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An abelianization of SU(2) WZW model

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1. Introduction

The purpose of this paper is to carry out the abelianization program proposed by Atiyah [1] and Hitchin [9] for the geometric quantization of SU(2) Wess-Zumino-Witten model.

Let C be a Riemann surface of genus g. Let M_g be the moduli space of semi-stable rank 2 holomorphic vector bundles on C with trivial determinant. For a positive integer k, let $\Gamma(M_g, \mathcal{L}^k)$ be the space of holomorphic sections of the k-th tensor product of the determinant line bundle $\mathcal{L}$ on M_g. An element of $\Gamma(M_g, \mathcal{L}^k)$ is called a rank 2 theta function of level k.

The main result of our abelianization is to give an explicit representation of a base of $\Gamma(M_g, \mathcal{L}^k)$ as well as its transformation formula in terms of classical Riemann theta functions with automorphic form coefficients defined on the Prym variety P associated with a two-fold branched covering surface $\tilde{C}$ of C.

$\Gamma(M_g, \mathcal{L}^k)$ can be identified with the conformal block of level k of the SU(2) WZW model ([5], [15]). The abelianization procedure enables us to deduce the various known results about the conformal block in a uniform way. Firstly, we construct a projectively flat connection on the vector bundle over the Teichmüller space with fibre $\Gamma(M_g, \mathcal{L}^k)$. Secondly, making use of our explicit representation of rank 2 theta functions we construct a Hermitian product on the vector bundle preserved by the connection. Also our explicit representation enables us to prove that $\Gamma(M_g, \mathcal{L}^k)$ has the predicted dimension from the Quantum Clebsh-Gordan conditions.

A natural connection on the said vector bundle for the SU(N) WZW model was first constructed by Hitchin [11]. It will turn out that the connection constructed in this paper coincides with the Hitchin connection.

Laszlo [16] showed that the Hitchin connection coincides with the connection constructed by Tsuchiya, Ueno and Yamada [21] through the above identification. On the other hand Kirillov [13], [14] constructed a Hermitian product on the conformal block compatible with the Tsuchiya-Ueno-Yamada connection using the representation theory of affine Lie algebras together with the theory of hermitian modular tensor categories; cf. [22]. Laszlo's result implies that the Hermitian product of Kirillov defines the one on $\Gamma(M_g, \mathcal{L}^k)$ compatible with the Hitchin connection. The author cannot figure out a relation between the Hermitian product constructed in this paper and the one found by Kirillov.

The paper is organized as follows. In Section 2 we study the topological properties of a family of 2-fold branched covering surfaces $\tilde{C}$ of a fixed Riemann surface C parametrized by the configuration space of $4g-4$ mutually distinct points on C.

In Sections 3 and 4 we study the Prym variety P of $\tilde{C}$ and the classical Riemann theta functions defined on it. Especially we will be concerned with their symmetric properties. That is, the fundamental group of the configuration space induces a finite group action on the space of Riemann theta functions on P. We call it global symmetry. There is a morphism $\pi : P \to M_g$ and a pulled back section of $\Gamma(M_g, \mathcal{L}^k)$ by π can be expressed by Riemann theta functions of level $2k$ on P. Then it should satisfy an invariance with respect to this group action.

In Sections 5 we study the branching divisor of $\pi : P \to M_g$. The square root (Pfaffian) of the determinant of π is given by a Riemann theta function Π of level 4 ([9]). Π plays a central role throughout the paper, and we give a precise formula for it.

In Section 6 we construct a differential operator D on the space of holomorphic sections of the line bundles on the family of Prym varieties P such that a family $\tilde{\psi}$ of holomorphic sections, which is a pull back by π of a section $\psi \in \Gamma(M_g, \mathcal{L}^k)$, satisfies the differential equation $D\tilde{\psi} = 0$.

In Section 7 we will show that the global symmetry and the differential equation $D\tilde{\psi} = 0$ characterize the pull back sections.

In Section 8 we construct a basis of $\Gamma(M_g, \mathcal{L}^k)$. It will be given in terms of classical Riemann theta functions with automorphic form coefficients. The result includes the fact that the dimension of $\Gamma(M_g, \mathcal{L}^k)$ is equal to the number of the 'admissible' spin weights attached to a pant decomposition of the Riemann surface (Quantum Clebsch-Gordan condition).

In Section 9 we construct a projectively flat connection and a hermitian product compatible with it on the vector bundle over Teichmüller space with fibre $\Gamma(M_g, \mathcal{L}^k)$.

In Section 10 we give the transformation formula of rank 2 theta functions. It involves a subtle but important aspect related to the Maslov index.

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2. A family of 2-fold branched covering surfaces

2.1. *A family of 2-fold branched covering surfaces.* Let C be a closed Riemann surface of genus g (≥ 2). Let $\mathrm{C}_{4g-4}(C)$ be the configuration space of $4g-4$ unordered mutually distinct points $b = \{x_j\}_{1\leq j\leq 4g-4}$ in C; that is,

$$\mathrm{C}_{4g-4}(C) = \left(C^{4g-4} - \Delta\right)/S_{4g-4}$$

where Δ denotes the big diagonal of C^{4g-4} and S_n is the symmetric group of degree n acting on C^{4g-4} by permutations of factors.

For $b = \{x_j\}$ in $\mathrm{C}_{4g-4}(C)$, let c_j denotes the class in $H_1(C-b, \mathbf{Z}_2)$ represented by the boundary circle of a small disc centered at x_j in C. Let

$$\hat{H}^1(C-b, \mathbf{Z}_2) \equiv \{\alpha \in H^1(C-b, \mathbf{Z}_2) \mid \langle \alpha, c_j\rangle = 1\ \} \tag{1}$$

where $\langle\ ,\ \rangle$ denotes the evaluation of cohomology classes on homology classes.

$\hat{H}^1(C-b, \mathbf{Z}_2)$ is in one-one correspondence with the set of topologically distinct 2-fold branched coverings of C with branch locus $b = \{x_j\}$. Here two branched coverings with branch locus $b = \{x_j\}$ are topologically distinct if and only if there is no diffeomorphism between them which is equivariant with respect to the covering involutions and covers the identity map of C.

Definition 2.1. We call an element of $\alpha \in \hat{H}^1(C-b, \mathbf{Z}_2)$ *a covering type of* C.

The family $\mathcal{H} = \{\hat{H}^1(C-b, \mathbf{Z}_2)\}_{b\in \mathrm{C}_{4g-4}(C)}$ forms a fiber bundle over $\mathrm{C}_{4g-4}(C)$ with finite discrete fiber. Choose a base point $b_o \in \mathrm{C}_{4g-4}(C)$ and let

$$\rho : \pi_1(\mathrm{C}_{4g-4}(C), b_o) \to \mathrm{Aut}(\hat{H}^1(C-b_o, \mathbf{Z}_2))$$

be the holonomy representation of the fiber bundle $\mathcal{H}$.

We can describe ρ as follows. For an oriented loop $l = \{b^t = \{x_j^t\}\}_{0\leq t\leq 1}$ based at b_o in $\mathrm{C}_{4g-4}(C)$, the union of oriented $4g-4$ arcs $\{x_j^t\}$ forms an oriented closed curve $\bar{l}$ in C. For $a \in H_1(C-b_o, \mathbf{Z}_2)$ we can define the $\mathbf{Z}_2$-intersection number $\bar{l}\cdot a \in \mathbf{Z}_2$. We obtain the following homomorphism ev which we call the evaluation map

$$\mathrm{ev} : \pi_1(\mathrm{C}_{4g-4}(C), b_o) \to H^1(C-b_o, \mathbf{Z}_2). \tag{2}$$

Clearly $\bar{l}\cdot c_j = 0$ for $1 \leq c_j \leq 4g-4$ and we have the following lemma:

Lemma 2.1. *Let $[l] \in \pi_1(\mathrm{C}_{4g-4}(C), b_o)$ be the homotopy class represented by a closed loop l based at b_o. Then $\rho([l]) \in \mathrm{Aut}(\hat{H}^1(C-b_o, \mathbf{Z}_2))$ is given by*

$$\rho([l])(\alpha) = \alpha + \mathrm{ev}([l]) \tag{3}$$

for $\alpha \in \hat{H}^1(C-b_o, \mathbf{Z}_2)$.

Definition 2.2. Let $q : \mathcal{B} \to \mathrm{C}_{4g-4}(C)$ be the covering space of $\mathrm{C}_{4g-4}(C)$ associated with the kernel of ρ. The set $\mathcal{B}$ can be identified with the set of pairs $\mathrm{C}_{4g-4}(C) \times \hat{H}^1(C-b, \mathbf{Z}_2)$ with q the projection to the first factor. We represent a point $\tilde{b}$ of $\mathcal{B}$ by a pair

$$\tilde{b} = (b, \alpha) \qquad for \;\; b \in \mathrm{C}_{4g-4}(C) \;\; and \;\; \alpha \in \hat{H}^1(C-b, \mathbf{Z}_2). \tag{4}$$

For $\tilde{b} = (b, \alpha) \in \mathcal{B}$, let $\tilde{C} = \tilde{C}_{\tilde{b}}$ be the associated two-fold branched covering surface of C with branch point set b of the covering type α. The genus $\tilde{g}$ of $\tilde{C}$ is $4g-3$. We denote the covering projection by $p : \tilde{C} \to C$ and the covering involution by $\sigma : \tilde{C} \to \tilde{C}$.

Definition 2.3. Let $\mathcal{C} \to \mathcal{B}$ be the fiber bundle over $\mathcal{B}$ whose fiber at $\tilde{b} = (b, \alpha) \in \mathcal{B}$ is the 2-fold branched covering surface $\tilde{C}_{\tilde{b}}$ of C.

Note that $\mathcal{B}$ and $\mathcal{C}$ are connected.

2.2. *Pant decompositions of surfaces.* Throughout the paper we use the following notation;

S_0: the three-holed 2-dimensional sphere

T_0: the one-holed 2-dimensional torus.

Definition 2.4. A pant decomposition $\Upsilon = \{e_l, C_i\}$ of a Riemann surface C of genus g is defined to be a set of simple closed curves $\{e_l\}_{l=1,\cdots,3g-3}$ and surfaces $\{C_i\}_{i=1,\cdots,2g-2}$ in C such that

(i) $\{e_l\}$ is a family of mutually disjoint and mutually freely nonhomotopic simple closed curves in C,

(ii) $C = \bigcup C_i$ where $C_i = S_0$ or $C_i = T_0$. If $C_i = S_0$, then ∂C_i is a union of three elements of $\{e_l\}$. If $C_i = T_0$, then ∂C_i is an element of $\{e_l\}$, and C_i contains an element of $\{e_l\}$ in its interior as an essential simple closed curve.

(iii) If we cut C along $\bigcup_l e_l$, then the resulting surface is a disjoint union of $\{C_i^*\}_{1 \le i \le 2g-2}$, where $C_i^* = S_0$ for $1 \le i \le 2g-2$ and, if $C_i = S_0$, then $C_i^* = C_i$ and, if $C_i = T_0$, then $\partial C_i^* = e_{l'} \cup e_l^+ \cup e_l^-$, where $e_{l'} = \partial C_i$ and $e_l^{\pm}$ are the two copies of the essential curve $e_l \subset C_i$.

Definition 2.5. Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C,

(i) We define $\mathrm{C}_{4g-4}(C)_\Upsilon$ to be the open subset of $\mathrm{C}_{4g-4}(C)$ consisting of those points $b \in \mathrm{C}_{4g-4}(C)$ such that $C_i^o = C_i - \bigcup_l e_l$ contains exactly two points $\{x_1^i, x_2^i\}$ of b.

(ii) We define $\mathcal{B}_\Upsilon$ to be the open subset of $\mathcal{B}$ consisting of those points $\tilde{b} = (b, \alpha) \in \mathcal{B}$ such that $b \in \mathrm{C}_{4g-4}(C)_\Upsilon$ and that $\langle \alpha, [e_l] \rangle = 0$ for $1 \leq l \leq 3g-3$, where $[e_l]$ is the $\mathbf{Z}_2$ homology class represented by e_l in $H_1(C - b, \mathbf{Z}_2)$.

Let $\mathcal{C}_\Upsilon \to \mathcal{B}_\Upsilon$ be the restriction of $\mathcal{C} \to \mathcal{B}$ to $\mathcal{B}_\Upsilon$.

Definition 2.6. For a pant decomposition Υ of C, let

$$W_\Upsilon = \pi_1(\mathcal{B}_\Upsilon, \tilde{b}) \;, \tag{5}$$

where $\tilde{b} = (b, \alpha)$ is a base point of $\mathcal{B}_\Upsilon$.

LEMMA 2.2. *There is an exact sequence of groups*

$$1 \to W_\Upsilon \to \pi_1\left(\mathrm{C}_{4g-4}(C)_\Upsilon, b\right) \to \mathbf{Z}_2^g \to 1. \tag{6}$$

Proof. If we set $\mathrm{C}_i = (C_i^o \times C_i^o - \{\text{diagonal}\})/S_2$ and $b \cap C_i^o = b_i$, where $C_i^o = C_i - \bigcup e_l$, the group W_Υ is the kernel of the composition map

$$\prod_i \pi_1(\mathrm{C}_i, b_i) \to \pi_1\left(\mathrm{C}_{4g-4}(C)_\Upsilon, b\right) \to H^1(C - b, \mathbf{Z}_2) \tag{7}$$

where the first map is induced by the inclusion and the second is the evaluation map ev. □

Now we choose and fix a pant decomposition Υ. We fix an orientation of e_l for each $l = 1, \cdots, 3g-3$. We write $e_l = C_i \cap -C_j$ if e_l is a common boundary of C_i and C_j and the orientation of e_l agrees with that of C_i.

We study the group W_Υ.

Let S_0 be a 3-holed sphere as before. Let e be a boundary circle of S_0. Let x_1,x_2 be two points in the interior of S_0. Let $p_e = \{p_e(s)\}_{0 \leq s \leq 1}$ be the embedded arc in S_0 connecting $p_e(0) = x_1$ and $p_e(1) = x_2$ as is depicted in Figure 1.

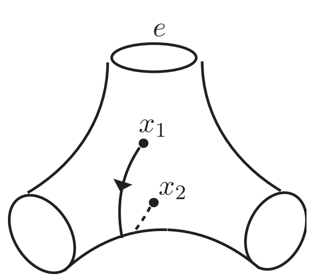

Figure 1: Arc p_e

Definition 2.7. Let e_1, e_2, e_3 be the three boundary circles of S_0. We define the following closed loops in the symmetric product $(S_0 \times S_0 - \Delta)/S_2$ in which the lower indices should be understood mod.3 (anti-clockwise in Figure 2),

(i) $t_{e_l} = \{ (p_{e_{l+1}}(s) \, , \, p_{e_{l-1}}(1-s)) \}_{0 \leq s \leq 1}$,

(ii) $k_{e_l} = t_{e_{l-1}} t_{e_l} t_{e_{l+1}}$.

Here in Figure 2 the left represents the curve t_{e_1} and the right represents the curve k_{e_1}. In the figure the curve with one arrow represents the trajectory of x_1 and one with double arrow does that of x_2 corresponding to the paths t_{e_l} and k_{e_l} respectively.

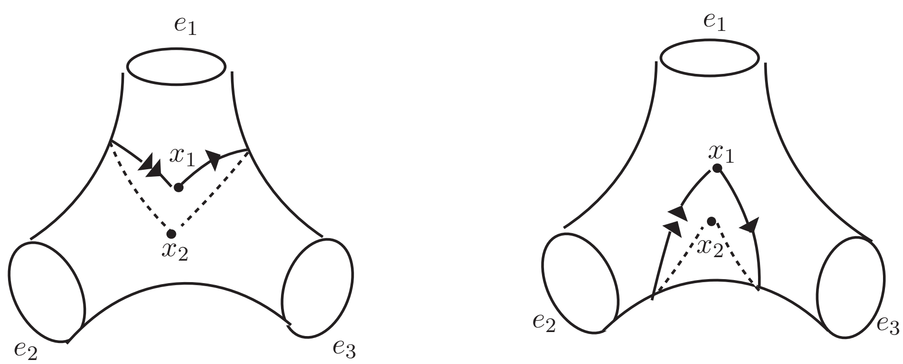

Figure 2: Curves

For a pant decomposition $\Upsilon = \{e_l, C_i\}$ of C, cutting out C along $\bigcup_l e_l$, we obtain the disjoint union $\bigcup_i C_i^*$ as in (iii) in Definition 2.4. Each C_i^* can be identified with S_0. Then the loops t_{e_l} and k_{e_l} in S_0 given in Definition 2.7 define the corresponding loops $t_{e_l}^{C_i^*}$ and $k_{e_l}^{C_i^*}$ respectively in C_i^* for $e_l \subset \partial C_i^*$.

Lemma 2.3. *Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Then W_Υ is generated by the following elements.*

(i) $\left\{ t_{e_l}^{C_i^*} (t_{e_l}^{C_j^*})^{\pm 1} \right\}$, *where $e_l = C_i \cap C_j$ $(i \neq j)$,*

(ii) $\left\{ t_{e_l^+}^{C_i^*} (t_{e_l^-}^{C_i^*})^{\pm 1} \right\}$, *where $C_i = T_0$ and $e_l^\pm$ is as in Definition* 2.4 (iii),

(iii) $\left\{ t_{e_l}^{C_i^*} \right\}$, *where $e_l \subset C_i$ is separating,*

(iv) $\left\{ \left(t_{e_l}^{C_i^*} \right)^2 \right\}$, *where $e_l \subset C_i$,*

(v) $\left\{ k_{e_l}^{C_i^*} \right\}$, *where $e_l \subset C_i$.*

Proof. Clearly the listed elements are in the kernel of the evaluation map ev. Let (C_i, b_i) be as in the proof of Lemma 2.2. The pure Braid group in the Braid group $\pi_1(\mathrm{C}_i, b_i)$ has index two and is generated by those homotopy classes represented by the loops such that x_1 moves once along the small circle centered at x_2 while x_2 is fixed and x_1 (or x_2 resp.) moves once along the loop parallel to one component of the boundary $\partial\mathrm{C}_i$ while x_2 (x_1 resp.) is fixed. It can be seen without difficulty that those homotopy classes can be represented by combinations of t_{e_l}. Hence the Braid group $\prod_i \pi_1(\mathrm{C}_i, b_i)$ is generated by the loops $\{t_{e_l}^{C_i}\}_{e_l \subset \partial C_i}$. It is not difficult to see that Ker(ev) is generated by the listed elements. □

2.3. *Holonomy action of* W_Υ. We study the holonomy diffeomorphisms of the fibre bundle $\mathcal{C}_\Upsilon \to \mathcal{B}_\Upsilon$ induced by moves of the branch points along simple closed curves in $\mathcal{B}_\Upsilon$.

Let S_0 be the 3-holed 2-sphere with $\partial S_0 = e_1 \cup e_2 \cup e_3$. Let $\tilde{S}_0$ be the 2-fold branched covering space of S_0 with branch locus $x_1 \cup x_2$ and covering involution σ.

For each e_l the curve t_{e_l} in S_0 induces a diffeomorphism τ_{e_l} of $\tilde{S}_0$ depicted in Figure 3 where the upper and the lower boundary circles are $\tilde{e}_l$ and $\sigma\tilde{e}_l$ respectively and $\tilde{e}_l \cup \sigma\tilde{e}_l$ represents the lifts of e_l. The diffeomorphism is a combination of the half Dehn twists along the four curves in the picture in the directions indicated by the arrows and the flip of the component of $\tilde{S}_0$ containing the branch points cutting along the two vertical circles which interchange the points x_1 and x_2 and the two components $\tilde{e}_l$ and $\sigma\tilde{e}_l$. The diffeomorphism is the identity on the lifts of the other boundary components.

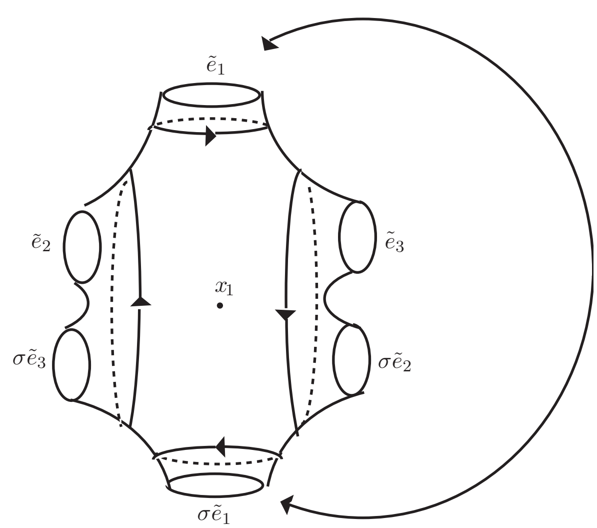

Figure 3: The induced diffeomorphism

Likewise the curve k_{e_l} induces the Dehn twist κ_{e_l} of $\tilde{S}_0$ along the simple closed curve which is the inverse image of the arc p_{e_l} (Figure 1) in $\tilde{S}_0$.

Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C.

Cutting out C along $\bigcup_l e_l$ to the disjoint union $\bigcup_i C_i^*$, where C_i^* is identified with S_0, let $\tilde{C}_i^*$ be the 2-fold branched cover of C_i^* branched at $x_1^i \cup x_2^i$. Then the above diffeomorphisms τ_{e_l} and κ_{e_l} of $\tilde{S}_0$ are converted to $\tilde{C}_i^*$; that is, for $e_l \subset \partial C_i^*$, the holonomy along the curve $t_{e_l}^{C_i^*}$ induces the diffeomorphism $\tau_{e_l}^{\tilde{C}_i^*}$ of $\tilde{C}_i^*$ which is τ_{e_l} under the identification $C_i^* = S_0$, and, for $e_l \subset \partial C_i^*$, the holonomy along the curve $k_{e_l}^{C_i}$ induces the Dehn twist $\kappa_{e_l}^{\tilde{C}_i^*}$ of $\tilde{C}_i^*$ which is κ_{e_l} under the identification $C_i^* = S_0$.

Definition 2.8. Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $b \in \mathcal{B}_\Upsilon$ and let $\tilde{C} = \tilde{C}_b$.

(i) For $e_l = \partial C_i \cap \partial C_j (i \neq j)$, we define a diffeomorphism of $\tilde{C}$ by

$$\tau(e_l) = \begin{cases} \tau_{e_l}^{\tilde{C}_i^*} & \text{on } \tilde{C}_i \\ \tau_{e_l}^{\tilde{C}_j^*} & \text{on } \tilde{C}_j \\ \mathrm{Id} & \text{on } \tilde{C} - \tilde{C}_i \cup \tilde{C}_j. \end{cases} \tag{8}$$

(ii) Let $C_i = T_0$ and let $e_l \in \Upsilon$ be the essential simple closed curve in C_i. We define a diffeomorphism $\tau(e_l)$ of $\tilde{C}$ by

$$\tau(e_l) = \begin{cases} \tau_{e_l^{\pm}}^{\tilde{C}_i^*} \tau_{e_l^{\mp}}^{\tilde{C}_i^*} & \text{on } \tilde{C}_i \\ \mathrm{Id} & \text{on } \tilde{C} - \tilde{C}_i. \end{cases} \tag{9}$$

(iii) For $e_l = \partial C_i \cap \partial C_j$ which is separating in C, let $C = C_- \cup C_i \cup C_+$ be the decomposition of C,where C_+ is the connected component of $C - e_l$ containing C_j. Let $\tilde{C} = \tilde{C}_- \cup \tilde{C}_i \cup \tilde{C}_+$ be the corresponding decomposition of $\tilde{C}$. We define a diffeomorphism $\nu(e_l)$ of $\tilde{C}$ by

$$\nu(e_l) = \begin{cases} \mathrm{Id} & \text{on } \tilde{C}_- \\ \tau_{e_l}^{\tilde{C}_i^*} & \text{on } \tilde{C}_i \\ \sigma & \text{on } \tilde{C}_+. \end{cases} \tag{10}$$

(iv) For $e_l \subset C_i$, $k_{e_l}^{C_i}$ induces a diffeomorphism $\kappa(e_l)$ of $\tilde{C}$ defined by

$$\kappa(e_l) = \begin{cases} \kappa_{e_l}^{\tilde{C}_i^*} & \text{on } \tilde{C}_i \\ \mathrm{Id} & \text{on } \tilde{C} - \tilde{C}_i. \end{cases} \tag{11}$$

LEMMA 2.4. *Let W_Υ^o be the subgroup of W_Υ generated by $\{\left(t_{e_l}^{C_i^*}\right)^2\}$ and $\{k_{e_l}^{C_i^*}\}$. Then there is an exact sequence of groups*

$$1 \to W_\Upsilon^o \to W_\Upsilon \to \mathbf{Z}_2^{3g-3} \to 1.$$

Proof. For $1 \le l \le 3g-3$, the inverse image $p^{-1}(e_l)$ consists of two connected components $\tilde{e}_l$ and $\sigma\tilde{e}_l$. The diffeomorphisms listed in (i) and (ii) in Definition 2.8 interchanges these two connected components. Hence the action of the holonomy diffeomorphisms on the homology classes $\{[\tilde{e}_l - \sigma\tilde{e}_l]\}$ (with $\tilde{e}_l$ suitably oriented) in $H_1(\tilde{C}, \mathbf{R})$ induces the homomorphism $W_\Upsilon \to \mathbf{Z}_2^{3g-3}$ in the above sequence in the lemma. Then the exactness of the sequence is an immediate consequence of the construction. □

2.4. *Marking and the universal cover of* $\mathcal{B}_\Upsilon$. Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $\mathcal{B}_\Upsilon$ be the space defined in Definition 2.5.

Let $\tilde{b} = (b, \alpha) \in \mathcal{B}_\Upsilon$ and let $p : \tilde{C} = \tilde{C}_{\tilde{b}} \to C$ be the corresponding two-fold branched covering surface of C with covering involution σ.

Since $\tilde{b} = (b, \alpha) \in \mathcal{B}_\Upsilon$, we may write $b = \{x_1^i, x_2^i\}_{1 \le i \le 2g-2}$ for $x_1^i, x_2^i \in C_i^o$ and $\tilde{C} = \cup\tilde{C}_i$,where $\tilde{C}_i$ is the 2-fold branched covering surface of C_i branched at $x_1^i \cup x_2^i$ for $1 \le i \le 2g-2$.

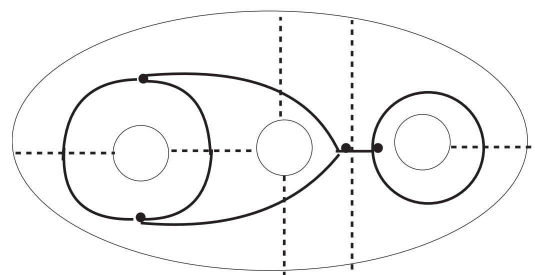

Figure 4: Marking

Definition 2.9. Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $\tilde{b} = (b, \alpha) \in \mathcal{B}_\Upsilon$.

We define a *marking* $\mathfrak{m} = \{f_l, e_l, T\}$ of C associated with Υ as follows:

(i) For $1 \le l \le 3g-3$ such that $e_l = C_i \cap C_j$ $(1 \le i \ne j \le 3g-3)$, f_l is an embedded arc in $C_i \cup C_j$ connecting x_1^i and x_1^j such that $f_l \cap e_l = \{a\ point\}$.

(ii) For $1 \le l \le 3g-3$ such that e_l is an essential curve in a 1-holed torus C_i, f_l is an *essential simple closed curve in* C_i such that $f_l \cap e_l = \{a\ point\}$.

(iii) For $1 \le l \ne l' \le 3g-3$, $f_l \cap f_{l'}$ is empty or x_1^i, where the latter case occurs exactly when $e_l \cup e_{l'} \subset C_i$.

(iv) T is a *maximal tree* which is a 1-complex whose vertices are $\{x_1^i\}_{1 \le i \le 2g-2}$ and $\{f_l \cap e_l\}_{1 \le l \le 3g-3}$ and whose edges are arcs in $\{f_l \cap C_i\}$ connecting x_1^i and $f_l \cap e_l$ in C_i for $1 \le i \le 2g-2$.

The set of pairs $(\tilde{b}, \mathfrak{m})$ for $\tilde{b} \in \mathcal{B}_\Upsilon$ and a marking $\mathfrak{m}$ associated with Υ serves as the universal covering space $\tilde{\mathcal{B}}_\Upsilon$ of $\mathcal{B}_\Upsilon$.

2.5. *The σ-anti-invariant homology group, the Lagrangian $\tilde{\ell}$ and the lattices Λ_0 and Λ.* Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. For the covering surface $p : \tilde{C} \to C$ associated with $\tilde{b} = (b, \alpha) \in \mathcal{B}_\Upsilon$, let

$$H_1(\tilde{C}, \mathbf{R}) = H_1(\tilde{C}, \mathbf{R})_+ \oplus H_1(\tilde{C}, \mathbf{R})_- \tag{12}$$

be the decomposition into the invariant $(+)$ and anti-invariant $(-)$ subspaces of the involution σ_* on $H_1(\tilde{C}, \mathbf{R})$ induced by the covering involution σ. Then $H_1(\tilde{C}, \mathbf{R})_+$ is isomorphic to $H_1(C, \mathbf{R})$ and $\dim_{\mathbf{R}} H_1(\tilde{C}, \mathbf{R})_- = 6g - 6$.

Definition 2.10. We define a symplectic form ω on $H_1(\tilde{C}, \mathbf{R})_-$, for $a, b \in H_1(\tilde{C}, \mathbf{R})_-$, by

$$\omega(a, b) = \frac{1}{2}\langle a, b\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the symplectic form induced by the intersection pairing on $\tilde{C}$.

Let $\tilde{e}_l$ be a connected component of $p^{-1}(e_l)$ $(1 \leq l \leq 3g - 3)$. Then $p^{-1}(e_l) = \tilde{e}_l \cup \sigma\tilde{e}_l$. We choose and fix an orientation of $\tilde{e}_l$.

Let $\tilde{\ell}$ be the subspace in $H_1(\tilde{C}, \mathbf{R})_-$ spanned by $\{[\tilde{e}_l - \sigma\tilde{e}_l]\}_{1\leq l\leq 3g-3}$. Then $\tilde{\ell}$ is Lagrangian with respect to ω.

Definition 2.11. Let $C_i \in \Upsilon$.

(i) Assume $C_i = S_0$ with $\partial C_i = e_{l_1^i} \cup e_{l_2^i} \cup e_{l_3^i}$. We set

$$\begin{aligned} E_1^i &= \frac{1}{2}\left[-(\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}) + (\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}) + (\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i})\right], \\ E_2^i &= \frac{1}{2}\left[(\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}) - (\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}) + (\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i})\right], \\ E_3^i &= \frac{1}{2}\left[(\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}) + (\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}) - (\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i})\right], \\ E_0^i &= \frac{1}{2}\left[(\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}) + (\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}) + (\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i})\right]. \end{aligned} \tag{13}$$

(ii) Assume $C_i = T_0$. Let $e_{l_1^i} = \partial C_i$ and let $e_{l_2^i} \in \Upsilon$ be the essential simple closed curve in C_i. We set

$$\begin{aligned} E_1^i &= -\frac{1}{2}[\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}] + [\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}], \\ E_2^i &= \frac{1}{2}[\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}], \\ E_0^i &= \frac{1}{2}[\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}] + [\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}]. \end{aligned} \tag{14}$$

Those classes are represented by the oriented simple closed curves which are the inverse images in $\tilde{C}_i$ of the arcs in C_i connecting the two branch points $\{x_1^i, x_2^i\}$ in it, and hence are contained in $\tilde{\ell} \cap H_1(\tilde{C}, \mathbf{Z})_-$. In fact $\tilde{\ell} \cap H_1(\tilde{C}, \mathbf{Z})_-$ is spanned by $\{E_1^i, E_2^i, E_3^i\}_{1 \le i \le 2g-2}$.

Associated with a marking, $\mathfrak{m} = \{f_l, e_l\}$, given in Definition 2.9, we have homology classes $\{[\tilde{f}_l - \sigma \tilde{f}_l]\}_{1 \le l \le 3g-3}$ in $H_1(\tilde{C}, \mathbf{R})_-$, where $\tilde{f}_l$ is a component of $p^{-1}(f_l)$ oriented in such a way that $\omega\left([\tilde{e}_l - \sigma \tilde{e}_l], [\tilde{f}_l - \sigma \tilde{f}_l]\right) = 1$.

For $1 \le l, k \le 3g-3$ we choose $d_{lk} \in \mathbf{Z}$ so that $d_{ll} = 0$, $\sum_{1 \le l \le 3g-3} d_{lk} \in 2\mathbf{Z}$, and $\tilde{f}_l^* \in H_1(\tilde{C}, \mathbf{Z})_-$ defined by

$$\tilde{f}_l^* = [\tilde{f}_l - \sigma \tilde{f}_l] + \sum_{1 \le k \le 3g-3} d_{lk} [\tilde{e}_k - \sigma \tilde{e}_k]$$

satisfies

$$\omega\left([\tilde{e}_l - \sigma \tilde{e}_l], \tilde{f}_k^*\right) = \delta_{lk} \quad , \quad \omega\left(\tilde{f}_l^*, \tilde{f}_k^*\right) = 0.$$

(We note that we can construct one such example of $\{d_{lk} \in \mathbf{Z}\}$ by using the notion of 'grouping' which will be defined in §8.1.)

We denote $\tilde{\ell}^*$ the Lagrangian spanned by $\{\tilde{f}_l^*\}$.

Definition 2.12. (i) Let Λ_0 be the integral lattice in $\tilde{\ell}$ generated by $\{[\tilde{e}_l - \sigma \tilde{e}_l]\}$. Let Λ_0^* be the integral lattice in $\tilde{\ell}^*$ spanned by $\{\tilde{f}_l^*\}_{1 \le l \le 3g-3}$, where $\{\tilde{f}_l^*\}_{1 \le l \le 3g-3}$ and $\tilde{\ell}^*$ are defined as above.

(ii) Let Λ be the integral lattice in $\tilde{\ell}$ generated by $\{E_1^i, E_2^i, E_3^i\}_{1 \le i \le 2g-2}$. Let Λ^* be the integral lattice in $\tilde{\ell}^*$ which is the symplectic dual of Λ. Now, Λ^* is a subset of Λ_0^* consisting of those vectors $\{\sum_l n_l \tilde{f}_l^* \in \Lambda_0^*\}$ such that, for each $C_i \in \Upsilon$ with $\partial C_i^* = e_{l_1^i} \cup e_{l_2^i} \cup e_{l_3^i}$,

$$n_{l_1^i} + n_{l_2^i} + n_{l_3^i} \in 2\mathbf{Z}, \tag{15}$$

where $n_{l_2^i} = n_{l_3^i}$ if $C_i = T_0$.

3. Family of Prym varieties

3.1. *Prym varieties and dominant maps to the moduli space of semistable rank two bundles on C.* Let $p : \tilde{C} \to C$ be a 2-fold branched covering, where $\tilde{C} = \tilde{C}_{\tilde{b}}$ for $\tilde{b} = (b, \alpha) \in \mathcal{B}$. Let J be the Jacobian of C.

Let $\mathfrak{d}$ be the line bundle over C of degree $2g-2$ such that $p_* \mathcal{O}_{\tilde{C}} = \mathcal{O}_C \oplus \mathfrak{d}^{-1}$. Let $\tilde{J}$ be the Jacobian of $\tilde{C}$, and let $\tilde{J}^{2g-2}$ be the variety which parametrizes the line bundles of degree $2g-2$ on $\tilde{C}$.

For a line bundle L on $\tilde{C}$, let $p_* L$ be the direct image of L which is a rank 2 bundle on C with determinant $\mathrm{Nm}(L) \otimes \mathfrak{d}^{-1}$. In particular for $L \in \tilde{J}^{2g-2}$, $p_* L$ is of degree 0. Let

$$P' = \{L \in \tilde{J}^{2g-2} \mid \mathrm{Nm}(L) = \mathfrak{d}\}. \tag{16}$$

Then for $L \in P'$, the determinant of $p_* L$ is trivial.

P' is an Abelian variety of dimension $3g-3$. Let P'_s (resp. P'_{ss}) be the subset of P' consisting of those $L \in P'$ such that p_*L is stable (resp. semistable).

LEMMA 3.1 ([4], [6]). *$P' - P'_{ss}$ (resp. $P' - P'_s$) is a subvariety of P' of codimension $\geq g+1$ (resp. $\geq g-1$).*

Proof. p_*L is not semistable (resp. stable) if it contains a line subbundle M of positive (resp. nonnegative) degree. Then there is a nonzero homomorphism $p^*M \to L$. Hence $L = p^*M(D)$ for an effective divisor D on $\tilde{C}$ such that $\mathrm{Nm}(M(D)) = \mathfrak{d}$. Let $u_p : J^r \times \tilde{C}^{2g-2-2r} \to P'$ be the morphism defined by $u_r(M, D) = p^*M(D)$, where J^r denotes the variety parametrizing the isomorphism classes of line bundles of degree r on C. The image of u_r restricted to those pairs (M, D) such that $\mathrm{Nm}(M(D)) = \mathfrak{d}$ is a subvariety of P' of codimension $\geq g-1+2r$. The subset of L such that p_*L is not semistable is the union of those subvarieties and the lemma follows. □

Let M_g be the moduli space of semistable, holomorphic, rank-two vector bundles on C with trivial determinant. Let M_{gs} be the subset of M_g consisting of the isomorphism classes of stable holomorphic rank 2 bundles. M_{gs} is Zariski dense in M_g.

From the above argument it follows that the map $L \to p_*L$ defines a morphism $\pi' : P'_{ss} \to M_g$ and $\pi' : P'_s \to M_{gs}$.

PROPOSITION 3.1 ([4], [6]). *The morphism $\pi' : P'_s \to M_{gs}$ is dominant.*

Proof. Let $L \in P'_s$. The sheaf p_*L has a structure of a $p_*\mathcal{O}_{\tilde{C}}$-module, and it induces a homomorphism $\nu : p_*\mathcal{O}_{\tilde{C}} \to \mathrm{End}(p_*L)$. On the other hand the tangent space $T_{p_*L}(M_g)$ is canonically identified with $H^1(C, \mathrm{End}(p_*L))$, and the space $T_L(P')$ with $H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})$ which is isomorphic to $H^1(C, p_*\mathcal{O}_{\tilde{C}})$. By functoriality the differential $d\pi'_L$ of π' at L is identified with $H^1(\nu)$.

Let N be the kernel of the canonical surjective homomorphism $p^*p_*L \to L$. We have an exact sequence

$$0 \to \mathrm{Hom}(L, L) \to \mathrm{Hom}(p^*p_*L, L) \to \mathrm{Hom}(N, L) \to 0. \tag{17}$$

Applying p_*, we have

$$0 \to p_*\mathcal{O}_{\tilde{C}} \to \mathrm{End}(p_*L) \to p_*(N^{-1} \otimes L) \to 0. \tag{18}$$

Hence the cokernel of $H^1(\nu)$ which is the first homomorphism of the above exact sequence is identified with $H^1(\tilde{C}, N^{-1} \otimes L)$. Since $\det(p_*L) = \mathrm{Nm}(L) \otimes \mathfrak{d}^{-1}$, we have $N = L^{-1} \otimes p^* \det(p_*L) = \sigma^*L \otimes p^*\mathfrak{d}^{-1}$, and $N^{-1} \otimes L = L \otimes \sigma^*L^{-1} \otimes p^*\mathfrak{d}$. Since the canonical bundle $K_{\tilde{C}}$ of $\tilde{C}$ is isomorphic to $p^*(K_C \otimes \mathfrak{d})$, by the duality, $T_L(\pi'_*)$ is surjective if and only if the space $H^0(\tilde{C}, \sigma^*L \otimes L^{-1} \otimes p^*K_C)$ is zero. Since the genus of $\tilde{C}$ is $4g-3$, $d\pi'_L$ is surjective on a Zariski open set. □

3.2. *A coordinate on a Prym variety.* Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $(\tilde{b}, \mathfrak{m}) \in \tilde{\mathcal{B}}_\Upsilon$, where $\mathfrak{m}$ is a marking of C associated with Υ and $\tilde{b} = (b, \alpha)$ for $b = \{x_1^i, x_2^i\}_{1 \leq i \leq 2g-2}$ such that $x_1^i, x_2^i \in C_i^o$ (§2.4).

Let η_0 be a divisor of degree 0 of $\tilde{C} = \tilde{C}_{\tilde{b}}$ such that $\sigma^* \eta_0 = \eta_0$ and

$$2\eta_0 = -\sum_{i=1}^{2g-2} [x_1^i] + \sum_{i=1}^{2g-2} [x_2^i]. \tag{19}$$

Let η be the divisor of degree $2g-2$ of $\tilde{C}$ defined by

$$\eta = \eta_0 + \sum_{i=1}^{2g-2} [x_1^i]. \tag{20}$$

Formally we may write $\eta = \frac{1}{2} \sum_{i=1}^{2g-2} \left([x_1^i] + [x_2^i] \right)$.

We denote the corresponding line bundle by the same letter η. Then clearly $\eta = \sigma^* \eta$ and $\eta \in P'$. We choose η as an origin of P'.

We write a line bundle L on $\tilde{C}$ of degree $2g-2$ as $L = \eta L_0$ for a degree 0 line bundle L_0 on $\tilde{C}$. Then, since $\sigma^* \eta \otimes \eta = [b]$, the condition that $\eta L_0 \in P'$ is equivalent to $\sigma^* L_0 \otimes L_0 = 1$, that is, L_0 is σ-anti-invariant.

Thus choosing η as the origin of the Prym variety, we see that P' can be identified with the set of the isomorphism classes of σ-anti-invariant degree 0 line bundles on $\tilde{C}$.

For $1 \leq i \leq 2g-2$ let $\tilde{C}_i$ be the 2-fold branched cover of C_i with branch set $\{x_1^i, x_2^i\}$. Then the set of the isomorphism classes of σ-anti-invariant degree 0 line bundles on $\tilde{C}$ can be coordinated by $(z_l)_{1 \leq l \leq 3g-3}$, where (z_l) represents the line bundle on $\tilde{C}$ constructed from the disjoint union of the trivial bundles $\bigcup \tilde{C}_i \times \mathbf{C}$ by attaching them by the transition functions $\exp(2\pi i z_l)$ at $\tilde{e}_l$ and $\exp(-2\pi i z_l)$ at $\sigma \tilde{e}_l$. We use (z_l) as the coordinate of the universal cover of P'.

Let $(\tilde{\ell}, \tilde{\ell}^*)$ be the Lagrangian pair in $H_1(\tilde{C}, \mathbf{R})_-$ given in Section 2.5, and let Λ_0 and Λ_0^* be the integral lattices in $\tilde{\ell}$ and $\tilde{\ell}^*$ respectively given there.

Then $H_1(\tilde{C}, \mathbf{Z})_- = \Lambda + \Lambda_0^*$, and as a real symplectic manifold we have

$$P' = H_1(\tilde{C}, \mathbf{R})_- / (\Lambda + \Lambda_0^*).$$

Here we make the following important remark; P' is difficult to manage for technical reasons and it is much more convenient for us to consider the covering space P of P' defined by

$$P = H_1(\tilde{C}, \mathbf{R})_- / (\Lambda_0 + \Lambda_0^*). \tag{21}$$

There is a covering map $P \to P'$ whose covering transformation is the translation by an element of Λ/Λ_0, and P is an abelian variety with the complex structure compatible with that of P'.

Instead of studying P' directly we consider everything as Λ-invariant objects on P, and from now on we call P as Prym variety. Also $\pi : P \to M_g$

denote the obvious map, and P_s and P_{ss} denote the set of the same meaning as P'_s and P'_{ss} respectively.

Let $\{[\tilde{e}_l - \sigma\tilde{e}_l], \tilde{f}_l^*\}_{1\leq l\leq 3g-3}$ be the symplectic basis of $H_1(\tilde{C}, \mathbf{R})_-$ given in Definition 2.12.

Let $\{w_l\}_{1\leq l\leq 3g-3}$ be the holomorphic 1-forms on $\tilde{C}$ such that $\sigma^* w_l = -w_l$ and that, for $1 \leq l, l' \leq 3g-3$,

$$\int_{\tilde{e}_l - \sigma\tilde{e}_l} w_{l'} = \delta_{ll'}. \tag{22}$$

The set $\{w_l\}_{1\leq l\leq 3g-3}$ forms a basis of the space of σ-anti-invariant holomorphic 1-forms on $\tilde{C}$.

Definition 3.1. The Riemann matrix associated with the lattice $\Lambda_0 + \Lambda_0^*$

$$\Omega = (\Omega_{ij})_{1\leq i,j\leq 3g-3} \tag{23}$$

is defined by

$$\Omega_{ij} = \int_{\tilde{f}_j^*} w_i. \tag{24}$$

Then Ω is a complex symmetric matrix and its imaginary part, $\operatorname{Im}\Omega$, is positive definite. $\Lambda_0 + \Omega\Lambda_0^*$ forms a lattice in $\mathbf{C}^{3g-3}$ and we have, as a complex variety,

$$P = \mathbf{C}^{3g-3} / (\Lambda_0 + \Omega\Lambda_0^*) \ . \tag{25}$$

The symplectic form ω on P is represented by the de Rham cohomology class

$$\omega = \frac{i}{2} \sum (\operatorname{Im}\Omega)^{-1}_{ij} dz_i \wedge d\bar{z}_j. \tag{26}$$

Definition 3.2. Let $\tilde{\mathcal{L}}$ be the holomorphic hermitian line bundle on P with nontrivial holomorphic section whose curvature form is ω.

4. Riemann theta functions on polarized Prym varieties

4.1. *Riemann theta functions on the polarized Prym variety.* Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $(\tilde{b}, \mathfrak{m}) \in \tilde{\mathcal{B}}_\Upsilon$ and let $P = P_{(\tilde{b},\mathfrak{m})}$ be the corresponding polarized Prym variety. Let $\pi : P_s \to M_g$ be the dominant map defined in Section 3.2.

Let $\mathcal{L}$ be the determinant line bundle on M_g; i.e., $\mathcal{L}$ corresponds to the divisor of M_g defined by the set of rank two semi-stable bundles E on C such that $H^0(C, E \otimes F) \neq 0$, where F is the line bundle on C satisfying $F^2 = K_C$ corresponding to the theta constant of C ([18]). Since the codimension of P_{ss} in P is greater than g, the pull-back of $\mathcal{L}^k$ to P_{ss} extends to a line bundle on

P which we denote by $\pi^*\mathcal{L}^k$. Also the pull-back of a holomorphic section of $\mathcal{L}^k$ extends to one of $\pi^*\mathcal{L}^k$ by Hartog's theorem.

LEMMA 4.1 ([4, Lemme 1.7]).

$$c_1(\pi^*\mathcal{L}) = [2\omega],$$

where the right-hand side denotes the de Rham cohomology class of 2ω.

Since an isomorphism class of a holomorphic line bundle with nontrivial holomorphic section on an abelian variety is determined by its first Chern class, $\pi^*\mathcal{L}$ is isomorphic to the line bundle $\tilde{\mathcal{L}}^2$, where $\tilde{\mathcal{L}}$ is the line bundle defined in Definition 3.2.

Thus the pull back by π of a holomorphic section of $\mathcal{L}^k$ is a holomorphic section of $\tilde{\mathcal{L}}^{2k}$, and it can be described as a Riemann theta function of level $2k$ on P.

For a positive integer k and

$$\vec{a} \in \Lambda_0^* \otimes \mathbf{Q}\ ,\ \vec{b} \in \Lambda_0 \otimes \mathbf{Q}, \tag{27}$$

we define $\vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (2k\vec{z}, 2k\Omega)$ by

(28)

$$\vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (2k\vec{z}, 2k\Omega) = \sum_{\vec{n} \in \Lambda_0^*} \exp\bigl(\pi i (\vec{n} + \vec{a})^t 2k\Omega(\vec{n} + \vec{a}) + 2\pi i (\vec{n} + \vec{a})^t (2k\vec{z} + \vec{b})\bigr),$$

where $\vec{a}$,$\vec{n}$ and $\vec{b}$ are thought of as column vectors with respect to the basis $\{[\tilde{f}_l^*]\}_{1\le l\le 3g-3}$ and $\{[\tilde{e}_l - \sigma\tilde{e}_l]\}_{1\le l\le 3g-3}$ respectively, $\vec{z}$ is a column vector in $\mathbf{C}^{3g-3}$ and $\vec{a}^t$ etc. denote their transposed vectors (we use the notation given in [18] for the Riemann theta function). The space Θ_{2k} of Riemann theta functions of level $2k$ on P associated with the lattice Λ_0 has a base given by $\left\{ \vartheta \begin{bmatrix} \vec{a} \\ \vec{0} \end{bmatrix} (2k\vec{z}, 2k\Omega) \right\}_{\vec{a} \in \frac{1}{2k}\Lambda_0^*}$.

4.2. *The heat equation.* For $(\tilde{b}, \mathfrak{m}) \in \tilde{\mathcal{B}}_\Upsilon$ let $P = P_{(\tilde{b},\mathfrak{m})}$ be the associated polarized Prym variety.

The complex structure $J = J_\Omega$ on $P = P_{(\tilde{b},\mathfrak{m})}$ is parametrized by Ω given in equation (23) in Definition 3.1 which is an element of the Siegel domain $\mathcal{S}$ of complex symmetric $(3g-3)\times(3g-3)$ matrices with positive definite imaginary part.

The map $\Omega \to J_\Omega$ is a holomorphic map. If we denote by δ the holomorphic derivative with respect to Ω, then

$$\delta J = -(\delta\Omega)(\mathrm{Im}\Omega)^{-1}. \tag{29}$$

As in [2], [18], the holomorphic derivatives on the sections of the line bundle $\tilde{\mathcal{L}}^{2k}$ become

$$\nabla_i \tilde{\psi}(z,\Omega) = \left(\frac{\partial}{\partial z_i} - 8k\pi(\mathrm{Im}\Omega)^{-1}_{ij}(z_j - \bar{z}_j)\right)\tilde{\psi}(z,\Omega), \tag{30}$$
$$\delta\tilde{\psi}(z,\Omega) = \left(\delta^{\Omega} + \frac{1}{2i}((\delta\Omega)(\mathrm{Im}\Omega)^{-1})_{ij}(z_j - \bar{z}_j)\frac{\partial}{\partial z_i}\right)\tilde{\psi}(z,\Omega)$$
$$+2k\frac{\pi}{i}((\mathrm{Im}\Omega)^{-1}(\delta\Omega)(\mathrm{Im}\Omega)^{-1})_{ij}(z_i - \bar{z}_i)(z_j - \bar{z}_j)\tilde{\psi}(z,\Omega),$$

where δ^{Ω} denotes the partial differential operator in the variables Ω_{ij}. The anti-holomorphic derivatives are given by

$$\bar{\nabla}_i = \frac{\partial}{\partial \bar{z}_i}, \qquad \bar{\delta} = \bar{\delta}^{\Omega}. \tag{31}$$

If we combine the equations (29),(30) and (31), the differential operator which gives the parallelism on the space of Riemann theta functions (which is a section of $\tilde{\mathcal{L}}^{2k}$) is

$$\delta + \frac{1}{8k}(\delta J\omega^{-1})_{ij}\nabla_i\nabla_j = \delta^{Th} + \frac{i}{4}\mathrm{tr}(\delta J), \tag{32}$$

where

$$\delta^{Th}\tilde{\psi}(z,\Omega) = \left(\delta^{\Omega} - \frac{1}{8\pi k i}(\delta\Omega)_{ij}\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\right)\tilde{\psi}(z,\Omega). \tag{33}$$

The differential operator acting on Θ_{2k}

$$\bar{\delta} + \delta + \frac{1}{8k}(\delta J\omega^{-1})_{ij}\nabla_i\nabla_j \tag{34}$$

gives a projectively flat connection on the bundle over the Siegel domain $\mathcal{S}$ with fibre Θ_{2k} whose curvature is central and which is given by the 2-form on $\mathcal{S}$, $\frac{i}{4}\mathrm{tr}(\bar{\delta}J\delta J)$. The differential operator $\bar{\delta}+\delta^{Th}$ gives the metaplectic correction of it on $\mathcal{S}$. Thus we represent the metaplectic correction on $\mathcal{S}$ by replacing the operator δ by $\delta - \frac{i}{4}\mathrm{tr}(\delta J)$ ([2], [18]).

4.3. *Actions of W_Υ on Riemann theta functions and automorphic forms.*

Definition 4.1. (i) For a positive integer k, let A_{2k} be the vector space of automorphic forms of level $2k$ associated with the lattice Λ, that is, an element of A_{2k} is a holomorphic function $q(\Omega_\Lambda)$ of Riemann matrix Ω_Λ of $\tilde{C}$ associated with Λ which has automorphy with respect to the Siegel modular group. Throughout this paper we only deal with the case where Ω_Λ is obtained from Ω by prescribed linear transformation. Hence we consider $q(\Omega_\Lambda)$ also as a holomorphic function of Ω.

(ii) Let $A_{2k}\cdot\Theta_{2k}$ be the space of Riemann theta functions of level $2k$ with coefficients in A_{2k} on the polarized Prym variety $P = P_{(\tilde{b},\mathfrak{m})}$ for $(\tilde{b},\mathfrak{m}) \in \tilde{\mathcal{B}}_\Upsilon$.

Let W_Υ be the group given in Definition 2.6 in Section 2.2. We consider the $\mathbf{Z}_2^{3g-3}$-action on $A_{2k} \cdot \Theta_{2k}$ induced by W_Υ.

From the description of the holonomy action of W_Υ in Section 2.3 and Lemma 2.4, it follows that W_Υ induces a $\mathbf{Z}_2^{3g-3} = \{\pm 1\}^{3g-3}$-action on $H_1(\tilde{C}, \mathbf{R})_-$ preserving $\tilde{\ell}$ given by, for $\vec{\varepsilon} = (\varepsilon_l)_{1 \le l \le 3g-3}$,

$$\vec{\varepsilon} \cdot [\tilde{e}_l - \sigma \tilde{e}_l] = \varepsilon_l [\tilde{e}_l - \sigma \tilde{e}_l] \quad , \quad \vec{\varepsilon} \cdot \tilde{f}_l^* = \varepsilon_l \tilde{f}_l^*. \tag{35}$$

In each $C_i \in \Upsilon (1 \le i \le 2g-2)$, the action is the combination of the following three involutions

$$\begin{cases} \iota_1^i : (E_1^i, E_2^i, E_3^i) \to (E_0^i, -E_3^i, -E_2^i) \\ \iota_2^i : (E_1^i, E_2^i, E_3^i) \to (-E_3^i, E_0^i, -E_1^i) \\ \iota_3^i : (E_1^i, E_2^i, E_3^i) \to (-E_2^i, -E_1^i, E_0^i). \end{cases} \tag{36}$$

These involutions correspond to the $\mathbf{Z}_2^{3g-3}$-action on $\Lambda_0^* \otimes \mathbf{Q}$ given by, for $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_{3g-3})^t \in \mathbf{Z}_2^{3g-3}$ and $\vec{a} = (a_1, \cdots, a_{3g-3})^t \in \Lambda_0^* \otimes \mathbf{Q}$,

$$\vec{\varepsilon} \cdot \vec{a} = (\varepsilon_1 a_1, \cdots, \varepsilon_{3g-3} a_{3g-3})^t. \tag{37}$$

Also we have the corresponding change of the coordinate $\vec{z} = (z_l)^t$ on the Prym variety

$$\vec{z} \to \vec{\varepsilon} \cdot \vec{z} = (\varepsilon_1 z_1, \cdots, \varepsilon_{3g-3} z_{3g-3})^t \tag{38}$$

and that of the Riemann matrix

$$\Omega \to \vec{\varepsilon} \cdot \Omega = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_{3g-3} \end{pmatrix} \Omega \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_{3g-3} \end{pmatrix}, \tag{39}$$

and a similar change of Ω_Λ.

For a Riemann theta function $\vartheta \begin{bmatrix} \vec{a} \\ \vec{0} \end{bmatrix} (2k\vec{z}, 2k\Omega) \in \Theta_{2k}$, this change of variables is equivalent to the substitution of characteristics $\vec{a} \to \vec{\varepsilon} \cdot \vec{a}$.

The diffeomorphism $\kappa(e_l)$ given in Definition 2.8 induces the endomorphism of the line bundle η of equation (20) covering $\kappa(e_l)$. It induces the change of the complex structure of η, and hence it induces the shift of the base point of P. From the fact that $\kappa(e_l)$ is half the Dehn twist on the homology class in the pant interchanging the two branch points, the resulting shift operator on the space of Riemann theta functions is the action as such given in (iii) in the next definition below. To summarize, we make the following definition.

Definition 4.2. (i) We define $\mathbf{Z}_2^{3g-3}$-action on A_{2k} by, for $\vec{\varepsilon} \in \mathbf{Z}_2^{3g-3}$ and $q(\Omega_\Lambda) \in A_{2k}$,

$$q(\Omega_\Lambda) \to q(\vec{\varepsilon} \cdot \Omega_\Lambda).$$

(ii) We define $\mathbf{Z}_2^{3g-3}$-action on Θ_{2k} by, for $\vec{\varepsilon} \in \mathbf{Z}_2^{3g-3}$

$$\vartheta \begin{bmatrix} \vec{a} \\ \vec{0} \end{bmatrix} (2k\vec{z}, 2k\Omega) \to \vartheta \begin{bmatrix} \vec{\varepsilon} \cdot \vec{a} \\ \vec{0} \end{bmatrix} (2k\vec{z}, 2k\Omega).$$

(iii) We define the shift operator $S^0_{\frac{1}{2}\Lambda^*/\Lambda_0^*}$ by

$$S^0_{\frac{1}{2}\Lambda^*/\Lambda_0^*} \left(\vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (2k\vec{z}, 2k\Omega) \right) = \sum_{\vec{\lambda} \in \frac{1}{2}\Lambda^*/\Lambda_0^*} \vartheta \begin{bmatrix} \vec{a} + \vec{\lambda} \\ \vec{b} \end{bmatrix} (2k\vec{z}, 2k\Omega),$$

where $\vec{a} \in \frac{1}{2k}\Lambda_0^*$, and $\vec{b} = \vec{0}$ or $\vec{\frac{1}{2}} = (\frac{1}{2}, \cdots, \frac{1}{2})^t$.

(iv) Also for later use we define the anti-invariant shift operator $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}$ by, for $\vec{a} \in \frac{1}{2k}\Lambda_0^*$,

$$S_{\frac{1}{2}\Lambda^*/\Lambda_0^*} \left(\vartheta \begin{bmatrix} \vec{a} \\ \vec{0} \end{bmatrix} (2k\vec{z}, 2k\Omega) \right) = \sum_{\vec{\lambda} \in \frac{1}{2}\Lambda^*/\Lambda_0^*} e^{2\pi i \left(\vec{\lambda}^t \vec{\frac{1}{2}} \right)} \vartheta \begin{bmatrix} \vec{a} + \vec{\lambda} \\ \vec{0} \end{bmatrix} (2k\vec{z}, 2k\Omega).$$

The group of the holonomy diffeomorphisms induced by W^o_Υ is generated by a Dehn twist of $\tilde{C}_{\tilde{b}}$ along simple closed curves each of which is contained in $\tilde{C}_i (1 \leq i \leq 2g-2)$. Those holonomy diffeomorphisms induce symplectic automorphisms of $H_1(\tilde{C}, \mathbf{R})_-$, and hence we have a projective action of W^o_Υ on $A_{2k} \cdot \Theta_{2k}$.

Definition 4.3. Let $\psi \in A_{2k} \cdot \Theta_{2k}$. Then ψ is called *projectively invariant* under W^o_Υ if, for $\gamma \in W^o_\Upsilon$,

$$\gamma\psi = c\psi$$

for a complex number c which depends on both of γ and ψ.

5. Branching divisor and theta function Π

Proposition 3.1 and its proof show that the dominant map $\pi : P \to M_g$ is a holomorphic branched covering whose branching locus is given by

$$\{L \in P \mid H^0(\tilde{C}, \sigma^* L \otimes L^{-1} \otimes p^* K_C) \neq 0\}, \tag{40}$$

where $p : \tilde{C} \to C$ is the covering map.

We write $L = \eta L_0 \in P$ for a degree 0 divisor L_0 as in Section 3.2. Then, since $\sigma^* L_0 = L_0^{-1}$, the above condition is equivalent to the condition $H^0(\tilde{C}, L_0^{-2} \otimes p^* K_C) \neq 0$. Furthermore, since $K_{\tilde{C}} = p^*\mathfrak{d} \otimes p^* K_C = [b] \otimes p^* K_C$, it is equivalent to the condition

$$H^0(\tilde{C}, L_0^2 \otimes [b]) \neq 0 \tag{41}$$

by the Serre duality and the Riemann-Roch theorem. Let $\Delta_{\tilde{C}}$ and Δ_C be the theta constants of $\tilde{C}$ and C respectively [18, Chap.II §3]. We define the 'relative' theta characteristic Δ_P by $\Delta_P = \Delta_{\tilde{C}} - \pi^*\Delta_C$.

Let $\vartheta(\vec{z}, \Omega)$ be the Riemann theta function on P defined by

$$\vartheta(\vec{z}, \Omega) = \sum_{\vec{n} \in \Lambda_0^*} \exp\left(\pi i \vec{n}^t \Omega \vec{n} + 2\pi i \vec{n}^t \vec{z}\right). \tag{42}$$

Then the locus of L_0 satisfying the condition (41) is given by the divisor of the Riemann theta function which is $S^0_{\frac{1}{2}\Lambda^*/\Lambda_0^*}$-image of the Riemann theta function obtained from $\vartheta(\vec{z}, \Omega)$ by the change of variables $\vec{z} \to 2\vec{z}$ and shifting by the characteristic Δ_P.

Proposition 5.1. *Let* $\vec{\frac{1}{2}} = (\frac{1}{2}, \cdots, \frac{1}{2})^t$.
Then

$$\Delta_P = \vec{\frac{1}{2}} + \Omega\vec{\frac{1}{2}} \in \frac{1}{2}\left(\Lambda_0 + \Omega\Lambda_0^*\right). \tag{43}$$

Proof. We calculate Δ_P in a similar way as in [18, Th. 3.1] and [19, Th. 5.3].

We give the proof under the assumption that $C_i = S_0$ for all $C_i \in \Upsilon$. In the case that there is a C_i such that $C_i = T_0$ a slight modification of the following calculation does well.

Let T be the 1-complex in C defined in Definition 2.9 (iv) in Section 2.4. Let $\tilde{T} = p^{-1}T$ be the inverse image of T in $\tilde{C}$. We cut open $\tilde{C}$ along $\tilde{T}$ and $\bigcup_l(\tilde{e}_l \cup \sigma\tilde{e}_l)$ to a disjoint union of simply connected surfaces $\tilde{\Delta} = \bigcup_{1 \le i \le 2g-2} \tilde{\Delta}_i$, where $\tilde{\Delta}_i$ is $\tilde{C}_i$ cut open along $\tilde{T}$. We use the notation

$$\partial\tilde{\Delta}_i = \bigcup_{e_l \subset C_i} (\tilde{e}^i_l \cup \sigma\tilde{e}^i_l) \cup (\tilde{f}^{i+}_l \cup (\sigma\tilde{f}^{i+}_l) \cup (\tilde{f}^{i-}_l \cup (\sigma\tilde{f}^{i-}_l),$$

where $\tilde{f}^{i+}_l$ corresponds to the endpoint of $\tilde{e}^i_l$.

Let $\vec{w} = (w_1, \cdots, w_{3g-3})^t$, where $\{w_l\}_{1 \le l \le 3g-3}$ is the basis of σ-anti-invariant holomorphic 1-forms on $\tilde{C}$ satisfying equation (22). As in [18, Th. 3.1], we define the function on $\tilde{C}_0$, for all $\vec{z} \in \mathbf{C}^{3g-3}$, $h(P) = \vartheta\left(\vec{z} + \int_{\sigma P}^{P} \vec{w}, \Omega\right)$, where, for $P \in \tilde{\Delta}_i$, the line integral is taken along a path in $\tilde{\Delta}_i$.

Although the function of P, $\vec{z} + \int_{\sigma P}^{P} \vec{w}$, has discontinuities across the boundaries $\partial\tilde{\Delta}$, the values of the discontinuities are contained in the lattice $\Lambda_0 + \Omega\Lambda_0$; hence the set of zeros of $h(P)$ is well defined by the quasi-periodicity of the theta function.

For $1 \le k \le 3g-3$, let g_k be the half of the indefinite integral of ω_k on $\tilde{\Delta}$ defined, for $x \in \tilde{\Delta}_i (1 \le i \le 2g-2)$, by

$$g_k(x) = \frac{1}{2}\int_{\sigma x}^{x} w_k, \tag{44}$$

where the right-hand side denotes the line integral along a path in $\tilde{\Delta}_i$ connecting σx and x.

In the same way as in the proof of [18, Th. 3.1] we see that there are exactly $6g-6$ points (counted with multiplicity if necessary) $\{Q_r\}_{1\le r\le 6g-6}$ such that $h(Q_r)=0$ and we may assume that $\cup_r Q_r$ are contained in the interior of $\tilde{\Delta}$. Let D_r be a small disc neighborhood of Q_r for $1\le r\le 6g-6$.

Then we have the equation

$$
\begin{aligned}
(45)\qquad 0 &= \int_{(\tilde{\Delta}-\cup D_r)} d\left(g_k \frac{dh}{h}\right) \\
&= -\sum_{r=1}^{6g-6}\int_{\partial D_r} g_k\frac{dh}{h} + \sum_{l,i}\int_{(\tilde{e}_l^i+\sigma\tilde{e}_l^i)} g_k\frac{dh}{h} \\
&\quad + \sum_{l,i}\int_{(\tilde{f}_l^{i+}+\sigma\tilde{f}_l^{i+})} g_k\frac{dh}{h} + \int_{(\tilde{f}_l^{i-}+\sigma\tilde{f}_l^{i-})} g_k\frac{dh}{h}.
\end{aligned}
$$

Taking these terms one at a time, we have

$$
(46)\qquad \sum_{r=1}^{6g-6}\int_{\partial D_r} g_k\frac{dh}{h} = \sum_{r=1}^{6g-6} 2\pi i g_k(Q_r) = 2\pi i\sum_{r=1}^{6g-6}\frac{1}{2}\int_{\sigma Q_r}^{Q_r} w_k.
$$

Next we consider the third and the fourth terms in the last line of equation (45).

In the following we use the notations $h^i = h|\tilde{\Delta}_i$ and $g_k^i = g_k|\tilde{\Delta}_i$.

For $e_l\subset C_i$, g_k^i on $(\tilde{f}_l^{i+}+\sigma\tilde{f}_l^{i+})$ is g_k^i on $(\tilde{f}_l^{i-}+\sigma\tilde{f}_l^{i-})$ plus $\frac{1}{2}\delta_{kl}$ because the path $\tilde{e}_l^i-\sigma\tilde{e}_l^i$ leads from $(\tilde{f}_l^{i-}+\sigma\tilde{f}_l^{i-})$ to $(\tilde{f}_l^{i+}+\sigma\tilde{f}_l^{i+})$ and $\int_{(\tilde{e}_l^i-\sigma\tilde{e}_l^i)} w_k=\delta_{kl}$. So for $e_l=C_i\cap C_j$, using the notation $\tilde{f}_l^{\pm}=\tilde{f}_l^{i\pm}\cup\tilde{f}_l^{j\pm}$, we have

$$
\begin{aligned}
(47)\qquad &\sum_l \int_{(\tilde{f}_l^{+}+\sigma\tilde{f}_l^{+})} g_k\frac{dh}{h} - \int_{(\tilde{f}_l^{-}+\sigma\tilde{f}_l^{-})} g_k\frac{dh}{h} \\
&= \sum_l \frac{1}{2}\delta_{kl}\int_{(\tilde{f}_l^{+}+\sigma\tilde{f}_l^{+})}\frac{dh}{h} \\
&= \sum_l \frac{1}{2}\delta_{kl}\left[\int_{(\tilde{f}_l^{i+}-\tilde{f}_l^{j+})}\frac{dh}{h} + \int_{(\sigma\tilde{f}_l^{i+}-\sigma\tilde{f}_l^{j+})}\frac{dh}{h}\right] \\
&\equiv -\pi i\Omega_{kk} - 2\pi i z_k \mod 2\pi i\mathbf{Z}.
\end{aligned}
$$

Next we consider the second term in the last line of equation (45).

Note that, for $e_l=C_i\cap -C_j$, $\frac{dh}{h}$ on $\tilde{e}_l^i$ ($\sigma\tilde{e}_l^j$ resp.) is equal to $\frac{dh}{h}$ on $\tilde{e}_l^j$ ($\sigma\tilde{e}_l^i$ resp.) minus $2w_l$.

Hence for $e_l = C_i \cap -C_j$,

$$\begin{aligned}
(48)\qquad &\int_{(\tilde{e}_l^i+\tilde{e}_l^j)} g_k \frac{dh}{h} + \int_{(\sigma\tilde{e}_l^i+\sigma\tilde{e}_l^j)} g_k \frac{dh}{h} \\
&= \int_{\tilde{e}_l^i} g_k^i \left(\frac{dh^j}{h^j} - 2\pi i(2w_l)\right) - g_k^j \frac{dh^j}{h^j} \\
&\quad + \int_{\sigma\tilde{e}_l^j} g_k^j \left(\frac{dh^i}{h^i} - 2\pi i(2w_l)\right) - g_k^i \frac{dh^i}{h^i} \\
&= -2\pi i \int_{\tilde{e}_l^i} (2g_k^i w_l) - 2\pi i \int_{\sigma\tilde{e}_l^j} (2g_k^j w_l) \\
&\quad + \int_{\tilde{e}_l^i} \left(g_k^i - g_k^j\right) \frac{dh^j}{h^j} + \int_{\sigma\tilde{e}_l^j} \left(g_k^j - g_k^i\right) \frac{dh^i}{h^i}.
\end{aligned}$$

First we consider the sum of the first and the second integrals of the four integrals of the last line of equation (48).

Using the σ-anti-invariance we see that the integral is equal to

$$2\pi i \int_{\tilde{f}_l' \cup -\sigma\tilde{f}_l'} w_k \int_{\tilde{e}_l} w_l,$$

where $\tilde{f}_l'$ is the curve in $\tilde{\Delta}_i \cup \tilde{\Delta}_j$ connecting x_2^i and x_2^j. Let $\{d_{lk}\}$ be the integers defined just before Definition 2.12 in Section 2.5. Then we have

$$(49)\qquad -2\pi i \int_{\tilde{e}_l^i} (2g_k^i w_l) - 2\pi i \int_{\sigma\tilde{e}_l^j} (2g_k^j w_l) = -\pi i \Omega_{kl} - \pi i d_{kl} + \frac{1}{4} r_{kl},$$

where $r_{kl} = \pm\langle \tilde{f}_k, \tilde{f}_l \rangle$ is the intersection number of the curves $\tilde{f}_k$ and $\tilde{f}_l$ arising from the pairs such that $e_k \cup e_l \subset C_i$.

Now we note here the following; the function g_k has discontinuities across each $\tilde{e}_l^i$ by values in $\frac{1}{2}\left(\mathbf{Z} + \sum_l \mathbf{Z}\Omega_{kl}\right)$. Hence to compute Δ_P, we compensate for these discontinuities.

The discontinuity of g_k yields at $\tilde{e}_l^i$ the compensations of the integrals (49) given by the integrals

$$(50)\qquad -2\pi i \int_{\tilde{e}_l^i} (2g_k^i w_l) + 2\pi i \int_{\sigma\tilde{e}_l^j} (2g_k^j w_l).$$

The integral (50) is given as follows.

First we assume $k = l$. Using the σ-anti-invariance and $dg_k = w_k$, we have

$$(51)\quad \begin{aligned} &-2\pi i \int_{\tilde{e}_k^i} (2g_k^i w_k) + 2\pi i \int_{\sigma\tilde{e}_k^j} (2g_k^j w_k) \\ &= -2\pi i \int_{\tilde{e}_k^i} d(g_k^i)^2 + 2\pi i \int_{\sigma\tilde{e}_k^j} d(g_k^j)^2 \\ &= -2\pi i \left[\left(g_k^i(0) + \frac{1}{2} \right)^2 - g_k^i(0)^2 + \left(g_k^j(1) - \frac{1}{2} \right)^2 - g_k^j(1)^2 \right] \\ &= -2\pi i (g_k^i(0) - g_k^j(1)) - \pi i \\ &= -\pi i \Omega_{kk} - \pi i, \end{aligned}$$

where 0 and 1 denote the initial and end points of $\tilde{e}_l^i$ and $\tilde{e}_l^j$ respectively and they are equal.

Next we assume $k \neq l$. Since $\int_{\tilde{e}_l^i} w_k = 0$, g_k^i and g_k^j have the same values at the two endpoints of $\tilde{e}_l^i$ and $\tilde{e}_l^j$ respectively, and hence by partial integration we can see that the two integrals cancel out and we have

$$(52)\quad -2\pi i \int_{\tilde{e}_l^i} (2g_k^j w_l) + 2\pi i \int_{\sigma\tilde{e}_l^j} (2g_k^i w_l) = \frac{1}{4} r'_{kl},$$

where $r'_{kl} = \pm\langle \tilde{f}_k, \tilde{f}_l \rangle$ is similar to r_{kl} in equation (49). Note that, from the curve configuration in each $C_i \in \Upsilon$, we have $\sum_l (r_{kl} + r'_{kl}) \in 4\mathbf{Z}$.

Next we consider the sum of the third and fourth integrals of the last line of equation (48).

By the quasi-periodicity of the theta function, $\frac{dh^j}{h^j}$ at $\tilde{e}_l^j$ differs from $\frac{dh^i}{h^i}$ at $\sigma\tilde{e}_l^i$ by $2\pi i(4w_l)$. Hence, by similar calculations in (49), (50), (51) and (52), we have

$$(53)\quad \begin{aligned} &\int_{\tilde{e}_l^i} \left(g_k^i - g_k^j \right) \frac{dh^j}{h^j} + \int_{\sigma\tilde{e}_l^j} \left(g_k^j - g_k^i \right) \frac{dh^i}{h^i} \\ &= 2\pi i \int_{\tilde{e}_l^i} \left(g_k^i - g_k^j \right) 4w_l \\ &\equiv 0 \qquad \bmod 2\pi i\mathbf{Z} + 2\pi i \mathbf{Z}\Omega_{kl}. \end{aligned}$$

Putting equations (45), (46), (47), (48), (49), (50) and (51) together and using $\sum_l d_{lk} \in 2\mathbf{Z}$ and $\frac{1}{4}\sum_l (r_{kl} + r'_{kl}) \in \mathbf{Z}$, we find

$$\frac{1}{2} \sum_{r=1}^{6g-6} \int_{\sigma Q_r}^{Q_r} w_k \equiv -z_k - \left[\frac{1}{2} \sum_{l=1}^{3g-3} \Omega_{kl} + \frac{1}{2} \right] \qquad \bmod \mathbf{Z} + \sum_l \mathbf{Z}\Omega_{kl}.$$

Here we regard the left-hand side to be compensated by the discontinuity of g_k.

It follows that the k-th component of the vector Δ_P is given by

$$-\frac{1}{2}\sum_{l=1}^{3g-3}\Omega_{kl}-\frac{1}{2} \qquad \mod \mathbf{Z}+\sum_{l}\mathbf{Z}\Omega_{kl}.$$

This proves Proposition 5.1. □

THEOREM 5.1. *Let $\tilde{C}=\tilde{C}_{(\tilde{b},\mathfrak{m})}$ be the 2-fold branched covering surface of C with marking associated to $(\tilde{b},\mathfrak{m})\in\tilde{\mathcal{B}}_\Upsilon$. Coordinate the Prym variety $P=P_{(\tilde{b},\mathfrak{m})}$ as in Section 3.2. Let Π be a Riemann theta function of level 4 on P defined by*

$$\Pi(\vec{z},\Omega)=\sum_{\vec{\lambda}\in\frac{1}{2}\Lambda^*/\Lambda_0^*}e^{2\pi i(\vec{\lambda}^t\frac{\vec{1}}{2})}\sum_{\vec{\varepsilon}\in\mathbf{Z}_2^{3g-3}}w(\vec{\varepsilon})\vartheta\begin{bmatrix}\frac{\vec{\varepsilon}}{4}+\frac{\vec{\lambda}}{2}\\ \vec{0}\end{bmatrix}(4\vec{z},4\Omega),$$

where $\vec{\lambda}^t\frac{\vec{1}}{2}$ denotes the scalar product of the two column vectors $\vec{\lambda}$ and $\frac{\vec{1}}{2}$ and $w(\vec{\varepsilon})=\varepsilon_1\cdots\varepsilon_{3g-3}$ for $\vec{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_{3g-3})\in\mathbf{Z}_2^{3g-3}$.

Then the branching divisor of the map $\pi:P\to M_g$ is given by the divisor of Π, $\mathrm{Div}(\Pi)$.

Proof. The branching locus is the divisor of the Riemann theta function Π of level 4 obtained from $\vartheta(\vec{z},\Omega)$ by translating by Δ_P, substituting $\vec{z}$ with $2\vec{z}$ and making it $\frac{1}{2}\Lambda^*/\Lambda_0^*$ invariant. Hence it is the divisor of

$$S^0_{\frac{1}{2}\Lambda^*/\Lambda_0^*}\left(\vartheta\begin{bmatrix}\frac{\vec{1}}{2}\\ \frac{\vec{1}}{2}\end{bmatrix}(2\vec{z},\Omega)\right)$$

which is equal to $e^{-\frac{(3g-3)}{2}\pi i}\Pi$. □

6. Differential equations satisfied by pull back sections

In this section we construct a differential equation which characterizes locally the pull back of holomorphic sections of $\mathcal{L}^k$ by the dominant map $\pi:P\to M_g$. Throughout this section we fix a pant decomposition $\Upsilon=\{e_l,C_i\}$ of C.

6.1. *The point-inverse vector field.* Let $\mathcal{P}_\Upsilon\to\tilde{\mathcal{B}}_\Upsilon$ be the bundle of the polarized Prym varieties over the universal cover $\tilde{\mathcal{B}}_\Upsilon$ of $\mathcal{B}_\Upsilon$. Then the morphism $\pi:P=P_{(\tilde{b},\mathfrak{m})}\to M_g$ at each fiber combines to define a morphism $\pi:\mathcal{P}_\Upsilon\to M_g$. Let P_s and M_{gs} be the subsets of P and M_g respectively corresponding to the stable bundles as in Section 3.2.

Definition 6.1 (Point-inverse vector field). For a holomorphic tangent vector $v\in T^{(1,0)}_{(\tilde{b},\mathfrak{m})}\tilde{\mathcal{B}}_\Upsilon$, the morphism $\pi:\mathcal{P}_\Upsilon\to M_g$ induces a holomorphic tangent vector field $\mathcal{V}$ of $\mathcal{P}_\Upsilon$ defined on $P_s=P_{bs}$ with singularity along $\mathrm{Div}(\Pi)\cap P_s$

such that it is mapped to v by the projection $\mathcal{P}_\Upsilon \to \tilde{\mathcal{B}}_\Upsilon$ and $\pi_*\mathcal{V} = 0$; i.e. it is tangent to the inverse images of points of $M_{gs} - \mathrm{Div}(\Pi)$ by π at P_s.

Let $\tilde{P} = \tilde{P}_{(\tilde{b},\mathfrak{m})}$ be the universal cover of the Prym variety $P = P_{(\tilde{b},\mathfrak{m})}$ and let $\tilde{\mathcal{P}}_\Upsilon \to \tilde{\mathcal{B}}_\Upsilon$ be the fibre bundle on $\tilde{\mathcal{B}}_\Upsilon$ whose fibre at b is $\tilde{P} = \tilde{\mathcal{P}}_{(\tilde{b},\mathfrak{m})}$. Then the vector field $\mathcal{V}_b$ can be pulled back to a vector field $\tilde{\mathcal{V}}_b$ on $\tilde{\mathcal{P}}_\Upsilon$. Let $\tilde{P}_s$ be the inverse image of P_s under the covering projection $\tilde{P} \to P$.

Theorem 6.1. *For $v \in T^{(1,0)}_{(\tilde{b},\mathfrak{m})}\tilde{\mathcal{B}}_\Upsilon$, the corresponding vector field $\tilde{\mathcal{V}}$ on $\tilde{P}_s$ is given by*

$$\tilde{\mathcal{V}} = \delta^\Omega + \frac{1}{8}(\delta J\omega^{-1})_{ij}\Pi^{-1}\partial_i\Pi\frac{\partial}{\partial z_j}, \tag{54}$$

where δ^Ω denotes the tangent vector on the Siegel domain $\mathcal{S}$ induced by v. It descends to the vector field $\mathcal{V}$ on P_s given by

$$\mathcal{V} = \delta + \frac{1}{8}(\delta J\omega^{-1})_{ij}\Pi^{-1}\nabla_i\Pi\frac{\partial}{\partial z_j} \tag{55}$$

where δ and ∇_i are the covariant derivatives given in equation (30) *of Section* 4.2.

Proof. Let $\tilde{\pi} : \tilde{P}_s \to M_{gs}$ be the composition of the covering map $\tilde{P}_s \to P$ and π. If we choose a local holomorphic coordinate $(y_j)_{1\le j\le 3g-3}$ of M_{gs} at a point and we write $\tilde{\pi}(\Omega, z_i) = (f_j(\Omega, z_i))$, then the meromorphic vector field $\tilde{\mathcal{V}}$ is given by

$$\tilde{\mathcal{V}} = -\left(\frac{\partial f_j}{\partial z_i}\right)^{-1}_{1\le i,j\le 3g-3}\left(\frac{\partial f_j}{\partial \Omega}\right)_{1\le j\le 3g-3}. \tag{56}$$

Hence if we write

$$\tilde{\mathcal{V}} = \delta^\Omega + \sum_{i=1}^{3g-3}\nu_j\frac{\partial}{\partial z_j} \tag{57}$$

then $\vec{\nu} = (\nu_j)_{1\le j\le 3g-3}$ is a meromorphic vector with singularity along the divisor of $\det\left(\frac{\partial f_j}{\partial z_i}\right)^{-1}$ which is $\mathrm{Div}(\Pi)$.

Let $\wedge^{3g-3}TM_g$ and $\wedge^{3g-3}TP$ be the top exterior bundle of the holomorphic tangent bundles of M_g and P respectively. Then the morphism $\pi : P \to M_g$ gives a holomorphic section s of the bundle $(\wedge^{3g-3}T^*P)\otimes\pi^*\wedge^{3g-3}TM_g$ which is isomorphic to $\pi^*\mathcal{L}^4 = \tilde{\mathcal{L}}^8$ ([4]). As was mentioned in Section 5 the branching locus of the map $\pi_{\tilde{b}} : P_{\tilde{b}} \to M_g$ is given by the divisor of the Riemann theta function Π of level 4. Hence we have $s = c(b)\Pi^2$ for a holomorphic function $c(b)$ on $\tilde{\mathcal{B}}_\Upsilon$.

The divergence $\operatorname{div}_{\tilde{\mathcal{V}}}(c(b)\mathit{\Pi}^2)$ of the vector field $\tilde{\mathcal{V}}$ with respect to the volume form $c(b)\mathit{\Pi}^2$ is given by

$$\operatorname{div}_{\tilde{\mathcal{V}}}(c(b)\mathit{\Pi}^2) = \sum_{1\le j\le 3g-3} (c(b)\mathit{\Pi}^2)^{-1}\partial_j(c(b)\mathit{\Pi}^2\nu_j) - (c(b)\mathit{\Pi}^2)^{-1}\delta^{\Omega}(c(b)\mathit{\Pi}^2) \tag{58}$$

where the second term in the right-hand side of the above equation is incorporated since δ acts on the complex volume form (the canonical bundle) on $P_{\tilde{b}}$ by the said amount.

Since $\tilde{\mathcal{V}}$ is defined by the tangent vector field of the point inverses, $c(b)\mathit{\Pi}^2$ satisfies the tautological relation

$$\operatorname{div}_{\tilde{\mathcal{V}}}(c(b)\mathit{\Pi}^2) = (c(b)\mathit{\Pi}^2)^{-1}\delta^{\Omega}(c(b)\mathit{\Pi}^2). \tag{59}$$

From the equations (58) and (59) it follows that

$$2(c(b)\mathit{\Pi}^2)^{-1}\delta^{\Omega}(c(b)\mathit{\Pi}^2) = \sum_{1\le j\le 3g-3} (c(b)\mathit{\Pi}^2)^{-1}\partial_j(c(b)\mathit{\Pi}^2\nu_j) \tag{60}$$

which is equivalent to

$$\begin{aligned} &\mathit{\Pi}^{-1}\delta^{\Omega}(\mathit{\Pi}) \\ &= \frac{1}{2}\sum_{1\le j\le 3g-3}(\mathit{\Pi}^{-1}\partial_j\mathit{\Pi})\nu_j + \frac{1}{4}\sum_{1\le j\le 3g-3}\partial_j\nu_j - c(b)^{-1}\delta c(b). \end{aligned} \tag{61}$$

On the other hand by the heat equation satisfied by $\mathit{\Pi}$,

$$\mathit{\Pi}^{-1}\delta^{\Omega}\mathit{\Pi} = -\sum_{1\le i,j\le 3g-3}\frac{1}{16}(\delta J\omega^{-1})_{ij}\mathit{\Pi}^{-1}\partial_i\partial_j\mathit{\Pi}. \tag{62}$$

The right-hand sides of the two equations (61) and (62) should coincide with each other. Hence from the equation

$$\mathit{\Pi}^{-1}\partial_i\partial_j\mathit{\Pi} = \left(\mathit{\Pi}^{-1}\partial_j\mathit{\Pi}\right)\left(\mathit{\Pi}^{-1}\partial_i\mathit{\Pi}\right) + \partial_j\left(\mathit{\Pi}^{-1}\partial_i\mathit{\Pi}\right), \tag{63}$$

we can deduce $c(b) = 1$ and obtain the equation

$$\nu_j = \frac{1}{8}(\delta J\omega^{-1})_{ij}\mathit{\Pi}^{-1}\partial_i\mathit{\Pi} \tag{64}$$

which proves (54) in the theorem.

To obtain equation (55) we substitute the terms in (54) by covariant derivatives

$$\begin{aligned} \delta\mathit{\Pi} &= \left(\delta^{\Omega} - \frac{1}{2i}\sum_{ij}\delta J_{ij}(z_i-\bar{z}_i)\partial_j\right)\mathit{\Pi}, \\ \nabla_i\mathit{\Pi} &= \left(\partial_i + \sum_j\frac{2\omega_{ij}}{i}(z_j-\bar{z}_j)\right)\mathit{\Pi}. \end{aligned} \tag{65}$$

Then $\mathcal{V}$ can be expressed as in (55) and it defines a meromorphic vector field on P_s with singularity along $\operatorname{Div}(\mathit{\Pi})$. □

The meromorphic vector field $\mathcal{V}$ in the above theorem is defined on P_s. However the resulting equation can have a meaning on the whole P as a meromorphic vector field with singularity along $\mathrm{Div}(\Pi)$. Hence from now on we regard $\mathcal{V}$ to be defined on P.

6.2. *The differential equation satisfied by pull back sections.* We construct a differential equation satisfied by pull-back sections of $\mathcal{L}^k$ by lifting the vector field $\mathcal{V}$ to a differential operator acting on sections of the vector bundle $\tilde{\mathcal{L}}^{2k}$ over $\mathcal{B}_\Upsilon$.

Definition 6.2. Let

$$P_\Pi : \Theta_{2(k+2)} \to \Pi\Theta_{2k} \tag{66}$$

be the *orthogonal projection* onto the subspace $\Pi\Theta_{2k}$ with respect to the usual Hermitian inner product on $\Theta_{2(k+2)}$.

THEOREM 6.2. *Let D be the differential operator on $\tilde{\mathcal{L}}^{2k}$, for $\tilde{\psi} \in \Gamma(\tilde{\mathcal{L}}^{2k})$, given by*

$$D\tilde{\psi} = \Pi^{-1}\left(\delta + \frac{1}{8(k+2)}(\delta J\omega^{-1})_{ij}\partial_i\partial_j - \delta P_\Pi\right)\left(\Pi\tilde{\psi}\right) \tag{67}$$

where δP_Π is the derivative of P_Π.

Then for a holomorphic section ψ of $\mathcal{L}^k$, its pull back section $\tilde{\psi}$ of $\tilde{\mathcal{L}}^{2k}$ satisfies the differential equation

$$D\tilde{\psi} = 0. \tag{68}$$

Proof. The differential equation satisfied by pull back sections can be derived from the fact that the pull back section should be invariant along the vector field $\mathcal{V}$. Hence we must lift the differential operator defined by $\mathcal{V}$ to a differential operator on the space of holomorphic sections of $\tilde{\mathcal{L}}^{2k}$.

The required differential operator must have several necessary properties. We construct it step by step so that it may satisfy all the necessary conditions.

Step 1. To derive the correct differential equation we must take into account the parallelism of the Riemann theta functions. The bundle Θ_{2k} has the natural basis and hence the natural framing consisting of parallel Riemann theta functions. With respect to this framing of Θ_{2k} the differential along $\mathcal{V}$ is expressed by the differential operator

$$D_1(\tilde{\psi}) = \delta\tilde{\psi} + \frac{1}{8(k+2)}(\delta J\omega^{-1})_{ij}\left(\Pi^{-1}\partial_i\Pi\partial_j + \partial_i\partial_j\right)\tilde{\psi}. \tag{69}$$

Here we note that the factor $\frac{1}{8(k+2)}$ is incorporated to eliminate $\bar{\partial}\delta\tilde{\psi}$ (holomorphicity preservation).

Step 2. Let $\vec{n} = (n_i) \in \mathbf{Z}^{3g-3}$. Using the relation

$$\delta\left(\tilde{\psi}(\vec{z}+\Omega\vec{n})\right) = (\delta\tilde{\psi})(\vec{z}+\Omega\vec{n}) + ((\delta\Omega)\vec{n})_i\,(\partial_i\tilde{\psi})(\vec{z}+\Omega\vec{n}) \tag{70}$$

and the quasi-periodicity of $\tilde{\psi}$, we have

$$\begin{aligned}(D_1\tilde{\psi})(\vec{z}+\vec{n}) = {} & e^{\pi i({}^t\vec{n}\Omega\vec{n}+2{}^t\vec{n}\vec{z})}(D_1\tilde{\psi})(\vec{z}) \\ & + \frac{k}{k+2}(\delta J\omega^{-1})_{ij}\left(\Pi^{-1}\partial_i\Pi\right)(\vec{z})n_j\tilde{\psi}(\vec{z}).\end{aligned} \tag{71}$$

Therefore the differential operator D_1 does not preserve the quasi-periodicity of $\tilde{\psi}$ because of the second term of the right-hand side of the above equation. Consequently it does not preserve the sections of $\tilde{\mathcal{L}}^{2k}$. This can be remedied by adding to D_1 the multiplication operator

$$\tilde{\psi}(\vec{z}) \;\to\; \left(\frac{k}{k+2}(\Pi^{-1}\delta\Pi)(\vec{z})\right)\tilde{\psi}(\vec{z}). \tag{72}$$

By the heat equation (33) we have

$$\frac{k}{k+2}\Pi^{-1}\delta\Pi = \Pi^{-1}\delta\Pi + \frac{1}{8(k+2)}\Pi^{-1}(\delta J\omega^{-1})_{ij}\partial_i\partial_j\Pi. \tag{73}$$

Hence adding the multiplication operator (72), D_1 changes to

$$D_2\tilde{\psi} = \Pi^{-1}\left(\delta + \frac{1}{8(k+2)}(\delta J\omega^{-1})_{ij}\partial_i\partial_j\right)\Pi\tilde{\psi}. \tag{74}$$

Another explanation of the reason for adding the multiplication operator (72) is the necessity for the incorporation of the difference of trivializations of fibres of $\tilde{\mathcal{L}}^{2k}$. For $x \in M_g$ and a nonzero vector $a \in \mathcal{L}_x^k$, $\pi^{-1}(a)$ defines a holomorphic function on the point-inverse orbit $\pi^{-1}(x)$. The derivative $v_x = \delta_{\mathcal{V}} \log \pi^{-1}a$ does not depend on the choice of a and the family $\{v_x\}_{x\in M_g}$ combines together to give a holomorphic function on the universal cover $\tilde{P}$. To obtain the lift of the differential operator $\mathcal{V}$ to a differential operator on the space of holomorphic sections of $\tilde{\mathcal{L}}^{2k}$, we must take a covariant derivative with respect to the holomorphic connection defined by $\{v_x\}$.

Step 3. We note that the expression (74) is formal as it stands because the differential operator in the parentheses on the right-hand side of (74) does not keep the subbundle $\Pi\Theta_{2k}$ invariant in general and hence we cannot divide by Π. We may remedy this by adding the term of the operator

$$\tilde{\psi} \;\to\; -\Pi^{-1}\delta P_{\Pi}\left(\Pi\tilde{\psi}\right). \tag{75}$$

As a result we obtain the operator D given in equation (67) in the theorem and this completes the construction of the desired differential operator D. □

7. The pull back sections; a characterization

7.1. *A characterization of pull back sections.* Let $\Upsilon = \{e_l, C_i\}$ be a decomposition of C into 3-holed spheres.

The results so far obtained in the preceding sections are sufficient to give a characterization of holomorphic sections of $\tilde{\mathcal{L}}^{2k}$ which are pull-backs of holomorphic sections of $\mathcal{L}^k$.

We have the following three conditions for the pull back sections:

(i) local invariance,

(ii) global invariance,

(iii) automorphy.

(i) Local invariance. Theorem 6.2 states that a pull back section $\tilde{\psi}$ satisfies the differential equation given in equation (68).

(ii) Global invariance. A pull back section $\tilde{\psi}$ should be a linear combination of $S^0_{\frac{1}{2}\Lambda^*/\Lambda^*_0}$ image of Riemann theta functions which are invariant under the action of $\mathbf{Z}_2^{3g-3}$ and projectively invariant under the W^o_Υ-action (Definition 4.3).

(iii) Automorphy. A pull back section $\tilde{\psi}$ should be a linear combination of Riemann theta functions of level $2k$ with coefficients of holomorphic functions of Riemann matrices Ω_Λ.

Consequently a pull back section should be a linear combination of sections of the form

$$\tilde{\psi} = S^0_{\frac{1}{2}\Lambda^*/\Lambda^*_0}\left(\sum_{\vec{a}} c_{\vec{a}}(\Omega_\Lambda)\vartheta\begin{bmatrix}\vec{a}\\ \vec{0}\end{bmatrix}(2k\vec{z}, 2k\Omega)\right), \tag{76}$$

where $c_{\vec{a}}(\Omega_\Lambda) \in A_*$ and the section in the parenthesis is projectively invariant under W^o_Υ action.

To summarize we have the following characterization of pull back sections,

Proposition 7.1. *Let Υ be a pant decomposition of C. A family of holomorphic sections of $\tilde{\mathcal{L}}^{2k}_b$ on $\tilde{\mathcal{B}}_\Upsilon$, $\{\tilde{\psi} = \tilde{\psi}_{(\tilde{b},\mathfrak{m})}\}_{(\tilde{b},\mathfrak{m})\in\tilde{\mathcal{B}}_\Upsilon}$, is a family of pull back sections of a holomorphic section of $\mathcal{L}^k$ if and only if it satisfies the above conditions* (i), (ii) *and* (iii), *and it has the form of equation* (76).

Proof. It is obvious that the said conditions are necessary for $\{\tilde{\psi}\}$ to be a family of pull back sections of a holomorphic section of $\mathcal{L}^k$.

We prove the converse. Assume that $\{\tilde{\psi}\}$ satisfies the local condition. Since the differential equation is constructed from the point-inverse vector field, its solution is locally a pull-back of a holomorphic section. Hence $\{\tilde{\psi}\}$ is locally a pull-back of a holomorphic section.

The map $\pi : P_{\tilde{b}} \to M_g$ factors the map $P' \to M_g$ and the degree of the latter is 2^{3g-3} ([6] and see Section 3 for P'). Since the order of W_Υ/W^o_Υ is equal to $\mathbf{Z}_2^{3g-3}$ stated as in Lemma 2.4, W_Υ-invariance implies that the family descends to a holomorphic section of $\mathcal{L}^k$, that is, it is globally a family of pull back sections. □

8. A construction of a basis of $\Gamma(M_g, \mathcal{L}^k)$

In this section we construct a set of holomorphic sections $\{\tilde{\psi}\}$ of $\tilde{\mathcal{L}}^{2k}$ satisfying the conditions stated in Section 7. Throughout this section we choose and fix a pant decomposition $\Upsilon = \{e_l, C_i\}$ of C.

In the defining equation (67) of the differential operator D in Theorem 6.2, the operator in the big parenthesis coincides with the usual heat operator of the Riemann theta function of level $2k+4$ (see equation (33)) except for the last term $-\delta P_\Pi$. Hence we expect that $\Pi\tilde{\psi}$ might be expressed in much simpler form than $\tilde{\psi}$ itself. Hence we look for ϕ such that $\tilde{\psi} = \frac{\phi}{\Pi}$.

From the formula of Π given in Theorem 5.1, Π is anti-invariant with respect to the $\mathbf{Z}_2^{3g-3}$-action of Definition 4.2. Since $\tilde{\psi}$ is $\mathbf{Z}_2^{3g-3}$-invariant, $\phi = \Pi\tilde{\psi}$ is $\mathbf{Z}_2^{3g-3}$-anti-invariant. Also $\tilde{\psi}$ is $S^0_{\frac{1}{2}\Lambda^*/\Lambda_0^*}$-invariant, and from the formula of Π given in Theorem 5.1 again, we see that $\phi = \Pi\tilde{\psi}$ is a $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}$ image, where $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}$ is the anti-invariant shift operator defined in Definition 4.2 (iv).

Taking this into account, from the characterization of pull back sections in the previous section, we make the following definition.

Definition 8.1. Let $(A_{2(k+2)} \cdot \Theta_{2(k+2)})_-^{W^o_\Upsilon}$ be the subspace of $A_{2(k+2)} \cdot \Theta_{2(k+2)}$ spanned by those elements each of which is W^o_Υ-projectively invariant and is of the form

$$\sum_{\vec{\varepsilon} \in \mathbf{Z}_2^{3g-3}} w(\vec{\varepsilon}) q(\vec{\varepsilon} \cdot \Omega_\Lambda) \vartheta \begin{bmatrix} \vec{\varepsilon} \cdot \frac{(2\vec{j}+\vec{1})}{2(k+2)} \\ \vec{0} \end{bmatrix} (2(k+2)\vec{z}, 2(k+2)\Omega),$$

where $w(\vec{\varepsilon}) = \varepsilon_1 \cdots \varepsilon_{3g-3}$ and $\vec{1} = (1, \cdots, 1)^t \in \mathbf{Z}^{3g-3}$.

Since our object is to find $\phi \in (A_{2(k+2)} \cdot \Theta_{2(k+2)})_-^{W^o_\Upsilon}$ such that $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi) = \Pi\tilde{\psi}$, we begin by studying the condition for $\phi \in (A_{2(k+2)} \cdot \Theta_{2(k+2)})_-^{W^o_\Upsilon}$ so that $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi)$ may be divisible by Π. For this purpose we use the following notions:

Energy weight, spin weight and weight lattice. Let $\phi \in A_{2(k+2)} \cdot \Theta_{2k+2}$. Then by Fourier expansion $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi)$ can be expressed by an infinite linear combination of exponential functions each of which has the form $e^{\pi i s(\vec{a})} e^{2\pi i (\vec{m}^t \vec{z})}$, where $s(\vec{a})$ is a function of $\vec{a}$, $\vec{m} \in \mathbf{Q}^{3g-3}$ and $\vec{z} = (z_1, \cdots, z_{3g-3})^t$.

Definition 8.2. The *weight lattice* L_ϕ of ϕ with respect to the coordinate $\vec{z}$ is defined as the point set $L_\phi = \{(\mathrm{Im}s(\vec{a}), \vec{m})\}$ in $\mathbf{R} \oplus \mathbf{R}^{3g-3}$. $s(\vec{a})$ and $\vec{m}$ are called the *energy and spin weight* respectively. For a vector $\vec{a} = (a_l)$ such that $|a_l| < 1$ for $1 \leq l \leq 3g-3$, $s(\vec{a})$ is called a minimal energy weight.

8.1. *Grouping and decomposition of minimal energy weight part of Π.* We introduce the notion of grouping for the construction of the basis of $\Gamma(M_g, \mathcal{L}^k)$.

Definition 8.3. Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. We set $\mathcal{E} = \{e_l \in \Upsilon\}_{1 \leq l \leq 3g-3}$ and define a *grouping* g to be a decomposition of $\mathcal{E}$ into three disjoint subsets

$$g : \mathcal{E} = \mathcal{E}_1^g \cup \mathcal{E}_2^g \cup \mathcal{E}_3^g \tag{77}$$

such that, for $1 \leq m \leq 3$, $\mathcal{E}_m^g = \{e_{l(m)}\}$ consists of $g-1$ elements, and, for each $C_i \in \Upsilon$,

- (i) if $C_i = S_0$ and all the three boundary curves are simultaneously separating or nonseparating, then $\sharp\{e_{l(m)} \in \mathcal{E}_m^g \,|\, e_{l(m)} \subset C_i\} = 1$;
- (ii) if $C_i = S_0$ and one boundary curve is separating and the other two boundary curves are nonseparating, then
  - (a) $\sharp\{e_{l(m)} \in \mathcal{E}_m^g \,|\, e_{l(m)} \subset C_i\} = 0$, or
  - (b) $\sharp\{e_{l(m)} \in \mathcal{E}_m^g \,|\, e_{l(m)} \subset C_i\} = 1$ and it is the separating boundary component, or
  - (c) $\sharp\{e_{l(m)} \in \mathcal{E}_m^g \,|\, e_{l(m)} \subset C_i\} = 2$ and they are the two nonseparating boundary components,
- (iii) if $C_i = T_0$, then $\sharp\{e_{l(m)} \in \mathcal{E}_m^g \,|\, e_{l(m)} \subset C_i\} = 0$, or 1.

We denote the set of all the groupings by $\mathcal{E}_\Upsilon$; here the numbering m is irrelevant; that is, $\mathcal{E}_{\tau(1)}^g \cup \mathcal{E}_{\tau(2)}^g \cup \mathcal{E}_{\tau(3)}^g$ defines the same grouping for any permutation τ of m.

The following can be proved by induction on the genus g without difficulty.

LEMMA 8.1. *Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $\mathcal{E}_\Upsilon$ be the set of all the groupings.*

Then

$$1 \leq \sharp\mathcal{E}_\Upsilon \leq 2^{g-2}. \tag{78}$$

We relate the concept of grouping to a more geometric one, that is, a system of simple closed curves each of which is transverse to $\bigcup_l e_l$ as follows.

Associated to the marking $\mathfrak{m} = \{f_l, e_l\}$, there is a dual graph G of the pant decomposition Υ; that is, G is the trivalent graph in C whose vertices

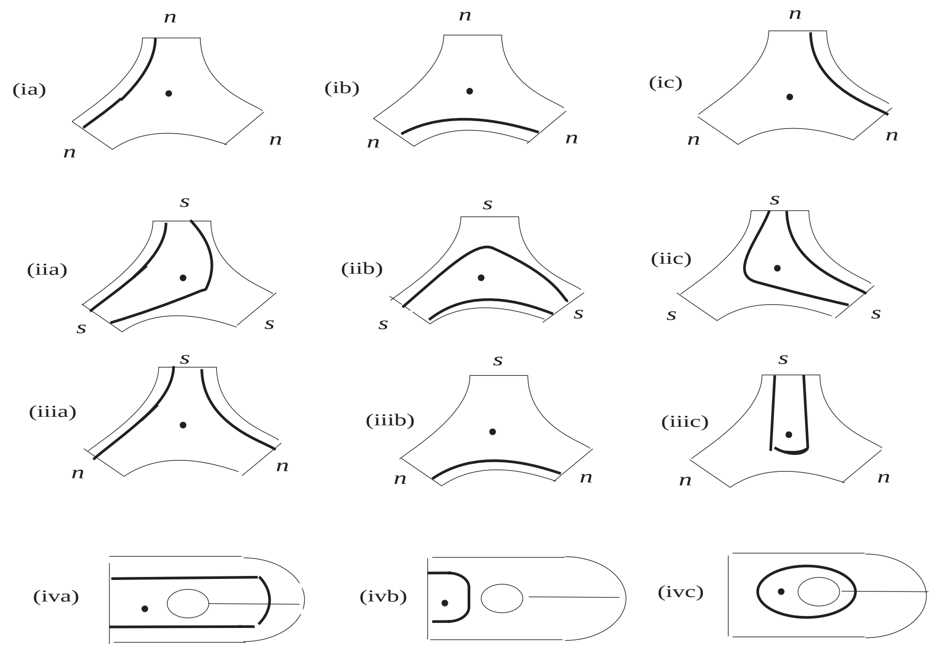

Figure 5: curve system

are $\{x_i^1\}_{1\leq i\leq 2g-2}$ and whose edges are $\{f_l\}_{1\leq l\leq 3g-3}$. Let $N(G)$ be a regular neighborhood of G in C. Then C is homeomorphic to the double of $N(G)$.

For each $C_i \in \Upsilon$, $N(G)\cap C_i$ is a hexagon with three disjoint boundary arcs in ∂C_i. In Figure 5 we define the arcs in $N(G)\cap C_i$, where the symbol $\cdot$ denotes the branch point x_i^1 and the letters n or s attached to the boundary component $N(G)\cap\partial C_i$ indicate that the corresponding boundary component of C_i is nonseparating or separating respectively.

Definition 8.4. Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. A triple of maximal curve system is defined to be a triple $s = (s_1, s_2, s_3)$ satisfying the following conditions: for $1 \leq m \leq 3$, s_m is a disjoint union of simple closed curves in $N(G) - \bigcup_i x_i^1$ such that each $e_l \in \Upsilon$ intersects with at most one component of s_m and exactly $2g-2$ e_l's intersects s_m, $s_m \cap C_i (C_i \in \Upsilon)$ is one of the arcs listed in Figure 5, and $\{s_m \cap C_i\}_{m=1,2,3}$ runs through all the three types of arcs listed there corresponding to the type of C_i.

We denote the set of all the triples of maximal curve system by $\mathcal{S}_\Upsilon$.

LEMMA 8.2. *There is a one-one correspondence between the sets $\mathcal{E}_\Upsilon$ and $\mathcal{S}_\Upsilon$.*

Proof. Let $g : \mathcal{E} = \mathcal{E}_1^g \cup \mathcal{E}_2^g \cup \mathcal{E}_3^g$ be a grouping. For $1 \leq m \leq 3$, we construct from $\mathcal{E}_m^g = \{e_{l(m)}\}$ the corresponding curve system s_m^g as follows.

For $C_i \in \Upsilon$, $s_m^g \cap C_i$ is defined so that:

(i) If $C_i = S_0$ and all the boundary components are simultaneously nonseparating, then $s^g_m \cap C_i$ is one of (ia), (ib) and (ic) in which $e_{l(m)}$ corresponds to the boundary component of $N(G) \cap \partial C_i$ disjoint from the arcs.

(ii) If $C_i = S_0$ and all the boundary components are simultaneously separating, then $s^g_m \cap C_i$ is one of (iia), (iib) and (iic) in which $e_{l(m)}$ corresponds to the boundary component of $N(G) \cap \partial C_i$ disjoint from the arcs.

(iii) If $C_i = S_0$ and exactly one boundary component is separating, then

(a) if $\sharp\{e_{l(m)} \subset C_i\} = 0$, then $s^g_m \cap C_i$ is (iiia),
(b) if $\sharp\{e_{l(m)} \subset C_i\} = 1$, then $s^g_m \cap C_i$ is (iiib),
(c) if $\sharp\{e_{l(m)} \subset C_i\} = 2$, then $s^g_m \cap C_i$ is (iiic).

(iv) If $C_i = T_0$, then

(a) if $\sharp\{e_{l(m)} \subset C_i\} = 0$, then $s^g_m \cap C_i$ is (iva),
(b) if $\sharp\{e_{l(m)} \subset C_i\} = 1$ and $e_{l(m)}$ is the nonseparating simple closed curve, then $s^g_m \cap C_i$ is (ivb),
(c) if $\sharp\{e_{l(m)} \subset C_i\} = 1$ and $e_{l(m)} = \partial C_i$, then $s^g_m \cap C_i$ is (ivc).

From such a chosen subset $\{s^g_m \cap C_i\}_{1\le i\le 2g-2}$ we can construct a curve system s^g_m satisfying the condition in Definition 8.4 and the construction gives a one-one correspondence between $\mathcal{E}_\Upsilon$ and S_Υ. □

Definition 8.5. For a grouping $g \in \mathcal{E}_\Upsilon$, let $s^g = (s^g_1, s^g_2, s^g_3)$ denote the curve system constructed in Lemma 8.2.

By formula Π in Theorem 5.1 we can write Π as

$$\Pi = \Pi_0 + \Pi_1, \tag{79}$$

where Π_0 is the sum of the terms of minimal energy weights and is equal to

$$\sum_{\vec\lambda \in \frac{1}{2}\Lambda^*/\Lambda^*_0} e^{2\pi i (\vec\lambda^t \frac{\vec 1}{2})} \sum_{\vec\varepsilon \in \mathbf{Z}_2^{3g-3}} w(\vec\varepsilon) e_{\vec\varepsilon,\vec\lambda},$$

for

$$e_{\vec\varepsilon,\vec\lambda} = \exp\left(\pi i \left(\frac{\vec\varepsilon}{2} + \vec\lambda\right)^t \Omega \left(\frac{\vec\varepsilon}{2} + \vec\lambda\right) + 4\pi i \left(\frac{\vec\varepsilon}{2} + \vec\lambda\right)^t \vec z\right), \tag{80}$$

and Π_1 is the sum of those of higher energy weights.

For each $\tilde C_i \in \Upsilon$ we make a change of variables in the following way:

First we assume that $C_i = S_0$. Let $\partial C_i = e_{l^i_1} \cup e_{l^i_2} \cup e_{l^i_3}$. Let $\{z_{l^i_1}, z_{l^i_2}, z_{l^i_3}\}$ be the corresponding variables defined in Section 3.2.

In $\tilde{C}_i \subset \tilde{C}$ we make the base-change of $\tilde{\ell} \cap H_1(\tilde{C}_i, \mathbf{R})_-$ as

$$\left\{[\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}], [\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}], [\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i}]\right\} \rightarrow \{E_1^i, E_2^i, E_3^i\},$$

where $\{E_r^i\}$ are as in Definition 2.11.

This induces the change of variables

$$\left\{\begin{array}{l} w_{l_1^i} = z_{l_2^i} + z_{l_3^i} \\ w_{l_2^i} = z_{l_1^i} + z_{l_3^i} \\ w_{l_3^i} = z_{l_1^i} + z_{l_2^i}, \end{array}\right. \tag{81}$$

and for vectors, $\vec{n}_i = (n_{l_1^i}, n_{l_2^i}, n_{l_3^i})^t$, $\vec{m}_i = (m_{l_1^i}, m_{l_2^i}, m_{l_3^i})^t$, $\vec{z}_i = (z_{l_1^i}, z_{l_2^i}, z_{l_3^i})^t$ and $\vec{w}_i = (w_{l_1^i}, w_{l_2^i}, w_{l_3^i})^t$, satisfying $\vec{n}_i^t\vec{z}_i = \vec{m}_i^t\vec{w}_i$, we have

$$\left\{\begin{array}{l} m_{l_1^i} = \frac{1}{2}\left(-n_{l_1^i} + n_{l_2^i} + n_{l_3^i}\right) \\ m_{l_2^i} = \frac{1}{2}\left(n_{l_1^i} - n_{l_2^i} + n_{l_3^i}\right) \\ m_{l_3^i} = \frac{1}{2}\left(n_{l_1^i} + n_{l_2^i} - n_{l_3^i}\right). \end{array}\right. \tag{82}$$

Next assume that $C_i = T_0$. Let $e_{l_1^i} = \partial C_i$ and let $e_{l_2^i} \in \Upsilon$ be the essential simple closed curve in C_i. Then we have

$$\Lambda \cap H_1(\tilde{C}_i, \mathbf{R}) = \mathbf{Z}\frac{1}{2}[\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}] \oplus \mathbf{Z}[\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}]. \tag{83}$$

In $\tilde{C}_i$ we make the base change of $\tilde{\ell} \cap H_1(\tilde{C}_i, \mathbf{R})_-$ as

$$\left\{[\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}], [\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}]\right\} \rightarrow \{E_1^i, E_2^i\},$$

where $\{E_r^i\}_{r=1,2}$ are as in Definition 2.11.

This induces the changes of variables

$$\left\{\begin{array}{l} w_{l_1^i} = 2z_{l_2^i} \\ w_{l_2^i} = z_{l_1^i} + z_{l_2^i}, \end{array}\right. \tag{84}$$

and for vectors, $\vec{n}_i = (n_{l_1^i}, n_{l_2^i})^t$, $\vec{m}_i = (m_{l_1^i}, m_{l_2^i})^t$, $\vec{z}_i = (z_{l_1^i}, z_{l_2^i})^t$ and $\vec{w}_i = (w_{l_1^i}, w_{l_2^i})^t$, satisfying, $\vec{n}_i^t\vec{z}_i = \vec{m}_i^t\vec{w}_i$, we have

$$\left\{\begin{array}{l} m_{l_1^i} = -\frac{1}{2}n_{l_1^i} + n_{l_2^i} \\ m_{l_2^i} = \frac{1}{2}n_{l_1^i}. \end{array}\right. \tag{85}$$

Let $\Pi_0^{\vec{\lambda}=0}$ be the part of Π_0 with $\vec{\lambda} = 0$. Then using the above change of variables we decompose each term of $\Pi_0^{\vec{\lambda}=0}$ as follows.

Extracting the part of $\Pi_0^{\vec{\lambda}=0}$ only involving the variables concerning $\tilde{C}_i$ and using the variable changes (81) and (84), we set

$$\Pi(i)_0^{\vec{\lambda}=0} = \sum_{\vec{\varepsilon}_i \in \mathbf{Z}_2^3} w(\vec{\varepsilon}_i) e_{\vec{\varepsilon}}(\Omega) e^{2\pi i(\varepsilon'_{l_1^i} w_{l_1^i} + \varepsilon'_{l_2^i} w_{l_2^i} + \varepsilon'_{l_3^i} w_{l_3^i})}, \tag{86}$$

for $C_i = S_0$, and

$$\Pi(i)_0^{\vec{\lambda}=0} = \sum_{\vec{\varepsilon}_i \in \mathbf{Z}_2^2} w(\vec{\varepsilon}_i) e_{\vec{\varepsilon}}(\Omega) e^{2\pi i(\varepsilon'_{l_1^i} w_{l_1^i} + \varepsilon'_{l_2^i} w_{l_2^i})}, \tag{87}$$

for $C_i = T_0$, where $e_{\vec{\varepsilon}}(\Omega)$ is an elementary exponential function of Ω, the vector $(\varepsilon'_{l_1^i}, \varepsilon'_{l_2^i}, \varepsilon'_{l_3^i})^t$ in (86) is obtained from $(\varepsilon_{l_1^i}, \varepsilon_{l_2^i}, \varepsilon_{l_3^i})^t$ by the transformation (82), and the vector $(\varepsilon'_{l_1^i}, \varepsilon'_{l_2^i})^t$ in (87) is obtained from $(\varepsilon_{l_1^i}, \varepsilon_{l_2^i})^t$ by the transformation (85).

The variable $w_{l_m^i}$ is a linear combination of variables in $\{z_{l_1^i}, z_{l_2^i}, z_{l_3^i}\}$ $(C_i = S_0)$ or $\{z_{l_1^i}, z_{l_2^i}\}(C_i = T_0)$. Hence, at each $e_l = \partial C_i \cap \partial C_j$, the equality of the spin weight of the variable z_l $(= z_{l_r^i} = z_{l_s^j}$ for some r and $s)$ determines a coincident relation between the terms of $\Pi(i)_0^{\vec{\lambda}=0}$ and $\Pi(j)_0^{\vec{\lambda}=0}$ appearing on the right-hand side of equation (86) and equation (87) such that two terms in $\Pi(j)_0^{\vec{\lambda}=0}$ are related to one term in $\Pi(i)_0^{\vec{\lambda}=0}$.

We make the following correspondence between the set of arcs in C_i listed in Figure 5, all but the ones in (iiic) and (ivb) and the set of elementary exponential functions in $\vec{w}_i$, each of which is a factor of a term on the right-hand side of equations (86) and (87);

(i) The arc of (ia), (ib) or (ic) connecting $e_{l_r^i}$ and $e_{l_s^i}$ corresponds to the term $e^{2\pi i \varepsilon'_{l_m^i} w_{l_m^i}}$, where $w_{l_m^i} = z_{l_r^i} + z_{l_s^i}$.

(ii) The arcs of (iia),(iib) or (iic) connecting $e_{l_r^i}$ and $e_{l_s^i}$ correspond to the term $e^{2\pi i \varepsilon'_{l_m^i} w_{l_m^i}}$, where $w_{l_m^i} = z_{l_r^i} + z_{l_s^i}$.

(iii) The arcs of (iiia) connecting $\{e_{l_r^i}, e_{l_s^i}\}$ and $\{e_{l_r^i}, e_{l_t^i}\}$ correspond to the term $e^{(2\pi i \varepsilon'_{l_m^i} w_{l_m^i} - 2\pi i \varepsilon'_{l_n^i} w_{l_n^i})}$, where $w_{l_m^i} = z_{l_r^i} + z_{l_s^i}$ and $w_{l_n^i} = z_{l_r^i} + z_{l_t^i}$.

(iv) The arc of (iiib) connecting $e_{l_r^i}$ and $e_{l_s^i}$ corresponds to the term $e^{2\pi i \varepsilon'_{l_m^i} w_{l_m^i}}$, where $w_{l_m^i} = z_{l_r^i} + z_{l_s^i}$.

(v) The arc of (iva) corresponds to the term $e^{2\pi i \varepsilon'_{l_2^i} w_{l_2^i}}$, where $w_{l_2^i} = 2z_{l_1^i} + z_{l_2^i}$.

(vi) The arc of (ivc) corresponds to the term $e^{2\pi i \varepsilon'_{l_1^i} w_{l_1^i}}$, where $w_{l_1^i} = z_{l_2^i}$.

This correspondence, if there is a maximal set of mutually incident terms involved in a subset of terms in $\{\Pi(i)_0^{\vec{\lambda}=0}\}_{1 \le i \le 2g-2}$, defines a simple closed curve or an arc in C which is a combination of those arcs listed in Figure 5. If it is not closed, then its endpoints lie in a separating $e_l \notin \mathcal{E}_m^g$ at which $\mathcal{E}_m^g$ realizes the case (ii) with $\sharp\{e_{l(m)} \subset C_i\} = 2$ or the case (iii) with $\sharp\{e_{l(m)} \subset C_i\} = 1$ in Definition 8.3. By adding to it the arc (iiic) or (ivb) in Figure 5 in the respective case, we obtain a simple closed curve. Hence the choice of a maximal family of mutually coincident terms corresponds to a triple of a maximal curve system as in Definition 8.4, and hence by Lemma 8.2 it corresponds to a grouping $g \in \mathcal{E}_\Upsilon$.

Thus, if we choose and fix a grouping $g \in \mathcal{E}_\Upsilon$, then each term of $\Pi_0^{\vec{\lambda}=0}$ is a product of three elementary exponential functions each of which corresponds to one of the curves s_m of the corresponding curve system and

$$\Pi_0 = \left(S_{\frac{1}{2}\Lambda^*/\Lambda_0^*} \Pi^{\vec{\lambda}=0} \right)_0 . \tag{88}$$

Finally we note the following. Using the notation of equation (80) we consider the following terms contained in Π_0^{-1},

$$u_{\vec{\varepsilon},\vec{\lambda}} = e^{-1}_{\vec{\varepsilon},\vec{0}} e_{\vec{\varepsilon},\vec{\lambda}} \quad , \quad v_{\vec{\varepsilon},\vec{\lambda}} = e^{-1}_{-\vec{\varepsilon},\vec{\lambda}} e_{\vec{\varepsilon},\vec{\lambda}} \qquad \left(\vec{\lambda} \neq \vec{0} \in \frac{1}{2}\Lambda^*/\Lambda_0^* \right).$$

Then all the shifts of the spin weights induced from the multiplication by Π_0^{-1} are generated by those induced from the mutiplications by $\{u_{\vec{\varepsilon},\vec{\lambda}}\}_{\vec{\varepsilon},\vec{\lambda}}$. For example in $C_i = S_0$,in the above coordinate of spin weight $\vec{n}_i = (n_{l_1^i}, n_{l_2^i}, n_{l_3^i})^t$, the shifts of the spin weights induced from the multiplication of Π_0^{-1} are generated by the translations by the vectors $\left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $\left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Also $\{u_{\vec{\varepsilon},\vec{\lambda}} u_{-\vec{\varepsilon},\vec{\lambda}}\}$ and $\{v_{\vec{\varepsilon},\vec{\lambda}} v_{-\vec{\varepsilon},\vec{\lambda}}\}$ generate the shifts of the energy weights without spin weight shifts through their multiplications.

8.2. *Divisibility by Π and the Quantum-Clebsh-Gordan condition.*

Definition 8.6. Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. For a positive integer k, an *admissible weight* $\vec{j} = (j_l)_{1 \leq l \leq 3g-3} \in \frac{1}{2}\mathbf{Z}^{3g-3}$ of level k is defined as a function

$$\vec{j} = (j_l) : \{e_l\} \to \left\{ 0, \frac{1}{2}, \cdots, \frac{k}{2} \right\} \tag{89}$$

satisfying the so-called Quantum-Clebsch-Gordan condition of level k: For each $C_i \in \Upsilon (1 \leq i \leq 2g-2)$ with $\partial C_i = e_{l_1^i} \cup e_{l_2^i} \cup e_{l_3^i}$, the corresponding weights $\{j_{l_1^i}, j_{l_2^i}, j_{l_3^i}\}$ satisfy

$$\begin{cases} j_{l_1^i} + j_{l_2^i} + j_{l_3^i} \in \mathbf{Z} \\ |j_{l_1^i} - j_{l_2^i}| \leq j_{l_3^i} \leq j_{l_1^i} + j_{l_2^i} \\ j_{l_1^i} + j_{l_2^i} + j_{l_3^i} \leq k, \end{cases} \tag{90}$$

where if $C_i = T_0$, then $e_{l_2^i} = e_{l_3^i}$ and $j_{l_2^i} = j_{l_3^i}$ are to be understood.

We denote the set of all the admissible weights of level k by QCG_k.

THEOREM 8.1. *Let $\vec{j} \notin \mathrm{QCG}_k$. Then there does not exist a nontrivial Riemann theta function $\phi_{\vec{j}} \in (A_{2(k+2)} \cdot \Theta_{2k})_-^{W_\Upsilon^\circ}$ of the form*

$$\phi_{\vec{j}} = \sum_{\vec{\varepsilon} \in \mathbf{Z}_2^{3g-3}} w(\vec{\varepsilon})\, q_{\vec{j}}^{\vec{\varepsilon}} \cdot \vartheta \begin{bmatrix} \vec{\varepsilon} \cdot \left(\frac{2\vec{j}+\vec{1}}{2(k+2)} \right) \\ \vec{0} \end{bmatrix} (2(k+2)\vec{z},\, 2(k+2)\Omega), \tag{91}$$

such that $\tilde{\phi}_{\vec{j}} = S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi_{\vec{j}})$ is divisible by Π.

Proof. Let $\mathit{\Pi} = \mathit{\Pi}_0 + \mathit{\Pi}_1$ be the decomposition given in equation (79).

We formally expand $\mathit{\Pi}^{-1}\tilde{\phi}_{\vec{j}}$ as

$$\mathit{\Pi}^{-1}\tilde{\phi}_{\vec{j}} = \mathit{\Pi}_0^{-1}\tilde{\phi}_{\vec{j}} + \left(\sum_{m\geq 1}(-1)^m\left(\mathit{\Pi}_0^{-1}\mathit{\Pi}_1\right)^m\right)\tilde{\phi}_{\vec{j}}. \tag{92}$$

Then $\tilde{\phi}_{\vec{j}}$ is divisible by $\mathit{\Pi}$ if and only if the above series converges. If the first term $\mathit{\Pi}_0^{-1}\tilde{\phi}_{\vec{j}}$ converges, then there are constants $C, q > 0$ such that the second term satisfies the inequality

$$\left|\left(\sum_{m\geq 1}(-1)^m\left(\mathit{\Pi}_0^{-1}\mathit{\Pi}_1\right)^m\right)\tilde{\phi}_{\vec{j}}\right| \leq C(\eta(q)^{-2})^{(3g-3)},$$

where $\eta(q) = \prod_{n\geq 1}(1-q^n)$ is the Dedekind η function.

Thus $\tilde{\phi}_{\vec{j}}$ is divisible by $\mathit{\Pi}$ if and only if it is divisible by $\mathit{\Pi}_0$.

Choosing an elementary exponential function summand q_0 of $\mathit{\Pi}_0$, we write $\mathit{\Pi}_0 = q_0(1+\mathit{\Pi}_0')$. Then $\tilde{\phi}_{\vec{j}}$ can be divisible by $\mathit{\Pi}_0$ if and only if the formal power series

$$\mathit{\Pi}_0^{-1}\tilde{\phi}_{\vec{j}} = q_0^{-1}\left(1 + \sum_{m\geq 1}(-1)^m \mathit{\Pi}_0'^m\right)\tilde{\phi}_{\vec{j}} \tag{93}$$

defines a convergent series. Moreover it defines a convergent series if and only if the right-hand side of the above equation involves only finitely many terms with the same energy weight at each energy level; actually all the terms must be contained in the convex hull of the weight lattice L_ϕ of $\phi = \tilde{\phi}_{\vec{j}}$ (Definition 8.2).

Case with $C_i = S_0$. For $C_i \in \Upsilon$ with $\partial C_i = e_{l_1^i} \cup e_{l_2^i} \cup e_{l_3^i}$, we consider the three-vectors $(n_{l_1^i}, n_{l_2^i}, n_{l_3^i})$ representing the spin weights at the three boundary circles.

The first condition of (90) follows from the Λ invariance of $\mathit{\Pi}^{-1}\tilde{\phi}_{\vec{j}}$; i.e. it is invariant by the change of variables $w_{l_j^i} \to w_{l_j^i} + 1$ where $w_{l_j^i}$ is as in equation (81).

By the description of $\mathit{\Pi}_0$ given in equation (88), for $\tilde{C}_i$ $(1 \leq i \leq 2g-2)$, multiplication of each elementary function term in the parenthesis on the right-hand side of equation (93) affects the spin weight $(n_{l_1^i}, n_{l_2^i}, n_{l_3^i})$ by a shift given by a linear combination of the vectors $(1,1,0), (1,0,1)$ and $(0,1,1)$.

Hence if the right-hand side of (93) converges, the spin weights $(n_{l_1^i}, n_{l_2^i}, n_{l_3^i})$ of all the terms with minimal energy weights appearing on the right-hand side of (93) must be contained in a convex hull spanned by the vectors $\{\pm(2j_{l_1^i}, 2j_{l_2^i}, 2j_{l_3^i})\}$ and their shift by the vectors $\{\pm(1,1,0), \pm(1,0,1), \pm(0,1,1)\}$

in $\mathbf{R}^3$ which is of finite volume and has $\{\pm(2j_{l_1^i}, 2j_{l_2^i}, 2j_{l_3^i})\}$ as its vertices. Now, we have the following equation

$$(4j_{l_1^i}, 4j_{l_2^i}, 4j_{l_3^i}) = p(1,1,0) + q(1,0,1) + r(0,1,1) \tag{94}$$

for nonnegative integers p, q and r. It is equivalent to the second condition of (90).

By the periodicity of the Riemann theta function the same argument can be applied to the convex hull in $\mathbf{R}^3$ spanned by the spin weights

$$\{\pm\left((2k-2j_{l_1^i}), (2k-2j_{l_2^i}), (2k+2j_{l_3^i})\right)\}$$

and their shift by the vectors $\{\pm(1,1,0), \pm(1,0,1), \pm(0,1,1)\}$. It follows that $\tilde{\phi}_{\vec{j}}$ is divisible by Π_0 only if the vector $\left((2k-2j_{l_1^i}), (2k-2j_{l_2^i}), (2k+2j_{l_3^i})\right)$ satisfies the first and second conditions of (90). This yields the inequality

$$j_{l_1^i} + j_{l_2^i} + j_{l_3^i} \leq k \tag{95}$$

which is the third condition of (90).

Case with $C_i = T_0$. Let $e_{l_1^i} = \partial C_i$ and let $e_{l_2^i}$ be the essential simple closed curve in C_i. The second and the third conditions of (90) are

$$\begin{cases} -j_{l_1^i} + 2j_{l_2^i} \geq 0 \\ j_{l_1^i} + 2j_{l_2^i} \leq k. \end{cases} \tag{96}$$

Changing the variables as given in equation (84), we see that essentially the same argument as in the former case proves the claim of the theorem. □

8.3. *Divisibility by* Π *and* QCG: *sufficiency.* We proceed to prove the existence theorem which claims that, for $\vec{j} \in \mathrm{QCG}_k$, there is a nontrivial Riemann theta function $\phi_{\vec{j}}$ of the form (91) in Theorem 8.1 which is projectively invariant under W^o_Υ and is such that $\tilde{\phi}_{\vec{j}} = S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi_{\vec{j}})$ can be divided by Π.

Definition 8.7 (Marking of the lattice Λ). Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $g : \mathcal{E} = \mathcal{E}_1^g \cup \mathcal{E}_2^g \cup \mathcal{E}_3^g \in \mathcal{E}_\Upsilon$ be a grouping.

We define a *triple of markings* (*bases*) of Λ, $\{\mathfrak{m}_m^g\}_{m=1,2,3}$, such that

$$\mathfrak{m}_m^g = \{-[\tilde{e}_l - \sigma\tilde{e}_l]\}_{e_l \in \mathcal{E}_m^g} \cup \{E^i_{s(i)}\}_{1 \leq i \leq 2g-2},$$

where $E^i_{s(i)}$ is one of the vectors given in Definition 2.11 for $C_i \in \Upsilon$.

Likewise for $\vec{\varepsilon} = (\varepsilon_l) \in \mathbf{Z}_2^{3g-3}$ we define a triple of markings of Λ, $\{\mathfrak{m}_m^{g,\vec{\varepsilon}}\}_{m=1,2,3}$ in the same way by replacing $\{\tilde{e}_l\}$ by $\{\varepsilon_l \tilde{e}_l\}$ (and hence replacing $\{E^i_{s(i)}\}$ by the corresponding elements).

Let $\Omega_m^{g,\vec{\varepsilon}}$ be the *Riemann matrix* for $H_1(\tilde{C}, \mathbf{R})_-$ associated with the marking $\mathfrak{m}_m^{g,\vec{\varepsilon}}$ and its symplectic dual.

Remark 8.1. In the above definition we neglect to fix an ordering of the basis $\mathfrak{m}_m^{g,\vec{\varepsilon}}$. What we need actually is a coherent orientation of the Lagrangian $\tilde{\ell}$. We choose and fix, arbitrarily, ordering of $\mathfrak{m}_m^{g,\vec{\varepsilon}}$ so that it defines the orientation of $\tilde{\ell}$ which differs from that of $\{\varepsilon_l[\tilde{e}_l - \sigma\tilde{e}_l]\}$ by $(-1)^{g-1}$.

Now we define automorphic forms which are of fundamental importance in the subsequent arguments.

Definition 8.8 (Basic automorphic forms). Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C.

For a grouping $g : \mathcal{E} = \mathcal{E}_1^g \cup \mathcal{E}_2^g \cup \mathcal{E}_3^g \in \mathcal{E}_\Upsilon$, we define a *triple of vectors*, $\{\vec{\delta}_m^g = (\delta_l^m)_{1\le l\le 3g-3}\}_{m=1,2,3}$, by

$$\delta_l^m = \begin{cases} -1 & e_l \in \mathcal{E}_m^g \\ +1 & e_l \notin \mathcal{E}_m^g. \end{cases}$$

For $\vec{j} = (j_l) \in \mathrm{QCG}_k$ and $g \in \mathcal{E}_\Upsilon$, let $A_{\vec{j}}^g$ be the set of triples $a = (\vec{a}_1, \vec{a}_2, \vec{a}_3)$ of vectors in $\mathbf{Z}^{3g-3}$ such that, for $1 \le m \le 3$, $\vec{a}_m = (a_l^m)$ with $0 \le a_l^m \le 2k+3$ satisfies the following two conditions:

(i) For $1 \le l \le 3g-3$,

$$\delta_l^1 \left(a_l^1\right)^2 + \delta_l^2 \left(a_l^2\right)^2 + \delta_l^3 \left(a_l^3\right)^2 \equiv 0 \qquad \mod 4(k+2).$$

(ii) For each pair $e_l \cup e_{l'} \subset C_i$ $(1 \le i \le 2g-2)$,

$$a_l^1 a_{l'}^1 + a_l^2 a_{l'}^2 + a_l^3 a_{l'}^3 \equiv -(2j_l+1)(2j_{l'}+1) \quad \mod 2(k+2).$$

For $\vec{j} = (j_l) \in \mathrm{QCG}_k$, $g \in \mathcal{E}_\Upsilon$, $a = (\vec{a}_m)_{1\le m\le 3} \in A_{\vec{j}}^g$ and $\vec{\varepsilon} = (\varepsilon_l) \in \mathbf{Z}_2^{3g-3}$, we define a theta series $\vartheta_{\nu_{\vec{j},m}^{g,a,\vec{\varepsilon}}}(\Omega_m^{g,\vec{\varepsilon}})$ associated with the marking $\mathfrak{m}_m^{g,\vec{\varepsilon}}$ by

$$\vartheta_{\nu_{\vec{j},m}^{g,a,\vec{\varepsilon}}}(\Omega_m^{g,\vec{\varepsilon}}) = \sum_{\vec{\lambda}\in\Lambda^*} \exp\left\{2(k+2)\pi i \left(\vec{\lambda} + \nu_{\vec{j},m}^{g,a,\vec{\varepsilon}}\right)^t \Omega_m^{g,\vec{\varepsilon}} \left(\vec{\lambda} + \nu_{\vec{j},m}^{g,a,\vec{\varepsilon}}\right)\right\},$$

where

$$\nu_{\vec{j},m}^{g,a,\vec{\varepsilon}} = \sum_{l=1}^{l=3g-3} \frac{\varepsilon_l a_l^m}{2(k+2)} [\tilde{f}_l^{\,*}] \in \frac{1}{2(k+2)}\Lambda^* \tag{97}$$

and on the right-hand side the vectors $\vec{\lambda}$ and $\nu_{\vec{j},m}^{g,\vec{\varepsilon}}$ are understood to be written as column vectors with respect to the basis $\mathfrak{m}_m^{g,\vec{\varepsilon}}$.

Finally, for $\vec{j} = (j_l) \in \mathrm{QCG}_k$, we define

$$q_{\vec{j}}^{g,\vec{\varepsilon}} = \sum_{a\in A_{\vec{j}}^g} \vartheta_{\nu_{\vec{j},1}^{g,a,\vec{\varepsilon}}}(\Omega_1^{g,\vec{\varepsilon}}) \vartheta_{\nu_{\vec{j},2}^{g,a,\vec{\varepsilon}}}(\Omega_2^{g,\vec{\varepsilon}}) \vartheta_{\nu_{\vec{j},3}^{g,a,\vec{\varepsilon}}}(\Omega_3^{g,\vec{\varepsilon}}). \tag{98}$$

The following existence theorem is the main result of this section.

THEOREM 8.2. *Let* $\Upsilon = \{e_l, C_i\}$ *be a pant decomposition of* C. *Choose a grouping* $g \in \mathcal{E}_\Upsilon$. *For* $\vec{j} \in \mathrm{QCG}_k$ *and* $\vec{\varepsilon} \in \mathbf{Z}_2^{3g-3}$, *let* $q_{\vec{j}}^{g,\vec{\varepsilon}}$ *be the automorphic form of level* $2(k+2)$ *defined in equation* (98) *of Definition* 8.8. *Now,*

(99)
$$\tilde{\psi}_{\vec{j}} = \Pi^{-1} S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}\left(\sum_{\vec{\varepsilon}} w(\vec{\varepsilon})\, q_{\vec{j}}^{g,\vec{\varepsilon}} \cdot \vartheta\begin{bmatrix}\vec{\varepsilon}\cdot\frac{2\vec{j}+\vec{1}}{2(k+2)}\\ \vec{0}\end{bmatrix}(2(k+2)\vec{z},\, 2(k+2)\Omega)\right).$$

Then $\tilde{\psi}_{\vec{j}}$ *defines a Riemann theta function which is an image of the shift operator of a projectively invariant section under the action of* W_Υ^o.

Proof. First we prove that the Riemann theta function

(100)
$$\phi_{\vec{j}} = \sum_{\vec{\varepsilon}} w(\vec{\varepsilon})\, q_{\vec{j}}^{g,\vec{\varepsilon}} \cdot \vartheta\begin{bmatrix}\vec{\varepsilon}\cdot\frac{2\vec{j}+\vec{1}}{2(k+2)}\\ \vec{0}\end{bmatrix}(2(k+2)\vec{z},\, 2(k+2)\Omega)$$

is projectively invariant under the action of W_Υ^o.

W_Υ^o is generated by $\left(\tau_{e_l}^{C_i^*}\right)^2$ and $\kappa_{e_l}^{C_i^*}$ for $e_l \subset C_i \in \Upsilon$. We prove the projective invariance of $\phi_{\vec{j}}$ under $\kappa_{e_l}^{C_i^*}$. The corresponding statement for $\left(\tau_{e_l}^{C_i^*}\right)^2$ can be proved similarly, and we omit the details.

For $\partial C_i = e_{l_1^i} \cup e_{l_2^i} \cup e_{l_3^i}$, let $\{E_1^i, E_2^i, E_3^i\}$ be as in Definition 2.11.

The eigenvalue of the Dehn twist along the direction E_r^i $(r = 1,2,3)$ of $\vartheta\begin{bmatrix}\vec{\varepsilon}\cdot\frac{2\vec{j}+\vec{1}}{2(k+2)}\\ \vec{0}\end{bmatrix}$ is

$$\exp\left\{\pi i\frac{\left(\delta_1(2j_{l_1^i}+1)+\delta_2(2j_{l_2^i}+1)+\delta_3(2j_{l_3^i}+1)\right)^2}{2(k+2)}\right\}$$

for some $\delta_s = \pm 1$ $(s = 1,2,3)$. The conditions (i) and (ii) in Definition 8.8 ensure that the corresponding eigenvalues of $q_{\vec{j}}^{g,\vec{\varepsilon}}$ cancel the off-diagonal terms of the expansion of the square in the above eigenvalue, and hence the corresponding eigenvalue of $q_{\vec{j}}^{g,\vec{\varepsilon}} \cdot \vartheta\begin{bmatrix}\vec{\varepsilon}\cdot\frac{2\vec{j}+\vec{1}}{2(k+2)}\\ \vec{0}\end{bmatrix}$ does not depend on $\vec{\varepsilon}$. Therefore $\phi_{\vec{j}}$ is projectively invariant under the Dehn twists along the curves $\kappa_{e_l}^{C_i}$.

Secondly we prove that $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi_{\vec{j}})$ can be divided by Π. As noted in the proof of Theorem 8.1, $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi_{\vec{j}})$ can be divided by Π if and only if it can be divided by Π_0. Moreover the divisibility by Π_0 is equivalent to that; if we expand Π_0^{-1} into a formal power series as in equation (93), the formal power series $\Pi_0^{-1} S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(\phi_{\vec{j}})$ involves only finitely many terms at each energy level. The multiplication by $(\Pi_0)^{-1}$ induces the shifts both of the spin weight and the energy weight. Let P_Υ be the group of shifts of the spin weights generated by all the transformations induced by multiplication by $(\Pi_0)^{-1}$. Then

P_Υ is generated by the shifts of the spin weights corresponding to the vectors in $\frac{1}{2}\Lambda^*/\Lambda_0^*$. Hence by the condition QCG_k, for each fixed $\vec{\varepsilon} \in \mathbf{Z}_2^{3g-3}$, the sum of $\left\{ w(\pm\vec{\varepsilon}) q_{\vec{j}}^{g,\pm\vec{\varepsilon}} \cdot \vartheta \begin{bmatrix} \pm\vec{\varepsilon} \cdot \frac{2\vec{j}+\vec{1}}{2(k+2)} \\ \vec{0} \end{bmatrix} \right\}$ and their transformed image by $S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}$ supplies one set of subsums divisible by Π_0. Thus the right-hand side of equation (99) in Theorem 8.2 defines a well defined Riemann theta function.

□

THEOREM 8.3. *The set of families* $\{\tilde{\psi}_{\vec{j}}\}_{b \in \mathcal{B}_\Upsilon}$ *indexed by* $\vec{j} \in \mathrm{QCG}_k$ *constructed in Theorem* 8.2 *forms a basis of the vector space consisting of pull back sections of holomorphic sections of* $\mathcal{L}^k$. *Hence this set of families can be canonically identified with a basis of* $\Gamma(M_g, \mathcal{L}^k)$.

Proof. $\{\tilde{\psi}_{\vec{j}}\}_{\tilde{b} \in \mathcal{B}_\Upsilon}$ satisfies the global invariance (ii) in Section 7.1. As was mentioned at the end of Section 8.1, the terms $\{u_{\vec{\varepsilon},\vec{\lambda}} u_{-\vec{\varepsilon},\vec{\lambda}}\}$ and $\{v_{\vec{\varepsilon},\vec{\lambda}} v_{-\vec{\varepsilon},\vec{\lambda}}\}$ in Π^{-1} generate the energy- shifts without spin-shifts through their multiplication, where $u_{\vec{\varepsilon},\vec{\lambda}}$ and $v_{\vec{\varepsilon},\vec{\lambda}}$ are given below equation (88).

To make the multiplication by Π^{-1} well defined, we must eliminate by adding the counter terms the negative energy-weight terms produced from the minimal energy weight term of $\vartheta \begin{bmatrix} \pm\vec{\varepsilon} \cdot \frac{2\vec{j}+\vec{1}}{2(k+2)} \\ \vec{0} \end{bmatrix}$ by producting by the combinations of the above generators and their inverses.

Let (s_1^g, s_2^g, s_3^g) be the maximal curve system corresponding to the grouping $g \in \mathcal{E}_\Upsilon$. Each segment $s_m^g \cap C_i (C_i \in \Upsilon)$ corresponds to an element $\vec{\lambda} \in \frac{1}{2}\Lambda^*/\Lambda_0^*$ such that $\vec{\lambda} = \frac{1}{2}(\tilde{f}_l^* + \tilde{f}_{l'}^*)$ where the two endpoints of $s_m^g \cap C_i$ lies on $e_l \cup e_{l'}$. The above generators of energy-weights shift operators are assembled into three groups each of which belongs to the subgroup generated by those elements of $\frac{1}{2}\Lambda^*/\Lambda_0^*$ corresponding to $\{s_m^g \cap C_i\}_{C_i} (1 \le m \le 3)$.

By the description of the minimal energy part of $\Pi^{\vec{\lambda}=0}$ given in Section 8.1, for each s_m^g, we may perform a simultaneous coherent coordinate change in all the pants C_i so that $\{z_l\}_{e_l \in \mathcal{E}_m^g}$ and $\{w_{l_i}\}$ forms a coordinate associated with the lattice Λ. For such three coordinate systems, the necessary counter terms are coherently assembled to the sum of the product of three theta constants $q_{\vec{j}}^{g,\vec{\varepsilon}}$ given in Definition 8.8 equation (98). We note that those counter terms correspond to the Dehn twists along the basis vectors of Λ given in Definition 8.7, and the conditions (i) and (ii) in Definition 8.8 come from the effects of those Dehn twists.

Thus the coefficients $q_{\vec{j}}^{g,\vec{\varepsilon}}$ of $\tilde{\psi}_{\vec{j}}$ in equation (99) automatically appear for the well-definedness of the multiplication by the formal power series Π^{-1}. It shows that the $\tilde{\psi}_{\vec{j}}$ are minimal,that is, any other element of $\Pi^{-1} S_{\frac{1}{2}\Lambda^*/\Lambda_0^*}(A_{2(k+2)} \cdot \Theta_{2(k+2)})_-^{W_\Upsilon^o}$ is a linear combination of the form $\sum_{\vec{j}} f(\Omega)_{\vec{j}} \tilde{\psi}_{\vec{j}}$ for some functions

$f(\Omega)_{\vec{j}}$. It implies that the operator δP_Π in the differential operator D in Theorem 6.2 is given by the differential of $q_{\vec{j}}^{g,\vec{\varepsilon}}$. Hence each of $\{\tilde{\psi}_{\vec{j}}\}_{\tilde{b}\in\mathcal{B}_\Upsilon}$ satisfies the local invariance (i) in Section 7.1. Hence we obtain the theorem. □

9. Projectively flat connection and unitarity

Up to the previous section our argument has been done for a fixed Riemann surface C. In this section we vary the complex structure of C and we extend the results in previous sections to the whole family on the Teichmüller space of genus g Riemann surfaces.

9.1. *Projectively flat connection.* Let $\mathcal{T}$ be the Teichmüller space of genus g Riemann surfaces. We denote the point of $\mathcal{T}$ represented by a Riemann surface C by $[C]$. Then we have a fibre bundle $\mathcal{C} \to \mathcal{T}$ whose fibre over $[C]$ is $\mathcal{C}_C$.

There is a fibre bundle $\mathcal{B}_\mathcal{T} \to \mathcal{T}$ whose fibre on $[C]$ is $\mathcal{B}_{[C]}$, where $\mathcal{B}_{[C]}$ is the space $\mathcal{B}$ introduced in Definition 2.2 (Section 2) associated to the surface C. Also we have a fibre bundle $\mathcal{P}_\mathcal{T} \to \mathcal{B}_\mathcal{T}$ whose fibre over $[C]$ is the family of the Prym varieties $P_{\tilde{b}}$ with $\tilde{b} \in \mathcal{B}$.

The family $\{\Gamma(M_g, \mathcal{L}^k)\}_{[C]}$ combines together to form a holomorphic vector bundle over $\mathcal{T}$. We denote it by $\Gamma(M_g, \mathcal{L}^k)_\mathcal{T}$.

We consider the pull back of a section in $\{\Gamma(M_g, \mathcal{L}^k)\}_{[C]}$ as a holomorphic section in the family $\{\Gamma(P_{\tilde{b}}, \tilde{\mathcal{L}}^{2k})_\mathcal{T}\}$.

Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C. Let $\tilde{\mathcal{B}}_{\Upsilon\mathcal{T}}$ be the fibre bundle over $\mathcal{T}$ whose fibre over $[C]$ is the universal cover $\tilde{\mathcal{B}}_\Upsilon$ of $\mathcal{B}_\Upsilon$ for C. Let $\Theta_{k\mathcal{T}} \to \tilde{\mathcal{B}}_{\Upsilon\mathcal{T}}$ be the fibre bundle over $\mathcal{T}$ whose fibre over $[C]$ is the family of the vector spaces of Riemann theta functions of level k.

The Riemann theta functions Π combine together to give a global holomorphic section Π of the bundle $\Theta_{4\mathcal{T}}$ which is parallel with respect to the usual connection on the level-4 Riemann theta functions.

For $\vec{j} \in \mathrm{QCG}_k$, the holomorphic sections $\tilde{\psi}_{\vec{j}}$ given in Theorem 8.2 combine together to define a global holomorphic section $\tilde{\psi}_{\vec{j}}$ of $\Gamma(\mathcal{P}, \tilde{\mathcal{L}}^{2k})_\mathcal{T}$ on $\tilde{\mathcal{B}}_{\Upsilon\mathcal{T}}$.

The operator δP_Π (P_Π is the projection operator given in Definition 6.2) can be defined also on $\Gamma(\mathcal{P}, \tilde{\mathcal{L}}^{2k})_\mathcal{T}$.

Theorem 9.1 (projectively flat connection). *The differential operator D acting on $\Gamma(\mathcal{P}, \tilde{\mathcal{L}}^{2k})_\mathcal{T}$ given by*

$$D\tilde{\psi} = \Pi^{-1}\left(\delta + \frac{1}{8(k+2)}(\delta J\omega^{-1})_{ij}\partial_i\partial_j - \delta P_\Pi\right)\left(\Pi\tilde{\psi}\right) \tag{101}$$

defines a projectively flat connection on the vector bundle $\Gamma(M_g, \mathcal{L}^k)_\mathcal{T}$. The curvature of D is central and equal to the multiplication

(102) $$\frac{k}{8(k+2)}\mathrm{tr}\delta\Omega(\mathrm{Im}\Omega)^{-1}.$$

Moreover, for a decomposition of C, $\Upsilon = \{e_l, C_i\}$, $\{\tilde{\psi}_{\vec{j}}\}_{\vec{j}\in\mathrm{QCG}_k}$ *forms a basis of parallel sections in* $\Gamma(M_g, \mathcal{L}^k)_{\mathcal{T}}$ *with respect to this connection.*

Proof. The curvature of D is equal to $\bar{\delta}D$ and it coincides with

(103) $$-\bar{\delta}\delta P_{\Pi} + \bar{\delta}\delta.$$

This is equal to the central curvature of the determinant line bundle on the Grassamann varieties consisting of $\Pi\Theta_{2k}$ which is the sum of the central curvature of $\Theta_{2(k+2)}$ and the curvature which comes from the variation of Π. Hence it equals

$$\frac{1}{8}\mathrm{tr}\delta\Omega(\mathrm{Im}\Omega)^{-1} - \frac{2}{8(k+2)}\mathrm{tr}\delta\Omega(\mathrm{Im}\Omega)^{-1} = \frac{k}{8(k+2)}\mathrm{tr}\delta\Omega(\mathrm{Im}\Omega)^{-1}$$

and we obtain the result. □

In [11] Hitchin showed that a connection on $\Gamma(M_g, \mathcal{L}^k)_{\mathcal{T}}$ defined by a differential operator on $\mathcal{L}^k$ is uniquely determined by the holomorphicity-preserving property of the differential operator up to a constant multiplication operator. The differential operator D defining the connection in Theorem 9.1 is also constructed basically from the holomorphicity-preserving condition (Step 1 in the proof of Theorem 6.2). D is a family of differential operators on $\{\Gamma(P_{\tilde{b}}, \tilde{\mathcal{L}}^{2k})_{\mathcal{T}}\}$ which are invariant under the variation of $\tilde{b}$. Hence D defines a differential operator on $\mathcal{L}^k$ which induces a connection on $\Gamma(M_g, \mathcal{L}^k)_{\mathcal{T}}$. Thus the Hitchin's result implies

THEOREM 9.2. *The connection in Theorem* 9.1 *coincides with the Hitchin connection in* [11].

9.2. *Unitarity.* Next we prove the existence of a hermitian product on $\Gamma(M_g, \mathcal{L}^k)$ which is invariant with respect to the projectively flat connection defined in the above theorem.

The space $\Theta_{2(k+2)}$ has the usual Hermitian inner product such that $\left\{\vartheta\begin{bmatrix}\vec{a}\\ \vec{0}\end{bmatrix}\right\}$ forms an orthonormal bases. Likewise the space $A_{2(k+2)}$ has the Hermitian product given by the Petersson scalar product ([8]).

THEOREM 9.3 (The invariant Hermitian product). *Let* $\Upsilon = \{e_l, C_i\}$ *be a pant decomposition of* C. *Let* $(\ ,\ )_{A\cdot\Theta} = (\ ,\ )_{A_{2(k+2)}\cdot\Theta_{2(k+2)}}$ *be the Hermitian product on* $A_{2(k+2)}\cdot\Theta_{2(k+2)}$ *defined by the tensor product of the usual Hermitian product on the space of Riemann theta functions and the Petersson scalar product on the space of automorphic forms. Let* V_k *be the vector space generated by* $\{\tilde{\psi}_{\vec{j}}\}_{\vec{j}\in\mathrm{QCG}_k}$. *We define a Hermitian inner product* $\langle\ ,\ \rangle_{V_k}$ *on* V_k, *for*

$\tilde{\psi}, \tilde{\psi}' \in V_k$, *by*

$$\left\langle \tilde{\psi}, \tilde{\psi}' \right\rangle_{V_k} = \left(\mathit{\Pi}\tilde{\psi}, \mathit{\Pi}\tilde{\psi}' \right)_{A \cdot \Theta}.$$

This defines a Hermitian inner product $\langle\ ,\ \rangle$ *on the space* $\Gamma(M_g, \mathcal{L}^k)$ *which is invariant under the projectively flat connection* D *given in Theorem* 9.1.

10. The transformation formula

10.1. *General Remarks.* A symplectic linear transformation of the Prym variety P induces a metaplectic transformation of theta functions and theta constants in the usual way.

The metaplectic correction on the Prym variety is defined by incorporating the bundle of half-volume form on Prym varieties ([23]). What we should actually consider is the half-volume form of M_g which is a holomorphic section of the square root $\kappa_{M_g} = \sqrt{K_{M_g}}$ of the canonical bundle of M_g. Now $\pi^* \kappa_{M_g}$ is isomorphic to $\tilde{\mathcal{L}}^4$, and $\mathit{\Pi}$ is a holomorphic section of it (see Theorem 5.1). Thus $\mathit{\Pi}\tilde{\psi}_{\vec{j}}$ incorporates the half-volume form of M_g, and we consider its transformation law.

10.2. *Transformation formula.* Let $(\tilde{b}, \mathfrak{m}) \in \tilde{\mathcal{B}}_\Upsilon$ where $\tilde{b} = (b, \alpha)$.

Let h be a diffeomorphism of C. By isotopy we may assume that h fixes b pointwise.

Let $\Upsilon = \{e_l, C_i\}$ be a pant decomposition of C and let $\Upsilon' = h_* \Upsilon = \{e'_l, C'_i\}$ be the transformed image of Υ by h. Likewise let $\alpha' = (h_*^{-1})\alpha$ and $\mathfrak{m}' = h_* \mathfrak{m}$ be the transformed images by h.

Let $\{\tilde{\psi}_{\vec{j}}\}$ and $\{\tilde{\psi}'_{\vec{j}'}\}$ be the corresponding bases of pull back sections as constructed in Theorem 8.2.

We consider the transformation law between $\{\tilde{\psi}_{\vec{j}}\}$ and $\{\tilde{\psi}'_{\vec{j}'}\}$.

First we note that h alone cannot determine a lifting of it to a diffeomorphism $\tilde{h}$ of $\tilde{C}$.

To determine $\tilde{h}$, we need extra information attached to h. Let $\{\Upsilon, h, h_*\Upsilon\}$ be a triple of h, a pant decomposition Υ and its transformed image $h_*\Upsilon$. Then such a triple determines a lifting of h, $\tilde{h}_\Upsilon$, such that $\tilde{h}_\Upsilon$ maps the decomposition of $\tilde{C}$ induced by Υ to the decomposition of $\tilde{C}$ induced by $h_*\Upsilon$. The set of such triples $\{\Upsilon, h, h_*\Upsilon\}$ forms a groupoid and it defines a projective group action of a central extension of the mapping class group of C on the conformal block by making use of the projectively flat connection.

Definition 10.1. The diffeomorphism $\tilde{h}_\Upsilon$ is called a *lifting of* h *associated with* Υ.

Let $\tilde{\ell}$ be the Lagrangian in $H_1(\tilde{C}, \mathbf{R})_-$ spanned by $\{[\tilde{e}_l - \sigma\tilde{e}_l]\}_{1 \le l \le 3g-3}$. We choose and fix a complementary Lagrangian $\tilde{\ell}^*$. Let $\{\tilde{f}^*_l\}$ be the basis of $\tilde{\ell}^*$ which is the symplectic dual of $\{[\tilde{e}_l - \sigma\tilde{e}_l]\}$. See Definition 2.12.

Let $\tilde{h}_{\Upsilon *} : H_1(\tilde{C}, \mathbf{R})_- \to H_1(\tilde{C}, \mathbf{R})_-$ be the symplectic linear transformation induced by the diffeomorphism $\tilde{h}_\Upsilon : \tilde{C} \to \tilde{C}$.

Let

$$T_{\tilde{h}_\Upsilon} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{104}$$

be the matrix representation of the symplectic transformation $\tilde{h}_*$ with respect to the basis $\{[\tilde{e}_l - \sigma \tilde{e}_l]\} \cup \{\tilde{f}_l^*\}$, where

$$A : \tilde{\ell} \to \tilde{\ell} \quad , \quad B : \tilde{\ell}^* \to \tilde{\ell} \quad , \quad C : \tilde{\ell} \to \tilde{\ell}^* \quad , \quad D : \tilde{\ell}^* \to \tilde{\ell}^* . \tag{105}$$

The symplectic matrix $T_{\tilde{h}_\Upsilon}$ acts on the pair $(\Omega, \vec{z})$ by

$$T_{\tilde{h}_\Upsilon}(\Omega, \vec{z}) = \left((A\Omega + B)(C\Omega + D)^{-1}, (C\Omega + D)^{-1} \vec{z} \right).$$

Likewise for the groupings $g \in \mathcal{E}_\Upsilon$, $g' = h_* g \in \mathcal{E}_{h_* \Upsilon}$ and $\gamma \in S_3$, we have a matrix, for $1 \le m \le 3$,

$$(T_{\tilde{h}_\Upsilon})^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} = \begin{pmatrix} A^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} & B^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} \\ C^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} & D^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} \end{pmatrix} \tag{106}$$

which transforms the corresponding Riemann matrix $\Omega^{g,\vec{\varepsilon}}_m$ to

$$\Omega^{g',\vec{\varepsilon}'}_{\gamma(m)} = (T_{\tilde{h}_\Upsilon})^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} (\Omega^{g,\vec{\varepsilon}}_m);$$

that is,

$$\Omega^{g',\vec{\varepsilon}'}_{\gamma(m)} = \left(A^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} \Omega^{g,\vec{\varepsilon}}_m + B^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} \right) \left(C^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} \Omega^{g,\vec{\varepsilon}}_m + D^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} \right)^{-1}.$$

Let $\mathrm{Sp}(H_1(\tilde{C}, \mathbf{R})_-)$ be the group of all the symplectic transformations of $H_1(\tilde{C}, \mathbf{R})_-$. Then a choice of a path in $\mathrm{Sp}(H_1(\tilde{C}, \mathbf{R})_-)$ connecting $\tilde{h}_{\Upsilon *}$ to the identity determines the lifts of $T_{\tilde{h}_\Upsilon}$ and $(T_{\tilde{h}_\Upsilon})^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma}$ to metaplectic transformations

$$\hat{T}_{\tilde{h}_\Upsilon} : \Theta_{2(k+2)} \to \Theta_{2(k+2)} \quad , \quad (\hat{T}_{\tilde{h}_\Upsilon})^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} : A_{2(k+2)} \to A_{2(k+2)}$$

both of which are isometries with respect to the usual Hermitian product on $\Theta_{2(k+2)}$ and the Petersson scalar product on $A_{2(k+2)}$ respectively.

The transformation $\hat{T}_{\tilde{h}_\Upsilon} : \Theta_{2(k+2)} \to \Theta_{2(k+2)}$ is represented as (for $\mu, \mu' \in \frac{1}{2(k+2)} \Lambda_0^*$ and $\vec{\lambda}, \vec{\lambda}' \in \frac{1}{2} \Lambda^* / \Lambda_0^*$)

$$\begin{aligned} \hat{T}_{\tilde{h}_\Upsilon} &\left(\vartheta \begin{bmatrix} \mu + \vec{\lambda} \\ \vec{0} \end{bmatrix} (2(k+2)\vec{z}, 2(k+2)\Omega) \right) \\ &= \sum_{\mu', \vec{\lambda}'} a^{\mu' + \vec{\lambda}'}_{\mu + \vec{\lambda}} \vartheta \begin{bmatrix} \mu' + \vec{\lambda}' \\ \vec{0} \end{bmatrix} (2(k+2)\vec{z}, 2(k+2)\Omega). \end{aligned} \tag{107}$$

The transformation $(\hat{T}_{\tilde{h}_\Upsilon})^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} : A_{2(k+2)} \to A_{2(k+2)}$ preserves the subspace spanned by the theta constants and is represented as

$$(\hat{T}_{\tilde{h}_\Upsilon})^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma} \left(\vartheta_\nu(\Omega^{g,\vec{\varepsilon}}_m)\right) = \sum_{\nu'} \left(b^{g,g',\vec{\varepsilon},\vec{\varepsilon}'}_{m,\gamma}\right)_{\nu,\nu'} \vartheta_{\nu'}(\Omega^{g',\vec{\varepsilon}'}_{\gamma(m)}), \tag{108}$$

where for $\nu, \nu' \in \frac{1}{2(k+2)}\Lambda^*$,

$$\vartheta_\nu(\Omega^{g,\vec{\varepsilon}}_m) = \sum_{\vec{\lambda}\in\Lambda^*} \exp\left\{2(k+2)\pi i \left(\vec{\lambda}+\nu\right)^t \Omega^{g,\vec{\varepsilon}}_m \left(\vec{\lambda}+\nu\right)\right\},$$

and similarly for $\vartheta_{\nu'}(\Omega^{g',\vec{\varepsilon}'}_{\gamma(m)})$.

THEOREM 10.1. *Let V_k be the subspace in $A_{2(k+2)} \cdot \Theta_{2(k+2)}$ spanned by $\{\Pi\tilde{\psi}_{\vec{j}}\}_{\vec{j}\in \mathrm{QCG}_k}$. For a diffeomorphism h of C, let $\tilde{h}_\Upsilon$ be the lifting of h to a diffeomorphism of $\tilde{C}$ associated with Υ. There is a fixed grouping $g \in \mathcal{E}_\Upsilon$ and $g' = h_* g$.*

(i) *Choose a path in $\mathrm{Sp}(H_1(\tilde{C},\mathbf{R})_-)$ connecting $\tilde{h}_{\Upsilon *}$ to the identity, and there are metaplectic transformations $\hat{T}_{\tilde{h}_\Upsilon}$ and $(\hat{T}_{\tilde{h}_\Upsilon})^{g,g',\varepsilon,\varepsilon'}_{m,\gamma}$ as above.*

Then those data define a linear transformation

$$\mathbb{T}_{\tilde{h}_\Upsilon} : V_k \to V_k,$$

and this gives an action of a central extension of the mapping class group of C on V_k.

(ii) *Let $\vec{\varepsilon} \in \mathbf{Z}_2^{3g-3}$, $\vec{j}, \vec{j}' \in \mathrm{QCG}_k$ and $\vec{\lambda} \in \frac{1}{2}\Lambda^*/\Lambda_0^*$. Let $a^{\mu_{\vec{j}'}}_{\vec{\varepsilon}\cdot\mu_{\vec{j}}+\vec{\lambda}}$ be the coefficient on the right-hand side of equation (107) where $\mu_{\vec{j}} = \frac{2\vec{j}+\vec{1}}{2(k+2)}$ and $\mu_{\vec{j}'} = \frac{2\vec{j}'+\vec{1}}{2(k+2)}$. Likewise for $g \in \mathcal{E}_\Upsilon$, $g' \in \mathcal{E}_{h_*\Upsilon}$, $a \in A^g_{\vec{j}}$, $a' \in A^{g'}_{\vec{j}'}$ and $\gamma \in S_3$, set*

$$\left(b^{g,g',\vec{\varepsilon},\vec{1}}_{m,\gamma}\right)^{a,a'}_{\vec{j},\vec{j}'} = \left(b^{g,g',\vec{\varepsilon},\vec{1}}_{m,\gamma}\right)_{\nu^{g,a,\vec{\varepsilon}}_{\vec{j},m},\,\nu^{g',a',\vec{1}}_{\vec{j}',\gamma(m)}}, \tag{109}$$

where $\left(b^{g,g',\vec{\varepsilon},\vec{1}}_{m,\gamma}\right)_{\nu^{g,a,\vec{\varepsilon}}_{\vec{j},m},\nu^{g',a',\vec{1}}_{\vec{j}',\gamma(m)}}$ is the coefficient on the right-hand side of equation (108) for $\nu = \nu^{g,a,\vec{\varepsilon}}_{\vec{j},m}$ and $\nu' = \nu^{g',a',\vec{1}}_{\vec{j}',\gamma(m)}$ which are given by equation (97) in Definition 8.8. We define

(110)

$$c^g_{\vec{j},\vec{j}'} = \sum_{\vec{\lambda}\in\frac{1}{2}\Lambda^*/\Lambda_0^*} \sum_{\vec{\varepsilon}\in\mathbf{Z}_2^{3g-3}} \sum_{a\in A^g_{\vec{j}}} w(\vec{\varepsilon}) e^{2\pi i(\vec{\lambda}^t\frac{\vec{1}}{2})} a^{\mu_{\vec{j}'}}_{\vec{\varepsilon}\cdot\mu_{\vec{j}}+\vec{\lambda}} \left[\sum_{\gamma\in S_3}\prod_{m=1}^{3} \left(b^{g,g',\vec{\varepsilon},\vec{1}}_{m,\gamma}\right)^{a,a'}_{\vec{j},\vec{j}'}\right].$$

Then $c^{g}_{\vec{j},\vec{j}'}$ does not depend on the choices of $g \in \mathcal{G}$ and $a' \in A^{g'}_{\vec{j}'}$, and when $c_{\vec{j},\vec{j}'} = c^{g}_{\vec{j},\vec{j}'}$, the transformation $\mathbb{T}_{\tilde{h}_\Upsilon} : V_k \to V_k$ is given by

$$\mathbb{T}_{\tilde{h}_\Upsilon}(\Pi\tilde{\psi}_{\vec{j}}) = \sum_{\vec{j}' \in \mathrm{QCG_k}} c_{\vec{j},\vec{j}'}\Pi\tilde{\psi}_{\vec{j}'}. \tag{111}$$

Proof. The existence of the linear transformation $\mathbb{T}_{\tilde{h}_\Upsilon}$ follows from the fact that a pull back section transforms to a pull back section.

The coefficients $a^{\mu_{\vec{j}'}}_{\vec{\varepsilon}\cdot\mu_{\vec{j}}+\vec{\lambda}}$ and $\left(b^{g,g',\vec{\varepsilon},\vec{1}}_{m,\gamma}\right)^{a,a'}_{\vec{j},\vec{j}'}$ are calculated as sums of elementary exponential functions of the symplectic pairings of the involved theta characteristics and Maslov index, and it can be seen that $c^{g}_{\vec{j},\vec{j}'}$ does not depend on the choice of $g \in \mathcal{G}$ and $a' \in A^{g'}_{\vec{j}'}$. □

11. The case of genus one

So far we have been working on Riemann surfaces of genus $g \geq 2$. We briefly comment on the case of genus one curves in this section.

Although all of the results in the genus one case in this section are well known ([2], [12]), we deduce them here by using arguments similar to our arguments in the genus $g \geq 2$ case.

Let C be a Riemann surface of genus one. The moduli space M_1 of holomorphic rank 2 bundles on C is the quotient space C/σ where σ is the hyperelliptic involution of C. Thus M_1 is the complex line $\mathbf{P}^1$.

The space of holomorphic sections of the line bundle $\mathcal{L}^k$ on M_1 can be identified with the space of the σ-invariant holomorphic sections of the pull-back line bundles $\tilde{\mathcal{L}}^{2k}$ on C.

Let $\Upsilon = \{e, f\}$ be a marking of C; that is, Υ is a pair of two oriented simple closed curves e and f in C such that $e \cap f$ is a point and the orientation of C coincides with the one defined by $\{e, f\}$. Let $b = \{x_1, x_2\}$ be a pair of points in $C - e$. Let $\alpha \in \hat{H}^1(C - b, \mathbf{Z}_2)$ be a covering type and let $\tilde{C}_b$ be the two-fold branched covering surface of C with branch set b associated to α.

The Prym variety P_b of $\tilde{C}_b$ is an elliptic curve which is isomorphic to C. Let $\tilde{e}$ and $\tilde{f}$ be a lifting of e and f to $\tilde{C}$ respectively such that $\tilde{e} \cap \tilde{f} = \{a\ point\}$ and $\tilde{e} \cap \sigma\tilde{f} = \emptyset$. Let w be the holomorphic 1-forms on $\tilde{C}$ such that $\int_{\tilde{e}} w = 1$ and $\int_{\tilde{e}} \sigma w = 0$. Then $\{w_l, \sigma w_l\}$ forms a basis of the space of holomorphic 1-forms on $\tilde{C}$.

The Riemann matrix $\Omega_{\tilde{C}}$ of $\tilde{C}$ with respect to the above basis is a 2×2-matrix whose entries are the line integrals of these holomorphic 1-forms along $\{[\tilde{f}_l], [\sigma\tilde{f}_l]\}$,

$$\tilde{\Omega} = \begin{pmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} \\ \tilde{\Omega}_{21} & \tilde{\Omega}_{22} \end{pmatrix} \tag{112}$$

where $\tilde{\Omega}$ is a complex symmetric matrix such that

$$\tilde{\Omega}_{11} = \int_{\tilde{f}_l} w_{l'} \quad , \quad \tilde{\Omega}_{12} = \int_{\tilde{f}_l} \sigma w_{l'} \tag{113}$$

and $\tilde{\Omega}_{22} = \tilde{\Omega}_{11}$, $\tilde{\Omega}_{21} = \tilde{\Omega}_{12}$.

The Jacobian $\tilde{J}$ of $\tilde{C}$ is given by $\mathbf{C}^2/(\mathbf{Z}^2 + \tilde{\Omega}\mathbf{Z}^2)$. The Prym variety P is the subspace of $\tilde{J}$ consisting of σ-anti-invariant elements. The Prym variety is the subspace of $\tilde{J}$ spanned by $[\tilde{f} - \sigma\tilde{f}]$.

Let Λ be the lattice in $H_1(\tilde{C}, \mathbf{R})_-$ generated by $\left\{[\tilde{e} - \sigma\tilde{e}]\ ,\ \frac{1}{2}[\tilde{f} - \sigma\tilde{f}]\right\}$.

We set

$$\Omega = \tilde{\Omega}_{11} - \tilde{\Omega}_{12}. \tag{114}$$

Then as a symplectic manifold $P = H_1(\tilde{C}, \mathbf{R})_-/\Lambda$ and as an elliptic curve $P = \mathbf{C}/\mathbf{Z} + \Omega\mathbf{Z}$.

We have a family of holomorphic maps $\pi : \tilde{P}_b \to M_1$ parametrized by $b \in \mathcal{B} = C \times C - \Delta$.

Both of the spaces $\tilde{\mathcal{B}}_\Upsilon$ and $\mathcal{T}$ are complex one-dimensional and the space $\tilde{\mathcal{B}}_{\Upsilon\mathcal{T}}$ can be identified with $\tilde{\mathcal{B}}_\Upsilon \times \mathcal{T}$. It is parametrized by the set of pairs $(\tilde{\Omega}_{11}, \tilde{\Omega}_{12})$ and $\tilde{\Omega}_{11} + \tilde{\Omega}_{12}$ is constant along $\mathcal{B}_\Upsilon$.

We apply the arguments of the previous sections to the genus one case. In the genus one case Π is the σ-anti-invariant Riemann theta function of level 4 on the Prym variety and the multiplication operator

$$\Pi\ :\ (\Theta_{2k})_+ \to \left(\Theta_{2(k+2)}\right)_- \tag{115}$$

is an isomorphism between the linear spaces over $\mathbf{C}$, where $\pm$ denote the σ-invariant and σ-anti-invariant subspaces. Hence the subspace of $\Theta_{2(k+2)}$ consisting of those elements divisible by Π coincides with the σ-anti-invariant subspace and we have

$$\delta P_\Pi = 0. \tag{116}$$

Moreover,for $\varepsilon = \pm 1$ and for each half integer $0 \leq j \leq \frac{k}{2}$, we can set

$$q_{\vec{j}}^{\varepsilon} = 1, \tag{117}$$

where $q_{\vec{j}}^{\varepsilon}$ is the automorphic form corresponding to $q_{\vec{j}}^{g,\varepsilon}$ in Theorem 8.2 in the case of genus one.

Thus we obtain the following:

Theorem 11.1. *For a Riemann surface of genus* 1 *the differential operator*

$$D\tilde{\psi} = \Pi^{-1}\left(\delta + \frac{1}{8(k+2)}(\delta J\omega^{-1})_{ij}\partial_i\partial_j\right)\Pi\tilde{\psi} \tag{118}$$

gives a projectively flat connection on the vector bundle of conformal blocks $\Gamma(M_1, \tilde{\mathcal{L}}^k)_{\mathcal{T}}$ *of level* k.

The set of Riemann theta functions defined by

$$\tilde{\psi}_j = \Pi^{-1} \sum_{\varepsilon \in \mathbf{Z}_2} w(\varepsilon)\vartheta \begin{bmatrix} \varepsilon \frac{(2j+1)}{2(k+2)} \\ 0 \end{bmatrix} (2(k+2)z, 2(k+2)\Omega), \tag{119}$$

where $j \in \frac{1}{2}\mathbf{Z}$ *such that* $0 \leq j \leq k/2$, *forms a parallel orthonormal basis of* $\Gamma(\mathcal{P}, \tilde{\mathcal{L}}^{2k})_{\mathcal{T}}$ *with respect to the above connection and the invariant hermitian form given by*

$$\langle \tilde{\psi}\ ,\ \tilde{\psi}' \rangle = \Big(\Pi\tilde{\psi}\ ,\ \Pi\tilde{\psi}' \Big)_{\Theta_{2(k+2)}}. \tag{120}$$

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References

[1] M. F. Atiyah, *The Geometry and Physics of Knots*, Cambridge Univ. Press, Cambridge, 1990.

[2] S. Axelrod, S. D. Pietra, and E. Witten, Geometric quantization of Chern-Simons gauge theory, *J. Differential Geom.* **33** (1991), 787–902.

[3] S. Bates and A. Weinstein, *Lectures on the Geometry of Quantization*, *Berkeley Math. Lecture Note*, Vol. 8, A. M. S., Providence, RI, 1997.

[4] A. Beauville, Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta, *Bull. Soc. Math. France* **116** (1988), 431–448.

[5] A. Beauville and Y. Laszlo, Conformal blocks and generalized theta functions, *Comm. Math. Phys*. **164** (1994), 385–419.

[6] A. Beauville, M. S. Narasimhan, and S. Ramanan, Spectral curves and the generalised theta divisor, *J. reine angew. Math*. **398** (1989), 169–179.

[7] F. Falceto, K. Gawedzki, and A. Kupianien, Scalar product of current blocks in WZW theory, *Phys. Lett*. **B260** (1991), 101–108.

[8] H. M. Farkas and I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group*, *Grad. Studies Math*. **37**, A. M. S., Providence, RI, 2001.

[9] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc*. **55** (1987), 59–126.

[10] ———, Stable bundles and integrable systems, *Duke Math. J*. **54** (1987), 91–114.

[11] ———, Flat connections and geometric quantization, *Comm. Math. Phys*. **131** (1990), 347–380.

[12] V. G. Kac, *Infinite Dimensional Lie Algebras*, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[13] A. A. Kirillov, Jr., On inner product in modular tensor categories. I, *J. Amer. Math. Soc*. **9** (1996), 1135–1170.

[14] ———, On inner product in modular tensor categories. II. Inner product on conformal blocks and affine inner product identities, *Adv. Theory Math. Phys*. **2** (1998), 155–180.

[15] S. Kumar, M. S. Narasimhan, and A. Ramanathan, Infinite Grassmannians and moduli spaces of G-bundles, *Math. Ann.* **300** (1994), 41–75.

[16] Y. Laszlo, Hitchin's and WZW connections are the same, *J. Differential Geom.* **49** (1998), 547–576.

[17] G. Lion and M. Vergne, *The Weil Representation*, *Maslov Index and Theta Series*, Birkhäuser Boston, Inc., Boston, MA, 1980.

[18] D. Mumford, *Tata Lectures on Theta* I, *Progr. Math.* **28**, Birkhäuser Boston, Inc., Boston, MA, 1983.

[19] ———, *Tata Lectures on Theta* II, *Progr. Math.* **43**, Birkhäuser Boston, Inc., Boston, MA, 1983.

[20] M. S. Narasimhan and C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–567.

[21] A. Tsuchiya, K. Ueno, and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmetries, *Adv. Studies Pure Math.* **19** (1989), 459–566.

[22] V. G. Turaev, *Quantum Invariants of Knots and* 3-*Manifolds*, *de Gruyter Studies in Math.* **18**, Walter de Gruyter, 1994.

[23] N. M. J. Woodhouse, *Geometric Quantization*, Oxford Univ. Press, Oxford, 1992.

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