Annals of Mathematics, **163** (2006), 809–899

On a class of type II_1 factors with Betti numbers invariants

By Sorin Popa*

Abstract

We prove that a type II_1 factor M can have at most one Cartan subalgebra A satisfying a combination of rigidity and compact approximation properties. We use this result to show that within the class $\mathcal{HT}$ of factors M having such Cartan subalgebras $A \subset M$, the Betti numbers of the standard equivalence relation associated with $A \subset M$ ([G2]), are in fact isomorphism invariants for the factors M, $\beta_n^{\mathrm{HT}}(M), n \geq 0$. The class $\mathcal{HT}$ is closed under amplifications and tensor products, with the Betti numbers satisfying $\beta_n^{\mathrm{HT}}(M^t) = \beta_n^{\mathrm{HT}}(M)/t$, $\forall t > 0$, and a Künneth type formula. An example of a factor in the class $\mathcal{HT}$ is given by the group von Neumann factor $M = L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$, for which $\beta_1^{\mathrm{HT}}(M) = \beta_1(\mathrm{SL}(2,\mathbb{Z})) = 1/12$. Thus, $M^t \not\simeq M, \forall t \neq 1$, showing that the fundamental group of M is trivial. This solves a long standing problem of R. V. Kadison. Also, our results bring some insight into a recent problem of A. Connes and answer a number of open questions on von Neumann algebras.

Contents

*Supported in part by a NSF Grant 0100883.

0. Introduction

We consider in this paper the class of type II_1 factors with maximal abelian *-subalgebras satisfying both a weak rigidity property, in the spirit of Kazhdan, Margulis ([Ka], [Ma]) and Connes-Jones ([CJ]), and a weak amenability property, in the spirit of Haagerup's compact approximation property ([H]). Our main result shows that a type II_1 factor M can have at most one such maximal abelian *-subalgebra $A \subset M$, up to unitary conjugacy. Moreover, we prove that if $A \subset M$ satisfies these conditions then A is automatically a Cartan subalgebra of M, i.e., the normalizer of A in N, $\mathcal{N}(A) = \{u \in M \mid uu^* = 1, uAu^* = A\}$, generates all the von Neumann algebra M. In particular, $\mathcal{N}(A)$ implements an ergodic measure-preserving equivalence relation on the standard probability space (X, μ), with $A = L^\infty(X, \mu)$ ([FM]), which up to orbit equivalence only depends on the isomorphism class of M.

We call HT the Cartan subalgebras satisfying the combination of the rigidity and compact approximation properties and denote by $\mathcal{HT}$ the class of factors having HT Cartan subalgebras. Thus, our theorem implies that if $M \in \mathcal{HT}$, then there exists a unique (up to isomorphism) ergodic measure-preserving equivalence relation $\mathcal{R}_M^{\mathrm{HT}}$ on (X, μ) associated with it, implemented by the HT Cartan subalgebra of M. In particular, any invariant for $\mathcal{R}_M^{\mathrm{HT}}$ is an invariant for $M \in \mathcal{HT}$.

In a recent paper ([G2]), D. Gaboriau introduced a notion of ℓ^2-Betti numbers for arbitrary countable measure-preserving equivalence relations $\mathcal{R}$, $\{\beta_n(\mathcal{R})\}_{n\geq 0}$, starting from ideas of Atiyah ([A]) and Connes ([C4]), and generalizing the notion of L^2-Betti numbers for measurable foliations defined in [C4]. His notion also generalizes the ℓ^2-Betti numbers for discrete groups Γ_0 of Cheeger-Gromov ([ChGr]), $\{\beta_n(\Gamma_0)\}_{n\geq 0}$, as Gaboriau shows that $\beta_n(\Gamma_0) = \beta_n(\mathcal{R}_{\Gamma_0})$, for any countable equivalence relation $\mathcal{R}_{\Gamma_0}$ implemented by a free, ergodic, measure-preserving action of the group Γ_0 on a standard probability space (X, μ) ([G2]).

We define in this paper the Betti numbers $\{\beta_n^{\mathrm{HT}}(M)\}_{n\geq 0}$ of a factor M in the class $\mathcal{HT}$ as the ℓ^2-Betti numbers ([G2]) of the corresponding equivalence relation $\mathcal{R}_M^{\mathrm{HT}}$, $\{\beta_n(\mathcal{R}_M^{\mathrm{HT}})\}_n$.

Due to the uniqueness of the HT Cartan subalgebra, the general properties of the Betti numbers for countable equivalence relations proved in [G2] entail similar properties for the Betti numbers of the factors in the class $\mathcal{HT}$. For instance, after proving that $\mathcal{HT}$ is closed under amplifications by arbitrary $t > 0$, we use the formula $\beta_n(\mathcal{R}^t) = \beta_n(\mathcal{R})/t$ in [G2] to deduce that $\beta_n^{\mathrm{HT}}(M^t) = \beta_n^{\mathrm{HT}}(M)/t, \forall n$. Also, we prove that $\mathcal{HT}$ is closed under tensor products and that a Künneth type formula holds for $\beta_n^{\mathrm{HT}}(M_1 \overline{\otimes} M_2)$ in terms of the Betti numbers for $M_1, M_2 \in \mathcal{HT}$, as a consequence of the similar formula for groups and equivalence relations ([B], [ChGr], [Lu], [G2]).

Our main example of a factor in the class $\mathcal{HT}$ is the group von Neumann algebra $L(G_0)$ associated with $G_0 = \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$, regarded as the group-measure space construction $L^\infty(\mathbb{T}^2, \mu) = A_0 \subset A_0 \rtimes_{\sigma_0} \mathrm{SL}(2,\mathbb{Z})$, where $\mathbb{T}^2$ is regarded as the dual of $\mathbb{Z}^2$ and σ_0 is the action implemented by $\mathrm{SL}(2,\mathbb{Z})$ on it. More generally, since our HT condition on the Cartan subalgebra A requires only part of A to be rigid in M, we show that any crossed product factor of the form $A \rtimes_\sigma \mathrm{SL}(2,\mathbb{Z})$, with $A = A_0 \overline{\otimes} A_1$, $\sigma = \sigma_0 \otimes \sigma_1$ and σ_1 an arbitrary ergodic action of $\mathrm{SL}(2,\mathbb{Z})$ on an abelian algebra A_1, is in the class $\mathcal{HT}$. By a recent result in [Hj], based on the notion and results on tree-ability in [G1], all these factors are in fact amplifications of group-measure space factors of the form $L^\infty(X,\mu) \rtimes \mathbb{F}_n$, where $\mathbb{F}_n$ is the free group on n generators, $n = 2, 3, \ldots$.

To prove that M belongs to the class $\mathcal{HT}$, with A its corresponding HT Cartan subalgebra, we use the Kazhdan-Margulis rigidity of the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ ([Ka], [Ma]) and Haagerup's compact approximation property of $\mathrm{SL}(2,\mathbb{Z})$ ([Ha]). The same arguments are actually used to show that if $\alpha \in \mathbb{C}, |\alpha| = 1$, and $L_\alpha(\mathbb{Z}^2)$ denotes the corresponding "twisted" group algebra (or "quantized" 2-dimensional thorus), then $M_\alpha = L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ is in the class $\mathcal{HT}$ if and only if α is a root of unity.

Since the orbit equivalence relation $\mathcal{R}_M^{\mathrm{HT}}$ implemented by $\mathrm{SL}(2,\mathbb{Z})$ on A has exactly one nonzero Betti number, namely $\beta_1(\mathcal{R}_M^{\mathrm{HT}}) = \beta_1(\mathrm{SL}(2,\mathbb{Z})) = 1/12$ ([B], [ChGr], [G2]), it follows that the factors $M = A \rtimes_\sigma \mathrm{SL}(2,\mathbb{Z})$ satisfy $\beta_1^{\mathrm{HT}}(M) = 1/12$ and $\beta_n^{\mathrm{HT}}(M) = 0, \forall n \neq 1$. More generally, if α is an n^{th} primitive root of 1, then the factors $M_\alpha = L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ satisfy $\beta_1^{\mathrm{HT}}(M_\alpha) = n/12, \beta_k^{\mathrm{HT}}(M_\alpha) = 0, \forall k \neq 1$. We deduce from this that if α, α' are primitive roots of unity of order n respectively n' then $M_\alpha \simeq M_{\alpha'}$ if and only if $n = n'$.

Other examples of factors in the class $\mathcal{HT}$ are obtained by taking discrete groups Γ_0 that can be embedded as arithmetic lattices in $\mathrm{SU}(n,1)$ or $\mathrm{SO}(m,1)$, together with suitable actions σ of Γ_0 on abelian von Neumann algebras $A \simeq L(\mathbb{Z}^N)$. Indeed, these groups Γ_0 have the Haagerup approximation property by [dCaH], [CowH] and their action σ on A can be taken to be rigid by a recent result of Valette ([Va]). In each of these cases, the Betti numbers have been calculated in [B]. Yet another example is offered by the action of $\mathrm{SL}(2,\mathbb{Q})$ on $\mathbb{Q}^2$: Indeed, the rigidity of the action of $\mathrm{SL}(2,\mathbb{Z})$ (regarded as a subgroup of $\mathrm{SL}(2,\mathbb{Q})$) on $\mathbb{Z}^2$ (regarded as a subgroup of $\mathbb{Q}^2$), as well as the property H of $\mathrm{SL}(2,\mathbb{Q})$ proved in [CCJJV], are enough to insure that $L(\mathbb{Q}^2 \rtimes \mathrm{SL}(2,\mathbb{Q}))$ is in the class $\mathcal{HT}$.

As a consequence of these considerations, we are able to answer a number of open questions in the theory of type II_1 factors. Thus, the factors $M = A \rtimes_\sigma \mathrm{SL}(2,\mathbb{Z})$ (more generally, $A \rtimes_\sigma \Gamma_0$ with Γ_0, σ as above) provide the first class of type II_1 factors with trivial fundamental group, i.e.

$$\mathscr{F}(M) \stackrel{\mathrm{def}}{=} \{t > 0 \mid M^t \simeq M\} = \{1\}.$$

Indeed, we mentioned that $\beta_n^{\mathrm{HT}}(M^t) = \beta_n^{\mathrm{HT}}(M)/t, \forall n$, so that if $\beta_n^{\mathrm{HT}}(M) \neq 0$ or ∞ for some n then $\mathscr{F}(M)$ is forced to be equal to $\{1\}$.

In particular, the factors M are not isomorphic to the algebra of n by n matrices over M, for any $n \geq 2$, thus providing an answer to Kadison's Problem 3 in [K1] (see also Sakai's Problem 4.4.38 in [S]). Also, through appropriate choice of actions of the form $\sigma = \sigma_0 \otimes \sigma_1$, we obtain factors of the form $M = A \rtimes_\sigma \mathrm{SL}(2, \mathbb{Z})$ having the property Γ of Murray and von Neumann, yet trivial fundamental group.

The fundamental group $\mathscr{F}(M)$ of a II_1 factor M was defined by Murray and von Neumann in the early 40's, in connection with their notion of continuous dimension. They noticed that $\mathscr{F}(M) = \mathbb{R}_+^*$ when M is isomorphic to the hyperfinite type II_1 factor R, and more generally when M "splits off" R.

The first examples of type II_1 factors M with $\mathscr{F}(M) \neq \mathbb{R}_+^*$, and the first occurrence of rigidity in the von Neumann algebra context, were discovered by Connes in [C1]. He proved that if G_0 is an infinite conjugacy class discrete group with the property (T) of Kazhdan then its group von Neumann algebra $M = L(G_0)$ is a type II_1 factor with countable fundamental group. It was then proved in [Po1] that this is still the case for factors M which contain some irreducible copy of such $L(G_0)$. It was also shown that there exist type II_1 factors M with $\mathscr{F}(M)$ countable and containing any prescribed countable set of numbers ([GoNe], [Po4]). However, the fundamental group $\mathscr{F}(M)$ could never be computed exactly, in any of these examples.

In fact, more than proving that $\mathscr{F}(M) = \{1\}$ for $M = A \rtimes_\sigma \mathrm{SL}(2, \mathbb{Z})$, the calculation of the Betti numbers shows that $M^{t_1} \overline{\otimes} M^{t_2} \ldots \overline{\otimes} M^{t_n}$ is isomorphic to $M^{s_1} \overline{\otimes} M^{s_2} \ldots \overline{\otimes} M^{s_m}$ if and only if $n = m$ and $t_1 t_2 \ldots t_n = s_1 s_2 \ldots s_m$. In particular, all tensor powers of M, $M^{\overline{\otimes} n}, n = 1, 2, 3, \ldots$, are mutually nonisomorphic and have trivial fundamental group. (N.B. The first examples of factors having nonisomorphic tensor powers were constructed in [C4]; another class of examples was obtained in [CowH]). In fact, since $\beta_k^{\mathrm{HT}}(M^{\overline{\otimes} n}) \neq 0$ if and only if $k = n$, the factors $\{M^{\overline{\otimes} n}\}_{n \geq 1}$ are not even stably isomorphic.

In particular, since $M^t \simeq L^\infty(X, \mu) \rtimes \mathbb{F}_n$ for $t = (12(n-1))^{-1}$ (cf. [Hj]), it follows that for each $n \geq 2$ there exists a free ergodic action σ_n of $\mathbb{F}_n$ on the standard probability space (X, μ) such that the factors $M_n = L^\infty(X, \mu) \rtimes_{\sigma_n} \mathbb{F}_n, n = 2, 3, \ldots$, satisfy $M_{k_1} \overline{\otimes} \cdots \overline{\otimes} M_{k_p} \simeq M_{l_1} \overline{\otimes} \ldots \overline{\otimes} M_{l_r}$ if and only if $p = r$ and $k_1 k_2 \ldots k_p = l_1 l_2 \ldots l_r$. Also, since $\beta_1^{\mathrm{HT}}(M_n) \neq 0$, the Künneth formula shows that the factors M_n are prime within the class of type II_1 factors in $\mathcal{HT}$.

Besides being closed under tensor products and amplifications, the class $\mathcal{HT}$ is closed under finite index extensions/restrictions, i.e., if $N \subset M$ are type II_1 factors with finite Jones index, $[M : N] < \infty$, then $M \in \mathcal{HT}$ if and only if $N \in \mathcal{HT}$. In fact, factors in the class $\mathcal{HT}$ have a remarkably rigid "subfactor picture".

Thus, if $M \in \mathcal{HT}$ and $N \subset M$ is an irreducible subfactor with $[M:N] < \infty$ then $[M:N]$ is an integer. More than that, the graph of $N \subset M$, $\Gamma = \Gamma_{N,M}$, has only integer weights $\{v_k\}_k$. Recall that the weights v_k of the graph of a subfactor $N \subset M$ are given by the "statistical dimensions" of the irreducible M-bimodules $\mathcal{H}_k$ in the Jones tower or, equivalently, as the square roots of the indices of the corresponding irreducible inclusions of factors, $M \subset M(\mathcal{H}_k)$. They give a Perron-Frobenius type eigenvector for Γ, satisfying $\Gamma\Gamma^t\vec{v} = [M:N]\vec{v}$. We prove that if $\beta_n^{\scriptscriptstyle\mathrm{HT}}(M) \neq 0$ or ∞ then

$$v_k = \beta_n^{\scriptscriptstyle\mathrm{HT}}(M(\mathcal{H}_k))/\beta_n^{\scriptscriptstyle\mathrm{HT}}(M), \quad \forall k;$$

i.e., the statistical dimensions are proportional to the Betti numbers. As an application of this subfactor analysis, we show that the non-Γ factor $L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$ has two nonconjugate period 2-automorphims.

We also discuss invariants that can distinguish between factors in the class $\mathcal{HT}$ which have the same Betti numbers. Thus, we show that if $\Gamma_0 = \mathrm{SL}(2,\mathbb{Z}), \mathbb{F}_n$, or if Γ_0 is an arithmetic lattice in some $\mathrm{SU}(n,1), \mathrm{SO}(n,1)$, for some $n \geq 2$, then there exist three nonorbit equivalent free ergodic measure-preserving actions σ_i of Γ_0 on (X,μ), with $M_i = L^\infty(X,\mu) \rtimes_{\sigma_i} \Gamma_0 \in \mathcal{HT}$ nonisomorphic for $i = 1,2,3$. Also, we apply Gaboriau's notion of approximate dimension to equivalence relations of the form $\mathcal{R}_M^{\scriptscriptstyle\mathrm{HT}}$ to distinguish between $\mathcal{HT}$ factors of the form $M_k = L^\infty(X,\mu) \rtimes \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_k} \times S_\infty$, with S_∞ the infinite symmetric group and $k = 1,2,\ldots$, which all have only 0 Betti numbers.

As for the "size" of the class $\mathcal{HT}$, note that we could only produce examples of factors $M = A \rtimes_\sigma \Gamma_0$ in $\mathcal{HT}$ for certain property H groups Γ_0, and for certain special actions σ of such groups. We call $\mathrm{H}_{\mathcal{T}}$ the groups Γ_0 for which there exist free ergodic measure-preserving actions σ on the standard probability space (X,μ) such that $L^\infty(X,\mu) \rtimes_\sigma \Gamma_0 \in \mathcal{HT}$. Besides the examples $\Gamma_0 = \mathrm{SL}(2,\mathbb{Z}), \mathrm{SL}(2,\mathbb{Q}), \mathbb{F}_n$, or Γ_0 an arithmetic lattice in $\mathrm{SU}(n,1), \mathrm{SO}(n,1), n \geq 2$, mentioned above, we show that the class of $\mathrm{H}_{\mathcal{T}}$ groups is closed under products by arbitrary property H groups, crossed product by amenable groups and finite index restriction/extension.

On the other hand, we prove that the class $\mathcal{HT}$ does not contain factors of the form $M \simeq M \overline{\otimes} R$, where R is the hyperfinite II_1 factor. In particular, $R \notin \mathcal{HT}$. Also, we prove that the factors $M \in \mathcal{HT}$ cannot contain property (T) factors and cannot be embedded into free group factors (by using arguments similar to [CJ]). In the same vein, we show that if $\alpha \in \mathbb{T}$ is not a root of unity, then the factors $M_\alpha = L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z}) = R \rtimes \mathrm{SL}(2,\mathbb{Z})$ cannot be embedded into any factor in the class $\mathcal{HT}$. In fact, such factors M_α belong to a special class of their own, that we will study in a forthcoming paper.

Besides these concrete applications, our results give a partial answer to a challenging problem recently raised by Alain Connes, on defining a notion of Betti numbers $\beta_n(M)$ for type II_1 factors M, from similar conceptual grounds as in the case of measure-preserving equivalence relations in [G2] (simplicial structure, ℓ^2 homology/cohomology, etc), a notion that should satisfy $\beta_n(L(G_0)) = \beta_n(G_0)$ for group von Neumann factors $L(G_0)$. In this respect, note that our definition is not the result of a "conceptual approach", relying instead on the uniqueness result for the HT Cartan subalgebras, which allows reduction of the problem to Gaboriau's work on invariants for equivalence relations and, through it, to the results on ℓ^2-cohomology for groups in [ChGr], [B], [Lu]. Thus, although they are invariants for "global factors" $M \in \mathcal{HT}$, the Betti numbers $\beta_n^{\text{HT}}(M)$ are "relative" in spirit, a fact that we have indicated by adding the upper index $^{\text{HT}}$. Also, rather than satisfying $\beta_n(L(G_0)) = \beta_n(G_0)$, the invariants β_n^{HT} satisfy $\beta_n^{\text{HT}}(A \rtimes \Gamma_0) = \beta_n(\Gamma_0)$. In fact, if $A \rtimes \Gamma_0 = L(G_0)$, where $G_0 = \mathbb{Z}^N \rtimes \Gamma_0$, then $\beta_n(G_0) = 0$, while $\beta_n^{\text{HT}}(L(G_0)) = \beta_n(\Gamma_0)$ may be different from 0.

The paper is organized as follows: Section 1 consists of preliminaries: we first establish some basic properties of *Hilbert bimodules* over von Neumann algebras and of their associated *completely positive maps*; then we recall the *basic construction* of an inclusion of finite von Neumann algebras and study their *compact ideal space*; we also recall the definitions of *normalizer* and *quasi-normalizer* of a subalgebra, as well as the notions of *regular, quasi-regular, discrete* and *Cartan subalgebras*, and discuss some of the results in [FM] and [PoSh]. In Section 2 we consider a relative version of Haagerup's compact approximation property for inclusions of von Neumann algebras, called *relative property* H (cf. also [Bo]), and prove its main properties. In Section 3 we give examples of property H inclusions and use [PoSh] to show that if a type II_1 factor M has the property H relative to a maximal abelian subalgebra $A \subset M$ then A is a Cartan subalgebra of M. In Section 4 we define a notion of rigidity (or relative property (T)) for inclusions of algebras and investigate its basic properties. In Section 5 we give examples of rigid inclusions and relate this property to the co-rigidity property defined in [Zi], [A-De], [Po1]. We also introduce a new notion of property (T) for equivalence relations, called *relative property* (T), by requiring the associated Cartan subalgebra inclusion to be rigid.

In Section 6 we define the class $\mathcal{HT}$ of factors M having HT *Cartan subalgebras* $A \subset M$, i.e., maximal abelian *-subalgebras $A \subset M$ such that M has the property H relative to A and A contains a subalgebra $A_0 \subset A$ with $A_0' \cap M = A$ and $A_0 \subset M$ rigid. We then prove the main technical result of the paper, showing that HT Cartan subalgebras are unique. We show the stability of the class $\mathcal{HT}$ with respect to various operations (amplification, tensor product), and prove its rigidity to perturbations. Section 7 studies the lattice of subfactors of $\mathcal{HT}$ factors: we prove the stability of the class $\mathcal{HT}$ to finite index, obtain a canonical decomposition for subfactors in $\mathcal{HT}$ and prove that the index is always an integer. In Section 8 we define the *Betti numbers* $\{\beta_n^{\text{HT}}(M)\}_n$

for $M \in \mathcal{HT}$ and use the previous sections and [G2] to deduce various properties of this invariant. We also discuss some alternative invariants for factors $M \in \mathcal{HT}$, such as the *outomorphism group* $\mathrm{Out}_{\mathrm{HT}}(M) \overset{\mathrm{def}}{=} \mathrm{Aut}(\mathcal{R}_M^{\mathrm{HT}})/\mathrm{Int}(\mathcal{R}_M^{\mathrm{HT}})$, which we prove is discrete countable, or $\mathrm{ad}_{\mathrm{HT}}(M)$, defined to be Gaboriau's *approximate dimension* ([G2]) of $\mathcal{R}_M^{\mathrm{HT}}$. We end with applications, as well as some remarks and open questions. We have included an appendix in which we prove some key technical results on unitary conjugacy of von Neumann subalgebras in type II_1 factors. The proof uses techniques from [Chr], [Po2,3,6], [K2].

Acknowledgement. I want to thank U. Haagerup, V. Lafforgue and A. Valette for useful conversations on the properties H and (T) for groups. My special thanks are due to Damien Gaboriau, for keeping me informed on his beautiful recent results and for useful comments on the first version of this paper. I am particularly grateful to Alain Connes and Dima Shlyakhtenko for many fruitful conversations and constant support. I want to express my gratitude to MSRI and the organizers of the Operator Algebra year 2000–2001, for their hospitality and for a most stimulating atmosphere. This article is an expanded version of a paper with the same title which appeared as MSRI preprint 2001/0024.

1. Preliminaries

1.1. *Pointed correspondences.* By using the GNS construction as a link, a representation of a group G_0 can be viewed in two equivalent ways: as a group morphism from G_0 into the unitary group of a Hilbert space $\mathcal{U}(\mathcal{H})$, or as a positive definite function on G_0.

The discovery of the appropriate notion of representations for von Neumann algebras, as so-called *correspondences*, is due to Connes ([C3,7]). In the vein of group representations, Connes introduced correspondences in two alternative ways, both of which use the idea of "doubling" - a genuine conceptual breakthrough. Thus, correspondences of von Neumann algebras N can be viewed as *Hilbert N-bimodules* $\mathcal{H}$, the quantized version of group morphisms into $\mathcal{U}(\mathcal{H})$; or as *completely positive maps* $\phi : N \to N$, the quantized version of positive definite functions on groups (cf. [C3,7] and [CJ]). The equivalence of these two points of view is again realized via a version of the GNS construction ([CJ], [C7]).

We will in fact need "pointed" versions of Connes's correspondences, adapted to the case of inclusions $B \subset N$, as introduced in [Po1] and [Po5]. In this section we detail the two alternative ways of viewing such pointed correspondences, in the same spirit as [C7]: as "B-pointed bimodules" or as "B-bimodular completely positive maps". This is a very important idea, to appear throughout this paper.

1.1.1. *Pointed Hilbert bimodules.* Let N be a finite von Neumann algebra with a fixed normal faithful tracial state τ and $B \subset N$ a von Neumann subalgebra of N. A *Hilbert* $(B \subset N)$*-bimodule* $(\mathcal{H}, \xi)$ is a Hilbert N-bimodule with a fixed unit vector $\xi \in \mathcal{H}$ satisfying $b\xi = \xi b, \forall b \in B$. When $B = \mathbb{C}$, we simply call $(\mathcal{H}, \xi)$ a *pointed Hilbert* N*-bimodule.*

If $\mathcal{H}$ is a Hilbert N*-bimodule* then $\xi \in \mathcal{H}$ is a *cyclic vector* if $\overline{\mathrm{sp}} N\xi N = \mathcal{H}$.

To relate Hilbert $(B \subset N)$-bimodules and B-bimodular completely positive maps on N one uses a generalized version of the GNS construction, due to Stinespring, which we describe below:

1.1.2. *From completely positive maps to Hilbert bimodules.* Let ϕ be a normal, completely positive map on N, normalized so that $\tau(\phi(1)) = 1$. We associate to it the pointed Hilbert N-bimodule $(\mathcal{H}_\phi, \xi_\phi)$ in the following way:

Define on the linear space $\mathcal{H}_0 = N \otimes N$ the sesquilinear form $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_\phi = \tau(\phi(x_2^* x_1) y_1 y_2^*), x_{1,2}, y_{1,2} \in N$. The complete positivity of ϕ is easily seen to be equivalent to the positivity of $\langle \cdot, \cdot \rangle_\phi$. Let $\mathcal{H}_\phi$ be the completion of $\mathcal{H}_0 / \sim$, where $\sim$ is the equivalence modulo the null space of $\langle \cdot, \cdot \rangle_\phi$ in $\mathcal{H}_0$. Also, let ξ_ϕ be the class of $1 \otimes 1$ in $\mathcal{H}_\phi$. Note that $\|\xi_\phi\|^2 = \tau(\phi(1)) = 1$.

If $p = \Sigma_i x_i \otimes y_i \in \mathcal{H}_0$, then by use again of the complete positivity of ϕ it follows that $N \ni x \to \Sigma_{i,j} \tau(\phi(x_j^* x x_i) y_i y_j^*)$ is a positive normal functional on N of norm $\langle p, p \rangle_\phi$. Similarly, $N \ni y \to \Sigma_{i,j} \tau(\phi(x_j^* x_i) y_i y y_j^*)$ is a positive normal functional on N of norm $\langle p, p \rangle_\phi$. Note that the latter can alternatively be viewed as a functional on the opposite algebra N^{op} (which is the same as N as a vector space but has multiplication inverted, $x \cdot y = yx$). Moreover, N acts on $\mathcal{H}_0$ on the left and right by $xpy = x(\Sigma_i x_i \otimes y_i)y = \Sigma_i x x_i \otimes y_i y$. These two actions clearly commute and the complete positivity of ϕ entails:

$$\langle xp, xp \rangle_\phi = \langle x^* xp, p \rangle_\phi \leq \|x^* x\| \langle p, p \rangle_\phi = \|x\|^2 \langle p, p \rangle_\phi.$$

Similarly

$$\langle py, py \rangle_\phi \leq \|y\|^2 \langle p, p \rangle_\phi.$$

Thus, the above left and right actions of N on $\mathcal{H}_0$ pass to $\mathcal{H}_0 / \sim$ and then extend to commuting left-right actions on $\mathcal{H}_\phi$. By the normality of the forms $x \to \langle xp, p \rangle_\phi$ and $y \to \langle py, p \rangle_\phi$, these left-right actions of N on $\mathcal{H}_\phi$ are normal (i.e., weakly continuous).

This shows that $(\mathcal{H}_\phi, \xi_\phi)$ with the above N-bimodule structure is a pointed, Hilbert N-bimodule, which in addition is clearly cyclic. Moreover, if $B \subset N$ is a von Neumann subalgebra and the completely positive map ϕ is B-bimodular, then it is immediate to check that $b\xi_\phi = \xi_\phi b, \forall b \in B$. Thus, if ϕ is B-bimodular, then $(\mathcal{H}_\phi, \xi_\phi)$ is a Hilbert $(B \subset N)$-bimodule.

Let us end this paragraph with some useful inequalities which show that elements that are almost fixed by a B-bimodular completely positive map ϕ on N are almost commuting with the associated vector $\xi_\phi \in \mathcal{H}_\phi$:

Lemma. *1°. $\|\phi(x)\|_2 \leq \|\phi(1)\|_2, \forall x \in N, \|x\| \leq 1$.*

2°. If $a = 1 \vee \phi(1)$ and $\phi'(\cdot) = a^{-1/2}\phi(\cdot)a^{-1/2}$, then ϕ' is completely positive, B-bimodular and satisfies $\phi'(1) \leq 1$, $\tau \circ \phi' \leq \tau \circ \phi$ and the estimate:

$$\|\phi'(x) - x\|_2 \leq \|\phi(x) - x\|_2 + 2\|\phi(1) - 1\|_1^{1/2}\|x\|, \forall x \in N.$$

3°. Assume $\phi(1) \leq 1$ and define $\phi''(x) = \phi(b^{-1/2}xb^{-1/2})$, where $b = 1 \vee (\mathrm{d}\tau \circ \phi/\mathrm{d}\tau) \in L^1(N,\tau)_+$. Then ϕ'' is completely positive, B-bimodular and satisfies $\phi''(1) \leq \phi(1) \leq 1, \tau \circ \phi'' \leq \tau$, as well as the estimate:

$$\|\phi''(x) - x\|_2^2 \leq 2\|\phi(x) - x\|_2 + 5\|b - 1\|_1^{1/2}, \forall x \in N, \|x\| \leq 1.$$

4°. $\|x\xi_\phi - \xi_\phi x\|_2^2 \leq 2\|\phi(x) - x\|_2^2 + 2\|\phi(1)\|_2\|\phi(x) - x\|_2, \forall x \in N, \|x\| \leq 1$.

Proof. 1°. Since any $x \in N$ with $\|x\| \leq 1$ is a convex combination of two unitary elements, it is sufficient to prove the inequality for unitary elements $u \in N$. By continuity, it is in fact sufficient to prove it in the case the unitary elements u have finite spectrum. If $u = \Sigma_i \lambda_i p_i$ for some scalars λ_i with $|\lambda_i| = 1$, $1 \leq i \leq n$, and some partition of the identity exists with projections $p_i \in N$, then $\tau(\phi(p_i)\phi(p_j)) \geq 0, \forall i,j$. Taking this into account, we get:

$$\begin{aligned}\tau(\phi(u)\phi(u^*)) &= \Sigma_{i,j}\lambda_i\overline{\lambda_j}\tau(\phi(p_i)\phi(p_j)) \leq \Sigma_{i,j}|\lambda_i\overline{\lambda_j}|\tau(\phi(p_i)\phi(p_j)) \\ &= \Sigma_{i,j}\tau(\phi(p_i)\phi(p_j)) = \tau(\phi(1)\phi(1)).\end{aligned}$$

2°. Since $a \in B' \cap N$, ϕ' is B-bimodular. We clearly have $\phi'(1) = a^{-1/2}\phi(1)a^{-1/2} \leq 1$. Since $a^{-1} \leq 1$, for $x \geq 0$ we get $\tau(\phi'(x)) = \tau(\phi(x)a^{-1}) \leq \tau(\phi(x))$. Also, we have:

$$\begin{aligned}\|\phi'(x) - x\|_2 &\leq \|a^{-1/2}\phi(x)a^{-1/2} - a^{-1/2}xa^{-1/2}\|_2 + \|a^{-1/2}xa^{-1/2} - x\|_2 \\ &\leq \|\phi(x) - x\|_2 + 2\|a^{-1/2} - 1\|_2\|x\|.\end{aligned}$$

But

$$\begin{aligned}\|a^{-1/2} - 1\|_2 &\leq \|a^{-1} - 1\|_1^{1/2} = \|a^{-1} - aa^{-1}\|_1 \\ &\leq \|a - 1\|_1\|a^{-1}\| \leq \|a - 1\|_1 \leq \|\phi(1) - 1\|_1.\end{aligned}$$

Thus,

$$\|\phi'(x) - x\|_2 \leq \|\phi(x) - x\|_2 + 2\|\phi(1) - 1\|_1^{1/2}\|x\|.$$

3°. The first properties are clear by the definitions. Then note that $\|y\|_2^2 \leq \|y\|\|y\|_1$ and $\|\phi''(y)\|_1 \leq \|y\|_1$. (Indeed, because if ϕ''^* is as defined in Lemma 1.1.5, then for $z \in N$ with $\|z\| \leq 1$ we have $\|\phi''^*(z)\| \leq 1$ so that $\|\phi''(y)\|_1 = \sup\{|\tau(\phi''(y)z)| \mid z \in N, \|z\| \leq 1\} = \sup\{|\tau(y\phi''^*(z))| \mid z \in N, \|z\| \leq 1\} \leq \sup\{|\tau(yz))| \mid z \in N, \|z\| \leq 1\} = \|y\|_1$.) Note also that $\tau(b) \leq$

$1+\tau(\phi(1)) \leq 2$. Thus, for $x \in N$, $\|x\| \leq 1$, we get:

$$\begin{aligned}\|\phi''(x)-x\|_2^2 &\leq 2\|\phi''(x)-x\|_1\\ &\leq 2\|\phi''(x)-\phi''(b^{1/2}xb^{1/2})\|_1+2\|\phi(x)-x\|_1\\ &\leq 2\|x-b^{1/2}xb^{1/2}\|_1+2\|\phi(x)-x\|_1\\ &\leq 2\|x-xb^{1/2}\|_1+2\|xb^{1/2}-b^{1/2}xb^{1/2}\|_1+2\|\phi(x)-x\|_1.\end{aligned}$$

But $\|x\|_2 \leq 1$ and $\|xb^{1/2}\|_2^2 \leq \tau(b) \leq 2$, so by the Cauchy-Schwartz inequality the above is majorized by:

$$\begin{aligned}&2\|x\|_2\|1-b^{1/2}\|_2+2\|1-b^{1/2}\|_2\|xb^{1/2}\|_2+2\|\phi(x)-x\|_2\\ &\quad\leq (2+2^{3/2})\|b^{1/2}-1\|_2+2\|\phi(x)-x\|_2 \leq 5\|b-1\|_1^{1/2}+2\|\phi(x)-x\|_2.\end{aligned}$$

4°. Since by the Cauchy-Schwartz inequality we have

$$\pm\mathrm{Re}\tau(\phi(x)(\phi(x)^*-x^*)) \leq \|\phi(x)\|_2\|\phi(x^*)-x^*\|_2,$$

it follows that

$$\begin{aligned}\|\phi(x)-x\|_2^2 &= \tau(\phi(x)\phi(x)^*)+1-2\mathrm{Re}\tau(\phi(x)x^*)\\ &= \mathrm{Re}\tau(\phi(x)x^*)+\mathrm{Re}\tau(\phi(x)(\phi(x)^*-x^*))+1-2\mathrm{Re}\tau(\phi(x)x^*)\\ &\geq 1-\mathrm{Re}\tau(\phi(x)x^*)-\|\phi(x)-x\|_2\|\phi(x)\|_2\\ &= \|x\xi_\phi-\xi_\phi x\|_2^2/2-\|\phi(x)-x\|_2\|\phi(x)\|_2,\end{aligned}$$

which by part 1° proves the statement. □

The inequalities in the previous lemmas show in particular that if ϕ almost fixes some $u \in \mathcal{U}(N)$, then $\phi(ux)$ is close to $u\phi(x)$, uniformly in $x \in N$, $\|x\| \leq 1$, whenever we have control over $\|\phi\|$:

Corollary. *For any unitary element $u \in N$ and $x \in N$,*

$$\begin{aligned}\|\phi(ux)-u\phi(x)\|_2 &\leq \|\phi\|^{1/2}\|x\|\|[u,\xi_\phi]\|_2\\ &\leq \|\phi\|^{1/2}\|x\|(2\|\phi(u)-u\|_2^2+2\|\phi(1)\|_2\|\phi(u)-u\|_2)^{1/2}.\end{aligned}$$

Proof. By using the fact that

$$\|\phi(ux)-u\phi(x)\|_2 = \sup\{|\tau((\phi(ux)-u\phi(x))y)| \mid y \in N, \|y\|_2 \leq 1\},$$

we get:

$$\begin{aligned}\|\phi(ux)-u\phi(x)\|_2 &= \sup\{|\langle ux\xi_\phi y,\xi_\phi\rangle-\langle x\xi_\phi yu,\xi_\phi\rangle| \mid y \in N, \|y\|_2 \leq 1\}\\ &= \sup\{|\langle x\xi_\phi y,[u^*,\xi_\phi]\rangle| \mid y \in N, \|y\|_2 \leq 1\}\\ &\leq \sup\{\|x\xi_\phi y\|_2 \mid y \in N, \|y\|_2 \leq 1\}\|[u^*,\xi_\phi]\|_2\\ &= \|\phi(x^*x)\|^{1/2}\|[u,\xi_\phi]\|_2 \leq \|\phi\|^{1/2}\|x\|\|[u,\xi_\phi]\|_2.\end{aligned}$$

□

1.1.3. *From Hilbert bimodules to completely positive maps.* Conversely, let $(\mathcal{H}, \xi)$ be a pointed Hilbert $(B \subset N)$-bimodule, with $\langle \xi \cdot, \xi \rangle \leq c\tau$, for some $c > 0$. Let $T : L^2(N, \tau) \to \mathcal{H}$ be the unique bounded operator defined by $T\hat{y} = \xi y, y \in N$. Then $\langle \xi y, \xi y \rangle \leq c\tau(yy^*) = c\|\hat{y}\|_2^2$, so that $\|T\| \leq c^{1/2}$.

It is immediate to check that if for clarity we denote by $L(x)$ the operator of left multiplication by x on $\mathcal{H}$, then T satisfies:

$$\begin{aligned}\langle T^* L(x) T(J_N y J_N(\hat{y_1})), \hat{y_2} \rangle_\tau &= \langle L(x)(\xi y_1 y^*), \xi y_2 \rangle_{\mathcal{H}} \\ &= \langle L(x)\xi y_1, \xi y_2 y \rangle_{\mathcal{H}} = \langle J_N y J_N (T^* L(x) T)\hat{y_1}, \hat{y_2} \rangle_\tau.\end{aligned}$$

This shows that the operator $\phi_{(\mathcal{H},\xi)}(x) \overset{\text{def}}{=} T^* L(x) T$ commutes with the right multiplication on $L^2(N, \tau)$ by elements $y \in N$. Thus, $\phi_{(\mathcal{H},\xi)}(x)$ belongs to $(J_N N J_N)' \cap \mathcal{B}(L^2(N, \tau)) = N$, showing that $\phi_{(\mathcal{H},\xi)}$ defines a map from N into N, which is obviously completely positive and B-bimodular, by the definitions. Furthermore, if we denote by $\mathcal{H}'$ the closed linear span of $N\xi N$ in $\mathcal{H}$, then $U : \mathcal{H}_\phi \to \mathcal{H}', U(x \otimes y) = x\xi y$ is easily seen to be an isomorphism of Hilbert $(B \subset N)$-bimodules.

The assumption that ξ is "bounded from the right" by c is not really a restriction for this construction, since if we put $\mathcal{H}^0 = \{\xi \in \mathcal{H} \mid b\xi = \xi b, \forall b \in B$, ξ bounded from the left and from the right $\}$, then it is easy to see that $\mathcal{H}^0$ is dense in the Hilbert space $\mathcal{H}_0 \subset \mathcal{H}$ of all B-central vectors in $\mathcal{H}$. This actually implies that any $(B \subset N)$ Hilbert bimodule $(\mathcal{H}, \xi)$ is a direct sum of some $(B \subset N)$ Hilbert bimodules $(\mathcal{H}_i, \xi_i)$ with ξ_i bounded both from left and right (hint: just use the above density and a maximality argument).

Note that if $(\mathcal{H}, \xi)$ comes itself from a completely positive B-bimodular map ϕ, i.e., $(\mathcal{H}, \xi) = (\mathcal{H}_\phi, \xi_\phi)$ as in 1.1.2, then $\phi_{(\mathcal{H},\xi)} = \phi$. Similarly, if $(\mathcal{H}, \xi)$ is a cyclic pointed $(B \subset N)$-Hilbert bimodule and $\phi = \phi_{(\mathcal{H},\xi)}$, then $(\mathcal{H}_\phi, \xi_\phi) \simeq (\mathcal{H}, \xi)$.

Let us also note a converse to Lemma 1.1.3, showing that if ξ almost commutes with a unitary element $u \in N$ then u is almost fixed by $\phi = \phi_{(\mathcal{H},\xi)}$, provided we have some control over $\|\phi(1)\|_2$:

Lemma. *Let $\xi \in \mathcal{H}$ be a vector bounded from the right and denote* $\phi = \phi_{(\mathcal{H},\xi)}$.

1°. *Let $a_0, b_0 \in L^1(N, \tau)_+$ be such that $\langle \cdot\xi, \xi \rangle = \tau(\cdot b_0), \langle \xi \cdot, \xi \rangle = \tau(\cdot a_0)$ and put $a = 1 \vee a_0, b = 1 \vee b_0$, $\xi' = b^{-1/2}\xi a^{-1/2}$. Then $\phi(1) = a_0$ and*

$$\|\xi - \xi'\|^2 \leq 4\|a_0 - 1\|_1 + 4\|b_0 - 1\|_1.$$

2°. *If $u \in \mathcal{U}(N)$, then*

$$\|\phi(u) - u\|_2^2 \leq \|[u, \xi]\|_2^2 + (\|\phi(1)\|_2^2 - 1).$$

Proof. 1°. We have:

$$\begin{aligned}\|\xi-\xi'\|^2 &\le 2\|\xi-b^{-1/2}\xi\|^2+2\|\xi-\xi a^{-1/2}\|^2\\ &=2\tau((1-b^{-1/2})^2b_0)+2\tau((1-a^{-1/2})^2a_0)\\ &\le 4\|b_0-1\|_1+4\|a_0-1\|_1.\end{aligned}$$

2°. By part 1° of Lemma 1.1.2 we have $\tau(\phi(u^*)\phi(u))\le\tau(\phi(1)\phi(1))$, so that:

$$\begin{aligned}\|\phi(u)-u\|_2^2 &=\tau(\phi(u)\phi(u^*))+1-2\mathrm{Re}\tau(\phi(u)u^*)\\ &\le\tau(\phi(1)\phi(1))+1-2\mathrm{Re}\tau(\phi(u)u^*)\\ &=2-2\mathrm{Re}\tau(\phi(u)u^*)+(\tau(\phi(1)\phi(1))-1)\\ &=\|[u,\xi]\|_2^2+(\|\phi(1)\|_2^2-1).\end{aligned}$$

□

1.1.4. *Correspondences from representations of groups.* Let Γ_0 be a discrete group, (B,τ_0) a finite von Neumann algebra with a normal faithful tracial state and σ a cocycle action of Γ_0 on (B,τ_0) by τ_0-preserving automorphisms. Denote by $N=B\rtimes_\sigma\Gamma_0$ the corresponding crossed product algebra and by $\{u_g\}_g\subset N$ the canonical unitaries implementing the action σ on B.

Let $(\pi_0,\mathcal{H}_0,\xi_0)$ be a pointed, cyclic representation of the group Γ_0. We denote by $(\mathcal{H}_{\pi_0},\xi_{\pi_0})$ the pointed Hilbert space $(\mathcal{H}_0,\xi_0)\overline{\otimes}(L^2(N,\tau),\hat{1})$. We let N act on the right on $\mathcal{H}_{\pi_0}$ by $(\xi\otimes\hat{x})y=\xi\otimes(\hat{xy}),x,y\in N,\xi\in\mathcal{H}_0$ and on the left by $b(\xi\otimes\hat{x})=\xi\otimes\hat{bx}$, $u_g(\xi\otimes\hat{x})=\pi_0(g)(\xi)\otimes\hat{u_gx},b\in B,x\in N,g\in\Gamma_0,\xi\in\mathcal{H}_0$.

It is easy to check that these are indeed mutually commuting left-right actions of N on $\mathcal{H}_{\pi_0}$. Moreover, the vector $\xi_{\pi_0}=\xi_0\otimes\hat{1}$ implements the trace τ on N, both from left and right. Also, ξ_{π_0} is easily seen to be B-central. Thus, $(\mathcal{H}_{\pi_0},\xi_{\pi_0})$ is a Hilbert $(B\subset N)$-bimodule.

Let now φ be a positive definite function on Γ_0 and denote by $(\pi_\varphi,\mathcal{H}_\varphi,\xi_\varphi)$ the representation obtained from it through the GNS construction. Let $(\mathcal{H},\xi)$ denote the $(B\subset B\rtimes\Gamma_0)$-Hilbert bimodule constructed out of the representation π_φ as above and ϕ the completely positive B-bimodular map associated with $(\mathcal{H},\xi)$ as in 1.1.3. An easy calculation shows that ϕ acts on $B\rtimes\Gamma_0$ by $\phi(\Sigma_g b_gu_g)=\Sigma_g\varphi(g)b_gu_g$.

Conversely, if $(\mathcal{H},\xi)$ is a $(B\subset N)$ Hilbert bimodule, then we can associate to it the representation π_0 on $\mathcal{H}_0=\overline{\mathrm{sp}}\{u_g\xi u_g^*\mid g\in\Gamma_0\}$ by $\pi_0(g)\xi'=u_g\xi'u_g^*,\xi'\in\mathcal{H}_0$. Equivalently, if ϕ is the B-bimodular completely positive map associated with $(\mathcal{H},\xi)$ then $\varphi(g)=\tau(\phi(u_g)u_g^*),g\in\Gamma_0$, is a positive definite function on Γ_0.

1.1.5. *The adjoint of a bimodule.* Let $(\mathcal{H},\xi_0)$ be a $(B\subset N)$ Hilbert bimodule. Let $\overline{\mathcal{H}}$ be the conjugate Hilbert space of $\mathcal{H}$, i.e., $\overline{\mathcal{H}}=\mathcal{H}$ as a set, the sum of vectors in $\overline{\mathcal{H}}$ is the same as in $\mathcal{H}$, but the multiplication by scalars is given by $\lambda\cdot\xi=\overline{\lambda}\xi$ and $\langle\xi,\eta\rangle_{\overline{\mathcal{H}}}=\langle\eta,\xi\rangle_{\mathcal{H}}$. Denote by $\overline{\xi}$ the element ξ regarded as a vector in the Hilbert space $\overline{\mathcal{H}}$. Define on $\overline{\mathcal{H}}$ the left and right multiplication operations by $x \cdot \overline{\xi} \cdot y = \overline{y^* \xi x^*}$, for $x, y \in N, \xi \in \mathcal{H}$. It is easy to see that they define an N Hilbert bimodule structure on $\overline{\mathcal{H}}$. Moreover, $\overline{\xi_0}$ is clearly B-central. We call $(\overline{\mathcal{H}}, \overline{\xi_0})$ the *adjoint of* $(\mathcal{H}, \xi_0)$. Note that we clearly have $(\overline{\overline{\mathcal{H}}}, \overline{\overline{\xi_0}}) = (\mathcal{H}, \xi_0)$.

LEMMA. *Let ϕ be a normal B-bimodular completely positive map on N. For each $x \in N$ let $\phi^*(x) \in L^1(N, \tau)$ denote the Radon-Nykodim derivative of $N \ni y \mapsto \tau(\phi(y)x)$ with respect to τ.*

1°. *$\phi^*(N) \subset N$ if and only if $\tau \circ \phi \le c\tau$ for some $c > 0$, i.e., if and only if the Radon-Nykodim derivative $b_0 = \mathrm{d}\tau \circ \phi / \mathrm{d}\tau$ is a bounded operator. Moreover, if the condition is satisfied then ϕ^* defines a normal, B-bimodular, completely positive map of N into N with $\phi^*(1) = b_0$ and*

$$\|\phi^*\| = \|b_0\| = \inf\{c > 0 \mid \tau \circ \phi \le c\tau\}.$$

2°. *If ϕ satisfies condition 1° then ϕ^* also satisfies it, and $(\phi^*)^* = \phi$. Also, $(\mathcal{H}_{\phi^*}, \xi_{\phi^*}) = (\overline{\mathcal{H}_\phi}, \overline{\xi_\phi})$.*

3°. *If $\tau \circ \phi \le \tau$ then for any unitary element $u \in N$,*

$$\|\phi^*(u) - u\|_2^2 \le 2\|\phi(u) - u\|_2.$$

Proof. Parts 1° and 2° are trivial by the definition of ϕ^*.

To prove 3°, note that by part 1°, $\tau \circ \phi \le \tau$ implies $\phi^*(1) \le 1$ and so by Lemma 1.1.2 we get:

$$\begin{aligned}\|\phi^*(u) - u\|_2^2 &= \tau(\phi^*(u)\phi^*(u)^*) + 1 - 2\mathrm{Re}\tau(\phi^*(u)u^*) \\ &\le \tau(\phi^*(1)\phi^*(1)) + 1 - 2\mathrm{Re}\tau(\phi(u)u^*) \le 2 - 2\mathrm{Re}\tau(\phi(u)u^*) \\ &= 2\mathrm{Re}\tau((u - \phi(u))u^*) \le 2\|\phi(u) - u\|_2. \qquad \square\end{aligned}$$

1.2. *Completely positive maps as Hilbert space operators.* We now show that if a completely positive map ϕ on the finite von Neumann algebra N is sufficiently smooth with respect to the normal faithful tracial state τ on N, then it can be extended to the Hilbert space $L^2(N, \tau)$. In case ϕ is B-bimodular, for some von Neumann subalgebra $B \subset N$, these operators belong to the algebra of the basic construction associated with $B \subset N$, defined in the next paragraph.

1.2.1. LEMMA. 1°. *If there exists $c > 0$ such that $\|\phi(x)\|_2 \le c\|x\|_2, \forall x \in N$, then there exists a bounded operator T_ϕ on $L^2(N, \tau)$ such that $T_\phi(\hat{x}) = \widehat{\phi(x)}$. The operator T_ϕ commutes with the canonical conjugation J_N. Also, if $B \subset N$ is a von Neumann subalgebra, then T_ϕ commutes with the operators of left and right multiplication by elements in B (i.e., $T_\phi \in B' \cap (JBJ)'$) if and only if the completely positive map ϕ is B-bimodular.*

2°. *If $\tau \circ \phi \le c_0\tau$, for some constant $c_0 > 0$, then ϕ satisfies condition 1° above, and so there exists a bounded operator T_ϕ on the Hilbert space $L^2(N, \tau)$*

such that $T_\phi(\hat{x}) = \hat{\phi(x)}$, *for* $x \in N$. *Moreover, if* $\phi^* : N \to N$ *is the adjoint of* ϕ, *as defined in* 1.1.5, *then* $\|T_\phi\|^2 \leq \|\phi(1)\|\|\phi^*(1)\|$. *Also,* ϕ^* *satisfies* $\tau \circ \phi^* \leq \|\phi(1)\|\tau$ *and so* $T_{\phi^*} = T_\phi^*$.

3°. *If* ϕ *is* B*-bimodular then* $\phi(1) \in B' \cap N$. *Thus, if* $B' \cap N = \mathcal{Z}(B)$ *then* $\phi(1) \in \mathcal{Z}(B)$, $\tau \circ \phi \leq \|\phi(1)\|\tau$ *and the bounded operator* T_ϕ *exists by* 2°. *If in addition* $\phi(1) = 1$, *then* ϕ *is trace-preserving as well.*

Proof. 1°. The existence of T_ϕ is trivial. Also, for $x \in N$ we have

$$T_\phi(J_N(\hat{x})) = \hat{\phi(x^*)} = \hat{\phi(x)}^* = J_N(T_\phi(\hat{x})).$$

If ϕ is B-bimodular and $b \in B$ is regarded as an operator of left multiplication by b on $L^2(N, \tau)$, then

$$bT_\phi(\hat{x}) = b\hat{\phi(x)} = \hat{\phi(bx)} = T_\phi(b\hat{x}).$$

Thus, $T_\phi \in B'$.

Similarly,

$$JbJ(T_\phi(\hat{x})) = \phi(x)b = \phi(xb) = T_\phi(JbJ(\hat{x}))$$

showing that $T_\phi \in JBJ'$ as well. Conversely, if $T_\phi \in B' \cap JBJ'$, then by exactly the same equalities, $\phi(bx) = b\phi(x), \phi(xb) = \phi(x)b, \forall x \in N, b \in B$.

2°. By Kadison's inequality, for $x \in M$,

$$\langle T_\phi(\hat{x}), T_\phi(\hat{x})\rangle = \tau(\phi(x)^*\phi(x)) \leq \|\phi(1)\|\tau(\phi(x^*x)), \forall x \in N.$$

Thus, by Lemma 1.1.5 we have $\|T_\phi\|^2 \leq \|\phi(1)\|\|\phi^*(1)\|$. The last part is now trivial, by 1.1.5 and the definitions of T_ϕ, ϕ^* and T_{ϕ^*}.

3°. The B-bimodularity of ϕ implies $u\phi(1)u^* = \phi(1), \forall u \in \mathcal{U}(B)$; thus $\phi(1) \in B' \cap N$.

Using again the bimodularity, as well as the normality of ϕ, for each fixed $x \in N$ we have

$$\tau(\phi(x)) = \tau(u\phi(x)u^*) = \tau(\phi(uxu^*)) = \tau(\phi(y))$$

for all $u \in \mathcal{U}(B)$ and all y in the weak closure of the convex hull of $\{uxu^* \mid u \in \mathcal{U}(N)\}$. The latter set contains $E_{B'\cap N}(x) \in B' \cap N \subset B$ (see e.g. [Po6]); thus

$$\tau(\phi(x)) = \tau(\phi(E_{B'\cap N}(x))) = \tau(E_{B'\cap N}(x)\phi(1)).$$

This shows that if $x \geq 0$ then $\tau(\phi(x)) \leq \|\phi(1)\|\tau(x)$. It also shows that in case $\phi(1) = 1$ then $\tau(\phi(x)) = \tau(x), \forall x \in N$. □

1.3. *The basic construction and its compact ideal space.* We now recall from [Chr], [J1], [Po2,3] some well known facts about the *basic construction* for an inclusion of finite von Neumann algebras $B \subset N$ with a normal faithful tracial state τ on it. Also, we establish some properties of the ideal generated by finite projections in the semifinite von Neumann algebra $\langle N, B\rangle$ of the basic construction.

1.3.1. *Basic construction for $B \subset N$.* We denote by $\langle N, B\rangle$ the von Neumann algebra generated in $\mathcal{B}(L^2(N,\tau))$ by N (regarded as the algebra of left multiplication operators by elements in N) and by the orthogonal projection e_B of $L^2(M,\tau)$ onto $L^2(B,\tau)$.

Since $e_B x e_B = E_B(x)e_B, \forall x \in N$, where E_B is the unique τ-preserving conditional expectation of N onto B, and $\vee\{x(e_B(L^2(N))) \mid x \in N\} = L^2(N)$, it follows that $\mathrm{sp}Ne_BN$ is a *-algebra with support equal to 1 in $\mathcal{B}(L^2(N,\tau))$. Thus, $\langle N, B\rangle = \overline{\mathrm{sp}}^{\mathrm{w}}\{xe_By \mid x, y \in N\}$ and $e_B\langle N, B, \rangle e_B = Be_B$.

One can also readily see that if $J = J_N$ denotes the canonical conjugation on the Hilbert space $L^2(N,\tau)$, given on $\hat{N}$ by $J(\hat{x}) = \hat{x^*}$, then $\langle N, B\rangle = JBJ' \cap \mathcal{B}(L^2(N,\tau))$. This shows in particular that $\langle N, B\rangle$ is a semifinite von Neumann algebra. It also shows that the isomorphism of $N \subset \langle N, B\rangle$ only depends on $B \subset N$ and not on the trace τ on N (due to the uniqueness of the standard representation).

As a consequence, if ϕ is a B-bimodular completely positive map on N satisfying $\|\phi(x)\|_2 \leq c\|x\|_2, \forall x \in N$, for some constant $c > 0$, as in Lemma 1.2.1, then the corresponding operator T_ϕ on $L^2(N,\tau)$ defined by $T_\phi(\hat{x}) = \hat{\phi(x)}, x \in N$ belongs to $B' \cap \langle N, B\rangle$.

We endow $\langle N, B\rangle$ with the unique normal semifinite faithful trace Tr satisfying $\mathrm{Tr}(xe_By) = \tau(xy), \forall x, y \in N$. Note that there exists a unique N bimodule map Φ from $\mathrm{sp}Ne_BN \subset \langle N, B\rangle$ into N satisfying $\Phi(xey) = xy, \forall x$, $y \in N$, and $\tau \circ \Phi = \mathrm{Tr}$. In particular this entails $\|\Phi(X)\|_1 \leq \|X\|_{1,\mathrm{Tr}}, \forall X \in \mathrm{sp}Ne_BN$. Note that the map Φ extends uniquely to an N-bimodule map from $L^1(\langle N, B\rangle, \mathrm{Tr})$ onto $L^1(N,\tau)$, still denoted Φ. This N-bimodule map satisfies the "pull down" identity $eX = e\Phi(eX), \forall X \in \langle N, B\rangle$ (see [PiPo], or [Po2]). Note that $\Phi(eX)$ actually belongs to $L^2(N,\tau) \subset L^1(N,\tau)$, for $X \in \langle N, B\rangle$.

1.3.2. *The compact ideal space of a semifinite algebra.* In order to define the compact ideal space of the semifinite von Neumann algebra $\langle N, B\rangle$, it will be useful to first mention some remarks about the compact ideal space of an arbitrary semifinite von Neumann algebra $\mathcal{N}$.

Thus, we let $\mathcal{J}(\mathcal{N})$ be the norm-closed two-sided ideal generated in $\mathcal{N}$ by the finite projections of $\mathcal{N}$, and call it the *compact ideal space* of $\mathcal{N}$ (see e.g., [KafW], [PoRa]). Note that $T \in \mathcal{N}$ belongs to $\mathcal{J}(\mathcal{N})$ if and only if all the spectral projections $e_{[s,\infty)}(|T|), s > 0$, are finite projections in $\mathcal{N}$. As a consequence, it follows that the set $\mathcal{J}^0(\mathcal{N})$ of all elements supported by finite projections (i.e., the *finite rank* elements in $\mathcal{J}(\mathcal{N})$) is a norm dense ideal in $\mathcal{J}(\mathcal{N})$.

Further, let $e \in \mathcal{N}$ be a finite projection with central support equal to 1 and denote by $\mathcal{J}_e(\mathcal{N})$ the norm-closed two-sided ideal generated by e in $\mathcal{N}$. It is easy to see that an operator $T \in \mathcal{N}$ belongs to $\mathcal{J}(\mathcal{N})$ if and only if there exists a partition of 1 with projections $\{z_i\}_i$ in $\mathcal{Z}(\mathcal{N})$ such that $Tz_i \in \mathcal{J}_e(\mathcal{N}), \forall i$. In particular, if $p \in \mathcal{N}$ is a finite projection then there exists a net of projections $z_i \in \mathcal{Z}(\mathcal{N})$ such that $z_i \uparrow 1$ and $pz_i \in \mathcal{J}_e(\mathcal{N}), \forall i$ (see e.g., 2.1 in [PoRa]). Also, $T \in \mathcal{J}_e(\mathcal{N})$ if and only if $e_{[s,\infty)}(|T|) \in \mathcal{J}_e(\mathcal{N}), \forall s > 0$. In turn, a projection $f \in \mathcal{N}$ lies in $\mathcal{J}_e(\mathcal{N})$ if and only if there exists a constant $c > 0$ such that $\mathrm{Tr}(fz) \leq c\mathrm{Tr}(ez)$, for any normal semifinite trace Tr on $\mathcal{N}$ and any projection $z \in \mathcal{Z}(\mathcal{N})$.

The next result, whose proof is very similar to some arguments in [Po7], shows that one can "push" elements of $\mathcal{J}(\mathcal{N})$ into the commutant of a subalgebra $\mathcal{B}$ of $\mathcal{N}$, while still staying in the ideal $\mathcal{J}(\mathcal{N})$, by averaging by unitaries in $\mathcal{B}$. We include a complete proof, for convenience.

PROPOSITION. *Let $\mathcal{B} \subset \mathcal{N}$ be a von Neumann subalgebra of $\mathcal{N}$. For $x \in \mathcal{N}$ denote $K_x = \overline{\mathrm{co}}^w\{uxu^* \mid u \in \mathcal{U}(\mathcal{B})\}$. If $x \in \mathcal{J}(\mathcal{N})$ then $\mathcal{B}' \cap K_x$ consists of exactly one element, denoted $\mathcal{E}_{\mathcal{B}'\cap\mathcal{N}}(x)$, which belongs to $\mathcal{J}(\mathcal{N})$. Moreover, the application $x \mapsto \mathcal{E}_{\mathcal{B}'\cap\mathcal{N}}(x)$ is a conditional expectation of $\mathcal{J}(\mathcal{N})$ onto $\mathcal{B}'\cap\mathcal{J}(\mathcal{N})$. Also, if $x \in \mathcal{J}_e(\mathcal{N})$ for some finite projection $e \in \mathcal{N}$ of central support* 1, *then $\mathcal{E}_{\mathcal{B}'\cap\mathcal{N}}(x) \in \mathcal{J}_e(\mathcal{N})$.*

Proof. If $x = f$ is a projection in $\mathcal{J}_e(\mathcal{N})$ then there exists $c > 0$ such that $\mathrm{Tr}(fz) \leq c\mathrm{Tr}(ez)$, for any normal semifinite trace Tr on $\mathcal{N}$ and any projection $z \in \mathcal{Z}(\mathcal{N})$. By averaging with unitaries and taking weak limits, this implies that $\mathrm{Tr}(yz) \leq c\mathrm{Tr}(ez), \forall y \in K_f$, so that $\mathrm{Tr}(pz) \leq s^{-1}c\mathrm{Tr}(ez)$, for any spectral projection $p = e_{[s,\infty)}(y), s > 0$ and $z \in \mathcal{Z}(\mathcal{N})$. Thus, $K_f \subset \mathcal{J}_e(\mathcal{N})$. Since any $x \in \mathcal{J}_e(\mathcal{N})$ is a norm limit of linear combinations of projections f in $\mathcal{J}_e(\mathcal{N})$, this shows that the very last part of the statement follows from the first part.

To prove the first part, consider first the case when $\mathcal{N}$ has a normal semifinite faithful trace Tr. Assume first that $x \in \mathcal{J}(\mathcal{N})$ actually belongs to $\mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$ ($\subset \mathcal{J}(\mathcal{N})$). Note that in this case all $K_x \subset \mathcal{N}$ is a subset of the Hilbert space $L^2(\mathcal{N}, \mathrm{Tr})$, where it is convex and weakly closed. Let then $x_0 \in K_x$ be the unique element of minimal Hilbert norm $\| \quad \|_{2,\mathrm{Tr}}$ in K_x. Since $\|ux_0u^*\|_{2,\mathrm{Tr}} = \|x_0\|_{2,\mathrm{Tr}}, \forall u \in \mathcal{U}(\mathcal{B})$, it follows that $ux_0u^* = x_0, \forall u \in \mathcal{U}(\mathcal{B})$. Thus, $x_0 \in \mathcal{B}' \cap \mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$. In particular, $x_0 \in \mathcal{B}' \cap \mathcal{J}(\mathcal{N})$.

If we now denote by p the orthogonal projection of $L^2(\mathcal{N}, \mathrm{Tr})$ onto the space of fixed points of the representation of $\mathcal{U}(\mathcal{B})$ on it given by $\xi \mapsto u\xi u^*$, then x_0 coincides with $p(x)$. Since $p(uxu^*) = p(x)$, this shows that $x_0 = p(x)$ is in fact the unique element y in K_x with $uyu^* = y, \forall u \in \mathcal{U}(\mathcal{B})$. Thus, if for each $x \in \mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$ we put $\mathcal{E}_{\mathcal{B}'\cap\mathcal{N}}(x) \stackrel{\mathrm{def}}{=} p(x)$, then we have proved the statement for the subset $\mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$.

Since $\|y\| \leq \|x\|, \forall y \in K_x$, it follows that if $\{x_n\}_n \subset \mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$ is a Cauchy sequence (in the uniform norm), then so is $\{\mathcal{E}_{\mathcal{B}'\cap\mathcal{N}}(x_n)\}_n$. Thus, $\mathcal{E}_{\mathcal{B}'\cap\mathcal{N}}$ extends uniquely by continuity to a linear, norm one projection from $\mathcal{J}(\mathcal{N})$

onto $\mathcal{B}' \cap \mathcal{J}(\mathcal{N})$, which by the above remarks takes the norm dense subspace $\mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$ into itself.

Let us now prove that $\mathcal{B}' \cap K_x \neq \emptyset, \forall x \in \mathcal{J}(\mathcal{N})$. To this end, let x be an arbitrary element in $\mathcal{J}(\mathcal{N})$ and $\varepsilon > 0$. Let $x_1 \in \mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$ with $\|x - x_1\| \leq \varepsilon$. Write $\mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x_1)$ as a weak limit of a net $\{T_{u_\alpha}(x_1)\}_\alpha$, for some finite tuples $u_\alpha = (u_1^\alpha, \dots, u_{n_\alpha}^\alpha) \subset \mathcal{U}(\mathcal{B})$, where $T_{u_\alpha}(y) = n_\alpha^{-1} \sum_i u_i^\alpha y u_i^{\alpha *}$, $y \in \mathcal{N}$. By passing to a subnet if necessary, we may assume $\{T_{u_\alpha}(x)\}_\alpha$ is also weakly convergent, to some element $x' \in K_x$. Since, $\|T_{u_\alpha}(x) - T_{u_\alpha}(x_1)\| \leq \|x - x_1\| \leq \varepsilon$, it follows that $\|x' - \mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x_1)\| \leq \varepsilon$. This shows that the weakly-compact set K_x contains elements which are arbitrarily close to $\mathcal{B}' \cap \mathcal{N}$. Since there is a weak limit of such elements it follows that $\mathcal{B}' \cap K_x \neq \emptyset$.

Finally, let $x \in \mathcal{J}(\mathcal{N})$ and assume x^0 is an element in $\mathcal{B}' \cap K_x$. To prove that $x^0 = \mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x)$, let $\varepsilon > 0$ and $x_1 \in \mathcal{N} \cap L^2(\mathcal{N}, \mathrm{Tr})$ with $\|x - x_1\| \leq \varepsilon$, as before. Write x^0 as a weak limit of a net $\{T_{v_\beta}(x)\}_\beta$, for some finite tuples $v_\beta = (v_1^\beta, \dots, v_{m_\beta}^\beta) \subset \mathcal{U}(\mathcal{B})$. By passing to a subnet if necessary, we may assume $\{T_{v_\beta}(x_1)\}_\beta$ is also weakly convergent, to some element $x_1^0 \in K_{x_1}$. Since, $\|T_{v_\beta}(x) - T_{v_\beta}(x_1)\| \leq \|x - x_1\| \leq \varepsilon$, it follows that $\|x^0 - x_1^0\| \leq \varepsilon$. But $p(x_1^0) = p(x_1) = \mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x_1)$, and $p(x_1^0)$ is obtained as a weak limit of averaging by unitaries in $\mathcal{B}$, which commute with x^0. Thus,

$$\begin{aligned} \|x^0 - \mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x)\| \leq & \|x^0 - \mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x_1)\| \\ & + \|\mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x_1) - \mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x)\| \leq \varepsilon + \|x_1 - x\| \leq 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows that $x^0 = \mathcal{E}_{\mathcal{B}' \cap \mathcal{N}}(x)$.

This finishes the proof of the case when $\mathcal{N}$ has a faithful trace Tr. The general case follows now readily, because if $\{z_i\}_i$ is an increasing net of projections in $\mathcal{Z}(\mathcal{N})$ such that $K_{z_i x} \cap (\mathcal{B} z_i)'$ consists of exactly one element, which belongs to $\mathcal{J}(\mathcal{N}) z_i = \mathcal{J}(\mathcal{N} z_i)$, $\forall x \in \mathcal{J}(\mathcal{N})$, then the same holds true for the projection $\lim_{i \to \infty} z_i$. □

1.3.3. *The compact ideal space of* $\langle N, B \rangle$. In particular, if $B \subset N$ is an inclusion of finite von Neumann algebras as in 1.3.1, then we denote by $\mathcal{J}(\langle N, B \rangle)$ the compact ideal space of $\langle N, B \rangle$. Noticing that e_B has central support 1 in $\langle N, B \rangle$, we denote $\mathcal{J}_0(\langle N, B \rangle)$ the norm closed two sided ideal $\mathcal{J}_{e_B}(\langle N, B \rangle)$ generated by e_B in $\langle N, B \rangle$. Note that if $B = \mathbb{C}$ then $\mathcal{J}(\langle N, B \rangle) = \mathcal{J}_0(\langle N, B \rangle)$ is the usual ideal of compact operators $\mathcal{K}(L^2(N))$.

It will be useful to have the following alternative characterizations of the compact ideal spaces $\mathcal{J}(\langle N, B \rangle), \mathcal{J}_0(\langle N, B \rangle)$.

Proposition. *Let N be a finite von Neumann algebra with countably decomposable center and $B \subset N$ a von Neumann subalgebra. Let $T \in \langle N, B \rangle$. The following conditions are equivalent*:

1°. $T \in \mathcal{J}(\langle N, B \rangle)$.

2°. *For any* $\varepsilon > 0$ *there exists a finite projection* $p \in \langle N, B\rangle$ *such that* $\|T(1-p)\| < \varepsilon$.

3°. *For any* $\varepsilon > 0$ *there exists* $z \in \mathcal{P}(\mathcal{Z}(J_N B J_N))$ *such that* $\tau(1-z) \leq \varepsilon$ *and* $Tz \in \mathcal{J}_0(\langle N, B\rangle)$.

4°. *For any given sequence* $\{\eta_n\}_n \in L^2(N)$ *with the properties* $E_B(\eta_n^*\eta_n) \leq 1, \forall n \geq 1$, *and* $\lim_{n\to\infty} \|E_B(\eta_n^*\eta_m)\|_2 = 0, \forall m$, $\lim_{n\to\infty} \|T\eta_n\|_2 = 0$.

5°. *For any given sequence* $\{x_n\}_n \in N$ *with the properties* $E_B(x_n^* x_n) \leq 1, \forall n \geq 1$, *and* $\lim_{n\to\infty} \|E_B(x_n^* x_m)\|_2 = 0, \forall m$, $\lim_{n\to\infty} \|Tx_n\|_2 = 0$.

Moreover, $T \in \mathcal{J}_0(\langle N, B\rangle)$ *if and only if condition* 2° *above holds true with projections* p *in* $\mathcal{J}_0(\langle N, B\rangle)$.

Proof. The equivalence of 1° and 2° (resp. the equivalence in the last part of the statement) is trivial by the following fact, noted in 1.3.2: $T \in \mathcal{J}(\langle N, B\rangle)$ (resp. $T \in \mathcal{J}_0(\langle N, B\rangle)$) if and only if $e_{[s,\infty)}(|T|) \in \mathcal{J}(\langle N, B\rangle)$ (resp. $\in \mathcal{J}_0(\langle N, B\rangle)$), $\forall s > 0$.

3° $\implies$ 2° is trivial by the general remarks in 1.3.2. To prove 2° $\implies$ 3°, for each $n \geq 1$ let T_n be a linear combination of finite projections in $\langle N, B\rangle$ such that $\|T - T_n\| \leq 2^{-n}$. We see that for any finite projection $e \in \langle N, B\rangle$ and $\delta > 0$ there exists a projection $z \in \mathcal{Z}(\langle N, B\rangle) = J_N \mathcal{Z}(B) J_N$ such that $\tau(1-z) \leq \delta$ and $ez \in \mathcal{J}_0(\langle N, B\rangle)$. It follows that for each n there exists a projection $z_n \in J_N \mathcal{Z}(B) J_N$ such that $\tau(1-z_n) \leq 2^{-n}\varepsilon$ and $T_n z_n \in \mathcal{J}_0(\langle N, B\rangle)$. Let $z = \wedge z_n$. Then $\tau(1-z) \leq \Sigma_n 2^{-n}\varepsilon \leq \varepsilon$, $T_n z \in \mathcal{J}_0(\langle N, B\rangle)$ and $\|(T - T_n)z\| \leq \|T - T_n\| \leq 2^{-n}, \forall n$. Thus, $Tz \in \mathcal{J}_0(\langle N, B\rangle)$ as well.

3° $\implies$ 4° is just a particular case of (2.5 in [PoRa]). To prove 4° $\implies$ 1°, assume by contradiction that there exists $s > 0$ such that the spectral projection $e = e_s(|T|)$ is properly infinite. It follows that there exist mutually orthogonal, mutually equivalent projections $p_1, p_2, \dots \in \langle N, B\rangle$ such that $\Sigma_n p_n \leq e$ with p_n majorised by $e_B, \forall n$. Thus, for each $n \geq 1$ there exists $\eta_n \in L^2(N)$ such that $p_n = \eta_n e_B \eta_n^*$. It then follows that $E_B(\eta_n^*\eta_m) = 0$ for $n \neq m$, with $E_B(\eta_n^*\eta_n)$ mutually equivalent projections in B. In particular, $\|\eta_n\|_2^2 = \tau(\eta_n^*\eta_n) = c > 0$ is constant, $\forall n$. Thus,

$$s^{-1}\|T\eta_n\|_2 \geq \|e(\eta_n)\|_2 \geq \|p_n(\eta_n)\|_2 = \|\eta_n\|_2 = c^{1/2}, \forall n,$$

a contradiction.

4° $\implies$ 5° is trivial. To prove 5° $\implies$ 4° assume 5° holds true and let η_n be a sequence satisfying the hypothesis in 4°. For each n let q_n be a spectral projection corresponding to some interval $[0, t_n]$ of $\eta_n\eta_n^*$ (the latter regarded as a positive, unbounded, summable operator in $L^1(N)$) such that $\|\eta_n - q_n\eta_n\|_2 < 2^{-n}$. Thus, $x_n = q_n\eta_n$ lies in N. One can easily check

$E_B(x_n^* x_n) \leq E_B(\eta_n^* \eta_n) \leq 1$ and

$$\lim_{n\to\infty} \|E_B(x_n^* x_m)\|_2^2 = \lim_{n\to\infty} \mathrm{Tr}((q_n \eta_n e_B \eta_n^* q_n)(q_m \eta_m e_B \eta_m^* q_m)) = 0.$$

Thus $\lim_{n\to\infty} \|Tx_n\|_2 = 0$. But

$$\|T\eta_n\|_2 \leq \|Tx_n\|_2 + \|T\| \|\eta_n - x_n\|_2 \leq \|Tx_n\|_2 + 2^{-n}\|T\|,$$

showing that $\lim_{n\to\infty} \|T\eta_n\|_2 = 0$ as well. □

1.4. *Discrete embeddings and bimodule decomposition.* If $B \subset N$ is an inclusion of finite von Neumann algebras with a faithful normal tracial state τ as before, then we often consider N as an (algebraic) (bi)module over B and $L^2(N, \tau)$ as a Hilbert (bi)module over B. In fact any vector subspace H of N which is invariant under left (resp. right) multiplication by B is a left (resp. right) module over B. Similarly, any Hilbert subspace of $L^2(N, \tau)$ which is invariant under multiplication to the left (resp. right) by elements in B is a left (resp. right) Hilbert module. Also, the closure in $L^2(N, \tau)$ of a B-module $H \subset N$ is a Hilbert B-module.

1.4.1. *Orthonormal basis.* An *orthonormal basis* for a right (respectively left) Hilbert B-module $\mathcal{H} \subset L^2(N, \tau)$ is a subset $\{\eta_i\}_i \subset L^2(N)$ such that $\mathcal{H} = \overline{\Sigma_k \eta_k B}$ (respectively $\mathcal{H} = \overline{\Sigma_k B\eta_k}$) and $E_B(\eta_i^* \eta_{i'}) = \delta_{ii'} p_i \in \mathcal{P}(B), \forall i, i'$, (respectively $E_B(\eta_{j'} \eta_j^*) = \delta_{j'j} q_j \in \mathcal{P}(B), \forall j, j'$). Note that if this is the case, then $\xi = \Sigma_i \eta_i E_B(\eta_i^* \xi), \forall \xi \in \mathcal{H}$ (resp. $\xi = \Sigma_j E_B(\xi \eta_j^*) \eta_j, \forall \xi \in \mathcal{H}$).

A set $\{\eta_j\}_j \subset L^2(N, \tau)$ is an orthonormal basis for $\mathcal{H}_B$ if and only if the orthogonal projection f of $L^2(N, \tau)$ on $\mathcal{H}$ satisfies $f = \Sigma_j \eta_j e_B \eta_j^*$ with $\eta_j e_B \eta_j^*$ projection $\forall j$. A simple maximality argument shows that any left (resp. right) Hilbert B-module $\mathcal{H} \subset L^2(N, \tau)$ has an orthonormal basis (see [Po2] for all this). The Hilbert module $\mathcal{H}_B$ (resp. ${}_B\mathcal{H}$) is *finitely generated* if it has a finite orthonormal basis.

1.4.2. *Quasi-regular subalgebras.* Recall from [D] that if $B \subset N$ is an inclusion of finite von Neumann algebras then the *normalizer* of B in N is the set $\mathcal{N}(B) = \mathcal{N}(B) = \{u \in \mathcal{U}(N) \mid uBu^* = B\}$. The von Neumann algebra B is called *regular* in N if $\mathcal{N}(B)'' = N$.

In the same spirit, the *quasi-normalizer* of B in N is defined to be the set $q\mathcal{N}(B) \overset{\text{def}}{=} \{x \in N \mid \exists\ x_1, x_2, \ldots, x_n \in N$ such that $xB \subset \sum_{i=1}^n Bx_i$ and $Bx \subset \sum_{i=1}^n x_i B\}$ (cf. [Po5], [PoSh]). The condition "$xB \subset \sum Bx_i$, $Bx \subset \sum x_i B$" is equivalent to "$BxB \subset (\sum_{i=1}^n Bx_i) \cap (\sum_{i=1}^n x_i B)$" and also to "sp$BxB$ is finitely generated both as a left and as a right B-module." It then follows readily that $\mathrm{sp}(q\mathcal{N}_N(B))$ is a *-algebra. Thus, $P \overset{\text{def}}{=} \overline{\mathrm{sp}}(q\mathcal{N}_N(B)) = q\mathcal{N}_N(B)''$ is a von Neumann subalgebra of N containing B. In case the von Neumann algebra $P = q\mathcal{N}_B(N)''$ is equal to all N, then B is *quasi-regular* in N ([Po5]).

The most interesting case of inclusions $B \subset N$ for which one considers the normalizer $\mathcal{N}(B)$ and the quasi-normalizer $q\mathcal{N}_N(B)$ of B in N is when the subalgebra B satisfies the condition $B' \cap N \subset B$, or equivalently $B' \cap N = \mathcal{Z}(B)$, notably when B and N are factors (i.e., when $B' \cap N = \mathbb{C}$) and when B is a maximal abelian *-subalgebra (i.e., when $B' \cap N = B$).

The next lemma lists some useful properties of $q\mathcal{N}(B)$. In particular, it shows that if a Hilbert B-bimodule $\mathcal{H} \subset L^2(N, \tau)$ is finitely generated both as a left and as a right Hilbert B module, then it is "close" to a bounded finitely generated B-bimodule $H \subset P$.

LEMMA. (i) *Let N be a finite von Neumann algebra with a normal finite faithful trace τ and $B \subset N$ a von Neumann subalgebra. Let $p \in B' \cap \langle N, B\rangle$ be a finite projection such that $J_N p J_N$ is also a finite projection. Let $\mathcal{H} \subset L^2(N, \tau)$ be the Hilbert space on which p projects* (*which is thus a Hilbert B-bimodule*). *Then there exists an increasing sequence of central projections $z_n \in \mathcal{Z}(B)$ such that $z_n \uparrow 1$ and such that the Hilbert B-bimodules $z_n \mathcal{H} z_n \subset L^2(N)$ are finitely generated both as left and as right Hilbert B-modules.*

(ii) *If $B \subset N$ are as in* (i) *and $\mathcal{H}^0 \subset L^2(N)$ is a Hilbert B-bimodule such that $\mathcal{H}^0_B$, ${}_B\mathcal{H}^0$ are finitely generated Hilbert modules, with $\{\xi_i \mid 1 \leq i \leq n\}$, $\{\zeta_j \mid 1 \leq j \leq m\}$ their corresponding orthonormal basis, then for any $\varepsilon > 0$ there exists a projection $q \in B' \cap N$ such that $\tau(1-q) < \varepsilon$ and $x_i = q\xi_i q \in N, y_j = q\zeta_j q \in N, \forall i, j$. In particular, $\Sigma_i x_i B = \Sigma_j B y_j = q\mathcal{H}^0 q \cap N$ is dense in $q\mathcal{H}^0 q$ and is finitely generated both as left and right B-module.*

(iii) *If p is a projection as in* (i) *then $p \leq e_P$. Also, B is quasiregular in N if and only if B is discrete in N, i.e., $B' \cap \langle N, B\rangle$ is generated by projections which are finite in $\langle N, B\rangle$* ([ILP]).

Proof. (i) and (ii) are trivial consequences of 1.4.1 and of the definitions.

The first part of (iii) is trivial by (i), (ii). Thus, e_P is the supremum of all projections $p \in B' \cap \langle N, B\rangle$ such that both p and $J_N p J_N$ are finite in $\langle N, B\rangle$. Thus, if $q \in \langle N, B\rangle$ is a nonzero finite projection orthogonal to e_P then any projection $q' \in B' \cap \langle N, B\rangle$ with $q' \leq J_N q J_N$ must be infinite (or else the maximality of e_P would be contradicted). But if q satisfies this property then $B' \cap \langle N, B\rangle$ cannot be generated by finite projections. $\square$

1.4.3. *Cartan subalgebras.* Recall from [D] that a maximal abelian *-subalgebra A of a finite von Neumann factor M is called *semiregular* if $\mathcal{N}(A)$ generates a factor, equivalently, if $\mathcal{N}(A)' \cap M = \mathbb{C}$. Also, while maximal abelian $*$-subalgebras A with $\mathcal{N}(A)'' = M$ were called *regular* in [D], as mentioned before, they were later called *Cartan subalgebras* in [FM], a terminology that seems to prevail and which we therefore adopt.

By results of Feldman and Moore ([FM]), in case a type II_1 factor M is *separable* in the norm $\| \|_2$ given by the trace, to each Cartan subalgebra $A \subset M$ corresponds a countable, measure-preserving, ergodic equivalence relation $\mathcal{R} = \mathcal{R}(A \subset M)$ on the standard probability space (X, μ), where $L^\infty(X,\mu) \simeq (A, \tau_{|A})$, given by orbit equivalence under the action of $\mathcal{N}(A)$. In fact, $\mathcal{N}(A)$ also gives rise to an A-valued 2-cocycle $v = v(A \subset M)$, reflecting the associativity mod A of the product of elements in the *normalizing pseudogroup* $\mathcal{GN} \overset{\text{def}}{=} \{pu \mid u \in \mathcal{N}(A), p \in \mathcal{P}(A)\}$.

Conversely, given any pair $(\mathcal{R}, v)$, consisting of a countable, measure-preserving, ergodic equivalence relation $\mathcal{R}$ on the standard probability space (X, μ) and an $L^\infty(X,\mu)$-valued 2-cocycle v for the corresponding pseudogroup action (N.B.: $v \equiv 1$ is always a 2-cocycle, $\forall \mathcal{R}$), there exists a type II_1 factor with a Cartan subalgebra $(A \subset M)$ associated with it, via a group-measure space construction “à la” Murray-von Neumann. The association $(A \subset M) \to (\mathcal{R}, v) \to (A \subset M)$ is one-to-one, modulo isomorphisms of inclusions $(A \subset M)$ and respectively measure-preserving orbit equivalence of $\mathcal{R}$ with equivalence of the 2-cocycles v (see [FM] for all this).

Examples of countable, measure-preserving, ergodic equivalence relations $\mathcal{R}$ are obtained by taking free ergodic measure-preserving actions σ of countable groups Γ_0 on the standard probability space (X, μ), and letting $x\mathcal{R}y$ whenever there exists $g \in \Gamma_0$ such that $y = \sigma_g(x)$.

If $t > 0$ then the *amplification of a Cartan subalgebra* $A \subset M$ *by* t is the Cartan subalgebra $A^t \subset M^t$ obtained by first choosing some $n \geq t$ and then compressing the Cartan subalgebra $A \otimes D \subset M \otimes M_{n \times n}(\mathbb{C})$ by a projection $p \in A \otimes D$ of (normalized) trace equal to t/n. (N.B. This Cartan subalgebra is defined up to isomorphism.) Also, the *amplification of a measurable equivalence relation* $\mathcal{R}$ by t is the equivalence relation obtained by reducing the equivalence relation $\mathcal{R} \times \mathcal{D}_n$ to a subset of measure t/n, where $\mathcal{D}_n$ is the ergodic equivalence relation on the n points set. Note that if $A \subset M$ induces the equivalence relation $\mathcal{R}$ then $A^t \subset M^t$ induces the equivalence relation $\mathcal{R}^t$. Also, $v_{A \subset M} \equiv 1$ implies $v_{A^t \subset M^t} \equiv 1, \forall t > 0$.

By using Lemma 1.4.2, we can reformulate a result from [PoSh], based on prior results in [FM], in a form that will be more suitable for us:

Proposition. *Let M be a separable type* II_1 *factor.*

(i) *A maximal abelian* **-subalgebra $A \subset M$ is a Cartan subalgebra if and only if $A \subset M$ is discrete, i.e., if and only if $A' \cap \langle M, A \rangle$ is generated by projections that are finite in $\langle M, A \rangle$.*

(ii) *Let $A_1, A_2 \subset M$ be two Cartan subalgebras of M. Then A_1, A_2 are conjugate by a unitary element of M if and only if $A_1' \cap \langle M, A_2 \rangle$ is generated by finite projections of $\langle M, A_2 \rangle$ and $A_2' \cap \langle M, A_1 \rangle$ is generated by finite projections of $\langle M, A_1 \rangle$. Equivalently, A_1, A_2 are unitary conjugate if and only if ${}_{A_1}L^2(M, \tau)_{A_2}$ is a direct sum $A_1 - A_2$ Hilbert bimodules that are finite dimensional both as left A_1-Hilbert modules and as right A_2-Hilbert modules.*

Proof. (i) By Lemma 1.4.2, the discreteness condition on A is equivalent to the quasi-regularity of A in N. By [PoSh], the latter is equivalent to A being Cartan.

(ii) If $A_i' \cap (J_N A_j J_N)'$ is generated by finite projections of the semifinite von Neumann algebra $(J_N A_j J_N)'$, for $i, j = 1, 2$, and we denote $M = M_2(N)$ the algebra of 2-by-2 matrices over N and $A = A_1 \oplus A_2$ then $A' \cap (J_M A J_M)'$ is also generated by finite projections of $J_M A J_M$. By part (i), this implies A is Cartan in M. By [Dy] this implies there exists a partial isometry $v \in M$ such that $vv^* = e_{11}, v^*v = e_{22}$, where $\{e_{ij}\}_{i,j=1,2}$, is a system of matrix units for $M_2(\mathbb{C})$. Thus, if $u \in N$ is the unitary element with $ue_{12} = v$ then $uA_1u^* = A_2$. $\square$

2. Relative Property H: Definition and examples

In this section we consider a "co-type" relative version of Haagerup's compact approximation property for inclusions of von Neumann algebras. This property can be viewed as a "weak co-amenability" property; see the next section (see 3.5, 3.6). It is a property that excludes "co-rigidity", as later explained (see 5.6, 5.7). We first recall the definition for groups and for single von Neumann algebras, for completeness.

2.0.1. *Property* H *for groups.* In [H1] Haagerup proved that the free groups $\Gamma_0 = \mathbb{F}_n, 2 \leq n \leq \infty$, satisfy the following condition: There exist positive definite functions φ_n on Γ_0 such that

$$(2.0.1') \qquad \lim_{g\to\infty} \varphi_n(g) = 0, \quad \forall n, \text{ (equivalently, } \varphi_n \in c_0(\Gamma_0)).$$

$$(2.0.1'') \qquad \lim_{g\to\infty} \varphi_n(g) = 1, \quad \forall g \in \Gamma_0.$$

Many more groups Γ_0 were shown to satisfy conditions (2.0.1) in [dCaH], [CowH], [CCJJV]. This property is often refered to as *Haagerup's approximation property*, or *property* H (see e.g., [Cho], [CJ], [CCJJV]). By a result of Gromov, a group has property H if and only if it satisfies a certain embeddability condition into a Hilbert space, a property he called a-T-*menability* ([Gr]). There has been a lot of interest in studying these groups lately. We refer the reader to the recent book ([CCJJV]) for a comprehensive account on this subject. Note that property H is a hereditary property, so if a group Γ_0 has it, then any subgroup $\Gamma_1 \subset \Gamma_0$ has it as well.

2.0.2. *Property* H *for algebras.* A similar *property* H, has been considered for finite von Neumann algebras N ([C3], [Cho], [CJ]): It requires the existence of a net of normal completely positive maps ϕ_α on N satisfying the conditions:

$$(2.0.2') \qquad \tau \circ \phi_\alpha \leq \tau \text{ and } \phi_\alpha(\{x \in N \mid \|x\|_2 \leq 1\}) \text{ is } \| \quad \|_2\text{-precompact}, \forall\alpha,$$

$$(2.0.2'') \qquad \lim_{\alpha\to\infty} \|\phi_\alpha(x) - x\|_2 = 0, \forall x \in N,$$

with respect to some fixed normal faithful trace τ on N. The net can of course be taken to be a sequence in case N is separable in the $\| \quad \|_2$-topology.

It was shown in [Cho] that if N is the group von Neumann algebra $L(\Gamma_0)$ associated to some group Γ_0, then $L(\Gamma_0)$ has the property H (as a von Neumann algebra) if and only if Γ_0 has the property H (as a group). It was further shown in [Jo1] that the set of properties (2.0.2) does not depend on the normal faithful trace τ on N, i.e., if there exists a net of completely positive maps ϕ_α on N satisfying conditions $(2.0.2'), (2.0.2'')$ with respect to some faithful normal trace τ, then given any other faithful normal trace τ' on N there exists a net of completely positive maps ϕ'_α on N satisfying the conditions with respect to τ'. It was also proved in [Jo1] that if N has property H then given any faithful normal trace τ on N the completely positive maps ϕ_α on N satisfying (2.0.2) with respect to τ can be taken τ-preserving and unital.

We now extend the definition of the property H from the above single algebra case to the relative ("co-type") case of inclusions of von Neumann algebras, by using a similar strategy to the way the notions of amenabilty and property (T) were extended from single algebras to inclusions of algebras in [Po1,10]; see Remarks 3.5, 3.6, 5.6 hereafter.

2.1. *Definition.* Let N be a finite von Neumann algebra with countable decomposable center and $B \subset N$ a von Neumann subalgebra. N has *property* H *relative to* B if there exists a normal faithful tracial state τ on N and a net of normal completely positive B-bimodular maps ϕ_α on N satisfying the conditions:

$$(2.1.0) \qquad \tau \circ \phi_\alpha \leq \tau;$$

$$(2.1.1) \qquad T_{\phi_\alpha} \in J(\langle N, B\rangle), \forall \alpha;$$

$$(2.1.2) \qquad \lim_{\alpha\to\infty} \|\phi_\alpha(x) - x\|_2 = 0, \forall x \in N,$$

where T_{ϕ_α} are the operators in the semifinite von Neumann algebra $\langle N, B\rangle \subset \mathcal{B}(L^2(N,\tau))$ defined out of ϕ_α and τ, as in 1.2.1.

Following [Gr], one can also use the terminology: N is a-T-*menable relative to* B.

Note that the finite von Neumann algebra N has the property H as a single von Neumann algebra if and only if N has the property H relative to $B = \mathbb{C}$.

Note that a similar notion of "relative Haagerup property" was considered by Boca in [Bo], to study the behaviour of the Haagerup property under amalgamated free products. The definition in [Bo] involved a fixed trace and it required the completely positive maps to be unital and trace preserving.

The next proposition addresses some of the differences between his definition and 2.1:

2.2. Proposition. *Let N be a finite von Neumann algebra with countably decomposable center and $B \subset N$ a von Neumann subalgebra.*

1°. *If N has the property* H *relative to B and $\{\phi_\alpha\}_\alpha$ satisfy* (2.1.0)–(2.1.2) *with respect to the trace τ on N, then there exists a net of completely positive maps $\{\phi'_\alpha\}_\alpha$ on N, which still satisfy* (2.1.0)–(2.1.2) *with respect to the trace τ, but also $T_{\phi'_\alpha} \in J_0(\langle N, B\rangle)$ and $\phi'_\alpha(1) \leq 1, \forall \alpha$.*

2°. *Assume $B' \cap N \subset B$. Then the following conditions are equivalent*:

(i) *N has the property* H *relative to B.*

(ii) *Given any faithful normal tracial state τ_0 on N, there exists a net of unital, τ_0-preserving, B-bimodular completely positive maps ϕ_α on N such that $T_{\phi_\alpha} \in \mathcal{J}_0(\langle N, B\rangle), \forall \alpha$, and such that condition* (2.1.2) *is satisfied for the norm $\| \quad \|_2$ given by τ_0.*

(iii) *There exists a normal faithful tracial state τ and a net of normal, B-bimodular completely positive maps ϕ_α on N such that ϕ_α can be extended to bounded operators T_{ϕ_α} on $L^2(N, \tau)$, such that $T_{\phi_\alpha} \in \mathcal{J}(\langle N, B\rangle)$ and* (2.1.2) *is satisfied for the trace τ.*

Moreover, in case N is countably generated as a B-module, i.e., there exists a countable set $S \subset N$ such that $\overline{\text{sp}} SB = N$, the closure being taken in the norm $\| \quad \|_2$, then the net ϕ_α in either 1°, 2° or 3° can be taken to be a sequence.

Proof. 1°. By part 3° of Proposition 1.3.3, we can replace if necessary ϕ_α by $\phi_\alpha(z_\alpha \cdot z_\alpha)$, for some $z_\alpha \in \mathcal{P}(\mathcal{Z}(B))$ with $z_\alpha \uparrow 1$, so that the corresponding operators on $L^2(N, \tau)$ belong to $\mathcal{J}_0(\langle N, B\rangle), \forall \alpha$.

By using continuous functional calculus for $\phi_\alpha(1)$, let $b_\alpha = (1 \vee \phi_\alpha(1))^{-1/2} \in B' \cap N$. Then $b_\alpha \leq 1, \|b_\alpha - 1\|_2 \to 0$ and

$$\phi'_\alpha(x) = b_\alpha \phi_\alpha(x) b_\alpha, x \in N,$$

still defines a normal completely positive map on N with $\|\phi'_\alpha(x) - x\|_2 \to 0$, $\forall x \in N$. Moreover, if $x \geq 0$ then

$$\tau(\phi'_\alpha(x)) = \tau(\phi_\alpha(x) b_\alpha^2) \leq \tau(\phi_\alpha(x)).$$

Also, since $T_{\phi'_\alpha} = L(b_\alpha) R(b_\alpha) T_{\phi_\alpha}$ and $L(b_\alpha) \in N \subset \langle N, B\rangle$, $R(b_\alpha) \in J(B' \cap N)J \subset \langle N, B\rangle$ and $T_{\phi_\alpha} \in \mathcal{J}(\langle N, B\rangle)$, it follows that $T_{\phi'_\alpha} \in \mathcal{J}(\langle N, B\rangle)$.

2°. We clearly have (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii).

Assume now (iii) holds true for the trace τ and let τ_0 be an arbitrary normal, faithful tracial state on N. Thus, $\tau_0 = \tau(\cdot a_0)$, for some $a_0 \in \mathcal{Z}(N)_+$

with $\tau(a_0) = 1$. Since $B' \cap N = \mathcal{Z}(B)$, by part 3° of Lemma 1.2.1 we have $a_\alpha = \phi_\alpha(1) \in \mathcal{Z}(B)$. Also, (2.1.2) implies

$$\lim_{\alpha\to\infty} \|a_\alpha - 1\|_2 = 0, \tag{2.2.2$'$}$$

where $\| \quad \|_2$ denotes the norm given by τ.

Let p_α be the spectral projection of a_α corresponding to $[1/2, \infty)$. Since $a_\alpha \in \mathcal{Z}(B)$, $p_\alpha \in \mathcal{Z}(B)$. Also, condition (2.2.2′) implies $\lim_{\alpha\to\infty} \|p_\alpha - 1\|_2 = \lim_{\alpha\to\infty} \|a_\alpha^{-1} p_\alpha - p_\alpha\|_2 = 0$. Furthermore, by condition 3° of Proposition 1.3.3, there exists $p'_\alpha \in \mathcal{Z}(\mathcal{B})$ with $p'_\alpha \leq p_\alpha$, such that $T_{\phi_\alpha} p'_\alpha \in \mathcal{J}_0(\langle N, B\rangle)$ and

$$\lim_{\alpha\to\infty} \|p'_\alpha - 1\|_2 = 0, \qquad \lim_{\alpha\to\infty} \|a_\alpha^{-1} p_\alpha - p'_\alpha\|_2 = 0. \tag{2.2.2$''$}$$

Define ϕ'_α on N by

$$\phi'_\alpha(x) = a_\alpha^{-1/2} p'_\alpha \phi_\alpha(x) p'_\alpha a_\alpha^{-1/2} + (1 - p'_\alpha) E_B(x)(1 - p'_\alpha), x \in N.$$

Then we clearly have $\phi'_\alpha(1) = 1$, ϕ'_α are B-bimodular and $T_{\phi'_\alpha} \in \mathcal{J}_0(\langle N, B\rangle)$. Since $B' \cap N \subset B$, by part 2° in Lemma 1.2.1, this also implies $\tau \circ \phi'_\alpha = \tau$, $\tau_0 \circ \phi'_\alpha = \tau_0$. Moreover, since $a_\alpha^{-1} p_\alpha \leq 2$, it follows that for each $x \in N$,

$$\begin{aligned}
\|\phi'_\alpha(x) - x\|_2 \leq & \|a_\alpha^{-1/2} p'_\alpha \phi_\alpha(x) a_\alpha^{-1/2} p'_\alpha - p'_\alpha x p'_\alpha\|_2 \\
& + \|(1 - p'_\alpha) x p'_\alpha\|_2 + \|p'_\alpha x (1 - p'_\alpha)\|_2 \\
& + \|(1 - p'_\alpha)(x - E_B(x))(1 - p'_\alpha)\|_2 \\
\leq & 2\|\phi_\alpha(x) - x\|_2 + 2\|a_\alpha^{-1/2} p'_\alpha x a_\alpha^{-1/2} p'_\alpha - p'_\alpha x p'_\alpha\|_2 \\
& + 3\|1 - p'_\alpha\|_2 \|x\|,
\end{aligned}$$

with the latter tending to 0 for all $x \in N$, by (2.2.2″). Since this convergence holds true for one faithful normal trace, it holds true in the s-topology, thus for the normal trace τ_0 as well.

The last part of 2° is trivial. □

We now prove some basic properties of the relative property H, showing that it is well behaved to simple operations such as tensor products, amplifications, finite index extensions/restrictions.

2.3. Proposition. 1°. *If N has property* H *relative to B and $B_0 \subset N_0$ is embedded into $B \subset N$ with commuting squares, i.e., $N_0 \subset N$, $B_0 \subset B$, $B_0 = N_0 \cap B$ and $E_{N_0} \circ E_B = E_B \circ E_{N_0} = E_{B_0}$, then N_0 has property* H *relative to B_0.*

2°. *If $B_1 \subset N_1$ and $B_2 \subset N_2$ then $N_1 \overline{\otimes} N_2$ has property* H *relative to $B_1 \overline{\otimes} B_2$ if and only if N_i has property* H *relative to $B_i, i = 1, 2$.*

3°. *Let $B \subset N_0 \subset N$. If N has property* H *relative to B, then N_0 has property* H *relative to B. Conversely, if $N_0 \subset N$ has a finite orthonormal basis $\{u_j\}_j$ with u_j unitary elements such that $u_j B u_j^* = B, \forall j$, and N_0 has property*

H *relative to* B, *with respect to* $\tau_{|N_0}$ *for some normal faithful trace* τ *on* N, *then* N *has from property* H *relative to* B, *with respect to* τ.

4°. *Assume* $B \subset B_0 \subset N$ *and* $B \subset B_0$ *has a finite orthonormal basis. If* N *has from property* H *relative to* B_0 *then* N *has property* H *relative to* B. *If in addition* $B_0' \cap N \subset B_0$ *then, conversely, if* N *has from property* H *relative to* B, *then* N *has property* H *relative to* B_0.

Proof. 1°. If $\phi_\alpha : N \to N$ are B-bimodular completely positive maps approximating the identity on N, then by the commuting square relation $E_{N_0} \circ E_B = E_B \circ E_{N_0} = E_{B_0}$, it follows that $\phi'_\alpha = E_{N_0} \circ \phi_{\alpha|N_0}$ approximate the identity on N_0 and are B_0-bimodular. Moreover, by commuting squares, if T_{ϕ_α} satisfy condition 5° in 1.3.3 then so do $T_{\phi'_\alpha}$.

2°. The implication from left to right follows by applying 1° to $(B \subset N) = (B_1 \overline{\otimes} B_2 \subset N_1 \overline{\otimes} N_2)$ and $(B_0 \subset N_0) = (B_i \otimes \mathbb{C} \subset N_i \otimes \mathbb{C}), i = 1, 2$. The implication from right to left follows from the fact that $T_{\phi^i_\alpha} \in \mathcal{J}(\langle N_i, B_i \rangle)$, $i = 1, 2$, implies $T_{\phi^1_\alpha \otimes \phi^2_\alpha} \in \mathcal{J}(\langle N_1 \overline{\otimes} N_2, B_1 \overline{\otimes} B_2)$ (since the tensor product of finite projections is a finite projection).

3°. For the first implication, let ϕ_α be completely positive maps on N that satisfy (2.1.0)–(2.1.2) for $B \subset N$ and for the trace τ on N. Define $\phi^0_\alpha(x) = E_{N_0}(\phi_\alpha(x)), x \in N_0$. Then ϕ^0_α are completely positive, B-bimodular maps which still satisfy $\tau \circ \phi^0_\alpha \leq \tau$. Moreover, since T_{ϕ_α} satisfy condition 5° in Proposition 1.3.3, then clearly ϕ^0_α do as well.

For the converse, assuming ϕ^0_α are completely positive maps on N_0 that satisfy (2.1.0)–(2.1.2) for $B \subset N_0$, define $\tilde{\phi}_\alpha$ on $\langle N, e_{N_0} \rangle$ by

$$\tilde{\phi}_\alpha(\Sigma_{i,j} u_i x_{ij} e_{N_0} u_j^*) = \Sigma_{i,j} u_i \phi^0_\alpha(x_{ij}) e_{N_0} u_j^*,$$

where $x_{ij} \in N_0$. It is then immediate to check that $\tilde{\phi}_\alpha$ are completely positive, B-bimodular and check (2.1.0)–(2.1.2) with respect to the canonical trace $\tilde{\tau}$ on $\langle N, e_{N_0} \rangle$ implemented by the trace τ on N (which is clearly Markov by hypothesis). Thus, $\langle N, e_{N_0} \rangle$ has property H relative to B, so that by the first part N has property H relative to B as well (with respect to $\tilde{\tau}_{|N} = \tau$).

4°. For the first implication, note that the condition that B_0 has a finite orthonormal basis over B implies $\mathcal{J}_0(\langle N, B_0 \rangle) \subset \mathcal{J}_0(\langle N, B \rangle)$. Indeed, this follows by first approximating $T \in \mathcal{J}_0(\langle N, B_0 \rangle)$ by linear combination of projections in $J_0(\langle N, B_0 \rangle)$ then noticing that if $\dim({}_{B_0}\mathcal{H}) < \infty$ (respectively, $\dim(\mathcal{H}_{B_0}) < \infty$), then $\dim({}_B\mathcal{H}) < \infty$ (respectively, $\dim(\mathcal{H}_B) < \infty$).

For the opposite implication, let $\{m'_j\}_j$ be a finite orthonormal basis of B_0 over B and recall from ([Po2]) that $b = \Sigma_j m'_j m'^*_j \in \mathcal{Z}(B_0)$ and $b \geq 1$. Also, since for any $T \in B' \cap \langle N, B \rangle$,

$$\Sigma_{i,j} L(m'_j) R(m'^*_i) \circ T \circ L(m'^*_j) R(m'_i) \in B'_0 \cap \langle N, B_0 \rangle$$

(cf. [Po2]), it follows that if we put $m_j = b^{-1/2} m'_j$ then

$$T^0 = \Sigma_{i,j} L(m_j) R(m_i^*) \circ T \circ L(m_j^*) R(m_i) \in B_0' \cap \langle N, B_0 \rangle.$$

This shows that if $\phi^0_\alpha = \Sigma_{i,j} m_j \phi_\alpha(m_j^* \cdot m_i) m_i^*$, then $T^0 = T_{\phi^0_\alpha} \in B_0' \cap \langle N, B_0 \rangle$. Also, if in the above we take T to be a projection with the property that $\mathcal{H} = e(L^2(N,\tau))$ is a finitely generated left-right Hilbert B-module, then the support projection of the corresponding operator T^0 is contained in $\mathcal{H}^0 = \overline{\Sigma_{i,j} m_i \mathcal{H} m_j^*}$. To prove that T^0 is contained in $\mathcal{J}_0(\langle N, B_0 \rangle)$ it is sufficient to show that $\mathcal{H}^0$ is a finitely generated left-right Hilbert B_0-bimodule.

To do this, write first $\mathcal{H}$ as the closure of a finite sum $\Sigma_k \eta_k B$. Then $\mathcal{H}^0$ follows the closure of

$$\Sigma_{i,j} m_i (\Sigma_k \eta_k B) m_j^* = \Sigma_{i,k} (m_i \eta_k (\Sigma_j B m_j^*) = \Sigma_{i,k} m_i \eta_k B_0.$$

This shows that $\dim_{B_0} \mathcal{H}^0 < \infty$. Similarly, $\dim \mathcal{H}^0_{B_0} < \infty$.

Taking linear combinations and norm limits, we get that $T \in \mathcal{J}_0(\langle N, B \rangle)$ implies $T^0 \in \mathcal{J}_0(\langle N, B_0 \rangle)$.

Finally, since $\Sigma_j m_j m_j^* = 1$, by Corollary 1.1.2 the convergence to id_N of ϕ_α implies convergence to id_N of ϕ^0_α. By condition (iii) in 2.3.2°, this implies N has the property H relative to B_0. □

2.4. Proposition. 1°. *If N has property* H *relative to B and $p \in \mathcal{P}(B)$ or $p \in \mathcal{P}(B' \cap N)$, then pNp has property* H *relative to pBp.*

2°. *If $\{p_n\}_n \subset \mathcal{P}(B)$ or $\{p_n\}_n \subset \mathcal{P}(B' \cap N)$ are such that $p_n \uparrow 1$ and $p_n N p_n$ has property* H *relative to $p_n B p_n$, $\forall n$, then N has property* H *relative to B.*

3°. *Assume there exist partial isometries $\{v_n\}_{n \geq 0} \subset N$ such that $v_n^* v_n \in pBp$, $v_n v_n^* \in B$, $v_n B v_n^* = v_n v_n^* B v_n v_n^*, \forall n \geq 0, \Sigma_n v_n v_n^* = 1$ and $B \subset (\{v_n\}_n \cup pBp)''$. If pNp has property* H *relative to pBp then N has property* H *relative to B.*

4°. *If $B \subset N_0 \subset N_1 \subset \ldots$, then $N = \overline{\cup_k N_k}$ has property* H *relative to B (with respect to a trace τ on N) if and only if N_k has property* H *relative to B (with respect to $\tau_{|N_k}$), $\forall k$.*

Proof. 1°. In both cases, if ϕ is B-bimodular completely positive on N then $p\phi(p \cdot p)p$ is a pBp-bimodular completely positive map on pNp. Also, $\tau \circ \phi \leq \tau$ implies $\tau_p \circ (p\phi(p \cdot p)p) \leq \tau_p$, where $\tau_p(x) = \tau(x)/\tau(p), x \in pNp$. Finally, if T_ϕ satisfies condition 5° in 1.3.3 as an element in $\langle N, B \rangle$ then clearly $T_{p\phi(p \cdot p)p}$ satisfies the condition as an element in $\langle pNp, pBp \rangle$.

The case $\{p_n\}_n \subset \mathcal{P}(B' \cap N)$ of 2° follows because if $p \in \mathcal{P}(B' \cap N)$ and ϕ_p is Bp-bimodular completely positive map on pNp, with $\tau_p \circ \phi_p \leq \tau_p$, $\tau(1-p) \leq \delta$, $\|\phi_p(x) - x\|_2 \leq \delta$, $\forall x \in pFp$, for some finite set $F \subset N$, and $T_{\phi_p} \in \mathcal{J}_0(\langle pNp, Bp \rangle)$, then $\phi(y) \overset{\text{def}}{=} \phi_p(pyp) + E_B((1-p)y(1-p))$, $\forall y \in N$

is B-bimodular and satisfies $\tau \circ \phi \leq \tau$, $\|\phi(x) - x\|_2 \leq \varepsilon(\delta), \forall x \in F$ and $T_\phi \in \mathcal{J}_0(\langle N, B\rangle)$, where $\lim_{\delta\to 0}\varepsilon(\delta) = 0$.

To prove 3°, let ϕ_α^p be pBp-bimodular, completely positive maps on pNp with $\tau_p \circ \phi_\alpha^p \leq \tau_p$, $T_{\phi_\alpha^p} \in \mathcal{J}_0(\langle pNp, pBp\rangle)$ and $\phi_\alpha^p \to \mathrm{id}_{pNp}$. Define ϕ_α on N by

$$\phi_\alpha(x) = \Sigma_{i,j} v_i \phi_\alpha^p(v_i^* x v_j) v_j^*, x \in N.$$

Now, $\tau \circ \phi_\alpha \leq \tau$ and $\phi_\alpha \to \mathrm{id}_N$. Also, if $b \in pBp$ or $b = v_i v_j^*$ then $b\phi_\alpha(x) = \phi_\alpha(bx), \phi_\alpha(x)b = \phi_\alpha(xb), \forall x \in N$. Thus, if we denote by B_1 the von Neumann algebra generated by pBp and $\{v_n\}_n$ then ϕ_α is B_1-bimodular.

Also, the same argument as in the last part of the proof of 2.3.4° shows that $T_{\phi_\alpha^p} \in \mathcal{J}_0(\langle pNp, pBp\rangle)$ implies $T_{\phi_\alpha(p_n \cdot p_n)} \in \mathcal{J}_0(\langle p_n N p_n, p_n B_1 p_n\rangle)$, where $p_n = \Sigma_{0\leq k\leq n} v_k^* v_k$. Thus, $p_n N p_n$ has property H relative to $p_n B_1 p_n$. Since $p_n B p_n \subset p_n B_1 p_n$ and $p_n B_1 p_n$ has finite orthonormal basis over $p_n B p_n$, by 2.4.1° above and the first implication in 2.3.4°, it follows that $p_n N p_n$ has property H relative to $p_n B p_n, \forall n$.

For each n let $\{z_k^n\}_k$ be a partition of the identity with projections in $\mathcal{Z}(B)$ such that z_k^n has a finite partition into projections in B that are majorized by $p_n z_k^n$. Thus, there exist finitely many partial isometries $v_0^n = p_n z_k^n, v_1^n, v_2^n, \ldots$ in B such that $v_i^{n*} v_i^n \geq {v_{i+1}^n}^* v_{i+1}^n, \forall i \geq 0$ and such that $\Sigma_i v_i^n v_i^{n*} = z_k^n$. By the first part of the proof, $z_k^n N z_k^n$ has property H relative to $B z_k^n$. By the case of 2° that we have already proved, it follows that N has property H relative to B.

The case $\{p_n\}_n \subset B$ in 2° now follows by using 3°, to reduce the problem to the case where p_n are central in B (as in the proof of the last part of 3°).

4°. The implication $\Longrightarrow$ follows by condition 2.3.3°. The reverse implication follows immediately once we note that if ϕ is a completely positive map on N_k such that $\tau \circ \phi \leq \tau$ and $T_\phi \in \mathcal{J}(\langle N_k, B\rangle)$, then the completely positive map $\phi^k = \phi \circ E_{N_k}$ on N satisfies $\tau \circ \phi^k \leq \tau$ and $T_{\phi^k} \in \mathcal{J}(\langle N, B\rangle)$ (for instance, by 5° in 1.3.3). □

2.5. COROLLARY. *Let $A \subset M$ be a Cartan subalgebra of the type* II_1 *factor M. If $t > 0$ then M^t has property* H *relative to A^t if and only if M has property* H *relative to A* (*see* 1.4 *for the definition of the amplification by t of a Cartan subalgebra*).

Proof. Since the amplification by $1/t$ of $A^t \subset M^t$ is $A \subset M$, it is sufficient to prove one of the implications. Assume M has property H relative to A and let $n \geq t$. By 2.3.2° it follows that $M \otimes M_{n\times n}(\mathbb{C})$ has property H relative to $A \otimes D_n$, where D_n is the diagonal algebra in $M_{n\times n}(\mathbb{C})$. If $p \in A \otimes D_n$ is a projection with $\tau(p) = t/n$ then, by 2.4.1°, $M^t = p(M \otimes M_{n\times n}(\mathbb{C}))p$ has property H relative to $A^t = (A \otimes D_n)p$. □

2.6. Remark. We do not know whether the "smoothness" condition (2.1.0) on the B-bimodular, completely positive, compact maps ϕ_n approximating the identity on N in Definition 2.1 can be removed. This is not known even in the case $B = \mathbb{C}1_N$. In this respect, we mention that in fact, for all later applications, the following weaker "property H"-type condition will be sufficient:

(2.6.1) There exists a net of completely positive B-bimodular maps ϕ_α on N which satisfy condition (2.2.2) so that for all $\{u_n\}_n \subset \mathcal{U}(N)$ with $\lim_{n\to\infty} \|E_B(u_n^* u_m)\|_2 = 0, \forall m$, we have $\lim_{n\to\infty} \|\phi_\alpha(u_n)\|_2 = 0$.

We do not know whether (2.6.1) implies conditions (2.1.0)–(2.1.2), not even in the case N is a factor and $B = \mathbb{C}1_N$.

We mention however that for type II_1 factors N without the property Γ of Murray and von Neumann ([MvN]), the smoothness condition (2.1.0) is automatically satisfied, in case the completely positive map ϕ is sufficiently close to the identity, thus making condition (2.1.0) redundant. Indeed, we have the following observation, essentially due to Connes and Jones ([CJ]):

2.7. Lemma. *If N is a non-Γ type II_1 factor then for any $\varepsilon > 0$ there exist $\delta > 0$ and a finite subset $F \subset \mathcal{U}(N)$ such that the following conditions hold true*:

1°. *If ϕ is a completely positive map satisfying $\|\phi(u) - u\|_2 \leq \delta, \forall u \in F$, then there exists a normal completely positive map ϕ'' on N such that $\phi''(1) \leq 1$, $\tau \circ \phi'' \leq \tau$, $\|\tau \circ \phi'' - \tau\| \leq \varepsilon$, $\Phi'' \leq a_0 \Phi(b_0 \cdot b_0) a_0$, for some $0 \leq a_0, b_0 \leq 1$ in N, and $\|\phi''(x) - x\|_2 \leq \|\phi(x) - x\|_2 + \varepsilon$, $\forall x \in N$, $\|x\| \leq 1$. Moreover, if ϕ is B-bimodular for some $B \subset N$, then ϕ'' can be taken B-bimodular.*

2°. *If $(\mathcal{H}, \xi)$ is a $(B \subset N)$ Hilbert bimodule with $\|u\xi - \xi u\| \leq \delta$, $\forall u \in F$ then $\|\langle \cdot \xi, \xi\rangle - \tau\| \leq \varepsilon$, $\|\langle \xi \cdot, \xi\rangle - \tau\| \leq \varepsilon$.*

Proof. 1°. Since N is non-Γ, by [C2] there exist unitary elements $u_1, u_2, \ldots, u_n$ in N such that if a state $\varphi \in N^*$ satisfies $\|\varphi - \varphi(u \cdot u^*)\| \leq \delta$ then $\|\varphi - \tau\| \leq \varepsilon^2/9$.

Let $F = \{1\} \cup \{u_i\}_i$. Assume ϕ is a completely positive map on N such that $\|\phi(u) - u\|_2 \leq \delta^4/200, \forall u \in F$. Let $a = 1 \vee \phi(1)$ and first define ϕ' on N as in part 2° of Lemma 1.1.2, i.e., $\phi'(x) = a^{-1/2}\phi(x)a^{-1/2}, x \in N$. By 1.1.2, $\phi'(1) \leq 1$ and

$$\|\phi'(x) - x\|_2 \leq \|\phi(x) - x\|_2 + 2\|\phi(1) - 1\|_2^{1/2}\|x\|.$$

Thus, by Corollary 1.1.2 we have for all $x \in N$ with $\|x\| \leq 1$ the estimates:

$$\|\phi'(uxu^*) - u\phi'(x)u^*\|_2 \leq 2(2\|\phi'(u) - u\|_2^2 + 2\|\phi'(u) - u\|_2)^{1/2} \leq \delta.$$

Thus, if $\varphi = \tau \circ \phi'$ then $\|\varphi - \varphi(u_i \cdot u_i^*)\| \leq \delta, \forall i$, implying that $\|\varphi - \tau\| \leq \varepsilon^2/9$.

Thus, if we now take ϕ_1 to be the normal part of ϕ' then we still have $\phi_1(1) \leq 1$, $\|\tau \circ \phi_1 - \tau\| \leq \varepsilon^2/9$ and

$$\|\phi_1(x) - x\|_2 \leq \|\phi(x) - x\|_2 + 2\|\phi(1) - 1\|_2^{1/2} \leq \|\phi(x) - x\|_2 + \delta^2/6,$$

for all $x \in N, \|x\| \leq 1$. Finally, let $b_1 \in L^1(N, \tau)$ be the Radon-Nykodim derivative of $\tau \circ \phi_1$ with respect to τ and define $b = 1 \vee b_1$, $\phi'' = \phi_1(b^{-1/2} \cdot b^{-1/2})$, as in Lemma 1.1.2. Thus, by part 3° of that lemma, all the required conditions are satisfied, by letting $a_0 = a^{-1/2}$, $b_0 = b^{-1/2}$.

2°. This part is now trivial, by part 1° above and 1.1.3. □

3. More on property H

In this section we provide examples of inclusions of finite von Neumann algebras with property H. We also prove that if a type II_1 factor N has property H relative to a maximal abelian $*$-subalgebra B then B is necessarily a Cartan subalgebra of N. Finally, we relate relative property H with notions of relative amenability considered in [Po1,5].

The examples we construct arise from crossed product constructions, being a consequence of the following relation between groups and inclusions of algebras with property H:

3.1. Proposition. *Let Γ_0 be a discrete group and (B, τ_0) a finite von Neumann algebra with a normal faithful tracial state. Let σ be a cocycle action of Γ_0 on (B, τ_0) by τ_0-preserving automorphisms. Then $N = B \rtimes_\sigma \Gamma_0$ has property* H *relative to B if and only if the group Γ_0 has property* H.

Proof. First assume that Γ_0 has property H and let $\varphi_\alpha : \Gamma_0 \to \mathbb{C}$ be unital positive definite functions such that $\varphi_\alpha \in c_0(\Gamma_0)$ and $\varphi_\alpha(g) \to 1, \forall g \in \Gamma_0$. Also, without loss of generality, we may assume $\varphi_\alpha(e) = 1, \forall \alpha$. For each α, let ϕ_α be the associated completely positive map on $N = B \rtimes \Gamma_0$ defined as in Section 1.4, by $\phi(\Sigma_g b_g u_g) = \Sigma_g \varphi(g) b_g u_g$. Note that ϕ_α are unital, trace-preserving and B-bimodular (cf. 1.4).

Also, since $T_{\phi_\alpha} = \Sigma \varphi(g) u_g e_B u_g^*$, it follows that $T_{\phi_\alpha} \in \mathcal{J}(\langle N, B\rangle)$ if and only if $\varphi_\alpha \in c_0(\Gamma_0)$. Finally, since $|1 - \varphi_\alpha(g)| = \|\phi(u_g) - u_g\|_2$, it follows that $\lim_{\alpha\to\infty} \varphi_\alpha(g) = 1, \forall g \in \Gamma_0$, if and only if $\lim_{\alpha\to\infty} \|\phi_\alpha(x) - x\|_2 = 0$, $\forall x \in N$.

In particular, this shows that N has property H relative to B.

Conversely, assume N has property H relative to B and let $\phi_\alpha : N \to N$ be a net of completely positive maps satisfying (2.3.0)–(2.3.2). Let $\varphi_\alpha : \Gamma_0 \to \mathbb{C}$ be defined out of ϕ_α, as in Section 1.4, i.e., by $\varphi_\alpha(g) = \tau(\phi_\alpha(u_g)u_g^*), \forall g \in \Gamma_0$. By 2.6.1°,

$$\lim_{g\to\infty} \|\phi_\alpha(u_g)\|_2 = 0, \forall \alpha.$$

Thus, by the Cauchy-Schwartz inequality,

$$\lim_{g\to\infty} \varphi_\alpha(g) = 0, \forall \alpha.$$

Similarly, $\lim_\alpha \|\phi_\alpha(u_g) - u_g\|_2 = 0$ implies $\lim_\alpha \varphi_\alpha(g) = 1$, thus showing that Γ_0 has property H. □

3.2. *Examples of groups with property* H. The following groups Γ_0 (and thus, by heredity, any of their subgroups as well) are known to have property H, thus giving rise to property H inclusions $B \subset B \rtimes \Gamma_0$ whenever acting (possibly with a cocycle) on a finite von Neumann algebra (B, τ_0), by trace-preserving automorphisms, as in 3.1:

3.2.0. Any amenable group Γ_0 (cf. [BCV]; see also 3.5 below).

3.2.1. $G = \mathbb{F}_n$, for some $2 \leq n \leq \infty$, more generally $\mathbb{F}_S$, for S an arbitrary set of generators (cf. [H]).

3.2.2. Γ_0 a discrete subgroup of $\mathrm{SO}(n, 1)$, for some $n \geq 2$ (cf. [dCaH]).

3.2.3. Γ_0 a discrete subgroup of $\mathrm{SU}(n, 1)$, for some $n \geq 2$ (cf. [CowH]).

3.2.4. $\mathrm{SL}(2, \mathbb{Q})$, more generally $\mathrm{SL}(2, \mathbb{K})$ for any field $\mathbb{K} \subset \mathbb{R}$ which is a finite extension over $\mathbb{Q}$ (by a result of Jolissaint, Julg and Valette, cf. [CCJJV]).

3.2.5. $\Gamma_0 = G_1 *_H G_2$, where G_1, G_2 have property H and $H \subset G_1, H \subset G_2$ is a common finite subgroup (cf. [CCJJV]). In particular $\Gamma_0 = \mathrm{SL}(2, \mathbb{Z})$.

3.2.6. $\Gamma = \Gamma_0 \times \Gamma_1$, with Γ_0, Γ_1 property H groups. Also, $\Gamma = \Gamma_0 \rtimes_\gamma \Gamma_1$, with Γ_0 a property H group and Γ_1 an amenable group acting on it (cf. [CCJJV]).

We refer the reader to the book ([CCJJV]) for a more comprehensive list of groups with the property H. As pointed out there, the only known examples of groups which do not have the Haagerup property are the groups G_0 containing infinite subgroups $G \subset G_0$ such that (G_0, G) has the relative property (T) in the sense of ([Ma, dHVa]; see also the next section).

3.3. *Examples of actions.* We are interested in constructing examples of cocycle actions σ of (property H) groups Γ_0 on finite von Neumann algebras (B, τ) (see e.g. [CJ] for the def. of cocycle actions) that are *ergodic* (i.e., $\sigma_g(b) = b, \forall g \in \Gamma_0$ implies $b \in \mathbb{C}1$) and *properly outer* (i.e., $\sigma_g(b)b_0 = b_0 b$, $\forall b \in B$, implies $g = e$ or $b_0 = 0$). Also, we consider the condition of *weak mixing*, which requires that $\forall F \subset B$ finite and $\forall \varepsilon > 0$, $\exists g \in \Gamma_0$ such that $|\tau(\sigma_g(x)y) - \tau(x)\tau(y)| \leq \varepsilon, \forall x, y \in F$. Weakly mixing actions are clearly ergodic.

Recall that the proper outernes of σ is equivalent to the condition $B' \cap B \rtimes_\sigma \Gamma_0 = \mathcal{Z}(B)$. Also, if σ is a properly outer action, then σ acts ergodically on the center of B if and only if $B \rtimes_\sigma \Gamma_0$ is a factor. Finally, weak-mixing is equivalent to the fact that $L^2(B, \tau)$ has no σ-invariant finite dimensional subspaces other than $\mathbb{C}1$.

Yet another property of actions to be considered is the action σ of Γ_0 on (B, τ) which is *strongly ergodic* if B has no nontrivial approximately σ-invariant sequences; i.e., if $(b_n)_n \in \ell^\infty(\mathbb{N}, B)$ satisfies $\lim_{n\to\infty} \|\sigma_g(b_n) - b_n\|_2 = 0$,

$\forall g \in \Gamma_0$ then $\lim\limits_{n\to\infty} \|b_n - \tau(b_n)1\|_2 = 0$. Note that if we denote $N = B \rtimes_\sigma \Gamma_0$ and take ω to be a free ultrafilter on $\mathbb{N}$, then this condition is equivalent to $N' \cap B^\omega = \mathbb{C}$.

3.3.1. *Bernoulli shifts.* Given any countable discrete group Γ_0 and any finite von Neumann algebra (B_0, τ_0), Γ_0 acts on

$$(B, \tau) = (B, \tau) = \bar{\bigotimes}_{g\in\Gamma_0} (B_0, \tau_0)_g$$

by Bernoulli shifts σ_g; namely, $\sigma_g(\otimes_h x_h) = x'_h$, where $x'_h = x_{g^{-1}h}$.

If B_0 has no atoms or if Γ_0 is an infinite group, then σ is known to be properly outer. Also, if Γ_0 is infinite, then σ is ergodic, in fact even mixing. A Bernoulli shift action is strongly ergodic if and only if Γ_0 is nonamenable (cf. [J2]).

3.3.2. *Actions induced by automorphisms of groups.* Let γ be an action of an infinite group Γ_0 on a group G, by automorphisms. Let also ν be a (normalized) scalar 2-cocycle on G such that $\nu_{\gamma_h(g_1),\gamma_h(g_2)} = \nu_{g_1,g_2}, \forall g_1, g_2 \in G$, $h \in \Gamma_0$. Then γ implements an action of Γ_0 on the "twisted" group von Neumann algebra $L_\nu(G)$, denoted σ_γ, defined by $\sigma_\gamma(h)(\lambda(g)) = \lambda(\gamma_h(g))$, $\forall g \in G$, $h \in \Gamma_0$. Note that σ_γ preserves the canonical trace τ of $L_\nu(G)$.

LEMMA. (i) *The following conditions are equivalent*:

(a) *σ_γ is ergodic*;

(b) *σ_γ is weakly mixing*;

(c) *γ has no finite invariant subsets $\neq \{e\}$*;

(d) *For any finite subset $S \subset G$ there exists $h \in \Gamma_0$ such that $\gamma_h(S) \cap S = \emptyset$.*

(ii) *If $G_1 \subset G$ is so that $\{g_1^{-1} g_0 \gamma_h(g_1) \mid g_1 \in G_1\}$ is infinite, $\forall h \in \Gamma_0 \setminus \{e\}$, $\forall g_0 \in G$ then $L_\nu(G_1)' \cap L_\nu(G) \rtimes_{\sigma_\gamma} \Gamma_0 \subset L_\nu(G)$. In particular, if this holds true for $G_1 = G$ then σ_γ is properly outer. If $\nu = 1$ then the converse holds true as well.*

(iii) *Let $\Gamma_1 \subset \Gamma_0, G_1 \subset G$ be subgroups of finite index such that G_1 is invariant to the restriction of γ to Γ_1. If γ, Γ_0, G satisfy either of the conditions* (c), (d) *in* (i), *or* (ii) *then $\gamma_{|\Gamma_1}, \Gamma_1, G_1$ satisfy that condition as well.*

Proof. (i). (b) $\Longrightarrow$ (a) is trivial.

(a) $\Longrightarrow$ (c). If $\gamma_h(S) = S$, $\forall h \in \Gamma_0$ for some finite set $S \subset G$ with $e \notin S$, then $x = \Sigma_{g\in S}\lambda(g) \notin \mathbb{C}1$ satisfies $\sigma_\gamma(h)(x) = x$, $\forall h \in \Gamma_0$, implying that σ_γ is not ergodic.

(c) $\Longrightarrow$ (d). If $\gamma_h(S) \cap S \neq \emptyset, \forall h \in \Gamma_0$, for some finite set $S \subset G \setminus \{e\}$, then denote by f the characteristic function of S regarded as an element of $\ell^2(G)$.

If we denote by $\tilde{\gamma}$ the action (=representation) of Γ_0 on $\ell^2(G)$ implemented by γ, then $\langle \tilde{\gamma}_h(f), f \rangle \geq 1/|S|, \forall h \in \Gamma_0$. Thus, the element a of minimal norm $\| \ \|_2$ in the weak closure of $\mathrm{co}\{\tilde{\gamma}_h(f) \mid h \in \Gamma_0\} \subset \ell^2(G)$ is nonzero. But then any "level set" of $a \geq 0$ is invariant to γ, showing that (c) doesn't hold true.

(d) $\Longrightarrow$ (b). Let E_0 be a finite set in the unit ball of $L_\nu(G)$, $\varepsilon > 0$ and $F_0 \subset \Gamma_0 \backslash \{e\}$ a finite set as well. Let $S_0 \subset G \backslash \{e\}$ be finite and such that $\|(x - \tau(x)1) - x_{S_0}\|_2 \leq \varepsilon/2, \forall x \in E_0$. By applying the hypothesis to $S = \cup\{\gamma_h(S_0) \mid h \in F_0\}$, we see that there exists $h \in \Gamma_0$ such that $\gamma_h(S) \cap S = \emptyset$. But then $h \notin F_0$ and $\gamma_h(S_0) \cap S_0 = \emptyset$. Also, by Cauchy-Schwartz, for each $x, y \in E_0$,

$$\begin{aligned}
&|\tau(\sigma_\gamma(h)(x)y) - \tau(x)\tau(y)| \\
&\quad \leq \|(x - \tau(x)1) - x_{S_0}\|_2 \|y\|_2 \\
&\qquad + \|(y - \tau(y)1) - y_{S_0}\|_2 \|x\|_2 + |\tau(\sigma_\gamma(h)(x_{S_0}) y_{S_0})| \\
&\quad = \|(x - \tau(x)1) - x_{S_0}\|_2 \|y\|_2 + \|(y - \tau(y)1) - y_{S_0}\|_2 \|x\|_2 \leq \varepsilon.
\end{aligned}$$

(ii) If $y_0 \in L_\nu(G) \rtimes_\sigma \Gamma_0$ satisfies $y_0 x = y_0 x, \forall x \in L_\nu(G_1)$ and $y_0 \notin L_\nu(G)$ then there exists $h \in \Gamma_0, h \neq e$, such that $\sigma_\gamma(h)(x)a = ax, \forall x \in L_\nu(G)$, for some $a \in L_\nu(G), a \neq 0$. This implies $\lambda(\gamma_h(g_1))a\lambda(g_1^{-1}) = a, \forall g_1 \in G_1$. But if this holds true then $\{\gamma_h(g_1)g' g_1^{-1} \mid g_1 \in G_1\}$ must be finite, for any $g' \in G$ in the support of a. When $G_1 = G$ and $\nu = 1$, reversing the implications proves the converse.

(iii) Note first that if $S \subset G_1$ is a finite subset such that $\gamma_h(S) = S$, $\forall h \in \Gamma_1$, the set $\cup_{h \in \Gamma_0} \gamma_h(S)$ is finite as well. Thus, if γ, Γ_0, G check with (c) in (i) so are $\gamma_{|\Gamma_1}, \Gamma_1, G_1$.

Then note that if γ, Γ_0, G verify (ii) and for some $g_1 \in G_1$ the set

$$\{\gamma_h(g) g_1 g^{-1} \mid g \in G_1\}$$

is finite, then the set $\{\gamma_h(g) g_1 g^{-1} \mid g \in G\}$ is finite, a contradiction. □

COROLLARY. *Let $\tilde{\gamma}$ be the action of the group* $\mathrm{SL}(2, \mathbb{R})$ *on* $\mathbb{R}^2$. *For each* $\alpha = e^{2\pi i t} \in \mathbb{T}$, *let* $\tilde{\nu} = \tilde{\nu}(\alpha)$ *be the unique normalized scalar* 2*-cocycle on* $\mathbb{R}^2$ *satisfying the relation* $u_x v_y = \exp(2\pi i t x y) v_y u_x$, *where* $u_x = (x, 0), v_y = (0, y)$ *for* $x, y \in \mathbb{R}$. *Then* $\tilde{\nu}$ *is* $\tilde{\gamma}$*-invariant. Moreover, the following restrictions* $(\gamma, \Gamma_0, G, \nu)$ *of* $(\tilde{\gamma}, \mathrm{SL}(2, \mathbb{R}), \mathbb{R}^2, \tilde{\nu})$ *are strongly ergodic and satisfy conditions* (i), (ii) *in the previous lemma* (*so the corresponding actions* σ_γ *of* Γ_0 *are free and weakly mixing on* $L_\nu(G)$):

(a) $\Gamma_0 = \mathrm{SL}(2, \mathbb{Z}), G = \mathbb{Z}^2$, *or any other subgroup* G *of* $\mathbb{R}^2$ *which is* $\mathrm{SL}(2, \mathbb{Z})$*-invariant, with* γ *the appropriate restriction of* $\tilde{\gamma}$ (*and of* $\tilde{\nu}$).

(b) $\Gamma_0 = \mathrm{SL}(2, \mathbb{Q})$, $G = \mathbb{Q}^2$ (*or any other* $\mathrm{SL}(2, \mathbb{Q})$*-invariant subgroup of* $\mathbb{R}^2$), *with* γ *the appropriate restriction of* $\tilde{\gamma}$.

(c) $\Gamma_0 \simeq \mathbb{F}_n$, *regarded as a subgroup of finite index in* $\mathrm{SL}(2, \mathbb{Z})$ (*see e.g.*, [dHVa]), *and* $G = L((k\mathbb{Z})^2)$, *for some* $k \geq 1$.

Proof. Both conditions (i) and (ii) of the lemma are trivial in cases (a) and (b). Then (c) is just a simple consequence of part (iii) of the lemma. The strong ergodicity of these actions was proved in [S1]. □

3.3.3. *Tensor products of actions.* We often need to take tensor products of actions σ_i of the same group Γ_0 on (B_i, τ_i), $i = 1, 2, \ldots$, thus getting an action $\sigma = \sigma_1 \otimes \sigma_2 \otimes \ldots$ of Γ_0 on $(B, \tau) = (B_1, \tau_1)\overline{\otimes}(B_2, \tau_2)\overline{\otimes} \ldots$.

It is easy to see that the tensor product of a properly outer action σ of a group Γ_0 with any other action σ_0 of Γ_0 gives a properly outer action. In fact, if σ is an action of Γ_0 on (B, τ) and $A_0 \subset B$ is so that $A_0' \cap B \rtimes_\sigma \Gamma_0 \subset B$ then given any action σ_0 of Γ_0 on some (B_0, τ_0), we have $(A_0 \otimes 1)' \cap (B\overline{\otimes}B_0 \rtimes_{\sigma\otimes\sigma_0} \Gamma_0) = (A_0' \cap B)\overline{\otimes}B_0$.

While ergodicity does not always behave well with respect to tensor products, weak-mixing does: If σ is weakly mixing and σ_0 is ergodic then $\sigma \otimes \sigma_0$ is ergodic. If $\sigma_i, i \geq 1$, are weakly mixing then $\otimes_i \sigma_i$ is weakly mixing.

If σ_0 is not strongly ergodic, then $\sigma \otimes \sigma_0$ is not strongly ergodic for all σ. Note that by [CW], if Γ_0 is an infinite property H group then there always exist free ergodic measure-preserving actions σ_0 of Γ_0 on $L^\infty(X, \mu)$ which are not strongly ergodic. Thus, given any σ, $\sigma \otimes \sigma_0$ is not strongly ergodic either.

The following combination of Bernoulli shifts and tensor products of actions will be of interest to us: Let σ_0 be an action of Γ_0 on (B_0, τ_0). Let also Γ_1 be another discrete group and γ an action of Γ_1 on Γ_0 by group automorphisms. (N.B. The action γ may be trivial.) Let σ_1 be the Bernoulli shift action of Γ_1 on $(B, \tau) = \bar{\otimes}_{g_1 \in \Gamma_1} (B_0, \tau_0)_{g_1}$. Let also σ_0^γ be the action of Γ_0 on (B, τ) given by $\sigma_0^\gamma = \otimes_{g_1} \sigma_0 \circ \gamma(g_1)$.

Lemma. 1°. *$\sigma_1(g_1)\sigma_0^\gamma(g_0)\sigma_1(g_1^{-1}) = \sigma_0^\gamma(\gamma(g_1)(g_0))$, for any $g_0 \in \Gamma_0$ and $g_1 \in \Gamma_1$. Thus, $(g_0, g_1) \mapsto \sigma_0^\gamma(g_0)\sigma_1(g_1)$ implements an action $\sigma = \sigma_0 \rtimes_\gamma \sigma_1$ of $\Gamma_0 \rtimes_\gamma \Gamma_1$ on (B, τ).*

2°. *If the group Γ_0 is infinite and the action σ_0 is properly outer then the action σ defined in 1° is properly outer. Moreover, if $B_1 \subset B_0$ satisfies $B_1' \cap (B_0 \rtimes_{\sigma_0} \Gamma_0) \subset B_0$, and B_1 is identified with $\cdots \otimes \mathbb{C} \otimes B_1 \otimes \mathbb{C} \cdots \subset B$, then $B_1' \cap (B \rtimes_\sigma (\Gamma_0 \rtimes \Gamma_1)) = B_1' \cap B$.*

3°. *If the action σ_0 is weakly mixing, or if the group Γ_1 is infinite, then σ is weakly mixing (thus ergodic).*

4°. *If the group Γ_1 is nonamenable, then σ is strongly ergodic.*

Proof. 1° is a straightforward direct calculation.

2° follows once we notice that if Γ_0 is infinite and σ_0 is properly outer, it automatically follows that B_0 has no atomic part. This in turn implies that the Bernoulli shift of Γ_1 on $(B_0, \tau_0)^{\otimes\Gamma_1}$ is a properly outer action, even when Γ_1 is a finite group.

3°. This follows by the observations at the beginning of 3.3.3 and 3.3.1.

4°. This follows from the properties of the Bernoulli shift listed in 3.3.1 (cf. [J2]). □

3.4. PROPOSITION. *If the finite von Neumann algebra N has property* H *relative to its von Neumann subalgebra $B \subset N$, then B is quasiregular in N. If in addition N is a type* II_1 *factor M and $B = A$ is maximal abelian in M, then A is a Cartan subalgebra of M.*

Proof. By Proposition 2.3, given any $x_1, x_2, \ldots, x_n \in N$, with $\|x_i\|_2 \leq 1$, and any $\varepsilon > 0$, there exists an operator $T \in B' \cap J(\langle N, B\rangle)$ such that $\|T\| \leq 1$ and $\|T(\hat{x}_i) - \hat{x}_i\|_2 < \varepsilon^2/32, \forall i$. Since $\|T\| \leq 1$, this implies

$$\begin{aligned}\|T^*(\hat{x}_i) - \hat{x}_i\|_2^2 &= \|T^*(\hat{x}_i)\|_2^* - 2\mathrm{Re}\langle T^*(\hat{x}_i), \hat{x}_i\rangle + \|x_i\|_2^2 \\ &\leq 2\|x_i\|_2^2 - 2\mathrm{Re}\langle T^*(\hat{x}_i), \hat{x}_i\rangle = 2\mathrm{Re}\langle \hat{x}_i, (\hat{x}_i - T(\hat{x}_i))\rangle \\ &\leq 2\|x_i\|_2\|\hat{x}_i - T(\hat{x}_i)\|_2 < \varepsilon^2/16.\end{aligned}$$

As a consequence, we get:

$$\|T^*T(\hat{x}_i) - \hat{x}_i\|_2 \leq \|T^*\|\|T(\hat{x}_i) - \hat{x}_i\|_2 + \|T^*(\hat{x}_i) - \hat{x}_i\|_2 < \varepsilon/2.$$

Thus, if we let e be the spectral projection of T^*T corresponding to $[1-\delta, 1]$ then $\|T^*T - T^*Te\| \leq \delta$, yielding

$$\begin{aligned}\|e(\hat{x}_i) - \hat{x}_i\|_2 &\leq \|T^*T(\hat{x}_i) - \hat{x}_i\|_2 + \|e(T^*T(\hat{x}_i) - (\hat{x}_i))\|_2 + \|T^*T - T^*Te\| \\ &\leq 2\|T^*T(\hat{x}_i) - \hat{x}_i\|_2 + \delta.\end{aligned}$$

But for δ sufficiently small the latter follows less than $\varepsilon, \forall i$. Since the projection e lies in $B' \cap J(\langle N, B\rangle)$, this proves that $\vee\{f \mid f \in \mathcal{P}(B' \cap \langle N, B\rangle), f$ finite projection in $\langle N, B\rangle\} = 1$. By part (iii) of Lemma 1.4.2, this implies B is quasiregular in N. If in addition B is a maximal abelian subalgebra then B follows Cartan by ([PoSh]; see also part (i) in Proposition 1.4.3). □

3.5. *Remarks.* 0°. It is interesting to note that in most known examples of groups Γ_0 with property H, the positive definite functions $\varphi_n \in c_0(\Gamma_0)$ approximating the identity can be chosen in $\ell^p(\Gamma_0)$, for some $p = p(n)$. This is the case, for instance, with the free groups $\mathbb{F}_m$ (cf. [H]), the arithmetic lattices in $\mathrm{SO}(m, 1), \mathrm{SU}(m, 1)$, etc. It is a known fact that if all φ_n can be taken in the same $\ell^p(\Gamma_0)$ (which is easily seen to imply they can be taken in $\ell^2(\Gamma_0), \forall n$), then Γ_0 is amenable. This fact, along with many other similar observations, justifies regarding Haagerup's approximating property as a "weak amenability" property.

1°. The same proof as in [Cho] shows that if $G \subset G_0$ is an inclusion of discrete groups with the property that there exists a net of positive definite functions φ_α on G_0 which are constant on double cosets $Gg_0G, \forall g_0 \in G_0$ (thus factoring out to bounded functions on $G\backslash G_0/G$) and satisfy

(3.5.1′) G is quasi-normal in G_0 and $\varphi_\alpha \in c_0(G\backslash G_0/G), \forall \alpha$;

(3.5.1″) $\lim_{\alpha\to\infty} \varphi_\alpha(g_0) = 1, \forall g_0 \in \Gamma_0$,

then $L_\nu(G_0)$ has property H relative to $L_\nu(G)$ for any scalar 2-cocycle ν for G_0.

When $G \subset G_0$ satisfies the set of conditions (3.5.1) we say that G_0 has *property* H *relative to* G. Note that in the case G is normal in G_0 this is equivalent to G_0/G having property H as a group. (See 3.18–3.20 in [Bo] for similar considerations).

2°. The relative property H for inclusions of finite von Neumann algebras is related to the following notion of relative amenability considered in [Po1,5]: If $B \subset N$ is an inclusion of finite von Neumann algebras then N is *amenable relative to* B if there exists a norm-one projection of $\langle N, B\rangle = (J_N B J_N)' \cap \mathcal{B}(L^2(N))$ onto N, where $L^2(N)$ is the standard representation of N and J_N is the corresponding canonical conjugation.

It is easy to see that if $B \subset N$ is a crossed product inclusion $B \subset B \rtimes_\sigma \Gamma_0$ for some cocycle action σ of a discrete group Γ_0 on (B, τ_0), with τ_0 a faithful normal trace on B, then N is amenable relative to B in the above sense if and only if Γ_0 is amenable, a fact that justifies the terminology. Thus, in this case N amenable relative to B implies N has the property H relative to B.

If N is an arbitrary finite von Neumann algebra with a normal faithful tracial state τ and $B \subset N$ is a von Neumann subalgebra, then the amenability of N relative to B is equivalent to the existence of an N-*hypertrace* on $\langle N, B\rangle$, i.e., a state φ on $\langle N, B\rangle$ with N in its centralizer: $\varphi(xT) = \varphi(Tx), \forall x \in N$, $T \in \langle N, B\rangle$ (cf. [Po1]). It is also easily seen to be equivalent (by using the standard Day-Namioka-Connes trick) to the following Følner type condition: $\forall F \subset \mathcal{U}(N)$ finite and $\varepsilon > 0$, $\exists\ e \in \mathcal{P}(\langle N, B\rangle)$ with $\mathrm{Tr} e < \infty$ such that

(3.5.2) $$\|u_0 e - e u_0\|_{2,\mathrm{Tr}} < \varepsilon \|e\|_{2,\mathrm{Tr}}, \forall u_0 \in F.$$

Note that in case $(B \subset N) = (L_\nu(G) \subset L_\nu(G_0))$ for some inclusion of discrete groups $G \subset G_0$ and a scalar 2-cocycle ν on G_0, condition (3.5.2) amounts to the following: $\forall F \subset G_0$ finite and $\varepsilon > 0$, $\exists E \subset G_0/G$ finite such that

(3.5.2′) $$|g_0 E - E| < \varepsilon |E|, \forall g_0 \in F.$$

This condition for inclusions of groups, for which the terminology used is "*G co-Følner in G_0*", was first considered in [Ey]. It has been used in [CCJJV] to prove that if $G \subset G_0$ is an inclusion of groups, G_0 is amenable relative to G and G has the Haagerup property, then G_0 has Haagerup's property. It would be interesting to know whether a similar result holds true in the case of inclusions of finite von Neumann algebras $B \subset N$.

3°. A stronger version of relative amenability for inclusions of finite von Neumann algebras $B \subset N$ was considered in [Po5], as follows: N is *s-amenable relative to* B if given any finite set of unitaries $F \subset \mathcal{U}(N)$ and any $\varepsilon > 0$ there exists a projection $e \in B' \cap \langle N, B\rangle$, with $\mathrm{Tr} e < \infty$, such that e satisfies the Følner condition (3.5.2) and $\|\mathrm{Tr}(\cdot e)/\mathrm{Tr}(e) - \tau\| \leq \varepsilon$. (No specific terminology is in fact used in [Po5] to nominate this amenability property.) Note that in case $B' \cap N = \mathbb{C}$, we actually have $\mathrm{Tr}(\cdot e)/\mathrm{Tr}(e) = \tau$ for any finite projection e in $B' \cap \langle N, B\rangle$, so the second condition is redundant. The s-amenability of N relative to B is easily seen to be equivalent to: There exists a net of B-bimodular completely positive maps ϕ_α on N such that $\tau \circ \phi_\alpha \leq \tau$, T_{ϕ_α} belong to the (algebraic) ideal generated in $\langle N, B\rangle$ by e_B and

$$\lim_{\alpha\to\infty} \|\phi_\alpha(x) - x\|_2 = 0, \quad \forall x \in N.$$

Thus, N s-amenable relative to B implies N has property H relative to B. Also, one can check that if $N = B \rtimes_\sigma \Gamma_0$ for some cocycle action σ of a discrete group Γ_0 on (B, τ), then N is s-amenable relative to B if and only if N is amenable relative to B and if and only if Γ_0 is an amenable group.

4°. Let $N \subset M$ be an extremal inclusion of type II_1 factors with finite Jones index and let $T = M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}} = S$ be its associated *symmetric enveloping inclusion*, as defined in [Po5]. It was shown in [Po5] that T is quasiregular in S. It was also shown that S is amenable relative to T if and only if S is s-amenable relative to T and if and only if $N \subset M$ has amenable graph $\Gamma_{N,M}$ (or, equivalently, $N \subset M$ has amenable standard invariant $\mathcal{G}_{N,M}$).

By [Po5, §3], if $N \subset M$ is the subfactor associated to a properly outer cocycle action σ of a finitely generated group Γ_0 on a factor $\simeq M$, then the corresponding symmetric enveloping inclusion

$$T = M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}} = S$$

is isomorphic to

$$M \overline{\otimes} M^{\mathrm{op}} \subset M \overline{\otimes} M^{\mathrm{op}} \rtimes_{\sigma\otimes\sigma^{\mathrm{op}}} \Gamma_0,$$

so that T is regular in S. But if $N \subset M$ has index $\lambda^{-1} \geq 4$ and Temperley-Lieb-Jones (TLJ) standard invariant $\mathcal{G}_{N,M} = \mathcal{G}^\lambda$, then the corresponding symmetric enveloping inclusion $T \subset S$ is quasi-regular but not regular. In particular, if $\lambda^{-1} = 4$ then $[S : T] = \infty$ and S has property H relative to T (because $\mathcal{G}_{N,M}$ is amenable by [Po3]), while T is quasi-regular but not regular in S.

5°. By exactly the same arguments as in the case of property (T) for standard lattices considered in [Po5], it can be shown that for an extremal standard lattice $\mathcal{G}$ the following conditions are equivalent: (i). There exists an irreducible subfactor $N \subset M$ with $\mathcal{G}_{N,M} = \mathcal{G}$ such that $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has property

H relative to $M \vee M^{\mathrm{op}}$; (ii). Given any subfactor $N \subset M$ with $\mathcal{G}_{N,M} = \mathcal{G}$, $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has property H relative to $M \vee M^{\mathrm{op}}$. If $\mathcal{G}$ satisfies either of these conditions, we say that the standard lattice $\mathcal{G}$ *has property* H. By 4° above, any amenable $\mathcal{G}$ has property H. We will prove in a forthcoming paper that TLJ standard lattices $\mathcal{G}^{\lambda}$ have the property H, $\forall \lambda^{-1} \geq 4$, while they are known to be amenable if and only if $\lambda^{-1} = 4$ ([Po2], [Po5]).

6°. When applied to the case of Cartan subalgebras $A \subset M$ coming from standard equivalence relations $\mathcal{R}$ (i.e., countable, free, ergodic, measure-preserving) and having trivial 2-cocycle $v \equiv 1$, Definition 2.2 gives the following: A standard equivalence relation $\mathcal{R}$ has *property* H (or is of *Haagerup-type*) if M has property H relative to A. Note that in case $\mathcal{R}$ comes from an action σ of a group Γ_0 then property H of the corresponding $\mathcal{R}$ depends entirely on the group Γ_0, and not on the action (cf. 3.1). Since in addition $A \rtimes \Gamma_0$ has property H relative to A if and only if $p(A \rtimes \Gamma_0)p$ has the property H relative to Ap, for $p \in \mathcal{P}(A)$ (cf. 2.5), it follows that property H for groups is invariant to stable orbit equivalence (this fact was independently noticed by Jolissaint; see [Fu] for a reformulation of stable orbit equivalence as Gromov's "measure equivalence", abbreviated ME).

4. Rigid embeddings: Definitions and properties

In this section we consider a notion of rigid embeddings for finite von Neumann algebras, inspired by the Kazhdan-Margulis example of the rigid embedding of groups $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ ([Ka], [Ma]). Our definition will be the operator algebraic version of the notion of property (T) for pairs of groups in [Ma], in the same spirit Connes and Jones defined property (T) for single von Neumann algebras starting from property (T) of groups, in [CJ]. Thus, as in [CJ], to formulate the definition we use Connes's idea ([C3]) of regarding Hilbert bimodules as an operator algebra substitute for unitary representations of groups, and completely positive maps as an operator algebra substitute for positive definite functions on groups (see Section 1.1). For convenience (and comparison), we first recall the definition of property (T) for inclusions of groups and for single II_1 factors:

4.0.1. *Relative property* (T) *for pairs of groups.* The key part in Kazhdan's proof that the groups $\mathrm{SL}(n,\mathbb{R})$ (resp. $\mathrm{SL}(n,\mathbb{Z})$), $n \geq 3$, have the property (T) consists in showing that representations of $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ that are close to the trivial representation contain copies of the trivial representation of $\mathbb{R}^2$. This type of "relative rigidity" property was later emphasized as a notion in its own right by Margulis ([Ma]; see also [dHVa]), as follows:

Let $G \subset G_0$ be an inclusion of discrete groups. *The pair* (G_0, G) *has relative property* (T) if the following condition holds true:

(4.0.1) There exist finitely many elements $g_1, g_2, \ldots, g_n \in G_0$ and $\varepsilon > 0$, such that if $\pi : G_0 \to \mathcal{U}(\mathcal{H})$ is a unitary representation of the group G_0 on the Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi(g_i)\xi - \xi\| < \varepsilon, \forall i$, then there exists a nonzero vector $\xi_0 \in \mathcal{H}$ such that $\pi(h)\xi_0 = \xi_0, \forall h \in G$.

Due to a recent result of Jolissaint ([Jo2]), the above condition is equivalent to:

(4.0.1′) For any $\varepsilon > 0$, there exist a finite subset $E' \subset G_0$ and $\delta' > 0$ such that if $(\pi, \mathcal{H})$ is a unitary representation of G_0 on the Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a unit vector satisfying $\|\pi(h)\xi - \xi\| \leq \delta', \forall h \in E'$, then $\|\pi(g)\xi - \xi\| \leq \varepsilon, \forall g \in G$.

Note that the equivalence of (4.0.1) and (4.0.1′) is easy to establish in case G is a normal subgroup of G_0 (exactly the same argument as in [DeKi] will do), but it is less simple in general (cf. [Jo2]). On the other hand, condition (4.0.1′) is easily seen to be equivalent to:

(4.0.1″) For any $\varepsilon > 0$, there exist a finite subset $E \subset G_0$ and $\delta > 0$ such that if φ is a positive definite function on G_0 with $|\varphi(h) - 1| \leq \delta, \forall h \in E$ then $|\varphi(g) - 1| \leq \varepsilon, \forall g \in G$.

Note that in the case $G = G_0$, condition (4.0.1) amounts to the usual *property* T *of Kazhdan* for the group G_0 ([Ka]; see also [DeKi], [Zi]). We will in fact also use the following alternative terminologies to designate property (T) pairs: $G \subset G_0$ *is a property* (T) (or *rigid*) *embedding*, or G is a *relatively rigid subgroup* of G_0.

4.0.2. *Property* (T) *for factors.* The abstract definition of property (T) for a single von Neumann factor is due to Connes and Jones ([CJ]): A type II_1 factor N *has property* (T) if the following condition holds true:

(4.0.2) There exist finitely many elements $x_1, x_2, \ldots, x_n \in N$ and $\varepsilon_0 > 0$ such that if $\mathcal{H}$ is an N Hilbert bimodule with a unit vector $\xi \in \mathcal{H}$ such that $\|x_i\xi - \xi x_i\| \leq \varepsilon_0, \forall i$, then $\mathcal{H}$ contains a nonzero vector ξ_0 such that $x\xi_0 = \xi_0 x, \forall x \in N$.

Connes and Jones have also proved that the fixed vector ξ_0 can be taken close to the initial ξ, if the "critical set" in N is taken sufficiently large and the "commutation constant" sufficiently small ([CJ]), by showing that (4.0.2) is equivalent to the following:

(4.0.2′) For any $\varepsilon > 0$, there exist a finite subset $F' \subset N$ and $\delta' > 0$ such that if $\mathcal{H}$ is a Hilbert N-bimodule and $\xi \in \mathcal{H}$ is a unit vector satisfying $\|y\xi - \xi y\| \leq \delta', \forall y \in F'$, then there exists $\xi_0 \in \mathcal{H}$ such that $x\xi_0 = \xi_0 x, \forall x \in N$ and $\|\xi - \xi_0\| \leq \varepsilon$.

For inclusions of finite von Neumann algebras, we first establish the equivalence of several conditions:

4.1. Proposition. *Let N be a finite von Neumann algebra with countable decomposable center* (*i.e.*, *with normal faithful tracial states*). *Let $B \subset N$ be a von Neumann subalgebra. The following conditions are equivalent*:

1°. *There exists a normal faithful tracial state τ on N such that*: $\forall \varepsilon > 0$, $\exists F' \subset N$ *finite and* $\delta' > 0$ *such that if $\mathcal{H}$ is a Hilbert N-bimodule with a vector $\xi \in \mathcal{H}$ satisfying the conditions* $\|\langle \cdot \xi, \xi \rangle - \tau\| \leq \delta', \|\langle \xi \cdot, \xi \rangle - \tau\| \leq \delta'$ *and* $\|y\xi - \xi y\| \leq \delta', \forall y \in F'$ *then* $\exists \xi_0 \in \mathcal{H}$ *such that* $\|\xi_0 - \xi\| \leq \varepsilon$ *and* $b\xi_0 = \xi_0 b$, $\forall b \in B$.

2°. *There exists a normal faithful tracial state τ on N such that*: $\forall \varepsilon > 0$, $\exists F \subset N$ *finite and* $\delta > 0$ *such that if $\phi : N \to N$ is a normal*, *completely positive map with* $\tau \circ \phi \leq \tau, \phi(1) \leq 1$ *and* $\|\phi(x) - x\|_2 \leq \delta, \forall x \in F$, *then* $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B$, $\|b\| \leq 1$.

3°. *Condition* 1° *above is satisfied for any normal faithful tracial state τ on N.*

4°. *Condition* 2° *above is satisfied for any normal faithful tracial state τ on N.*

Proof. We first prove that condition 1° holds true for a specific normal faithful tracial state τ if and only if condition 2° holds true for that same trace. Then we prove $1^\circ \Leftrightarrow 3^\circ$, which due to the equivalence of 1° and 2° ends the proof of the proposition.

$2^\circ \implies 1^\circ$. By part 1° of Lemma 1.1.3, we may assume the vectors $\xi \in \mathcal{H}$ in condition 4.1.1° also satisfy $\langle \cdot \xi, \xi \rangle \leq \tau$ and $\langle \xi \cdot, \xi \rangle \leq \tau$, in addition to the given properties. We take $x_1, x_2, \ldots, x_n$ to be an enumeration of the finite set F and for any given $\varepsilon' > 0$ let δ' be the δ given by condition 2° for $\varepsilon = {\varepsilon'}^2/4$. By part 2° of Lemma 1.1.3, such a vector ξ gives rise to a completely positive map $\phi = \phi_{(\mathcal{H},\xi)}$ on N which satisfies condition 4.1.2°. Thus, $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B, \|b\| \leq 1$. By Lemma 1.1.2, this implies that ξ (which is equal to ξ_ϕ) satisfies $\|u\xi u^* - \xi\| \leq 2\varepsilon^{1/2} \leq \varepsilon', \forall u \in \mathcal{U}(B)$. By averaging over the unitaries $u \in \mathcal{U}(B)$, we see that there exists $\xi_0 \in \mathcal{H}$ such that $\|\xi_0 - \xi\| \leq \varepsilon'$ and ξ_0 commutes with B.

$1^\circ \implies 2^\circ$. Let $\varepsilon > 0$. Define $F(\varepsilon) = F'(\varepsilon^2/8), \delta(\varepsilon) = \delta'(\varepsilon^2/8)^2/4$. Let then $\phi : N \to N$ be a completely positive map satisfying the conditions 2° for this $F(\varepsilon)$ and $\delta(\varepsilon)$. Let $(\mathcal{H}_\phi, \xi_\phi)$ be constructed as in 1.1.2. By part 4° of Lemma 1.1.2, we have for $x \in F(\varepsilon)$ the inequality

$$\|x\xi_\phi - \xi_\phi x\| \leq 2\|\phi(x) - x\|_2^{1/2} \leq \delta'(\varepsilon^2/8).$$

Thus, there exists $\xi_0 \in \mathcal{H}_\phi$ such that $\|\xi_\phi - \xi_0\| \leq \varepsilon^2/8$ and $b\xi_0 = \xi_0 b$, $\forall b \in B$. But then, if $u \in \mathcal{U}(B)$ we get

$$\begin{aligned}\|\phi(u)-u\|_2^2 &\leq 2-2\mathrm{Re}\langle u\xi_\phi u^*, \xi_\phi\rangle \\ &\leq 2-2\|\xi_0\|^2+4\|\xi_0-\xi_\phi\| \leq 2-2(1-\varepsilon^2/8)^2+4\varepsilon^2/8 < \varepsilon^2.\end{aligned}$$

Since any $b \in B$, $\|b\| \leq 1$, is a convex combination of unitary elements, we are done.

$3^\circ \implies 1^\circ$ is trivial. To prove $1^\circ \implies 3^\circ$, let τ_0 be a normal faithful tracial state on N. We have to show that $\forall\varepsilon > 0$, $\exists F_0 \subset N$ finite and $\delta_0 > 0$ such that if $\mathcal{H}$ is a Hilbert N-bimodule with $\eta \in \mathcal{H}$ satisfying $\|\langle \cdot\eta, \eta\rangle - \tau_0\| \leq \delta_0, \|\langle \eta\cdot, \eta\rangle - \tau_0\| \leq \delta_0$ and $\|y\eta - \eta y\| \leq \delta_0, \forall y \in F_0$ then $\exists\eta_0 \in \mathcal{H}$ such that $\|\eta_0 - \eta\| \leq \varepsilon$ and $b\eta_0 = \eta_0 b, \forall b \in B$.

By Sakai's Radon-Nykodim theorem, τ_0 is of the form $\tau_0 = \tau(\cdot a_0)$ for some $a_0 \in L^1(\mathcal{Z}(N), \tau)_+$ with $\tau(a_0) = 1$. It is clearly sufficient to prove the statement in the case a_0 is bounded and with finite spectrum (thus bounded away from 0 as well). Also, by taking the spectral projections of a_0 to be in F_0 and slightly perturbing η, we may assume η commutes with a_0. We take $F_0 = F'(\varepsilon/\|a_0\|)$ and $\delta_0 = \delta'(\varepsilon/\|a_0\|)/\|a_0^{-1}\|$, as given by condition 1° for τ.

Let $\xi = a_0^{-1/2}\eta = \eta a_0^{-1/2}$. Then

$$\|\langle\cdot\xi, \xi\rangle - \tau\| = \|\langle\cdot a_0^{-1}\eta, \eta\rangle - \tau_0(\cdot a_0^{-1})\| \leq \|a_0^{-1}\|(\delta'/\|a_0^{-1}\|) = \delta'.$$

Similarly, $\|\langle\xi\cdot, \xi\rangle - \tau\| \leq \delta'$. Also, for $y \in F_0$,

$$\|[y, \xi]\| = \|[y, a_0^{-1/2}\eta]\| \leq \|a_0^{-1/2}\|(\delta'/\|a_0^{-1}\|) \leq \delta'.$$

Thus, by 1°, there exists $\xi_0 \in \mathcal{H}$ such that $b\xi_0 = \xi_0 b, \forall b \in B$ and $\|\xi_0 - \xi\| \leq \varepsilon/\|a_0\|$. In addition, since ξ commutes with a_0, we may assume ξ_0 also does. Let $\eta_0 = a_0^{1/2}\xi_0$. Then η_0 still commutes with B and we have the estimates:

$$\|\eta_0 - \eta\| = \|a_0^{1/2}\xi_0 - a_0^{1/2}\xi\| \leq \|a_0^{1/2}\|\|\xi_0 - \xi\| \leq \|a_0^{1/2}\|(\varepsilon/\|a_0\|) \leq \varepsilon. \quad \square$$

4.2. *Definitions.* Let N be a countable decomposable finite von Neumann algebra and $B \subset N$ a von Neumann subalgebra.

4.2.1. $B \subset N$ is *a rigid* (or *property* (T)) *embedding* (or, B is a *relatively rigid subalgebra* of N, or *the pair* (N, B) *has the relative property* (T)) if $B \subset N$ satisfies the equivalent conditions 4.1.

4.2.2. If N is a finite factor and $\varepsilon_0 > 0$ then $B \subset N$ is ε_0-*rigid* if $\exists F \subset N$ finite and $\delta > 0$ such that if ϕ is a completely positive map on N with $\phi(1) \leq 1$, $\tau \circ \phi \leq \tau$ and $\|\phi(x) - x\|_2 \leq \delta, \forall x \in F$ then $\|\phi(b) - b\|_2 \leq \varepsilon_0, \forall b \in B, \|b\| \leq 1$.

Note that if N is a finite factor then an embedding $B \subset N$ is rigid if and only if it is ε_0-rigid $\forall\varepsilon_0 > 0$. We see that if some additional conditions are satisfied (e.g., B regular, in N, in 4.3.2°; B, N group algebras coming from a group-subgroup situation, in 5.1) then $B \subset N$, ε_0-rigid, for $\varepsilon_0 = 1/3$, is in fact sufficient to insure that $B \subset N$ is rigid.

4.3. Theorem. *Let N be a separable type II_1 factor and $B \subset N$ a von Neumann subalgebra.*

1°. *Assume $B \subset N$ is either rigid or ε_0-rigid, for some $\varepsilon_0 < 1$, with B semi-regular. Then $N' \cap N^\omega = N' \cap (B' \cap N)^\omega$, for any free ultrafilter ω on $\mathbb{N}$. If in addition to either of the above conditions B also satisfies $B' \cap N = \mathcal{Z}(B)$ (resp. $B' \cap N = \mathbb{C}$) then N is non-McDuff (resp. non-Γ).*

2°. *Assume that either B is regular in N or that $\mathcal{N}_N(B)' \cap N^\omega = \mathbb{C}$. Then $B \subset N$ is rigid if and only if it is ε_0-rigid for some $\varepsilon_0 \leq 1/3$.*

Proof. 1°. Assume first that $B \subset N$ is rigid. By applying 4.1.2° to the completely positive maps $\phi = \mathrm{Ad}u$ for $u \in \mathcal{U}(N)$, it follows that for any $\varepsilon > 0$ there exist $\delta > 0$ and $x_1, x_2, \ldots, x_n \in N$ such that if $u \in \mathcal{U}(N)$ satisfies

$$\|ux_i - x_iu\|_2 \leq \delta, \forall i,$$

then

$$\|ub - bu\|_2 \leq \varepsilon, \forall b \in B, \|b\| \leq 1.$$

In particular, $\|vuv^* - u\|_2 \leq \varepsilon, \forall v \in \mathcal{U}(B)$. Thus, by taking averages over the unitaries $v \in B$, we see that $\|E_{B' \cap N}(u) - u\|_2 \leq \varepsilon$. Thus, if $(u_n) \subset \mathcal{U}(N)$ is a central sequence of unitary elements in N, i.e.,

$$\lim_{n \to \infty} \|[x, u_n]\|_2 = 0, \forall x \in N,$$

then

$$\lim_{n \to \infty} \|u_n - E_{B' \cap N}(u_n)\|_2 = 0.$$

Assume now that $B \subset N$ is ε_0-rigid, with $\varepsilon_0 < 1$, and that $\mathcal{N}(B)' \cap N = \mathbb{C}$. We proceed by contradiction, assuming there exists $u = (u_n)_n \in \mathcal{U}(N' \cap N^\omega)$ such that $u \notin (B' \cap N)^\omega$. By taking a suitable subsequence of (u_n), we see that there exists $(v_n)_n \subset \mathcal{U}(N)$ such that $\lim_{n \to \infty} \|[v_n, x]\|_2 = 0$, $\forall x \in N$, and $\|E_{B' \cap N}(v_n)\|_2 \leq c, \forall n$, for some $c < 1$. It further follows that given any separable von Neumann subalgebra $P \subset N^\omega$ there exist $k_1 \ll k_2 \ll \ldots$ such that $\lim_{n \to \infty} \|[v_{k_n}, y_n]\|_2 = 0$, $\forall y = (y_n)_n \in P$.

Moreover, if $P \subset \mathcal{N}_{N^\omega}(B^\omega)''$, then the subsequence $v' = (v_{k_n})_n$ can be taken so that we also have $[E_{B^{\omega\prime} \cap N^\omega}(v'), y] = 0$, $\forall y \in P$. To see this, let $S \subset \mathcal{N}(B^\omega)$ be a countable set such that the von Neumann algebra P_0 generated by S contains P. Choose $k_n \uparrow \infty$ so that $\lim_{n \to \infty} \|[v_{k_n}, w_n]\|_2 = 0$, $\forall w = (w_n)_n \in S$. We then have

$$wE_{B^{\omega\prime} \cap N^\omega}(v')w^* = wE_{B^{\omega\prime} \cap N^\omega}(w^*v'w)w^* = E_{B^{\omega\prime} \cap N^\omega}(v'), \quad \forall w \in S.$$

Thus $[E_{B^{\omega\prime} \cap N^\omega}(v'), S] = 0$, implying $[E_{B^{\omega\prime} \cap N^\omega}(v'), P_0] = 0$ as well.

Now notice that $(B' \cap N)^\omega = B^{\omega\prime} \cap N^\omega$ (see e.g. [Po2]). As a consequence, since $E_{B^{\omega\prime} \cap N^\omega}(x)$ is the element of minimal norm $\| \quad \|_2$ in $\overline{\mathrm{co}}^w\{wxw^* \mid w \in \mathcal{U}(B^\omega)\}$, which in turn can be realized as a $\| \quad \|_2$-limit of convex combinations of the form wxw^* with w in a suitable countable subset of $\mathcal{U}(B^\omega)$, it follows that for any $x \in N^\omega$ there exists a separable von Neumann subalgebra $P \in B^\omega$ such that $E_{P' \cap N^\omega}(x) = E_{B^{\omega\prime} \cap N^\omega}(x)$. Also, since $\mathcal{N}_{N^\omega}(B^\omega) \supset \prod_{n\to\infty} \mathcal{N}_N(B)$, $\mathcal{N}(B^\omega)''$ follows a factor and for any $x' \in N^\omega$ there exists a separable von Neumann subalgebra P_0 generated by a countable subset in $\mathcal{N}(B^\omega)$ such that $P_0 \supset P$ and $E_{P_0' \cap N^\omega}(x') = \tau(x')1$.

Using all the above, we prove the following statement:

(4.3.1$'$) If $x \in N^\omega$ then there exists a subsequence $(v_{k_n})_n$ of $(v_n)_n$ such that $v' = (v_{k_n})_n \in N^\omega$ satisfies $\|E_{B^{\omega\prime}\cap N^\omega}(xv')\|_2 = \|E_{B^{\omega\prime}\cap N^\omega}(x)\|_2 \|E_{B^{\omega\prime}\cap N^\omega}(v')\|_2$.

To see this, take first a separable von Neumann subalgebra $P \subset B^\omega$ such that $E_{B^{\omega\prime}\cap N^\omega}(x) = E_{P'\cap N^\omega}(x)$. Then take P_0 a von Neumann algebra generated by a countable subset in $\mathcal{N}(B^\omega)$ such that $P_0 \supset P$ and $E_{P_0'\cap N^\omega}(x') = \tau(x')1$ where $x' = E_{B^{\omega\prime}\cap N^\omega}(x)^* E_{B^{\omega\prime}\cap N^\omega}(x)$. Since $B^{\omega\prime} \cap N^\omega \subset P' \cap N^\omega$, if the subsequence $(v_{k_n})_n$ is chosen such that $[v', P_0] = 0$ then $[v', P] = 0$ and

$$\begin{aligned} E_{B^{\omega\prime}\cap N^\omega}(xv') &= E_{B^{\omega\prime}\cap N^\omega}(E_{P'\cap N^\omega}(xv')) = E_{B^{\omega\prime}\cap N^\omega}(E_{P'\cap N^\omega}(x)v') \\ &= E_{B^{\omega\prime}\cap N^\omega}(E_{B^{\omega\prime}\cap N^\omega}(x)v') = E_{B^{\omega\prime}\cap N^\omega}(x)E_{B^{\omega\prime}\cap N^\omega}(v'). \end{aligned}$$

Also, since $y' = E_{B^{\omega\prime}\cap N^\omega}(v')E_{B^{\omega\prime}\cap N^\omega}(v')^*$ satisfies $[y', P_0] = 0$,

$$\begin{aligned} \|E_{B^{\omega\prime}\cap N^\omega}(xv')\|_2^2 &= \|E_{B^{\omega\prime}\cap N^\omega}(x)E_{B^{\omega\prime}\cap N^\omega}(v')\|_2^2 = \tau(x'y') = \tau(E_{P_0'\cap N^\omega}(x'y')) \\ &= \tau(E_{P_0'\cap N^\omega}(x')y') = \tau(x')\tau(y') = \|E_{B^{\omega\prime}\cap N^\omega}(x)\|_2^2 \|E_{B^{\omega\prime}\cap N^\omega}(v')\|_2^2. \end{aligned}$$

Now, by applying (4.3.1$'$) recursively, we can choose a subsequence v^1 of $v = (v_n)_n$, then v^2 of v^1, etc, such that

$$\|E_{B^{\omega\prime}\cap N^\omega}(\Pi_{j=1}^m v^j)\|_2 = \Pi_{j=1}^m \|E_{B^{\omega\prime}\cap N^\omega}(v^j)\|_2 = \|E_{B^{\omega\prime}\cap N^\omega}(v)\|_2^m \leq c^m.$$

Take m so that $c^m < 1 - \varepsilon_0$ and put $w = v^1 v^2 \ldots v^m$, $w = (w_n)_n$, with $w_n \in \mathcal{U}(N)$, and $\phi_n = \mathrm{Ad}(w_n)$. Then,

$$\text{(4.3.1}''\text{)} \qquad \lim_{n\to\omega} \|E_{B'\cap N}(w_n)\|_2 < 1 - \varepsilon_0, \lim_{n\to\infty} \|\phi_n(x) - x\|_2 = 0, \forall x \in N.$$

By the ε_0-rigidity of $B \subset N$ the second condition in (4.3.1$''$) implies that for large enough n we have

$$\|uw_n u^* - w_n\|_2 = \|w_n u w_n^* - u\|_2 = \|\phi_n(u) - u\|_2 \leq \varepsilon_0, \forall u \in \mathcal{U}(B).$$

After we take convex combinations over u, this yields $\|E_{B'\cap N}(w_n) - w_n\|_2 \leq \varepsilon_0$. Thus $\|E_{B'\cap N}(w_n)\|_2 \geq 1 - \varepsilon_0$ for all large enough n, contradicting the first condition in (4.3.1$''$).

2°. We need to show that if $(\psi_n)_n$ are completely positive maps on N satisfying

$$\text{(a)} \qquad \tau \circ \psi_n \leq \tau, \psi_n(1) \leq 1, \forall n, \lim_{n\to\infty} \|\psi_n(x) - x\|_2 = 0, \forall x \in N,$$

then $\lim_{n\to\infty}\sup(\{\|\psi_n(b) - b\|_2 \mid b \in B, \|b\| \leq 1\}) = 0$. Assume by contradiction that there exist $(\psi_n)_n$ satisfying (a) but

$$\text{(b)} \qquad \inf_n(\sup\{\|\psi_n(b) - b\|_2 \mid b \in B, \|b\| \leq 1\}) > 0.$$

Note that by the ε_0-rigidity of $B \subset N$, (a) implies

$$\text{(c)} \qquad \limsup_{n\to\infty}(\sup\{\|\psi_n(b) - b\|_2 \mid b \in B, \|b\| \leq 1\}) \leq \varepsilon_0.$$

If $(\psi_n)_n$ satisfies $\tau \circ \psi_n \leq \tau, \psi_n(1) \leq 1, \forall n$ in (a) then

$$\text{(d)} \qquad \Psi((x_n)_n) \overset{\text{def}}{=} (\psi_n(x_n))_n, (x_n)_n \in N^\omega,$$

gives a well defined completely positive map Ψ on N^ω with $\tau \circ \Psi \leq \tau$, $\Psi(1) \leq 1$. Thus, the fixed point set $(N^\omega)^\Psi \overset{\text{def}}{=} \{x \in N^\omega \mid \Psi(x) = x\}$ is a von Neumann algebra. If $(\psi_n)_n$ also satisfies the last condition in (a), then $N \subset (N^\omega)^\Psi$. In particular $\Psi(1) = 1$ which together with $\|T_\Psi\| \leq 1$ implies $T_\Psi{}^*(\hat{1}) = \hat{1}$; equivalently $\Psi^*(1) = 1$, i.e., $\tau \circ \Psi = \tau$.

If in addition to (a) the sequence $(\psi_n)_n$ satisfies (b), then $B^\omega \not\subset (N^\omega)^\Psi$. Let us prove that the ε_0-rigidity of $B \subset N$ entails

$$\text{(e)} \qquad B^\omega \subset_{\varepsilon_0} (N^\omega)^\Psi.$$

For ψ a map on an algebra denote by ψ^m the m-time composition $\psi \circ \psi \cdots \circ \psi$. Then note that for each $m \geq 1$ the sequence $(\psi_n^m)_n$ still satisfies (a), and thus, by ε_0-rigidity, (c) as well. Thus

$$\|\Psi^k(b) - b\|_2 \leq \varepsilon_0, \quad \forall b \in B^\omega, \|b\| \leq 1.$$

But by von Neumann's ergodic theorem applied to Ψ and $x \in N^\omega$, we have

$$\text{(f)} \qquad \lim_{n\to\infty} \|m^{-1}\Sigma_{k=1}^m \Psi^k(x) - E_{(N^\omega)^\Psi}(x)\|_2 = 0,$$

which together with the previous estimate shows that for $x = b \in B^\omega, \|b\| \leq 1$, we have $\|E_{(N^\omega)^\Psi}(b) - b\|_2 \leq \varepsilon_0$, i.e., (e).

The assumption $\mathcal{N}(B)' \cap N^\omega = \mathbb{C}$ implies in particular that $N' \cap N^\omega = \mathbb{C} \subset (N^\omega)^\Psi$. We next prove that B regular in N implies $N' \cap N^\omega \subset (N^\omega)^\Psi$ as well, for any Ψ on N^ω associated as in (d) to a sequence $(\psi_n)_n$ satisfying (a). Denote $P = (N^\omega)^\Psi$ and assume by contradiction that $N' \cap N^\omega \not\subseteq P$. Since $N' \cap N^\omega$ and P make a commuting square, this implies there exists $x \in N' \cap N^\omega$, $x \neq 0$, such that $E_P(x) = 0$. Moreover, we may assume $x = (x_n)_n$ satisfies $x_n = x_n^*, \|x_n\|_2 = 1, \forall n$.

By using (f), we can choose "rapidly" increasing $k_1 \ll k_2 \ll \dots$ and "slowly" nondecreasing $m_1 \leq m_2 \leq \dots$ such that the sequence of completely positive maps $\psi_n' = (m_n)^{-1}\Sigma_{j=1}^{m_n}\psi_{k_n}^j$ satisfies (a) and $\lim_{n\to\infty}\|\psi_n'(x_n')\|_2 = 0$, with $\lim_{n\to\infty}\|[x_n', y]\|_2 = 0, \forall y \in N$, $\lim_{n\to\infty}\tau((x_n')^k) = \tau(x^k), \forall k$, where $x_n' = x_{k_n}$.

Denote by Ψ_1 the completely positive map on N^ω associated with $(\psi'_n)_n$, as in (d), and put $X = X_1 = (x'_n)_n \in N^\omega$. Since each separable von Neumann subalgebra of N^ω is contained in a separable factor and since for each separable $Q \subset N^\omega$ there exists $j_1 \ll j_2 \ll \dots$ such that $X' = (x'_{j_n})_n \in Q' \cap N^\omega$, it follows that there exist separable factors $Q_0 = N \subset Q_1 \subset \cdots \subset Q_{m-1}$ in N^ω and consecutive subsequences of indices $(j, 1) < (j, 2) < \dots$, for $j = 1, 2, \dots, m$, with $(1, n) = n$, such that $X_j = (x'_{j,n})_n \in N^\omega$ satisfy $X_1, X_2, \dots, X_j \in Q_j$, $[Q_j, X_{j+1}] = 0$, for $0 \leq j \leq m-1$. Denote by Ψ_j the completely positive map on N^ω associated with $(\psi'_{j,n})_n$, noticing that each one of these sequences checks (a). Thus for each $j = 1, 2, \dots, m$ we have $\Psi_j(x) = x, \forall x \in N$ and $\Psi_j(X_j) = 0$. Moreover, the von Neumann algebra generated by $X_1, X_2, \dots, X_m$ in N^ω is isomorphic to the tensor power $(A(X), \tau)^{\otimes m}$, where $A(X)$ is the von Neumann algebra generated by $X \in N^\omega$.

Let $\tilde{X} = m^{-1/2}\Sigma_{j=1}^m X_j$ and $\tilde{\Psi} = m^{-1}\Sigma_{j=1}^m \Psi_j$. Let $P_j = (N^\omega)^{\Psi_j}, 1 \leq j \leq m$, and $\tilde{P} = (N^\omega)^{\tilde{\Psi}}$. By $(a)-(e)$, $\tilde{P}, P_j$ are von Neumann algebras containing N and $B^\omega \subset_{\varepsilon_0} P_j, \tilde{P}$. Moreover, since by convexity we have $\tilde{\Psi}(Y) = Y$ if and only if $\Psi_j(Y) = Y$, $\forall j$, it follows that $\tilde{P} = \cap_j P_j$. Thus, since $\Psi_j(X_j) = 0$ implies $E_{P_j}(X_j) = 0$, it follows that $E_{\tilde{P}}(\tilde{X}) = 0$.

But by the central limit theorem, as $m \to \infty$, $\tilde{X}$ gets closer and closer (in distribution) to an element $Y = Y^*$ with Gaussian spectral distribution, independently of X. Let $Y' = Ye_{[-2,2]}(Y)$ and $\|Y\|_2^2 = t$. By using Mathematica, one finds $t > 0.731$. Thus, for large enough m, $\tilde{X}' = \tilde{X}e_{[-2,2]}(\tilde{X})$ satisfies $\|\tilde{X}'\|_2^2 = t_-$ with t_- close to t. Let $\tilde{X}'' = \tilde{X} - \tilde{X}'$ and note that $\tilde{X}'\tilde{X}'' = 0$, so that $\|\tilde{X}'\|_2^2 + \|\tilde{X}''\|_2^2 = \|\tilde{X}\|_2^2 = 1$. Also,

$$E_{\tilde{P}}(\tilde{X}') = E_{\tilde{P}}(\tilde{X} - \tilde{X}'') = -E_{\tilde{P}}(\tilde{X}''),$$

implying that $\|E_{\tilde{P}}(\tilde{X}')\|_2^2 \leq \|\tilde{X}''\|_2^2 = 1 - t_-$. Altogether

$$\|\tilde{X}' - E_{\tilde{P}}(\tilde{X}')\|_2^2 = \|\tilde{X}'\|_2^2 - \|E_{\tilde{P}}(\tilde{X}')\|_2^2 \geq 2t_- - 1.$$

Since $\tilde{X}_1 \in N' \cap N^\omega \subset B^\omega$ and $\|\tilde{X}_1\| = 2$, if we take $\tilde{X}_0 = \tilde{X}'/2$ then $\|\tilde{X}_0\| = 1$ and $\|\tilde{X}_0 - E_{\tilde{P}}(\tilde{X}_0)\|_2^2 = (2t_- - 1)/4 > (1/3)^2$, this contradicts $B^\omega \subset_{1/3} \tilde{P}$.

This finishes the proof of the fact that $N' \cap N^\omega \subset (N^\omega)^\Psi$, independently of Ψ, for arbitrary $(\psi_n)_n$ checking (a). Thus $P = \cap_i (N^\omega)^{\Psi_i}$, where $\Psi_i, i \in \mathcal{I}$, is the family of all completely positive maps on N^ω coming from sequences $(\psi_{i,n})_n$ satisfying (a), still satisfies $N, N' \cap N^\omega \subset P$. Let us show that this newly designated P still satisfies $B^\omega \subset_{\varepsilon_0} P$. To see this, take a finite subset $I \subset \mathcal{I}$ and consider the sequence $\psi_{I,n} = |I|^{-1}\Sigma_i \psi_{i,n}$, which clearly satisfies (a). Thus, the associated completely positive map Ψ_I on N^ω satisfies

$$\|E_{P_I}(b) - b\|_2 \leq \varepsilon_0, \forall b \in B^\omega, \|b\| \leq 1.$$

where $P_I \overset{\text{def}}{=} (N^\omega)^{\Psi_I}$. Since $|I|^{-1}\Sigma_i \Psi_i(x) = x$ if and only if $\Psi_i(x) = x, \forall i \in I$, we have $P_I = \bigcap_{i\in I} (N^\omega)^{\Psi_i}$. But $P_I \downarrow P$ as $I \uparrow \mathcal{I}$, implying that $\|E_P(b) - b\|_2 \le \varepsilon_0, \forall b$, as well.

Denote $\mathcal{U}_0 = \mathcal{N}(B) \cup \mathcal{U}(\mathcal{N}(B)' \cap (B^\omega)' \cap N^\omega)$, $N_0 = \mathcal{U}_0''$ and notice that $v(B^\omega)v^* = B^\omega, \forall v \in \mathcal{U}_0$. Also, if we let $M = N^\omega$, $Q = B^\omega$, then by 1° both the assumption $\mathcal{N}(B)' \cap N^\omega = \mathbb{C}$ and $\mathcal{N}_N(B)'' = N$ imply that $\mathcal{U}_0 \subset P$ and $N_0' \cap M = \mathcal{Z}(N_0)$, $[\mathcal{Z}(N_0), Q] = 0$ are satisfied. Thus, A.3 applies and we get a nonzero projection $p \in \mathcal{Z}(N_0)$ such that $Qp \subset P$. In the case $\mathcal{N}(B)' \cap N^\omega = \mathbb{C}$, this implies $p = 1$ and we get $B^\omega \subset P$, a contradiction which finishes the proof under this assumption.

If B is regular in N, then the group $\mathcal{N}(B) = \mathcal{N}(B \vee B' \cap N)$ generates the factor N, a fact that is easily seen to imply $\mathcal{N}_{N^\omega}(B^\omega)' \cap N^\omega = \mathbb{C}$. This implies there exists a countable subgroup $\mathcal{U}_1 \subset \mathcal{N}(B^\omega)$ such that $\tau(p)1$ is a limit in the norm-$\| \quad \|_2$ of convex combinations of elements of the form $u_1 p u_1^*$, $u_1 \in \mathcal{U}_1$. Let then $(\psi_n)_n$ be the sequence of completely positive maps satisfying $(a)-(b)$ at the beginning of the proof, with $b_n \in B, \|b_n\| \le 1$, $\|\psi_n(b_n) - b_n\|_2 \ge c > 0$, $\forall n$. If we choose a sufficiently rapidly increasing $k_1 \ll k_2 \ll \dots$, then the completely positive map Ψ' associated with $(\psi_{k_n})_n$ as in (d) has both N and $\mathcal{U}_1$ in the fixed point algebra $(N^\omega)^{\Psi'}$. But since $P \subset (N^\omega)^{\Psi'}$, it follows that $(N^\omega)^{\Psi'}$ contains $B^\omega p$, and thus $u_1(B^\omega p)u_1^* = B^\omega(u_1 p u_1^*), \forall u_1 \in \mathcal{U}_1$ as well. This implies $B^\omega \subset (N^\omega)^{\Psi'}$, contradicting $\|\Psi'(b') - b'\|_2 \ge c > 0$, where $b' = (b_{k_n})_n \in B^\omega$. □

4.4. Theorem. *Let N be a type* II$_1$ *factor and $B \subset N$ a von Neumann subalgebra such that $B' \cap N = \mathcal{Z}(B)$ and such that the normalizer of B in N, $\mathcal{N}(B)$, acts ergodically on the center of B. Let $\mathcal{G}_B \subset$* Aut$N$ *be the group generated by* Int$N$ *and by the automorphisms of N that leave all elements of B fixed. If $B \subset N$ is ε_0-rigid for some $\varepsilon_0 < 1$ then $\mathcal{G}_B$ is open and closed in* Aut$N$. *Thus,* Aut$N/\mathcal{G}_B$ *is countable.*

Proof. By applying condition 4.2.2° to the completely positive maps $\theta \in$ Aut$N$, it follows that there exist $\delta > 0$ and $x_1, x_2, \dots, x_n \in N$ such that if $\|\theta(x_i) - x_i\|_2 \le \delta$ then

$$\|\theta(u) - u\|_2 \le \varepsilon_0, \forall u \in \mathcal{U}(B).$$

Thus, if k denotes the unique element of minimal norm $\| \quad \|_2$ in $K = \overline{\text{co}}^w\{\theta(u)u^* \mid u \in \mathcal{U}(B)\}$ then $\|k - 1\|_2 \le \varepsilon_0$ and thus $k \ne 0$. Also, since $\theta(u)Ku^* \subset K$ and $\|\theta(u)ku^*\|_2 = \|k\|_2, \forall u \in \mathcal{U}(B)$, by the uniqueness of k it follows that $\theta(u)ku^* = u$, or equivalently $\theta(u)k = ku$, for all $u \in \mathcal{U}(B)$. By a standard trick, if $v \in N$ is the (nonzero) partial isometry in the polar decomposition of k, then $\theta(u)v = vu, \forall u \in \mathcal{U}(B)$, $v^*v \in B' \cap N = \mathcal{Z}(B), vv^* \in \theta(B)' \cap N = \theta(\mathcal{Z}(B))$. Since $\mathcal{N}(B)$ acts ergodically on $\mathcal{Z}(B)$ (equivalently, $\mathcal{N}(B)' \cap N = \mathbb{C}$), there exist finitely many partial isometries $v_0 = v^*v, v_1, v_2, \dots, v_n \in N$ such that $v_i^* v_i = v^* v, 0 \leq i \leq n-1$, $v_n^* v_n \in \mathcal{Z}(B) v^* v$ and $v_i v_i^* \in \mathcal{Z}(B), v_i B v_i^* = B v_i v_i^*, \forall i$.

If we then define $w = \Sigma_i \theta(v_i) v v_i^*$, an easy calculation shows that w is a unitary element and $wbw^* = \theta(b), \forall b \in B$. □

4.5. Proposition. *Let N be a type II_1 factor and $B \subset N$ a rigid embedding.*

1°. *For any $\varepsilon_0 > 0$ there exist $F_0 \subset N$ and $\delta_0 > 0$ such that if $N_0 \subset N$ is a subfactor with $B \subset N_0$ and $F_0 \subset_{\delta_0} N_0$, then $B \subset N_0$ is ε_0-rigid. In particular, if $N_k \subset N, k \geq 1$ is an increasing sequence of subfactors such that $B \subset N_k, \forall k$, and $\overline{\cup_k N_k} = N$, then for any $\varepsilon_0 > 0$ there exists k_0 such that $B \subset N_k$ is ε_0-rigid $\forall k \geq k_0$.*

2°. *Assume in addition that B is regular in N and $B' \cap N = \mathcal{Z}(B)$. For any $\varepsilon > 0$ there exist a finite subset $F \subset N$ and $\delta > 0$ such that if $N_0 \subset N$ is a subfactor with $N_0' \cap N = \mathbb{C}$ and $F \subset_\delta N_0$ then there exists $u \in \mathcal{U}(N)$ such that $\|u-1\|_2 \leq \varepsilon$ and $uBu^* \subset N_0$, with $uBu^* \subset N_0$ rigid embedding. If in addition $N_0 \supset B$ then one can take $u = 1$. In particular, if $N_k \subset N$ is an increasing sequence of subfactors with $N_k' \cap N = \mathbb{C}$ and $N_k \uparrow N$ then there exist k_0 such that $u_k B u_k^* \subset N_k$ rigid, $\forall k \geq k_0$, for some $u_k \in \mathcal{U}(N)$, $\|u_k - 1\|_2 \to 0$, and such that if $N_k \supset B, \forall k$, then $B \subset N_k$ rigid $\forall k \geq k_0$.*

Proof. 1°. With the notation of 4.1.2°, for the critical sets $F(\varepsilon')$ and constants $\delta(\varepsilon')$ for $B \subset N$, let $F_0 = F(\varepsilon_0)$ and $\delta_0 = \delta(\varepsilon_0)/2$. Let $N_0 \subset N$ be a von Neumann algebra with $B \subset N_0$, $\|E_{N_0}(y) - y\|_2 \leq \delta_0, \forall y \in F_0$. We want to prove that $B \subset N_0$ is ε_0-rigid by showing that if ϕ_0 is a completely positive map on N_0 with $\phi_0(1) \leq 1, \tau \circ \phi_0 \leq \tau$ and

$$\|\phi_0(y_0) - y_0\|_2 \leq \delta(\varepsilon_0)/2, \forall y_0 \in E_{N_0}(F_0),$$

then $\|\phi_0(b) - b\|_2 \leq \varepsilon_0, \forall b \in B, \|b\| \leq 1$. To this end let $\phi = \phi_0 \circ E_{N_0}$, which we regard as a completely positive map from N into N ($\supset N_0$). Clearly $\phi(1) \leq 1, \tau \circ \phi \leq \tau$. Also, for $y \in F(\varepsilon_0)$ we have

$$\|\phi(y) - y\|_2 \leq \|\phi_0(E_{N_0}(y)) - E_{N_0}(y)\|_2 + \|E_{N_0}(y) - y\|_2 \leq \delta(\varepsilon_0).$$

Thus, $\|\phi(b) - b\|_2 \leq \varepsilon_0, \forall b \in B, \|b\| \leq 1$. Since for $b \in B$ we have $\phi(b) = \phi_0(b)$, we are done.

2°. By application of condition 4.1.2° to the completely positive maps E_{N_0}, it follows that if we denote $\varepsilon(N_0) = \sup\{\|E_{N_0}(b) - b\|_2 \mid b \in B, \|b\| \leq 1\}$, then $\varepsilon(N_0) \to 0$ as $E_{N_0} \to \mathrm{id}_N$. Thus, by Theorem A.2 it follows that there exist unitary elements $u = u(N_0) \in N$ such that $uBu^* \subset N_0$ and $\|u-1\|_2 \to 0$. Moreover, by 1° above and 4.3.2°, it follows that $uBu^* \subset N_0$ (equivalently, $B \subset uN_0u^*$) is a rigid embedding when N_0 is close enough to N on an appropriate finite set of elements. The fact that B is still regular in N_0 is a consequence of ([JPo]). The last part is now trivial. □

4.6. PROPOSITION. 1°. *($B_i \subset N_i$) are rigid embeddings for $i = 1, 2$ if and only if ($B_1 \overline{\otimes} B_2 \subset N_1 \overline{\otimes} N_2$) is a rigid embedding.*

2°. *Let $B \subset N_0 \subset N$. If $B \subset N_0$ is a rigid embedding then $B \subset N$ is a rigid embedding. Conversely, if we assume $N_0 \subset N$ is a λ-Markov inclusion* ([Po2]), *i.e., N has an orthonormal basis $\{m_j\}_j$ with $\Sigma m_j m_j^* = \lambda^{-1}$ for some constant $\lambda > 0$ (e.g., if N, N_0 are factors and $[N : N_0] < \infty$) then $B \subset N$ is a rigid embedding, implies $B \subset N_0$ is a rigid embedding.*

3°. *Let $B \subset B_0 \subset N$. If $B_0 \subset N$ is a rigid embedding, then $B \subset N$ is a rigid embedding. Conversely, if B_0 has a finite orthonormal basis with respect to B and $B \subset N$ is a rigid embedding, then $B_0 \subset N$ is a rigid embedding.*

Proof. 1°. Assume first that ($B_i \subset N_i$) are rigid embeddings, τ_i, for $i = 1, 2$. Let $\varepsilon > 0$ and $F_i'(\varepsilon/2), \delta_i'(\varepsilon/2)$ be the critical sets and constants for $B_i \subset N_i$, as given by 4.1.1°, for $\varepsilon/2$. Define $F' = F_1' \otimes 1 \cup 1 \otimes F_2', \delta' = \min\{\delta_1', \delta_2'\}$.

Put $N = N_1 \overline{\otimes} N_2, B = B_1 \overline{\otimes} B_2$. Let $\mathcal{H}$ be a Hilbert N-bimodule with a vector $\xi \in \mathcal{H}$ which satisfies conditions 4.1.1° with respect to the trace $\tau_1 \otimes \tau_2$, for F', δ'. In particular, $\mathcal{H}$ is a Hilbert N_i bimodule, for $i = 1, 2$. Thus, if we denote by p_i the orthogonal projection of $\mathcal{H}$ onto the Hilbert subspace of all vectors in $\mathcal{H}$ that commute with B_i, then $\|\xi - p_i(\xi)\|_2 \leq \varepsilon/2, i = 1, 2$, for any vector $\xi \in \mathcal{H}$ that satisfies 4.1.1° for the above F', δ'. But p_1 and p_2 are commuting projections and $p_1 p_2$ projects onto the Hilbert subspace of vectors commuting with both B_1 and B_2, i.e., onto the Hilbert subspace of vectors commuting with B. Since

$$\begin{aligned}\|\xi - p_1 p_2(\xi)\| &\leq \|\xi - p_1(\xi)\| + \|p_1(\xi) - p_1(p_2(\xi))\| \\ &\leq \|\xi - p_1(\xi)\| + \|\xi - p_2(\xi)\| \leq \varepsilon,\end{aligned}$$

it follows that $B \subset N$ satisfies 4.1.1°.

Assume now that $B \subset N$ satisfies 4.1.2° for some trace τ. Since $N_1 \otimes N_2$ is a dense $*$-subalgebra in N, by using Kaplanski's density theorem and the fact that in 4.1.2° we only have to deal with completely positive maps ϕ satisfying $\tau \circ \phi \leq \tau, \phi(1) \leq 1$, it follows that we may assume the critical set $F'(\varepsilon)$ is contained in $N_1 \otimes N_2$ (by diminishing if necessary the corresponding $\delta'(\varepsilon)$).

Let $F_i' \subset N_i$ be finite subsets such that $F' \subset \mathrm{sp} F_1' \otimes F_2'$. There clearly exist $\delta_i' > 0$ such that if ϕ_i are completely positive maps on N_i with $\tau \circ \phi_i \leq \tau$, $\phi_i(1) \leq 1$ and $\|\phi_i(x_i) - x_i\|_2 \leq \delta_i', \forall x_i \in F_i', i = 1, 2$, then $\phi = \phi_1 \otimes \phi_2$ satisfies $\|\phi(x) - x\|_2 \leq \delta', \forall x \in F'$. Thus, $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B, \|b\| \leq 1$. Taking $b \in B_i$, we get $\|\phi_i(b) - b\|_2 \leq \varepsilon, \forall b \in B_i, \|b\| \leq 1, i = 1, 2$.

2°. The implication $\Longrightarrow$ follows by noticing that if ϕ is a completely positive map on N such that $\phi(1) \leq 1$ and $\tau \circ \phi \leq \tau$ then for $x \in N_0$ we have $\|E_{N_0}(\phi(x)) - x\|_2 \leq \|\phi(x) - x\|_2$ while for $b \in B, \|b\| \leq 1$, we have

$$\|\phi(b) - b\|_2^2 \leq \|E_{N_0}(\phi(b)) - b\|_2^2 + 2\|E_{N_0}(\phi(b)) - b\|_2.$$

Thus, if 4.1.2° is satisfied for $B \subset N_0$ with critical set $F_0(\varepsilon)$ and constant $\delta_0(\varepsilon)$, then 4.1.2° holds true for $B \subset N$ for the same set F_0 but constant $\delta(\varepsilon) = \delta_0(\varepsilon)^2/3$.

To prove the opposite implication, let $e = e_{N_0}$ be the Jones projection corresponding to $N_0 \subset N$ and $N_1 = \langle N, e\rangle$ the basic construction. Since $N_0 \subset N$ is λ-Markov, there exists a unique trace τ on N_1 extending the trace τ of N and such that $E_N^\tau(e) = \lambda 1$.

We may assume 1 belongs to the orthonormal basis $\{m_j\}_j$ of N over N_0. Note that $x = \Sigma_j m_j E_N(m_j{}^* x), \forall x \in N$. Any element $X \in N_1$ can be uniquely written in the form $X = \Sigma_{i,j} m_i x_{ij} e m_j^*$ for some $x_{ij} \in p_i N_0 p_j$, where $p_i = E_{N_0}(m_i^* m_i) \in \mathcal{P}(N_0)$. Also, if $x \in N$ then

$$x = (\Sigma_i m_i e m_i^*) x (\Sigma_j m_j e m_j^*) = \Sigma_{i,j} m_i E_{N_0}(m_i^* x m_j) e m_j{}^*. \tag{4.6.2$'$}$$

For each completely positive map ϕ on N_0 define $\tilde\phi$ on N_1 by

$$\tilde\phi(\Sigma_{i,j} m_i x_{ij} e m_j^*) = \Sigma_{i,j} m_i \phi(x_{ij}) e m_j^*. \tag{4.6.2$''$}$$

Note that if $X = \Sigma_{i,j} m_i x_{ij} e m_j^* \geq 0$ and $\tau \circ \phi \leq \tau$ then

$$\begin{aligned}\tau(\tilde\phi(X) &= \tau(\tilde\phi(\Sigma_{i,j} m_i x_{ij} e m_j^*)) = \lambda \Sigma_{i,j} \tau(m_i \phi(x_{ij}) m_j^*)\\ &= \lambda \Sigma_{i,j} \tau(m_i \phi(x_{ij}) m_j^*) = \lambda \Sigma_i \tau(\phi(x_{ii}) p_i)\\ &\leq \lambda \Sigma_i \tau(\phi(x_{ii})) \leq \lambda \Sigma_i \tau(x_{ii}) = \tau(X).\end{aligned}$$

Similarly, if $\phi(1) \leq 1$ then $\tilde\phi(1) \leq 1$.

Let now $\varepsilon > 0$ be given. Let $F = F(\lambda\varepsilon^2/3), \delta = \delta(\lambda\varepsilon^2/3)$ be the critical set and constant for $B \subset N$, corresponding to $\lambda\varepsilon^2/3$. Let $F_0 = \{E_{N_0}(m_i^* x m_j) \mid \forall i, j, \forall x \in F\}$. Formulas $(4.6.2'), (4.6.2'')$ above show that there exists $\delta_0 > 0$ such that if $\|\phi(x) - x\|_2 \leq \delta_0, \forall x \in F_0$ then $\|\tilde\phi(x) - x\|_2 \leq \delta, \forall x \in F$.

We claim that F_0, δ_0 give the critical set and constant for $B \subset N_0$, corresponding to ε. To see this, note first that by the proof of $\Longrightarrow$ above we get $\|\tilde\phi(b) - b\|_2 \leq \lambda^{1/2}\varepsilon, \forall b \in B, \|b\| \leq 1$. By $(4.6.2'')$ this gives

$$\begin{aligned}\lambda^{1/2}\|\phi(b) - b\|_2 &= \|(\phi(b) - b)e\|_2\\ &\leq \|\tilde\phi(b) - b\|_2 \leq \lambda^{1/2}\varepsilon.\end{aligned}$$

3°. The first implication is trivial. The opposite implication is equally evident, if we take the critical set $F_0(\varepsilon)$ and constant $\delta_0(\varepsilon)$ for $B_0 \subset N$ to be defined as follows: We first choose $\delta_1 > 0$ with the property that if ϕ is a completely positive map on N with $\tau\circ\phi \leq \tau, \phi(1) \leq 1$ and $\|\phi(b) - b\|_2 \leq \delta_1, \forall b \in B, \|b\| \leq 1$ and $\|\phi(b_j^0) - b_j^0\|_2 \leq \delta_1$, then $\|\phi(b_0) - b_0\|_2 \leq \varepsilon, \forall b_0 \in B_0, \|b_0\| \leq 1$ ($\{b_j^0\}_j$ denotes here the orthonormal basis of B_0 over B). We then define $F_0(\varepsilon) = F(\delta_1) \cup \{b_j^0\}_j$ and put $\delta_0(\varepsilon) = \delta_1$. $\square$

4.7. Proposition. 1°. *If $B \subset N$ and $\{p_n\}_n$ is an increasing sequence of projections in N, with $p_n \uparrow 1$, which lie either in B or in $B' \cap N$, and with* *the property that* $p_nBp_n \subset p_nNp_n$ *are rigid embeddings*, $\forall n$, *then* $B \subset N$ *is a rigid embedding. In particular*, *if* B *is atomic then* $B \subset N$ *is rigid.*

2°. *If* $B \subset N$ *is a rigid embedding and* $p \in \mathcal{P}(B)$ *or* $p \in \mathcal{P}(B' \cap N)$ *then* $pBp \subset pNp$ *is a rigid embedding.*

3°. *Let* $B \subset N$ *and* $p \in \mathcal{P}(B)$. *Assume there exist partial isometries* $\{v_n\}_{n\geq 0} \subset N$ *such that* $v_n^* v_n \in pBp$, $v_n v_n^* \in B$, $v_n B v_n^* = v_n v_n^* B v_n v_n^*, \forall n \geq 0$, $\Sigma_n v_n v_n^* = 1$ *and* $B \subset (\{v_n\}_n \cup pBp)''$. *If* $pBp \subset pNp$ *is a rigid embedding then* $B \subset N$ *is a rigid embedding.*

Proof. 1°. Notice first that if ϕ is completely positive on N and $\tau \circ \phi \leq \tau, \phi(1) \leq 1$ then $\tau(p_n\phi(p_nxp_n)p_n) \leq \tau(\phi(p_nxp_n)) \leq \tau(p_nxp_n), \forall x \geq 0$, and $p_n\phi(p_n)p_n \leq p_n$. Then we simply take the critical set and constant for $B \subset N$ to be the critical set and constant for $p_nBp_n \subset p_nNp_n$, with n sufficiently large, and apply the above to deduce that for ϕ satisfying the conditions for this set and constant, $p_n\phi(p_n \cdot p_n)p_n$ follows uniformly close to the identity on the unit ball of p_nBp_n.

The case when B is atomic is now trivial, when we first apply 4.6.3° and then the first part of the proof.

2°. The statement is clearly true in case $p \in \mathcal{Z}(N)$. Assume next that $p \in \mathcal{P}(B)$. By part 1° above, we may suppose pBp has some nonatomic part.

Since there exist projections $z_n \in \mathcal{Z}(N)$ with $z_n \uparrow 1$ such that each z_n is a sum of finitely many projections in Bz_n which are majorized by pz_n in B, by 1° above it is sufficient to prove the case when there exist partial isometries $v_0 = p, v_1, v_2, \ldots, v_n \in B$ such that $v_i^*v_i \leq p, \forall i$, $\Sigma_i v_i v_i^* = 1$.

Let then $\varepsilon > 0$. Let $F = F(\varepsilon\tau(p))$ and $\delta = \delta(\varepsilon\tau(p))$ be given by 4.1.2° for the inclusion $B \subset N$. Let also $F_0 = \{v_i^* x v_j \mid 1 \leq i, j \leq n, x \in F\}$. We show that F_0 and $\delta_0 = \delta$ are good for $pBp \subset pNp$. Thus, let ϕ be a completely positive map on pNp such that $\phi(p) \leq p$, $\tau_p \circ \phi \leq \tau_p$ and $\|\phi(y) - y\|_2 \leq \delta_0$, $\forall y \in F_0$. Define $\tilde{\phi}(x) = \Sigma_{i,j} v_i \phi(v_i^* x v_j) v_j^*$. As in the proof of 4.6.1°, we get $\tau \circ \tilde{\phi}(x) \leq \tau(x), \forall x \in N$ and $\tilde{\phi}(1) \leq 1$.

An easy calculation shows that $\|\tilde{\phi}(x) - x\|_2 \leq \delta$ for $x \in F$. Thus,

$$\|\tilde{\phi}(b) - b\|_2 \leq \varepsilon\tau(p), \quad \forall b \in B, \|b\| \leq 1.$$

But this implies $\|\phi(pbp) - pbp\|_2 \leq \varepsilon\|p\|_2, \forall b \in B, \|b\| \leq 1$ as well.

If the projection p lies in $B' \cap N$ then by the last part of 4.6.3° the subalgebra $B_0 \subset N$ generated by B and $\{1, p\}$ is rigid in N. But then we apply the first part to get $pBp = pB_0p$ is rigid in pNp.

3°. By 1° above, it is sufficient to prove the case when the set $\{v_i\}_i$ is finite. Let $\varepsilon > 0$ and $F_p = F(\varepsilon'), \delta_p = \delta(\varepsilon')$ be given by condition 4.1.2°, for $pBp \subset pNp$ and $\varepsilon' = \varepsilon(\min_i \tau(v_i v_i^*)/2)^2$. Then define $F_0 = F_p \cup \{v_i\}_{0 \leq i \leq n}$. If ϕ is a completely positive map on N such that $\|\phi(x) - x\|_2 \leq \delta_0$ with $\delta_0 \leq \delta_p \tau(p)^{1/2}, \forall x \in F_0$, then in particular we have $\|\phi(x) - x\|_{2,p} \leq \delta_p, \forall x \in F_p$.

Thus, $\|p\phi(b)p - b\|_2 \leq \varepsilon(\min_i\|v_iv_i^*\|_2/2)^2, \forall b \in pBp, \|pbp\| \leq 1$. This easily gives $\|\phi(b) - b\|_2 \leq \varepsilon$ for all b in the von Neumann algebra $B_0 = \Sigma_{i,j}v_iBv_j^*$, generated by pBp and $\{v_i\}_{0\leq i\leq n}$, with $\|b\| \leq 1$ (in fact, even for all $b \in B_0$ that satisfy $\|v_i^*bv_j\| \leq 1, \forall i,j$). Thus, $B_0 \subset N$ is rigid, so that by 4.6.3°, $B \subset N$ is rigid as well. ☐

5. More on rigid embeddings

In this section we produce examples of rigid inclusions of algebras, by using results of Kazhdan ([Ka]) and Valette ([Va]), which provide examples of property (T) inclusions of groups, and the result below, which establishes the link between property (T) for an inclusion of groups and property (T) (rigidity) for the inclusion of the corresponding group von Neumann algebras (as defined in (4.2)).

5.1. PROPOSITION. *Let $G \subset G_0$ be an inclusion of discrete groups and ν a scalar* 2*-cocycle for G_0. Denote $(B \subset N) = (L_\nu(G) \subset L_\nu(G_0))$. Conditions* (a)–(d) *are equivalent. If in addition $L_\nu(G_0)$ is a factor then* (a)–(e) *are equivalent.*

(a) *(G_0, G) is a property* (T) *pair, i.e., $G \subset G_0$ satisfies the equivalent conditions* (4.0.1), (4.0.1′), (4.0.1″).

(b) *$B \subset N$ is a rigid embedding of algebras.*

(c) *For any $\varepsilon > 0$ there exist a finite set $F' \subset N$ and $\delta' > 0$ such that if $\mathcal{H}$ is a Hilbert N-bimodule with a unit vector $\xi \in \mathcal{H}$ satisfying $\|x_i\xi - \xi x_i\| \leq \delta', \forall i$ then there exists a vector $\xi_0 \in \mathcal{H}$ such that $\|\xi_0 - \xi\| \leq \varepsilon$ and $b\xi_0 = \xi_0 b, \forall b \in B$.*

(d) *For any $\varepsilon > 0$ there exist a finite set $F \subset N$ and $\delta > 0$ such that if $\phi : N \to N$ is a normal completely positive map with $\|\phi(x) - x\|_2 \leq \delta, \forall x \in F$, then $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B$, $\|b\| \leq 1$.*

(e) *$L_\nu(G) \subset L_\nu(G_0)$ is ε_0-rigid for some $\varepsilon_0 < 1$.*

Proof. To prove (a) $\Longrightarrow$ (c), we prove (4.0.1′) $\Longrightarrow$ (c). Let $\varepsilon > 0$ and let $E \subset G_0$, $\delta' > 0$ be given by (4.0.1′), for this ε. Let $\mathcal{H}$ be a Hilbert N bimodule with $\xi \in \mathcal{H}$, $\|\xi\| = 1$, $\|u_h\xi - \xi u_h\| \leq \delta', \forall h \in E'$. Taking $\pi(g)\eta = u_g\eta u_g^*$, $\eta \in \mathcal{H}$, $g \in G_0$, gives a representation of G_0 on $\mathcal{H}$, with $\|\pi(h)\xi - \xi\| = \|u_h\xi - \xi u_h\| \leq \delta'$. Thus, there exists $\xi_0 \in \mathcal{H}$ fixed by $\pi(G)$ (equivalently, $u_g\xi_0 = \xi_0u_g, \forall g \in G$) and such that $\|\xi_0 - \xi\| \leq \varepsilon$.

(b) $\Longrightarrow$ (a). We prove that 4.1.1° implies (4.0.1′). Let $\varepsilon > 0$. By part 1° in Lemma 1.1.3 and by Kaplanski's density theorem (which implies that the unit ball of the group algebra $\mathbb{C}_\nu G_0$ is dense in the unit ball of $L_\nu(G_0)$ in the norm $\| \quad \|_2$), it follows that given any ε there exist a finite set $E_0 \subset G_0$ and $\delta_0 > 0$, $\delta_0 \leq \varepsilon$, such that if $\mathcal{H}$ is an $L_\nu(G_0)$ Hilbert bimodule with $\xi \in \mathcal{H}$ a unit vector which is left and right δ_0-tracial and satisfies $\|u_h\xi - \xi u_h\| \leq \delta_0, \forall h \in E_0$, then there exists $\xi_1 \in \mathcal{H}$ such that $\|\xi_1 - \xi\| \leq \varepsilon/2$ and $b\xi_1 = \xi_1 b, \forall b \in L_\nu(G), \|b\| \leq 1$.

Let then $(\pi_0, \mathcal{H}_0, \xi_0)$ be a cyclic representation of G_0 such that $\|\pi_0(h)\xi_0 - \xi_0\| \leq \delta_0, \forall h \in E_0$. Let $(\mathcal{H}_{\pi_0}, \xi_{\pi_0})$ be the pointed Hilbert $L_\nu(G_0)$ bimodule, as defined in 1.4. We clearly have $\|u_h\xi_{\pi_0} - \xi_{\pi_0}u_h\| = \|\pi_0(h)\xi_0 - \xi_0\| \leq \delta_0, \forall h \in E_0$, by the definitions. Thus, there exists $\xi_1 \in \mathcal{H}_{\pi_0}$ such that $\|\xi_1 - \xi_{\pi_0}\| \leq \varepsilon/2$ and ξ_1 commutes with $L_\nu(G)$. But this implies that for all $g \in G$

$$\begin{aligned}\|\pi_0(g)\xi_0 - \xi_0\| &= \|u_g\xi_{\pi_0} - \xi_{\pi_0}u_g\| \\ &\leq \|[u_g, (\xi_{\pi_0} - \xi_1)]\| + \|[u_g, \xi_1]\| \leq 2\varepsilon/2 = \varepsilon.\end{aligned}$$

Taking the element of minimal norm ξ_2 in the weak closure of $\mathrm{co}\{\pi_0(g)\xi_1 \mid g \in G\}$, it follows that ξ_2 is fixed by π_0 and $\|\xi_2 - \xi_0\| \leq \varepsilon$.

The implications (c) $\implies$ (b), (d) $\implies$ (b), (b) $\implies$ (e) (the latter for factorial $L_\nu(G_0)$) are trivial.

To prove (a) $\implies$ (d), we prove $(4.0.1')$ $\implies$ (d). Let $\varepsilon > 0$ and let $E' \subset G_0$, $\delta' > 0$ be given by $(4.0.1')$, for $\varepsilon/2$. Also, we take E' to contain the unit e of the group G_0.

Let ϕ be a completely positive map on $L_\nu(G_0)$ such that $\|\phi(u_h) - u_h\|_2 \leq \delta'$, $\forall h \in E'$, where the norm $\| \quad \|_2$ is given by some trace τ on $L_\nu(G_0)$. Let $F = \{u_h \mid h \in E'\}$.

Let $(\mathcal{H}_\phi, \xi_\phi)$ be the pointed Hilbert N-bimodule defined out of ϕ as in 1.1.2. Let π be the associated representation of G_0 on $\mathcal{H}_\phi$, as in the last part of 1.1.4. It follows that there exists $\xi_0 \in \mathcal{H}_\phi$ such that $b\xi_0 = \xi_0 b, \forall b \in L_\nu(G)$ and $\|\xi_\phi - \xi_0\| \leq \varepsilon/2$. Since $1 \in F$, part 2° of Lemma 1.1.2 shows that we may assume $\phi(1) \leq 1$. By part 1° of Lemma 1.1.2 it then follows that for any $u \in \mathcal{U}(B)$

$$\begin{aligned}\|\phi(u) - u\|_2^2 &\leq 2 - 2\mathrm{Re}\tau(\phi(u)u^*) = \|u\xi_\phi - \xi_\phi u\|^2 \\ &= \|u(\xi_\phi - \xi_0) - (\xi_\phi - \xi_0)u\|^2 \leq 4\|\xi_\phi - \xi_0\|^2 \leq \varepsilon^2.\end{aligned}$$

(e) $\implies$ (a). As in the proof of (b) $\implies$ (a), by Kaplanski's density theorem, there exists $\delta > 0$ and $E \subset G_0$ such that if ϕ is completely positive on $N = L_\nu(G_0)$, with $\phi(1) \leq 1, \tau \circ \phi \leq \tau$ and $\|\phi(u_h) - u_h\|_2 \leq \delta, \forall h \in E$, then $\|\phi(b) - b\|_2 \leq \varepsilon_0$, for all b in the unit ball of $B = L_\nu(G)$.

Let $(\pi_0, \mathcal{H}_0, \xi_0)$ be a cyclic representation of G_0 such that $\|\pi_0(h)\xi_0 - \xi_0\| \leq \delta$, $\forall h \in E$. Define ϕ_0 on N by $\phi_0(\Sigma_g \alpha_g u_g) = \Sigma_g \langle \pi_0(g)\xi_0, \xi_0\rangle \alpha_g u_g$. We clearly have $\phi_0(1) = 1$, $\tau \circ \phi_0 = \tau$, $\|\phi_0(u_h) - u_h\|_2 \leq \delta$, $\forall h \in E$. Thus, $\|\phi_0(u_g) - u_g\|_2 \leq \varepsilon_0$, $\forall g \in G$, yielding $|\langle \pi_0(g)\xi_0, \xi_0\rangle - 1| \leq \varepsilon_0 < 1, \forall g \in G$. Taking the vector ξ of minimal norm in $\overline{\mathrm{co}}\{\pi_0(g) \mid g \in G\} \subset \mathcal{H}_0$, it follows that $\xi \neq 0$ and $\pi_0(g)(\xi) = \xi, \forall g \in G$. This shows that the pair (G_0, G) satisfies (4.0.1), i.e., it has relative property (T). $\square$

For the first part of the next corollary recall that any (normalized, unitary, multiplicative) scalar 2-cocycle ν on $\mathbb{Z}^2$ is given by a bicharacter, and it is uniquely determined by a relation of the form $uv = \alpha vu$ between the generators

$u = (1, 0), v = (0, 1)$ of $\mathbb{Z}^2$, where α is some scalar with $|\alpha| = 1$. We already considered such 2-cocycles in Corollary 3.3.2, where we pointed out that they are $\mathrm{SL}(2, \mathbb{Z})$-invariant. Thus, if we denote by $L_\alpha(\mathbb{Z}^2)$ the twisted group algebra $L_\nu(\mathbb{Z}^2)$, then the action σ of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{Z}^2$ induces an action still denoted σ of $\mathrm{SL}(2, \mathbb{Z})$ on $L_\alpha(\mathbb{Z}^2)$, preserving the canonical trace (cf. 3.3.2). We have:

5.2. Corollary. 1°. *The inclusion* $\mathbb{Z}^2 \subset Z^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ *is rigid. Thus, given any* $\alpha \in \mathbb{T}$, $L_\alpha(\mathbb{Z}^2) \subset L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2, \mathbb{Z})$ *is a rigid embedding of algebras. Moreover, if* α *is not a root of unity, then the "2-dimensional noncommutative torus"* $L_\alpha(\mathbb{Z}^2)$ *is isomorphic to the hyperfinite* II_1 *factor* R, *thus giving rigid embeddings* $R \subset R \rtimes_\sigma \mathrm{SL}(2, \mathbb{Z})$. *If* α *is a primitive root of unity of order* n, *then*

$$\begin{aligned}(L_\alpha(\mathbb{Z}^2) &\subset L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2, \mathbb{Z}))\\ &= (L((n\mathbb{Z})^2) \subset L((n\mathbb{Z})^2) \rtimes \mathrm{SL}(2, \mathbb{Z})) \otimes M_{n\times n}(\mathbb{C})\\ &\simeq (L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2) \rtimes \mathrm{SL}(2, \mathbb{Z})) \otimes M_{n\times n}(\mathbb{C})\\ &= (L^\infty(\mathbb{T}^2, \lambda) \subset L^\infty(\mathbb{T}^2, \lambda) \rtimes \mathrm{SL}(2, \mathbb{Z})) \otimes M_{n\times n}(\mathbb{C}).\end{aligned}$$

2°. *If* $n \geq 2$ *and* $\mathbb{F}_n \subset \mathrm{SL}(2, \mathbb{Z})$ *has finite index, then the restriction to* $\mathbb{F}_n$ *of the canonical action of* $\mathrm{SL}(2, \mathbb{Z})$ *on* $\mathbb{T}^2 = \hat{\mathbb{Z}}^2$ (*resp. on* $L_\alpha(\mathbb{Z}^2) \simeq R$, *for* α *not a root of unity*) *is free, weakly mixing, measure-preserving, with* $L^\infty(\mathbb{T}^2, \mu) \subset L^\infty(\mathbb{T}^2, \lambda) \rtimes \mathbb{F}_n$ *rigid* (*resp.* $R \subset R \rtimes \mathbb{F}_n$ *rigid*).

3°. *For each* $n \geq 2$ *and each arithmetic lattice* Γ_0 *in* $\mathrm{SO}(n, 1)$ (*resp. in* $\mathrm{SU}(n, 1)$) *there exist free weakly mixing, measure-preserving actions of* Γ_0 *on* $A \simeq L^\infty(X, \mu)$ *such that the corresponding crossed product inclusions* $A \subset A \rtimes \Gamma_0$ *are rigid.*

4°. *Let* σ_0 *be a properly outer, weakly mixing action of some group* Γ_0 *on* (B_0, τ_0) *such that* $B_0 \subset B_0 \rtimes_{\sigma_0} \Gamma_0$ *is rigid* (*e.g., as in* 1°, 2° *or* 3°). *Let* σ_1 *be any action of* Γ_0 *on some finite von Neumann algebra* (B_1, τ_1), *which acts ergodically on the center of* B_1. *If* $B = B_0 \overline{\otimes} B_1$ *and* $M = (B_0 \overline{\otimes} B_1) \rtimes_{\sigma_0 \otimes \sigma_1} \Gamma_0$, *then* M *is a factor,* $B_0' \cap M \subset B$, *and* $B_0 \subset M$ *is a rigid embedding.*

Proof. 1°. The rigidity of $\mathbb{Z}^2 \subset Z^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ is a well known result in [Ka], [Ma]; (see also [Bu], [Sha] for more elegant proofs). The fact that $L_\alpha(\mathbb{Z}^2) \simeq R$ if α is not a root of unity and that $L_\alpha(\mathbb{Z}^2) \simeq A \otimes M_{n\times n}(\mathbb{C})$, with $A = \mathcal{Z}(L_\alpha(\mathbb{Z}^2)) \simeq L((n\mathbb{Z})^2)$, if α is a primitive root of order n, are folklore type results (see [Ri] and [HkS]).

In the latter case, if $p \in 1 \otimes M_{n\times n}(\mathbb{C}) \subset L_\alpha(\mathbb{Z}^2)$ is a projection of central trace $1/n$ then $\sigma_g(p)$ has central trace $1/n$ as well, so there exists $v_g \in \mathcal{U}(L_\alpha(\mathbb{Z}^2))$ such that $v_g \sigma_g(p) v_g^* = p$. Thus, since v_g commutes with the center A, if we denote by σ_g' the action implemented by the restriction of $\mathrm{Ad} v_g \circ \sigma_g$ to $p(L_\alpha(\mathbb{Z}^2))p = Ap \simeq A \simeq L((n\mathbb{Z})^2)$, then σ_g' coincides with the restriction of σ_g to $A \simeq L((n\mathbb{Z})^2)$.

Moreover, if $u_g \in L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ are the canonical unitaries implementing σ_g on $L_\alpha(\mathbb{Z}^2)$, then $u'_g = v_g u_g p$ implement the action $\sigma'_g = \sigma_{|A}$ on A, but with an A-valued 2-cocycle v', i.e., $p(L_\alpha(\mathbb{Z}^2) \subset L_\alpha(\mathbb{Z}^2) \rtimes_\sigma \mathrm{SL}(2,\mathbb{Z}))p \simeq (A \subset A \rtimes_{\sigma',v'} \mathrm{SL}(2,\mathbb{Z}))$. But by [Hj], $A \subset A \rtimes_{\sigma',v'} \mathrm{SL}(2,\mathbb{Z})$ is the amplification by 12 of an inclusion of the form $A_0 \subset A_0 \rtimes \mathbb{F}_2$, for some free ergodic action of $\mathbb{F}_2$ on A_0. Since any action by the free group has trivial cocycle, $A_0 \subset A_0 \rtimes \mathbb{F}_2$ is associated with the bare equivalence relation it induces on the probability space, with trivial cocycle. Thus, so does its $1/12$ reduction (see 1.4); i.e., $(A \subset A \rtimes_{\sigma'} \mathrm{SL}(2,\mathbb{Z})) = (L^\infty(\mathbb{T}^2,\lambda) \subset L^\infty(\mathbb{T}^2,\lambda) \rtimes_\sigma \mathrm{SL}(2,\mathbb{Z}))$.

The rest of the statement follows from part (a) of Corollary 3.3.2°.

2° follows from part 1° above, Proposition 4.6.2° and part (c) of Corollary 3.3.2°.

3° follows by a recent result in [Va], showing that there exist actions γ of such Γ_0 on some appropriate $\mathbb{Z}^N$ which give rise to rigid embeddings $\mathbb{Z}^N \subset \mathbb{Z}^N \rtimes \Gamma_0$. It is easy to see that the actions γ in [Va] can be taken to satisfy conditions (i), (ii) in Lemma 3.3.2.

4°. By 3.3.3, since σ_0 is properly outer, it follows that $\sigma_0 \otimes \sigma_1$ is properly outer and $B'_0 \cap M = \mathcal{Z}(B_0) \otimes B_1$. Also, since σ_0 is weakly mixing and σ_1 is ergodic, $\sigma_0 \otimes \sigma_1$ is ergodic and M is a factor. □

5.3. Corollary. 1°. *Let Γ_0 be an arbitrary discrete, countable group. Denote by σ_1 the Bernoulli shift action of Γ_0 on $(A_1,\tau_1) = \overline{\otimes}_{g\in\Gamma_0}(L^\infty(\mathbb{T},\lambda))_g$ and let σ_0 be an ergodic action of Γ_0 on an abelian von Neumann algebra (A_0,τ_0). If $A = A_0\overline{\otimes}A_1, \sigma = \sigma_0 \otimes \sigma_1$ then σ is free ergodic and the inclusion $A \subset A \rtimes_\sigma \Gamma_0$ is not rigid.*

2°. *$L(\mathbb{Q}^2) = A \subset M = L(\mathbb{Q}^2) \rtimes \mathrm{SL}(2,\mathbb{Q})$ is not a rigid inclusion but $A_0 = L(\mathbb{Z}^2) \subset A$ satisfies $A_0 \subset M$ rigid and $A'_0 \cap M = A$.*

3°. *If Γ_0 is equal to $\mathrm{SL}(2,\mathbb{Z})$, or to $\mathbb{F}_n$, for some $n \geq 2$, or to an arithmetic lattice in some $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $n \geq 2$, then there exist three non orbit equivalent free ergodic measure-preserving actions $\sigma_i, 1 \leq i \leq 3$, of Γ_0 on the probability space (X,μ). Moreover, each σ_i can be taken such that $A = L^\infty(X,\mu)$ contains a subalgebra A_i with $A_i \subset A \rtimes_{\sigma_i} \Gamma_0$ rigid and $A'_i \cap A \rtimes_{\sigma_i} \Gamma_0 = A$.*

Proof. 1°. Write $L^\infty(\mathbb{T},\lambda) = \overline{\cup_n A^n}$, with A^n an increasing sequence of finite dimensional subalgebras and denote $A_1^n = \overline{\otimes}_g(A^n)_g \subset A_1$. Then $A_1^n \uparrow A_1$ and $\sigma_g(A_1^n) = A_1^n, \forall g \in \Gamma_0, \forall n$. Thus, if $N_n = (A_0\overline{\otimes}A_1^n \cup \{u_g\}_g)''$ then $N_n \uparrow N = A \rtimes_\sigma \Gamma_0$. So if we assume $A \subset N$ is rigid, then by 4.5 there exists n such that $\|E_{N_n}(a) - a\|_2 \leq 1/2, \forall a \in A, \|a\| \leq 1$. But if $a \in 1 \otimes A_1$ then $E_{N_n}(a) = E_{A_1^n}(a)$. Or, since A^n is finite dimensional and $L^\infty(\mathbb{T},\lambda)$ is diffuse, there exists a unitary element $u_0 \in L^\infty(\mathbb{T},\lambda)$ such that $E_{A_1^n}(u_0) = 0$. Taking $u = \cdots \otimes 1 \otimes u_0 \otimes 1 \cdots \in A$, it follows that $E_{A_n}(u) = 0$, so that $1 = \|E_{A_1^n}(u) - u\|_2 = \|E_{N_n}(u) - u\|_2 \leq 1/2$, a contradiction.

2°. For each n let $\mathbb{Q}_n$ be the ring of rationals with the denominator having prime decomposition with only the first n prime numbers appearing. Then $A \supset A_n = L(\mathbb{Q}_n) \subset L(\mathbb{Q}_n) \rtimes \mathrm{SL}(2, \mathbb{Q}_n) = M_n \subset M$ and we have $E_{M_n} \circ E_A = E_{A_n}, \forall n$. If $A \subset M$ were rigid, then by 4.5 there would exist n such that $\|E_{M_n}(a) - a\|_2 \leq 1/2, \forall a \in A, \|a\| \leq 1$. But any unitary element $u \in A = L(\mathbb{Q}^2)$ corresponding to a group element in $\mathbb{Q} \backslash \mathbb{Q}_n$ satisfies $E_{A_n}(u) = 0$, a contradiction.

3°. We take σ_1 to be the action of Γ_0 on $A = L^\infty(X, \mu)$ given by 5.2.1°–5.2.3° and σ_2 to be the tensor product of σ_1 with the Bernoulli shift action of Γ_0 on $\overline{\otimes}_{g \in \Gamma_0} (L^\infty(\mathbb{T}, \lambda))_g$.

Finally, we take σ_3 to be the tensor product of σ_1 with a free ergodic measure-preserving action of Γ_0 which is not strongly ergodic, as provided by the Connes-Weiss Theorem ([CW]; this is possible because Γ_0 has property H, so it does not have property (T)).

By part 1° we have $(A \subset A \rtimes_{\sigma_1} \Gamma_0) \not\simeq (A \subset A \rtimes_{\sigma_2} \Gamma_0)$. By results of Klaus Schmidt ([Sc]; see also [J2]), σ_1, σ_2 are strongly ergodic, while σ_3 is not. Thus, $(A \subset A \rtimes_{\sigma_3} \Gamma_0) \not\simeq (A \subset A \rtimes_{\sigma_i} \Gamma_0), i = 1, 2$.

Since all these Cartan subalgebras have trivial 2-cocycle by construction, their nonisomorphism implies the nonequivalence of the corresponding orbit equivalence relations.

The existence of "large" subalgebras $A_i \subset A$ with $A_i \subset A \rtimes_{\sigma_i} \Gamma_0$ rigid follows by construction and by 3.3.3. □

5.4. Theorem. 1°. *If N is a type* II_1 *factor with property* H (*as defined in* 2.0.2), *then N contains no diffuse relatively rigid subalgebras $B \subset N$.*

2°. *If N has property* H *relative to a type* I *von Neumann algebra $B_0 \subset N$ then N contains no relatively rigid type* II_1 *von Neumann subalgebras $B \subset N$.*

Proof. 1°. Let ϕ_n be completely positive maps on N such that $\phi_n \to \mathrm{id}_N$, $\tau \circ \phi_n \leq \tau$ and $T_{\phi_n} \in \mathcal{K}(L^2(N, \tau))$. If $B \subset N$ is a rigid inclusion then by 4.1.2°, there exists n such that $\phi = \phi_n$ satisfies $\|\phi(u) - u\|_2 \leq 1/2, \forall u \in \mathcal{U}(B)$. If in addition B has no atoms, then any maximal abelian subalgebra A of B is diffuse. Thus, such A contains unitary elements v with $\tau(v^m) = 0, \forall m \neq 0$. Since the sequence $\{\hat{v^m}\}_m \subset L^2(N, \tau)$ is weakly convergent to 0 and T_ϕ is compact, $\|\phi(v^m)\|_2 = \|T_\phi(\hat{v^m})\|_2 \to 0$. Thus,

$$\lim_{m \to \infty} \|\phi(v^m) - v^m\|_2 = \lim_{m \to \infty} \|v^m\|_2 = 1,$$

contradicting $\|\phi(v^m) - v^m\|_2 \leq 1/2, \forall m$.

2°. Assume N does contain a relatively rigid type II_1 von Neumann subalgebra $B \subset N$. Let ϕ_n contain completely positive B_0 bimodular maps on N such that $\phi_n \to \mathrm{id}_N$, $\tau \circ \phi_n \leq \tau$ and $T_{\phi_n} \in \mathcal{J}_0(\langle N, B_0 \rangle)$. By the rigidity of

$B \subset N$ it follows that $\varepsilon_n = \sup\{\|\phi_n(u) - u\|_2 \mid u \in \mathcal{U}(B)\} \to 0$. Since

$$\|u^* T_{\phi_n} u(\hat{1}) - \hat{1}\|_2 = \|u^* \phi_n(u) - 1\|_2 = \|\phi_n(u) - u\|_2,$$

by taking convex combinations and weak limits of elements of the form $uT_{\phi_n}u^*$, by Proposition 1.3.2 we see that there exists $T_n \in K_{T_{\phi_n}} \cap (B' \cap \mathcal{J}(\langle N, B\rangle))$ such that $\|T_n(\hat{1}) - \hat{1}\|_2 \to 0$. Thus, $T_n \neq 0$ for n large enough, so that $B' \cap \langle N, B_0\rangle$ contains nonzero projections of finite trace. By [Chr], this implies there exist nonzero projections $p \in B, q \in B_0$ and a unital isomorphism θ of pBp into qB_0q. But qB_0q is type I and pBp is not, a contradiction. □

5.5. Corollary. 1°. *If N has a diffuse relatively rigid subalgebra $B \subset N$ then N cannot be embedded into a free group factor $L(\mathbb{F}_n)$. In particular, the factors constructed in Corollary* 5.2 *cannot be embedded into $L(\mathbb{F}_n)$.*

2°. *The factors $L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$, constructed in* 5.2.1° *for α irrational, cannot be embedded into $L_{\alpha'}(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ for α' rational.*

Proof. Part 1° is a consequence of 5.4.1, while part 2° follows trivially from 5.4.2. □

5.6. *Remarks.* 1°. In the case when N is a finite factor, a different notion of "relative property T" for inclusions $B \subset N$, was considered in [A-De], [Po1], as follows:

(5.6.1). *N has property* T *relative to B* (or *B is co-rigid in N*) if there exists a finite set $F_1 \subset N$ and $\varepsilon_1 > 0$ such that if $(\mathcal{H}, \xi)$ is a $(B \subset N)$ Hilbert bimodule (recall that by definition this requires $[B, \xi] = 0$) such that $\|x\xi - \xi x\| \leq \varepsilon, \forall x \in F$, then there exists $\xi_0 \in \mathcal{H}, \xi_0 \neq 0$, with $x\xi_0 = \xi_0 x$, $\forall x \in N$.

In the case B is a Cartan subalgebra A of a type II$_1$ factor $N = M$, this definition is easily seen to be equivalent to Zimmer's property (T) ([Zi2]) for the countable, measurable, measure-preserving equivalence relation $\mathcal{R}_{A \subset M}$, which it thus generalizes to the case of arbitrary inclusions of von Neumann algebras (cf. Section 4.8 in [Po1]). Thus, in this re-formulation, a standard equivalence relation $\mathcal{R}$ satisfies Zimmer's relative property (T) if and only if the Cartan subalgebra $A \subset M$, constructed as in [FM] out of $\mathcal{R}$ and the trivial 2-cocycle $v \equiv 1$, is co-rigid in the sense of [Po1], [A-De]. We will in fact call such equivalence relations $\mathcal{R}$ *co-rigid*.

2°. It is easy to see that in case $(B \subset N) = (B \subset B \rtimes_\sigma \Gamma_0)$, for some cocycle action σ of a group Γ_0 on (B, τ) then N has property (T) relative to B (i.e., B is co-rigid in N) if and only if Γ_0 has the property (T) of Kazhdan (cf. [A-De], [Po1]; also [Zi] for the Cartan subalgebra case). In particular, if $H \subset G_0$ is a normal subgroup of G_0 then $L(G_0)$ has property (T) relative to $L(H)$ if and only if the quotient group G_0/H has property (T). In fact, it is easy to see that if $H \subset G_0$ is an inclusion of discrete groups then $L(G_0)$ has property (T) relative to $L(H)$ if and only if the following holds true:

(5.6.2). There exist a finite set $E \subset G_0$ and $\varepsilon > 0$ such that if π is a unitary representation of G_0 on a Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ such that $\pi(h)\xi = \xi, \forall h \in H$ and $\|\pi(g)\xi - \xi\| \leq \varepsilon, \forall g \in E$, then $\mathcal{H}$ contains a nonzero vector ξ_0 such that $\pi(g)\xi_0 = \xi_0, \forall g \in G_0$.

A sufficient condition for an inclusion of groups $H \subset G_0$ to satisfy 5.6.2° exists when G_0 has *finite length over* H, i.e., when the following holds true:

(5.6.2′). There exist $n \geq 1$ and a finite set $E \subset G_0$ such that any element $g \in G_0$ can be written as $g = h_1 f_1 h_2 f_2 \dots h_n f_n$, for some $f_i \in E, h_j \in H$.

Indeed, because then $\pi(h)\xi = \xi, \forall h \in H$ and ξ is almost fixed by $\pi(f)$, $f \in E$, implies that ξ is almost fixed by $\pi(g)$, uniformly for all $g \in G_0$. This, of course, shows that $\mathcal{H}$ has a nonzero vector fixed by $\pi(G_0)$. (N.B. Finite length was exploited in relation to rigidity in [Sha].)

An example of inclusion of groups $H \subset G_0$ satisfying (5.6.2′) is obtained by taking G_0 to be the group of all affine transformations of $\mathbb{Q}$ and H to be the subgroup of all homotheties of $\mathbb{Q}$. Indeed, because if we take E to be the single element set consisting of the translation by 1 on $\mathbb{Q}$, then we clearly have $G_0 = HEH$. Thus, $L(G_0)$, which is isomorphic to the hyperfinite type II_1 factor R, has property (T) relative to $L(H)$, which is a singular maximal abelian subalgebra in $L(G_0)$ (cf. [D]).

5.7. Proposition. *Let N be a finite factor and $B \subset N$ a von Neumann subalgebra.*

1°. *If $\langle N, B\rangle$ is finite then N has both property* (T) *relative to B* (*in the sense of* (5.6.1)) *and property* H *relative to B.*

2°. *If N has both property* (T) *and* H *relative to B then there exists a nonzero $q \in \mathcal{P}(B' \cap N)$ such that qNq is a finitely generated Bq-module. Thus, if in addition B is a subfactor with $B' \cap N = \mathbb{C}$ then $[N : B] < \infty$ and if B is a maximal abelian von Neumann subalgebra in N then* $\dim N < \infty$.

Proof. 1°. If $\langle N, B\rangle$ is finite, then there exists a sequence of projections $p_n \in \mathcal{Z}(B), p_n \uparrow 1$, such that $p_n N p_n$ has finite orthonormal basis over Bp_n. By 2.3.4°, this implies $p_n N p_n$ has property H relative to Bp_n and by 4.6.3°, $Bp_n \subset p_n N p_n$ is rigid. By 2.4.2° this implies N has property H relative to B and by 4.7.1°, $B \subset N$ is rigid.

2°. Note first that if there exist no $q \in \mathcal{P}(B' \cap N)$ such that qNq is a finitely generated Bq-module, then $N' \cap \langle N, B\rangle$ contains no finite projections of $\langle N, B\rangle$.

On the other hand, if N has property H relative to B then by 2.2.1° there exist unital completely positive, B-bimodular maps ϕ_n on N such that

$\tau \circ \phi_n \leq \tau$, $\phi_n(1) \leq 1$, $\phi_n \to \mathrm{id}_N$ and $T_{\phi_n} \in \mathcal{J}_0(\langle N, B\rangle)$. If in addition N has the property (T) relative to B, then $\exists n$ such that $\|\phi_n(u) - u\|_2 \leq 1/4, \forall u \in \mathcal{U}(N)$. By 1.3.3, $\exists$ a spectral projection $p \in B' \cap \mathcal{J}_0(\langle N, B\rangle)$ of $T^*_{\phi_n} T_{\phi_n}$ such that $\|T_{\phi_n}(1-p)\| < 1/4$. If we now assume $N' \cap \langle N, B\rangle$ has no finite projections, then there exists a unitary element $u \in \mathcal{U}(N)$ such that $\mathrm{Tr}(pue_Bu^*) < 1/4$. But $\mathrm{Tr}(pue_Bu^*) = \|p(\hat{u})\|_2^2$ (see the proof of 6.2 in the next section). Altogether, since $\|p(\hat{u})\|_2 \geq \|T_{\phi_n}(\hat{u})\|_2 - \|T_{\phi_n}((1-p)(\hat{u}))\|_2 \geq 1/2$, it follows that $1/4 > \mathrm{Tr}(pue_Bu^*) \geq 1/4$, a contradiction. The last part of 2° follows trivially from [PiPo]. □

5.8. *Remarks.* 1°. Both the notion 4.2 considered here and the notion considered in [A-De], [Po1] are in some sense "relative property (T)" notions for an inclusion $B \subset N$; but while the notion in [A-De], [Po1] means "N has the property (T) relative to B", thus being a "co"-type property (T), the notion considered in this paper is a "property (T) of B relative to its embedding into N". The two notions are complementary, and together they imply (and are implied by) property (T) of the global factor (see Proposition 5.9 below).

2°. An interesting relation between these two complementary notions of property (T) is the following: If a group Γ_0 acts on (B, τ) such that $B \subset N = B \rtimes \Gamma_0$ is a rigid embedding, then N has property (T) relative to its group von Neumann subalgebra $L(\Gamma_0)$ (i.e., $L(\Gamma_0)$ is co-rigid in N). Indeed, because if $(\mathcal{H}, \xi)$ is an $(L(\Gamma_0) \subset N)$-Hilbert bimodule with ξ almost commuting with all $u \in \mathcal{U}(B)$, uniformly, then ξ almost commutes with the group of elements $\mathcal{G} = \{uu_g \mid u \in \mathcal{U}(B), g \in \Gamma_0\}$. Thus ξ is close to a vector commuting with all $v \in \mathcal{G}$, thus with all $x \in N$. For instance, the factor $L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}))$ has property T relative to its subalgebra $L(\mathrm{SL}(2, \mathbb{Z}))$ (in the sense of definition (5.6.1)).

5.9. Proposition. *Let N be a type* II_1 *factor and $B \subset N$ a von Neumann subalgebra. The following conditions are equivalent*:

1°. *N has property* (T) *in the sense of Connes and Jones* (*i.e., of the equivalent conditions* (4.0.2), (4.0.2′)).

2°. *The identity embedding $N \subset N$ is rigid, i.e., for any $\varepsilon > 0$ there exists a finite subset $x_1, x_2, \ldots, x_n \in N$ and $\delta > 0$ such that if $\mathcal{H}$ is a Hilbert N-bimodule with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\langle \cdot \xi, \xi\rangle - \tau\| \leq \delta$, $\|\langle \xi, \xi \cdot\rangle - \tau\| \leq \delta$ and $\|x_i\xi - \xi x_i\| \leq \delta, \forall i$, then there exists a vector $\xi_0 \in \mathcal{H}$ such that $\|\xi - \xi_0\| \leq \varepsilon$ and $x\xi_0 = \xi_0 x, \forall x \in N$.*

3°. *$B \subset N$ is a rigid embedding* (*in the sense of Definition* 4.2) *and N has property* (T) *relative to B* (*in the sense of* (5.6.1)).

Proof. 1° $\implies$ 3° and 1° $\implies$ 2° are trivial, by the characterization (4.0.2′) of property (T) for N.

To prove $3^\circ \implies 1^\circ$ let $F_1 \subset N$ and ε_1 give the critical set and constant for property (T) of N relative to B and $F' \subset N, \delta' > 0$ be the critical set and constant for the rigidity of $B \subset N$, corresponding to $\varepsilon_1/4$. Let $F = F' \cup F_1$ and let $\mathcal{H}$ be a Hilbert N bimodule with a unit vector ξ which is left and right δ'-tracial and satisfies $\|y\xi - \xi y\| \leq \delta', \forall y \in F$. By the rigidity of $B \subset N$ it follows that there exists $\xi_0 \in \mathcal{H}$ such that $b\xi_0 = \xi_0 b, \forall b \in B$ and $\|\xi_0 - \xi\| \leq \varepsilon_1/4$. Thus, if we assume $\varepsilon_1 \leq 1/4$ from the beginning and denote $\xi_1 = \xi_0/\|\xi_0\|$, then $\|\xi_1\| = 1$, $b\xi_1 = \xi_1 b, \forall b \in B$, and $\|y\xi_1 - \xi_1 y\| \leq \varepsilon_1, \forall y \in F$, in particular for all $y \in F_1$. Thus, by the property (T) of N relative to B, $\mathcal{H}$ has a nonzero N-central vector.

$2^\circ \implies 1^\circ$. By part 1° of Theorem 4.3, N follows non-Γ. Thus, by Lemma 2.9 it is sufficient to check that any Hilbert N bimodule with a vector that is almost left-right tracial and almost central has a nonzero central vector for N. But this does hold true by the fact that N satisfies condition 2°. $\square$

5.10. *Remark.* When applied to the case of Cartan subalgebras coming from standard equivalence relations with trivial 2-cocycle, the definition of rigid embeddings 4.2 gives the following new property for equivalence relations:

5.10.1. *Definition.* A countable, ergodic, measure-preserving equivalence relation $\mathcal{R}$ has the *relative property* (T) if its associated Cartan subalgebra $A \subset M$, constructed out of $\mathcal{R}$ and the trivial 2-cocycle $v \equiv 1$ as in [FM], is a rigid embedding (Definition 4.2).

Since the rigidity for Cartan subalgebras is an invariant for the isomorphism class of $A \subset M$, this relative property (T) is an orbit equivalence invariant for equivalence relations $\mathcal{R}$. Also, when applied to the particular case of Cartan subalgebras with trivial 2-cocycle, all the results on rigid embeddings of algebras in Sections 4 and 5 translate into corresponding results about standard equivalence relations $\mathcal{R}$. For instance, by 4.6, 4.7, if $\mathcal{R}$ has relative property (T) then $\mathcal{R}^t$ has relative property (T), $\forall t > 0$, and if $\mathcal{R}_1, \mathcal{R}_2$ have relative property (T) then so does $\mathcal{R}_1 \times \mathcal{R}_2$. Also, if $\mathcal{R}$ has relative property (T) then $\mathrm{Out}(\mathcal{R}) \overset{\mathrm{def}}{=} \mathrm{Aut}(\mathcal{R})/\mathrm{Int}(\mathcal{R})$ is discrete (cf. 4.4) and if we further have $\mathcal{R} = \cup_n \mathcal{R}_n$ for some increasing sequence of ergodic sub-equivalence relations, then $\mathcal{R}_n$ have relative property (T) for all large enough n.

We have proved that equivalence relations implemented by Bernoulli shift actions of a group Γ_0 cannot have relative property (T), no matter the group Γ_0 (cf. 5.3). Thus, equivalence relations coming from actions of the same group Γ_0 may or may not have relative property (T), depending on the action. While by [Zi] (see also [A-De], [Po1]), $A \rtimes_\sigma \Gamma_0$ has property (T) relative to A, in the sense of definition (5.6.1) if and only if Γ_0 has Kazhdan's property (T), thus being a property entirely depending on the group. Even more: since by [Po1] if $A \subset M$ is a Cartan subalgebra in a II_1 factor and $p \in \mathcal{P}(A)$ then pMp has property (T) relative to Ap if and only if M has property (T) relative to A, it follows that property (T) for groups is invariant to stable orbit equivalence, or equivalently, it is an ME invariant (see [Fu] for an "ergodic theory" proof of this fact).

Proposition 5.9 shows that when the relative property (T) (5.10.1) for $\mathcal{R}$ is combined with the co-rigidity property (5.6.1) for $\mathcal{R}$ they imply, and are implied by, the "full" *property* T *of* $\mathcal{R}$, which by definition requires that the finite factor $M = M(\mathcal{R})$ have property (T) in the sense (4.0.2), of Connes-Jones. It is thus of great interest to answer the following:

5.10.2. *Problem.* Characterize the countable discrete groups Γ_0 that can act rigidly on the probability space, i.e., for which there exist free ergodic measure-preserving actions σ on (X, μ) such that $L^\infty(X, \mu) \subset L^\infty(X, \mu) \rtimes_\sigma \Gamma_0$ is a rigid embedding. Do all property (T) groups Γ_0 admit such rigid actions (i.e., in view of the above, actions σ with property that the II_1 factor $L^\infty(X, \mu) \rtimes_\sigma \Gamma_0$ has property (T) in the sense of (4.0.2))?

6. HT subalgebras and the class $\mathcal{HT}$

6.1. *Definition.* Let N be a finite von Neumann algebra with a normal faithful tracial state and $B \subset N$ a von Neumann subalgebra. B is an HT *subalgebra* of N (or $B \subset N$ is an HT *inclusion*) if the following two conditions are met:

(6.1.1). N has property H relative to B (as defined in Section 2).

(6.1.2). There exists a von Neumann subalgebra $B_0 \subset B$ such that $B_0' \cap N \subset B$ and $B_0 \subset N$ is a rigid (or property (T)) embedding.

Also, B is an HT_s *subalgebra* of N if conditions (6.1.1) and (6.1.2) hold true with $B_0 = B$, i.e., if N has the property H relative to B and $B \subset N$ is itself a rigid embedding.

If $A \subset M$ is a Cartan subalgebra of a finite factor M and $A \subset M$ satisfies the conditions (6.1.1) and (6.1.2), then we call it an HT *Cartan subalgebra*. Similarly, if a Cartan subalgebra $A \subset M$ satisfies (6.1.1) and is a rigid embedding then it is called an HT_s *Cartan subalgebra*.

Note that condition (6.1.2) implies that $B' \cap N \subset B$ and (6.1.1) implies B is quasi-regular in N (cf. 3.4). In particular, by Proposition 3.4, for $A \subset M$ a maximal abelian $*$-subalgebra of type II_1 factor M, the condition that A is an HT (resp. HT_s) subalgebra of M is sufficient to insure that A is an HT (resp. HT_s) Cartan subalgebra of M.

6.2. THEOREM. *Let M be a type II_1 factor with two abelian von Neumann subalgebras A, A_0 such that A, $A_0' \cap M$ are maximal abelian in M. Assume that M has property* H *relative to A and that $A_0 \subset M$ is a rigid inclusion. Then both A and $A_0' \cap M$ are* HT *Cartan subalgebras of M and there exists a* *unitary element u in M such that $uA_0u^* \subset A$, and thus $u(A_0' \cap M)u^* = A$. In particular, if A_1, A_2 are* HT *Cartan subalgebras of a type* II_1 *factor M then there exists a unitary element $u \in \mathcal{U}(M)$ such that $uA_1u^* = A_2$.*

Proof. We first prove that there exists a nonzero partial isometry $v \in M$ such that $v^*v \in A_0' \cap M$, $vv^* \in A$ and $vA_0v^* \subset Avv^*$. If we assume by contradiction that this is not the case, then Theorem A.1 implies $0 \in K_{\mathcal{U}(A_0)}(e_A) \subset \langle M, A\rangle$. This in turn implies that given any finite projection $f \in \langle M, A\rangle$, with $\mathrm{Tr}(f) < \infty$, and any $\varepsilon > 0$, there exists a unitary element $u \in \mathcal{U}(A_0)$ such that $\mathrm{Tr}(fue_Au^*) < \varepsilon$. Indeed, if for some $c_0 > 0$ we had $\mathrm{Tr}(fue_Au^*) \geq c_0, \forall u \in \mathcal{U}(A_0)$, then by taking appropriate convex combinations and weak limits, we would get that $0 = \mathrm{Tr}(f0) \geq c_0 > 0$, a contradiction.

By property H of M relative to A, there exist completely positive, unital, A-bimodular maps $\phi_n : M \to M$ which tend strongly to the identity and satisfy $\phi_n(1) \leq 1, \tau \circ \phi_n \leq \tau$, $T_{\phi_n} \in \mathcal{J}_0(\langle M, A\rangle)$.

Let $0 < \varepsilon_0 < 1$. By the rigidity of the embedding $A_0 \subset M$, there exists n large enough such that $\phi = \phi_n$ satisfies

$$\|\phi(v) - v\|_2 \leq \varepsilon_0, \forall v \in \mathcal{U}(A_0). \tag{6.2.1}$$

On the other hand, since $T_\phi \in \mathcal{J}_0(\langle M, A\rangle)$, it follows that there exists a finite projection $f \in \mathcal{J}_0(\langle M, A\rangle)$ such that $\mathrm{Tr}(f) < \infty$ and

$$\|T_\phi(1 - f)\| \leq (1 - \varepsilon_0)/2. \tag{6.2.2}$$

Let then $u \in \mathcal{U}(A_0)$ satisfy the condition

$$\mathrm{Tr}(fue_Au^*) < (1 - \varepsilon_0)^2/4. \tag{6.2.3}$$

Let $\{m_j\}_j \subset L^2(M, \tau)$ be such that $\Sigma_j m_j e_A m_j^* = f$. Equivalently, $\oplus_j L^2(m_jA) = f(L^2(M, \tau))$. Thus, if $x \in N = \hat{N} \subset L^2(M, \tau)$ then $f(\hat{x}) = \Sigma_j m_j E_A(m_j^* x)$ and $\|f(\hat{x})\|_2^2 = \Sigma_j \|m_j E_A(m_j^* x)\|_2^2$.

It follows that

$$\begin{aligned}\mathrm{Tr}(fue_Au^*) &= \mathrm{Tr}(fue_Au^*f) \\ &= \mathrm{Tr}((\Sigma_j m_j e_A m_j^*)ue_Au^*(\Sigma_i m_i e_A m_i^*)) \\ &= \Sigma_{i,j}\tau(m_j E_A(m_j^* u)E_A(u^* m_i)m_i^*) = \|f(\hat{u})\|_2^2.\end{aligned}$$

By (6.2.3) this implies

$$\|f(\hat{u})\|_2 < (1 - \varepsilon_0)/2. \tag{6.2.4}$$

Thus, since $\|T_\phi\| \leq 1$, (6.2.2) and (6.2.4) entail:

$$\begin{aligned}\|T_\phi(\hat{u})\|_2 &\leq \|T_\phi((1 - f)(\hat{u}))\|_2 + \|f(\hat{u})\|_2 \\ &\leq (1 - \varepsilon_0)/2 + \|f(\hat{u})\|_2 < 1 - \varepsilon_0.\end{aligned}$$

But by (6.2.1), this implies:

$$\begin{aligned}\|u\|_2 &\le \|T_\phi(\hat{u})\|_2 + \|\phi(u) - u\|_2\\ &< 1 - \varepsilon_0 + \varepsilon_0 = 1.\end{aligned}$$

Thus $1 = \tau(uu^*) < 1$, a contradiction.

Let now $(\mathcal{V}, \le)$ denote the set of partial isometries $v \in M$ with $v^*v \in A_0' \cap M$, $vv^* \in A$ and $vA_0v^* \subset Avv^*$, endowed with the order $\le$ given by restriction, i.e., $v \le v'$ if $v = vv^*v'$. $(\mathcal{V}, \le)$ is clearly inductively ordered. Let $v_0 \in \mathcal{V}$ be a maximal element. Assume v_0 is not a unitary element.

By 2.4.1°, $(1-v_0v_0^*)M(1-v_0v_0^*)$ has property H relative to $A(1-v_0v_0^*)$ and by 4.7.2° the inclusion $A_0(1-v_0^*v_0) \subset (1-v_0^*v_0)M(1-v_0^*v_0)$ is rigid. Let $u_0 \in M$ be a unitary element extending v_0 and denote $M^0 = (1 - v_0v_0^*)M(1 - v_0v_0^*)$, $A_0^0 = u_0(A_0(1-v_0^*v_0))u_0^*$, $A^0 = A(1-v_0v_0^*)$. Thus, M^0 has property H relative to A^0 and $A_0^0 \subset M_0$ is rigid. By the first part it follows that there exists a nonzero partial isometry $v \in M^0$ such that $v^*v \in (A_0^0)' \cap M$, $vv^* \in A^0$ and $vA_0^0v^* \subset A^0vv^*$. But then $v' = v_0 + vu_0^* \in \mathcal{V}$, $v' \ge v_0$ and $v' \ne v_0$, contradicting the maximality of v_0.

We conclude that v_0 is a unitary element, so that $A, A_0' \cap M$ are conjugate in M. The last part follows now by Proposition 3.4. □

6.3. *Remarks.* 1°. If in the last part of Theorem 6.2 we restrict ourselves to the case where A_1, A_2 are HT_s Cartan subalgebras of the type II_1 factor M, then we can give the following alternative proof of the statement, by using part (ii) of Proposition 1.4.3 in lieu of Theorem A.1 and an argument similar to the proof of 5.4.2°: By property H of M relative to A_1 there exists completely positive A_1 bimodular trace-preserving unital maps ϕ_n on M such that $\phi_n \to \mathrm{id}_M$ and $T_{\phi_n} \in \mathcal{J}_0(\langle M, A_1\rangle)$. By the rigidity of $A_2 \subset M$ it follows that $\varepsilon_n = \sup\{\|\phi_n(u) - u\|_2 \mid u \in \mathcal{U}(A_2)\} \to 0$. Fix $x \in M$ and note that by Corollary 1.1.2,

$$\begin{aligned}\|u^*T_{\phi_n}u(\hat{x}) - \hat{x}\|_2 &= \|\phi_n(ux) - ux\|_2 \le \|\phi_n(ux) - u\phi_n(x)\|_2\\ &\quad + \|\phi_n(x) - x\|_2 \le 2\varepsilon_n^{1/2} + \|\phi_n(x) - x\|_2.\end{aligned}$$

Thus, by taking weak limits of appropriate convex combinations of elements of the form $u^*T_{\phi_n}u$ with $u \in \mathcal{U}(A_2)$, and using Proposition 1.3.2 we see that $T_n = \mathcal{E}_{A_2' \cap \langle M, A_1\rangle}(T_{\phi_n}) \in K_{T_{\phi_n}} \cap (A_2' \cap J_0(\langle M, A_1\rangle))$ satisfy $\lim_{n\to\infty} \|T_n(\hat{x}) - \hat{x}\|_2 = 0$. But $x \in M$ was arbitrary. This shows that the right supports of T_n span the identity of $\langle M, A_1\rangle$. Since T_n are compact, this shows that $A_2' \cap \langle M, A_1\rangle$ is generated by finite projections of $\langle M, A_1\rangle$. Thus, A_2 is discrete over A_1. Similarly, A_1 is discrete over A_2 and A_1 is conjugate to A_2 by a result in [PoSh]; see part (ii) of Proposition 1.4.3.

2°. The above argument uses the fact that two Cartan subalgebras A_1, A_2 in M are unitarily equivalent in M if and only if the $A_1 - A_2$ Hilbert bimodule $L^2(M,\tau)$ is a direct sum of Hilbert bimodules that are finite dimensional both as left A_1 modules and as right A_2 modules. The proof of Theorem 6.2 uses, instead, Theorem A.1, which shows that in order for an abelian von Neumann algebra $A_0 \subset M$ to be unitarily conjugate to a subalgebra of a semi-regular maximal abelian $*$-subalgebra A of M it is sufficient that $A_0' \cap M$ be semi-regular abelian and that ${}_{A_0}L^2(M,\tau)_A$ contain a nonzero $A_0 - A$ Hilbert bimodule which is finite dimensional as a right A-module (a much weaker requirement).

3°. Note that by 3.4 and 4.3.2°, $A \subset M$ is HT_{s} Cartan if and only if $A \subset M$ is maximal abelian, M has property H relative to A and $A \subset M$ is ε_0-rigid for some $\varepsilon_0 \leq 1/3$.

4°. Note that the proof of Theorem 6.2 shows in fact that if A, A_0 are abelian von Neumann subalgebras of a finite factor M such that A is maximal abelian, M has property H relative to A, $A_0' \cap M$ is semi-regular abelian and $A_0 \subset M$ is ε_0-rigid, for some $\varepsilon_0 < 1$, then there exists $u \in \mathcal{U}(M)$ such that $u(A_0' \cap M)u^* = A$. In particular, if one calls HT_w the Cartan subalgebras $A \subset M$ with the properties that M has property H relative to A and there exists $A_0 \subset A$ with $A_0' \cap M = A$, $A_0 \subset M$ ε_0-rigid, for some $\varepsilon_0 < 1$, then any two HT_w Cartan subalgebras of a II_1 factor are unitary conjugate.

6.4. *Notation.* We denote by $\mathcal{HT}$ the class of finite separable (in norm $\| \ \|_2$) factors with HT Cartan subalgebras and by $\mathcal{HT}_{s}$ the class of finite separable factors with HT_{s} Cartan subalgebras. Note that $\mathcal{HT}_{s} \subset \mathcal{HT}$ and that Theorem 6.2 shows the uniqueness up to unitary conjugacy of HT Cartan subalgebras in factors $M \in \mathcal{HT}$.

6.5. Corollary. *If $A_i \subset M_i, i = 1, 2,$ are* HT *Cartan subalgebras and θ is an isomorphism from M_1 onto M_2 then there exists a unitary element $u \in M_2$ such that $u\theta(A_1)u^* = A_2$. Thus, there exists a unique* (*up to isomorphism*) *standard equivalence relation $\mathcal{R}_M^{\mathrm{HT}}$ on the standard probability space, implemented by the normalizer of the* HT *Cartan subalgebra of M.*

The next result shows that $\mathcal{HT}$ is closed to amplifications and tensor products and that it has good "continuity" properties. The proof of part 3° below, like the proof of 4.5.2°, uses A.2 and is inspired by the proofs of 4.5.1, 4.5.6 in [Po1].

6.6. Theorem. 1°. *If $M \in \mathcal{HT}$ (resp. $M \in \mathcal{HT}_{s}$) and $t > 0$ then $M^t \in \mathcal{HT}$ (resp. $M^t \in \mathcal{HT}_{s}$).*

2°. *If $M_1, M_2 \in \mathcal{HT}$ (resp. $M_1, M_2 \in \mathcal{HT}_{s}$) then $M_1 \overline{\otimes} M_2 \in \mathcal{HT}$ (resp. $M_1 \overline{\otimes} M_2 \in \mathcal{HT}_{s}$).*

3°. *If $M \in \mathcal{HT}_{s}$ then there exist a finite set $F \subset M$ and $\delta > 0$ such that if $N \subset M$ is a subfactor with $F \subset_\delta N$ then $N \in \mathcal{HT}_{s}$. In particular, if $N_k \subset M$*

are subfactors with $N_k \uparrow M$, then there exists k_0 such that $N_k \in \mathcal{HT}_{_s}, \forall k \geq k_0$. If in addition $N_k' \cap M = \mathbb{C}$, then all the $N_k, k \geq k_0$, contain the same $\mathrm{HT}_{_s}$ *Cartan subalgebra of M.*

Proof. 1°. Let $A \subset M$ be an HT Cartan subalgebra and $A_0 \subset A$ be so that $A_0 \subset M$ is a rigid embedding and $A_0' \cap M = A$. Choose some integer $n \geq t$. By 2.3.2° it follows that if D denotes the diagonal of $M_0 = M_{n\times n}(\mathbb{C})$ then $A\otimes D \subset M_n(M)$ has property H. Also, $(A_0 \otimes D)' \cap M \otimes M_{n\times n}(\mathbb{C}) = A \otimes D$ and by 4.6.1°, $A_0 \otimes D \subset M \otimes M_{n\times n}(\mathbb{C})$ is a rigid embedding.

If we now take $p \in A_0 \otimes D$ to be a projection of trace $\tau(p) = t/n$, then by 2.4.1° and 4.7.2°, it follows that $A_0^t = (A_0 \otimes D)p \subset M^t = pM_{n\times n}(\mathbb{C})p$ is rigid and M^t has property H relative to A^t. Thus, $M^t \in \mathcal{HT}$. In case $A_0 = A$, then $A_0^t = A^t$, so that M^t is in $\mathcal{HT}_{_s}$.

2°. This follows trivially by application of 2.3.2° and 4.6.1°, once we notice that if $A_i \subset M_i$ are maximal abelian $*$-subalgebras and $A_0^i \subset A_i$ satisfy $(A_o^i)' \cap M_i = A_i$, then $(A_0^1 \overline{\otimes} A_0^2)' \cap M_1 \overline{\otimes} M_2 = A_1 \overline{\otimes} A_2$.

3°. Let $A \subset M$ be a fixed $\mathrm{HT}_{_s}$ Cartan subalgebra of M. By 4.5.2°, it follows that there exist a finite subset F in the unit ball of M and $\varepsilon > 0$ such that if a subfactor $N_0 \subset M$ satisfies $F \subset_\varepsilon N_0$ and $N_0' \cap M = \mathbb{C}$ then N_0 contains a unitary conjugate $A_0 = uAu^*$ of A with $A_0 \subset N_0$ rigid and Cartan. Moreover, N_0 has property H relative to A_0 by 2.3.3° (since M has property H relative to A_0). Thus, $A_0 \subset N_0$ is $\mathrm{HT}_{_s}$ Cartan, proving the statement in the case of subfactors with trivial relative commutant.

To prove the general case, note first that by Step 1 in the proof of A.2, for the above given $\varepsilon > 0$ there exists $\delta_0 > 0$, with $\delta_0 \leq \varepsilon/4$, such that if $N \subset M$ is a subfactor with $A \subset_{\delta_0} N$ then there exist projections $p \in A$, $q \in N$, a unital isomorphism $\theta : Ap \to qNq$ and a partial isometry $v \in M$ such that $\tau(p) \geq 1 - \varepsilon/4$, $v^*v = p$, $vv^* = qq'$, for some projection $q' \in \theta(Ap)' \cap qMq$, and $va = \theta(a)v, \forall a \in Ap$.

Since Ap is maximal abelian in pMp, by spatiality it follows that $\theta(Ap)q'$ is maximal abelian in $q'qMq'q$. Thus, if $x \in \theta(Ap)' \cap qMq$ then $q'xq' \in \theta(Ap)q' \simeq \theta(Ap)$. Thus, there exists a unique normal conditional expectation E of $\theta(Ap)' \cap qMq$ onto $\theta(Ap)$ satisfying $q'xq' = E(x)q', \forall x \in \theta(Ap)' \cap qMq$.

Let $q_0' \in N' \cap M$ be the support projection of $E_{N'\cap M}(q')$. Thus, $q_0' \geq q'$ and if $b \in q_0'(N' \cap M)q_0'$ is so that $q'b = 0$ then $b = 0$. Since E is implemented by q', E is faithful on $q_0'(N' \cap M)q_0'q$, implying that if $b \in q_0'(N' \cap M)q_0'q$ and $a \in \theta(Ap)$ are positive elements with $E(b)a = 0$ then $ba = 0$. But if $ba = 0$ then $0 = E_N(ba) = E_N(b)a = (\tau(b)/\tau(q))a$ (because b commutes with the factor qNq). This shows that $E(b) \in \theta(Ap)$ must have support equal to q for any $b \in q_0'(N' \cap M)q_0'q$, with $b \geq 0, b \neq 0$. Thus, if f is a nonzero projection in $q_0'(N' \cap M)q_0'q$ then $q'fq' = E(f)q'$ has support q'. This implies that any projection $f \neq 0$ in $q_0'(N' \cap M)q_0'q$ must have trace $\tau(f) \geq \tau(q') \geq 1 - \varepsilon/4$, showing that $N' \cap M$ has an atom q_1' of trace $\tau(q_1') \geq 1 - \varepsilon/4$.

An easy calculation shows that if we denote by $\tilde{N} \subset M$ a unital subfactor with $q_1' \in \tilde{N}$ and $q_1'\tilde{N}q_1' = Nq_1'$ (N.B.: $\tilde{N}$ is obtained by amplifying Nq_1' by $1/\tau(q_1')$), then $F \subset_\varepsilon \tilde{N}$. Also, $\tilde{N}' \cap M = \mathbb{C}$ by construction. Thus, by the first part of the proof, $\tilde{N} \in \mathcal{HT}_s$. Since N is isomorphic to a reduction of $\tilde{N}$ by a projection, by part 1° it follows that $N \in \mathcal{HT}_s$ as well. $\square$

6.7. Corollary. 1°. *If $A \subset M$ is an* HT *Cartan subalgebra then any automorphism of M can be perturbed by an inner automorphism to an automorphism that leaves A invariant*; *i.e.*,

$$\mathrm{Aut}M/\mathrm{Int}M = \mathrm{Aut}(M, A)/\mathrm{Int}(M, A).$$

2°. *Let $M \in \mathcal{HT}_{_s}$ with $A \subset M$ its* $\mathrm{HT}_{_s}$ *Cartan subalgebra. Denote by $\mathcal{G}_{_\mathrm{HT}}(M)$ the subgroup of* $\mathrm{Aut}(M)$ *generated by the inner automorphisms and by the automorphisms leaving all elements of A fixed. Then $\mathcal{G}_{_\mathrm{HT}}(M)$ is an open-closed normal subgroup of* $\mathrm{Aut}(M)$, *the quotient group*

$$\mathrm{Out}_{_\mathrm{HT}}(M) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(M)/\mathcal{G}_{_\mathrm{HT}}(M)$$

is countable and is naturally isomorphic to the group of outer automorphisms of $\mathcal{R}_M^{^\mathrm{HT}}$, $\mathrm{Out}(\mathcal{R}_M^{^\mathrm{HT}}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\mathcal{R}_M^{^\mathrm{HT}})/\mathrm{Int}(\mathcal{R}_M^{^\mathrm{HT}})$.

Proof. 1°. If $\theta \in \mathrm{Aut}(M)$ then $\theta(A)$ is HT Cartan, so by Theorem 6.2 there exists a unitary element $u \in M$ such that $u\theta(A)u^* = A$.

2°. This is trivial by 4.4. $\square$

6.8. Corollary. *If $M \in \mathcal{HT}$ then any central sequence of M is contained in the* HT *Cartan subalgebra of M. Thus, $M' \cap M^\omega$ is always abelian and M is non-*Γ *if and only if the equivalence relation $\mathcal{R}_M^{^\mathrm{HT}}$ is strongly ergodic. In particular, $M \not\simeq M \overline{\otimes} R$.*

Proof. If $A \subset M$ is the HT Cartan subalgebra of M and $A_0 \subset A$ is so that $A_0 \subset M$ is rigid and $A_0' \cap M = A$ then by 4.3.1° we have $M' \cap M^\omega = M' \cap (A_0' \cap M)^\omega = M' \cap A^\omega$. $\square$

6.9. *Examples.* We now give a list of examples of HT inclusions of the form $B \subset B \rtimes_\sigma \Gamma_0$ and of factors in the class $\mathcal{HT}$ of the form $L^\infty(X, \mu) \rtimes \Gamma_0$, based on the examples in 5.2, 5.3.2°, 5.3.3°. Note that if $B \subset B \rtimes_\sigma \Gamma_0$ is an HT inclusion then Γ_0 must have the property H (cf. 3.1), but that in Section 5 we were able to provide examples of inclusions $B \subset B \rtimes_\sigma \Gamma_0$ satisfying the rigidity condition (6.1.2) only for certain property H groups Γ_0 and for certain actions of such groups (see Problem 6.12 below). Note also that by Theorem 6.2 if $M = L^\infty(X, \mu) \rtimes_\sigma \Gamma_0$ belongs to the class $\mathcal{HT}$ and Γ_0 is a property H group then $A = L^\infty(X, \mu)$ is automatically the (unique) HT Cartan subalgebra of M; i.e., $A \subset M$ must satisfy the rigidity condition (6.1.2) as well.

6.9.1. Let $\Gamma_0 = \mathrm{SL}(2,\mathbb{Z})$, $B_0 = L_\alpha(\mathbb{Z}^2)$, for some $\alpha \in \mathbb{T} \subset \mathbb{C}$, and σ_0 be the action of the group $\mathrm{SL}(2,\mathbb{Z})$ on B_0 induced by its action on $\mathbb{Z}^2$. Then $B_0 \subset M_\alpha \overset{\text{def}}{=} B_0 \rtimes_{\sigma_0} \mathrm{SL}(2,\mathbb{Z})$ is an $\mathrm{HT}_{_s}$ inclusion with M_α a type II_1 factor. In case α is not a root of 1, this gives $\mathrm{HT}_{_s}$ inclusions $R = B_0 \subset M_\alpha$ and when α is a n^{th} primitive root of 1, this gives $\mathrm{HT}_{_s}$ inclusions $B_0 \subset M_\alpha$, with B_0 homogeneous of type I_n and diffuse center. Indeed, in all these examples the property (6.1.1) is satisfied by 3.2, and property (6.1.2) is satisfied by 5.1. Moreover, by the isomorphism in 5.2.1°, if α is a root of 1 then $M_\alpha \in \mathcal{HT}_{_s}$ and any maximal abelian subalgebra of $B_0 = L_\alpha(\mathbb{Z}^2)$ is Cartan in M_α.

6.9.1′. If we take the inclusion $A = L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z}) = M$ from the previous example, which we regard as the group measure space construction $L^\infty(\mathbb{T}^2,\lambda) \subset L^\infty(\mathbb{T}^2,\lambda) \rtimes \mathrm{SL}(2,\mathbb{Z})$, through the usual identification of $\mathbb{T}^2$ with the dual of $\mathbb{Z}^2$ and of $L^\infty(\mathbb{T}^2,\lambda)$ with $L(\mathbb{Z}^2)$, and we "cut it in half" with a projection $p \in A$ of trace $1/2$, then we obtain the inclusion $(Ap \subset pMp) \simeq (L^\infty(\mathbb{S}^2,\lambda) \subset L^\infty(\mathbb{S}^2,\lambda) \rtimes \mathrm{PSL}(2,\mathbb{Z}))$, where $\mathbb{S}^2$ is the 2-sphere. Thus, by 6.9.1 and Theorem 6.6, it follows that $L^\infty(\mathbb{S}^2,\lambda) \rtimes \mathrm{PSL}(2,\mathbb{Z}) \in \mathcal{HT}_{_s}$.

6.9.2. If $\mathbb{F}_n \subset \mathrm{SL}(2,\mathbb{Z})$ is an embedding with finite index and σ_0 is the restriction to $\mathbb{F}_n$ of the action σ_0 on $B_0 = L_\alpha(\mathbb{Z}^2)$ considered in 1°, then $B_0 \subset B_0 \rtimes_{\sigma_0} \mathbb{F}_n$ is an $\mathrm{HT}_{_s}$ inclusion, which in case $\alpha = 1$ is an $\mathrm{HT}_{_s}$ Cartan subalgebra. Also, if $p \in L(\mathbb{Z}^2)$ has trace $(12(n-1))^{-1}$ then the inclusion $(L(\mathbb{Z}^2)p \subset p(L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))p)$ is an $\mathrm{HT}_{_s}$ Cartan subalgebra of the form $(A \subset A \rtimes \mathbb{F}_n)$. In all these cases, again, property (6.1.1) is satisfied by 3.2, and property (6.1.2) is satisfied by 5.2.2°.

6.9.3. If Γ_0 is an arithmetic lattice in $\mathrm{SU}(n,1), \mathrm{SO}(n,1), n \geq 2$, then there exist free weakly mixing trace-preserving actions σ_0 of Γ_0 on $A = L^\infty(X,\mu)$ such that $A \subset M = A \rtimes_{\sigma_0} \Gamma_0$ is $\mathrm{HT}_{_s}$ Cartan (cf. 3.2 and 5.2.3°).

6.9.4. If $\Gamma_0 = \mathrm{SL}(2,\mathbb{Q})$, $A = L(\mathbb{Q}^2)$ and $M = L(\mathbb{Q}^2 \rtimes \mathrm{SL}(2,\mathbb{Q})) = A \rtimes \mathrm{SL}(2,\mathbb{Q})$, then $A \subset M$ is HT Cartan but not $\mathrm{HT}_{_s}$ Cartan (cf. 3.2 and 5.3.2°).

6.9.5. Let $\Gamma_0, \sigma_0, (B_0,\tau)$ be as in 6.9.1, 6.9.2 or 6.9.3. Let $n \geq 1$ and $B = B_0^{\otimes n}, \sigma = \sigma_0^{\otimes n}$. Then $B \subset B \rtimes_\sigma \Gamma_0$ is an $\mathrm{HT}_{_s}$ inclusion (cf. 3.2, 3.3.3 and 5.2). Moreover, if $B_0 = A_0$ is abelian, then $A_0^{\otimes n} = A \subset A \rtimes_\sigma \Gamma_0$ is $\mathrm{HT}_{_s}$ Cartan.

6.9.6. Let $\Gamma_0, \sigma_0, (B_0,\tau)$ be any of the actions considered above. Let σ_1 be an ergodic action of Γ_0 on a von Neumann algebra $B_1 \simeq L^\infty(X,\mu)$. If $B = B_0 \overline{\otimes} B_1$ and $M = B \rtimes_{\sigma_0 \otimes \sigma_1} \Gamma_0$, then $B \subset M$ is an HT inclusion (cf. 3.2 and 5.2.4°). In particular, if $B_0 = A_0, B_1 = A_1$ are abelian and $A = A_0 \overline{\otimes} A_1$, then $A \subset M$ is an HT Cartan subalgebra. If σ_1 is taken to be a Bernoulli shift, then $A \subset M$ is not $\mathrm{HT}_{_s}$ Cartan. For any such group Γ_0 the action σ_1 can be taken nonstrongly ergodic by ([CW]). In this case, the resulting factor M has the property Γ of Murray and von Neumann, with $M' \cap M^\omega = M' \cap A^\omega$

abelian. Note that for each of the groups Γ_0 this gives three distinct HT Cartan subalgebras of the form $A \subset A \rtimes \Gamma_0$ (cf. 5.3.3°).

6.9.7. Let $\Gamma_0, \sigma_0, (B_0, \tau)$ be any of the actions considered above (so that $B_0 \subset B_0 \rtimes_{\sigma_0} \Gamma_0$ is an HT inclusion). Let also Γ_1 be a property H group and γ an action of Γ_1 on Γ_0 such that $\Gamma = \Gamma_0 \rtimes_\gamma \Gamma_1$ has property H (for instance, if Γ_1 is amenable or if γ is the trivial action, giving $\Gamma = \Gamma_0 \times \Gamma_1$). Let σ denote the Γ-action $\sigma_0 \rtimes \sigma_1$ on $B = \overline{\otimes}_{g\in\Gamma_1}(B_0, \tau_0)_g$ constructed in 3.3.3. Then $B \subset B \rtimes_\sigma \Gamma$ is an HT inclusion, which follows an HT Cartan subalgebra in case B_0 is abelian (cf. 3.1, 3.3.3, and the definitions).

6.10. Corollary. 1°. *If M is a McDuff factor, i.e., $M \simeq M\overline{\otimes}R$, then $M \notin \mathcal{HT}$. In particular, $R \notin \mathcal{HT}$.*

2°. *If M contains a relatively rigid type* II_1 *von Neumann subalgebra then $M \notin \mathcal{HT}$. In particular, if M contains $L(G)$ for some infinite property* T *group G, or if M contains a property* T *factor, then $M \notin \mathcal{HT}$.*

3°. *If M contains a copy of some $L_\alpha(\mathbb{Z}^2) \rtimes_\sigma \Gamma_1$, with Γ_1 a subgroup of finite index in* $\mathrm{SL}(2,\mathbb{Z})$ *and α an irrational rotation, then $M \notin \mathcal{HT}$.*

4°. *If M has property* H *(e.g., $M \simeq L(\mathbb{F}_n)$ for some $2 \leq n \leq \infty$) then $M \notin \mathcal{HT}$. In fact such factors do not even contain subfactors in the class $\mathcal{HT}$.*

Proof. 1° is trivial by 6.8, 2° and 3° are clear by 5.4.2° and 4° follows from 5.4.1°. □

6.11. *Definition.* A countable discrete group Γ_0 is an $\mathrm{H}_{\scriptscriptstyle T}$ (resp. $\mathrm{H}_{\scriptscriptstyle T_s}$) *group* if there exists a free ergodic measure-preserving action σ_0 of Γ_0 on the standard probability space (X, μ) such that $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes_{\sigma_0} \Gamma_0$ is an HT (resp. $\mathrm{HT}_{\scriptscriptstyle s}$) Cartan subalgebra. Note that an $\mathrm{H}_{\scriptscriptstyle T}$ group has property H but is not amenable.

6.12. *Problems.* 1°. Characterize the class of all $\mathrm{H}_{\scriptscriptstyle T}$ (resp. $\mathrm{H}_{\scriptscriptstyle T_s}$) groups.

2°. Construct examples of free ergodic measure-preserving actions σ of $\Gamma_0 = \mathbb{F}_n$ (or of any other noninner amenable property H group Γ_0) on $A = L^\infty(X,\mu)$ such that $A \subset M = A \rtimes_\sigma \Gamma_0$ is not HT Cartan. Is this the case if σ is a Bernoulli shift?

6.13. Corollary. 1°. $\mathrm{SL}(2,\mathbb{Z}), \mathbb{F}_n, n \geq 2$, *as well as any arithmetic lattice in* $\mathrm{SU}(n,1)$ *or* $\mathrm{SO}(n,1), n \geq 2$, *are* $\mathrm{H}_{\mathrm{T}_s}$ *groups.*

2°. *Let $\Gamma \subset \Gamma_0$ be an inclusion of groups with $[\Gamma_0 : \Gamma] < \infty$. Then Γ_0 is an* H_T (*resp.* $\mathrm{H}_{\mathrm{T}_s}$) *group if and only if Γ is an* H_T (*resp.* $\mathrm{H}_{\mathrm{T}_s}$) *group.*

3°. *If Γ_0 is an* H_T *group and Γ_1 has the property* H (*for instance, if Γ_1 is amenable*) *then $\Gamma_0 \times \Gamma_1$ is an* H_T *group.*

4°. *If Γ_0 is an* H_T *group and Γ_1 is amenable and acts on Γ_0 then $\Gamma_0 \rtimes \Gamma_1$ is an* H_T *group.*

Proof. Part 1° follows from $6.9.1^\circ - 3^\circ$, while parts 3° and 4° follow from 6.9.7.

To prove 2° note first that by 3.1 and $2.3.3^\circ$, Γ_0 has the property H if and only if Γ has the property H (this result can be easily proved directly, see e.g. [CCJJV]).

If Γ_0 is an H_τ group and $A \subset A \rtimes_\sigma \Gamma_0$ is HT Cartan and $A_0 \subset A$ is so that $A_0 \subset M$ is rigid and $A_0' \cap M = A$ then $A_0 \subset A \rtimes_\sigma \Gamma$ is also rigid, by $4.6.2^\circ$. Moreover, the fixed point algebra A^Γ is atomic (because $[\Gamma_0 : \Gamma] < \infty$), so if p is any minimal projection in A^Γ then $p(A \rtimes_\sigma \Gamma)p$ is a factor and $Ap \subset p(A \rtimes_\sigma \Gamma)p$ is an HT Cartan subalgebra. Thus, Γ is an H_τ group.

Conversely, if Γ is an H_τ group, then let $\Gamma_1 \subset \Gamma$ be a subgroup of finite index so that $\Gamma_1 \subset \Gamma_0$ is normal. By the first part, Γ_1 is an H_τ group. By part 4°, it follows that Γ_0 is an H_τ group. ☐

7. Subfactors of an $\mathcal{HT}$ factor

In this section we prove that the class $\mathcal{HT}$ is closed under extensions and restrictions of finite Jones index. More than that, we show that the lattice of subfactors of finite index of a factor in the class $\mathcal{HT}$ is extremely rigid.

7.1. LEMMA. *Let $N \subset M$ be an irreducible inclusion of factors with $[M : N] < \infty$ and $A \subset N$ a Cartan subalgebra of N. Denote by $\mathcal{N} = \mathcal{N}_N(A)$ the normalizer of A in N. Then*

1°. *$A' \cap M$ is a homogeneous type I_m algebra, for some $1 \leq m < \infty$, and if $A_1 = \mathcal{Z}(A' \cap M)$ then there exists a partition of the identity $q_1, q_2, \ldots, q_n \in \mathcal{P}(A_1)$ such that $A_1 = \Sigma_i A q_i$ and $E_N(q_i) = E_A(q_i) = 1/n, \forall i$.*

2°. *$\mathcal{N}$ normalizes A_1 and $Q \overset{\text{def}}{=} \mathrm{sp} A_1 N = \overline{\mathrm{sp}} A_1 \mathcal{N}$ is a type II_1 factor containing N, with $[Q : N] = n$. Moreover, $A_1 \subset Q$ is a Cartan subalgebra and the following is a nondegenerate commuting square*:

$$\begin{array}{ccc} N & \subset & Q \\ \cup & & \cup \\ A & \subset & A_1. \end{array}$$

3°. *$\mathcal{N}$ normalizes $A' \cap M = A_1' \cap M \simeq M_{m \times m}(A_1)$ and $P \overset{\text{def}}{=} \mathrm{sp}(A_1' \cap M)N = \overline{\mathrm{sp}}(A_1' \cap M)\mathcal{N}$ is a type II_1 factor containing Q, with $[P : Q] = m^2$. Moreover, the following is a nondegenerate commuting square*

$$\begin{array}{ccc} Q & \subset & P \\ \cup & & \cup \\ A_1 & \subset & A_1' \cap M. \end{array}$$

4°. *Any maximal abelian $*$-subalgebra A_2 of $A' \cap M = A_1' \cap M$ is a Cartan subalgebra in P, with $A_2 p \subset pPp$ implementing the same equivalence relation as $A_1 \subset Q$, $\forall p \in \mathcal{P}(A_2)$, $\tau(p) = 1/m$; i.e., $\mathcal{R}_{A_2 p \subset pPp} \simeq \mathcal{R}_{A_1 \subset Q}$ (equivalently,*

$\mathcal{R}_{A_2\subset P} \simeq (\mathcal{R}_{A_1\subset Q})^m$), *but with the two Cartan subalgebras possibly differing by their* 2*-cocycles.*

Proof. Since $\mathcal{N}$ normalizes A, it also normalizes $A' \cap M$, and thus $\mathcal{Z}(A' \cap M) = A_1$ as well. In particular, $A_1\mathcal{N} = \mathcal{N}A_1$ and $(A' \cap M)\mathcal{N} = \mathcal{N}(A' \cap M)$, showing that $\mathrm{sp}A_1\mathcal{N}$ and $\mathrm{sp}(A' \cap M)\mathcal{N}$ are $*$-algebras. Since $\mathcal{N}' \cap M = N' \cap M = \mathbb{C}$, this implies that Q, P are factors. In particular, this shows that the squares of inclusions in 2° and 3° are commuting and nondegenerate. Also, by definitions, A_1 is Cartan in Q.

Since $N \subset Q$ is a λ-Markov inclusion, for $\lambda^{-1} = [Q : N]$ (see e.g., [Po2] for the definition), it follows that $A \subset A_1$, with the trace τ inherited from M, is λ-Markov. Thus, $e = e_N^Q$ implements the conditional expectation $E_A^{A_1}$ and $A_1 \subset B = \langle A_1, A\rangle = \langle A_1, e\rangle$ gives the basic construction for $A \subset A_1$. Moreover, since A, A_1 are abelian, it follows that $\mathcal{Z}(B) = A = J_{A_1}AJ_{A_1}$ and that

$$A_1' \cap B = J_{A_1}A_1J_{A_1} \cap (J_{A_1}AJ_{A_1})' = J_{A_1}(A_1 \cap A')J_{A_1} = J_{A_1}A_1J_{A_1} = A_1.$$

Thus, A_1 is maximal abelian in B, implying that the Markov expectation of B onto A_1 given by $E(xey) = \lambda xy$, for $x, y \in A_1$, is the unique expectation of B onto A_1.

Also, for each $u \in \mathcal{N}$, $\mathrm{Ad}u$ acts on $A \subset A_1$ τ-preservingly. Thus, $\mathrm{Ad}u$ extends uniquely to an automorphism θ_u on $B = \langle A_1, e_A^{A_1}\rangle = \langle A_1, e_N^Q\rangle$ by $\theta_u(e_A^{A_1}) = e_A^{A_1}$. This automorphism leaves invariant the Markov trace on B. Also, since $\theta_u, u \in \mathcal{N}$, act ergodically on $A = \mathcal{Z}(B)$, it follows that B is homogeneous of type I_n, for some n. By [K2], it follows that there exists a matrix units system $\{e_{ij}\}_{1\le i,j\le n}$ in B such that $B = A \vee \mathrm{sp}\{e_{ij}\}_{i,j}$ with $A_1 = \Sigma_i Ae_{ii}$.

By the uniqueness of the conditional expectation E of B onto A_1, if we put $q_i = e_{ii}$ then $E(X) = \Sigma_i q_iXq_i, \forall X \in B$. In particular, the index of $A_1 \subset B$ is given by $\lambda^{-1} = n = \tau(e)^{-1}$ and by the Markov property we have $1/n = E(e) = \Sigma_i q_ieq_i$. Thus, $q_ieq_i = n^{-1}q_i$, and so $eq_ie = n^{-1}e = E(q_i)e$ as well, since $\tau(e) = \tau(q_i)$. This ends the proof of 1° and 2°.

Now, since A_1 is the center of $B_1 = A' \cap M = A_1' \cap M$ and $\mathrm{Ad}u, u \in \mathcal{N}$, act ergodically on A_1, it also follows that B_1 is homogeneous of type I_m, for some $m \ge 1$. This clearly implies 3°.

To prove 4°, let $\{f_{ij}\}_{1\le i,j\le m} \subset B_1$ be a matrix units system in B_1 such that $A_2 = \Sigma_j A_1f_{jj}$ and $B_1 = \Sigma_{i,j}A_1f_{ij}$ (cf. [K2]). A_2 is Cartan in P because by construction f_{ij} are in the normalizing pseudogroup of A_2 in P.

For each $u \in \mathcal{N}$ let $v(u)$ be a unitary element in B_1 such that

$$v(u)(uf_{jj}u^*)v(u)^* = f_{jj}, \quad \forall j$$

(this is possible because $uf_{jj}u^*$ and f_{jj} have the same central trace $1/m$ in B_1). Since $v(u)$ commute with $A_1 = \mathcal{Z}(B_1)$, $\forall u \in \mathcal{N}$, it follows that

A_1f_{11} with the action implemented on it by $\{v(u)u \mid u \in \mathcal{N}\}$ is isomorphic to A_1 with the action implemented on it by $\mathcal{N}$. Thus, the equivalence relation $\mathcal{R}_{A_1f_{11}\subset f_{11}Mf_{11}}$ is the same as the equivalence relation $\mathcal{R}_{A_1\subset Q}$, but with the 2-cocycle coming from the multiplication between the unitaries $v(u)u, u \in \mathcal{N}$ (for $A_1f_{11} \subset f_{11}Mf_{11}$) possibly differing from the 2-cocycle given by the multiplication of the corresponding $u \in \mathcal{N}$ (for $A_1 \subset Q$). □

7.2. Lemma. 1°. *Let $A^1 \subset M_1$ be a maximal abelian $*$-subalgebra in the type* II_1 *factor M_1. If there exists a von Neumann subalgebra $A^0 \subset A^1$ such that $A^0 \subset M_1$ is rigid and $A^1 \subset A^{0\prime} \cap M_1$ has finite index* (*in the sense of* [PiPo]), *then A^1 contains a von Neumann subalgebra A^1_0 such that $A^1_0 \subset M_1$ is rigid and $A^{1\prime}_0 \cap M_1 = A^1$.*

2°. *Let $M_0 \subset M_1$ be a subfactor of finite index with an* HT (*resp.* HT_s) *Cartan subalgebra $A \subset M_0$. If $A^1 \subset M_1$ is a maximal abelian $*$-subalgebra of M_1 such that $A^1 \supset A$ and M_1 has property* H *relative to A^1 then $A^1 \subset M_1$ is an* HT (*resp.* HT_s) *Cartan subalgebra.*

Proof. 1°. Since $A^1 \subset A^{0\prime}\cap M_1$ has finite index, it follows that $A^{0\prime}\cap M_1$ is a type $\mathrm{I_{fin}}$ von Neumann algebra and A^1 is maximal abelian in it (see e.g., [Po7]). It follows that there exists a finite partition of the identity with projections $\{f_k\}_k$ in A^1 such that $\{f_k\}'_k \cap A^{0\prime} \cap M_1 \subset A^1$. Thus, if we let $A^1_0 \overset{\text{def}}{=} \Sigma_k A^0 f_k$, then $A^{1\prime}_0 \cap M_1 \subset A^1$. By 4.6.3° it follows that $A_0 \subset M_1$ is a rigid embedding.

2°. This is an immediate application of 1°, once we notice that if $A^0 \subset A$ is so that $A^0 \subset M_0$ is rigid and $A^{0\prime} \cap M_0 = A$ then $A \subset A^{0\prime} \cap M_1$ has index majorized by $[M_1 : M_0]$, implying that $A^1 \subset A^{0\prime} \cap M_1$ has finite index as well. □

7.3. Theorem. *Let $N \subset M$ be an inclusion of type* II_1 *factors with $[M : N] < \infty$. Then*

1°. *$N \in \mathcal{HT}$ (resp. $N \in \mathcal{HT}_s$) if and only if $M \in \mathcal{HT}$ (resp. $M \in \mathcal{HT}_s$).*

2°. *Assume $N' \cap M = \mathbb{C}$ and $N, M \in \mathcal{HT}$. If $Q, P \subset M$ are the intermediate subfactors constructed out of an* HT *Cartan subalgebra of N, as in* 7.1, *then $Q, P \in \mathcal{HT}$ and the triple inclusion $N \subset Q \subset P \subset M$ is canonical. Moreover, the* HT *Cartan subalgebra of P is an* HT *Cartan subalgebra in M.*

3°. *If $M \in \mathcal{HT}$ and $N \subset M$ is an irreducible subfactor then $[M : N]$ is an integer. Moreover, the canonical weights of the graph $\Gamma_{N,M}$ of $N \subset M$ are integers.*

Proof. 1°. Since the algebra $\langle M, N\rangle$ in the basic construction $N \subset M \subset \langle M, N\rangle$ is an amplification of N, by Theorem 6.6 it follows that it is sufficient to prove that if $N \in \mathcal{HT}$ (resp. $N \in \mathcal{HT}_s$) then $M \in \mathcal{HT}$ (resp. $M \in \mathcal{HT}_s$). By 6.6.1°, it is in fact sufficient to prove this implication in the case $N' \cap M = \mathbb{C}$.

Let $A \subset N$ be an HT Cartan subalgebra and $A_1 = \mathcal{Z}(A' \cap M) \subset Q$ be constructed out of $A \subset N$ as in Lemma 7.1. We begin by showing that $A_1 \subset Q$ is an HT Cartan subalgebra. Let $q_1, q_2, \ldots, q_n \in A_1 \subset Q$ be so that $A_1 = \Sigma_i A q_i$, $E_N(q_i) = E_A(q_i) = 1/n$, as in Lemma 7.1. By the last part of 2.3.3°, it follows that Q has property H relative to A. But by the last part of 2.3.4° this implies Q has property H relative to A_1. Also, $A_1 \subset Q$ satisfies the conditions in part 2° of Lemma 7.2, implying that it is HT Cartan.

Next we prove that if A_2 is constructed as in part 3° of Lemma 7.1, then $A_2 \subset P$ is an HT Cartan subalgebra. Let $\{e_{ij}\}_{1 \leq i,j \leq m} \subset A_1' \cap M$ be a matrix units system which together with A_1 generates $A_1' \cap M$ and such that $A_2 = \Sigma_j A_1 e_{jj}$. Since P has an orthonormal basis made up of unitary elements commuting with A_1, by the last part of 2.3.3° it follows that P has property H relative to A_1. By applying the last part of 2.3.4°, we see that P has property H relative to A_2. Then 7.2.2° applies and we deduce that $A_2 \subset P$ is an HT Cartan subalgebra, which is even HT_s when $A \subset N$ (and thus $A_1 \subset Q$) is HT_s.

Having proved that $A_2 \subset P$ is an HT Cartan subalgebra, we now prove that A_2 is HT Cartan in M as well. Since A_2 is maximal abelian in M, 7.2.2° shows that it is sufficient to prove that M has property H relative to A_2. To do this, we prove that if A_3 is any maximal abelian subalgebra in $A_2' \cap M_1$, where $M_1 = \langle M, P \rangle$, then $A_3 \subset M_1$ is HT Cartan. This would finish the proof, because by the first part of 2.3.4° M_1 would have the property H relative to A_2, and then by the first part of 2.3.3° this would imply M has the property H relative to A_2.

Since M_1 is an amplification of $P \in \mathcal{HT}$, by Theorem 6.6 it follows that M_1, as well as any reductions of M_1 by projections in M_1, belong to $\mathcal{HT}$. Let $\mathcal{N}_1$ be the normalizer of A_2 in P. Since A_2 is regular in P, $\mathcal{N}_1'' = P$ and $\mathcal{N}_1' \cap M_1 = P' \cap M_1$. Let $\{p_t'\}_t$ be a partition of the identity with minimal projections in $P' \cap M_1$. For each t, the inclusion $A_2 p_t' \subset P p_t' \subset p_t' M_1 p_t'$ satisfies the hypothesis of Lemma 7.1. Thus, if A_2^t is a maximal abelian $*$-subalgebra of $(A_2 p_t')' \cap p_t' M_1 p_t'$, then $A_2 p_t'$ is included in A_2^t and by 7.1.4°, A_2^t is semi-regular in $p_t' M_1 p_t'$. In addition, by 7.2.1° it follows that A_2^t contains a von Neumann subalgebra A_0^t with ${A_0^t}' \cap p_t' M_1 p_t' = A_2^t$ and $A_0^t \subset p_t' M_1 p_t'$ rigid. Since $p_t' M_1 p_t' \in \mathcal{HT}$, by Theorem 6.2 it follows that $A_2^t \subset p_t' M_1 p_t'$ is HT Cartan. Moreover, $M_1 \in \mathcal{HT}$ implies $A_3 = \Sigma_t A_2^t$ is HT Cartan in M_1, while clearly $A_2 \subset A_3$, by construction.

2°. The triple inclusion $(N \subset Q \subset P \subset M)$ depends on the choice of the Cartan subalgebra $A \subset N$. But such A is unique up to conjugacy by unitaries in N, which leave Q and P fixed. The fact that the HT Cartan subalgebra of P is HT Cartan in M was proved in part 1°.

3°. With the notation in 1°, we have $[M : N] = nm^2[M : P]$, with $[M : P]$ being itself an integer, since P contains a Cartan subalgebra of M (see e.g., [Po8]).

The weights v_k of $\Gamma = \Gamma_{N,M}$ are square roots of indices of irreducible subfactors appearing in the Jones tower for $N \subset M$. Thus, v_k are square roots of integers. Since $v_* = 1$, $[M : N] \in \mathbb{N}$ and Γ is irreducible and has nonnegative integral entries, by the relations coming from $\Gamma\Gamma^t\vec{v} = [M : N]\vec{v}$, it follows recursively that all v_k must be integers. $\square$

7.4. *Definitions.* Let $N \subset M$ be an irreducible inclusion of factors in the class $\mathcal{HT}$ with $[M : N] < \infty$ and let $N \subset Q \subset P \subset M$ be the canonical triple inclusion defined in part 2° of Theorem 7.3.

7.4.1. $N \subset Q \subset P \subset M$ is called the *canonical decomposition* of $N \subset M$.

7.4.2. If $M = Q$, i.e., if the HT Cartan subalgebra A of N is so that $A' \cap M$ is abelian (thus HT Cartan in M) and $M = \mathrm{sp}AN = M$, then $N \subset M$ is a *type* C_- inclusion (or subfactor). If $N = P$, i.e., if $A' \cap M = A$ (so that A is Cartan in both N and M) then $N \subset M$ is of *type* C_+. If $P = Q$, i.e., if $A' \cap M$ is abelian, then $N \subset M$ is of *type* $C_\pm$.

7.4.3. If $N = Q, P = M$ then $N \subset M$ is of *type* C_0. More generally, an extremal inclusion $N \subset M$ of factors in the class $\mathcal{HT}$ is of *type* C_0 if the HT Cartan subalgebra A of N satisfies $A' \cap M = A \vee P_0$, with $P_0 \simeq M_{m\times m}(\mathbb{C}), m = [M : N]^{1/2}$, and $M = \mathrm{sp}(A' \cap M)N = \mathrm{sp}P_0N$.

7.5. Theorem. 1°. *Let $N \subset M$ be an irreducible inclusion of factors in the class $\mathcal{HT}$, with $[M : N] < \infty$. $N \subset M$ is of type C_- (resp. $C_+, C_\pm, C_0$) if and only if its dual inclusion $M \subset \langle M, N\rangle$ is of type C_+ (resp. $C_-, C_\pm, C_0$).*

2°. *If $N \subset M$ and $M \subset L$ are irreducible inclusions of factors in the class $\mathcal{HT}$ with finite index and both of type C_- (resp. C_+), then $N \subset L$ is an irreducible inclusion of type C_- (resp. C_+).*

3°. *If $N \subset M$ and $M \subset L$ are extremal inclusions of factors in the class $\mathcal{HT}$, both of type C_0, then $N \subset L$ is of type C_0 and so are all subfactors of the form $Np \subset pLp$, with $p \in \mathcal{P}(N' \cap L)$.*

4°. *Let $N \subset M$ and $M \subset L$ be irreducible inclusions of factors in the class $\mathcal{HT}$ with finite index and such that $N \subset M$ is of type C_+ and $M \subset L$ is of type C_-. If $A \subset N$ is an* HT *Cartan subalgebra then $A' \cap L$ is abelian and each irreducible inclusion $Np \subset pLp$ for p minimal projection in $N' \cap L$ is of type $C_\pm$. In particular this is the case if $(M \subset L) = (M \subset \langle M, N\rangle)$.*

5°. *Let $N \subset M$ be an inclusion of factors in the class $\mathcal{HT}$ with $[M : N] < \infty$. If $N \subset M$ is either irreducible of type C_- or extremal of type C_0 then $N \subset \langle M, N\rangle$ is a type C_0 inclusion, and so are all subfactors of the form $Np \subset p\langle M, N\rangle p$, for p projection in $N' \cap \langle M, N\rangle$.*

Proof. 1°. Let $A \subset N$ be an HT Cartan subalgebra of N. If $N \subset M$ is of type C_- then let $A' \cap M = \Sigma_i Aq_i$, where $\{q_i\}_{1\leq i\leq n} \subset A' \cap M$ is a partition of the identity with projections satisfying $E_N(q_i) = 1/n, \forall i$. Let $\alpha = e^{2\pi i/n}$ and denote $u = n\Sigma_i q_i e_N q_{i+1}$. We clearly have $[u, A] = 0$, $uq_iu^* = q_{i+1}$ and $E_N(u^j) = 0, \forall j \leq n-1$. Thus, the HT Cartan subalgebra $A_1 = A' \cap M$ of M is maximal abelian in $\langle M, N\rangle$ and is normalized by u^j, with $\langle M, N\rangle = \Sigma_j u^j M$; i.e., A_1 is the HT Cartan subalgebra in $\langle M, N\rangle$ as well, showing that $M \subset \langle M, N\rangle$ is of type C_+.

If $N \subset M$ is of type C_+, $A \subset N \subset M$ is HT Cartan in both factors and $u_1, u_2, \ldots, u_n \in \mathcal{N}_M(A)$ are unitary elements such that $M = \Sigma_i u_i N$ and $E_N(u_i^* u_j) = \delta_{ij}$ then $q_j = u_j e_N u_j^*$ is a partition of the identity with projections in $\langle M, N\rangle$ and we have $A' \cap \langle M, N\rangle = \Sigma_j q_j A, \langle M, N\rangle = \Sigma_j q_j M$. Thus, $M \subset \langle M, N\rangle$ is of type C_-.

If $N \subset P \subset M$ is so that $N \subset P$ is C_-, $P \subset M$ is C_+ then we have the irreducible inclusions $M \subset \langle M, P\rangle$, which is C_-, and $\langle M, P\rangle \subset \langle M, N\rangle$, which is an amplification of $P \subset \langle P, N\rangle$, thus of type C_+. This shows that $M \subset \langle M, N\rangle$ is $C_\pm$.

If $N \subset M$ is of type C_0 and $A \subset N$ is an HT Cartan subalgebra with $A' \cap M = \Sigma_{i,j} e_{ij} A$ for some matrix units system $\{e_{ij}\}_{1 \leq i,j \leq m} \subset A' \cap M$, then denote $e'_{ij} = m\Sigma_k e_{ki} e_N e_{jk}, 1 \leq i, j \leq m$. It is immediate to show that $\{e'_{ij}\}_{i,j}$ is a matrix units system which commutes with A and with $\{e_{kl}\}_{k,l}$, that $\{e'_{ij}\}_{i,j}$ is an orthonormal basis of $\langle M, N\rangle$ over M and that $\{e'_{ij} e_{kl}\}_{i,j,k,l}$ is an orthonormal basis of $\langle M, N\rangle$ over N. It follows that $A' \cap \langle M, N\rangle = \mathrm{sp}\{e'_{ij} e_{kl}\}_{i,j,k,l} A$. Thus, if $A_2 \subset A' \cap M$ is a maximal abelian subalgebra, then $A_2' \cap \langle M, N\rangle = \Sigma_{i,j} e'_{ij} A_2$. This shows that $M \subset \langle M, N\rangle$ is of type C_0.

2°. By duality in the Jones tower ([PiPo]) and part 1°, it is sufficient to prove that if $N \subset M, M \subset L$ are of type C_+ then so is $N \subset L$. But this is trivial, since if $A \subset N$ is HT Cartan in N then it first follows that N is Cartan in M, then in L.

3°. Let $\{e_{ij}\}_{1 \leq i,j \leq m} \subset A' \cap M$ be a matrix units system such that $A' \cap M = \Sigma_{i,j} e_{ij} A$, as in the proof of the last part of 1° (thus, $[M : N] = m^2$). Let $A_2 = \Sigma_j e_{jj} A$, which is HT Cartan in M. Let $\{f'_{kl}\}_{1 \leq k,l \leq m'} \subset A_2' \cap L$ be a matrix units system such that $A_2' \cap L = \Sigma_{k,l} f'_{kl} A_2$, with $m'^2 = [L : M]$. Then $\{f_{ts}\}_{t,s} = \{e_{i1} f'_{kl} e_{1j} \mid 1 \leq i, j \leq m, 1 \leq k, l \leq m'\}$ is a matrix units system in $A' \cap L$ and if we denote $P_0 \simeq M_{mm' \times mm'}(\mathbb{C})$ the algebra it generates, then clearly $E_N(f_{st}) = \delta_{st}/mm'$. Since $[L : N] = (mm')^2$, and since we have the commuting square

$$\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ A & \subset & A' \cap L \end{array}$$

as well as

$$\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ A & \subset & A \vee P_0 \end{array}$$

with $A \vee P_0 \subset A' \cap L$ and with P_0 containing an orthonormal system of L over N made up of mm' elements, it follows that $A \vee P_0 = A' \cap L$, thus showing that $N \subset L$ is of type C_0.

Finally, if $p \in \mathcal{P}(N' \cap L)$ then in particular $p \in A \vee P_0$. By the above commuting squares, we have $E_A(p) = E_N(p) = \tau(p)1$. But $A = \mathcal{Z}(A \vee P_0)$, implying that p has scalar central trace in $A \vee P_0$. Thus, $(Ap)' \cap pLp = p(A \vee P_0)p$ is homogeneous of type I. Since we also have $pLp = p(\mathrm{sp}P_0N)p = p(\mathrm{sp}P_0)pNp$, this shows that $Np \subset pLp$ is of type C_0.

4°. Let $A \subset N$ be the HT Cartan subalgebra of N, which is thus HT Cartan in M as well. Thus $A_1 = A' \cap L$ is abelian with $L = \mathrm{sp}A_1M$. Since any irreducible projection $p \in N' \cap L$ lies in A_1, by cutting these relations with p we obtain that $(Ap)' \cap pLp$ is abelian, which by Lemma 7.1 means that $Np \subset pLp$ has only type C_- and C_+ components in its canonical decomposition.

5°. This is immediate from the proofs in 1° and the last part of 3°. $\square$

7.6. *Examples.* 1°. Let Γ_0 be a property H group and σ a free, weakly mixing measure-preserving action of Γ_0 on the probability space (X, μ) such that the Cartan subalgebra $L^\infty(X, \mu) = A \subset N = L^\infty(X, \mu) \rtimes_\sigma \Gamma_0$ contains a von Neumann subalgebra $A_1 \subset A$ with $A_1' \cap N = A$ and $A_1 \subset N$ rigid. Let $\Gamma_1 \subset \Gamma_0$ be a subgroup of finite index and σ_0 the left action of Γ_0 on Γ_0/Γ_1. Let $A_0 = \ell^\infty(\Gamma_0/\Gamma_1)$ and $M = A \otimes A_0 \rtimes_{\sigma\otimes\sigma_0} \Gamma_0$.

Then $N, M \in \mathcal{HT}$ and if we identify N with the subfactor of M generated by $A = A \otimes \mathbb{C}$ and by the canonical unitaries $\{u_g\}_g \subset M$ implementing the action $\sigma \otimes \sigma_0$ on $A \otimes A_0$, then $N \subset M$ is an irreducible type C_- inclusion. Moreover, if we denote $N_1 = A \vee \{u_g\}_{g\in\Gamma_1} \simeq A \rtimes_\sigma \Gamma_1 \subset N$ then $N_1 \subset N$ is a type C_+ inclusion and $N_1 \subset N \subset M$ is a basic construction.

We have $[M : N] = [N : N_1] = [\Gamma_0 : \Gamma_1]$, the standard invariant of $N_1 \subset N$ coincides with the standard invariant $\mathcal{G}_{\Gamma_1\subset\Gamma_0}$ of $R \rtimes \Gamma_1 \subset R \rtimes \Gamma_0$ studied in [KoYa] and the standard invariant of $N \subset M$ is the dual of $\mathcal{G}_{\Gamma_1\subset\Gamma_0}$. In particular, $N_1 \subset N \subset M$ are finite depth inclusions.

2°. Let Γ_0, σ, A be as in example 1° above and let π_0 be a finite-dimensional irreducible projective representation of Γ_0 on the Hilbert space $\mathcal{H}_0$, with scalar 2-cocycle v. Let $B_0 = \mathcal{B}(\mathcal{H}_0)$ and $\sigma_0(g) = \mathrm{Ad}\pi_0(g)$ be the action of Γ_0 on B_0 implemented by π_0. Denote $M = M_{\pi_0} = A \otimes B_0 \rtimes_{\sigma\otimes\sigma_0} \Gamma_0$ and let N be the subfactor of M generated by $A \otimes 1 = \mathcal{Z}(A \otimes B_0)$ and by the canonical unitaries $\{u_g\}_{g\in\Gamma_0} \subset M$ implementing the action $\sigma \otimes \sigma_0$. Thus, $N \simeq A \rtimes_\sigma \Gamma_0$, $M \simeq M_{n\times n}(A \rtimes_{\sigma,v} \Gamma_0)$ and both belong to the class $\mathcal{HT}$.

Moreover, $N \subset M$ is an irreducible type C_0 inclusion and its standard invariant coincides with the standard invariant of the generalized Wassermann-type subfactor corresponding to the projective representation π_0, i.e.:

$$\begin{array}{ccccccc} \mathbb{C} & \subset & \mathrm{End}(\mathcal{H}_0)^{\sigma_0} & \subset & \mathrm{End}(\mathcal{H}_0 \otimes \overline{\mathcal{H}_0})^{\sigma_0 \otimes \overline{\sigma_0}} & \subset & \dots \\ & & \cup & & \cup & & \\ & & \mathbb{C} & \subset & \mathbb{C} \otimes \mathrm{End}(\mathcal{H}_0)^{\overline{\sigma_0}} & \subset & \dots . \end{array}$$

3°. Let σ be the action of $\mathrm{SL}(2,\mathbb{Z})$ on $L_\alpha(\mathbb{Z}^2)$ implemented by the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$, as in 5.2.1° and 6.9.1°, for α a primitive root of 1 of order n. Let $M_\alpha = L_\alpha(\mathbb{Z}^2) \rtimes_\sigma \mathrm{SL}(2,\mathbb{Z})$, $A = \mathcal{Z}(L_\nu(\mathbb{Z}^2))$ and $N = A \vee \{u_g\}_g$ be the von Neumann algebra generated by A and the canonical unitaries in M_α implementing the action σ. Then $N, M_\alpha \in \mathcal{HT}_s$ and $N \subset M_\alpha$ is an irreducible inclusion of type C_0 with $[M_\alpha : N] = n^2$. Indeed, we have already noticed in 6.9.1° that $N \in \mathcal{HT}_s$, so that by 7.3 we have $M_\alpha \in \mathcal{HT}_s$. Also, by construction we have $A' \cap M_\alpha = L_\alpha(\mathbb{Z}^2) = A \otimes B_0$, with $B_0 \simeq M_{n\times n}(\mathbb{C})$, and $M_\alpha = \mathrm{sp} L_\alpha(\mathbb{Z}^2) N$.

One can show that $N \subset M_\alpha$ is isomorphic to a type C_0 inclusion $N \subset M_{\pi_0}$ as in example 2°, when taking $\Gamma_0 = \mathrm{SL}(2,\mathbb{Z})$, with σ, σ_0 the actions of $\mathrm{SL}(2,\mathbb{Z})$ on $A = \mathcal{Z}(L_\alpha(\mathbb{Z}^2)) \simeq L((n\mathbb{Z})^2)$, $B_0 = L_\alpha((\mathbb{Z}/n\mathbb{Z})^2) \simeq M_{n\times n}(\mathbb{C})$. Note that the standard invariant ([Po3]) of $N \subset M_\alpha$ depends only on the order n of α, because if π_0, π_0' are representations corresponding to primitive roots α, α' of order n then there exists an automorphism γ of the group $(\mathbb{Z}/n\mathbb{Z})^2$ such that $\pi' = \pi \circ \gamma$. But we do not know whether the isomorphism class of $N \subset M_\alpha$ depends only on n.

We now reformulate the results in Theorem 7.5 in terms of correspondences. For the definition of Connes' general $N - M$ *correspondences* (or $N - M$ *Hilbert bimodules*) $\mathcal{H} = {}_N\mathcal{H}_M$, of the *adjoint* $\overline{\mathcal{H}} = {}_M\overline{\mathcal{H}}_N$ of $\mathcal{H}$, as well as for the definition of the *composition* $\mathcal{H} \circ \mathcal{K}$ (also called *tensor product*, or *fusion*) of correspondences $\mathcal{H} = {}_N\mathcal{H}_M, \mathcal{K} = {}_M\mathcal{K}_P$ see [C7], [Po1], [Sa].

7.7. *Definition.* Let $N, M \in \mathcal{HT}$ and $\mathcal{K}$ be an $N - M$ correspondence, viewed as a Hilbert $N-M$ bimodule. Assume that $\dim_N \mathcal{K}_M \overset{\text{def}}{=} \dim_N \mathcal{K} \cdot \dim \mathcal{K}_M < \infty$ and that $\mathcal{K}$ is *irreducible*, i.e., $N \vee (M^{\mathrm{op}})' = \mathcal{B}(\mathcal{K})$. We say that $\mathcal{K}$ is of *type* C_- (resp. $C_+, C_\pm, C_0$) if the inclusion $N \subset (M^{\mathrm{op}})'$ is of type C_- (resp. $C_+, C_\pm, C_0$), in the sense of Definitions 7.4.

Finite index correspondences (resp. bimodules) between factors in the class $\mathcal{HT}$ will also be called HT *correspondences* (resp. HT *bimodules*).

7.8. Corollary. *Let* ${}_N\mathcal{H}_M, {}_M\mathcal{K}_L$ *be irreducible* HT *bimodules.*

1°. *$\mathcal{H}$ is of type C_- (resp. $C_+, C_\pm, C_0$) if and only if $\overline{\mathcal{H}}$ is of type C_+ (resp. $C_-, C_\pm, C_0$).*

2°. *If both $\mathcal{H}, \mathcal{K}$ are of type C_- (resp. C_+, resp. C_0) then $\mathcal{H} \circ \mathcal{K}$ is irreducible of type C_- (resp. irred. C_+, resp. a sum of irreducible C_0). In particular, the class of* HT *bimodules (or correspondences) of type C_0 over an* HT *factor forms a selfadjoint tensor category.*

3°. *If $\mathcal{H}$ is of type C_+ and $\mathcal{K}$ is of type C_- then $\mathcal{H} \circ \mathcal{K}$ is a direct sum of irreducible type $C_\pm$ bimodules. Also, $\mathcal{K} \circ \overline{\mathcal{K}}$ is a direct sum of irreducible C_0 bimodules.*

Proof. Part 1° is a reformulation of 7.5.1°, while 2° and 3° are direct consequences of 7.5.2° − 5°. ☐

7.9. *Definition.* Let $M \in \mathcal{HT}$ and $\theta \in \mathrm{Aut}M$ be a periodic automorphism of M, with $\theta^n = \mathrm{id}$ and θ^k outer $\forall 0 < k < n$. Then θ *is of type* C_- (resp. C_+) if the inclusion $M \subset M \rtimes_\theta \mathbb{Z}/n\mathbb{Z}$ is of type C_- (resp. C_+). By the uniqueness of the HT Cartan subalgebra, this property is clearly a conjugacy invariant for θ.

7.10. Corollary. *The factor $N = L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$ has two nonconjugate, period-two automorphisms, one of type C_- and one of type C_+.*

Proof. In example 7.6.1°, take $\Gamma_1 \subset \Gamma_0 = \mathrm{SL}(2,\mathbb{Z})$ a subgroup of index 2 and $(X,\mu) = (\mathbb{T}^2,\mu)$ with $\mathrm{SL}(2,\mathbb{Z})$ acting on it in the usual way. Then $N = L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$ and the resulting type C_- inclusion $N \subset M$ given by the construction 7.6.1° is of index 2. Thus, by Goldman's theorem, it is given by a period 2 automorphism θ_-, which is thus of type C_-. Alternatively, we can take θ_- to be the automorphism given by the nontrivial character γ of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ with $\gamma^2 = 1$, defined by $\gamma(a) = -a$, $\gamma(b) = b$, on the generators a, b of period 4, resp. 6 of $\mathrm{SL}(2,\mathbb{Z})$, and $\gamma(\mathbb{Z}^2) = 1$.

Now take θ_+ to be the automorphism of N implemented by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. Thus, $N \subset M = N \rtimes_{\theta_+} \mathbb{Z}/2\mathbb{Z}$ coincides with $L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})) \subset L(\mathbb{Z}^2 \rtimes \mathrm{GL}(2,\mathbb{Z}))$, and since $\mathrm{GL}(2,\mathbb{Z})$ acts freely on $\mathbb{Z}^2$, it follows that $L(\mathbb{Z}^2)' \cap M = L(\mathbb{Z}^2)$, so that $N \subset M$ is of type C_+. ☐

7.11. *Question.* Let $N \simeq L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$. Is, then, any irreducible type C_-, C_+ or C_0 inclusion of factors $N \subset M$ isomorphic to a "model" inclusion 7.6.1°–7.6.2°?

8. Betti numbers for $\mathcal{HT}$ factors

8.1. *Definition.* Let $M \in \mathcal{HT}$ and $\mathcal{R}_M^{\mathrm{HT}}$ be the standard equivalence relation implemented by the normalizer of the HT Cartan subalgebra of M, as in Corollary 6.5. Let $\{\beta_n(\mathcal{R}_M^{\mathrm{HT}})\}_{n\geq 0}$ be the ℓ^2-Betti numbers of $\mathcal{R}_M^{\mathrm{HT}}$, as defined by Gaboriau in [G2]. For each $n = 0, 1, 2, \ldots$, we denote $\beta_n^{\mathrm{HT}}(M) \stackrel{\mathrm{def}}{=} \beta_n(\mathcal{R}_M^{\mathrm{HT}})$ and call it the n^{th} ℓ^2_{HT}-*Betti number* (or simply the n^{th} Betti number) of M. By 6.5, $\beta_n^{\mathrm{HT}}(M)$ are isomorphism invariants for M.

From the results in Section 6 and the properties proved by Gaboriau for ℓ^2-Betti numbers of standard equivalence relations, one immediately gets:

8.2. Corollary. 0°. *If M is of type* II_1 *then* $\beta_0^{\mathrm{HT}}(M) = 0$ *and if* $M = M_{n\times n}(\mathbb{C})$ *then* $\beta_0^{\mathrm{HT}}(M) = 1/n$.

1°. *If* $A \subset M = A \rtimes_\sigma \Gamma_0$ *is a* HT *Cartan subalgebra, for some countable discrete group* Γ_0 *acting freely and ergodically on* $A \simeq L^\infty(X,\mu)$, *then* $\beta_n^{\mathrm{HT}}(M)$ *is equal to the* n^{th} ℓ^2*-Betti number of* Γ_0, $\beta_n(\Gamma_0)$, *as defined in* [ChG]).

2°. *If* $M \in \mathcal{HT}$ *and* $t > 0$ *then* $\beta_n^{\mathrm{HT}}(M^t) = \beta(M)/t, \forall n$.

3°. *If* $M_1, M_2 \in \mathcal{HT}$ *then for each* $n \geq 0$ *the following Künneth-type formula holds*:

$$\beta_n^{\mathrm{HT}}(M_1 \overline{\otimes} M_2) = \sum_{i+j=n} \beta_i^{\mathrm{HT}}(M_1)\beta_j^{\mathrm{HT}}(M_2),$$

where $0 \cdot \infty = 0$ *and* $b \cdot \infty = \infty$ *if* $b \neq 0$.

4°. *Let* $M \in \mathcal{HT}_{_s}$ *and let* $N_k \subset M, k \geq 1$, *be an increasing sequence of subfactors with* $N_k \uparrow M$ (*so that* $N_k \in \mathcal{HT}_{_s}$, *for* k *large enough, by* 6.8.3°). *Then* $\liminf_{k\to\infty} \beta_n^{\mathrm{HT}}(N_k) \geq \beta_n^{\mathrm{HT}}(M)$.

Proof. 0°. This is trivial by the definitions and [G2].

1°. By 8.1, we have $\beta_n^{\mathrm{HT}}(M) = \beta_n(\mathcal{R}_M^{\mathrm{HT}})$. But $\mathcal{R}_M^{\mathrm{HT}} = \mathcal{R}_{\Gamma_0}$, and by Gaboriau's theorem the latter has Betti numbers $\beta_n(\mathcal{R}_{\Gamma_0})$ equal to the Cheeger-Gromov ℓ^2-Betti numbers $\beta_n(\Gamma_0)$ of the group Γ_0.

2°. By Section 6 we know that the class $\mathcal{HT}$ is closed under amplifications and tensor products. Moreover, by 1.4.3 the "amplification" by t of a Cartan subalgebra $A \subset M$ has a normalizer that gives rise to the standard equivalence relation $(\mathcal{R}_M^{\mathrm{HT}})^t$. Then formula 2° is a consequence of Gaboriau's similar result for standard equivalence relations.

Part 3° follows similarly, by taking into account that if $A_1 \subset M_1, A_2 \subset M_2$ are Cartan subalgebras then $\mathcal{N}(A_1 \otimes A_2)'' = (\mathcal{N}(A_1) \otimes \mathcal{N}(A_2))''$.

4°. By 6.8.3°, there exists k_0 and an $\mathrm{HT}_{_s}$ Cartan subalgebra A of M such that $A \subset N_k, \forall k \geq k_0$. Then the statement follows from Theorem 5.13 in [G2]. $\square$

8.3. Corollary. 1°. *If* $M \in \mathcal{HT}$ *has at least one nonzero, finite Betti number then* $\mathscr{F}(M) = \{1\}$ *and in fact* $M^{t_1} \overline{\otimes} \cdots \otimes M^{t_n}$ *is isomorphic to* $M^{s_1} \overline{\otimes} \cdots \otimes M^{s_m}$ *if and only if* $n = m$ *and* $t_1 \ldots t_n = s_1 \ldots s_m$. *Equivalently,* $\{M^{\overline{\otimes} m}\}_{m \geq 1}$ *are stably nonisomorphic and all the tensor powers* $M^{\overline{\otimes} m}$ *have trivial fundamental group,* $\mathscr{F}(M^{\overline{\otimes} m}) = \{1\}, \forall m \geq 1$.

2°. *If* $M \in \mathcal{HT}$ *and* $\beta_1^{\mathrm{HT}}(M) \neq 0$ *or* ∞, *then* M *is not the tensor product of two factors* M_1, M_2 *in the class* $\mathcal{HT}$. *More generally if* $\beta_k^{\mathrm{HT}}(M)$ *is the first nonzero finite Betti number for* M, *then* $M^{\overline{\otimes} m}$ *cannot be expressed as the tensor product of* $km + 1$ *or more factors in the class* $\mathcal{HT}$.

Proof. 1°. First note that if M has $\beta_k^{\text{HT}}(M)$ as first nonzero Betti number, then the formula $\beta_k^{\text{HT}}(M^t) = \beta_k^{\text{HT}}(M)/t$ implies that $M \not\simeq M^t$ if $t \neq 1$. Thus, $\mathscr{F}(M) = \{1\}$.

Also, by the Künneth formula 8.2.2°, if $\beta_{n_i}^{\text{HT}}(M_i)$ is the first nonzero finite Betti number for $M_i \in \mathcal{HT}, i = 1, 2$, and we put $n = n_1 + n_2$, then $\beta_n^{\text{HT}}(M_1 \overline{\otimes} M_2) = \beta_n^{\text{HT}}(M_1)\beta_n^{\text{HT}}(M_2)$, is the first nonzero finite Betti number for $M_1 \overline{\otimes} M_2$.

Thus, $\beta_{km}^{\text{HT}}(M^{\overline{\otimes} m})$ is the first nonzero finite Betti number for $M^{\overline{\otimes} m}, m \geq 1$, showing that $\{M^{\overline{\otimes} m}\}_{m \geq 1}$ are stably nonisomorphic.

2°. This is trivial by the first part of the proof and the Künneth formula 8.2.2°. □

8.4. Corollary. 1°. *Let $N \subset M$ be an irreducible inclusion of factors in the class $\mathcal{HT}$ with $[M : N] < \infty$. If $N \subset M$ is of type C_- then $\beta_n^{\text{HT}}(M) = \beta_n^{\text{HT}}(N), \forall n$. If $N \subset M$ is of type C_+ then $\beta_n^{\text{HT}}(M) = [M : N]\beta_n^{\text{HT}}(N)$.*

2°. *Let $N \subset M$ be an extremal inclusion of factors in the class $\mathcal{HT}$. If $N \subset M$ is of type C_0 then $\beta_n^{\text{HT}}(M) = [M : N]^{1/2}\beta_n^{\text{HT}}(N), \forall n$.*

3°. *If $N \subset Q \subset P \subset M$ is the canonical decomposition of an irreducible inclusion of factors $N \subset M$ in the class $\mathcal{HT}$, then $\beta_n^{\text{HT}}(Q) = \beta_n^{\text{HT}}(N)$, $\beta_n^{\text{HT}}(P) = [P : Q]^{1/2}\beta_n^{\text{HT}}(N)$ and $\beta_n^{\text{HT}}(M) = [M : P]\beta_n^{\text{HT}}(P)$.*

4°. *Let $M \in \mathcal{HT}$, $N \subset M$ be a subfactor of finite index, $(\Gamma_{N,M}, (v_k)_k)$ be the graph of $N \subset M$, with its standard weights. Let also $\{\mathcal{H}_k\}_k$ be the list of irreducible Hilbert M-bimodules appearing in some $L^2(M_n, \tau), n = 0, 1, 2, \dots$, with $\{M \subset M(\mathcal{H}_k)\}_k$ the corresponding irreducible inclusions of factors. If $\beta_n^{\text{HT}}(M) \neq 0$ or ∞ for some $n \geq 1$ then $v_k = \beta_n^{\text{HT}}(M(\mathcal{H}_k))/\beta_n^{\text{HT}}(M), \forall k$. Thus,*

$$\Gamma_{N,M}\Gamma_{N,M}^t(\beta_n^{\text{HT}}(M(\mathcal{H}_k)))_k = [M : N](\beta_n^{\text{HT}}(M(\mathcal{H}_k)))_k.$$

Proof. 1°. If $N \subset M$ is of type C_+ then $\mathcal{R}_N^{\text{HT}}$ is a subequivalence relation of index $[M : N]$ in $\mathcal{R}_M^{\text{HT}}$, so that by [G2] we have

$$\beta_n^{\text{HT}}(M) = \beta_n(\mathcal{R}_M^{\text{HT}}) = [M : N]\beta_n(\mathcal{R}_N^{\text{HT}}) = [M : N]\beta_n^{\text{HT}}(N).$$

If $N \subset M$ is of type C_- then by part 1° of Theorem 7.5, $M \subset \langle M, N\rangle$ is of type C_+. Since $\langle M, N\rangle$ is the $[M : N]$-amplification of N, by the first part and by formula 8.2.2, we get:

$$\beta_n^{\text{HT}}(N) = [M : N]^{-1}\beta_n^{\text{HT}}(\langle M, N\rangle) = [M : N]^{-1}[M : N]\beta_n^{\text{HT}}(M).$$

2°. If $N \subset M$ is of type C_0 then by 7.1 the equivalence relation $\mathcal{R}_M^{\text{HT}}$ is an $[M : N]^{1/2}$-amplification of $\mathcal{R}_N^{\text{HT}}$. Thus, $\beta_n^{\text{HT}}(M) = [M : N]^{1/2}\beta_n^{\text{HT}}(N)$.

3°. This is just a combination of 1° and 2°.

4°. Note that all subfactors $M \subset M(\mathcal{H}_k)$ appear as irreducible inclusions of factors in some $M \subset M_{2n}$. By Jones' formula for the local indices ([J1]), if

p is a minimal projection in $M' \cap M_{2n}$ with $(Mp \subset pM_{2n}p) \simeq (M \subset M(\mathcal{H}_k))$ then $[M(\mathcal{H}_k) : M]/\tau(p)^2 = [M_{2n} : M]$. On the other hand, since M_{2n} is the $[M : N]^n$-amplification of M and since $M(\mathcal{H}_k) \simeq pM_{2n}p$, it follows that $M(\mathcal{H}_k)$ is the $\tau(p)[M : N]^n$- amplification of M. By 8.2.2°, this yields $\beta_n^{\text{HT}}(M(\mathcal{H}_k)) = [M(\mathcal{H}_k) : M]^{1/2}\beta_n^{\text{HT}}(M) = v_k\beta_n^{\text{HT}}(M)$. $\square$

Using the inventory of examples 6.9 of factors in the class $\mathcal{HT}$, and the calculations of ℓ^2-Betti numbers for groups in [ChGr], [B], from 8.2.1° above we get the following list of Betti numbers for factors:

8.5. Corollary. 1°. *If $\alpha \in \mathbb{T}$ is a primitive root of unity of order n, then $M_\alpha = L_\alpha(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z}) \in \mathcal{HT}_{s}$ (cf. 6.9.1) and $\beta_1^{\text{HT}}(M_\alpha) = (12n)^{-1}$, while $\beta_k^{\text{HT}}(M_\alpha) = 0, \forall k \neq 1$.*

2°. *If α, α' are primitive roots of unity of order n respectively n' then $M_\alpha \simeq M_{\alpha'}$ if and only if $n = n'$.*

Proof. 1°. By 5.2.1°, 8.2.1° and 8.2.2°, $\beta_k^{\text{HT}}(M_\alpha) = \beta_k(\mathrm{SL}(2,\mathbb{Z}))/n$. But by [B] we have $\beta_1(\mathrm{SL}(2,\mathbb{Z})) = 1/12$, $\beta_k(\mathrm{SL}(2,\mathbb{Z})) = 0$ if $k \neq 1$.

2°. By 5.2.1°, if $n = n'$ then $M_\alpha \simeq M_{\alpha'}$, while if $n \neq n'$ then $\beta_1^{\text{HT}}(M_\alpha) \neq \beta_1^{\text{HT}}(M_{\alpha'})$, and so $M_\alpha \not\simeq M_{\alpha'}$. $\square$

8.6. Corollary. 1°. *If $M = L^\infty(\mathbb{S}^2, \lambda) \rtimes \mathrm{PSL}(2,\mathbb{Z})$ as in 6.9.1′ then $\beta_1^{\text{HT}}(M) = 1/6$ and $\beta_n^{\text{HT}}(M) = 0, \forall n \neq 1$.*

2°. *Let σ be any of the actions 6.9.2 or 6.9.6 of the free group $\mathbb{F}_n$ on the diffuse abelian von Neumann algebra (A, τ), and $M = A \rtimes_\sigma \mathbb{F}_n$ the corresponding factor in the class $\mathcal{HT}$. Then $\beta_1^{\text{HT}}(M) = (n-1)$, $\beta_k^{\text{HT}}(M) = 0, \forall k \neq 1$.*

3°. *Let Γ_0 be an arithmetic lattice in $\mathrm{SU}(n,1), n \geq 2$, or in $\mathrm{SO}(2n,1)$, $n \geq 1$, and σ a free ergodic trace-preserving action of Γ_0 on the diffuse abelian von Neumann algebra A as in 6.9.3 or 6.9.6. Let $M = A \rtimes_\sigma \Gamma_0 \in \mathcal{HT}$ be the corresponding $\mathcal{HT}$ factor. Then $\beta_n^{\text{HT}}(M) \neq 0$ and $\beta_k^{\text{HT}}(M) = 0, \forall k \neq n$. Also, if Γ_0 is an arithmetic lattice in some $\mathrm{SO}(2n+1,1), n \geq 1$, then the corresponding $\mathcal{HT}$ factors constructed in 6.9.3 satisfy $\beta_k^{\text{HT}}(M) = 0, \forall k \geq 0$.*

4°. *Let Γ_0 be an $\mathrm{H_T}$ group (in the sense of Definition 6.11; e.g., any of the groups listed in 6.13) and Γ_1 an infinite amenable group. Let $M \in \mathcal{HT}$ be of the form $M = L^\infty(X,\mu) \rtimes (\Gamma_0 \times \Gamma_1)$ (cf. 6.13.3°). Then $\beta_k^{\text{HT}}(M) = 0, \forall k \geq 0$.*

Proof. For each of the groups in 1°, 2° the ℓ^2-Betti numbers for certain specific co-compact actions were calculated in [B]. Then the statements follow by [G2], [ChGr] and 8.2.1°, similarly for 3°. $\square$

8.7. Corollary. *If $\Gamma_0 = \mathrm{SL}(2,\mathbb{Z}), \mathbb{F}_n$ or if Γ_0 is an arithmetic lattice in $\mathrm{SU}(n,1), \mathrm{SO}(n,1)$, for some $n \geq 2$, then there exist three nonisomorphic factors $M_i = L^\infty(X,\mu) \rtimes_{\sigma_i} \Gamma_0, 1 \leq i \leq 3$, in the class $\mathcal{HT}$, with $M_1 \in \mathcal{HT}_{s}$, $M_{2,3} \notin \mathcal{HT}_{s}$, $M_{1,2}$ non-Γ and M_3 with the property Γ.*

Proof. All the groups mentioned have property H (see 3.2). The statement then follows from the last part of 5.3.3°. □

8.8. Corollary. *There exist both property* Γ *and non-*Γ *type* II_1 *factors* M *with trivial fundamental group*, $\mathscr{F}(M) = \{1\}$. *Moreover, such factors* M *can be taken to have non stably-isomorphic tensor powers, all with trivial fundamental group.*

8.9. Definition. Let $M \in \mathcal{HT}$. The HT-*approximate dimension* of M, denoted $\mathrm{ad}_{\mathrm{HT}}(M)$, is by definition Gaboriau's approximate dimension ([G2]) of the equivalence relation $\mathcal{R}_M^{\mathrm{HT}}$ associated with the HT Cartan subalgebra of M. Note that $\mathrm{ad}_{\mathrm{HT}}(M^t) = \mathrm{ad}_{\mathrm{HT}}(M), \forall t > 0$.

8.10. Corollary. *Let* $M \in \mathcal{HT}$ *be of the form* $M_k = L^\infty(X,\mu) \rtimes \Gamma_k$, *where* $\Gamma_k = \Gamma_0 \times \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_k}$, *for some* $2 \leq n_i < \infty, \forall 1 \leq i \leq k$, *with* Γ_0 *an increasing union of finite groups. Then* $\mathrm{ad}_{\mathrm{HT}}(M_k) = k$, *so the factors* $M_k, k \geq 1$, *are non stably-isomorphic.*

Proof. By 5.17, 5.13 and 5.16 in [G2], the approximate dimension of the group Γ_k, and thus of $\mathcal{R}_{M_k}^{\mathrm{HT}}$, is equal to k. □

8.11. Definition. Let $M \in \mathcal{HT}_s$ and $\mathrm{Out}_{\mathrm{HT}}(M)$ be the countable discrete group defined in Corollary 6.7.2°. We call it the HT-*outomorphism group* of M. As noted in 6.7, $\mathrm{Out}_{\mathrm{HT}}(M)$ can be identified with the outer automorphism group of the equivalence relation $\mathcal{R}_M^{\mathrm{HT}}$, $\mathrm{Out}(\mathcal{R}_M^{\mathrm{HT}}) = \mathrm{Aut}(\mathcal{R}_M^{\mathrm{HT}})/\mathrm{Int}(\mathcal{R}_M^{\mathrm{HT}})$. Note that $\mathrm{Out}_{\mathrm{HT}}(M^t) = \mathrm{Out}_{\mathrm{HT}}(M), \forall t > 0$. The outer automorphism group of an equivalence relation $\mathcal{R}$ was first considered by I. M. Singer in [Si], and was also studied in [FM]. By 6.7 this group is discrete (with the quotient topology) and countable. Thus, it seems likely that $\mathrm{Out}_{\mathrm{HT}}(M)$ can be computed in certain specific examples. In this respect we mention the following:

8.12. *Problem.* Calculate $\mathrm{Out}_{\mathrm{HT}}(M)$ for $M = L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$, more generally for $M_n = L((\mathbb{Z}^2)^n \rtimes \mathrm{SL}(2,\mathbb{Z}))$, with $\mathrm{SL}(2,\mathbb{Z})$ acting diagonally on $(\mathbb{Z}^2)^n = \mathbb{Z}^2 \oplus \cdots \oplus \mathbb{Z}^2$. Let $\mathcal{G}_n$ be the normalizer of $\mathrm{SL}(2,\mathbb{Z})$ in $\mathrm{GL}(2n,\mathbb{Z})$, where $\mathrm{SL}(2,\mathbb{Z})$ is embedded in $\mathrm{GL}(2n,\mathbb{Z})$ block-diagonally. Is $\mathrm{Out}_{\mathrm{HT}}(M_n)$ equal to the quotient group $\mathcal{G}_n/\mathrm{SL}(2,\mathbb{Z})$, in particular is $\mathrm{Out}_{\mathrm{HT}}(M_1)$ equal to $\{\theta_+, \mathrm{id}\}$, for θ_+ the C_+ period 2 automorphism in Corollary 7.7?

8.13. *Remarks.* 1°. Note that the above Corollary 8.8 (and also 8.5–8.7) solves Problem 3 from Kadison's Baton Rouge list, providing lots of examples of factors M with the property that the algebra of n by n matrices over M is not isomorphic to M, for any $n \geq 2$.

2°. We could extend the definition of $\beta_n^{\mathrm{HT}}(M)$ to arbitrary II_1 factors M, by simply letting $\beta_n^{\mathrm{HT}}(M) = 0, \forall n$, whenever M does not belong to the class

$\mathcal{HT}$. This definition would still be consistent with the property $\beta_n^{\text{HT}}(M^t) = \beta_n^{\text{HT}}(M)/t, \forall t > 0$. However, in order for this definition to also satisfy the Künneth formula (an imperative!), one needs to solve the following:

8.13.2. *Problem.* Does $M_1 \overline{\otimes} M_2 \in \mathcal{HT}$ imply $M_1, M_2 \in \mathcal{HT}$?

Note that if this problem were to have an affirmative answer, our factors $A \rtimes \mathbb{F}_n \in \mathcal{HT}$ would be prime, i.e., $A \rtimes \mathbb{F}_n$ would not be expressible as a tensor product of type II_1 factors $M_1 \overline{\otimes} M_2$. Indeed, this is because $\beta_1^{\text{HT}}(M_1 \overline{\otimes} M_2) = 0$ for $M_1, M_2 \in \mathcal{HT}$, by the Künneth formula, while $\beta_1^{\text{HT}}(A \rtimes \mathbb{F}_n) = n - 1 \neq 0$.

3°. It would be interesting to extend the class of factors in the "good class" for which a certain uniqueness result can be proved for some special type of Cartan subalgebras, beyond the HT factors considered here. Such generalizations can go two ways: by either extending the class of groups Γ_0 for which $A \subset A \rtimes_\sigma \Gamma_0$ works, for certain σ, or by showing that for the groups Γ_0 already considered here (e.g., the free groups) any action σ works (see Problems 6.12.1° and respectively 6.12.2°, in this respect).

4°. During a conference at MSRI in May 2001 ([C6]), Alain Connes posed the problem of constructing ℓ^2-type Betti number invariants $\beta_k(M)$ for type II_1 factors M, building on similar conceptual grounds as in [A], [C4], [ChGr], [G2,3], through appropriate definitions of simplicial complexes, ℓ^2-homology/cohomology for M, which should satisfy $\beta_k(L(G_0)) = \beta_k(G_0)$ for von Neumann factors $M = L(G_0)$ associated to discrete groups G_0. Thus, since $\beta_k(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})) = 0, \forall k$ (cf. [ChGr]), such Betti numbers would give $\beta_k(L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))) = 0, \forall k$.

Instead, our approach to defining ℓ^2-Betti number invariants was to restrict our attention to a class of factors M having a special type of Cartan subalgebras A, the HT ones, for which we could prove a uniqueness result, thus being able to use the notion of Betti numbers for equivalence relations in [G2]. Thus, our Betti numbers are defined "relative" to HT Cartan subalgebras, a fact we emphasized by using the terminology "ℓ^2_{HT}-Betti numbers" and the notation "$\beta_n^{\text{HT}}(M)$". When $M = A \rtimes G_0$ these ℓ^2_{HT}-Betti numbers satisfy $\beta_k^{\text{HT}}(M) = \beta_k(G_0)$. In particular, if $M = L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$ then $\beta_1^{\text{HT}}(M) = \beta_1(\mathrm{SL}(2,\mathbb{Z})) \neq 0$. Thus $\beta_1^{\text{HT}}(M) \neq \beta_1(M)$, if $\beta_k(M)$ could be defined as asked in [C6].

Moreover, if such $\beta_k(M)$ are possible, then according to Voiculescu's formula ([V1]) for the number of generators of the amplifications/compressions M^t of the free group factors $M = L(\mathbb{F}_n)$ (cf. also [Ra], [Dy], [Sh]), the first Betti number $\beta_1(M^t)$ (= (number of generators of M^t) -1) should satisfy a formula of the type $\beta_1(M^t) = \beta_1(M)/t^2$, rather than $\beta_1^{\text{HT}}(M^t) = \beta_1^{\text{HT}}(M)/t$, as we have in this paper!

Appendix: Some conjugacy results

We prove here several conjugacy results for subalgebras in type II_1 factors. The first one, Theorem A.1, plays a key role in the proof of 6.2. The starting point in its proof is the following simple observation: If B_0, B are finite von Neumann algebras for which there exists a $B_0 - B$ Hilbert bimodule $\mathcal{H}$ with $\dim \mathcal{H}_B < \infty$ then a suitable reduction algebra of B_0 is isomorphic to a subalgebra of some reduced of B. In the context of C*-algebras, this is reminiscent of the fact that imprimitivity bimodules entail Morita equivalence. In the von Neumann context, if both B_0, B are subalgebras in some finite factor M then existence of Hilbert $B_0 - B$ bimodules $\mathcal{H} \subset L^2(M, \tau)$ with $\dim \mathcal{H}_B < \infty$ amounts to existence of finite projections in $B_0' \cap \langle M, B \rangle$ ($\langle M, B \rangle$ being the basic construction algebra) and the corresponding isomorphism of B_0 into B is implemented by an element in M.

The basic construction was first used in conjugacy problems by Christensen ([Chr]), to study "small perturbations" of subalgebras of type II_1 factors. Although in A.1 we deal with conjugacy of subalgebras for which no "small distance" assumption is made, we still use the basic construction as a set-up for the proof. This framework allows us to use a trick inspired from [Chr], and then to utilize techniques from "subfactor theory", notably the pull down identity ([PiPo], [Po2,3]). We also use von Neumann algebra analysis of projections, with repeated use of results from [K2]. For notation and elementary properties of the basic construction, see Section 1.3 and [J1], [PiPo], [Po2,3].

To state A.1, let M be a finite factor, $B \subset M$ a von Neumann subalgebra and $\mathcal{U}_0 \subset M$ be a subgroup of unitary elements. Let $B_0 = \mathcal{U}_0''$ be the von Neumann algebra it generates in M. For each $b \in \langle M, B \rangle$, $\mathrm{Tr}(b^* b) < \infty$, we denote by $K_{\mathcal{U}_0}(b)$ the weak closure of the convex hull of $\{u_0 b u_0^* \mid u_0 \in \mathcal{U}_0\}$, i.e., $K_{\mathcal{U}_0}(b) = \overline{\mathrm{co}}^{\mathrm{w}}\{u_0 b u_0^* \mid u_0 \in \mathcal{U}_0\}$. Note that $K_{\mathcal{U}_0}(b)$ is also contained in the Hilbert space $L^2(\langle M, B \rangle, \mathrm{Tr})$, where it is still weakly closed.

Let $h = h_{\mathcal{U}_0}(b) \in K_{\mathcal{U}_0}(b)$ be the unique element of minimal norm $\| \ \|_{2,\mathrm{Tr}}$ in $K_{\mathcal{U}_0}(b)$. Since $u K_{\mathcal{U}_0}(b) u^* = K_{\mathcal{U}_0}(b)$ and $\|u h u^*\|_{2,\mathrm{Tr}} = \|h\|_{2,\mathrm{Tr}}, \forall u \in \mathcal{U}_0$, by the uniqueness of h it follows that $u h u^* = h, \forall u \in \mathcal{U}_0$. Thus $h \in \mathcal{U}_0' \cap \langle M, B \rangle = B_0' \cap \langle M, B \rangle$. Moreover, by the definitions, we see that if $0 \leq b \leq 1$ then $0 \leq k \leq 1$ and $\mathrm{Tr}(k) \leq \mathrm{Tr}(b)$, for all $k \in K_{\mathcal{U}_0}(b)$.

A.1. Theorem. *Let $M, B, B_0, \mathcal{U}_0$ be as above. Assume the von Neumann subalgebra $B \subset M$ is maximal abelian in M and B_0 is abelian with $B_{01} \overset{\mathrm{def}}{=} B_0' \cap M$ still abelian (thus maximal abelian in M). Then the following conditions are equivalent:*

1°. *There exists a nonzero projection $e_0 \in B_0' \cap \langle M, B \rangle$ with* $\mathrm{Tr}(e_0) < \infty$.

2°. *There exist nonzero projections* $q_0 \in B_0' \cap M$, $q \in B$ *and a partial isometry* $v \in M$ *such that* $v^*v = q_0, vv^* = q$ *and* $vB_0v^* \subset Bq$.

Proof. 2° $\implies$ 1°. If v satisfies condition 3° then B_0q_0 is contained in v^*Bv. Since e_B commutes with B, it follows that $e_0 = v^*e_Bv$ commutes with B_0, i.e., $e_0 \in B_0' \cap \langle M, B\rangle$. Also, $\mathrm{Tr}e_0 = \mathrm{Tr}(v^*e_Bv) \leq \mathrm{Tr}(e_B) = 1$.

1° $\implies$ 2°. Denote $M_1 = \langle M, B\rangle$. Since B_0e_0 is abelian, it is contained in a maximal abelian subalgebra B_1 of $e_0M_1e_0$. Since $M_1 = (JBJ)' \cap \mathcal{B}(L^2M)$, it is a type I von Neumann algebra. Thus, by a result of Kadison ([K2]), B_1 contains a nonzero abelian projection e_1 of M_1 (i.e., $e_1M_1e_1$ is abelian). Since e_B is a maximal abelian projection in M_1 and has central support 1 in M_1, it follows that e_B majorizes e_1. Thus, e_1 satisfies $e_1(L^2(M, \tau)) = \overline{\xi B}$ for some $\xi \in L^2(M, \tau)$.

Let $V \in M_1$ be a partial isometry such that $V^*V = e_1 \leq e_0$ and $VV^* \leq e_B$. It follows that $VB_1e_1V^*$ is a subalgebra of $e_BM_1e_B = Be_B$. Since e_1 commutes with B_0, if we denote by f' the maximal projection in B_0 such that $f'e_1 = 0$ and let $f_0 = 1 - f'$, then there exists a unique isomorphism α from B_0f_0 into B such that $\alpha(b)e_B = VbV^*, \forall b \in B_0f_0$. Let $f = \alpha(f_0) \in B$.

Then $\alpha(b)e_BV = e_BVb, \forall b \in B_0f_0$. By applying Φ to both sides and denoting a the square integrable operator $a = \Phi(e_BV) \in L^2(M, \tau)$, we see that $\alpha(b)a = ab, \forall b \in B_0$. Since $e_Ba = e_BV = V$, it follows that $a \neq 0$.

By the usual trick, if we denote by $v_0 \in M$ the unique partial isometry in the polar decomposition of a such that the right supports of a and v_0 coincide, then $p_0 = v_0^*v_0$ belongs to the algebra $B_0' \cap M = B_{01}$, which is abelian by hypothesis, $p = v_0v_0^*$ belongs to $(\alpha(B_0)f)' \cap fMf$ and $\alpha(b)v_0 = v_0b, \forall b \in B_0f_0$.

But $B_{01} = B_0' \cap M$ maximal abelian in M implies $B_{01}f_0$ maximal abelian in f_0Mf_0. Moreover, since $v_0B_0v_0^* = \alpha(B_0)p$, if we denote $B_{11} = v_0B_{01}v_0^*$, then by spatiality,

$$\begin{aligned} B_{11} &= v_0B_{01}v_0^* = v_0(B_0' \cap M)v_0^* = v_0B_0v_0^{*\prime} \cap pMp \\ &= (\alpha(B_0)p)' \cap pMp = p((\alpha(B_0)f)' \cap fMf)p. \end{aligned}$$

This implies that p is an abelian projection in $(\alpha(B_0)f)' \cap fMf$. Thus, if z is the central projection of p in $(\alpha(B_0)f)' \cap fMf$ then $((\alpha(B_0)f)' \cap fMf)z = ((\alpha(B_0)z)' \cap zMz$ is finite of type I.

Since Bf is maximal abelian in fMf it follows that $z \in Bf$ and Bz is maximal abelian in the type $\mathrm{I}_{\mathrm{fin}}$ algebra $((\alpha(B_0)z)' \cap zMz$. By [K2], there exists a projection $f_{11} \in Bz$ such that f_{11} is equivalent to p in $(\alpha(B_0)z)' \cap zMz$. Let $v_1 \in (\alpha(B_0)z)' \cap zMz$ be such that $v_1v_1^* = f_{11}$, $v_1^*v_1 = p$ and denote $v = v_1v_0 \in M$. Then $v^*v = p_0 \in B_0', vv^* = f_{11} \in B$ and $vB_0v^* = \alpha(B_0)f_{11} \subset Bf_{11}$. □

Our second conjugacy result, A.2, is a "small perturbation"-type result, needed in the proofs of 4.5 and 6.6.3°. The starting point in its proof is a trick from [Chr]. Then, as in A.1, we use techniques from [Po2,3,7], [PiPo]. Note that the proof of Step 1 below is a refinement of the proof of 4.4.2 in [Po1], while the proof of Step 2 is a refinement of an argument used in proving 4.5.1, 4.5.6 and 4.7.3 in [Po1].

A.2. THEOREM. *For any* $\varepsilon_0 > 0$ *there exists* $\delta > 0$ *such that if* M *is a type* II_1 *factor*, $B \subset M$ *is a subfactor with* $B' \cap M = \mathbb{C}$, $B_0 \subset M$ *is a von Neumann subalgebra with* $B_0' \cap M = \mathcal{Z}(B_0)$, $\mathcal{N}_M(B_0)'' = M$ *and* $B_0 \subset_\delta B$ *then there exists a unitary element* $u \in M$ *such that* $\|u-1\|_2 \leq \varepsilon_0$ *and* $uB_0u^* \subset B$.

Proof. Step 1. Let $\varepsilon = \varepsilon_0^2/4$. We first prove that $\exists\delta > 0$ such that if $B_0, B \subset M$ satisfy $B_0' \cap M = \mathcal{Z}(B_0)$ and $B_0 \subset_\delta B$ then $\exists p_0 \in \mathcal{P}(B_0), p \in \mathcal{P}(B)$, a unital isomorphism θ of $p_0B_0p_0$ into pBp, a projection $q \in \theta(p_0B_0p_0)' \cap pMp$ and a partial isometry $v \in M$ such that $v^*v = p_0, vv^* = q \leq p$, $\|v-1\|_2 \leq \varepsilon$, $\tau(q) \geq 1-\varepsilon$ and $vb_0 = \theta(b_0)v, \forall b_0 \in p_0B_0p_0$.

To do this note first that if $u_0 \in \mathcal{U}(B_0)$ then $\|u_0e_Bu_0^* - e_B\|_{2,\mathrm{Tr}}^2/2 = 1 - \mathrm{Tr}(e_Bu_0e_Be_0^*) = \|u_0 - E_B(u_0)\|_2^2$ (see e.g., line 17 on page 322 in [Po9]). So if $\|u_0 - E_B(u_0)\|_2 \leq \delta$, $\forall u_0 \in \mathcal{U}_0 = \mathcal{U}(B_0)$, then with the notation in A.1 we get $h = h_{\mathcal{U}_0}(e_B) \in B_0' \cap \langle M, B\rangle$, with $h \leq 1$, $\mathrm{Tr}(h) \leq 1$ and $\|h - e_B\|_{2,\mathrm{Tr}} \leq 2^{1/2}\delta$. Thus, by (1.1 in [C2]) there exists $s > 0$ such that the spectral projection e of h corresponding to the interval $[s,\infty)$ satisfies $\|e - e_B\|_{2,\mathrm{Tr}} \leq (2\delta)^{1/2}$. Note that $e \in B_0' \cap \langle M, B\rangle$ as well. We next want to show that by slightly shrinking e we may assume in addition $(B_0e)' \cap e\langle M, B\rangle e = \mathcal{Z}(B_0)e$.

So let $u \in \mathcal{U}(C)$, where $C = (B_0e)' \cap e\langle M, B\rangle e$. Since $e_B\langle M, B\rangle e_B = Be_B$ and e is $(2\delta)^{1/2}$-close to e_B in the norm $\| \quad \|_{2,\mathrm{Tr}}$, if we denote by b the unique element in B with $be_B = e_Bue_B$, then u is close to ebe in the norm $\| \quad \|_{2,tr}$ implemented by the normalized trace $\mathrm{tr} = \mathrm{Tr}(e)^{-1}\mathrm{Tr}$ on $e\langle M, B\rangle e$. This implies that $\|[ebe, v]\|_{2,tr} \leq \varepsilon(\delta)$, $\forall v \in \mathcal{U}(B_0e)$, in which $\varepsilon(\delta)$ denotes from now on a constant depending on δ, with $\lim\limits_{\delta\to 0}\varepsilon(\delta) = 0$ (but $\varepsilon(\delta)$ possibly changing in each of the subsequent estimates). Since $B_0' \cap M = \mathcal{Z}(B_0)$, if we average ebe by unitaries in B_0e, we see that u is $\varepsilon(\delta)$-close to an element in $\mathcal{Z}(B_0)e$. Thus $C \subset_{\varepsilon(\delta)} A_0$, where $A_0 = \mathcal{Z}(B_0)e$. Noticing that $A_0 \subset \mathcal{Z}(C)$, we infer that this implies $\exists e' \in \mathcal{Z}(C)$, with $\mathrm{tr}(e') \geq 1 - \varepsilon(\delta)$ and $Ce' = A_0e'$; i.e., $(B_0e)' \cap e\langle M, B\rangle e = \mathcal{Z}(B_0)e$. Indeed, for if $q' \in \mathcal{Z}(C)$ is the maximal projection with Cq' abelian and $A \subset C$ is a maximal abelian *-subalgebra with $A_0 \subset A$ then $q' \in A$ and there exists $u \in \mathcal{U}(B(1-q'))$ with $E_A(u) = 0$. Since $q' + u \in \mathcal{U}(C)$ we have:

$$\begin{aligned}\mathrm{tr}(1-q') &= \|u\|_{2,tr}^2 = \|(q'+u) - E_A(q'+u)\|_{2,tr}^2 \\ &\leq \|(q'+u) - E_{A_0}(q'+u)\|_{2,tr}^2 \leq \varepsilon(\delta)^2.\end{aligned}$$

This reduces the problem to the case C is abelian, which is an easy exercise (e.g., use the argument on page 745 in [Po7]).

Taking e' for e in the above, this shows that if $B_0 \subset_\delta B$ then $\exists e \in B_0' \cap \langle M, B\rangle$ finite projection with $\|e - e_B\|_{2,\mathrm{Tr}} \leq \varepsilon(\delta)$ and $(B_0 e)' \cap e\langle M, B\rangle e = \mathcal{Z}(B_0)e$. But by ([Po6]) the latter condition implies there exists $A_1 \subset B_0$ abelian such that $A_1 e$ is maximal abelian in $e\langle M, B\rangle e$. By [K2] there exists a projection $P \in A_1 = A_1 e$ such that P is equivalent to the support projection of $e e_B e \in e\langle M, B\rangle e$. In particular, P is majorized by e_B. Also, P, e and e_B are $\varepsilon(\delta)$-close one to another. By 1.2 in [C2], there exists a partial isometry $V \in \langle M, B\rangle$ such that V is $\varepsilon(\delta)$-close to e_B, $V^*V = P \in A_1 \subset B_0'$ and $VV^* \leq e_B$. As in [Chr] and in the proof of A.1, if $p_0 \in B_0$ and $p \in B$ denote the support projections of V^*V in B_0 and respectively VV^* in B then there exists a unital isomorphism θ of $p_0 B_0 p_0$ into pBp such that $Vb_0 = \theta(b_0)V, \forall b_0 \in p_0 B_0 p_0$. If we now take the partial isometry $v = \Phi(V)|\Phi(V)|^{-1} \in M$, then we still have $vb_0 = \theta(b_0)v, \forall b_0 \in p_0 B_0 p_0$ and v is $\varepsilon(\delta)$-close to 1 (using $\|\Phi(V) - 1\|_1 \leq \|V - e_B\|_{1,\mathrm{Tr}}$ and applying 2.1 in [C2]). Since $v^*v \in (p_0 B_0 p_0)' \cap p_0 M p_0 = \mathcal{Z}(B_0)p_0$ and $vv^* \in \theta(p_0 B_0 p_0)' \cap pMp$, letting $q = vv^*$, we are done.

Step 2. If p_0, p, q, v, θ are as in Step 1, then $vB_0v^* = \theta(p_0 B_0 p_0)q$, so by spatiality we have:

$$\begin{aligned} q(\theta(p_0B_0p_0)' \cap pMp)q &= (vB_0v^*)' \cap qMq \\ &= v(p_0B_0p_0' \cap p_0Mp_0)v^* = v\mathcal{Z}(B_0)v^* = \mathcal{Z}(\theta(p_0B_0p_0))q. \end{aligned}$$

In particular, $q(\theta(p_0B_0p_0)' \cap pBp)q = \mathcal{Z}(\theta(p_0B_0p_0))q$. Since $Z(\theta(p_0B_0p_0)) \subset \theta(p_0B_0p_0)' \cap pBp$ this implies that there exists a normal conditional expectation E of $\theta(p_0B_0p_0)' \cap pBp$ onto $Z(\theta(p_0B_0p_0))$ such that $qxq = E(x)q, \forall x \in \theta(p_0B_0p_0)' \cap pBp$.

Let $p' \in \theta(p_0B_0p_0)' \cap pBp$ be the minimal projection such that $qp' = q$. By replacing if necessary θ by $\theta(\cdot)q'$ (while leaving v unchanged), we may assume $p' = p$. Thus, if $a \in \theta(p_0B_0p_0)' \cap pBp$ satisfies $aq = 0$ then the support of a^*a is majorized by $p - p' = 0$, implying that $a = 0$ and showing that E is faithful. Since q implements the normal faithful conditional expectation E of $\theta(p_0B_0p_0)' \cap pBp$ onto $Z(\theta(p_0B_0p_0))$, the weak closure of $\mathrm{sp}\{xqy \mid x, y \in \theta(p_0B_0p_0)' \cap pBp\}$ is a finite von Neumann subalgebra Q of pMp with $qQq \simeq Z(\theta(p_0B_0p_0))$. Since q has support 1 in Q, this shows that Q is type $\mathrm{I_{fin}}$. But Q contains $(\theta(p_0B_0p_0)' \cap pBp)1_Q$, which is isomorphic to $\theta(p_0B_0p_0)' \cap pBp$. Thus, the latter follows type $\mathrm{I_{fin}}$ as well.

Let $q' \in Z(\theta(p_0B_0p_0))(\subset \mathcal{Z}(\theta(p_0B_0p_0)' \cap pBp))$ be the maximal projection with

$$q'Z(\theta(p_0B_0p_0)) = q'(\theta(p_0B_0p_0)' \cap pBp).$$

It follows that there exists $b \in L^2(\theta(p_0B_0p_0)' \cap pBp)(p - q')$ with $E(b) = 0$ and $E(b^*b) = p - q'$ (see e.g., [Po2]). This shows that bqb^* is a projection orthogonal to $q(p - q')$ and equivalent to $q(p - q')$, while still under $p - q'$.

Thus

$$\tau(q(p-q')) = \tau(bq(p-q')b^*) \leq \tau((1-q)(p-q')) \leq \tau(1-q) \leq \varepsilon.$$

Thus, $1-\varepsilon-\tau(q') \leq \tau(p-q') \leq 2\varepsilon$, implying that $\tau(q') \geq 1-3\varepsilon$. This shows that by "cutting everything" by q' we may assume $\theta(p_0B_0p_0)' \cap pBp = \mathcal{Z}(\theta(p_0B_0p_0))$.

Since B_0 is regular in M, $p_0B_0p_0$ is regular in p_0Mp_0 (see e.g. [JPo]) and thus, by spatiality, $\theta(p_0B_0p_0)q$ is regular in qMq. Since $\theta(p_0B_0p_0) \ni b \to bq \in \theta(p_0B_0p_0)q$ is an isomorphism, for each $u \in \mathcal{N}_{qMq}(\theta(p_0B_0p_0)q)$ there exists an automorphism σ_u of $\theta(p_0B_0p_0)$ such that $ubqu^* = \sigma_u(b)q, \forall b \in \theta(p_0B_0p_0)$. Thus, $ub = \sigma_u(b)u, \forall b \in \theta(p_0B_0p_0)$.

By applying E_B to both sides of this equality, it follows that $E_B(u)b = \sigma_u(b)E_B(u), \forall b \in \theta(p_0B_0p_0)$. By also taking into account that $\theta(p_0B_0p_0)' \cap pBp \subset \theta(p_0B_0p_0)$, we see that if $B_1 \subset pBp$ denotes the von Neumann algebra generated by the normalizer of $\theta(p_0B_0p_0)$ in pBp then $E_B(\mathcal{N}_{qMq}(\theta(p_0B_0p_0)q) \subset B_1$. By the regularity of $\theta(p_0B_0p_0)q$ in qMq, this entails $E_B(qMq) \subset B_1$ as well. Since $q \leq p$ and $\tau(q) \geq 1-\varepsilon$, we thus have $pBp \subset_\varepsilon B_1 \subset pBp$. Since pBp is a factor, this implies there exists a projection $p'' \in \mathcal{Z}(B_1)$ with $\tau(p'') \geq 1-2\varepsilon$ such that $B_1p'' = p''Bp''$.

By cutting with p'' we may thus also assume $\theta(p_0B_0p_0)$ is regular in pBp. Since $pBp' \cap pMp = \mathbb{C}p$, this implies $\mathcal{N}_1 = \mathcal{N}_{pBp}(\theta(p_0B_0p_0))$ satisfies $\mathcal{N}_1' \cap pMp = \mathbb{C}p$. Since $\mathcal{N}_1$ also normalizes the algebras $\mathcal{Z}(\theta(p_0B_0p_0)) = \theta(p_0B_0p_0)' \cap pBp$ and $\theta(p_0B_0p_0)' \cap pMp$, it acts ergodically on both. By ergodicity, $\theta(p_0B_0p_0)' \cap pMp$ is either homogeneous of type $\mathrm{I}_{\mathrm{fin}}$ or of type II_1. Since $q(\theta(p_0B_0p_0)' \cap pMp)q = \mathcal{Z}(\theta(p_0B_0p_0))q$ is abelian and $\tau(q) > 1/2$ (for ε chosen sufficiently small), $\theta(p_0B_0p_0)' \cap pMp$ is abelian.

Denote $A_0 = \mathcal{Z}(\theta(p_0B_0p_0))$, $A_1 = \theta(p_0B_0p_0)' \cap pMp$, $N_0 = pBp$ and Q_0 the factor generated by $\mathcal{N}_1$ and A_1 in pMp. Thus, we have $N_0' \cap Q_0 = \mathbb{C}$ and the nondegenerate commuting square

$$\begin{array}{ccc} N_0 & \subset & Q_0 \\ \cup & & \cup \\ A_0 & \subset & A_1. \end{array}$$

(Recall that we also have $q \in A_1, A_1q = A_0q$ and $\tau(q) \geq 1-\varepsilon$.)

Thus, if $e = e_{N_0}^{Q_0}$ denotes the Jones projection corresponding to the inclusion $N_0 \subset Q_0$ then $A_0 \subset A_1 \subset \langle A_1, e\rangle$ is the basic construction for $A_0 \subset A_1$. Since $\mathcal{Z}(\langle A_1, e\rangle) = A_0$ and since $\mathcal{N}_1$ acts on $A_0 \subset A_1$ with the action on A_0 being ergodic, it follows that $\langle A_1, e\rangle$ is homogeneous of type I. But $q(A_1eA_1)q = A_0(qeq)A_0$, and since $[A_0, qeq] = 0$ this implies $q\langle A_1, e\rangle q = A_0qeq$. Thus, $q\langle A_1, e\rangle q$ is abelian. Equivalently, q is an abelian projection in $\langle A_1, e\rangle$. But then q is majorised by e in $\langle A_1, e\rangle$. Thus q is majorised by e in $\langle Q_0, e\rangle$ as well, showing that q is finite in $\langle Q_0, e\rangle$.

But q enters finitely many times in 1_Q, in the factor Q_0, which is a subalgebra of $\langle Q_0, e\rangle$. Thus $\langle Q_0, e\rangle$ is a finite factor and $\tau(e) \geq \tau(q) \geq 1-\varepsilon > 1/2$. By Jones' theorem, $e = 1$ and $N_0 = Q_0$. In particular, $q \in \theta(p_0B_0p_0)$, so that $q = p$. Thus, $v^*v = p_0 \in B_0, vv^* = p \in B$ and $v(p_0B_0p_0)v^* \subset pBp$. Since the normalizer of B_0 acts ergodically on the center of B_0 and B is a factor, there exists a unitary element $u \in M$ such that $up_0 = v$ and $uB_0u^* \subset B$. But then $\|1-u\|_2 \leq \|1-v\|_2 + \|v-u\|_2 \leq 2\varepsilon^{1/2} = \varepsilon_0$. □

Our last conjugacy result, somewhat technical, is needed in the proof of 4.3.2°.

A.3. Theorem. *Let M be a type II_1 factor and $P, Q \subset M$ von Neumann subalgebras. Assume there exists a group of unitary elements $\mathcal{U}_0 \subset P$ that normalizes Q and satisfies $N_0' \cap M = \mathcal{Z}(N_0)$ and $[\mathcal{Z}(N_0), Q] = 0$, where $N_0 = \mathcal{U}_0''$. If $Q \subset_{\varepsilon_0} P$, for some $\varepsilon_0 < 1/2$, then there exists a nonzero projection $p \in \mathcal{Z}(N_0)$ such that $Qp \subset P$.*

Proof. Let $M \subset^{e_P} \langle M, e_P\rangle$ be the basic construction for $P \subset M$, with Tr and Φ the canonical trace and weight, respectively, as in 1.3.1. The statement is equivalent to proving that there exists $p \in Q' \cap \mathcal{Z}(N_0)$, $p \neq 0$, such that $[Qp, e_P] = 0$.

Let k be the unique element of minimal norm $\| \quad \|_{2,\mathrm{Tr}}$ in $K = \overline{\mathrm{co}}^w\{ue_Pu^* \mid u \in \mathcal{U}(Q)\}$. Note that $0 \leq k \leq 1, \mathrm{Tr}(k) \leq 1$. Also, since for $u \in \mathcal{U}(Q)$ we have

$$\|e_P - ue_Pu^*\|_{2,\mathrm{Tr}}^2 = 2 - 2\|E_P(u)\|_2^2 = 2\|u - E_P(u)\|_2^2 \leq 2\varepsilon_0^2,$$

by taking convex combinations and weak limits $\|k - e_P\|_{2,\mathrm{Tr}}^2 \leq 2\varepsilon_0^2 < 1/2$.

Since $uKu^* = K$ and $\|uku^*\|_{2,\mathrm{Tr}} = \|k\|_{2,\mathrm{Tr}}, \forall u \in \mathcal{U}(Q)$, by the uniqueness of k as the element of minimal norm $\| \quad \|_{2,\mathrm{Tr}}$ in K, it follows that $uku^* = k, \forall u \in \mathcal{U}(Q)$. Thus $[k, Q] = 0$. Moreover, if $v \in \mathcal{U}_0 \subset P$ then $[v, e_P] = 0$ and $vQv^* = Q$, implying that $v(ue_Pu^*)v^* = (vuv^*)e_P(vu^*v^*) \subset K$, $\forall u \in \mathcal{U}(Q)$. Thus, $vKv^* = K$ and so, by the uniqueness of k, $[k, v] = 0$. Since $\mathcal{U}_0$ generates N_0, it follows that k and all its spectral projections commute with both Q and $N_0 = \mathcal{U}_0''$.

Together with $[e_P, N_0] = 0$ this yields $[ke_P, N_0] = 0$ and further on, by applying the operator valued weight Φ of $\langle M, e_P\rangle$ on M (which is M-bimodular, thus N_0-bimodular as well) and letting $a = \Phi(ke_P)$, we see that $[a, N_0] = 0$. Equivalently, $a \in N_0' \cap M = \mathcal{Z}(N_0)$. Since $\mathcal{Z}(N_0) \subset N_0 \subset P$, $a \in P$ and so $[a, e_P] = 0$. Together with $ae_P = ke_P$, this entails $ae_P = e_Pae_P = e_Pke_P \geq 0$, and so $a \geq 0$. In particular, $a = a^*$. Thus, $ke_P = ae_P = (ae_P)^* = (ke_P)^* = e_Pk$, showing that $[k, e_P] = 0$.

Let now e_1 be the spectral projection of k corresponding to the set $\{1\}$. Thus $e_1 = e_1k \in \overline{\mathrm{co}}^w\{u(e_1e_P)u^* \mid u \in \mathcal{U}_0\}$, showing that $e_1 \leq e_P$. Thus, if $p = \Phi(e_1)$ then p is a projection in P with $e_1 = pe_P$, $[p, Q \vee N_0] = 0$ and $[e_P, Qp] = 0$. Thus, we are done, provided we can show that $p \neq 0$.

Assume by contradiction that $e_1 = 0$. We show that this implies that for any spectral projection e of k, ee_P is majorized by $e(1-e_P)$ in $\langle M, e_P\rangle$. Indeed, for if this is not the case then there exists a projection z in $\mathcal{Z}(\langle M, e_P\rangle)$ and a partial isometry $w \in \langle M, e_P\rangle$ such that $w^*w \lneqq zee_P$, $ww^* = ze(1-e_P)$. If we denote $b = \Phi(w)$, then $be_P = w$ and so

$$bb^* = \Phi(ww^*) = \Phi(ze(1-e_P)) \in N_0' \cap M = Q' \cap \mathcal{Z}(N_0).$$

Similarly, $q = \Phi(ezе_P)$ is a projection in P which commutes with N_0, thus lying in $\mathcal{Z}(N_0) \subset P$. Since $bb^* \geq be_Pb^* = ze(1-e_P)$ and the morphism $\mathcal{Z}(N_0) \ni x \mapsto xze(1-e_P)$ has support q (because $e_1 = 0$), it follows that $bb^* \geq q$. Thus

$$\tau(q) = \mathrm{Tr}(zee_P) \gneqq \mathrm{Tr}(w^*w) = \mathrm{Tr}(ww^*) = \tau(bb^*) \geq \tau(q),$$

a contradiction.

In particular, since $ee_P \prec e(1-e_P)$ for any spectral projection e of k, we have $\|k(1-e_P)\|_{2,\mathrm{Tr}} \geq \|ke_P\|_{2,\mathrm{Tr}}$. By Pythagoras, this gives

$$\tau((1-k)^2) + \tau(k^2) \leq \|ke_P - e_P\|_{2,\mathrm{Tr}}^2 + \|k(1-e_P)\|_{2,\mathrm{Tr}}^2 = \|k - e_P\|_{2,\mathrm{Tr}}^2 < 1/2.$$

Thus $0 > \tau(2(1-k)^2 + 2k^2 - 1) = \tau(1 - 4k + 4k^2) = \tau((1-2k)^2)$. This final contradiction ends the proof of the theorem. □

University of California, Los Angeles, CA
E-mail address: popa@math.ucla.edu

References

[A-De] C. Anantharam-Delaroche, On Connes' property (T) for von Neumann algebras, *Math. Japon.* **32** (1987), 337–355.

[At] M. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, in *Colloque "Analyse et Topologie", en l'Honneur de Henri Cartan* (Orsay, 1974), *Astérisque* **32**, 43–72, Soc. Math. France, Paris, 1976.

[Bi] D. Bisch, A note on intermediate subfactors, *Pacific J. Math.* **163** (1994), 201–216.

[Bo] F. Boca, On the method of constructing irreducible finite index subfactors of Popa, *Pacific J. Math.* **161** (1993), 201–231.

[B] A. Borel, The L^2-cohomology of negatively curved Riemannian symmetric spaces, *Ann. Acad. Sci. Fenn. Ser. A Math.* **10** (1985), 95–105.

[Bu] M. Burger, Kazhdan constants for $\mathrm{SL}(3,\mathbb{Z})$, *J. Reine Angew. Math.* **413** (1991), 36–67.

[dCaH] J. de Canniére and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, *Amer. J. Math.* **107** (1984), 455–500.

[ChGr] J. Cheeger and M. Gromov, L^2-cohomology and group cohomology, *Topology* **25** (1986), 189–215.

[CCJJV] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette, *Groups with the Haagerup Property* (*Gromov's a-*T*-menability*), *Progr. in Math.* **197**, Birkhäuser Verlag, Basel, 2001.

[Cho] M. Choda, Group factors of the Haagerup type, *Proc. Japan Acad. Ser. A Math. Sci*. **59** (1983), 174–177.

[Chr] E. Christensen, Subalgebras of a finite algebra, *Math. Ann.* **243** (1979), 17–29.

[C1] A. Connes, A factor of type II_1 with countable fundamental group, *J. Operator Theory* **4** (1980), 151–153.

[C2] ———, Classification of injective factors, *Ann. of Math.* **104** (1976), 73–115.

[C3] ———, Classification des facteurs, *Proc. Sympos. Pure Math*. **38** (1982), 43–109.

[C4] ———, Sur la théorie non commutative de l'intégration, in *Algèbres d'Opérateurs* (Seḿ. Les Plans-sur-Bex, 1978), 19–143, *Lecture Notes in Math*. **725**, Springer-Verlag, New York, 1979.

[C5] ———, Sur la classification des facteurs de type II, *C. R. Acad. Sci. Paris Sér*. I *Math*. **281** (1975), A13–A15.

[C6] ———, Factors and geometry, lecture at MSRI, May 1, 2001.

[C7] ———, Correspondences, handwritten notes, 1980.

[CJ] A. Connes and V. F. R. Jones, Property (T) for von Neumann algebras, *Bull. London Math. Soc*. **17** (1985), 57–62.

[CW] A. Connes and B. Weiss, Property T and asymptotically invariant sequences, *Israel J. Math*. **37** (1980), 209–210.

[CowH] M. Cowling and U. Haagerup, Completely bounded multipliers and the Fourier algebra of a simple Lie group of real rank one, *Invent. Math.* **96** (1989), 507–549.

[DeKi] C. Delaroche and A. Kirillov, Sur les relations entre l'espace dual d'un groupe et la structure de ses sous-groupes fermés, *Sém. Bourbaki* **10**, Exp. No. 343, 507–528, Soc. Math. France, Paris, 1995.

[D] J. Dixmier, Sous-anneaux abéliens maximaux dans les facteurs de type fini, *Ann. of Math*. **59** (1954), 279–286.

[Dy] H. Dye, On groups of measure-preserving transformations, I, II, *Amer. J. Math*. **81** (1959), 119–159, and **85** (1963), 551–576.

[Dyk] K. Dykema, Interpolated free group factors, *Pacific J. Math.* **163** (1994), 123–135.

[Ey] P. Eymard, *Moyennes Invariantes et Représentations Unitaires*, *Lecture Notes in Math*. **300**, Springer-Verlag, New York, 1972.

[FM] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras I, II, *Trans. Amer. Math. Soc*. **234** (1977), 289–324, 325–359.

[Fu] A. Furman, Gromov's measure equivalence and rigidity of higher rank lattices, *Ann. of Math.* **150** (1999), 1059–1081.

[G1] D. Gaboriau, Coût des rélations d'équivalence et des groupes, *Invent. Math*. **139** (2000), 41–98.

[G2] ———, Invariants ℓ^2 de relations d'équivalence et de groupes, *Publ. Math. IHES* **95** (2002), 93–150.

[Ge] L. Ge, Prime factors, *Proc. Natl. Acad. Sci. USA* **93** (1996), 12762–12763.

[GoNe] V. Y. Golodets and N. I. Nessonov, T-property and nonisomorphic full factors of type II and III, *J. Funct. Anal*. **70** (1987), 80–89.

[Gr] M. Gromov, Rigid transformation groups, in *Géometrie Différentielle* (Paris 1986), 65–139, *Travaux en Cours* **33**, Hermann, Paris, 1988.

[H] U. Haagerup, An example of a nonnuclear C*-algebra which has the metric approximation property, *Invent. Math*. **50** (1979), 279–293.

[dHVa] P. de la Harpe and A. Valette, *La propriété T de Kazhdan pour les groupes localement compacts*, *Astérisque* **175**, Soc. Math. France, Paris (1989).

[Hj] G. Hjorth, A lemma for cost attained, UCLA preprint, 2002.

[HkS] R. Hoegh-Krohn and T. Skjelbred, Classification of C^*-algebras admitting ergodic actions of the two-dimensional torus, *J. Reine Angew. Math.* **328** (1981), 1–8.

[ILP] M. Izumi, R. Longo, and S. Popa, A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras, *J. Funct. Anal.* **155** (1998), 25–63.

[Jo1] P. Jolissaint, Haagerup approximation property for finite von Neumann algebras, *J. Operator Theory* **48** (2002), 549–571.

[Jo2] ———, On the property (T) for pairs of topological groups, *Enseign. Math.* **51** (2005), 31–45.

[J1] V. F. R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[J2] ———, A converse to Ocneanu's theorem, *J. Operator Theory* **10** (1983), 61–63.

[JPo] V. F. R. Jones and S. Popa, Some properties of MASAs in factors, in *Invariant Subspaces and other Topics* (Timişoara/Herculane, 1981), *Operator Theory*: *Adv. Appl.* **6** (1982), 89–102, Birkhäuser, Boston, MA, 1982.

[K1] R. V. Kadison, Problems on von Neumann algebras, Baton Rouge Conference, 1967.

[K2] ———, Diagonalizing matrices, *Amer. Math. J.* **106** (1984), 1451–1468.

[KafW] V. Kaftal and G. Weiss, Compact derivations relative to semifinite von Neumann algebras, *J. Funct. Anal.* **62** (1985), 202–220.

[Ka] D. Kazhdan, Connection of the dual space of a group with the structure of its closed subgroups, *Funct. Anal. Appl.* **1** (1967), 63–65.

[KoY] H. Kosaki and S. Yamagami, Irreducible bimodules associated with crossed product algebras, *Internat. J. Math.* **3** (1992), 661–676.

[Lu] W. Lück, Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers II. Applications to Grothendieck groups, L^2-Euler characteristics and Burnside groups, *J. Reine Angew. Math.* **496** (1998), 213–236.

[Ma] G. Margulis, Finitely-additive invariant measures on Euclidian spaces, *Ergodic Theory Dynam. Systems* **2** (1982), 383–396.

[McD] D. McDuff, Central sequences and the hyperfinite factor, *Proc. London Math. Soc.* **21** (1970), 443–461.

[MvN] F. Murray and J. von Neumann, On rings of operators. IV, *Ann. of Math.* **44** (1943), 716–808.

[PiPo] M. Pimsner and S. Popa, Entropy and index for subfactors, *Ann. Sci. École Norm. Sup.* **19** (1986), 57–106.

[Po1] S. Popa, Correspondences, INCREST preprint 1986, unpublished.

[Po2] ———, *Classification of Subfactors and their Endomorphisms*, CBMS *Lecture Notes Series in Math.* **86** (1994), A. M. S., Providence, RI, 1995.

[Po3] ———, Classification of amenable subfactors of type II, *Acta Math.* **172** (1994), 163–255.

[Po4] ———, Free independent sequences in type II_1 factors and related problems, in *Recend Advances in Operator Algebras* (Orléans 1992), *Astérisque* **232** (1995), 187–202.

[Po5] S. Popa, Some properties of the symmetric enveloping algebra of a subfactor with applications to amenability and property T, *Doc. Math.* **4** (1999), 665–744.

[Po6] ———, On a problem of R. V. Kadison on maximal abelian $*$-subalgebras in factors, *Invent. Math.* **65** (1981), 269–281.

[Po7] ———, The relative Dixmier property for inclusions of von Neumann algebras of finite index, *Ann. Sci. École Norm. Sup.* **32** (1999), 743–767.

[Po8] ———, Notes on Cartan subalgebras in type II_1 factors, *Math. Scand.* **57** (1985), 171–188.

[Po9] ———, On the distance between MASA's in type II_1 factors, in *Mathematical Physics in Mathematics and Physics* (Siena, 2000), *Fields Inst. Commun.* **30**, 321–324, A. M. S., Providence, RI, 2001.

[PoRa] S. Popa and F. Radulescu, Derivations of von Neumann algebras into the compact ideal space of a semifinite algebra, *Duke Math. J.* **57** (1988), 485–518.

[PoSh] S. Popa and D. Shlyakhtenko, Cartan subalgebras and bimodule decomposition of II_1 *factors*, *Math. Scand.* **92** (2003), 93–102.

[Ra] F. Radulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group of noninteger index, *Invent. Math.* **115** (1994), 347–389.

[Ri] M. Rieffel, C^*-algebras associated with irrational rotations, *Pacific J. Math.* **93** (1981), 415–429.

[S] S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer-Verlag, New York, 1971.

[Sa] J.- L. Sauvageot, Sur le produit tensoriel relatif d'espaces de Hilbert, *J. Operator Theory* **9** (1983), 237–252.

[Sc] K. Schmidt, Asymptotically invariant sequences and an action of $\mathrm{SL}(2,\mathbb{Z})$ on the 2-sphere, *Israel J. Math.* **37** (1980), 193–208.

[Sha] Y. Shalom, Bounded generation and Kazhdan's property (T), *Publ. Math. I.H.E.S.* **90** (2001), 145–168.

[Sh] D. Shlyakhtenko, Free quasi-free states, *Pacific J. Math.* **177** (1997), 329–368.

[Si] I. M. Singer, Automorphisms of finite factors, *Amer. J. Math.* **77** (1955), 117–133.

[Va] A. Valette, Group pairs with the relative property (T) from arithmetic lattices, *Geom. Dedicata* **112** (2005), 183–196.

[V1] D. Voiculescu, Circular and semicircular systems and free product factors, *Progr. in Math.* **92**, Birkhauser, Boston, 1990, 45–60.

[V2] ———, The analogues of entropy and of Fisher's information measure in free probability. II, *Invent. Math.* **118** (1994), 411–440 .

[W] A. Wassermann, Coactions and Yang-Baxter equations for ergodic actions and subfactors, *London Math. Soc. Lecture Note Series* **136**, Cambridge Univ. Press, Cambridge, 1988, 203–236.

[Zi] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhaüser-Verlag, Boston, 1984.

(Received September 11, 2002)
(Revised October 28, 2004)