\} \ , \end{aligned}$$

since $x \in \mathrm{supp}\,\theta_\nu$ for at most C distinct ν. In view of Lemma 10.1(d), it follows that $\lim_{\delta_1,\delta_2 \to 0+} \| F^{[\delta_1]} - F^{[\delta_2]} \|_{C^m(\mathbb{R}^n)} = 0$. Consequently, $F^{[\delta]}$ converges in C^m norm to a function $F^{[0]} \in C^m(\mathbb{R}^n)$, as $\delta \to 0+$. In particular, $J_x(F^{[\delta]}) \longrightarrow J_x(F^{[0]})$ as $\delta \to 0+$, for each x. Hence, (2), (5), (7) show that

$$J_x(F^{[0]}) = 0 \text{ for all } x \in E_1 \ , \tag{8}$$

$$J_x(F^{[0]}) \in [f(x) - J_x(\tilde{F})] + I(x) \text{ for all } x \in E \smallsetminus E_1 \ , \text{ and} \tag{9}$$

$$\| F^{[0]} \|_{C^m(\mathbb{R}^n)} \leq C \ . \tag{10}$$

Although we will not use the fact, the reader may readily verify that $F^{[0]} = \sum_\nu \theta_\nu F_\nu$ on $\mathbb{R}^n$. Thus, the results in this section agree with the description of Step 3 of the plan of our proof, given in the introduction.

Next, we recall from Section 10 that $\tilde{F} \in C^m(\mathbb{R}^n)$, with

$$J_x(\tilde{F}) \in f(x) + I(x) \text{ for all } x \in E_1, \text{ and} \tag{11}$$

$$\| \tilde{F} \|_{C^m(\mathbb{R}^n)} \leq C \ . \tag{12}$$

Finally, we set

$$F = F^{[0]} + \tilde{F} \in C^m(\mathbb{R}^n) \ . \tag{13}$$

From (8) and (11), we have $J_x(F) \in f(x) + I(x)$ for all $x \in E_1$; and from (9), we have $J_x(F) \in f(x) + I(x)$ for all $x \in E \smallsetminus E_1$. Thus,

$$J_x(f) \in f(x) + I(x) \text{ for all } x \in E \ . \tag{14}$$

From (10) and (12) we have

$$\| F \|_{C^m(\mathbb{R}^n)} \leq C' \ . \tag{15}$$

Thus, we have exhibited a C^m-function F satisfying (14) and (15). However, the existence of such an F is precisely the conclusion of Theorem 3. Thus, Theorem 3 holds for $E, f(x), I(x)$.

This completes our induction on the number of strata, and proves Theorem 3. □

Princeton University, Princeton, NJ
E-mail address: cf@math.princeton.edu

References

[BMP1] E. Bierstone, P. D. Milman, and W. Pawłucki, Differentiable functions defined on closed sets. A problem of Whitney, *Invent. Math.* **151** (2003), 329–352.

[BMP2] ———, Higher-order tangents and Fefferman's paper on Whitney's extension problem, *Ann. Math.* **164** (2006), 361–370.

[B] Y. Brudnyi, On an extension theorem, *Funk. Anal. i Prilzhen.* **4** (1970), 97–98; English transl. in *Func. Anal. Appl.* **4** (1970), 252–253.

[BS1] Y. Brudnyi and P. Shvartsman, A linear extension operator for a space of smooth functions defined on closed subsets of $\mathbf{R}^n$, *Dokl. Akad. Nauk SSSR* **280** (1985), 268–270.

[BS2] ———, Generalizations of Whitney's extension theorem, *Internat. Math. Res. Notices* **3** (1994), 129–139.

[BS3] ———, The Whitney problem of existence of a linear extension operator, *J. Geometric Anal.* **7** (1997), 515–574.

[BS4] ———, Whitney's extension problem for $C^{1,\omega}$ multivariate functions, *Trans. A.M.S.* **353** (2001), 2487–2512.

[F1] C. Fefferman, Interpolation and extrapolation of smooth functions by linear operators, *Revista Matematica Iberoamericana* **21** (2005) no. 1, 313–348.

[F2] ———, A sharp form of Whitney's extension theorem, *Ann. of Math.* **69** (2005), 509–577.

[F3] ———, A generalized sharp Whitney theorem for jets, *Revista Mathematica Iberoamericana* **21** (2005) no. 2, 577–688.

[F4] ———, Whitney's extension problem in certain function spaces, preprint.

[G] G. Glaeser, Etudes de quelques algebrès tayloriennes, *J. d'Analyse* **6** (1958), 1–124.

[M] B. Malgrange, *Ideals of Differentiable Functions*, Oxford Univ. Press, Oxford, 1966.

[S1] P. Shvartsman, Lipschitz selections of multivalued mappings and traces of the Zygmund class of functions to an arbitrary compact, *Dokl. Acad. Nauk SSSR* **276** (1984), 559–562; English transl. in *Soviet Math. Dokl.* **29** (1984), 565–568.

[S2] ———, On traces of functions of the Zygmund class, *Sib. Mat. Zh.* **28** (1987), 203–215; English transl. in *Siberian Math. J.* **28** (1987), 853–863.

[S3] ———, Lipschitz selections of set-valued functions and Helly's theorem, *J. Geom. Anal.* **12** (2002), 289–324.

[emS] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, *Princeton Math. Series* **30**, Princeton Univ. Press, Princeton, NJ, 1970.

[rW] R. Webster, *Convexity*, *Oxford Science Publications*, Oxford Univ. Press, New York, 1994.

[hW1] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, *Trans. Amer. Math. Soc.* **36** (1934), 63–89.

[hW2] ———, Differentiable functions defined in closed sets. I, *Trans. Amer. Math. Soc.* **36** (1934), 369-387.

[hW3] ———, Functions differentiable on the boundaries of regions, *Ann. of Math.* **35** (1934), 482–485.

[Z1] N. Zobin, Whitney's problem on extendability of functions and an intrinsic metric, *Adv. Math.* **133** (1998), 96–132.

[Z2] ———, Extension of smooth functions from finitely connected planar domains, *J. Geom. Anal.* **9** (1999), 489–509.

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