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Divisibility of anticyclotomic L-functions and theta functions with complex multiplication

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1. Introduction

The divisibility properties of Dirichlet L-functions in infinite families of characters have been studied by Iwasawa, Ferrero and Washington. The families considered by them are obtained by twisting an arbitrary Dirichlet character with all characters of p-power conductor for some prime p. One has to distinguish divisibility by p (the case considered by Iwasawa and Ferrero-Washington [FeW]) and by a prime $\ell \neq p$ (considered by Washington [W1], [W2]). Ferrero and Washington proved the vanishing of the Iwasawa μ-invariant of any branch of the Kubota-Leopoldt p-adic L-function. This means that each of the power series, which p-adically interpolate the nontrivial L-values of twists of a fixed Dirichlet character by characters of p-power conductor, has some coefficient that is a p-adic unit.

In the case $\ell \neq p$ Washington [W2] obtained the following theorem on divisibility of L-values by ℓ: given an integer $n \geq 1$ and a Dirichlet character χ, for all but finitely many Dirichlet characters ψ of p-power conductor with $\chi\psi(-1) = (-1)^n$,

$$v_\ell(\frac{1}{2}L(1-n, \chi\psi)) = 0.$$

Here v_ℓ denotes the ℓ-adic valuation of an element in $\mathbb{C}_\ell$, and we apply v_ℓ to algebraic numbers in $\mathbb{C}$ after fixing embeddings $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_\ell : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$.

By the class number formula these theorems are related to divisibility properties of class numbers in the cyclotomic $\mathbb{Z}_p$-extension of an abelian number field. One obtains the following qualitative picture: let F be an abelian number field, and $F_\infty = F\mathbb{Q}_\infty$ its cyclotomic $\mathbb{Z}_p$-extension with unique intermediate extensions F_n/F of degree p^n. The vanishing of the μ-invariant of F_∞/F implies by a well-known result of Iwasawa that the p-part of the class number h_n of F_n grows linearly with n for $n \to \infty$. Washington's theorem allows to control divisibility of h_n by primes $\ell \neq p$: his result implies that in this case the sequence of valuations $v_\ell(h_n)$ gets stationary for $n \to \infty$ [W1].

This paper considers the case of an imaginary quadratic field K and a prime p split in K. In this situation one can consider several possible

$\mathbb{Z}_p$-extensions and families of characters. Gillard [Gi] proved the analogue of Washington's theorem for the $\mathbb{Z}_p$-extensions in which precisely one of the primes of K lying above p is ramified. Here we are considering anticyclotomic $\mathbb{Z}_p$-extensions and families of anticyclotomic characters.

The main result will be phrased in terms of Hecke L-functions for the field K. For a prime ℓ fix embeddings i_∞ and i_ℓ as above. We consider K as a subfield of $\bar{\mathbb{Q}}$. Let D be the absolute value of the discriminant of K, and $\delta \in \mathfrak{o}_K$ the unique square root of $-D$ with $\operatorname{Im} i_\infty(\delta) > 0$. To define periods, consider an elliptic curve E with complex multiplication by $\mathfrak{o}_K$, defined over some number field $M \subseteq \bar{\mathbb{Q}}$, and a nonvanishing invariant differential ω on E. Given a pair (E, ω), we may extend the field of definition to $\mathbb{C}$ via i_∞, and (after replacing E by a Galois conjugate, if necessary) obtain a nonzero complex number Ω_∞, uniquely determined up to units in K, such that the period lattice of ω on E is given by $\Omega_\infty \mathfrak{o}_K$. Since we will be looking at L-values modulo ℓ, we need to normalize the pair (E, ω) by demanding that E has good reduction at the ℓ-adic place $\mathfrak{L}$ of M defined by i_ℓ (we are always able to find such a curve E after possibly enlarging M), and that ω reduces modulo $\mathfrak{L}$ to a nonvanishing invariant differential on the reduced curve $\bar{E}$. Fix the pair (E, ω) and the resulting period Ω_∞.

Consider (in general nonunitary) Hecke characters λ of K. If the infinity component of λ is $\lambda_\infty(x) = x^{-k}\bar{x}^{-j}$ for integers k and j, we say that λ has infinity type (k, j). Precisely for $k < 0$ and $j \geq 0$ or $k \geq 0$ and $j < 0$ the L-value $L(0, \lambda)$ is critical in the sense of Deligne. In this case it is known that $\pi^{\max(j,k)}\Omega_\infty^{-|k-j|}L(0, \lambda)$ is an algebraic number in $\mathbb{C}$.

The functional equation relates $L(0, \lambda)$ to $L(0, \lambda^*)$, where the dual λ^* of λ is defined by $\lambda^*(x) = \lambda(\bar{x})^{-1}|x|_{\mathbb{A}_K}$. We call a Hecke character λ anticyclotomic if $\lambda = \lambda^*$. This implies that its infinity type (k, j) satisfies $k+j = -1$, and that its restriction to $\mathbb{A}_{\mathbb{Q}}^\times$ is $\omega_{K/\mathbb{Q}} |\cdot|_{\mathbb{A}}$ for the quadratic character $\omega_{K/\mathbb{Q}}$ associated to the extension $K/\mathbb{Q}$. These will be the characters considered in this paper. Let $W(\lambda)$ be the root number appearing in the functional equation for $L(0, \lambda)$. For an anticyclotomic character we have $W(\lambda) = \pm 1$. We also need to introduce local root numbers. For this, define for a prime ideal $\mathfrak{q}$ and an element $d_\mathfrak{q} \in K_\mathfrak{q}^\times$ with $d_\mathfrak{q}\mathfrak{o}_{K_\mathfrak{q}} = \delta\mathfrak{o}_{K_\mathfrak{q}}$ the local Gauss sum at $\mathfrak{q}$ by

$$G(d_\mathfrak{q}, \lambda_\mathfrak{q}) = \lambda(\varpi_\mathfrak{q}^{-e(\mathfrak{q})}) \sum_{u \in (\mathfrak{o}_K/\mathfrak{q}^{e(\mathfrak{q})})^\times} \lambda_\mathfrak{q}(u) e_K(\varpi_\mathfrak{q}^{-e(\mathfrak{q})} d_\mathfrak{q}^{-1} u),$$

if $\lambda_\mathfrak{q}$ is ramified, and set $G(d_\mathfrak{q}, \lambda_\mathfrak{q}) = 1$ otherwise. Here $e(\mathfrak{q})$ is the exponent of $\mathfrak{q}$ in the conductor of λ, $\varpi_\mathfrak{q}$ is a prime element of $K_\mathfrak{q}$, and e_K is the additive character of $\mathbb{A}_K/K$ defined by $e_K = e_{\mathbb{Q}} \circ \operatorname{Tr}_{K/\mathbb{Q}}$ in terms of the standard additive character $e_{\mathbb{Q}}$ of $\mathbb{A}/\mathbb{Q}$ normalized by $e_{\mathbb{Q}}(x_\infty) = e^{2\pi i x_\infty}$. The ℓ-adic root number of λ is then

$$W_\ell(\lambda) = \mathrm{N}(\mathfrak{l})^{-e(\mathfrak{l})} G(\delta, \lambda_\mathfrak{l}),$$

where $\mathfrak{l}$ is the prime ideal of K determined by i_ℓ. In the same way set $W_q(\lambda) = W_q(\lambda_q) = \mathrm{N}(\mathfrak{q})^{-e(\mathfrak{q})} G(-\delta, \lambda_\mathfrak{q})$ for all nonsplit primes q, where $\mathfrak{q}$ denotes the unique prime ideal of K above q. For anticyclotomic characters λ we have $W_q(\lambda) = \pm 1$ for all nonsplit q, $W_q(\lambda) = (-1)^{v_q(\mathfrak{f}_\lambda)}$ for all inert q, where $\mathfrak{f}_\lambda$ is the conductor of λ [MS, Prop. 3.7], and $W(\lambda) = \prod_q W_q(\lambda)$ if λ has infinity type $(-k, k-1)$ with $k \geq 1$ (cf. the proof of Corollary 2.3). Let $\mathcal{W}$ be the set of all systems of signs (w_q), q ranging over all nonsplit primes, with $w_q = 1$ for almost all q and $\prod_q w_q = 1$; to each anticyclotomic character λ of infinity type $(-k, k-1)$, $k \geq 1$, and root number $W(\lambda) = +1$ corresponds an element $w(\lambda) \in \mathcal{W}$. For an inert prime q and a character χ_q of $K_q^\times$ define $\mu_\ell(\chi_q)$ by $\mu_\ell(\chi_q) = 0$ if χ_q is unramified, and $\mu_\ell(\chi_q) = \min_{x \in \mathfrak{o}_{K_q}^\times} v_\ell(\chi_q(x) - 1)$ otherwise. Also, for ℓ inert or ramified in K, we will define in Equation (14) of Section 3 for each character χ_ℓ of $K_\ell^\times$ with $\chi_\ell|_{\mathbb{Q}_\ell^\times} = \omega_{K/\mathbb{Q},\ell} |\cdot|_\ell$ and each vector $w \in \mathcal{W}$ with $w_\ell = W_\ell(\chi_\ell)$ a rational number $b_\ell(\chi_\ell, w)$. If χ_ℓ is unramified (for ℓ inert) or has minimal conductor (for ℓ ramified), we have $b_\ell(\chi_\ell, w) = 0$. We are now able to state the main result.

THEOREM 1.1. *Let k and d be fixed positive integers, p an odd prime split in K, and ℓ an odd prime different from p. Fix a complex period Ω_∞ as above.*

1. *If ℓ splits in K, for all but finitely many anticyclotomic Hecke characters λ of K of conductor dividing dDp^∞, infinity type $(-k, k-1)$, and global root number $W(\lambda) = +1$ we have*

$$v_\ell\left(\Omega_\infty^{1-2k} (k-1)! \left(\frac{2\pi}{\sqrt{D}}\right)^{k-1} W_\ell(\lambda) L(0, \lambda)\right) = \sum_{q \text{ inert in } K} \mu_\ell(\lambda_q). \tag{1}$$

2. *If ℓ is inert or ramified in K and $k = 1$, for all but finitely many anticyclotomic Hecke characters λ of K of conductor dividing dDp^∞, infinity type $(-1, 0)$, and global root number $W(\lambda) = +1$,*

$$v_\ell(\Omega_\infty^{-1} D^{1/4} L(0, \lambda)) = \sum_{q \neq \ell \text{ inert in } K} \mu_\ell(\lambda_q) + b_\ell(\lambda_\ell, w(\lambda)). \tag{2}$$

Moreover, for all anticyclotomic characters λ of infinity type as above the left-hand side of these equations is bigger than or equal to the right-hand side (except possibly for $K = \mathbb{Q}(\sqrt{-3})$ and $\ell = 3$).

In the case $W(\lambda) = -1$ we have of course $L(0, \lambda) = 0$ from the functional equation. The inequality for all characters is much easier to prove than the equality assertion for almost all characters in an infinite family, which is the main content of the theorem.

Note that in contrast to the case of Dirichlet L-functions (and the case dealt with by Gillard) we do not obtain in general that almost all L-values are not divisible by ℓ, although this is true whenever the right-hand side vanishes, for example if we restrict to split ℓ and characters λ with no inert prime

$q \equiv -1\,(\ell)$ dividing the conductor of λ with multiplicity one. That a restriction of this type is necessary was indicated by examples of Gillard [Gi, §6].

The method used to obtain this result is based on ideas of Sinnott [Si1], [Si2], who gave an algebraic proof of Washington's theorem. Sinnott's strategy starts from the fact that Dirichlet L-values are closely connected to rational functions, which allows him to derive their nonvanishing modulo ℓ from an algebraic independence result. Gillard transfered this method to functions on an elliptic curve with complex multiplication by $\mathfrak{o}_K$. Here, we use a result of Yang [Y] which connects anticyclotomic L-values to special values of theta functions on such an elliptic curve. Section 2 of this paper, which is to a large part expository, reviews the theory of the Shintani representation [Shin] on theta functions, and reformulates Yang's result in this setting (see Proposition 2.4 below). Section 3 introduces arithmetic theta functions and reduces the main theorem to a nonvanishing result for theta functions in characteristic ℓ. This statement (Theorem 4.1), which may be regarded as the main result of this paper, is then established in Section 4. Sinnott's ideas have to be considerably modified in this situation, since we are dealing with sections of line bundles instead of functions on the curve.

Recently, Hida [Hid1], [Hid2] has considered the divisibility problem more generally for critical Hecke L-values of CM fields, using directly the connection to special values of Hilbert modular Eisenstein series at CM points. Although general proofs have not yet been worked out, it is likely that his methods are able to cover the first case of our result. On the other hand, to extend them to deal with divisibility by nonsplit primes (our second case) seems to require additional ideas. We hope that our completely different approach is of independent interest. In a forthcoming paper, we will apply it to the determination of the Iwasawa μ-invariant of anticyclotomic L-functions.[1]

This paper has its origins in a part of my 2000 Düsseldorf doctoral thesis [Fi]. I would like to thank Fritz Grunewald, Haruzo Hida, and Jon Rogawski for many interesting remarks and discussions. Special thanks to Don Blasius for some helpful discussions on some subtler aspects of Section 4.

We keep the notation introduced so far. In addition, let w_K denote the number of units in K, and $\nu(D)$ the number of distinct prime divisors of D.

2. Theta functions, Shintani operators and anticyclotomic L-values

This section reviews the theory of primitive theta functions and Shintani operators (mainly due to Shintani [Shin]), which amounts to a study of the dual pair $(\mathrm{U}(1), \mathrm{U}(1))$ in a "classical" setting. We do not touch here on the appli-

[1]See Tobias Finis, The μ-invariant of anticyclotomic L-functions of imaginary quadratic fields, to appear in *J. reine angew. Math.*

cations to the theory of automorphic forms on U(3). Building on Shintani's work, a complete description of the decomposition of Shintani's representation into characters is given as a consequence of the local results of Murase-Sugano [MS] (see also [Ro], [HKS]). Then we explain the connection between values of a certain linear functional on Shintani eigenspaces and anticyclotomic L-values for the field K, which is a reformulation of results of Yang [Y] (specialized to imaginary quadratic fields).

Generalized theta functions. We begin by defining spaces of generalized theta functions in the sense of Shimura [Shim2], [Shim3] (cf. also [I], [Mum2], [Mum3] for background on theta functions). Although only usual scalar valued theta functions will be used to prove the main result of this paper, we state the connection between theta functions and anticyclotomic L-values in the general case. A geometric reformulation of the theory will be given in Section 3. For an integer $\nu \geq 0$ let V_ν be a complex vector space of dimension $\nu + 1$ and $N \in \mathrm{End}(V_\nu)$ a nilpotent operator of exact order $\nu + 1$. We set $V_\nu = \mathbb{C}^{\nu+1}$ and normalize $N = (n_{ij})$ as a lower triangular matrix with $n_{i+1,i} = -i$, $1 \leq i \leq \nu$, and all other entries zero. Given a positive rational number r and a fractional ideal $\mathfrak{a}$ of K such that $r\mathrm{N}(\mathfrak{a})$ is integral, the space $T_{r,\mathfrak{a};\nu}$ of generalized theta functions is defined as the space of V_ν-valued holomorphic functions ϑ on $\mathbb{C}$ satisfying the functional equation

$$\vartheta(w + l) = \psi(l) e^{-2\pi i r \delta \bar{l}(w+l/2)} e^{\delta \bar{l} N} \vartheta(w), \quad l \in \mathfrak{a}, \tag{3}$$

where $\psi(l) = (-1)^{rD|l|^2}$ is a semi-character on $\mathfrak{a}$. The case $\nu = 0$ corresponds to ordinary scalar valued theta functions. It is not difficult to see that $\dim T_{r,\mathfrak{a};\nu} = rD\mathrm{N}(\mathfrak{a})(\nu + 1)$.

For $l \in \mathbb{C}$ and any V_ν-valued function f on $\mathbb{C}$ define

$$(A_l f)(w) = e^{2\pi i r \delta \bar{l}(w+l/2)} e^{-\delta \bar{l} N} f(w + l). \tag{4}$$

The operators A_l fulfill the basic commutation relation

$$A_{l_1} A_{l_2} = e^{\pi i r \mathrm{Tr}\,(\delta l_1 \bar{l}_2)} A_{l_1 + l_2}.$$

For $l \in \mathfrak{a}^* = (r\mathrm{N}(\mathfrak{a})D)^{-1}\mathfrak{a}$, the dual lattice of $\mathfrak{a}$, the operator A_l is an endomorphism of $T_{r,\mathfrak{a};\nu}$, and it acts by multiplication by $\psi(l)$ if $l \in \mathfrak{a}$. We may reformulate these facts in the language of group representations. Introduce a group structure on the set of pairs $(l, \lambda) \in \mathbb{C} \times \mathbb{C}^\times$ by setting $(l_1, \lambda_1)(l_2, \lambda_2) = (l_1 + l_2, \lambda_1 \lambda_2 e^{2\pi i r \mathrm{Re}\,(\delta l_1 \bar{l}_2)})$. The pairs $(l, \psi(l))$, $l \in \mathfrak{a}$, form a subgroup isomorphic to $\mathfrak{a}$, whose normalizer is the set of all pairs (l, λ) with $l \in \mathfrak{a}^*$. Define a group $G_{r,\mathfrak{a}}$ as the quotient of this normalizer by the subgroup $\{(l, \psi(l)) \,|\, l \in \mathfrak{a}\}$. The group $G_{r,\mathfrak{a}}$ is a Heisenberg group, i.e. it fits into an exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G_{r,\mathfrak{a}} \longrightarrow A \longrightarrow 0$$

with the abelian group $A = \mathfrak{a}^*/\mathfrak{a}$, and its center is precisely the image of $\mathbb{C}^\times$. Mapping (l, λ) to λA_l defines now clearly a representation of $G_{r,\mathfrak{a}}$ on $T_{r,\mathfrak{a};\nu}$. In the case $\nu = 0$ it is well-known that this representation is irreducible.

The standard scalar product on $T_{r,\mathfrak{a};\nu}$ is defined by

$$\langle \vartheta_1, \vartheta_2 \rangle = \frac{2}{\sqrt{D}\mathrm{N}(\mathfrak{a})} \int_{\mathbb{C}/\mathfrak{a}} \overline{(A_u \vartheta_1)(0)_{\nu+1}} (A_u \vartheta_2)(0)_{\nu+1} du. \tag{5}$$

The operators A_l are unitary with respect to this scalar product.

It will be necessary to deal simultaneously with all spaces $T_{r,\mathfrak{a};\nu}$ for $\mathfrak{a}$ ranging over the ideal classes of K. Let $\delta(x)$ be the operator on V_ν given by $\mathrm{diag}(x^\nu, \ldots, 1)$; then $\delta(x) N \delta(x)^{-1} = x^{-1} N$. Define for a positive integer d the space $\mathcal{T}_{d;\nu}$ as the space of families $(t_\mathfrak{a}) \in \prod_{\mathfrak{a} \in I_K} T_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\nu}$ satisfying

$$t_{\lambda\mathfrak{a}}(\lambda w) = \delta(\bar{\lambda}^{-1}) t_\mathfrak{a}(w), \quad \lambda \in K^\times.$$

After choosing a system of representatives $\mathcal{A}$ for the ideal classes of K we get an isomorphism $\mathcal{T}_{d;\nu} \simeq \bigoplus_{\mathfrak{a} \in \mathcal{A}} T^1_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\nu}$, where $T^1_{r,\mathfrak{a};\nu} \subseteq T_{r,\mathfrak{a};\nu}$ denotes the subspace of theta functions ϑ invariant under the action of the roots of unity in K: $\vartheta(\omega w) = \delta(\omega)\vartheta(w)$ for $\omega \in \mathfrak{o}_K^\times$. The standard scalar product on $\mathcal{T}_{d;\nu}$ is given by $\langle \vartheta, \vartheta' \rangle = \sum_{\mathfrak{a} \in \mathcal{A}} \langle \vartheta_\mathfrak{a}, \vartheta'_\mathfrak{a} \rangle$.

Finally, using the natural exact sequence of genus theory

$$1 \longrightarrow \mathrm{Cl}_K^2 \longrightarrow \mathrm{Cl}_K \xrightarrow{N} \mathrm{N}(I_K)/\mathrm{N}(K^\times) \longrightarrow 1,$$

for any class $C \in \mathrm{N}(I_K)/\mathrm{N}(K^\times)$ we define a subspace $\mathcal{V}_{d,C;\nu}$ of $\mathcal{T}_{d;\nu}$ by restricting $\mathfrak{a}$ to the preimage of C.

Review of Shintani theory. We now review the theory of primitive theta functions and Shintani operators. These operators give a description of the Weil representation for U(1) on the spaces of theta functions defined above. For more details see [Shin], [GlR], [MS].

For each pair of ideals $\mathfrak{b} \supseteq \mathfrak{a}$ such that $r\mathrm{N}(\mathfrak{b})$ is integral, there is a natural inclusion $T_{r,\mathfrak{b};\nu} \hookrightarrow T_{r,\mathfrak{a};\nu}$. Its adjoint with respect to the natural inner product is the trace operator $t_\mathfrak{b} : T_{r,\mathfrak{a};\nu} \longrightarrow T_{r,\mathfrak{b};\nu}$ defined by $t_\mathfrak{b} = \sum_{l \in \mathfrak{b}/\mathfrak{a}} \psi(l) A_l$. The space of primitive theta functions $T^{\mathrm{prim}}_{r,\mathfrak{a};\nu} \subseteq T_{r,\mathfrak{a};\nu}$ is then defined as

$$T^{\mathrm{prim}}_{r,\mathfrak{a};\nu} = \bigcap_{\mathfrak{b} \supset \mathfrak{a},\ r\mathrm{N}(\mathfrak{b}) \text{ integral}} \ker t_\mathfrak{b} = \bigcap_{\mathfrak{c} \subset \mathfrak{o}_K,\ \mathrm{N}(\mathfrak{c}) | r\mathrm{N}(\mathfrak{a})} \ker t_{\mathfrak{a}\mathfrak{c}^{-1}}.$$

It is the orthogonal complement of the span of the images of all inclusions $T_{r,\mathfrak{b};\nu} \hookrightarrow T_{r,\mathfrak{a};\nu}$ with $r\mathrm{N}(\mathfrak{b})$ integral. Correspondingly, the space $\mathcal{T}^{\mathrm{prim}}_{d;\nu}$ is the space of all families $(t_\mathfrak{a}) \in \mathcal{T}_{d;\nu}$ with $t_\mathfrak{a} \in T^{\mathrm{prim}}_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\nu}$ for all $\mathfrak{a}$, and in the same way one defines $\mathcal{V}^{\mathrm{prim}}_{d,C;\nu} \subseteq \mathcal{V}_{d,C;\nu}$.

Now let $\mathfrak{b} \in I_K^1$, the group of norm one ideals of K, and let $\mathfrak{c}$ be the unique integral ideal with $\mathfrak{c} + \bar{\mathfrak{c}} = \mathfrak{o}_K$ and $\mathfrak{b} = \bar{\mathfrak{c}}\mathfrak{c}^{-1}$. Then the composition

$$T_{r,\mathfrak{a};\nu} \hookrightarrow T_{r,\mathfrak{a}\bar{\mathfrak{c}};\nu} \xrightarrow{t_{\mathfrak{a}\bar{\mathfrak{c}}\mathfrak{c}^{-1}}} T_{r,\mathfrak{a}\bar{\mathfrak{c}}\mathfrak{c}^{-1};\nu}$$

is a linear operator called $\mathcal{E}(\mathfrak{b})$. Varying $\mathfrak{a}$, these operators induce an endomorphism of $\mathcal{V}_{d,C;\nu}$, also denoted by $\mathcal{E}(\mathfrak{b})$. We call these operators Shintani operators. For $\eta \in K^1$ we can construct an endomorphism $\mathcal{E}(\eta)$ of $T_{r,\mathfrak{a};\nu}$ by composing $\mathcal{E}((\eta)) : T_{r,\mathfrak{a};\nu} \to T_{r,\eta\mathfrak{a};\nu}$ with the isomorphism $T_{r,\eta\mathfrak{a};\nu} \simeq T_{r,\mathfrak{a};\nu}$ given by $\vartheta_{\mathfrak{a}}(w) = \delta(\bar{\eta})\vartheta_{\eta\mathfrak{a}}(\eta w)$ (for $\nu = 0$ these are the operators considered in [GlR]). The operators $\mathcal{E}(\eta)$ have the fundamental commutation property $\mathcal{E}(\eta)A_{\eta l} = A_l\mathcal{E}(\eta)$ for $l \in \mathfrak{a}^* \cap \eta^{-1}\mathfrak{a}^*$ [GlR, p. 72].

For all $\mathfrak{b}$ prime to $r\mathrm{N}(\mathfrak{a})$ we have the relation $\mathcal{E}(\mathfrak{b}^{-1})\mathcal{E}(\mathfrak{b}) = \mathrm{N}(\mathfrak{c})$, and in particular $\mathcal{E}(\mathfrak{b})$ is an isomorphism. Furthermore, $\mathcal{E}(\mathfrak{b}_1)\mathcal{E}(\mathfrak{b}_2) = \mathcal{E}(\mathfrak{b}_1\mathfrak{b}_2)$ if $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are prime to $r\mathrm{N}(\mathfrak{a})$ and the denominator of $\mathfrak{b}_1$ is prime to the denominator of $\mathfrak{b}_2^{-1}$ (cf. [GlR]). Therefore a slight modification of these operators gives a group representation. Any fractional ideal $\mathfrak{c}$ of K can be uniquely written as $\mathfrak{c} = c\mathfrak{c}'$ with a positive rational number c and an integral ideal $\mathfrak{c}'$ such that $p \nmid \mathfrak{c}'$ for any rational prime p. For a positive integer d let $\gamma_d(\mathfrak{c}) = \mathrm{N}(\mathfrak{c})^{-1}c\omega_{K/\mathbb{Q}}(c)$ for $\mathfrak{c}$ prime to dD, and extend the definition to all fractional ideals $\mathfrak{c}$ by stipulating that $\gamma_d(\mathfrak{c})$ depends only on the prime-to-dD part of $\mathfrak{c}$. Then define $\mathcal{F}^*(\mathfrak{c}) : T_{r,\mathfrak{a};\nu} \to T_{r,\mathfrak{a}\mathfrak{c}\bar{\mathfrak{c}}^{-1};\nu}$ by $\mathcal{F}^*(\mathfrak{c}) = \gamma_{r\mathrm{N}(\mathfrak{a})}(\mathfrak{c})\mathcal{E}(\mathfrak{c}\bar{\mathfrak{c}}^{-1})$ for all $\mathfrak{c}$ with $\mathfrak{c}\bar{\mathfrak{c}}^{-1}$ prime to $r\mathrm{N}(\mathfrak{a})$. These modified operators are multiplicative and yield in particular a representation of the group of all ideals $\mathfrak{c}$ with $\mathfrak{c}\bar{\mathfrak{c}}^{-1}$ prime to d on $\mathcal{V}_{d,C;\nu}$ which leaves the primitive subspace $\mathcal{V}_{d,C;\nu}^{\mathrm{prim}}$ invariant. This representation decomposes into Hecke characters of K [Shin], [GlR]; see Proposition 2.2 below for a complete description of the decomposition.

In the same way we obtain a representation of the group of all $z \in K^\times$ with $z/\bar{z}$ prime to $r\mathrm{N}(\mathfrak{a})$ on $T_{r,\mathfrak{a};\nu}$ by setting $\mathcal{F}^*(z) = \gamma_{r\mathrm{N}(\mathfrak{a})}((z))\mathcal{E}(z/\bar{z})$. These notions are clearly compatible: the action of $\mathcal{F}^*((z))$ on $\mathcal{V}_{d,C;\nu}$ is given by the action of $\mathcal{F}^*(z)$ on the components in $T_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\nu}$, therefore the components of Shintani eigenfunctions are eigenfunctions. On the other hand, if a Shintani eigenfunction in $T_{r,\mathfrak{a};\nu}$ is invariant under the roots of unity, it extends in $h_K/2^{\nu(D)-1}$ many ways to a Shintani eigenfunction in $\mathcal{V}_{r\mathrm{N}(\mathfrak{a}),\mathrm{N}(\mathfrak{a})\mathrm{N}(K^\times);\nu}$.

Classical and adelic theta functions. To apply the local results of Murase-Sugano to the study of the Shintani representation, we now introduce some adelic function spaces isomorphic to the classically defined spaces $T_{r,\mathfrak{a};\nu}$ and $\mathcal{V}_{d,C;\nu}$. This is a standard construction, and we follow Shintani with some modifications.

Let $e_{\mathbb{Q}}$ be the additive character of $\mathbb{A}/\mathbb{Q}$ normalized by $e_{\mathbb{Q}}(x_\infty) = e^{2\pi i x_\infty}$, as in the introduction. The Heisenberg group H is an algebraic group over $\mathbb{Q}$ which is $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{A}^1 \times \mathbb{A}^1$ as a variety, but has the modified non-abelian group law

$$(w_1,t_1)(w_2,t_2) = (w_1+w_2, t_1+t_2+\mathrm{Tr}_{K/\mathbb{Q}}(\delta\bar{w}_1 w_2)/2).$$

Adelic theta functions will be functions on the group $H(\mathbb{A})$ of adelic points of H. Define a differential operator D_- on smooth functions on $H(\mathbb{A})$ by

$$(D_-\theta)((w,t)) = \left(\frac{1}{2\pi i}\frac{\partial}{\partial\bar{w}_\infty}\right)(\theta((w,t))e^{-\pi ir\delta|w_\infty|^2})e^{\pi ir\delta|w_\infty|^2},$$

and let $T^{\mathbb{A}}_{r,\nu}$ be the space of all smooth functions $\theta : H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathbb{C}$ with $\theta((0,t)h) = e_{\mathbb{Q}}(rt)\theta(h)$ and $D_-^{\nu+1}\theta = 0$. This space comes with a natural right-$H(\mathbb{A}_f)$-action denoted by ρ. Given a fractional ideal $\mathfrak{a}$ of K, we define a subgroup $H(\mathfrak{a})_f$ of $H(\mathbb{A}_f)$ by

$$H(\mathfrak{a})_f = \{(w,t) \in H(\mathbb{A}_f) \,|\, w \in \hat{\mathfrak{a}},\, t+\delta w\bar{w}/2 \in \mathrm{N}(\mathfrak{a})\hat{\mathfrak{o}}_K\}$$

and denote by $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ the subspace of $H(\mathfrak{a})_f$-invariant functions in $T^{\mathbb{A}}_{r,\nu}$.

It is a basic fact that $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ is naturally isomorphic to the classically defined space $T_{r,\mathfrak{a};\nu}$. We give the construction of the isomorphism, leaving the details to the reader. First $T^{\mathbb{A}}_{r,\nu}$ is isomorphic (as a $H(\mathbb{A}_f)$-module) to the space $S^{\mathbb{A}}_{r,\nu}$ of all smooth functions $\Theta : H(\mathbb{Q})\backslash H(\mathbb{A}) \to V_\nu$ with $\Theta((0,t)h) = e_{\mathbb{Q}}(rt)\Theta(h)$ such that

$$\vartheta_{h_f}(w_\infty) = e^{-\pi ir\delta|w_\infty|^2}e^{\delta\bar{w}_\infty N}\Theta((w_\infty,0)h_f)$$

is holomorphic in $w_\infty \in \mathbb{C}$ for all $h_f \in H(\mathbb{A}_f)$. The isomorphism is obtained by mapping $\theta \in T^{\mathbb{A}}_{r,\nu}$ to the vector valued function $\Theta \in S^{\mathbb{A}}_{r,\nu}$ with

$$\Theta_j = \frac{(2\pi/\sqrt{D})^{\nu+1-j}}{\nu(\nu-1)\cdots j}D_-^{\nu+1-j}\theta, \quad 1 \le j \le \nu+1.$$

Then the space of $H(\mathfrak{a})_f$-invariants in $S^{\mathbb{A}}_{r,\nu}$ is identified with $T_{r,\mathfrak{a};\nu}$ by associating to Θ the holomorphic V_ν-valued function $\vartheta_{(0,0)}(w_\infty)$ defined above. Composing these two constructions gives the desired isomorphism.

We also introduce adelic counterparts of the spaces $\mathcal{V}_{d,C;\nu}$. Our definition is similar to Shintani's definition of the spaces $V_{d/c}(\rho,c)$, $c \in \mathbb{Q}^\times$ a representative for the class C [Shin, p. 29]. Consider the algebraic group R over $\mathbb{Q}$ obtained as the semidirect product of H with $\mathrm{U}(1) \subseteq \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$ (the group of norm one elements), where $\mathrm{U}(1)$ acts on H by $u(w,t)u^{-1} = (uw,t)$. Given r and $\mathfrak{a}$ let $V^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ be the space of all smooth functions

$$\varphi : R(\mathbb{Q})\backslash R(\mathbb{A})/\hat{\mathfrak{o}}^1_K K^1_\infty H(\mathfrak{a})_f \to \mathbb{C}$$

with $\varphi((0,t)g) = e_{\mathbb{Q}}(rt)\varphi(g)$ and $D_-^{\nu+1}\varphi = 0$. To every $\varphi \in V^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ we may associate functions $\varphi_u \in T^{\mathbb{A}}_{r,\nu}((u_f)\mathfrak{a})$ for $u \in \mathbb{A}^1_K$ by setting $\varphi_u(h) = \varphi(hu)$. By definition φ_u depends only on the norm one ideal (u_f) of K and $\varphi_{\lambda u}((\lambda w,t)) = \varphi_u((w,t))$ for $\lambda \in K^1$. Identifying the various functions φ_u for

$u \in \mathbb{A}_K^1$ with elements of $T_{r,(u_f)\mathfrak{a};\nu}$, we get an isomorphism between $V_{r,\nu}^{\mathbb{A}}(\mathfrak{a})$ and $\mathcal{V}_{r\mathrm{N}(\mathfrak{a}),\mathrm{N}(\mathfrak{a})\mathrm{N}(K^\times);\nu}$.

Using these isomorphisms, the classical Shintani operators $\mathcal{E}$ and $\mathcal{F}^*$ may be expressed directly in the adelic framework. It is not difficult to show (see [GlR, p. 92]),[2] that the operator $\mathcal{F}^*(z)$ on $T_{r,\mathfrak{a};\nu}$ corresponds to the operator $L^*(z) = \gamma_{r\mathrm{N}(\mathfrak{a})}((z))\mathrm{N}(\mathfrak{c})P_{\mathfrak{a}}l(z/\bar{z})$ on $T_{r,\nu}^{\mathbb{A}}(\mathfrak{a})$, where $\mathfrak{c}$ is the denominator ideal of $z/\bar{z}$,

$$P_{\mathfrak{a}} = \mathrm{vol}(H(\mathfrak{a})_f)^{-1} \int_{H(\mathfrak{a})_f} \rho(g)dg$$

is the projector onto the space of $H(\mathfrak{a})_f$-invariants, and we set $(l(\eta)\theta)((w,t)) = \theta((\eta w, t))$ for $\theta \in T_{r,\nu}^{\mathbb{A}}$ and $\eta \in K^1$. The operator on $V_{r,\nu}^{\mathbb{A}}(\mathfrak{a})$ corresponding to $\mathcal{F}^*(\mathfrak{b})$ on $\mathcal{V}_{d,C;\nu}$ is then $L^*(\mathfrak{b}) = \gamma_d(\mathfrak{b})\mathrm{N}(\mathfrak{c})P_{\mathfrak{a}}l(\mathfrak{b}\bar{\mathfrak{b}}^{-1})$, where $\mathfrak{c}$ denotes the denominator of $\mathfrak{b}\bar{\mathfrak{b}}^{-1}$, and $l(\mathfrak{b}\bar{\mathfrak{b}}^{-1})$ right translation by β^{-1} for any $\beta \in \mathbb{A}_K^1$ with $(\beta) = \mathfrak{b}\bar{\mathfrak{b}}^{-1}$.

Weil representation and theta functions. To construct theta functions in the adelic setting we use the Weil representation. By the Stone-von Neumann theorem there exists a unique irreducible smooth representation ρ of $H(\mathbb{A})$ on a space V such that $\rho((0,t))$ acts by the scalar $e_{\mathbb{Q}}(rt)$. The representation may be written as a (restricted) tensor product $V = \otimes_p V_p$ (p ranging over all places of $\mathbb{Q}$, including infinity).[3]

A standard realization of V_p is the lattice model $V_p \subseteq S(K_p)$ considered (among others) by Murase-Sugano [MS]. At the infinite place it may be supplemented by the Fock representation (cf. [I, Ch. 1, §8]): $V_\infty \subseteq S(K_\infty)$ (the space of Schwartz functions on $K_\infty \simeq \mathbb{C}$) is defined as

$$V_\infty = \{\phi : K_\infty \to \mathbb{C} \,|\, \phi(z)e^{-\pi i r\delta|z|^2} \text{ antiholomorphic}, \quad \int_{K_\infty} |\phi(z)|^2 dz < \infty\}.$$

It is a Hilbert space with the obvious scalar product. The action of $H(\mathbb{R})$ on V_∞ is given by

$$(\rho((w,t))\phi)(z) = e^{2\pi i r(\delta(\bar{z}w - z\bar{w})/2 + t)}\phi(z+w).$$

Denote by $V_\infty^{(\nu)} \subseteq V_\infty$ the subspace obtained by restricting $\phi(z)e^{-\pi i r\delta|z|^2}$ to polynomials in $\bar{z}$ of degree at most ν.

Putting everything together, we have a global lattice model $V \subseteq S(\mathbb{A}_K)$ with $H(\mathbb{A}_f)$-invariant subspaces $V^{(\nu)} \subseteq V$. The theta functional $V \to \mathbb{C}$ is given by $\theta(\phi) = \sum_{z\in K} \phi(z)$. To every $\phi \in V$ we associate the theta func-

[2]To be precise, the proof given there only considers the case $\nu = 0$, but carries over to the general case.

[3]For the following setup of the Weil representation until Proposition 2.1 I am indebted to Murase-Sugano.

tion $\theta = \theta_\phi : H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathbb{C}$ by $\theta(h) = \theta(\rho(h)\phi)$. Trivially $\theta((0,t)h) = e_{\mathbb{Q}}(rt)\theta(h)$.

We may now define an operator D_- on V compatible under the map $\phi \mapsto \theta_\phi$ with the operator D_- on smooth functions on $H(\mathbb{A})$ by setting

$$D_-(\phi)(z) = (2\pi i)^{-1}(\partial/\partial\bar{z}_\infty)(\phi(z)e^{-\pi i r\delta|z|^2})e^{\pi i r\delta|z|^2}.$$

It is then easy to see that for $\phi \in V^{(\nu)}$ we have $\theta_\phi \in T^{\mathbb{A}}_{r,\nu}$. In fact, the map $\phi \mapsto \theta_\phi$ is an $H(\mathbb{A}_f)$-equivariant isomorphism of these spaces.

Murase-Sugano define a (modified) Weil representation $\mathcal{M}_p$ of $K_p^\times$ on V_p for all primes p. A Weil representation $\mathcal{M}_\infty$ of $K_\infty^\times$ on V_∞ may be defined by exactly the same integral expression [MS, 2.1, 4.3] as in the nonarchimedian case. In this way we get a representation $\mathcal{M} = \bigotimes_p \mathcal{M}_p$ of $\mathbb{A}_K^\times$ on V fulfilling the commutation rule $\mathcal{M}(z)\rho(h) = \rho((\bar{z}/z)h(\bar{z}/z)^{-1})\mathcal{M}(z)$. Although the operators $\mathcal{M}(z)$ for $z \in K^\times$ act nontrivially on V, they leave the theta functional invariant: $\theta(\mathcal{M}(z)\phi) = \theta(\phi)$ for $z \in K^\times$. The structure of the representations $\mathcal{M}_p$ for finite p is described in detail by Murase-Sugano. Consideration of the infinite place does not pose any problems. We obtain here the eigenvectors

$$\phi_\infty^{(m)}(z) = \bar{z}^m e^{\pi i\delta r|z|^2}, \quad m \geq 0,$$

with eigencharacters

$$\mathcal{M}_\infty(z)\phi_\infty^{(m)} = \left(\frac{|z|}{z}\right)^{2m+1}\phi_\infty^{(m)}.$$

We are now able to relate the Shintani operators $\mathcal{F}^*(z)$ and $L^*(z)$ to the action of $\mathcal{M}$ on V. For a fractional ideal $\mathfrak{a}$ of K let $\mathfrak{a}_p = \mathfrak{a} \otimes \mathbb{Z}_p \subseteq K_p = K \otimes \mathbb{Q}_p$ be its completion at a prime p.

Proposition 2.1. *Under the isomorphism between $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ and the space*

$$V^{(\nu)}(\mathfrak{a}) \simeq V_\infty^{(\nu)} \otimes \bigotimes_{p|r\mathrm{N}(\mathfrak{a})D} V_p(\mathfrak{a}_p)$$

of $H(\mathfrak{a})_f$-invariants in $V^{(\nu)}$ induced by $T^{\mathbb{A}}_{r,\nu} \simeq V^{(\nu)}$, for all $z \in K^\times$ with $z/\bar{z}$ prime to $r\mathrm{N}(\mathfrak{a})$ the operator $L^*(z)$ on $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ corresponds to the operator

$$\mathcal{M}_\infty(z)|z|_{K_\infty}^{-1/2} \otimes \bigotimes_{p|r\mathrm{N}(\mathfrak{a})D} \mathcal{M}_p(z)|z|_{K_p}^{-1/2}$$

on $V^{(\nu)}(\mathfrak{a})$.

Proof. Take $\phi \in V^{(\nu)}(\mathfrak{a})$ corresponding to $\theta_\phi \in T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$. Since the theta functional is invariant under $\mathcal{M}(z)$ for $z \in K^\times$, we see that

$$\theta_{\mathcal{M}(z)\phi}(h) = \theta(\rho(h)\mathcal{M}(z)\phi) = \theta(\mathcal{M}(z)\rho((z/\bar{z})h(z/\bar{z})^{-1})\phi) = (l(z/\bar{z})\theta_\phi)(h).$$

Therefore the operator $L^*(z)$ on $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ corresponds to $\gamma_{r\mathrm{N}(\mathfrak{a})}((z))\mathrm{N}(\mathfrak{c})P_{\mathfrak{a}}\mathcal{M}(z)$ on $V^{(\nu)}(\mathfrak{a})$, where $\mathfrak{c}$ is the denominator ideal of $z/\bar{z}$. We may write $P_{\mathfrak{a}}\mathcal{M}(z)$ as a local product over all places p of $\mathbb{Q}$; if p is nonsplit or z is a unit at p, the space $V_p(\mathfrak{a}_p)$ is invariant under $\mathcal{M}_p(z)$, and the factor $P_{\mathfrak{a}_p}$ is superfluous. If $p\not| r\mathrm{N}(\mathfrak{a})D$ is inert, then $\mathcal{M}_p(z)$ acts via multiplication by $(-1)^{v_p(z)}$.

On the other hand, z can be a nonunit at a split place p only if $p\not| r\mathrm{N}(\mathfrak{a})D$, and the space $V_p(\mathfrak{a}_p)$ is then one-dimensional. We claim that in this case $P_{\mathfrak{a}_p}\mathcal{M}(z_p)$ acts on $V_p(\mathfrak{a}_p)$ via multiplication by $p^{-m_p/2}$, $m_p = |v_{\mathfrak{p}}(z_p) - v_{\mathfrak{p}}(\bar{z}_p)|$. This follows from the trace formula of Murase-Sugano [MS, Prop. 7.3]: one may easily verify that the formula given there holds actually for all $z_p \in K_p^\times$ and that it yields in our case

$$\operatorname{Tr} P_{\mathfrak{a}_p}\mathcal{M}(z_p)|_{V_p(\mathfrak{a}_p)} = \frac{|\mathrm{N}(z_p)|^{1/2}}{\max(|x_p|, |y_p|)},$$

where x_p and y_p are the coefficients of the expression of z_p with respect to some basis of $\mathfrak{o}_{K_p}/\mathbb{Z}_p$. Since $V_p(\mathfrak{a}_p)$ is one-dimensional, this is exactly what we need.

We see that the operator $L^*(z)$ corresponds to

$$\gamma_{r\mathrm{N}(\mathfrak{a})}((z))\mathrm{N}(\mathfrak{c})^{1/2} \prod_{p\not| r\mathrm{N}(\mathfrak{a}) \text{ inert}} (-1)^{v_p(z)}\mathcal{M}_\infty(z) \bigotimes_{p|r\mathrm{N}(\mathfrak{a})D} \mathcal{M}_p(z)$$

on $V^{(\nu)}(\mathfrak{a})$. An easy computation using the product formula for the absolute values $|z|_{K_p}$ finishes the proof.

Using the isomorphism $T_{r,\mathfrak{a};\nu} \simeq T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$, this proposition gives the existence of operators $\mathcal{F}^*_p(z_p)$ on $T_{r,\mathfrak{a};\nu}$ for $p = \infty$ or $p|r\mathrm{N}(\mathfrak{a})$, $z_p \in K_p^\times$ for p nonsplit and $z_p \in \mathbb{Q}_p\mathfrak{o}_{K_p}^\times$ for p split, which correspond to $\mathcal{M}_p(z_p^{-1})|z_p|_{K_p}^{1/2}$ on $V^{(\nu)}(\mathfrak{a})$, such that we have the factorization

$$\mathcal{F}^*(z)^{-1} = \mathcal{F}^*_\infty(z) \prod_{p|r\mathrm{N}(\mathfrak{a})D} \mathcal{F}^*_p(z), \quad z \in K^\times \cap \Lambda_{r\mathrm{N}(\mathfrak{a})D}. \tag{6}$$

Here we set $\Lambda_{dD} = \prod_{p|\infty d_{\text{inert}}D} K_p^\times \prod_{p|d_{\text{split}}} \mathbb{Q}_p^\times \mathfrak{o}_{K_p}^\times$. The restriction of $\mathcal{F}^*_p(z_p)$ to $\mathbb{Q}_p^\times$ is given by the scalar $\omega_{K/\mathbb{Q},p}(z)|z|_p$ [MS, 4.3]. We see that the action of $\mathcal{F}^*(z)^{-1}$, $z \in K^\times \cap \Lambda_{r\mathrm{N}(\mathfrak{a})D}$, on $T_{r,\mathfrak{a};\nu}$ extends to an action of $\Lambda_{r\mathrm{N}(\mathfrak{a})D}$, and that the characters $\lambda = \lambda_\infty \prod_p \lambda_p$ appearing in its decomposition are precisely those whose local components λ_p appear in the decomposition of $\mathcal{M}_p^{-1}\,|\cdot|_{K_p}^{1/2}$ on $V_p(\mathfrak{a}_p)$ (resp. $V^{(\nu)}$ if $p = \infty$). These decompositions and the decompositions of the primitive subspaces have been completely described by Murase-Sugano. A basic smoothness property is that $\mathcal{F}^*_p(z_p)$ becomes trivial for $z \in 1 + r\mathrm{N}(\mathfrak{a})D\mathfrak{o}_{K_p}$ [MS, Lemma 7.4]. From this we see already that $\mathcal{F}^*$ acts on an eigenfunction $\vartheta \in \mathcal{V}_{d,C;\nu}$ by a Hecke character of conductor dividing dD whose restriction to Λ_{dD} is given by the action of $\mathcal{F}^*_\infty \prod_{p|dD} \mathcal{F}^*_p$ on any component $\vartheta_{\mathfrak{a}}$ of ϑ. The following proposition and its corollary are now easy consequences.

Proposition 2.2. *A Hecke character λ of K with $\lambda|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}\,|\cdot|_{\mathbb{A}}$ appears in the representation $\mathcal{F}^*$ on $\mathcal{V}^{\mathrm{prim}}_{d,C;\nu}$ if and only if the following conditions are satisfied. If it appears, it has multiplicity one.*

1. *λ has infinity type $(-k, k-1)$ with $1 \le k \le \nu + 1$.*
2. *The conductor $\mathfrak{f}_\lambda$ of λ is equal to $dD\mathfrak{d}_\lambda^{-1}$, where $\mathfrak{d}_\lambda$ is a square-free product of ramified primes. (We then have automatically $\mathfrak{d}_\lambda + d\mathfrak{o}_K = \mathfrak{o}_K$.)*
3. *For each prime $q|D$ and a representative $c \in \mathbb{Q}^\times$ for the class C we have*

$$W_q(\lambda) = \omega_{K/\mathbb{Q},q}(d/c). \tag{7}$$

If we consider the whole space $\mathcal{V}_{d,C;\nu}$ instead of the primitive subspace, we have to change the second condition into $\mathfrak{f}_\lambda = (dt^{-1})D\mathfrak{d}_\lambda^{-1}$, where $t|d$ is the norm of an integral ideal of K. The multiplicity may then be greater than one.

Proof. Use the description of the decomposition of $\mathcal{M}_p$ on $V_p(\mathfrak{a}_p)$ at finite p given in [MS, Thm. 6.4, 6.6] and the description of $\mathcal{M}_\infty$ on $V^{(\nu)}$ stated above. For split p the characters appearing in the decomposition of $\mathcal{M}_p$ on $V_p(\mathfrak{a}_p)$ are precisely the characters of conductor dividing $d\mathfrak{o}_{K_p}$ extending $\omega_{K/\mathbb{Q},p}$, and for inert p they are the characters extending $\omega_{K/\mathbb{Q},p}$ of conductor $dp^{-2n}\mathfrak{o}_{K_p}$, $0 \le n \le v_p(d)/2$. At ramified primes q precisely those characters extending $\omega_{K/\mathbb{Q},q}$ appear that have conductor dividing $dD\mathfrak{o}_{K_q}$ and satisfy the epsilon condition $\varepsilon(\chi_q, e_{K,q})\chi_q(\delta)\omega_{K/\mathbb{Q},q}(d\mathrm{N}(\mathfrak{a})^{-1}) = +1$. A simple computation (cf. [T]) gives that $W_q(\lambda) = \varepsilon(|\cdot|_q^{1/2}\,\lambda_q^{-1}, e_{K,q})|\delta|_q^{1/2}\lambda_q^{-1}(\delta)$, which implies the result for the full space. The case of the primitive subspace is similar.

Corollary 2.3. *For fixed $\nu \ge 0$ a Hecke character λ of K with $\lambda|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}\,|\cdot|_{\mathbb{A}}$ occurs in the decomposition of $\mathcal{F}^*$ on one of the spaces $\mathcal{T}^{\mathrm{prim}}_{d;\nu}$, $d > 0$, if and only if λ has infinity type $(-k, k-1)$ with $1 \le k \le \nu + 1$, and the global root number $W(\lambda)$ is equal to $+1$. If these conditions are fulfilled, the character occurs with multiplicity one in precisely one of the spaces $\mathcal{V}^{\mathrm{prim}}_{d,C;\nu}$.*

Proof. For an anticyclotomic character λ of K the global root number $W(\lambda) = W(|\cdot|_{\mathbb{A}_K}\,\lambda^{-1})$ can be expressed as a product of local root numbers $W(\lambda) = \prod_p \varepsilon(|\cdot|_{K_p}^{1/2}\,\lambda_p^{-1}, e_{K,p}) = \prod_p \varepsilon(|\cdot|_{K_p}^{1/2}\,\lambda_p^{-1}, e_{K,p})|\delta|_{K_p}^{1/2}\lambda_p(\delta)^{-1}$ over all places p of K [T]. The term at infinity is $+1$ if the infinity type of λ is $(-k, k-1)$, $k \ge 1$, and the contributions of a pair of mutually conjugate split places cancel. Therefore in this case $W(\lambda) = \prod_q W_q(\lambda)$, q ranging over all nonsplit primes.

It is easy to see that the root number equation (7) holds for all nonsplit primes q if and only if it holds for all prime divisors of D (use [MS, Prop. 3.7]). Taking the product yields $W(\lambda) = \prod_q W_q(\lambda) = +1$, since $d/c > 0$. On the other hand, if this condition is true, there is always precisely one class $C \in \mathrm{N}(I_K)/\mathrm{N}(K^\times)$ which makes (7) true for all q dividing D. The assertions follow.

Connection to L-values (*results of Yang*). We review some results of Yang [Y] connecting theta functions with complex multiplication to special values of Hecke L-functions of anticyclotomic characters.

Yang considers a different model (V, ρ, ω) for the Weil representation, the standard Schrödinger model: here $V = S(\mathbb{A})$ with the standard scalar product

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{A}} \overline{\phi_1(x)} \phi_2(x) dx,$$

where we normalize the Haar measure on $\mathbb{A}$ by stipulating $\mathrm{vol}(\mathbb{Q}\backslash\mathbb{A}) = 1$. He defines a Weil representation of $\mathbb{A}_K^1$ on V by taking a splitting of the metaplectic group over $\mathrm{U}(1)$ (cf. [Ku]), which is determined by the choice of a unitary Hecke character χ of K with $\chi|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$. We denote the resulting Weil representation by ω_χ. The normalized theta functional on V is given by $\theta(\phi) = \sum_{x\in\mathbb{Q}} \phi(x)$.

We quote Yang's main result from [Y, p. 43, (2.19)]: choose local and global Haar measures on $\mathrm{U}(1)$ in a compatible way (no normalization required). For every character η of $\mathbb{A}_K^1/K^1$ whose local components η_p appear in the spaces V_p for all nonsplit p, there is an explicit function $\phi = \prod_p \phi_p \in V$ with

$$\frac{2}{\mathrm{vol}(K^1\backslash\mathbb{A}_K^1)^2} \left| \int_{K^1\backslash\mathbb{A}_K^1} \theta(\omega_\chi(g)\phi)\eta(g)dg \right|^2 = \mathrm{Tam}(K^1)c(0)\frac{L(1/2, \chi\tilde{\eta})}{L(1, \omega_{K/\mathbb{Q}})}. \tag{8}$$

Here $\tilde{\eta}$ is the "base change" of η to $\mathbb{A}_K^\times$ given by $\tilde{\eta}(z) = \eta(z/\bar{z})$,

$$\mathrm{Tam}(K^1) = \mathrm{vol}(K_\infty^1 \hat{\mathfrak{o}}_K^1)/\mathrm{vol}(K^1\backslash\mathbb{A}_K^1)$$

is the Tamagawa number of K^1, and

$$c(0) = \prod_{p\in S_1} (1 + p^{-1})^{-1} \prod_{p\in S_2} p^{-n_p}(1 - p^{-1})^{-2},$$

where S_1 (resp. S_2) is the set of inert (resp. split) primes at which $\chi\tilde{\eta}$ is ramified. For $p \in S_2$ let n_p be the maximum of the exponents of the conductors of χ and $\tilde{\eta}$ at p. Yang's choice of the function ϕ is as follows: at all nonsplit places p he takes ϕ_p to be a unitary eigenfunction of K_p^1 with eigencharacter $\bar{\eta}_p$. In the split case he defines ϕ_p in [Y, p. 48, (2.30)]: we have $\phi_p = \varrho(\mathrm{char}_{\mathbb{Z}_p})$ in case $\chi\tilde{\eta}$ is unramified at p, and $\phi_p = p^{n_p/2}\varrho(\mathrm{char}_{1+p^{n_p}\mathbb{Z}_p})$ in the ramified case, where char_S is the characteristic function of the set S, and ϱ the intertwining isometry between the "natural" and the "standard" Schrödinger models at p given by [Y, p. 47, (2.28)].[4]

[4]The printing error $|x_0^3\alpha|^{1/3}$ in this equation should be corrected to $|x_0^3\alpha|^{1/2}$.

It is not difficult to translate Yang's results to our situation. We define the linear functional l on $\mathcal{T}_{d;\nu}$ by

$$l(\vartheta) = \sum_{\mathfrak{a}} \vartheta_{\mathfrak{a}}(0)_{\nu+1}, \tag{9}$$

$\mathfrak{a}$ ranging over a system of representatives for the ideal classes of K.

Proposition 2.4. *Let $\vartheta \in \mathcal{V}_{d,C;\nu}^{\mathrm{prim}}$ be an eigenfunction of the Shintani operators $\mathcal{F}^*$ with associated Hecke character λ. Then*

$$\frac{|l(\vartheta)|^2}{\langle \vartheta, \vartheta \rangle} = \frac{w_K^2 \sqrt{D}}{4\pi h_K} \prod_{p|d} (1 - \omega_{K/\mathbb{Q}}(p) p^{-1})^{-1} L(0, \lambda). \tag{10}$$

Proof. Consider the Weil representation (V, ρ, ω_χ) as above. It is equivalent to the representation of $R(\mathbb{A})$ on V obtained by combining ρ and ω_χ. We denote this representation again by ω_χ. Take a character η of $\mathbb{A}_K^1/K^1$ with η_p appearing in V_p for all nonsplit p. Assume we are given a function $\phi' \in V$ which is an eigenfunction for the action of $\mathcal{K} = K_\infty^1 \hat{\mathfrak{o}}_K^1 \subseteq \mathbb{A}_K^1$ with eigencharacter $\bar{\eta}|_{\mathcal{K}}$. Consider the function $\varphi(g) = \eta(g)\theta(\omega_\chi(g)\phi')$ on $R(\mathbb{A})$ (here we extend η to $R(\mathbb{A})$ by the canonical map $R(\mathbb{A}) \to \mathbb{A}_K^1$). From the definition we see that φ is a nonzero element of $V_{r,\nu}^{\mathbb{A}}$ for a suitable ν.

We define ϕ' as the projection of Yang's function ϕ onto the $\bar{\eta}|_{\mathcal{K}}$-eigenspace of $\mathcal{K}$. We have $\phi' = \prod_p \phi'_p$, and ϕ'_p differs from ϕ_p only for $p \in S_2$. The integral in (8) remains unchanged if we replace ϕ by ϕ'.

On the other hand, by condition [Y, p. 43, (2.18)] for ϕ we have $\langle \phi, \phi \rangle = 1$. Using the description of the Weil representation at split places given in [Y, pp. 44–48], we may easily verify that for a split prime $p \in S_2$ projection onto the $\mathfrak{o}_{K_p}^1$-eigenspace induces multiplication of the scalar product by a factor $p^{-n_p}(1 - p^{-1})^{-1}$. Therefore

$$\langle \phi', \phi' \rangle = \prod_{p \in S_2} p^{-n_p} (1 - p^{-1})^{-1}.$$

Choosing a measure on $H(\mathbb{A})$ subject to $\mathrm{vol}(H(\mathbb{Q}) \backslash H(\mathbb{A})) = 1$, we obtain easily

$$\int_{R(\mathbb{Q}) \backslash R(\mathbb{A})} |\varphi(g)|^2 dg = \mathrm{vol}(K^1 \backslash \mathbb{A}_K^1) \langle \phi', \phi' \rangle.$$

Putting this together with (8) we get

$$\frac{\left| \int_{K^1 \backslash \mathbb{A}_K^1} \varphi(g) dg \right|^2}{\int_{R(\mathbb{Q}) \backslash R(\mathbb{A})} |\varphi(g)|^2 dg} = \frac{\mathrm{vol} \mathcal{K}}{2} \prod_{p \in S_1 \cup S_2} (1 - \omega_{K/\mathbb{Q}}(p) p^{-1})^{-1} \frac{L(1/2, \chi \tilde{\eta})}{L(1, \omega_{K/\mathbb{Q}})}.$$

Evidently, this identity remains valid if φ and ϕ' are multiplied by an arbitrary nonzero complex number.

Let $\mathfrak{a}$ be a fractional ideal such that φ and ϕ' are $H(\mathfrak{a})_f$-invariant. Using the isomorphism $V^{\mathbb{A}}_{r,\nu}(\mathfrak{a}) \simeq \mathcal{V}_{r\mathrm{N}(\mathfrak{a}),C;\nu}$, C the class of $\mathrm{N}(\mathfrak{a})$, we get from φ a theta function $\vartheta \in \mathcal{V}_{r\mathrm{N}(\mathfrak{a}),C;\nu}$. It is easily verified that

$$\int_{K^1\backslash\mathbb{A}^1_K} \varphi(g)dg = \frac{\mathrm{vol}\mathcal{K}}{w_K} l(\vartheta)$$

and

$$\int_{R(\mathbb{Q})\backslash R(\mathbb{A})} |\varphi(g)|^2 dg = \frac{\mathrm{vol}\mathcal{K}}{w_K} \langle\vartheta,\vartheta\rangle.$$

Furthermore, ϑ is an eigenfunction of the Shintani operators $\mathcal{F}^*$ with eigencharacter $\lambda = (\chi\tilde{\eta})^{-1}\,|\cdot|^{1/2}_{\mathbb{A}_K}$. To prove this, we have to show that $L^*(\mathfrak{p})$ acts on φ via multiplication by $p^{-1/2}(\chi\tilde{\eta})(\mathfrak{p})^{-1}$ for all but finitely many split prime ideals $\mathfrak{p}$ of K. Assuming that $\chi\tilde{\eta}$ is unramified at $\mathfrak{p}$, and that the space of $H(\mathfrak{a}_p)$-invariants in V_p is one-dimensional, we are reduced to proving that $p^{1/2}P_{\mathfrak{a}_p}\omega_\chi(\beta)^{-1}\phi'_p = \chi(\mathfrak{p})^{-1}\phi'_p$ for $\beta = (p, p^{-1}) \in K_p \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$. This may easily be checked using the definition of $\phi_p = \phi'_p$ cited above and the description of the "natural" Schrödinger model given at [Y, pp. 44–45, esp. Cor. 2.10].

Putting everything together, equation (10) follows for the function ϑ, since $L(1,\omega_{K/\mathbb{Q}}) = \frac{2\pi h_K}{w_K\sqrt{D}}$ by the well-known class number formula of Dirichlet. If we take $\eta = 1$, and choose χ accordingly, $\mathfrak{a}$ may be chosen to have norm d/r, where d is the unique positive integer such that the conductor of χ is equal to $dD\mathfrak{d}^{-1}$ for a square-free product of ramified primes $\mathfrak{d}$. This may be seen again by considering the definition of the "natural" Schrödinger model [Y, p. 44]. It follows that in this case ϑ belongs to the primitive subspace $\mathcal{V}^{\mathrm{prim}}_{d,C;\nu} \subseteq \mathcal{V}_{d,C;\nu}$. Proposition 2.2 implies that every primitive eigenfunction may be constructed this way, and we are done.

3. Integral theta functions and the main theorem

In this section, Proposition 2.4 will be used to reduce Theorem 1.1 to an assertion about arithmetic Shintani eigenfunctions. We define arithmetic and integral theta functions, and give an arithmetic variant of Proposition 2.4 as Proposition 3.6. After proving some auxiliary results, we can reduce the problem to the consideration of $l(\vartheta)$ modulo ℓ for primitive integral representatives ϑ of Shintani eigenspaces, which is the topic of the next section. Since we only consider scalar valued theta functions ($\nu = 0$), the results at first only pertain to anticyclotomic characters of infinity type $(-1,0)$ (the case $k = 1$), but for ℓ split in K they can be generalized to all $k \geq 1$ by using ℓ-adic L-functions. This finally yields the full statement of Theorem 1.1.

Integral theta functions. We begin by giving a geometric interpretation of theta functions, which implies the existence of integral structures on the spaces

$T_{r,\mathfrak{a}} = T_{r,\mathfrak{a};0}$ and $\mathcal{V}_{d,C} = \mathcal{V}_{d,C;0}$. Basic background references for the geometric theory of theta functions are [Mum1], [Mum2], [Mum4]. The construction easily extends to the case $\nu > 0$, but we skip this generalization here, since it will not be needed in the following. For a fractional ideal $\mathfrak{a}$ of $\mathfrak{o}_K$ fix an elliptic curve $E_\mathfrak{a}$ defined over a number field $M \subseteq \bar{\mathbb{Q}}$, which after extending scalars to $\mathbb{C}$ via i_∞ has period lattice $\Omega_{\infty,\mathfrak{a}}\mathfrak{a}$ for some complex period $\Omega_{\infty,\mathfrak{a}}$. Over the complex numbers there is an analytic parametrization $E_\mathfrak{a} \otimes_{i_\infty} \mathbb{C} \simeq \mathbb{C}/\mathfrak{a}$, and for any rational number r such that $r\mathrm{N}(\mathfrak{a})$ is integral we have a standard line bundle $L^{\mathrm{an}}_{r,\mathfrak{a}}$ of degree $rD\mathrm{N}(\mathfrak{a})$ over $\mathbb{C}/\mathfrak{a}$. It is defined as $L^{\mathrm{an}}_{r,\mathfrak{a}} = (\mathbb{C} \times \mathbb{C})/\mathfrak{a}$ with the action of $l \in \mathfrak{a}$ given by

$$l(w, x) = (w + l, \psi(l)e^{-2\pi i r\delta\bar{l}(w+l/2)}x).$$

Clearly, the space of global sections $\Gamma(\mathbb{C}/\mathfrak{a}, L^{\mathrm{an}}_{r,\mathfrak{a}})$ can be identified with $T_{r,\mathfrak{a}}$. There is a line bundle $L_{r,\mathfrak{a}}$ on $E_\mathfrak{a}$ defined over M, and unique up to isomorphism, such that after scalar extension to $\mathbb{C}$ we have $L_{r,\mathfrak{a}} \otimes_{i_\infty} \mathbb{C} \simeq L^{\mathrm{an}}_{r,\mathfrak{a}}$. We give $L_{r,\mathfrak{a}}$ a rigidification at the origin, i.e. identify the subscheme of points above the origin with the affine line. We fix the isomorphism of $L_{r,\mathfrak{a}} \otimes_{i_\infty} \mathbb{C}$ and $L^{\mathrm{an}}_{r,\mathfrak{a}}$ by demanding that it carries the rigidification of $L_{r,\mathfrak{a}}$ into the canonical one of the analytic line bundle which identifies the class of $(0, x)$ with x. These constructions give us an $i_\infty(M)$-vector space $i_\infty(\Gamma(E_\mathfrak{a}, L_{r,\mathfrak{a}}))$ of algebraic theta functions inside $T_{r,\mathfrak{a}}$.

Since the curve $E_\mathfrak{a} \otimes_{i_\ell} \mathbb{C}_\ell$ has good reduction, we can extend $E_\mathfrak{a} \otimes_{i_\ell} \mathbb{C}_\ell$ and $L_{r,\mathfrak{a}} \otimes_{i_\ell} \mathbb{C}_\ell$ canonically to an elliptic curve $\mathcal{E}_\mathfrak{a}$ over the ring of integers $\mathcal{O} = \mathcal{O}(\mathbb{C}_\ell)$ and a line bundle $\mathcal{L}_{r,\mathfrak{a}}$ on $\mathcal{E}_\mathfrak{a}$. In particular, we can consider the $\mathcal{O}$-module of integral sections $\Gamma(\mathcal{E}_\mathfrak{a}, \mathcal{L}_{r,\mathfrak{a}})$ inside the $\mathbb{C}_\ell$-vector space $\Gamma(E_\mathfrak{a} \otimes_{i_\ell} \mathbb{C}_\ell, L_{r,\mathfrak{a}} \otimes_{i_\ell} \mathbb{C}_\ell)$. Assume the rigidification normalized in such a way that the ℓ-integral elements of the stalk of $L_{r,\mathfrak{a}}$ over the origin correspond to the ℓ-integral points on the affine line. We then get an $i_\infty(i_\ell^{-1}(\mathcal{O}) \cap M)$-module of ℓ-integral theta functions inside $i_\infty(\Gamma(E_\mathfrak{a}, L_{r,\mathfrak{a}}))$. Since we will not deal with rationality questions, we extend scalars from M to $\bar{\mathbb{Q}}$, and denote the resulting module by $T^{\mathrm{int}}_{r,\mathfrak{a}}$, and the space of algebraic (or arithmetic) theta functions by $T^{\mathrm{ar}}_{r,\mathfrak{a}}$.

We recall the geometric construction of the Heisenberg group and its action on theta functions given by Mumford. Mumford's Heisenberg group $\mathcal{G}(L_{r,\mathfrak{a}})$ [Mum1, p. 289] fits into an exact sequence

$$1 \longrightarrow \bar{\mathbb{Q}}^\times \longrightarrow \mathcal{G}(L_{r,\mathfrak{a}}) \longrightarrow E_\mathfrak{a}[rD\mathrm{N}(\mathfrak{a})] \longrightarrow 0,$$

and acts on $\Gamma(E_\mathfrak{a} \otimes_M \bar{\mathbb{Q}}, L_{r,\mathfrak{a}} \otimes_M \bar{\mathbb{Q}})$ [Mum1, p. 295]. The set $\mathbb{C}^\times i_\infty(\mathcal{G}(L_{r,\mathfrak{a}}))$ can be identified with the analytically defined group $G_{r,\mathfrak{a}}$ of Section 2. On the other hand, $\mathbb{C}_\ell^\times i_\ell(\mathcal{G}(L_{r,\mathfrak{a}}))$ is the set of $\mathbb{C}_\ell$-points of a group scheme $\mathcal{G}(\mathcal{L}_{r,\mathfrak{a}})$ over $\mathcal{O}$ for which we have an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(\mathcal{L}_{r,\mathfrak{a}}) \longrightarrow \mathcal{E}_\mathfrak{a}[rD\mathrm{N}(\mathfrak{a})] \longrightarrow 0,$$

and a compatible action of $\mathcal{G}(\mathcal{L}_{r,\mathfrak{a}})$ on $\Gamma(\mathcal{E}_{\mathfrak{a}}, \mathcal{L}_{r,\mathfrak{a}})$. It is then clear that the set of points of finite order of $\mathcal{G}(\mathcal{L}_{r,\mathfrak{a}})$ over $\mathcal{O}$ is the same as the set of finite order elements of $i_\ell(\mathcal{G}(L_{r,\mathfrak{a}}))$. It follows that the action of the finite order elements of $G_{r,\mathfrak{a}}$ on $T_{r,\mathfrak{a}}$ preserves the space $T_{r,\mathfrak{a}}^{\mathrm{ar}}$ and the module $T_{r,\mathfrak{a}}^{\mathrm{int}}$. In particular, this applies to the operators A_x for $x \in \mathfrak{a}^*$.

We now give a simple characterization of the module of integral theta functions in the spirit of Shimura (cf. [Shim1], [Hic1]).

LEMMA 3.1. *The space $T_{r,\mathfrak{a}}^{\mathrm{ar}}$ of arithmetic theta functions inside $T_{r,\mathfrak{a}}$ consists out of all functions $\vartheta \in T_{r,\mathfrak{a}}$ such that all special values $(A_x\vartheta)(0)$ for $x \in K$ are algebraic numbers in $\mathbb{C}$. The module $T_{r,\mathfrak{a}}^{\mathrm{int}}$ of ℓ-integral theta functions consists out of those functions $\vartheta \in T_{r,\mathfrak{a}}$ for which the values $(A_x\vartheta)(0)$ for $x \in K$ are algebraic numbers whose images under $i_\ell \circ i_\infty^{-1}$ are integral in $\mathbb{C}_\ell$.*

Proof. We first show the statement that for arithmetic (resp. integral) $\vartheta \in T_{r,\mathfrak{a}}$ and $x \in K$ the special value $(A_x\vartheta)(0)$ is algebraic (resp. an element of $i_\infty(i_\ell^{-1}(\mathcal{O}))$). This follows from the following three facts: first, by definition for $\vartheta \in T_{r,\mathfrak{a}}^{\mathrm{ar}}$ (resp. $T_{r,\mathfrak{a}}^{\mathrm{int}}$) the value $\vartheta(0)$ is algebraic (resp. an element of $i_\infty(i_\ell^{-1}(\mathcal{O}))$). Also, for any fractional ideal $\mathfrak{b} \subseteq \mathfrak{a}$, the canonical inclusion $T_{r,\mathfrak{a}} \hookrightarrow T_{r,\mathfrak{b}}$ induces inclusions $T_{r,\mathfrak{a}}^{\mathrm{ar}} \hookrightarrow T_{r,\mathfrak{b}}^{\mathrm{ar}}$ and $T_{r,\mathfrak{a}}^{\mathrm{int}} \hookrightarrow T_{r,\mathfrak{b}}^{\mathrm{int}}$. Finally, as we have seen, for $x \in \mathfrak{b}^*$ the action of A_x preserves the sets of arithmetic and integral theta functions. To deduce the desired conclusion, let n be an integer with $nx \in \mathfrak{o}_K$, set $\mathfrak{b} = n\mathfrak{a}$, and apply A_x to ϑ viewed as an element of $T_{r,\mathfrak{b}}$.

To show the other implication, take $N = r\mathrm{N}(\mathfrak{a})D$ many points $x_1, \ldots, x_N \in K$, pairwise different modulo $\mathfrak{a}$, and consider the linear map $\Phi : T_{r,\mathfrak{a}} \to \mathbb{C}^N$ which associates to a function ϑ the vector $((A_{x_i}\vartheta)(0))_i$. If the sum of the x_i avoids a certain exceptional class in $2^{-1}\mathfrak{a}/\mathfrak{a}$, the map Φ is a bijection. Since it maps $T_{r,\mathfrak{a}}^{\mathrm{ar}}$ into the space of algebraic vectors, it follows that if the values $(A_{x_i}\vartheta)(0)$ are all algebraic, we need to have $\vartheta \in T_{r,\mathfrak{a}}^{\mathrm{ar}}$. Considering integral theta functions, Φ gives an inclusion of $T_{r,\mathfrak{a}}^{\mathrm{int}}$ into $i_\infty(i_\ell^{-1}(\mathcal{O}))^N$. If we can show that for a suitable choice of the x_i this map is an isomorphism after reduction modulo the maximal ideal, we are done. But this follows from the consideration of the reduction modulo ℓ of $\mathcal{E}_{\mathfrak{a}}$: one only has to chose the x_i in such a way that their reductions are pairwise different, and that the sum of these reductions avoids an exceptional point of order at most two. This completes the proof.

These concepts may be trivially extended to $\mathcal{T}_d$ and $\mathcal{V}_{d,C}$. One may observe that the Shintani operators $\mathcal{E}$ and $\mathcal{F}^*$ preserve the space of algebraic theta functions. Furthermore, the Shintani operator $\mathcal{F}^*(\mathfrak{c}) : T_{r,\mathfrak{a}} \to T_{r,\mathfrak{a}\mathfrak{c}\bar{\mathfrak{c}}^{-1}}$ induces an isomorphism of $T_{r,\mathfrak{a}}^{\mathrm{int}}$ and $T_{r,\mathfrak{a}\mathfrak{c}\bar{\mathfrak{c}}^{-1}}^{\mathrm{int}}$ if $\mathfrak{c}\bar{\mathfrak{c}}^{-1}$ is prime to $r\mathrm{N}(\mathfrak{a})$ and $\mathfrak{c}$ is prime to the prime ideal $\mathfrak{l}$ of K induced by i_ℓ. (This is clear for $\mathfrak{c}$ prime to ℓ, and the general case can be dealt with using the fact from Section 2 that $\mathcal{F}^*(z) = \mathcal{F}_\infty^*(z)^{-1} = z^{-1}\,\mathrm{id}$ for $z \in K^\times$ with $z \equiv 1(r\mathrm{N}(\mathfrak{a})D)$.) It is obvious that the linear functional l on $\mathcal{T}_d$ takes algebraic values on functions in $\mathcal{T}_d^{\mathrm{ar}}$, and that $i_\ell(i_\infty^{-1}(l(\vartheta)))$ falls into $\mathcal{O}$ for all $\vartheta \in \mathcal{T}_d^{\mathrm{int}}$.

Definition of the canonical bilinear forms. We now look at the arithmetic properties of the canonical scalar product. Consider the complex-antilinear maps $T_{r,\mathfrak{a}} \to T_{r,\bar{\mathfrak{a}}}$ defined by $\vartheta^\dagger(w) = \overline{\vartheta(\bar{w})}$. These maps fit together to a map from any space $\mathcal{V}_{d,C}$ to itself, also denoted by $\vartheta \mapsto \vartheta^\dagger$. In this way we may define nondegenerate bilinear forms

$$b : T_{r,\bar{\mathfrak{a}}} \times T_{r,\mathfrak{a}} \to \mathbb{C}, \quad b(\vartheta_1, \vartheta_2) = \langle \vartheta_1^\dagger, \vartheta_2 \rangle,$$

and a nondegenerate symmetric bilinear form b on $\mathcal{V}_{d,C}$ by summing over a system of representatives for the ideal classes of K.

Also, if $\mathfrak{a}\bar{\mathfrak{a}}^{-1}$ is prime to the integer $r\mathrm{N}(\mathfrak{a})$, it is not difficult to see that we obtain a nondegenerate symmetric bilinear form on the space $T_{r,\mathfrak{a}}$ by setting $b'(\vartheta_1, \vartheta_2) = b(\mathcal{F}^*(\bar{\mathfrak{a}})\vartheta_1, \vartheta_2)$.

We will establish that the bilinear forms b and b' have arithmetic counterparts b_{ar} and b'_{ar}, which take algebraic values on arithmetic theta functions, and that their values on ℓ-integral theta functions have ℓ-valuation bounded from below. Our method in obtaining these results will be rather rough: we consider usual standard bases of theta functions, whose integrality may be checked directly, and express the form b in these bases. The same method was used by Hickey [Hic2] to prove arithmeticity of the canonical scalar product.

Standard bases of theta functions. We give now the construction of special bases of the spaces $T_{r,\mathfrak{a}}$. These standard bases may be defined without assuming complex multiplication: for any lattice $L \subseteq \mathbb{C}$ let $a(L)$ be the area of $\mathbb{C}/L$, $H(x,y) = n\bar{x}y/a(L)$ for a positive integer n be a Riemann form, and ψ be a semicharacter associated to H. The space $T(H, \psi, L)$ of theta functions with respect to these choices is the space of all holomorphic functions ϑ on $\mathbb{C}$ satisfying

$$\vartheta(w + l) = \psi(l) e^{\pi H(l, w + l/2)} \vartheta(w)$$

for all $l \in L$. It has dimension n, as is well-known. The canonical Heisenberg group operation on $T(H, \psi, L)$ is given by $(A_l \vartheta)(w) = e^{-\pi H(l, w + l/2)} \vartheta(w + l)$ for $l \in n^{-1}L$. Let theta functions with characteristics be defined as usual by

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (w, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i (k+\alpha)^2 \tau + 2\pi i (k+\alpha)(w+\beta)},$$

and set

$$\phi_{\alpha\beta}(w, \tau) = e^{\pi w^2 / 2\mathrm{Im}(\tau)} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (w, \tau).$$

LEMMA 3.2. *Let L, H and ψ be as above, (ω_1, ω_2) a basis of L such that* $\mathrm{Im}(\tau) > 0$ *for $\tau = \omega_2/\omega_1$, and α_0 and β_0 real numbers with*

$$\psi(a\omega_1 + b\omega_2) = e^{\pi i n(ab+2a\alpha_0+2b\beta_0)}.$$

Then the functions

$$g_j(w) = \phi_{\alpha_0+j/n,-n\beta_0}(nw/\omega_1, n\tau),$$

where j ranges over the residue classes mod n, are a basis of $T(H, \psi, L)$, and the operation of the Heisenberg group on them is given by

$$A_{c\omega_1/n} g_j = e^{2\pi i c(\alpha_0+j/n)} g_j, \tag{11}$$

$$A_{c\omega_2/n} g_j = e^{2\pi i c\beta_0} g_{j+c}. \tag{12}$$

The g_j are orthogonal with respect to the standard scalar product, and we have $\langle g_j, g_j\rangle = |\omega_1|/(2na(L))^{1/2}$.

The proof is completely standard (see [I], [Hic2]). In the complex multiplication case it is then not difficult to show the following arithmeticity and integrality properties of these bases.

LEMMA 3.3. *For a fractional ideal $\mathfrak{a}$ of K, and a rational number r such that $r\mathrm{N}(\mathfrak{a})$ is a positive integer, choose a basis (ω_1, ω_2) of $\mathfrak{a}$ such that $\tau = \omega_2/\omega_1$ has positive imaginary part, and construct a basis (g_j) of $T_{r,\mathfrak{a}}$ as in Lemma 3.2. Then each of the functions $g'_j = \eta(rD\mathrm{N}(\mathfrak{a})\tau)^{-1} g_j \in T_{r,\mathfrak{a}}$ has the property that its special values $(A_x g'_j)(0)$ are integral algebraic in $\mathbb{C}$ for all $x \in K$, and units for some choice of $x \in K$. In particular, the g'_j form a basis of $T^{\mathrm{ar}}_{r,\mathfrak{a}}$ over $i_\infty(\bar{\mathbb{Q}})$. Furthermore, if G denotes the module generated over $i_\infty(i_\ell^{-1}(\mathcal{O}))$ by the g'_j, we have the inclusions*

$$G \subseteq T^{\mathrm{int}}_{r,\mathfrak{a}} \subseteq (rD\mathrm{N}(\mathfrak{a}))^{-1} G.$$

Proof. Note that $n = rD\mathrm{N}(\mathfrak{a})$. We use the classical Siegel functions [L, p. 262]. For $\mathrm{Im}(\tau) > 0$, and a, $b \in \mathbb{Q}$, they are defined by

$$g_{ab}(\tau) = -i\eta(\tau)^{-1} e^{\pi i a z} \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z, \tau), \quad z = a\tau + b.$$

Using this, an elementary calculation yields in the general situation of Lemma 3.2

$$\begin{aligned}(A_{a\omega_1+b\omega_2} g_j)(0) = {} & e^{\pi i((\alpha_0+j/n-1/2)(n(b-\beta_0)+1/2)+a(n\beta_0+1/2)+1/2)} \\ & \times \eta(n\tau) g_{a+\alpha_0+j/n-1/2, n(b-\beta_0)-1/2}(n\tau).\end{aligned}$$

Now the first assertion follows, since the Siegel functions g_{ab} take integral values at points in imaginary quadratic fields, and take units as values for suitable parameters a and b [Ra, p. 127].

It is then clear that $G \subseteq T^{\mathrm{int}}_{r,\mathfrak{a}}$ from Lemma 3.1. To show the other inclusion, let g be an integral theta function in $T_{r,\mathfrak{a}}$. If $g = \sum_j \lambda_j g'_j$, we have from (11)

$$\lambda_j g'_j = (rD\mathrm{N}(\mathfrak{a}))^{-1} \sum_{c \bmod rD\mathrm{N}(\mathfrak{a})} e^{-2\pi i c(\alpha_0 + j(rD\mathrm{N}(\mathfrak{a}))^{-1})} A_{c\omega_1(rD\mathrm{N}(\mathfrak{a}))^{-1}} g,$$

and the assertion follows from the above.

Arithmeticity and integrality theorem for the bilinear forms. We are now able to state and prove the following proposition on the bilinear forms b and b'. We introduce the arithmetic variant $b_{\mathrm{ar}} = (\Omega_\infty/2\pi)b$ of the form b. In the same manner we define $b'_{\mathrm{ar}} = (\Omega_\infty/2\pi)b'$ if $\mathfrak{a}\bar{\mathfrak{a}}^{-1}$ is prime to $r\mathrm{N}(\mathfrak{a})$.

PROPOSITION 3.4. *For r and $\mathfrak{a}$ such that $d = r\mathrm{N}(\mathfrak{a})$ is integral, the bilinear form $b_{\mathrm{ar}}(\vartheta_1, \vartheta_2)$ takes algebraic values at arithmetic theta functions $\vartheta_1 \in T^{\mathrm{ar}}_{r,\bar{\mathfrak{a}}}$ and $\vartheta_2 \in T^{\mathrm{ar}}_{r,\mathfrak{a}}$. Furthermore, for ℓ-integral functions ϑ_1 and ϑ_2 the value $(dD)^{5/2}D^{1/4}b_{\mathrm{ar}}(\vartheta_1, \vartheta_2)$ is ℓ-integral. If $\mathfrak{a}\bar{\mathfrak{a}}^{-1}$ is prime to d, the corresponding arithmeticity statement is true for the symmetric bilinear form b'_{ar}. The corresponding integrality statement for b'_{ar} is also true if in addition $\mathfrak{a}$ is prime to $\bar{\mathfrak{l}}$, where $\mathfrak{l}$ is the prime ideal of K above ℓ determined by i_ℓ.*

Proof. The statement for b'_{ar} reduces easily to the statement for b_{ar}, since under the stated assumption on $\mathfrak{a}$ the Shintani operator $\mathcal{F}^*(\bar{\mathfrak{a}})$ induces an isomorphism of $T^{\mathrm{ar}}_{r,\mathfrak{a}}$ and $T^{\mathrm{ar}}_{r,\bar{\mathfrak{a}}}$, and for $\mathfrak{a}$ prime to $\bar{\mathfrak{l}}$ also an isomorphism of the modules of integral theta functions.

To deal with the statement for b_{ar}, choose a basis (ω_1, ω_2) of $\mathfrak{a}$ such that $\tau = \omega_2/\omega_1$ has positive imaginary part, and construct a basis (g'_j) of $T_{r,\mathfrak{a}}$ as above. If $G_{\mathfrak{a}}$ is the $i_\infty(i_\ell^{-1}(\mathcal{O}))$-module generated by the g'_j, we have $T^{\mathrm{int}}_{r,\mathfrak{a}} \subseteq (dD)^{-1}G_{\mathfrak{a}}$. The functions ${g'_j}^\dagger$ form a basis of $T_{r,\bar{\mathfrak{a}}}$ and it is easily seen that we also have $T^{\mathrm{int}}_{r,\bar{\mathfrak{a}}} \subseteq (dD)^{-1}G_{\bar{\mathfrak{a}}}$ for the module $G_{\bar{\mathfrak{a}}}$ generated by them.

It is therefore enough to show that the numbers $(dD)^{1/2}D^{1/4}b_{\mathrm{ar}}({g'_j}^\dagger, g'_k) = (dD)^{1/2}D^{1/4}(\Omega_\infty/2\pi)\langle g'_j, g_k\rangle$ are algebraic and ℓ-integral. They are nonzero only for $j = k$, and then all equal to

$$\frac{\Omega_\infty|\omega_1|}{2\pi\mathrm{N}(\mathfrak{a})^{1/2}|\eta(dD\tau)|^2} = \frac{\Omega_\infty}{2\pi\mathrm{N}(\mathfrak{a})^{1/2}|\Delta(A_{dD})|^{1/12}},$$

where $A_{dD} = \mathbb{Z}\omega_1 + \mathbb{Z}dD\omega_2$ is a lattice of index dD in $\mathfrak{a}$. Since the number $(dD)^{12}\Delta(A_{dD})/\Delta(\mathfrak{a})$ is an algebraic integer dividing $(dD)^{12}$ [L, p. 164], this is equal to an algebraic integer times

$$\frac{\Omega_\infty}{2\pi\mathrm{N}(\mathfrak{a})^{1/2}|\Delta(\mathfrak{a})|^{1/12}}.$$

But it is well-known (see [L, p. 165, Th. 5]) that for an algebraic number α with $\alpha\bar{\mathbb{Z}} = \mathfrak{a}\bar{\mathbb{Z}}$ the number $\alpha^{12}\Delta(\mathfrak{a})/\Delta(\mathfrak{o}_K)$ is a unit. This means that

$\mathrm{N}(\mathfrak{a})^{1/2}|\Delta(\mathfrak{a})|^{1/12}$ is equal to a unit times $|\Delta(\mathfrak{o}_K)|^{1/12}$. But $\Omega_\infty/(2\pi|\Delta(\mathfrak{o}_K)|^{1/12})$ is (up to a root of unity) equal to $((2\pi i)^{12}\Delta(\Omega_\infty\mathfrak{o}_K))^{-1/12}$. By the definition of Ω_∞, the number in parentheses is the discriminant associated to an elliptic curve and an invariant differential with good reduction at ℓ, and is therefore an ℓ-adic unit, which shows the desired integrality statement.

Arithmetic variant of Yang's formula. We are now ready to give an arithmetic version of Proposition 2.4 and to establish the link between the valuations of anticyclotomic L-values and special values $l(\vartheta)$ of Shintani eigenfunctions. We first collect some simple observations in the following lemma.

LEMMA 3.5. 1. *For a fractional ideal $\mathfrak{c}$ with $\mathfrak{c}\bar{\mathfrak{c}}^{-1}$ prime to $r\mathrm{N}(\mathfrak{a})$, the isomorphism $\mathcal{F}^*(\mathfrak{c}) : T_{r,\mathfrak{a}} \to T_{r,\mathfrak{a}\mathfrak{c}\bar{\mathfrak{c}}^{-1}}$ induces multiplication of the standard inner product by the prime-to-$r\mathrm{N}(\mathfrak{a})D$-part of $\mathrm{N}(\mathfrak{c})^{-1}$.*

2. *For a Shintani eigenfunction $\vartheta \in \mathcal{V}_{d,C}$ and $\mathfrak{a} \in I_K$ with $\mathrm{N}(\mathfrak{a}) \in C$ we have $\langle\vartheta,\vartheta\rangle = (h_K/2^{\nu(D)-1})\langle\vartheta_\mathfrak{a},\vartheta_\mathfrak{a}\rangle$.*

3. *The Shintani operators $\mathcal{F}^*(\mathfrak{c})$ fulfill the relation $(\mathcal{F}^*(\mathfrak{c})\vartheta)^\dagger = \mathcal{F}^*(\bar{\mathfrak{c}})\vartheta^\dagger$.*

4. *For a Shintani eigenfunction $\vartheta \in \mathcal{V}_{d,C}$ we have $\vartheta^\dagger = \gamma\vartheta$ for a constant γ of absolute value one.*

PROPOSITION 3.6. *Let λ be an anticyclotomic Hecke character of infinity type $(-1,0)$, root number $W(\lambda) = +1$ and conductor $dD\mathfrak{d}^{-1}$, where $\mathfrak{d}$ is a square-free product of ramified prime ideals of K. Let ϑ_λ be an element of the associated one-dimensional Shintani eigenspace in $\mathcal{V}_{d,C}^{\mathrm{prim}}$ with the class C determined by λ, and let $\mathfrak{a}$ be a fractional ideal of K with $\mathrm{N}(\mathfrak{a}) \in C$. Then*

$$\Omega_\infty^{-1}L(0,\lambda) = \frac{2^{\nu(D)}}{\sqrt{D}}\prod_{q|d}(1-\omega_{K/\mathbb{Q}}(q)q^{-1})b_{\mathrm{ar}}(\vartheta_{\lambda,\bar{\mathfrak{a}}},\vartheta_{\lambda,\mathfrak{a}})^{-1}\left(\frac{l(\vartheta_\lambda)}{w_K}\right)^2. \tag{13}$$

If $\mathfrak{a}$ is prime to dD, we have here $b_{\mathrm{ar}}(\vartheta_{\lambda,\bar{\mathfrak{a}}},\vartheta_{\lambda,\mathfrak{a}}) = \lambda(\bar{\mathfrak{a}})^{-1}b'_{\mathrm{ar}}(\vartheta_{\lambda,\mathfrak{a}},\vartheta_{\lambda,\mathfrak{a}})$.

Proof. We know from Proposition 2.4 that

$$L(0,\lambda) = \frac{4\pi h_K}{w_K^2\sqrt{D}}\prod_{q|d}(1-\omega_{K/\mathbb{Q}}(q)q^{-1})\frac{|l(\vartheta_\lambda)|^2}{\langle\vartheta_\lambda,\vartheta_\lambda\rangle}.$$

By Lemma 3.5 we have $\vartheta_\lambda^\dagger = \gamma\vartheta_\lambda$, which implies

$$\frac{|l(\vartheta_\lambda)|^2}{\langle\vartheta_\lambda,\vartheta_\lambda\rangle} = \frac{l(\vartheta_\lambda^\dagger)l(\vartheta_\lambda)}{b(\vartheta_\lambda^\dagger,\vartheta_\lambda)} = \frac{l(\vartheta_\lambda)^2}{b(\vartheta_\lambda,\vartheta_\lambda)}.$$

The same lemma gives $\langle\vartheta_\lambda,\vartheta_\lambda\rangle = (h_K/2^{\nu(D)-1})\langle\vartheta_{\lambda,\mathfrak{a}},\vartheta_{\lambda,\mathfrak{a}}\rangle$, and we have therefore $b(\vartheta_\lambda,\vartheta_\lambda) = (h_K/2^{\nu(D)-1})b(\vartheta_{\lambda,\bar{\mathfrak{a}}},\vartheta_{\lambda,\mathfrak{a}})$. This yields the result.

The norm of a normalized integral eigenfunction. We call an integral theta function $\vartheta \in T^{\mathrm{int}}_{r,\mathfrak{a}}$ normalized, if a multiple $\gamma\vartheta$ lies in $T^{\mathrm{int}}_{r,\mathfrak{a}}$ precisely for $\gamma \in i_\infty(i_\ell^{-1}(\mathcal{O}))$. We will now determine the valuation $v_\ell(b'_{\mathrm{ar}}(\vartheta,\vartheta))$ for a normalized primitive Shintani eigenfunction ϑ.

Recall the definition, given in the introduction, of the local term $\mu_\ell(\lambda_q)$ associated to a character λ_q of $K_q^\times$ for inert primes q. We define now the local term $b_\ell(\lambda_\ell, w)$ for nonsplit ℓ used at the same place. Here λ_ℓ is a character of $K_\ell^\times$ with $\lambda_\ell|_{\mathbb{Q}_\ell^\times} = \omega_{K/\mathbb{Q},\ell}\,|\cdot|_\ell$ and $w = (w_q)_q$ an element of the set $\mathcal{W}$ defined in the introduction with $w_\ell = W_\ell(\lambda_\ell)$. The map from positive rational numbers r to elements of $\mathcal{W}$ defined by $r \mapsto (\omega_{K/\mathbb{Q},q}(r))_q$ gives an isomorphism between $\mathbb{Q}^+/N(K^\times)$ and $\mathcal{W}$. It is therefore equivalent to define $b_\ell(\lambda_\ell, r)$ for all positive rational numbers r with $W_\ell(\lambda_\ell) = \omega_{K/\mathbb{Q},\ell}(r)$ under the constraint that $b_\ell(\lambda_\ell, r\mathrm{N}(\alpha)) = b_\ell(\lambda_\ell, r)$ for $\alpha \in K^\times$. For split ℓ we set $b_\ell(\lambda_\ell, r) = 0$. Recall the factorization of the Shintani operator $\mathcal{F}^*(z)$ on a space $T_{r,\mathfrak{a}}$, where $r\mathrm{N}(\mathfrak{a})$ is integral, as a product $\mathcal{F}^*(z) = \mathcal{F}^*_\infty(z)^{-1}\prod_{p|r\mathrm{N}(\mathfrak{a})D}\mathcal{F}^*_p(z)^{-1}$ of local operators $\mathcal{F}^*_p$ described in Section 2. Since we are dealing with scalar valued theta functions, $\mathcal{F}^*_\infty(z) = z$. The condition $W_\ell(\lambda_\ell) = \omega_{K/\mathbb{Q},\ell}(r)$ implies that for any $\mathfrak{a}$ with $v_\ell(r\mathrm{N}(\mathfrak{a}))$ big enough the character λ_ℓ appears in the decomposition of $\mathcal{F}^*_\ell$ on $T_{r,\mathfrak{a}}$ and also on $T_{r,\bar{\mathfrak{a}}}$. Denote then by $T_{r,\mathfrak{a}}(\lambda_\ell)$ (resp. $T_{r,\bar{\mathfrak{a}}}(\lambda_\ell)$) the λ_ℓ-eigenspace, and by $T^{\mathrm{int}}_{r,\mathfrak{a}}(\lambda_\ell)$ (resp. $T^{\mathrm{int}}_{r,\bar{\mathfrak{a}}}(\lambda_\ell)$) the module of integral theta functions contained in it, and define

$$b_\ell(\lambda_\ell, r) = -\min_{\vartheta_1\in T^{\mathrm{int}}_{r,\bar{\mathfrak{a}}}(\lambda_\ell),\,\vartheta_2\in T^{\mathrm{int}}_{r,\mathfrak{a}}(\lambda_\ell)} v_\ell(D^{1/4} b_{\mathrm{ar}}(\vartheta_1,\vartheta_2)). \tag{14}$$

One may verify that this is well-defined, i.e. that the right-hand side does not depend on $\mathfrak{a}$, only on r. (In fact, it is enough to verify that it does not change if one replaces $\mathfrak{a}$ by $\mathfrak{b} \subseteq \mathfrak{a}$, where $\mathfrak{b}\mathfrak{a}^{-1}$ may be assumed to be prime to ℓ because of the multiplicity one theorem for $\mathcal{F}^*_\ell$. But $T^{\mathrm{int}}_{r,\mathfrak{b}}(\lambda_\ell)$ is generated over $i_\infty(i_\ell^{-1}(\mathcal{O}))$ by the functions $A_x\vartheta$ for $x \in d_1^{-1}\mathfrak{b}$, $\vartheta \in T^{\mathrm{int}}_{r,\mathfrak{a}}(\lambda_\ell)$, where d_1 is the prime-to-ℓ part of $r\mathrm{N}(\mathfrak{b})D$, and the same is true for $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{a}}$, which gives what we want.) Using the identifications $T_{r,\mathfrak{a}} \simeq T_{r\mathrm{N}(\alpha),\alpha^{-1}\mathfrak{a}}$ given by associating to a function ϑ the function $\vartheta(\alpha w)$, it is easy to see that $b_\ell(\lambda_\ell, r\mathrm{N}(\alpha)) = b_\ell(\lambda_\ell, r)$ for $\alpha \in K^\times$. If ℓ is inert and λ_ℓ is unramified, there exist ideals $\mathfrak{a}$ such that $r\mathrm{N}(\mathfrak{a})$ is an integer prime to ℓ and $T_{r,\mathfrak{a}}(\lambda_\ell) = T_{r,\mathfrak{a}}$ is the entire space. It is then clear from Proposition 3.4 that $b_\ell(\lambda_\ell, r) = 0$. If ℓ is ramified and λ_ℓ has minimal conductor, for ideals $\mathfrak{a}$ with $r\mathrm{N}(\mathfrak{a})$ prime to ℓ it follows from the local analysis of [MS] that $T_{r,\mathfrak{a}}(\lambda_\ell)$ is the space of theta functions invariant under the action of the $\mathfrak{l}$-division points, where $\mathfrak{l}$ is the prime ideal of K above ℓ. The arguments of Proposition 3.4 give $b_\ell(\lambda_\ell, r) = 0$ in this case also.

Finally, define for a Dirichlet character χ of conductor ℓ^m, $m \geq 1$, and a primitive ℓ^m-th root of unity μ the Gauss sum $g(\chi,\mu) = \sum_{k \bmod \ell^m}\chi(k)\mu^k$.

PROPOSITION 3.7. *Let d be a positive integer, and $\mathfrak{a}$ a fractional ideal of K prime to $dD\bar{\mathfrak{l}}$, where $\mathfrak{l}$ denotes the prime ideal of K above ℓ induced by i_ℓ. Let $\vartheta \in T^{\text{prim}}_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ be a normalized integral eigenfunction of the Shintani operators $\mathcal{F}^*(z)$ with associated eigencharacters λ_p of the operators $\mathcal{F}^*_p$ for all $p \mid dD$. Set $\tilde{W} = g(\lambda_{\mathfrak{l}}, \mu)$ for a root of unity μ of order $\ell^{v_\ell(d)}$, if ℓ divides d and splits in K, and $\tilde{W} = 1$ otherwise. Then*

$$(15) \qquad v_\ell\left(\frac{\tilde{W}}{D^{1/4} b'_{\text{ar}}(\vartheta,\vartheta)} \prod_{q \mid d} (1 - \omega_{K/\mathbb{Q}}(q) q^{-1})\right) = \sum_{q \mid d \text{ inert in } K,\ q \neq \ell} \mu_\ell(\lambda_q) + b_\ell(\lambda_\ell, d\mathrm{N}(\mathfrak{a})^{-1}).$$

The proof of this proposition consists out of two parts. In the first part we reduce to the case where d has no split prime factors, and in the second part we prove the statement in this special case. To accomplish the first task, it is enough to show that the statement for $d = d_0 p^m$, p split, d_0 prime to p, follows from the statement for d_0. Write $p = \mathfrak{p}\bar{\mathfrak{p}}$. In the case $p = \ell$ assume that $\mathfrak{p} = \mathfrak{l}$. Since the ideal $\mathfrak{a}$ was assumed to be prime to $dD\bar{\mathfrak{l}}$, we may write $\mathfrak{a} = \mathfrak{a}_0 \mathfrak{p}^m$ with $\mathfrak{a}_0$ prime to $d_0 D\bar{\mathfrak{l}}$.

The main task is now to construct a normalized integral Shintani eigenfunction $\vartheta_{\mathfrak{a}} \in T^{\text{prim}}_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ out of a normalized eigenfunction $\vartheta_{\mathfrak{a}_0} \in T^{\text{prim}}_{d_0/\mathrm{N}(\mathfrak{a}_0),\mathfrak{a}_0}$. The construction is done as follows: for r and $\mathfrak{b}$ such that $r\mathrm{N}(\mathfrak{b})$ is integral, a Dirichlet character χ modulo p^m, and an element $l \in \mathfrak{b}\mathfrak{p}^m\bar{\mathfrak{p}}^{-m}$ of order p^m modulo $\mathfrak{b}\mathfrak{p}^m$, define an operator

$$\Pi_{\mathfrak{p}^m,\chi;l} : T_{r,\mathfrak{b}} \longrightarrow T_{r,\mathfrak{b}\mathfrak{p}^m}$$

by

$$\Pi_{\mathfrak{p}^m,\chi;l}(\vartheta) = \sum_{x \bmod p^m} \chi(x)\psi(xl) A_{xl}\vartheta.$$

It is clear that $\Pi_{\mathfrak{p}^m,\chi;l}$ depends only on l modulo $\mathfrak{b}\mathfrak{p}^m$ and that $\Pi_{\mathfrak{p}^m,\chi;cl} = \chi(c)^{-1}\Pi_{\mathfrak{p}^m,\chi;l}$ for any integer c prime to p. Analogously we define $\Pi_{\bar{\mathfrak{p}}^m,\chi;l'}$ by exchanging the roles of $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. The role of this definition is explained by the following lemma.

LEMMA 3.8. *Let r and $\mathfrak{a}$ be such that $d = r\mathrm{N}(\mathfrak{a}) = d_0 p^m$ with $p \nmid d_0$, $p = \mathfrak{p}\bar{\mathfrak{p}}$ in K, and assume that $\mathfrak{p} = \mathfrak{l}$ if $p = \ell$. Also, let $l \in \mathfrak{a}\bar{\mathfrak{p}}^{-m}$ of order p^m modulo $\mathfrak{a}$. Then the operator $\Pi_{\mathfrak{p}^m,\chi;l} : T_{r,\mathfrak{a}\mathfrak{p}^{-m}} \to T_{r,\mathfrak{a}}$ has the following properties:*

1. *$\Pi_{\mathfrak{p}^m,\chi;l}$ commutes with the action of the Shintani operators $\mathcal{F}^*_q$ on $T_{r,\mathfrak{a}\mathfrak{p}^{-m}}$ and $T_{r,\mathfrak{a}}$ for $q \neq p$.*
2. *$\Pi_{\mathfrak{p}^m,\chi;l}$ maps $T^{\text{int}}_{r,\mathfrak{a}\mathfrak{p}^{-m}}$ to $T^{\text{int}}_{r,\mathfrak{a}}$, and if $\vartheta \in T^{\text{int}}_{r,\mathfrak{a}\mathfrak{p}^{-m}}$ is normalized, $\Pi_{\mathfrak{p}^m,\chi;l}(\vartheta)$ is also normalized.*

3. *If* $\lambda_p(z) = \lambda_{\mathfrak{p}}(z_{\mathfrak{p}}/z_{\bar{\mathfrak{p}}})$ *for* $z \in \mathbb{Q}_p^\times \mathfrak{o}_{K_p}^\times$, *with a primitive Dirichlet character* $\lambda_{\mathfrak{p}}$ *modulo* p^m, *the* λ_p*-eigenspace of* $\mathcal{F}_p^*$ *in* $T_{r,\mathfrak{a}}$ *is the image of* $T_{r,\mathfrak{a}\mathfrak{p}^{-m}}$ *under* $\Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}$.

Proof. From the commutation relation $\mathcal{E}(\eta)A_l = A_{\eta^{-1}l}\mathcal{E}(\eta)$, $l \in \mathfrak{a}^* \cap \eta\mathfrak{a}^*$, one deduces easily that $\mathcal{E}(\eta)\Pi_{\mathfrak{p}^m,\chi;l} = \chi(\eta_{\mathfrak{p}})^{-1}\Pi_{\mathfrak{p}^m,\chi;l}\mathcal{E}(\eta)$ for all $\eta \in K^1$ prime to d, which implies the first and third assertions. The first part of the second assertion is clear, and the whole assertion is easy to verify for $p \neq \ell$. For $p = \ell$ one has to show that for normalized ϑ the functions $\psi(xl)A_{xl}\vartheta$ stay linearly independent even after reduction modulo ℓ. But if one assumes the existence of a linear dependency modulo ℓ, by applying translations by the operators A_{yl} one could deduce that $\sum_{x \bmod \ell^m} \psi(xl)A_{xl}\vartheta = \mathcal{E}(\mathfrak{l}^m\bar{\mathfrak{l}}^{-m})\vartheta = \mathcal{F}^*(\bar{\mathfrak{l}}^{-m})\vartheta$ is congruent to zero modulo ℓ, which contradicts the fact that $\mathcal{F}^*(\bar{\mathfrak{l}}^{-m})$ is an isomorphism of $T^{\mathrm{int}}_{r,\mathfrak{a}\mathfrak{l}^{-m}}$ and $T^{\mathrm{int}}_{r,\mathfrak{a}\bar{\mathfrak{l}}^{-m}}$.

Continuing with the proof of the proposition, we can therefore write $\vartheta_{\mathfrak{a}} = \Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}(\vartheta_{\mathfrak{a}_0})$ with a normalized Shintani eigenfunction $\vartheta_{\mathfrak{a}_0} \in T^{\mathrm{prim}}_{d_0/\mathrm{N}(\mathfrak{a}_0),\mathfrak{a}_0}$ with the same eigencharacters λ_q for $q \neq p$, where $l \in \mathfrak{a}\bar{\mathfrak{p}}^{-m}$ has order p^m modulo $\mathfrak{a}$. To finish the inductive step, it remains to compute $b'_{\mathrm{ar}}(\vartheta_{\mathfrak{a}}, \vartheta_{\mathfrak{a}})$ in terms of $\vartheta_{\mathfrak{a}_0}$. For this we need the following lemma.

LEMMA 3.9. *Let* $p = \mathfrak{p}\bar{\mathfrak{p}}$, $m \geq 1$, $\vartheta \in T_{r,\mathfrak{b}}$, $v \in \mathfrak{b}$ *of order* p^m *modulo* $\mathfrak{b}\bar{\mathfrak{p}}^m$ *and* $w \in \mathfrak{b}\bar{\mathfrak{p}}^m\mathfrak{p}^{-m}$ *of order* p^m *modulo* $\mathfrak{b}\bar{\mathfrak{p}}^m$, *and* χ *a primitive Dirichlet character modulo* p^m. *Assume that* $r\mathrm{N}(\mathfrak{b})$ *is prime to* p. *Then we have the following identity of theta functions in* $T_{r,\mathfrak{b}\bar{\mathfrak{p}}^m}$:

$$\Pi_{\mathfrak{p}^m,\chi;v}(\mathcal{F}^*(\bar{\mathfrak{p}}^m)\vartheta) = p^{-m}g(\chi,\mu)\Pi_{\bar{\mathfrak{p}}^m,\chi^{-1};w}(\vartheta),$$

with the primitive p^m-th *root of unity* $\mu = e^{2\pi i r \mathrm{Tr}(\delta v\bar{w})}$.

Proof. Express

$$\mathcal{F}^*(\bar{\mathfrak{p}}^m)\vartheta = p^{-m}\sum_{x \bmod p^m} \psi(xw)A_{xw}\vartheta,$$

apply $\Pi_{\mathfrak{p}^m,\chi;v}$ to this, and note that $A_{yv}A_{xw} = e^{2\pi i r xy\mathrm{Tr}(\delta v\bar{w})}A_{xw}A_{yv}$ and $\psi(yv)A_{yv}\vartheta = \vartheta$ for all integers y. Then an elementary computation gives the result.

Write now $\bar{\mathfrak{a}}\mathfrak{a}^{-1} = \bar{\mathfrak{c}}\mathfrak{c}^{-1}$ with an integral ideal $\mathfrak{c}$ prime to its complex conjugate. Since $\mathfrak{c}$ is prime to p by assumption, we can take $l \in \mathfrak{a}\bar{\mathfrak{c}}\bar{\mathfrak{p}}^{-m}$ and have then

$$\mathcal{F}^*(\bar{\mathfrak{a}})\vartheta_{\mathfrak{a}} = \Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}(\mathcal{F}^*(\bar{\mathfrak{a}})\vartheta_{\mathfrak{a}_0}) = \Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}(\mathcal{F}^*(\bar{\mathfrak{p}}^m)\mathcal{F}^*(\bar{\mathfrak{a}}_0)\vartheta_{\mathfrak{a}_0}).$$

This implies

$$\begin{aligned} b'(\vartheta_{\mathfrak{a}},\vartheta_{\mathfrak{a}}) &= b(\mathcal{F}^*(\bar{\mathfrak{a}})\vartheta_{\mathfrak{a}},\vartheta_{\mathfrak{a}}) \\ &= b(\Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}(\mathcal{F}^*(\bar{\mathfrak{p}}^m)\mathcal{F}^*(\bar{\mathfrak{a}}_0)\vartheta_{\mathfrak{a}_0}),\Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}(\vartheta_{\mathfrak{a}_0})) \\ &= p^{-m}g(\lambda_{\mathfrak{p}},\mu)b(\Pi_{\bar{\mathfrak{p}}^m,\lambda_{\mathfrak{p}}^{-1};\bar{l}}(\mathcal{F}^*(\bar{\mathfrak{a}}_0)\vartheta_{\mathfrak{a}_0}),\Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}(\vartheta_{\mathfrak{a}_0})) \\ &= p^{-m}g(\lambda_{\mathfrak{p}},\mu)\langle\Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}((\mathcal{F}^*(\bar{\mathfrak{a}}_0)\vartheta_{\mathfrak{a}_0})^\dagger),\Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l}(\vartheta_{\mathfrak{a}_0})\rangle \\ &= (1-p^{-1})g(\lambda_{\mathfrak{p}},\mu)b(\mathcal{F}^*(\bar{\mathfrak{a}}_0)\vartheta_{\mathfrak{a}_0},\vartheta_{\mathfrak{a}_0}), \end{aligned}$$

where $\mu = e^{2\pi i d\mathrm{N}(\mathfrak{a})^{-1}\mathrm{Tr}(\delta l^2)}$. We can now conclude that in the case $p \neq \ell$ we have

$$v_\ell((1-p^{-1})^{-1}b'_{\mathrm{ar}}(\vartheta_{\mathfrak{a}},\vartheta_{\mathfrak{a}})) = v_\ell(b'_{\mathrm{ar}}(\vartheta_{\mathfrak{a}_0},\vartheta_{\mathfrak{a}_0})),$$

and for $p = \ell$ we get

$$v_\ell((1-\ell^{-1})^{-1}\tilde{W}^{-1}b'_{\mathrm{ar}}(\vartheta_{\mathfrak{a}},\vartheta_{\mathfrak{a}})) = v_\ell(b'_{\mathrm{ar}}(\vartheta_{\mathfrak{a}_0},\vartheta_{\mathfrak{a}_0})).$$

This finishes the first part of the proof.

We assume now that all prime divisors of d are inert or ramified in K. Let S be the set of prime divisors of dD, and assume we are given a partition of S into two disjoint sets S_1 and S_2, which corresponds to a factorization of dD as a product d_1d_2 of two coprime factors. Assume that $\ell \notin S_1$, i.e. that d_1 is prime to ℓ. Then we consider the space $T(\lambda; S_2) \subseteq T_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$, the eigenspace of the operators $\mathcal{F}_q^*$ for $q \in S_2$ with eigencharacters λ_q. It has dimension d_1, and is invariant under the action of the operators A_l, $l \in d_1^{-1}\mathfrak{a}$. Moreover, the subgroup $G_{r,\mathfrak{a};d_1}$ of the Heisenberg group $G_{r,\mathfrak{a}}$ that consists out of the classes of all pairs (l,λ) with $l \in d_1^{-1}\mathfrak{a}/\mathfrak{a} \subseteq \mathfrak{a}^*/\mathfrak{a}$, is again a Heisenberg group and it acts irreducibly on $T(\lambda; S_2)$. Since d_1 is prime to ℓ, it is easy to see that even $\mathrm{End}(T^{\mathrm{int}}(\lambda; S_2))$ is spanned over $i_\infty(i_\ell^{-1}(\mathcal{O}))$ by the operators A_l, $l \in d_1^{-1}\mathfrak{a}$. This implies that b'_{ar} on the module $T^{\mathrm{int}}(\lambda; S_2)$ of integral theta functions in $T(\lambda; S_2)$ is given in a suitable basis by a multiple of the identity matrix. (To see this, write $d_1^{-1}(\mathfrak{a} \cap \bar{\mathfrak{a}}) = \mathbb{Z}l \oplus \mathbb{Z}k$ with $\bar{l} = -l$, and take a normalized theta function $\vartheta \in T^{\mathrm{int}}(\lambda; S_2)$ that is an eigenfunction of A_l. Then the translates $A_{yk}\vartheta$, y ranging over a system of representatives for the residue classes modulo d_1, give such a basis of $T^{\mathrm{int}}(\lambda; S_2)$.) We define $v(S_2)$ as the minimum value of $v_\ell(b'_{\mathrm{ar}}(\vartheta,\vartheta))$ for $\vartheta \in T^{\mathrm{int}}(\lambda; S_2)$. By assumption, $T(\lambda; S)$ is the one-dimensional space generated by a full Shintani eigenfunction ϑ, and $v(S)$ is therefore equal to $v_\ell(b'_{\mathrm{ar}}(\vartheta,\vartheta))$ for a normalized eigenfunction ϑ as in the statement of the proposition. On the other hand, using the fact that b_ℓ on $T^{\mathrm{int}}(\lambda; S_2)$ can be represented by a constant multiple of the identity matrix, it is easy to see that $v(\{\ell\}) = -v_\ell(D^{1/4}) - b_\ell(\lambda_\ell, d\mathrm{N}(\mathfrak{a})^{-1})$. We claim the following inductive relations for the quantities $v(S_2)$: if $q \in S_1$ is ramified, $q \neq \ell$, we have $v(S_2 \cup \{q\}) = v(S_2)$, while if $q \neq \ell$ is inert, we have

$$v(S_2 \cup \{q\}) = v(S_2) + v_\ell(q+1) - \mu_\ell(\lambda_q).$$

Obviously, these relations imply the proposition.

We only give the proof of our claim in the case of inert primes q; the case of ramified q can be dealt with the same way, but is much easier. Note that for any $q \neq \ell$ the local operators $\mathcal{F}_q^*$ are ℓ-integral. Let q be an inert prime divisor of d_1, and $m = v_q(d)$. The representation $\mathcal{F}_q^*$ of $K_q^\times$ is equivalent to a representation of $(\mathfrak{o}_K/q^m\mathfrak{o}_K)^\times/(\mathbb{Z}/q^m\mathbb{Z})^\times$. The operator

$$P_{\lambda_q} = \sum_{z\in(\mathfrak{o}_K/q^m\mathfrak{o}_K)^\times/(\mathbb{Z}/q^m\mathbb{Z})^\times} \lambda_q(z)^{-1}\mathcal{F}_q^*(z)$$

acts on $T(\lambda; S_2)$ (and also on $T^{\mathrm{int}}(\lambda; S_2)$), and its image is the λ_q-eigenspace of $\mathcal{F}_q^*$ in $T(\lambda; S_2)$, i.e. the space $T(\lambda; S_2 \cup \{q\})$. Let $\beta \in i_\infty(i_\ell^{-1}(\mathcal{O}))$ be a nonzero element normalizing P_{λ_q}, i.e. an element such that $\beta^{-1}P_{\lambda_q}$ is an integral operator on $T^{\mathrm{int}}(\lambda; S_2)$ not congruent to zero modulo the maximal ideal. Then $\beta^{-1}P_{\lambda_q} : T^{\mathrm{int}}(\lambda; S_2) \to T^{\mathrm{int}}(\lambda; S_2 \cup \{q\})$ is surjective, since the operators A_l, $l \in (d_1q^{-m})^{-1}\mathfrak{a}$, commute with P_{λ_q}.

Since we know from Lemma 3.5 that $b'_{\mathrm{ar}}(\mathcal{F}_q^*(z)\vartheta_1, \vartheta_2) = b'_{\mathrm{ar}}(\vartheta_1, \mathcal{F}_q^*(z)\vartheta_2)$, it follows that

$$b'_{\mathrm{ar}}(\beta^{-1}P_{\lambda_q}\vartheta, \beta^{-1}P_{\lambda_q}\vartheta) = \frac{(q+1)q^{m-1}}{\beta} b'_{\mathrm{ar}}(\vartheta, \beta^{-1}P_{\lambda_q}\vartheta)$$

for $\vartheta \in T^{\mathrm{int}}(\lambda; S_2)$. The surjectivity of $\beta^{-1}P_{\lambda_q}$ implies that

$$v(S_2 \cup \{q\}) = v_\ell(q+1) - v_\ell(\beta) + \min_{\vartheta\in T^{\mathrm{int}}(\lambda;S_2)} v_\ell(b'_{\mathrm{ar}}(\vartheta, \beta^{-1}P_{\lambda_q}\vartheta)).$$

But since ℓ is assumed to be odd, and b'_{ar} on $T(\lambda; S_2)$ can be represented by a multiple of the identity matrix, the minimum value of $v_\ell(b'_{\mathrm{ar}}(\vartheta, \beta^{-1}P_{\lambda_q}\vartheta))$ is equal to $v(S_2)$.

It remains to prove that $v_\ell(\beta) = \mu_\ell(\lambda_q)$, i.e. to show that the valuation of the operator $P_{\lambda_q} \in \mathrm{End}(T^{\mathrm{int}}(\lambda; S_2))$ is equal to $\mu_\ell(\lambda_q)$. For this we need the following lemma.

Lemma 3.10. *The operators $\mathcal{F}_q^*(z)$ for $z \in K_q^\times$ span over $i_\infty(i_\ell^{-1}(\mathcal{O}))$ the same submodule of* $\mathrm{End}(T^{\mathrm{int}}(\lambda; S_2))$ *as the operators*

$$\sum_{x\in\mathfrak{a}/q^m\mathfrak{a},\, \mathrm{N}(x)=c\mathrm{N}(\mathfrak{a})} \psi(x)A_{q^{-m}x}$$

for q odd (resp. $\sum_{x\in\mathfrak{a}/2^m\mathfrak{a},\, \mathrm{N}(x)=c\mathrm{N}(\mathfrak{a})}(-1)^{2^{-m}(\mathrm{N}(x)/\mathrm{N}(\mathfrak{a})-c)}A_{2^{-m}x}$ for $q = 2$), c ranging over $\mathbb{Z}/q^m\mathbb{Z}$.

Proof. We prove only the case where q is odd, leaving the extension to $q = 2$ to the reader. Since an operator $\mathcal{F}_q^*(z)$, $z \in K_q^\times$, commutes with the Heisenberg group elements A_l, $l \in (dq^{-m})^{-1}\mathfrak{a}$, it can be written as a linear combination $\sum_{x\in\mathfrak{a}/q^m\mathfrak{a}} \gamma(x)\psi(x)A_{q^{-m}x}$ with a function $\gamma : \mathfrak{a}/q^m\mathfrak{a} \to \mathbb{C}$. Because of the integrality of $\mathcal{F}_q^*(z)$ at ℓ, the coefficients $\gamma(x)$ have to be ℓ-integral. That $\mathcal{F}_q^*(z)$ commutes with all operators $\mathcal{E}(\eta)$ translates into the fact that $\gamma(\eta x) = \gamma(x)$ for $\eta \in (\mathfrak{o}_K/q^m\mathfrak{o}_K)^1$. But this means that $\gamma(x) = \gamma'(\mathrm{N}(x))$ for a function γ'. Equivalently, the operator $\mathcal{F}_q^*(z)$ lies in the module generated by the operators $\sum_{x\in\mathfrak{a}/q^m\mathfrak{a},\,\mathrm{N}(x)=c\mathrm{N}(\mathfrak{a})}\psi(x)A_{q^{-m}x}$ for c modulo q^m.

To show the other inclusion, we can verify the identity

$$\sum_{x\in\mathfrak{a}/q^m\mathfrak{a}}\zeta^{c\mathrm{N}(x)/\mathrm{N}(\mathfrak{a})}\psi(x)A_{q^{-m}x} = (-q)^m\mathcal{F}_q^*(\delta(1+2c\delta))$$

for all $c \in \mathbb{Z}/q^m\mathbb{Z}$, where $\zeta = e^{-2\pi i d D/q^{2m}}$ is a primitive q^m-th root of unity. This can either be deduced from [MS], or one can verify the commutation relation with the $A_{q^{-m}x}$ for the left-hand side, which proves the identity up to a constant, and then determine the constant by comparing traces. Fourier analysis on the group $\mathbb{Z}/q^m\mathbb{Z}$ now implies the desired inclusion in the other direction, finishing the proof of the lemma.

Let now $F \subseteq \mathrm{End}(T^{\mathrm{int}}(\lambda;S_2))$ be the module generated over $i_\infty(i_\ell^{-1}(\mathcal{O}))$ by the operators $\mathcal{F}_q^*$. By the lemma, F is equal to the module of integral endomorphisms in the vector space it spans: $\mathbb{Q}F \cap \mathrm{End}(T^{\mathrm{int}}(\lambda;S_2)) = F$. Also, it is clear that it has rank q^m. A surjection f from the module E_m of $i_\infty(i_\ell^{-1}(\mathcal{O}))$-valued functions on $G_m = (\mathfrak{o}_K/q^m\mathfrak{o}_K)^\times/(\mathbb{Z}/q^m\mathbb{Z})^\times$ to the module F is obtained by associating to $\phi \in E$ the operator $\sum_{z\in G_m}\phi(z^{-1})\mathcal{F}_q^*(z) \in F$. The kernel $\mathcal{K}$ of this map may be explicitly described: for $1 \le n \le m$ there are trace maps from E_n to E_{n-1} associating to $\phi \in E_n$ the function $y \mapsto \sum_{z,\,\pi_{n-1}(z)=y}\phi(z)$, where π_{n-1} denotes the projection from G_n onto G_{n-1}. Denote by $E_n^{\mathrm{prim}} \subseteq E_n$ the kernel of the trace map on E_n, and set $E_0^{\mathrm{prim}} := E_0$. Via the canonical projections from G_m onto G_n, every function in E_n may be regarded as a function in E_m. Since $q \neq \ell$, the module E_m is the direct sum of E_n^{prim} for $2 \le n \le m$ and of E_1. The kernel $\mathcal{K}$ is now the direct sum of E_n^{prim} for all $0 \le n \le m$ with $n \equiv m+1\,(2)$. This follows from the fact that each function in this module can be expanded as a linear combination of characters not appearing in the decomposition of $\mathcal{F}_q^*$, and therefore belongs to the kernel. On the other hand, the kernel cannot be bigger, since this would contradict the fact that the image F of f has rank q^m.

Because of the crucial property $\mathbb{Q}F \cap \mathrm{End}(T^{\mathrm{int}}(\lambda;S_2)) = F$, for any $\phi \in E_m$ the operator $f(\phi)$ has nontrivial ℓ-valuation precisely if the reduction of ϕ modulo the maximal ideal falls into (the reduction of) the submodule $\mathcal{K}$. We may infer from this that for $m > 1$ and a primitive character λ_q the operator P_{λ_q} has nonzero reduction modulo the maximal ideal, i.e. that $v_\ell(\beta) = 0$ in this case. If $m = 1$, the kernel $\mathcal{K}$ consists out of the constant functions, and therefore the valuation of β is precisely $\mu_\ell(\lambda_q) = \min_{x\in\mathfrak{o}_{K_q}^\times} v_\ell(\lambda_q(x)-1)$ for any character λ_q of conductor q. So, we have $v_\ell(\beta) = \mu_\ell(\lambda_q)$ in all cases, which finishes the proof of the proposition.

Proof of Theorem 1.1. Propositions 3.6 and 3.7 allow us to reduce the proof of Theorem 1.1 in the case $k = 1$ to a divisibility statement about the values $l(\vartheta)$ for Shintani eigenfunctions. Let λ be a Hecke character of K of infinity type $(-1,0)$ and root number $W(\lambda)=+1$. Taken together the two equations established so far show that

$$v_\ell(\Omega_\infty^{-1}\tilde{W}D^{1/4}L(0,\lambda)) = \sum_{q\neq\ell \text{ inert in } K}\mu_\ell(\lambda_q) + b_\ell(\lambda_\ell, w(\lambda)) + 2v_\ell(l(\vartheta_\lambda)/w_K)$$

for a normalized integral element ϑ_λ of the λ-eigenspace. Note that $\tilde{W}=1$ if ℓ is nonsplit, and it is easy to see that $v_\ell(\tilde{W}) = v_\ell(W_\ell(\lambda))$ in the split case. Since $v_\ell(l(\vartheta_\lambda)/w_K)$ is nonnegative (except possibly in the case $\ell=3$ and $K=\mathbb{Q}(\sqrt{-3})$), this equation immediately implies a lower bound for $v_\ell(\Omega_\infty^{-1}\tilde{W}D^{1/4}L(0,\lambda))$, and Theorem 1.1 is seen to be equivalent to the statement that $v_\ell(l(\vartheta_\lambda)/w_K)=0$ for almost all λ with $W(\lambda)=+1$ and conductor dividing dDp^∞ for fixed d. The proof of this fact, which lies at the center of the argument, will be given as Theorem 4.1 in the next section.

It remains to finish the proof in the case ℓ splits in K and $k>1$. This is done by an argument based on the existence of an ℓ-adic L-function, as constructed by Manin-Vishik, Katz and others. Using the setup of [HidT] as a reference, we know that for any integer d there is a measure μ on the Galois group $G(dD\ell^\infty)=\mathrm{Gal}(K(dD\ell^\infty)/K)$ such that for anticyclotomic characters λ of infinity type $(-k,k-1)$, conductor dividing $dD\ell^\infty$ but divisible by all prime ideals dividing dD, we have

$$\Omega_\ell^{1-2k}\int_{G(dD\ell^\infty)}\hat{\lambda}d\mu = \Omega_\infty^{1-2k}(k-1)!\left(\frac{2\pi}{\sqrt{D}}\right)^{k-1}W_\ell(\lambda)(1-\lambda(\bar{\mathfrak{l}}))^2L(0,\lambda).$$

Here Ω_ℓ is a certain unit in $\mathbb{C}_\ell$, the ℓ-adic period, and $\hat{\lambda}$ the ℓ-adic avatar of λ defined by Weil.

Complex conjugation c acts on $G=G(dD\ell^\infty)$. Define its anticyclotomic quotient as $G^{\mathrm{ac}}=G/N(G)$, where $N(g)=gg^c$, and let W^{ac} the torsion-free part of G^{ac} (i.e. the quotient of G^{ac} by its torsion subgroup). As a topological group, $W^{\mathrm{ac}}\simeq\mathbb{Z}_\ell$. A measure μ_λ on W^{ac} can be defined by $\int_{W^{\mathrm{ac}}}fd\mu_\lambda = \int_G f\hat{\lambda}d\mu$ for continuous functions f on W^{ac}, using the projection $G\to W^{\mathrm{ac}}$. The Hecke characters σ of K for which $\hat{\sigma}$ factors through W^{ac} have conductor ℓ^m for some m. We know that

$$v_\ell(\int_{G(dD\ell^\infty)}\hat{\lambda}'d\mu) \geq \alpha(\lambda') = \sum_{q \text{ inert in } K}\mu_\ell(\lambda'_q)$$

for all anticyclotomic characters λ' of infinity type $(-1,0)$. Twists by characters of W^{ac} do not change the right-hand side. Using the ℓ-adic Weierstrass preparation theorem, it follows that for any character λ' the measure $\mu_{\lambda'}$ is divisible by an element of valuation $\alpha(\lambda')$.

Let now m be an integer such that there are no nontrivial units congruent to 1 modulo $\mathfrak{l}^m$. Then there is a Hecke character φ of conductor $\mathfrak{l}^m$ and infinity type $(k-1,0)$. Set $\sigma = \varphi/\varphi \circ c$, which gives a character of conductor ℓ^m and infinity type $(k-1, 1-k)$. After twisting by a character of finite order $\hat{\sigma}$ factors through W^{ac}, in particular it is then congruent to the trivial character. If we set $\lambda' = \lambda\sigma$, λ' has infinity type $(-1,0)$ and conductor dividing ℓ^m times the conductor of λ. We may write

$$\int (\hat{\lambda} - \hat{\lambda}')d\mu = \int (\hat{\sigma}^{-1} - 1)d\mu_{\lambda'},$$

and since we established divisibility of $\mu_{\lambda'}$ by an element of valuation $\alpha(\lambda') = \alpha(\lambda)$, it follows that $v_\ell(\int(\hat{\lambda} - \hat{\lambda}')d\mu) > \alpha(\lambda)$. Consequently,

$$v_\ell(\Omega_\infty^{1-2k}(k-1)!\left(\frac{2\pi}{\sqrt{D}}\right)^{k-1} W_\ell(\lambda)L(0,\lambda)) \geq \alpha(\lambda),$$

and the inequality is strict if and only if

$$v_\ell(\Omega_\infty^{-1} W_\ell(\lambda')(1-\lambda'(\bar{\mathfrak{l}}))^2 L(0,\lambda')) > \alpha(\lambda).$$

Note that $1 - \lambda(\bar{\mathfrak{l}})$ is an ℓ-adic unit, and is therefore omitted from the first inequality. An easy argument gives $W(\lambda') = W(\lambda) = +1$. Consider the infinitely many anticyclotomic characters λ of conductor dividing dDp^∞ and fixed infinity type $(-k, k-1)$. Every character λ violating the equality of Theorem 1.1 gives a character λ' of infinity type $(-1,0)$ and conductor dividing $dD\ell^m p^\infty$ violating the equality, except possibly in the cases where $\lambda'(\bar{\mathfrak{l}})$ is congruent to 1. However, this can happen only in finitely many cases, since the p-power roots of unity are all distinct modulo ℓ. Therefore the finiteness of the exceptional set for $k > 1$ follows from the finiteness for $k = 1$, which proves the theorem.

4. A nonvanishing result for theta functions in characteristic ℓ

The main theorem. The purpose of this section is to prove the following nonvanishing result for theta functions which is needed to complete our strategy.

THEOREM 4.1. *Let $\ell \neq p$ be primes, p odd and split in K, and d_0 be a positive integer not divisible by p. Then, if m is large enough, for every character λ appearing in the Shintani representation $\mathcal{F}^*$ on $\mathcal{T}_{d_0p^m}^{\mathrm{prim}}$ there is an ℓ-integral representative ϑ of the λ-eigenspace such that*

$$v_\ell(\frac{l(\vartheta)}{w_K}) = 0.$$

The proof of this theorem consists out of three main steps. In the first, rather elementary step, we fix the restriction of the character λ to the group Λ_{d_0D} and give a simple formula for the Shintani eigenfunctions corresponding to these characters, using some computations of Section 3. In the second step we set up an argument by contradiction, i.e. we assume the existence of infinitely many eigenspaces with $v_\ell(l(\vartheta)/w_K) > 0$ for all integral eigenfunctions ϑ. Following ideas of Sinnott, we derive from this assumption an algebraic relation in characteristic ℓ. For this purpose we use the formula obtained in the first step, reduce it modulo ℓ and apply Galois conjugation. After some further work we obtain the statement that for every large enough integer n_1 a certain subvariety $\mathcal{D}_{n_1}$ of a power E^r, where E is a characteristic ℓ elliptic curve with complex multiplication by $\mathfrak{o}_K$, contains an infinite set of p-power torsion points. In the third and final step we apply a result of Boxall to show that the Zariski closure of this infinite set contains a translate of an explicit abelian variety $\mathcal{A} \subseteq E^r$, and we finish by deducing a contradiction from the inclusion $\mathcal{A} + X \subseteq \mathcal{D}_{n_1}$ of algebraic varieties for large enough n_1.

Explicit expression for $l(\vartheta)$. For the rest of the paper we use the following notation: since p is split, we have $p\mathfrak{o}_K = \mathfrak{p}\bar{\mathfrak{p}}$, and to the primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ above p correspond the two embeddings $\iota_\mathfrak{p}$ and $\iota_{\bar{\mathfrak{p}}}$ of $\mathfrak{o}_K$ into $\mathbb{Z}_p$. Together they induce an isomorphism $\mathfrak{o}_{K_p} \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. We denote by $(x, y)_p$ or (x, y) the element of $\mathfrak{o}_{K_p}$ corresponding to the pair (x, y) on the right-hand side.

To begin with the first step of the proof, recall the description of the Shintani representation given in Proposition 2.2. The Hecke characters λ appearing in $\mathcal{T}^{\mathrm{prim}}_{d_0p^m}$ are precisely the characters with restriction to $\mathbb{A}_\mathbb{Q}^\times$ equal to $\omega_{K/\mathbb{Q}} \left|\cdot\right|_\mathbb{A}$, infinity type $(-1, 0)$, root number $W(\lambda) = +1$ and conductor $d_0p^mD\mathfrak{d}^{-1}$, where $\mathfrak{d}$ is a square-free product of ramified prime ideals. We fix the restriction β of λ to the group Λ_{d_0D} defined in Section 2, while $\lambda_p|_{\mathbb{Q}_p^\times \mathfrak{o}_{K_p}^\times}$ is free to vary over characters whose restriction to $\mathbb{Q}_p^\times$ is $\left|\cdot\right|_p$. Of course, to give $\lambda_p|_{\mathbb{Q}_p^\times \mathfrak{o}_{K_p}^\times}$ is equivalent to giving the character $\lambda_\mathfrak{p}(z) = \lambda_p((z, 1))$ of $\mathbb{Z}_p^\times$. The character λ itself is determined by these local data up to a twist by a character of $\mathrm{Cl}_K/\mathrm{Cl}_K^{\mathrm{inv}}$, where $\mathrm{Cl}_K^{\mathrm{inv}}$ denotes the group of ideal classes invariant under complex conjugation. The other way around, given a character β of Λ_{d_0D} with $\beta_q|_{\mathbb{Q}_q^\times} = \omega_{K/Q,q} \left|\cdot\right|_q$ for all q dividing d_0D, conductor $d_0D\mathfrak{d}^{-1}$, $\beta_\infty(z) = z$, root number $+1$, and some Dirichlet character of conductor p^m determining λ_p, there exists a Hecke character of K with these restrictions to Λ_{d_0D} and $\mathbb{Q}_p^\times \mathfrak{o}_{K_p}^\times$ if and only if $\beta(\xi)\lambda_p(\xi) = 1$ for all $\xi \in K^\times$ for which $\xi\mathfrak{o}_K$ contains only ramified prime factors. (For $w_K = 2$ it is enough to consider $\xi = \delta$.) By Proposition 2.2 the local components β_q at the ramified primes q determine via their root numbers $W_q(\beta_q)$ a unique class $C \in N(I_K)/N(K^\times)$ such that the character λ occurs (with multiplicity one) in the space $\mathcal{V}^{\mathrm{prim}}_{d_0p^m, Cp^m}$. If $\mathfrak{a}$ is an ideal with $\mathrm{N}(\mathfrak{a}) \in C$, the characters β as above that are compatible with the class C are in one-to-one correspondence with the Shintani eigenspaces in $T^{\mathrm{prim}}_{d_0/\mathrm{N}(\mathfrak{a}), \mathfrak{a}}$.

Let now $\mathcal{C}$ be a system of representatives for $\mathrm{Cl}_K/\mathrm{Cl}_K^{\mathrm{inv}}$ consisting out of integral ideals prime to their complex conjugates and to $p\ell d_0$. Then the ideals $\mathfrak{a}\bar{\mathfrak{c}}\mathfrak{c}^{-1}$, $\mathfrak{c} \in \mathcal{C}$, form a system of representatives for the classes of ideals with norms in the class C (the genus of $\mathfrak{a}$). A description of the λ-eigenspace is then given by the following lemma, which follows immediately from Lemma 3.8.

LEMMA 4.2. *The Shintani eigenspaces in* $\mathcal{V}^{\mathrm{prim}}_{d_0p^m,Cp^m}$ *may be described as follows. Let* $\vartheta_{\mathfrak{a}}$ *be a generator of the Shintani eigenspace in* $T^{\mathrm{prim}}_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ *determined by the restriction* $\lambda|\Lambda_{d_0D} = \beta$. *Also, let* $\mathfrak{C}$ *be the* l. c. m. *of all ideals* $\mathfrak{c} \in \mathcal{C}$, $\mathfrak{A} = \mathfrak{a}\bar{\mathfrak{C}}$, *and* l_0 *a generator of* $\mathfrak{A}\mathfrak{p}^m\bar{\mathfrak{p}}^{-m}/\mathfrak{A}\mathfrak{p}^m$. *Then* $\vartheta = (\vartheta_{\mathfrak{a}\mathfrak{p}^m\bar{\mathfrak{c}}\mathfrak{c}^{-1}})$ *with*

$$\vartheta_{\mathfrak{a}\mathfrak{p}^m\bar{\mathfrak{c}}\mathfrak{c}^{-1}} = \lambda(\mathfrak{c})\Pi_{\mathfrak{p}^m,\lambda_{\mathfrak{p}};l_0}(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}})$$

gives a generator of the λ*-eigenspace, and we see that*

$$l(\vartheta) = \varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C},\, z\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda((z,1)_p c)\psi(l_0z)A_{zl_0}(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}})\right), \tag{16}$$

where for every $\mathfrak{c} \in \mathcal{C}$ *we denote by* c *an idele with* $c\hat{\mathfrak{o}}_K = \mathfrak{c}\hat{\mathfrak{o}}_K$. *If* $\vartheta_{\mathfrak{a}}$ *is a normalized integral eigenfunction, the functions* $\vartheta_{\mathfrak{a}\mathfrak{p}^m\bar{\mathfrak{c}}\mathfrak{c}^{-1}}$ *are all normalized integral.*

Since for all $\xi \in K^\times$ such that $\xi\mathfrak{o}_K$ has only ramified prime factors the action of the unit $\xi/\bar{\xi}$ on $\vartheta_{\mathfrak{a}}$ is given by multiplication by $\beta(\xi)^{-1}$, and since furthermore $\beta(\xi)\lambda_p(\xi) = 1$, we see that the value at zero of a summand in (16) remains unchanged if z is multiplied by $\iota_{\bar{\mathfrak{p}}}(\xi/\bar{\xi}) = \iota_{\mathfrak{p}}(\bar{\xi}/\xi)$. Therefore we can divide by w_K and sum over z modulo the units of K in this equation, and we arrive at the following formula:

$$\frac{1}{w_K}l(\vartheta) = \varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C},\, z\in(\mathbb{Z}/p^m\mathbb{Z})^\times/\iota_{\mathfrak{p}}(\mathfrak{o}_K^\times)} \lambda((z,1)_p c)\psi(zl_0)A_{zl_0}(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}})\right). \tag{17}$$

We now consider a slightly different parametrization of the eigenspaces, which in a sense separates the wildly ramified part of the p-component. Namely, every Hecke character λ as above can be written as a product $\lambda = \lambda_0\phi$, where λ_0 is a character of the same type, but with conductor $d_0D\mathfrak{d}^{-1}$ or $d_0pD\mathfrak{d}^{-1}$, and ϕ a character of p-power order and conductor p^m, trivial on $\mathbb{A}_f^\times$. The finite order characters of K ramified only at p, and with trivial restriction to $\mathbb{Q}$, are the characters of the group $\Gamma_\infty = \mathbb{A}_{K,f}^\times/\mathbb{A}_f^\times K^\times\hat{\mathfrak{o}}_K^{(p)\times}$, which fits into an exact sequence

$$1 \longrightarrow \mathbb{Z}_p^\times/(\mathfrak{o}_K^\times)^2 \xrightarrow{z\mapsto(z,1)_p} \Gamma_\infty \longrightarrow \mathrm{Cl}_K \longrightarrow 1.$$

Since p is odd, we have $\mathbb{Z}_p^\times \simeq \mu_{p-1} \times U$ with $U = 1 + p\mathbb{Z}_p \simeq \mathbb{Z}_p$. Define $\tilde{\Gamma}_\infty$ as the pro-p part of Γ_∞. Then from the exact sequence above we get the exact sequence

$$1 \longrightarrow U \longrightarrow \tilde{\Gamma}_\infty \longrightarrow \mathrm{Cl}_{K,p} \longrightarrow 1,$$

where $\mathrm{Cl}_{K,p}$ is the p-part of the class group of K. The character ϕ is a character of $\tilde{\Gamma}_\infty$ of conductor p^m, i.e. it is trivial on $U_m = 1 + p^m\mathbb{Z}_p \subseteq U$, and nontrivial on U_{m-1}. In the following we denote by $\langle c\rangle$ the class of an idele $c \in \mathbb{A}_{K,f}^{\times}$ in the group $\tilde{\Gamma}_\infty$. Note that the factorization $\lambda = \lambda_0\phi$ is in general not unique, but the only ambiguity arises from ideal class characters of p-power order.

Since now for a given d_0 there are only finitely many choices of λ_0, it is enough to show that for fixed λ_0 the following is true: if m is large enough, for every character ϕ of $\tilde{\Gamma}_\infty$ of conductor p^m we can find an ℓ-integral representative ϑ in the $\lambda_0\phi$-eigenspace with $v_\ell(l(\vartheta)/w_K) = 0$. Equivalently, since λ_0 determines β, d_0 and C, we may also regard these data as fixed, and choose and fix an ideal $\mathfrak{a}$ with $\mathrm{N}(\mathfrak{a}) \in C$ and a normalized integral eigenfunction $\vartheta_{\mathfrak{a}} \in T^{\mathrm{prim}}_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$. Then we want to show that for m large enough for all characters ϕ of conductor p^m the value at the origin of the theta function

$$\sum_{\mathfrak{c}\in\mathcal{C},\,\eta\in\mu_{p-1}/\iota_{\mathfrak{p}}(\mathfrak{o}_K^\times)} \lambda_0((\eta,1)_p c) \sum_{z\in U/U_m} \phi(z\langle c\rangle)\psi(\eta z l_0) A_{\eta z l_0}(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}}) \tag{18}$$

has nonzero reduction modulo ℓ.

Application of an idea of Sinnott. We now proceed to the second step of the proof. First rephrase the situation in characteristic zero algebraically. Recall that in Section 3 we have introduced elliptic curves $E_{\mathfrak{A}\mathfrak{p}^m}$ and line bundles $L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}$ over a number field M such that $\Gamma(E_{\mathfrak{A}\mathfrak{p}^m}\otimes_{i_\infty}\mathbb{C}, L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}\otimes_{i_\infty}\mathbb{C})$ can be identified with the space $T_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}$. Also, the curves $E_{\mathfrak{A}\mathfrak{p}^m}\otimes_{i_\ell}\mathbb{C}_\ell$ have good reduction. There are isogenies $\varphi_m : E_{\mathfrak{A}\mathfrak{p}^m} \to E_{\mathfrak{A}}$ with $\ker\varphi_m = E_{\mathfrak{A}\mathfrak{p}^m}[\mathfrak{p}^m]$, and to the natural inclusion $T_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}} \to T_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}$ corresponds the associated map $\varphi_m^* : \Gamma(E_{\mathfrak{A}}, L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}}) \to \Gamma(E_{\mathfrak{A}\mathfrak{p}^m}, L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m})$.

Recall Mumford's geometric Heisenberg group $\mathcal{G}(L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m})$ from Section 3. Using that p is odd, we now associate to any $x \in E_{\mathfrak{A}\mathfrak{p}^m}[p^m]$ a canonical Heisenberg group element A'_x. There is a smallest subgroup $\mathcal{G}_{p,f}(L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m})$ of the Heisenberg group $\mathcal{G}(L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m})$ such that the projection to the p^m-division points is still surjective; it consists out of all elements of order dividing p^m in $\mathcal{G}(L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m})$ and is a part of an exact sequence

$$1 \longrightarrow \mu_{p^m} \longrightarrow \mathcal{G}_{p,f}(L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}) \longrightarrow E_{\mathfrak{A}\mathfrak{p}^m}[p^m] \longrightarrow 0.$$

Since the line bundle $L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}$ is symmetric, we may construct a canonical section of $\mathcal{G}_{p,f}$ by using the automorphism δ_{-1} of $\mathcal{G}$ defined in [Mum1, p. 308]. It is of order two, its restriction to the center is the identity, and its projection to $E_{\mathfrak{A}\mathfrak{p}^m}[p^m]$ induces the map $[-1]$. For $x \in E_{\mathfrak{A}\mathfrak{p}^m}[p^m]$ we now let $z = A'_x$ be the unique element in $\mathcal{G}_{p,f}$ with $\delta_{-1}(z) = z^{-1}$ projecting to x. We then have the multiplication law

$$A'_x A'_y = e(x,y)^{1/2} A'_{x+y},$$

where $e = e_{L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}} : E_{\mathfrak{A}\mathfrak{p}^m}[p^m] \times E_{\mathfrak{A}\mathfrak{p}^m}[p^m] \to \mu_{p^m}$ is the commutator pairing associated to $L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}$ and the square root is (uniquely) taken in μ_{p^m}. It is then easy to see that if $l \in p^{-m}\mathfrak{A}\mathfrak{p}^m$ corresponds to the point $x \in E_{\mathfrak{A}\mathfrak{p}^m}[p^m]$, the operator A'_x corresponds to $\psi(p^m l)A_l$ on $T_{r,\mathfrak{a}}$.

Consider the theta functions $\vartheta_{\mathfrak{a}}$ and $\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}}$ (or rather their images under i_∞^{-1}) as elements of $\Gamma(E_{\mathfrak{A}\mathfrak{p}^m}\otimes_M\bar{\mathbb{Q}}, L_{d_0/\mathrm{N}(\mathfrak{a}),\mathfrak{A}\mathfrak{p}^m}\otimes_M\bar{\mathbb{Q}})$. Let $P_0\in E_{\mathfrak{A}\mathfrak{p}^m}[\bar{\mathfrak{p}}^m]$ correspond to l_0. The theta function considered in (18) is then the element

$$\sum_{\mathfrak{c}\in\mathcal{C},\,\eta\in\mu_{p-1}/\iota_{\mathfrak{p}}(\mathfrak{o}_K^\times)}\lambda_0((\eta,1)_p c)\sum_{z\in U/U_m}\phi(z\langle c\rangle)A'_{\eta z P_0}\varphi_m^*(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}}),$$

of $\Gamma(E_{\mathfrak{A}\mathfrak{p}^m}\otimes_M\bar{\mathbb{Q}}, L_{\mathfrak{A}\mathfrak{p}^m}\otimes_M\bar{\mathbb{Q}})$, and we are interested in its value at the origin.

Let us now look at the situation modulo ℓ (i.e. for every object consider its base change to $\mathbb{C}_\ell$ via i_ℓ and reduce modulo the maximal ideal). For a certain finite field $k\subseteq\bar{\mathbb{F}}_\ell$ of characteristic ℓ we are given elliptic curves $E_m=\bar{E}_{\mathfrak{A}\mathfrak{p}^m}$ over k with complex multiplication by $\mathfrak{o}_K$ together with isogenies $\varphi_m: E_m\to E_0$ (over k) such that $\ker\varphi_m=E_m[\mathfrak{p}^m]$. For $0\le n\le m$ the isogeny φ_m may be factored as $\varphi_m=\varphi_n\psi_{mn}$ with an isogeny $\psi_{mn}:E_m\to E_n$ with kernel $\ker\psi_{mn}=E_m[\mathfrak{p}^{m-n}]$. In addition, we have a symmetric line bundle L_0 over E_0, defined over k, and induced bundles $L_m=\varphi_m^*L_0$ over E_m of degree $p^m\deg L_0$. We give the bundles L_m rigidifications along the zero sections compatible with the φ_m^*; this induces compatible maps $\Gamma_{\bar{\mathbb{F}}_\ell}(E_m,L_m):=\Gamma(E_m\otimes_k\bar{\mathbb{F}}_\ell, L_m\otimes_k\bar{\mathbb{F}}_\ell)\to\bar{\mathbb{F}}_\ell$ denoted by ε.

Observe now that $\vartheta_{\mathfrak{a}}$ and $\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}}$ are all ℓ-integral, which implies that the function (18) and in particular its value at the origin are ℓ-integral. Furthermore, because $\vartheta_{\mathfrak{a}}$ is normalized, it has nonzero reduction $\bar{\vartheta}_{\mathfrak{a}}$ modulo ℓ, and since the elements of $\mathcal{C}$ are assumed to be prime to ℓ, the functions $\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\vartheta_{\mathfrak{a}}$ reduce to nonzero sections $\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_{\mathfrak{a}}$ of the characteristic ℓ line bundle L_0 over the reduced curve E_0. (Simply note that the operators $\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})$ and $\mathcal{E}(\mathfrak{c}\bar{\mathfrak{c}}^{-1})$ both preserve integrality, and $\mathcal{E}(\mathfrak{c}\bar{\mathfrak{c}}^{-1})\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})=\mathrm{N}(\mathfrak{c})$, which is prime to ℓ.) Choose isomorphisms $i_m:\mathbb{Z}/p^m\mathbb{Z}\to E_m[\mathfrak{p}^m]$ and $j_m:\mathbb{Z}/p^m\mathbb{Z}\to E_m[\bar{\mathfrak{p}}^m]$; together they determine a primitive p^m-th root of unity ξ_m in $\bar{\mathbb{F}}_\ell$ by $e_{L_m}(i_m(x),j_m(y))=\xi_m^{xy}$. We write $i(x,y)=i(x)+j(y)$. Because of the symmetry of L_m and L_0, it is easily seen that the "level subgroup" in the sense of Mumford associated to L_m, L_0 and φ_m consists out of the A'_X for $X\in E[\mathfrak{p}^m]$. Therefore for $\vartheta\in\Gamma_{\bar{\mathbb{F}}_\ell}(E_0,L_0)$ the section $\varphi_m^*(\vartheta)\in\Gamma_{\bar{\mathbb{F}}_\ell}(E_m,L_m)$ is invariant under the operators $A'_{i(y)}$, $y\in\mathbb{Z}/p^m\mathbb{Z}$.

We assume now (contrary to the assertion of the theorem) that for infinitely many m there exists a character $\phi:\tilde{\Gamma}_\infty\to\bar{\mathbb{F}}_\ell^*$ of conductor p^m with $\varepsilon(\vartheta_\phi)=0$, where $\vartheta_\phi\in\Gamma_{\bar{\mathbb{F}}_\ell}(E_m,L_m)$ is defined by

$$\vartheta_\phi=\sum_{\mathfrak{c}\in\mathcal{C},\,\eta\in\mu_{p-1}/\iota_{\mathfrak{p}}(\mathfrak{o}_K^\times)}\bar{\lambda}_0((\eta,1)_p c)\sum_{z\in U/U_m}\phi(z\langle c\rangle)A'_{j(\eta z)}\varphi_m^*(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_{\mathfrak{a}}).\tag{19}$$

We are now in a position to transfer Sinnott's ideas to our situation (cf. [Si2, p. 215]). Enlarge the base field k, if necessary, such that $\bar{\vartheta}_{\mathfrak{a}}$ is defined over k and that it contains the values $\bar{\lambda}_0((z,1)_p c)$. Assume the existence of a character ϕ of conductor p^m with $\varepsilon(\vartheta_\phi)=0$. From this assumption we will derive certain algebraic relations for the theta function $\bar{\vartheta}_{\mathfrak{a}}$. We use the operation of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_\ell/k)$ on L_m and on the spaces of global sections. The first observation is the following.

LEMMA 4.3. *For any $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_\ell/k)$ and $\phi : \tilde{\Gamma}_\infty \to \bar{\mathbb{F}}_\ell^\times$ there is a constant $c(\sigma,\phi) \in \bar{\mathbb{F}}_\ell^\times$ with*

$$\vartheta_\phi^\sigma = c(\sigma,\phi)\vartheta_{\phi^\sigma}.$$

Proof. The action of σ on $E_m[\bar{\mathfrak{p}}^m]$ is given by multiplication with some $\alpha(\sigma) \in (\mathbb{Z}/p^m\mathbb{Z})^\times$. Moreover, we have in general ${A'_x}^\sigma = A'_{x^\sigma}$ for $x \in E_m[p^m]$, since A'_x is the unique lifting z of x to $\mathcal{G}_p(L)$ which is of order dividing p^m and fulfills $\delta^{-1}(z) = z^{-1}$. The assertion follows easily from these two facts.

Define n_0 by $p^{n_0} := \#(k \cap \mu_{p^\infty})$ and set $k_n := k(\mu_{p^{n_0+n}})$. Clearly we have for $\zeta \in \mu_{p^{n_0+n}}$:

$$\mathrm{Tr}_{k_n/k}(\zeta) = \begin{cases} p^n\zeta, & \zeta \in \mu_{p^{n_0}}, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Assume now that $n_0 \geq 1$, that the exponent of the group $\mathrm{Cl}_{K,p}$ divides p^{n_0}, and that $m \geq 2n_0$. Set $n = m - n_0$. Denote the torsion subgroup of $\tilde{\Gamma}_\infty$ by T. Then for $u \in U$ we have $\phi(u\langle c\rangle) \in \mu_{p^{n_0}}$ precisely if $\langle c\rangle = t_{\mathfrak{c}}u_{\mathfrak{c}} \in TU \subseteq \tilde{\Gamma}_\infty$ and $u \in u_{\mathfrak{c}}^{-1}U_n$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the set of $\mathfrak{c} \in \mathcal{C}$ with $\langle c\rangle \in TU$. It is elementary that there exists a primitive p^{n_0}-th root of unity $\gamma \in \bar{\mathbb{F}}_\ell^\times$ such that $\phi(u) = \gamma^{(u-1)/p^n}$ for $u \in U_n$. By Lemma 4.3, the assumption $\varepsilon(\vartheta_\phi) = 0$ implies that $\varepsilon(\vartheta_{\phi^\sigma}) = 0$ for $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_\ell/k)$. Conjugation by σ leaves the values $\bar{\lambda}_0((\eta,1)_p c)$ invariant. Consequently, we have for arbitrary $y \in U$ (using (20)):

$$\begin{aligned} 0 &= \varepsilon\left(p^{-n}\sum_{\sigma\in\mathrm{Gal}(k_n/k)} \phi^{-1}(y)^\sigma \vartheta_{\phi^\sigma}\right) \\ &= \varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C}',\,\eta} \bar{\lambda}_0((\eta,1)_p c) \sum_{z\in U_n/U_m} \phi(z) A'_{j(\eta u_{\mathfrak{c}}^{-1} yz)} \varphi_m^*(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_{\mathfrak{a}})\right). \end{aligned}$$

Since here $\phi(z) = \gamma^{(z-1)/p^n}$, we get

$$\varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C}',\,\eta} \bar{\lambda}_0((\eta,1)_p c) \sum_{u \bmod p^{n_0}} \gamma^u A'_{j(\eta u_{\mathfrak{c}}^{-1} y(1+p^n u))} \varphi_m^*(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_{\mathfrak{a}})\right) = 0. \tag{21}$$

We want to rewrite the inner sum in a more convenient form. Defining the projection operator

$$P_{n_0} = \sum_{u \bmod p^{n_0}} A'_{j(p^n u)} = \sum_{X\in E_m[\bar{\mathfrak{p}}^{n_0}]} A'_X,$$

we observe the following identity of operators on $\Gamma_{\bar{\mathbb{F}}_\ell}(E_m, L_m)$:

$$\begin{aligned} A'_{i(\eta^{-1}u_\mathfrak{c}v,\eta u_\mathfrak{c}^{-1}y)} P_{n_0} &= \sum_{u \bmod p^{n_0}} A'_{i(\eta^{-1}u_\mathfrak{c}v,\eta u_\mathfrak{c}^{-1}y)} A'_{j(p^n\eta u_\mathfrak{c}^{-1}yu)} \\ &= \xi_m^{2^{-1}yv} \sum_{u\in\mathbb{Z}/p^{n_0}\mathbb{Z}} \xi_m^{(p^n yv)u} A'_{j(\eta u_\mathfrak{c}^{-1}y(1+p^n u))} A'_{i(\eta^{-1}u_\mathfrak{c}v)}. \end{aligned}$$

If we choose $v \in \mathbb{Z}/p^m\mathbb{Z}$ with $\xi_m^{p^n vy} = \gamma$, and apply this identity to the function $\varphi_m^*(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_\mathfrak{a})$, we get, because of the invariance of this theta function under $A'_{i(y)}$, just $\xi_m^{2^{-1}yv}$ times the inner sum in (21). Therefore

$$\varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C}',\,\eta} \bar{\lambda}_0((\eta,1)_p c) A'_{i(\eta^{-1}u_\mathfrak{c}v,\eta u_\mathfrak{c}^{-1}y)} P_{n_0}\varphi_m^*(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_\mathfrak{a})\right) = 0.$$

Since we have the factorization $\varphi_m = \varphi_{n_0}\psi_{m,n_0}$, and

$$A'_X \psi^*_{m,n_0}(\vartheta) = \psi^*_{m,n_0}(A'_{\psi_{m,n_0}(X)}\vartheta)$$

for $\vartheta \in \Gamma_{\bar{\mathbb{F}}_\ell}(E_{n_0}, L_{n_0})$ and $X \in E_m[\bar{\mathfrak{p}}^{n_0}]$ (cf. [Mum1, Prop. 2]), we conclude

$$P_{n_0}\varphi_m^*(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_\mathfrak{a}) = \psi^*_{m,n_0}(P_{n_0}\varphi^*_{n_0}(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_\mathfrak{a})),$$

denoting by P_{n_0} on $\Gamma_{\bar{\mathbb{F}}_\ell}(E_{n_0}, L_{n_0})$ again the operator $\sum_{X\in E_{n_0}[\bar{\mathfrak{p}}^{n_0}]} A'_X$. Writing $\vartheta_{\mathfrak{c},n_0} = P_{n_0}\varphi^*_{n_0}(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_\mathfrak{a})$, we arrive at

$$\varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C}',\,\eta} \lambda_0((\eta,1)_p c) A'_{i(\eta^{-1}u_\mathfrak{c}v,\eta u_\mathfrak{c}^{-1}y)} \psi^*_{m,n_0}(\vartheta_{\mathfrak{c},n_0})\right) = 0. \tag{22}$$

Note that $\vartheta_{\mathfrak{c},n_0}$ is nonzero; in fact, we may compute

$$\left(\sum_{Y\in E_{n_0}[\mathfrak{p}^{n_0}]} A'_Y\right)\vartheta_{\mathfrak{c},n_0} = p^{n_0}\varphi^*_{n_0}(\mathcal{E}(\bar{\mathfrak{c}}\mathfrak{c}^{-1})\bar{\vartheta}_\mathfrak{a}).$$

We now want to deduce from these relations an algebraic identity saying that an algebraic variety contains a certain set of points. To this end, remember that $y \in U$ may be chosen arbitrarily, and that the condition $\xi_m^{p^n vy} = \gamma$ relates only the classes of y and v mod p^{n_0}. Consider therefore $y' = y + w$ with $w \equiv 0\,(p^{n_0})$ and take for both y and y' the same value of v. We have

$$A'_{i(\eta^{-1}u_\mathfrak{c}v,\eta u_\mathfrak{c}^{-1}y')} = \xi_m^{-2^{-1}vw} A'_{i(\eta^{-1}u_\mathfrak{c}v,\eta u_\mathfrak{c}^{-1}y)} A'_{j(\eta u_\mathfrak{c}^{-1}w)}.$$

Inserting this into (22) (with y replaced by y') yields

$$\varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C}',\,\eta} \bar{\lambda}_0((\eta,1)_p c) A'_{i(\eta^{-1}u_\mathfrak{c}v,\eta u_\mathfrak{c}^{-1}y)} A'_{j(\eta u_\mathfrak{c}^{-1}w)} \psi^*_{m,n_0}(\vartheta_{\mathfrak{c},n_0})\right) = 0.$$

Take a fixed integer $n_1 = n_0 + N > n_0$, assume $m \geq n_1 + n_0$, and set $w = p^{m-n_1}x$. Factorizing $\psi_{m,n_0} = \psi_{n_1,n_0}\psi_{m,n_1}$, we may write

(23)
$$\varepsilon\left(\sum_{\mathfrak{c},\,\eta} \bar{\lambda}_0((\eta,1)_p c) A'_{i(\eta^{-1}u_{\mathfrak{c}}v,\eta u_{\mathfrak{c}}^{-1}y)} \psi^*_{m,n_1}(A'_{[(\eta^{-1}u_{\mathfrak{c}},\eta u_{\mathfrak{c}}^{-1})]X} \psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0}))\right) = 0,$$

where $X = \psi_{m,n_1}(j_m(p^{m-n_1}x)) \in E_{n_1}[\bar{\mathfrak{p}}^{n_1}]$. Here we use the natural operation of $\mathfrak{o}_{K_p}$ on $E_{n_1}[p^\infty] \simeq K_p/\mathfrak{o}_{K,p}$. Because of the invariance of $\vartheta_{\mathfrak{c},n_0}$ under A'_Y, $Y \in E_{n_0}[\bar{\mathfrak{p}}^{n_0}]$, the value of X is only important modulo $E_{n_1}[\bar{\mathfrak{p}}^{n_0}]$.

Setting $\theta_{\mathfrak{c},X} = A'_X \psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0})$ for $X \in E_{n_1}[\bar{\mathfrak{p}}^{n_1}]$, we have morphisms $\Phi_{\mathfrak{c}} : E := E_{n_1} \to \mathbb{P}^{p^N-1}$ given by the global sections $\theta_{\mathfrak{c},j_{n_1}(x)}$, $x \in \mathbb{Z}/p^N\mathbb{Z}$, of L_{n_1}. Every element $\alpha \in (\mathbb{Z}/p^N\mathbb{Z})^\times$ defines an automorphism c_α of $\mathbb{P}^{p^N-1}$ by $(v_x)_x \mapsto (v_{\alpha x})_x$. Set $P_m := \psi_{m,n_1}(i_m(v,y)) \in E[\mathfrak{p}^{n_1}\bar{\mathfrak{p}}^m]$ for $y \in U$ and corresponding v, and look at the points $c_{\eta u_{\mathfrak{c}}^{-1}}(\Phi_{\mathfrak{c}}([(\eta^{-1}u_{\mathfrak{c}},\eta u_{\mathfrak{c}}^{-1})]P_m))$; they are $r = (p-1)\#\mathcal{C}'/w_K$ many in number. For $x \in E(\bar{\mathbb{F}}_\ell)$ define $L_{n_1}(x)$ as the tensor product with $\bar{\mathbb{F}}_\ell$ of the stalk of L_{n_1} at x (cf. [Mum1, p. 299]). Then the map $\Gamma_{\bar{\mathbb{F}}_\ell}(E, L_{n_1}) \to \bar{\mathbb{F}}_\ell$ given by $\vartheta \mapsto \varepsilon(A'_X\psi^*_{m,n_1}\vartheta)$ induces an identification of $L_{n_1}(\psi_{m,n_1}(X))$ with $\bar{\mathbb{F}}_\ell$. The relations (23) imply now the existence of a nontrivial linear dependency between the $c_{\eta u_{\mathfrak{c}}^{-1}}(\Phi_{\mathfrak{c}}([(\eta^{-1}u_{\mathfrak{c}},\eta u_{\mathfrak{c}}^{-1})]P_m))$: they have to lie in a projective space of dimension $r-2$.

Enumerate the elements of $\mu_{p-1}/\iota_{\mathfrak{p}}(\mathfrak{o}_K^\times) \times \mathcal{C}'$ as $(\eta_\nu, \mathfrak{c}_\nu)$ and define $\alpha_\nu = \eta_\nu u_{\mathfrak{c}_\nu}^{-1}$ for $0 \leq \nu \leq r-1$. Write $(\alpha^{-1},\alpha)P = ([(\alpha_\nu^{-1},\alpha_\nu)]P)_\nu \in E^r[p^\infty]$ for $P \in E[p^\infty]$. Summarizing, we have obtained the following intermediate result.

LEMMA 4.4. *Let $\mathcal{D} = \mathcal{D}_{n_1} \subseteq E^r$ be the subvariety defined by the relation*

$$\bigwedge_\nu c_{\alpha_\nu}(\Phi_{\mathfrak{c}_\nu}(P_\nu)) = 0.$$

Then for each $m \geq n_1 + n_0$ for which there exists a character $\phi : \tilde{\Gamma}_\infty \to \bar{\mathbb{F}}_\ell$ of conductor p^m with $\varepsilon(\vartheta_\phi) = 0$, we have $(\alpha^{-1},\alpha)P_m \in \mathcal{D}$ for all

$$P_m = \psi_{m,n_1}(i_m(v,y)) \in E[\mathfrak{p}^{n_1}\bar{\mathfrak{p}}^m]$$

with arbitrary $y \in U$ and $v \in \mathbb{Z}/p^m\mathbb{Z}$ such that $\xi_m^{p^n vy} = \gamma_m$ for a primitive p^{n_0}-th *root of unity $\gamma_m \in \bar{\mathbb{F}}_\ell^\times$ depending on ϕ.*

A geometric independence result. Assuming contrary to Theorem 4.1 the existence of infinitely many characters ϕ with $\varepsilon(\vartheta_\phi) = 0$, we proceed to derive a contradiction. Let us first consider the Zariski closure of the infinite set of all points $(\alpha^{-1},\alpha)P_m$ which under this assumption one obtains from the previous lemma. The following lemma uses a result of Boxall [B] for this purpose.

LEMMA 4.5. *Let $r \geq 1$ and $\beta_\nu = (\alpha_\nu, \alpha'_\nu) \in \mathfrak{o}_{K_p}$ for $0 \leq \nu \leq r-1$ be given, and $\beta : E[p^\infty] \to E^r[p^\infty]$ the map $P \mapsto ([\beta_\nu]P)_\nu$. Let*

$$\mathcal{R} = \{x \in \mathfrak{o}_K^r \mid \sum_{\nu=0}^{r-1} \iota_{\bar{\mathfrak{p}}}(x_\nu)\alpha'_\nu = 0\}$$

be the $\mathfrak{o}_K$-module of relations between the α'_ν, and

$$\mathcal{A} = \{P = (P_\nu) \in E^r \mid \sum_{\nu=0}^{r-1} [x_\nu]P_\nu = 0 \, \forall x \in \mathcal{R}\}$$

the abelian subvariety of E^r defined by these relations. If $\mathcal{M}$ is an infinite subset of $E[\mathfrak{p}^n\bar{\mathfrak{p}}^\infty]$ for some n, the Zariski closure B of $\beta(\mathcal{M})$ in E^r contains a translate $\mathcal{A} + X$ for some point $X \in E^r(\bar{\mathbb{F}}_\ell)$.

Proof. By definition $B \cap E^r[p^\infty]$ is Zariski dense in B, and the same is true for all irreducible components of B. Take a component B' such that $B' \cap \beta(\mathcal{M})$ is infinite. A theorem of Boxall [B, Thm. 1] shows that B' is a translate of its stabilizer $T_{B'} = \{X \mid B' + X = B'\}$, which is an algebraic subgroup of E^r. It is then immediate that $T_{B'}$ contains infinitely many points of $\beta(E[\bar{\mathfrak{p}}^\infty])$. But since $E[\bar{\mathfrak{p}}^\infty] \simeq \mathbb{Q}_p/\mathbb{Z}_p$, these points actually generate $\beta(E[\bar{\mathfrak{p}}^\infty])$. If we can show that $\mathcal{A}$ is the minimal algebraic subgroup of E^r containing $\beta(E[\bar{\mathfrak{p}}^\infty])$, we have $\mathcal{A} \subseteq T_{B'}$, which implies that B contains a translate of $\mathcal{A}$.

Let now A be an algebraic subgroup of E^r containing $\beta(E[\bar{\mathfrak{p}}^\infty])$. If A_0 is the connected component of zero (an abelian variety), it also contains $\beta(E[\bar{\mathfrak{p}}^\infty])$. If not $\mathcal{A} \subseteq A_0$, there exists a nontrivial homomorphism $\varphi : \mathcal{A}/(\mathcal{A} \cap A_0) \to E$. By Poincaré's complete reducibility theorem [Mum2, p. 173], we can extend $\varphi \circ [N]$ to E^r for some integer $N > 0$, and obtain a homomorphism $\varphi' : E^r \to E$ mapping $\beta(E[\bar{\mathfrak{p}}^\infty])$ to zero. Since φ' has to be of the form $P \mapsto \sum_\nu \xi_\nu(P_\nu)$ with $\xi_\nu \in \mathrm{End}E$, necessarily $\mathcal{A} \subseteq \ker\varphi'$, contradicting the assumption that φ is nontrivial on $\mathcal{A}$. (Observe that this holds true even if E is supersingular and $\mathrm{End}E$ is strictly bigger than $\mathfrak{o}_K$.) Therefore $\mathcal{A} \subseteq A$, and the lemma is proved.

Applying this lemma to our situation, we conclude that a translate $\mathcal{A}+X$, $X \in E^r(\bar{\mathbb{F}}_\ell)$, is contained in the subvariety $\mathcal{D}$. To derive a contradiction, we use the fact that translations by elements of $E[\mathfrak{p}^N\bar{\mathfrak{p}}^{n_1}]$ operate on $\Phi_\mathfrak{c}(E)$ via projective automorphisms. In fact, for $y \in \mathbb{Z}/p^N\mathbb{Z}$,

$$\begin{aligned} A'_{i(p^{n_0}y)}\theta_{\mathfrak{c},j(x)} &= A'_{i(p^{n_0}y)}A'_{j(x)}\psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0}) \\ &= e_{L_{n_1}}(i(p^{n_0}y), j(x))A'_{j(x)}A'_{i(p^{n_0}y)}\psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0}) \\ &= \zeta_{p^N}^{xy}\theta_{\mathfrak{c},j(x)} \end{aligned}$$

with the primitive p^N-th root of unity $\zeta_{p^N} = \xi_{n_1}^{p^{n_0}}$, and consequently

$$\Phi_\mathfrak{c}(X + i(p^{n_0}y)) = \tau_y(\Phi_\mathfrak{c}(X)),$$

where τ_y is the automorphism $(v_x)_x \mapsto (\zeta_{p^N}^{xy}v_x)_x$ of $\mathbb{P}^{p^N-1}$.

From $\mathcal{A}+X \subseteq \mathcal{D}$ trivially $P+\mathcal{A}[\mathfrak{p}^N] \subseteq \mathcal{D}$ for every $P \in \mathcal{A}+X$. Parametrize the elements Y of $\mathcal{A}[\mathfrak{p}^N]$ by writing $Y = (i(p^{n_0}x_\nu))_\nu$ with $x \in (\mathbb{Z}/p^N\mathbb{Z})^r$ satisfying $\rho^{\mathrm{tr}}x = 0$ for all $\rho \in \iota_\mathfrak{p}(\mathcal{R}) + p^N\mathbb{Z}_p^r/p^N\mathbb{Z}_p^r$. If for each ν the vector v_ν is a representative for $c_{\alpha_\nu}(\Phi_{\mathfrak{c}_\nu}(P_\nu))$ in $V = \bar{\mathbb{F}}_\ell^r$, the fact $P + Y \in \mathcal{D}$ translates into

$$\bigwedge_\nu \tau_{\alpha_\nu x_\nu}(v_\nu) = \bigwedge_\nu c_{\alpha_\nu}(\tau_{x_\nu}(c_{\alpha_\nu}^{-1}(v_\nu))) = 0. \tag{24}$$

We will get a contradiction by forming suitable linear combinations of these relations, which will force the vanishing of some coordinate of a v_ν, provided N was chosen large enough. Let $a : V^{\otimes r} \to \bigwedge^r V$ be the canonical map, and expand the vectors v_ν as $v_\nu = \sum_{i\in\mathbb{Z}/p^N\mathbb{Z}} v_{\nu i}e_i$ in terms of the standard basis (e_i). We have then

$$v_0 \otimes \ldots \otimes v_{r-1} = \sum_{i\in(\mathbb{Z}/p^N\mathbb{Z})^r} \prod_\mu v_{\mu i_\mu} \bigotimes_\mu e_{i_\mu}$$

and

$$\bigotimes_\nu \tau_{\alpha_\nu x_\nu} v_\nu = \sum_{i\in(\mathbb{Z}/p^N\mathbb{Z})^r} \zeta_{p^N}^{\sum_\mu \alpha_\mu x_\mu i_\mu} \prod_\mu v_{\mu i_\mu} \bigotimes_\mu e_{i_\mu}.$$

Applying a Fourier transform, we get

$$\sum_x \zeta_{p^N}^{-\lambda^{\mathrm{tr}}x} \bigotimes_\nu \tau_{\alpha_\nu x_\nu} v_\nu = (\#\mathcal{A}[\mathfrak{p}^N]) \sum_{i,\, \alpha i-\lambda\in\iota_\mathfrak{p}(\mathcal{R})+p^N\mathbb{Z}_p^r/p^N\mathbb{Z}_p^r} \prod_\mu v_{\mu i_\mu} \bigotimes_\mu e_{i_\mu}$$

for all $\lambda \in (\mathbb{Z}/p^N\mathbb{Z})^r$. From (24) we know that application of a to this equation yields zero. Since $\#\mathcal{A}[\mathfrak{p}^N]$ is a power of p, we conclude

$$\sum_{i,\, \alpha i-\lambda\in\iota_\mathfrak{p}(\mathcal{R})+p^N\mathbb{Z}_p^r/p^N\mathbb{Z}_p^r} \prod_\mu v_{\mu i_\mu} \bigwedge_\mu e_{i_\mu} = 0.$$

If we can find a summation index i with $i_\nu \neq i_\mu$ ($\nu \neq \mu$) such that no nontrivial permutation $\sigma(i)$, $\sigma \in \mathfrak{S}_r\backslash\{\mathrm{id}\}$, occurs in the sum for the same value of λ, a multiple of the multivector $\bigwedge_\mu e_{i_\mu}$ appears only once. Therefore

$$\prod_\mu v_{\mu i_\mu} = 0,$$

i.e. one of the coordinates $v_{\mu i_\mu}$ has to vanish. But it is easily seen that the subvariety of $\mathcal{A}+X$ cut out by the condition that one of the coordinates of the $\Phi_{\mathfrak{c}_\nu}(P_\nu)$ should vanish, has codimension one, and so choosing a point P outside of this exceptional set yields a contradiction. It remains to check the existence of an index i with the required property; this is provided by the following two lemmas, which finish the proof of the main theorem.

LEMMA 4.6. *The module of relations $\mathcal{R} \subseteq \mathfrak{o}_K^r$ does not contain any vectors* (*except zero*) *which have less than three nonzero entries.*

Proof. It is clear that no element of $\mathcal{R}$ can have exactly one nonzero coordinate. Assume there exists a vector in $\mathcal{R}$ with two nonzero entries. This implies $(\eta/\eta')(u_{\mathfrak{c}'}/u_{\mathfrak{c}}) \in K$ where either $\mathfrak{c} \neq \mathfrak{c}'$ or $\eta \neq \eta'$. In case $\mathfrak{c} = \mathfrak{c}'$ we get immediately a contradiction. If $\mathfrak{c} \neq \mathfrak{c}'$, let γ and γ' be generators of the principal ideals $\mathfrak{c}^{h_K}$ and ${\mathfrak{c}'}^{h_K}$. Then $u_{\mathfrak{c}}^{h_K} = (\gamma/\bar{\gamma})\zeta$ for some $\zeta \in \mu_{p-1}$, and the same for $u_{\mathfrak{c}'}$. From our assumption, $\gamma'\gamma^{-1}/\overline{\gamma'\gamma^{-1}}$ is an element of μ_{p-1} times the h_K-th power of an element of K, and therefore the product of a unit of K and a h_K-th power. By Hilbert 90, there is some $\alpha \in K^\times$ such that $\gamma'\gamma^{-1}\alpha^{h_K}$ generates an ideal of K invariant under complex conjugation. But this means that $\mathfrak{c}'\mathfrak{c}^{-1}\alpha$ has to be invariant under complex conjugation, which contradicts the fact that $\mathfrak{c}$ and $\mathfrak{c}'$ represent different classes in $\mathrm{Cl}_K/\mathrm{Cl}_K^{\mathrm{inv}}$.

LEMMA 4.7. *For N large enough, there exists an element $i \in (\mathbb{Z}/p^N\mathbb{Z})^r$ such that $i_\nu \neq i_\mu$ $(\nu \neq \mu)$ and*

$$\alpha(i - \sigma(i)) \notin \iota_{\mathfrak{p}}(\mathcal{R}) + p^N\mathbb{Z}_p^r/p^N\mathbb{Z}_p^r$$

for every $\sigma \in \mathfrak{S}_r \backslash \{\mathrm{id}\}$.

Proof. We use a simple counting argument. The number of $i \in (\mathbb{Z}/p^N\mathbb{Z})^r$ such that $i_\nu \neq i_\mu$ $(\nu \neq \mu)$ is simply

$$p^N(p^N - 1)\cdots(p^N - r + 1),$$

i.e. grows like p^{Nr}. We bound the number of i, for which there exists some $\sigma \in \mathfrak{S}_r \backslash \{\mathrm{id}\}$ with

$$\alpha(i - \sigma(i)) \in \iota_{\mathfrak{p}}(\mathcal{R}) + p^N\mathbb{Z}_p^r/p^N\mathbb{Z}_p^r,$$

by considering each σ separately. We have the linear map $f_\sigma : (\mathbb{Z}/p^N\mathbb{Z})^r \to (\mathbb{Z}/p^N\mathbb{Z})^r$ defined by $i \mapsto \alpha(i - \sigma(i))$ and want to count the number n_σ of elements in $f_\sigma^{-1}(\iota_{\mathfrak{p}}(\mathcal{R}) + p^N\mathbb{Z}_p^r/p^N\mathbb{Z}_p^r)$. If b_σ is the number of orbits of σ on $\{0, \ldots, r-1\}$, the kernel of f_σ has p^{Nb_σ} elements, and the image consists out of all $x \in (\mathbb{Z}/p^N\mathbb{Z})^r$ with

$$\sum_{\nu \in B} \alpha_\nu^{-1} x_\nu = 0$$

for all orbits B. Standard results on the number of solutions of a system of linear congruences imply that n_σ is equal to $p^{N(b_\sigma + r_\sigma) + c_\sigma}$ for N large enough, where c_σ is some integer independent of N, and r_σ is the rank of the $\mathbb{Z}_p$-module $\mathbb{Z}_p\iota_{\mathfrak{p}}(\mathcal{R}) \cap I_\sigma$,

$$I_\sigma = \{x \in \mathbb{Z}_p^r \,|\, \sum_{\nu \in B} \alpha_\nu^{-1} x_\nu = 0 \,\forall B\}.$$

Since the rank of I_σ is $r - b_\sigma$, we have $r_\sigma \leq r - b_\sigma$, and equality can occur only if $I_\sigma \subseteq \mathbb{Z}_p\iota_{\mathfrak{p}}(\mathcal{R})$. But I_σ is generated by vectors with only two nonzero components, and none of these generators can be contained in $\mathbb{Z}_p\iota_{\mathfrak{p}}(\mathcal{R})$. Therefore $r_\sigma < r - b_\sigma$ for every $\sigma \neq \mathrm{id}$, the number of excluded multiindices i is bounded by a constant times $p^{N(r-1)}$, and we see that for N large enough there will be a multiindex satisfying the assertion. The lemma is proved.

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