Runge approximation on convex sets implies the Oka property

By Franc Forstnerič*

Abstract

We prove that the classical Oka property of a complex manifold Y, concerning the existence and homotopy classification of holomorphic mappings from Stein manifolds to Y, is equivalent to a Runge approximation property for holomorphic maps from compact convex sets in Euclidean spaces to Y.

Introduction

Motivated by the seminal works of Oka [40] and Grauert ([24], [25], [26]) we say that a complex manifold Y enjoys the *Oka property* if for every Stein manifold X, every compact $\mathcal{O}(X)$-convex subset K of X and every continuous map $f_0 \colon X \to Y$ which is holomorphic in an open neighborhood of K there exists a homotopy of continuous maps $f_t \colon X \to Y$ ($t \in [0,1]$) such that for every $t \in [0,1]$ the map f_t is holomorphic in a neighborhood of K and uniformly close to f_0 on K, and the map $f_1 \colon X \to Y$ is holomorphic.

The Oka property and its generalizations play a central role in analytic and geometric problems on Stein manifolds and the ensuing results are commonly referred to as *the Oka principle*. Applications include the homotopy classification of holomorphic fiber bundles with complex homogeneous fibers (the *Oka-Grauert principle* [26], [7], [31]) and optimal immersion and embedding theorems for Stein manifolds [9], [43]; for further references see the surveys [15] and [39].

In this paper we show that the Oka property is equivalent to a Runge-type approximation property for holomorphic mappings from Euclidean spaces.

Theorem 0.1. *If Y is a complex manifold such that any holomorphic map from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) to Y can be approximated uniformly on K by entire maps $\mathbb{C}^n \to Y$ then Y satisfies the Oka property.*

The hypothesis in Theorem 0.1 will be referred to as the *convex approximation property* (CAP) of the manifold Y. The converse implication is obvious and hence the two properties are equivalent:

$$\text{CAP} \Longleftrightarrow \text{the Oka property.}$$

For a more precise result see Theorem 1.2 below. An analogous equivalence holds in the parametric case (Theorem 5.1), and CAP itself implies the one-parametric Oka propery (Theorem 5.3).

To our knowledge, CAP is the first known characterization of the Oka property which is stated purely in terms of holomorphic maps from Euclidean spaces and which does not involve additional parameters. The equivalence in Theorem 0.1 seems rather striking since linear convexity is not a biholomorphically invariant property and it rarely suffices to fully describe global complex analytic phenomenona. (For the role of convexity in complex analysis see Hörmander's monograph [33].)

In the sequel [19] to this paper it is shown that CAP of a complex manifold Y also implies the universal extendibility of holomorphic maps from closed complex submanifolds of Stein manifolds to Y (the *Oka property with interpolation*). A small extension of our method show that the CAP property of Y implies the Oka property for maps $X \to Y$ also when X is a reduced *Stein space* (Remark 6.6).

We actually show that a rather special class of compact convex sets suffices to test the Oka property (Theorem 1.2). This enables effective applications of the rich theory of holomorphic automorphisms of Euclidean spaces developed in the 1990's, beginning with the works of Andersén and Lempert [1], [2], thus yielding a new proof of the Oka property in several cases where the earlier proof relied on sprays introduced by Gromov [28]; examples are complements of thin (of codimension at least two) algebraic subvarieties in certain algebraic manifolds (Corollary 1.3).

Theorem 0.1 partly answers a question, raised by Gromov [28, p. 881, 3.4.(D)]: whether Runge approximation on a certain class of compact sets in Euclidean spaces, for example the balls, suffices to infer the Oka property. While it may conceivably be possible to reduce the testing family to balls by more careful geometric considerations, we feel that this would not substantially simplify the verification of CAP in concrete examples.

CAP has an essential advantage over the other known sufficient conditions when unramified holomorphic fibrations $\pi \colon Y \to Y'$ are considered. While it is a difficult problem to transfer a spray on Y' to one on Y and vice versa, lifting an individual map $K \to Y'$ from a convex (hence contractible) set $K \subset \mathbb{C}^n$ to a map $K \to Y$ is much easier — all one needs is the Serre fibration property of π and some analytic flexibility condition for the fibers (in order to find a holomorphic lifting). In such case the total space Y satisfies the Oka property if

and only if the base space Y' does; this holds in particular if π is a holomorphic fiber bundle whose fiber satisfies CAP (Theorems 1.4 and 1.8). This shows the Oka property for Hopf manifolds, Hirzebruch surfaces, complements of finite sets in complex tori of dimension > 1, unramified elliptic fibrations, etc.

The main conditions on a complex manifold which are known to imply the Oka property are complex homogeneity (Grauert [24], [25], [26]), the existence of a dominating spray (Gromov [28]), and the existence of a finite dominating family of sprays [13] (Def. 1.6 below). It is not difficult to see that each of them implies CAP — one uses the given condition to linearize the approximation problem and thereby reduce it to the classical Oka-Weil approximation theorem for sections of holomorphic vector bundles over Stein manifolds. (See also [21] and [23]. An analogous result for algebraic maps has recently been proved in Section 3 of [18].) The gap between these sufficient conditions and the Oka property is not fully understood; see Section 3 of [28] and the papers [18], [19], [37], [38].

Our proof of the implication CAP$\Rightarrow$Oka property (§3 below) is a synthesis of recent developments from [16] and [17] where similar methods have been employed in the construction of holomorphic submersions. In a typical inductive step we use CAP to approximate a family of holomorphic maps $A \to Y$ from a compact strongly pseudoconvex domain $A \subset X$, where the parameter of the family belongs to $\mathbb{C}^p$ ($p = \dim Y$), by another family of maps from a convex bump $B \subset X$ attached to A. The two families are patched together into a family of holomorphic maps $A \cup B \to Y$ by applying a generalized Cartan lemma proved in [16] (Lemma 2.1 below); this does not require any special property of Y since the problem is transferred to the source Stein manifold X. Another essential tool from [16] allows us to pass a critical level of a strongly plurisubharmonic Morse exhaustion function on X by reducing the problem to the noncritical case for another strongly plurisubharmonic function. The crucial part of extending a partial holomorphic solution to an attached handle (which describes the topological change at a Morse critical point) does not use any condition on Y thanks to a Mergelyan-type approximation theorem from [17].

1. The main results

Let $z = (z_1, \dots, z_n)$ be the coordinates on $\mathbb{C}^n$, with $z_j = x_j + iy_j$. Set

$$P = \{z \in \mathbb{C}^n : |x_j| \leq 1, \ |y_j| \leq 1, \ j = 1, \dots, n\}. \tag{1.1}$$

A *special convex set* in $\mathbb{C}^n$ is a compact convex subset of the form

$$Q = \{z \in P : y_n \leq h(z_1, \dots, z_{n-1}, x_n)\}, \tag{1.2}$$

where h is a smooth (weakly) concave function with values in $(-1, 1)$.

We say that a map is holomorphic on a compact set K in a complex manifold X if it is holomorphic in an unspecified open neighborhood of K in X; for a homotopy of maps the neighborhood should not depend on the parameter.

Definition 1.1. A complex manifold Y satisfies the *n-dimensional convex approximation property* (CAP_n) if any holomorphic map $f\colon Q \to Y$ on a special convex set $Q \subset \mathbb{C}^n$ (1.2) can be approximated uniformly on Q by holomorphic maps $P \to Y$. Y satisfies $\mathrm{CAP} = \mathrm{CAP}_\infty$ if it satisfies CAP_n for all $n \in \mathbb{N}$.

Let $\mathcal{O}(X)$ denote the algebra of all holomorphic functions on X. A compact set K in X is $\mathcal{O}(X)$-convex if for every $p \in X \backslash K$ there exists $f \in \mathcal{O}(X)$ such that $|f(p)| > \sup_{x \in K} |f(x)|$.

THEOREM 1.2 (The main theorem). *If Y is a p-dimensional complex manifold satisfying CAP_{n+p} for some $n \in \mathbb{N}$ then Y enjoys the Oka property for maps $X \to Y$ from any Stein manifold with $\dim X \leq n$. Furthermore, sections $X \to E$ of any holomorphic fiber bundle $E \to X$ with such fiber Y satisfy the* Oka principle: *Every continuous section $f_0\colon X \to E$ is homotopic to a holomorphic section $f_1\colon X \to E$ through a homotopy of continuous sections $f_t\colon X \to E$ ($t \in [0,1]$); if in addition f_0 is holomorphic on a compact $\mathcal{O}(X)$-convex subset $K \subset X$ then the homotopy $\{f_t\}_{t \in [0,1]}$ can be chosen holomorphic and uniformly close to f_0 on K.*

Note that the Oka property of Y is just the Oka principle for sections of the trivial (product) bundle $X \times Y \to X$ over any Stein manifold X.

We have an obvious implication $\mathrm{CAP}_n \implies \mathrm{CAP}_k$ when $n > k$ (every compact convex set in $\mathbb{C}^k$ is also such in $\mathbb{C}^n$ via the inclusion $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$), but the converse fails in general for $n \leq \dim Y$ (example 6.1). An induction over an increasing sequence of cubes exhausting $\mathbb{C}^n$ shows that CAP_n is equivalent to the Runge approximation of holomorphic maps $Q \to Y$ on special convex sets (1.2) by entire maps $\mathbb{C}^n \to Y$ (compare with the definition of CAP in the introduction).

We now verify CAP in several specific examples. The following was first proved in [28] and [13] by finding a dominating family of sprays (see Def. 1.6 below).

COROLLARY 1.3. *Let $p > 1$ and let Y' be one of the manifolds $\mathbb{C}^p$, $\mathbb{CP}_p$ or a complex Grassmanian of dimension p. If $A \subset Y'$ is a closed algebraic subvariety of complex codimension at least two then $Y = Y' \backslash A$ satisfies the Oka property.*

Proof. Let $f\colon Q \to Y$ be a holomorphic map from a special convex set $Q \subset P \subset \mathbb{C}^n$ (1.2). An elementary argument shows that f can be approximated

uniformly on a neighborhood of Q by algebraic maps $f' \colon \mathbb{C}^n \to Y'$ such that $f'^{-1}(A)$ is an algebraic subvariety of codimension at least two which is disjoint from Q. (If $Y' = \mathbb{C}^p$ we may take a suitable generic polynomial approximation of f, and the other cases easily reduce to this one by the arguments in [17].) By Lemma 3.4 in [16] there is a holomorphic automorphism ψ of $\mathbb{C}^n$ which approximates the identity map uniformly on Q and satisfies $\psi(P) \cap f'^{-1}(A) = \emptyset$. The holomorphic map $g = f' \circ \psi \colon \mathbb{C}^n \to Y'$ then takes P to $Y = Y' \backslash A$ and it approximates f uniformly on Q. This proves that Y enjoys CAP and hence (by Theorem 1.2) the Oka property. $\qquad \square$

By methods in [18] (especially Corollary 2.4 and Proposition 5.4) one can extend Corollary 1.3 to any algebraic manifold Y' which is a finite union of Zariski open sets biregularly equivalent to $\mathbb{C}^p$. Every such manifold satisfies an approximation property analogous to CAP for regular algebraic maps (Corollary 1.2 in [18]).

We now consider unramified holomorphic fibrations, beginning with a result which is easy to state (compare with Gromov [28, 3.3.C' and 3.5.B''], and Lárusson [37], [38]); the proof is given in Section 4.

THEOREM 1.4. *If $\pi \colon Y \to Y'$ is a holomorphic fiber bundle whose fiber satisfies* CAP *then Y enjoys the Oka property if and only if Y' does. This holds in particular if π is a covering projection, or if the fiber of π is complex homogeneous.*

COROLLARY 1.5. *Each of the following manifolds enjoys the Oka property*:

(i) *A Hopf manifold.*

(ii) *The complement of a finite set in a complex torus of dimension > 1.*

(iii) *A Hirzebruch surface.*

Proof. (i) A p-dimensional Hopf manifold is a holomorphic quotient of $\mathbb{C}^p \backslash \{0\}$ by an infinite cyclic group of dilations of $\mathbb{C}^p$ [3, p. 225]; since $\mathbb{C}^p \backslash \{0\}$ satisfies CAP by Corollary 1.3, the conclusion follows from Theorem 1.4. Note that Hopf manifolds are nonalgebraic and even non-Kählerian.

(ii) Every p-dimensional torus is a quotient $\mathbb{T}^p = \mathbb{C}^p / \Gamma$ where $\Gamma \subset \mathbb{C}^p$ is a lattice of maximal real rank $2p$. Choose finitely many points $t_1, \dots, t_m \in \mathbb{T}^p$ and preimages $z_j \in \mathbb{C}^p$ with $\pi(z_j) = t_j$ $(j = 1, \dots, m)$. The discrete set $\Gamma' = \cup_{j=1}^m (\Gamma + z_j) \subset \mathbb{C}^p$ is tame according to Proposition 4.1 in [5]. (The cited proposition is stated for $p = 2$, but the proof remains valid also for $p > 2$.) Hence the complement $Y = \mathbb{C}^p \backslash \Gamma'$ admits a dominating spray and therefore satisfies the Oka property [28], [21]. Since $\pi|_Y \colon Y \to \mathbb{T}^p \backslash \{t_1, \dots, t_m\}$ is a

holomorphic covering projection, Theorem 1.4 implies that the latter set also enjoys the Oka property.

The same argument applies if the lattice Γ has less than maximal rank.

(iii) A *Hirzebruch surface* H_l $(l = 0, 1, 2, \dots)$ is the total space Y of a holomorphic fiber bundle $Y \to \mathbb{P}_1$ with fiber $\mathbb{P}_1$ ([3, p. 191]; every Hirzebruch surface is birationally equivalent to $\mathbb{P}_2$). Since the base and the fiber are complex homogeneous, the conclusion follows from Theorem 1.4. $\square$

In this paper, an *unramified holomorphic fibration* will mean a surjective holomorphic submersion $\pi\colon Y \to Y'$ which is also a *Serre fibration* (i.e., it satisfies the homotopy lifting property; see [45, p. 8]). The latter condition holds if π is a topological fiber bundle in which the holomorphic type of the fiber may depend on the base point. (Ramified fibrations, or fibrations with multiple fibers, do not seem amenable to our methods and will not be discussed; see example 6.3 and problem 6.7 in [18].) In order to generalize Theorem 1.4 to such fibration we must assume that the fibers of π over small open subsets of the base manifold Y' satisfy certain condition, analogous to CAP, which allows holomorphic approximation of local sections. The weakest known sufficient condition is *subellipticity* [13], a generalization of Gromov's ellipticity [28]. We recall the relevant definitions.

Let $\pi\colon Y \to Y'$ be a holomorphic submersion onto Y'. For each $y \in Y$ let $VT_yY = \ker d\pi_y \subset T_yY$ (the *vertical tangent space* of Y with respect to π). A *fiber-spray* associated to $\pi\colon Y \to Y'$ is a triple (E, p, s) consisting of a holomorphic vector bundle $p\colon E \to Y$ and a holomorphic spray map $s\colon E \to Y$ such that for each $y \in Y$ we have $s(0_y) = y$ and $s(E_y) \subset Y_{\pi(y)} = \pi^{-1}(\pi(y))$. A spray on a complex manifold Y is a fiber-spray associated to the trivial submersion $Y \to \text{point}$.

Definition 1.6 ([13, p. 529]). A holomorphic submersion $\pi\colon Y \to Y'$ is *subelliptic* if each point in Y' has an open neighborhood $U \subset Y'$ such that the restricted submersion $h\colon Y|_U = h^{-1}(U) \to U$ admits finitely many fiber-sprays (E_j, p_j, s_j) $(j = 1, \dots, k)$ satisfying the domination condition

$$(ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) + \cdots + (ds_k)_{0_y}(E_{k,y}) = VT_yY \qquad (1.3)$$

for each $y \in Y|_U$; such a collection of sprays is said to be *fiber-dominating*. The submersion is *elliptic* if the above holds with $k = 1$. A complex manifold Y is *(sub-)elliptic* if the trivial submersion $Y \to \text{point}$ is such.

A holomorphic fiber bundle $Y \to Y'$ is *(sub-)elliptic* when its fiber is such.

Definition 1.7. A holomorphic map $\pi\colon Y \to Y'$ is a *subelliptic Serre fibration* if it is a surjective subelliptic submersion and a Serre fibration.

The following result is proved in Section 4 below (see also [38]).

THEOREM 1.8. *If $\pi\colon Y \to Y'$ is a subelliptic Serre fibration then Y satisfies the Oka property if and only if Y' does. This holds in particular if π is an unramified elliptic fibration (i.e., every fiber $\pi^{-1}(y')$ is an elliptic curve).*

Organization of the paper. In Section 2 we state a generalized Cartan lemma used in the proof of Theorem 1.2, indicating how it follows from Theorem 4.1 in [16]. Theorem 1.2 (which includes Theorem 0.1) is proved in Section 3. In Section 4 we prove Theorems 1.4 and 1.8. In Section 5 we discuss the parametric case and prove that CAP implies the one-parametric Oka property (Theorem 5.3). Section 6 contains a discussion and a list of open problems.

2. A Cartan type splitting lemma

Let A and B be compact sets in a complex manifold X satisfying the following:

(i) $A \cup B$ admits a basis of Stein neighborhoods in X, and

(ii) $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ (the separation property).

Such (A, B) will be called a *Cartan pair* in X. (The definition of a Cartan pair often includes an additional Runge condition; this will not be necessary here.) Set $C = A \cap B$. Let D be a compact set with a basis of open Stein neighborhoods in a complex manifold T. With these assumptions we have the following.

LEMMA 2.1. *Let $\gamma(x, t) = (x, c(x, t)) \in X \times T$ $(x \in X,\ t \in T)$ be an injective holomorphic map in an open neighborhood $\Omega_C \subset X \times T$ of $C \times D$. If γ is sufficiently uniformly close to the identity map on Ω_C then there exist open neighborhoods $\Omega_A, \Omega_B \subset X \times T$ of $A \times D$, respectively of $B \times D$, and injective holomorphic maps $\alpha\colon \Omega_A \to X \times T$, $\beta\colon \Omega_B \to X \times T$ of the form $\alpha(x, t) = (x, a(x, t))$, $\beta(x, t) = (x, b(x, t))$, which are uniformly close to the identity map on their respective domains and satisfy*

$$\gamma = \beta \circ \alpha^{-1}$$

in a neighborhood of $C \times D$ in $X \times T$.

In the proof of Theorem 1.2 (§3) we shall use Lemma 2.1 with D a cube in $T = \mathbb{C}^p$ for various values of $p \in \mathbb{N}$. Lemma 2.1 generalizes the classical Cartan lemma (see e.g. [29, p. 199]) in which A, B and $C = A \cap B$ are cubes in $\mathbb{C}^n$ and a, b, c are invertible linear functions of $t \in \mathbb{C}^p$ depending holomorphically on the base variable.

Proof. Lemma 2.1 is a special case of Theorem 4.1 in [16]. In that theorem we consider a Cartan pair (A, B) in a complex manifold X and a nonsingular

holomorphic foliation $\mathcal{F}$ in an open neighborhood of $A \cup B$ in X. Let $U \subset X$ be an open neighborhood of $C = A \cap B$ in X. By Theorem 4.1 in [16], every injective holomorphic map $\gamma \colon U \to X$ which is sufficiently uniformly close to the identity map on U admits a splitting $\gamma = \beta \circ \alpha^{-1}$ on a smaller open neighborhood of C in X, where α (resp. β) is an injective holomorphic map on a neighborhood of A (resp. B), with values in X. If in addition γ preserves the plaques of $\mathcal{F}$ in a certain finite system of foliation charts covering U (i.e., x and $\gamma(x)$ belong to the same plaque) then α and β can be chosen to satisfy the same property.

Lemma 2.1 follows by applying this result to the Cartan pair $(A \times D, B \times D)$ in $X \times T$, with $\mathcal{F}$ the trivial (product) foliation of $X \times T$ with leaves $\{x\} \times T$. □

Certain generalizations of Lemma 2.1 are possible (see [16]). First of all, the analogous result holds in the parametric case. Secondly, if Σ is a closed complex subvariety of $X \times T$ which does not intersect $C \times D$ then α and β can be chosen tangent to the identity map to a given finite order along Σ. Thirdly, shrinking of the domain is necessary only in the directions of the leaves of $\mathcal{F}$; an analogue of Lemma 2.1 can be proved for maps which are holomorphic in the interior of the respective set A, B, or C and of a Hölder class $\mathbb{C}^{k,\epsilon}$ up to the boundary. (The $\overline{\partial}$-problem which arises in the linearization is well behaved on these spaces.) We do not state or prove this generalization formally since it will not be needed in the present paper.

3. Proof of Theorem 1.2

The proof relies on Grauert's bumping method which has been introduced to the Oka-Grauert problem by Henkin and Leiterer [31] (their paper is based on a preprint from 1986), with several additions from [16] and [17].

Assume that Y is a complex manifold satisfying CAP. Let X be a Stein manifold, $K \subset X$ a compact $\mathcal{O}(X)$-convex subset of X and $f \colon X \to Y$ a continuous map which is holomorphic in an open set $U \subset X$ containing K. We shall modify f in a countable sequence of steps to obtain a holomorphic map $X \to Y$ which is homotopic to f and approximates f uniformly on K. (In fact, the entire homotopy will remain holomorphic and uniformly close to f on K.) The goal of every step is to enlarge the domain of holomorphicity and thus obtain a sequence of maps $X \to Y$ which converges uniformly on compacts in X to a solution of the problem.

Choose a smooth strongly plurisubharmonic Morse exhaustion function $\rho \colon X \to \mathbb{R}$ such that $\rho|_K < 0$ and $\{\rho \leq 0\} \subset U$. Set $X_c = \{\rho \leq c\}$ for $c \in \mathbb{R}$. It suffices to prove that for any pair of numbers $0 \leq c_0 < c_1$ such that c_0 and c_1 are regular values of ρ, a continuous map $f \colon X \to Y$ which is holomorphic on (an open neighborhood of) X_{c_0} can be deformed by a homotopy of maps

$f_t \colon X \to Y$ ($t \in [0,1]$) to a map f_1 which is holomorphic on X_{c_1}; in addition we require that f_t be holomorphic and uniformly as close as required to $f = f_0$ on X_{c_0} for every $t \in [0,1]$. The solution is then obtained by an obvious induction as in [21].

There are two main cases to consider:

The noncritical case. $d\rho \neq 0$ on the set $\{x \in X \colon c_0 \leq \rho(x) \leq c_1\}$.

The critical case.   There is a point $p \in X$ with $c_0 < \rho(p) < c_1$ such that $d\rho_p = 0$. (We may assume that there is a unique such p.)

A reduction of the critical case to the noncritical one has been explained in Section 6 of [17], based on a technique developed in the construction of holomorphic submersions of Stein manifolds to Euclidean spaces [16]. It is accomplished in the following three steps, the first two of which do not require any special properties of Y.

Step 1.   Let $f \colon X \to Y$ be a continuous map which is holomorphic in a neighborhood of $X_c = \{\rho \leq c\}$ for some $c < \rho(p)$ close to $\rho(p)$. By a small modification we make f smooth on a totally real handle E attached to X_c and passing through the critical point p. (In suitable local holomorphic coordinates on X near p, this handle is just the stable manifold of p for the gradient flow of ρ, and its dimension equals the Morse index of ρ at p.)

Step 2.   We approximate f uniformly on $X_c \cup E$ by a map which is holomorphic in an open neighborhood of this set (Theorem 3.2 in [17]).

Step 3.   We approximate the map in Step 2 by a map holomorphic on $X_{c'}$ for some $c' > \rho(p)$. This extension across the critical level $\{\rho = \rho(p)\}$ is obtained by applying the noncritical case for another strongly plurisubharmonic function constructed especially for this purpose.

After reaching $X_{c'}$ for some $c' > \rho(p)$ we revert back to ρ and continue (by the noncritical case) to the next critical level of ρ, thus completing the induction step. The details can be found in Section 6 in [16] and [17].

It remains to explain the noncritical case; here our proof differs from the earlier proofs (see e.g. [21] and [13]).

Let $z = (z_1, \dots, z_n)$, $z_j = u_j + iv_j$, denote the coordinates on $\mathbb{C}^n$, $n = \dim X$. Let P denote the open cube

$$P = \{z \in \mathbb{C}^n \colon |u_j| < 1, \ |v_j| < 1, \ j = 1, \dots, n\} \qquad (3.1)$$

and $P' = \{z \in P \colon v_n = 0\}$. Let A be a compact strongly pseudoconvex domain with smooth boundary in X. We say that a compact subset $B \subset X$ is a *convex bump* on A if there exist an open neighborhood $V \subset X$ of B, a

biholomorphic map $\phi\colon V \to P$ onto the set (3.1) and smooth strongly concave functions $h, \widetilde{h}\colon P' \to [-s, s]$ for some $0 < s < 1$ such that $h \le \widetilde{h}$, $h = \widetilde{h}$ near the boundary of P', and

$$\phi(A \cap V) = \{z \in P\colon v_n \le h(z_1, \dots, z_{n-1}, u_n)\},$$
$$\phi((A \cup B) \cap V) = \{z \in P\colon v_n \le \widetilde{h}(z_1, \dots, z_{n-1}, u_n)\}.$$

We also require that

(i) $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ (the separation condition), and

(ii) $C = A \cap B$ is Runge in A, in the sense that every holomorphic function in a neighborhood of C can be approximated uniformly on C by functions holomorphic in a neighborhood of A.

PROPOSITION 3.1. *Assume that $A, B \subset X$ are as above. Let Y be a p-dimensional complex manifold satisfying* CAP_{n+p}. *Choose a distance function d on Y induced by a Riemannian metric. For every holomorphic map $f_0\colon A \to Y$ and every $\epsilon > 0$ there is a holomorphic map $f_1\colon A \cup B \to Y$ satisfying* $\sup_{x \in A} d(f_0(x), f_1(x)) < \epsilon$. *The analogous result holds for sections of a holomorphic fiber bundle $Z \to X$ with fiber Y which is trivial over the set $V \supset B$.*

If f_0 and f_1 are sufficiently uniformly close on A, there clearly exists a holomorphic homotopy from f_0 to f_1 on A. If Y satisfies CAP_N with $N = p + [\frac{1}{2}(3n + 1)]$ then we may omit the hypothesis that C is Runge in A (Remark 3.3).

Assuming Proposition 3.1 we can complete the proof of the noncritical case (and hence of Theorem 1.2) as follows. By Narasimhan's lemma on local convexification of strongly pseudoconvex domains one can obtain a finite sequence $X_{c_0} = A_0 \subset A_1 \subset \dots \subset A_{k_0} = X_{c_1}$ of compact strongly pseudoconvex domains in X such that for every $k = 0, 1, \dots, k_0 - 1$ we have $A_{k+1} = A_k \cup B_k$ where B_k is a convex bump on A_k (Lemma 12.3 in [32]). Each of the sets B_k may be chosen sufficiently small so that it is contained in an element of a given open covering of X. The separation condition (i) for the pair (A_k, B_k), introduced just before Proposition 3.1, is trivial to satisfy while (ii) is only a small addition (one can use a local convexification of a strongly pseudoconvex domain A given by holomorphic functions defined in a neighborhood of A; see [10, p. 530, Prop. 1], or [30, Prop. 14]). It remains to apply Proposition 3.1 inductively to every pair (A_k, B_k), $k = 0, 1, \dots, k_0 - 1$. A more detailed exposition of this construction can be found in [21] and [17].

This completes the proof of Theorem 1.2 provided that Proposition 3.1 holds.

Proof of Proposition 3.1. Choose a pair of numbers r, r', with $0 < r' < r < 1$, such that $\phi(B) \subset r'\overline{P}$. The set

$$Q := \phi(A \cap V) \cap r\overline{P} = \{z \in r\overline{P} \colon v_n \le h(z', u_n)\}$$

is a special convex set in $\mathbb{C}^n$ (1.2) with respect to the closed cube $r\overline{P} \subset \mathbb{C}^n$, and the set $C = A \cap B$ is contained in $Q_0 := \phi^{-1}(Q) \subset X$.

By the hypothesis f_0 is holomorphic in an open neighborhood $U \subset X$ of A. Set $F_0(x) = (x, f_0(x)) \in X \times Y$ for $x \in U$.

LEMMA 3.2. *There are a neighborhood $U_1 \subset U$ of A in X, a neighborhood $W \subset \mathbb{C}^p$ of $0 \in \mathbb{C}^p$ and a holomorphic map $F(x, t) = (x, f(x, t)) \in X \times Y$, defined for $x \in U_1$ and $t \in W$, such that $f(\cdot, 0) = f_0$ and $f(x, \cdot) \colon W \to Y$ is injective holomorphic for every x in a neighborhood of $C = A \cap B$.*

Proof. The set $F_0(U)$ is a closed Stein submanifold of the complex manifold $U \times Y$ and hence it admits an open Stein neighborhood in $U \times Y$ according to [44]. Let $\pi_X \colon X \times Y \to X$ denote the projection $(x, y) \to x$. The set $E = \ker d\pi_X$ is a holomorphic vector subbundle of rank $p = \dim Y$ in the tangent bundle $T(X \times Y)$, consisting of all vectors $\xi \in T(X \times Y)$ which are tangent to the fibers of π_X.

Since the set Q_0 is contractible, the bundle E is trivial over a neighborhood of $F_0(Q_0)$ in $X \times Y$ and hence is generated there by p holomorphic sections, i.e., vector fields tangent to the fibers of π_X. Since C is Runge in A, these sections can be approximated uniformly on $F_0(C)$ by holomorphic sections $\xi_1, \dots, \xi_p$ of E, defined in a neighborhood of $F_0(A)$ in $X \times Y$, which still generate E over a neighborhood of $F_0(C)$. The flow θ_t^j of ξ_j is well defined for sufficiently small $t \in \mathbb{C}$. The map

$$F(x, t_1, \dots, t_p) = \theta_{t_1}^1 \circ \cdots \circ \theta_{t_p}^p \circ F_0(x) \in X \times Y,$$

defined and holomorphic for x in a neighborhood of A and for $t = (t_1, \dots, t_p)$ in a neighborhood of the origin in $\mathbb{C}^p$, satisfies Lemma 3.2. $\square$

Remark 3.3. The restriction of a rank p holomorphic vector bundle E to an n-dimensional Stein manifold is generated by $p + [\frac{1}{2}(n + 1)]$ sections (Lemma 5 in [11, p. 178]). Without assuming that C is Runge in A this gives a proof of Lemma 3.2 if Y satisfies CAP_N with $N = p + [\frac{1}{2}(3n + 1)]$.

We continue with the proof of Proposition 3.1. Let F and W be as in Lemma 3.2. Choose a closed cube D in $\mathbb{C}^p$ centered at 0, with $D \subset W$. The set $\widetilde{Q} := Q \times D \subset \mathbb{C}^{n+p}$ is a special convex set of the form (1.2) with respect to the closed cube $\widetilde{P} := r\overline{P} \times D \subset \mathbb{C}^{n+p}$, and the map $\widetilde{f}(z, t) := f(\phi^{-1}(z), t) \in Y$ is holomorphic in a neighborhood of $\widetilde{Q}$.

Since Y is assumed to satisfy CAP_{n+p}, we can approximate $\widetilde{f}$ uniformly on a neighborhood of $\widetilde{Q}$ by entire maps $\widetilde{g} \colon \mathbb{C}^{n+p} \to Y$. (This is the only place

in the proof where CAP is used.) The holomorphic map

$$g(x,t) := \widetilde{g}(\phi(x),t) \in Y, \qquad x \in V, \ t \in \mathbb{C}^p$$

then approximates f uniformly in a neighborhood of $Q_0 \times D$ in $X \times \mathbb{C}^p$. Since $f(x,\cdot) \colon W \to Y$ is injective holomorphic for every x in a neighborhood of C (Lemma 3.2), choosing g to approximate f sufficiently well we obtain a (unique) injective holomorphic map $\gamma(x,t) = (x,c(x,t)) \in X \times \mathbb{C}^p$, defined and uniformly close to the identity map in an open neighborhood $\Omega \subset X \times \mathbb{C}^p$ of $C \times D$, such that

$$f(x,t) = (g \circ \gamma)(x,t) = g(x,c(x,t)), \quad (x,t) \in \Omega. \tag{3.2}$$

If the approximation of f by g is sufficiently close then γ is so close to the identity map that we can apply Lemma 2.1 to obtain a decomposition $\gamma = \beta \circ \alpha^{-1}$, with $\alpha(x,t) = (x,a(x,t))$, $\beta(x,t) = (x,b(x,t))$ holomorphic and close to the identity maps in their respective domains $\Omega_A \supset A \times D$, $\Omega_B \supset B \times D$. From (3.2) we obtain

$$f(x,a(x,t)) = g(x,b(x,t)), \quad (x,t) \in C \times D.$$

When $t = 0$, the two sides define a holomorphic map $f_1 \colon A \cup B \to Y$ which approximates $f_0 = f(\cdot,0)$ uniformly on A (since $a(x,0) \approx 0$ for $x \in A$).

This proves Proposition 3.1 for maps $X \to Y$. The very same proof applies to sections of a holomorphic fiber bundle $Z \to X$ with fiber Y which is trivial over the set $V \supset B$; this is no restriction since all convex bumps in the inductive construction can be chosen small enough to insure this condition.

4. Proof of Theorems 1.4 and 1.8

We begin by proving Theorem 1.8. Let $\pi \colon Y \to Y'$ be a subelliptic Serre fibration (Definition 1.7). Assume first that Y' satisfies CAP. Let $f \colon U \to Y$ be a holomorphic map from an open convex subset $U \subset \mathbb{C}^n$. Let $K \subset L$ be compact convex sets in U, with $K \subset \operatorname{Int} L$. Set $g = \pi \circ f \colon U \to Y'$.

Since Y' satisfies CAP, there is an entire map $g_1 \colon \mathbb{C}^n \to Y'$ which approximates g uniformly on L.

By Lemma 3.4 in [17] there exists for every $x \in U$ a holomorphic retraction ρ_x of an open neighborhood of the fiber $R_x = \pi^{-1}(g_1(x)) \subset Y$ in the manifold Y onto R_x, with ρ_x depending holomorphically on $x \in U$. If g_1 is sufficiently uniformly close to g on L then for every $x \in L$ the point $f(x)$ belongs to the domain of ρ_x, and hence we can define $f_1(x) = \rho_x(f(x))$ for all $x \in L$. The map f_1 is then holomorphic on a neighborhood of K in X, it approximates f uniformly on K, and it satisfies $\pi \circ f_1 = g_1$ (i.e., f_1 is a lifting of g_1).

Since $\pi \colon Y \to Y'$ is a Serre fibration and the set $K \subset \mathbb{C}^n$ is convex, $f_1|_K$ extends to a continuous map $f_1 \colon \mathbb{C}^n \to Y$ which is holomorphic in a

neighborhood of K and satisfies $\pi \circ f_1 = g_1$ on $\mathbb{C}^n$ (hence f_1 is a global lifting of g_1).

Since g_1 is holomorphic and π is a subelliptic submersion, Theorem 1.3 in [14] shows that we can homotopically deform f_1 (through liftings of g_1) to a global holomorphic lifting $\widetilde{f} \colon \mathbb{C}^n \to Y$ of g_1 (i.e., $\pi \circ \widetilde{f} = g_1$) such that $\widetilde{f}|_K$ approximates $f_1|_K$, and hence $f|_K$. (In our case π is unramified and the quoted theorem from [14] is an immediate consequence of Theorem 1.5 in [22].) This shows that Y satisfies CAP and hence the Oka property.

Conversely, assume that Y satisfies CAP. Choose a holomorphic map $g \colon K \to Y'$ from a compact convex set $K \subset \mathbb{C}^n$. Since π is a Serre fibration and K is contractible, there is a continuous lifting $f_0 \colon K \to Y$ with $\pi \circ f_0 = g$. Since π is a subelliptic submersion, Theorem 1.3 in [14] gives a homotopy of liftings $f_t \colon K \to Y$ ($t \in [0,1]$), with $\pi \circ f_t = g$ for every $t \in [0,1]$, such that f_1 is holomorphic on K.

By CAP of Y we can approximate f_1 uniformly on K by entire maps $\widetilde{f} \colon \mathbb{C}^n \to Y$. The map $\widetilde{g} := \pi \circ \widetilde{f} \colon \mathbb{C}^n \to Y'$ is then entire and it approximates g uniformly on K. Thus Y' satisfies CAP.

Note that contractibility of K was essential in the last part of the proof.

Every unramified elliptic fibration $\pi \colon Y \to Y'$ without exceptional (and multiple) fibers is elliptic in the sense of Gromov [28] (Definition 1.6 above). Indeed, every fiber $Y_y = \pi^{-1}(y)$ ($y \in Y'$) is an elliptic curve, $Y_y = \mathbb{C}/\Gamma_y$, and the lattice $\Gamma_y \subset \mathbb{C}$ is defined over every sufficiently small open subset $U \subset Y'$ by a pair of generators $a(y)$, $b(y)$ depending holomorphically on y. A dominating fiber-spray on $Y|_U$ is obtained by pushing down to $Y|_U$ the Γ_y-equivariant spray on $U \times \mathbb{C}$ defined by $((y,t),t') \in U \times \mathbb{C} \times \mathbb{C} \to (y, t+t') \in U \times \mathbb{C}$.

The proof of Theorem 1.4 follows the same scheme; in this case we do not need to refer to [22] but can instead use Theorem 1.2 in this paper.

5. The parametric convex approximation property

We recall the notion of the *parametric Oka property* (POP) which is essentially the same as Gromov's Ell_∞ property ([28, §3.1]; see also Theorem 1.5 in [22]).

Let P be a compact Hausdorff space (the parameter space) and P_0 a closed subset of P (possibly empty) which is a strong deformation retract of some neighborhood in P. In applications P is usually a polyhedron and P_0 a subpolyhedron.

Given a Stein manifold X and a compact $\mathcal{O}(X)$-convex subset K in X, we consider a continuous map $f \colon X \times P \to Y$ such that for every $p \in P$ the map $f^p = f(\cdot\,,p) \colon X \to Y$ is holomorphic in an open neighborhood of K in X (independent of $p \in P$), and for every $p \in P_0$ the map f^p is holomorphic on X. We say that Y satisfies the *parametric Oka property* (POP) if for all such data

(X, K, P, P_0, f) there is a homotopy $f_t \colon X \times P \to Y$ $(t \in [0,1])$, consisting of maps satisfying the same properties as $f_0 = f$, such that

— the homotopy is fixed on P_0 (i.e., $f_t^p = f^p$ when $p \in P_0$ and $t \in [0,1]$),

— f_t approximates f uniformly on $K \times P$ for all $t \in [0,1]$, and

— $f_1^p \colon X \to Y$ is holomorphic for every $p \in P$.

Recall that POP is implied by ellipticity [28], [21] and subellipticity [14].

We say that a complex manifold Y satisfies the *parametric convex approximation property* (PCAP) if the above holds for every special convex set K of the form (1.2) in $X = \mathbb{C}^n$ for any $n \in \mathbb{N}$.

THEOREM 5.1. *If a complex manifold* Y *satisfies* PCAP *then it also satisfies the parametric Oka property* (*and hence* PCAP $\iff$ POP).

Theorem 5.1 is obtained by following the proof of Theorem 1.2 (§3) but using the requisite tools with continuous dependence on the parameter $p \in P$. Precise arguments of this kind can be found in [21], [22] and we leave out the details. For an additional equivalence involving interpolation conditions see Theorem 6.1 in [19].

An analogue of Theorem 1.8 holds for ascending/descending of the POP in a subelliptic Serre fibration $\pi \colon Y \to Y'$. The implication

$$\text{POP of } Y' \implies \text{POP of } Y$$

holds for any compact Hausdorff parameter space P and is proved as before by using the parametric versions of the relevant tools. However, we can prove the converse implication only for a *contractible* parameter space P, the reason being that we must lift a map $K \times P \to Y'$ (with K a compact convex set in $\mathbb{C}^n$) to a map $K \times P \to Y$. (See also Corollary 6.2 in [19].)

Question 5.2. To what extent does CAP imply PCAP?

We indicate how CAP$\Rightarrow$PCAP can be proved for sufficiently simple parametric spaces. For simplicity let P be a closed cube in $\mathbb{R}^k$ and $P_0 = \emptyset$, although the argument applies in more general situations. We identify $\mathbb{R}^k$ with $\mathbb{R}^k \times \{i0\}^k \subset \mathbb{C}^k$.

Let $K \subset \mathbb{C}^n$ be a special compact convex set, $U \subset \mathbb{C}^n$ an open neighborhood of K, and $f \colon U \times P \to Y$ a continuous map such that $f_p = f(\cdot, p)$ is holomorphic on U for every fixed $p \in P$. By the assumed CAP property of Y we can approximate f_p for every fixed $p \in P$ uniformly on K by a map with values in Y which is holomorphic in an open neighborhood of $K \times \{p\}$ in $\mathbb{C}^n \times \mathbb{C}^k$. Patching these holomorphic approximations by a smooth partition of unity in the p-variable we approximate the initial map f by another one,

still denoted f, which is smooth in all variables and is holomorphic in the x variable for every fixed $p \in P$.

The graph of f over $U \times P$ is a smooth CR submanifold of $\mathbb{C}^{n+k} \times Y$ foliated by n-dimensional complex manifolds, namely the graphs of $f_p \colon U \to Y$ for $p \in P$. By methods similar to those in [20] it can be seen that the graph of f over $K \times P$ admits an open Stein neighborhood Ω in $\mathbb{C}^{n+k} \times Y$. Embedding Ω into a Euclidean space $\mathbb{C}^N$ and applying standard approximation methods for CR functions (and a holomorphic retraction of a tube around the submanifold $\Omega \subset \mathbb{C}^N$ onto Ω) we can approximate f as closely as desired on $K \times P$ by a holomorphic map $\widetilde{f}$, defined in an open neighborhood of $K \times P$ in $\mathbb{C}^n \times \mathbb{C}^p$.

The cube $P \subset \mathbb{R}^k \subset \mathbb{C}^k$ admits a basis of cubic neighborhoods in $\mathbb{C}^k$. (By a 'cube' we mean a Cartesian product of intervals in the coordinate axes.) The product of K with a closed cube in $\mathbb{C}^k$ is a special compact convex set in $\mathbb{C}^{n+k}$. Applying the CAP property of Y to the map $\widetilde{f}$ we see that Y satisfies PCAP for the parameter space P.

If $P = [0,1] \subset \mathbb{R}$ and the maps $f_0 = f(\,\cdot\,, 0)$ and $f_1 = f(\,\cdot\,, 1)$ (corresponding to the endpoints of P) are holomorphic on $\mathbb{C}^n$, the above construction can be performed so that these two maps remain unchanged, thereby showing that the basic CAP implies the one-parametric CAP. Joined with Theorem 5.1 this gives

THEOREM 5.3. *If a complex manifold Y enjoys the* CAP *then a homotopy of maps $f_t \colon X \to Y$ ($t \in [0,1]$) from a Stein manifold X for which f_0 and f_1 are holomorphic can be deformed with fixed ends to a homotopy consisting of holomorphic maps.*

Theorem 5.3 also follows from Theorem 1.1 in [19] to the effect that CAP *implies the Oka property with interpolation.* Indeed, extending the homotopy $f_t \colon X \to Y$ ($t \in [0,1]$) to all values $t \in \mathbb{C}$ by precomposing with a retraction $\mathbb{C} \to [0,1] \subset \mathbb{C}$ we obtain a continuous map $F \colon X \times \mathbb{C} \to Y$, $F(x,t) = f_t(x)$, whose restriction to the complex submanifold $X_0 = X \times \{0,1\}$ of $X \times \mathbb{C}$ is holomorphic. By [19] there is a homotopy $F_s \colon X \times \mathbb{C} \to Y$ ($s \in [0,1]$), with $F_0 = F$, which remains fixed on X_0 and such that F_1 is holomorphic. The restriction of F_1 to $X \times [0,1]$ is a homotopy from f_0 to f_1 consisting of holomorphic maps $F_1(\,\cdot\,, t)$ ($t \in [0,1]$).

6. Discussion, examples and problems

It was pointed out by Gromov [28] that the existence of a dominating spray on a complex manifold Y is a precise way of saying that Y admits many holomorphic maps from Euclidean spaces; since every Stein manifold X embeds into a Euclidean space, this also implies the existence of many holomorphic maps $X \to Y$ and hence it is natural to expect that Y enjoys the Oka property (and it does).

The same philosophy justifies CAP which is another way of asserting the existence of many holomorphic maps $\mathbb{C}^N \to Y$. Indeed, CAP is the restriction of the Oka property (which refers to maps from *any* Stein manifold X to Y, with uniform approximation on *any* holomorphically convex subset K of X) to model pairs — the special compact convex sets in $X = \mathbb{C}^n$. For a discussion of this *localization principle* see Remark 1.10 in [18].

CAP is in a precise sense opposite to the hyperbolicity properties expressed by nonvanishing of Kobayashi-Eisenman metrics. More precisely, CAP_1 is an opposite property to *Kobayashi-Brody hyperbolicity* [34], [4] which excludes nonconstant entire maps $\mathbb{C} \to Y$; more generally, CAP_n for $n \leq \dim Y$ is opposite to the n-dimensional measure hyperbolicity [8]. The property CAP_p with $p = \dim Y$ implies the existence of *dominating* holomorphic maps $\mathbb{C}^p \to Y$; if such Y is compact, it is not of Kodaira general type [6], [36], [35]. For a further discussion see [18].

The property CAP_n for $n \geq \dim Y$ is also reminiscent of the Property S_n, introduced in [17], which requires that any holomorphic submersion $f \colon K \to Y$ from a special compact convex set $K \subset \mathbb{C}^n$ is approximable by entire submersions $\mathbb{C}^n \to Y$. By Theorem 2.1 in [17], Property S_n of Y implies that holomorphic submersions from any n-dimensional Stein manifold to Y satisfy the homotopy principle, analogous to the one which was proved for smooth submersions by Gromov [27] and Phillips [41]. The similarity is not merely apparent — our proof of Theorem 1.2 in this paper conceptually unifies the construction of holomorphic maps with the construction of holomorphic submersions in [16] and [17].

Example 6.1. For every $1 \leq k \leq p$ there exists a p-dimensional complex manifold which satisfies CAP_{k-1} but not CAP_k.

Indeed, for $k = p$ we can take $Y = \mathbb{C}^p \backslash A$ where A is a discrete subset of $\mathbb{C}^p$ which is *rigid* in the sense of Rosay and Rudin [42, p. 60], i.e., every holomorphic map $\mathbb{C}^p \to \mathbb{C}^p$ with maximal rank p at some point intersects A at infinitely many points. Thus CAP_p fails, but CAP_{p-1} holds since a generic holomorphic map $\mathbb{C}^{p-1} \to \mathbb{C}^p$ avoids A by dimension reasons. For $k < p$ we take $Y = \mathbb{C}^p \backslash \phi(\mathbb{C}^{p-k})$ where $\phi \colon \mathbb{C}^{p-k} \hookrightarrow \mathbb{C}^p$ is a proper holomorphic embedding whose complement is k-hyperbolic (every entire map $\mathbb{C}^k \to \mathbb{C}^p$ whose range omits $\phi(\mathbb{C}^{p-k})$ has rank $< k$; such maps were constructed in [12]); again CAP_k fails but CAP_{k-1} holds by dimension reasons. Another example is $Y = (\mathbb{C}^k \backslash A) \times \mathbb{C}^{p-k}$ where A is a rigid discrete set in $\mathbb{C}^k$.

We conclude by mentioning a few open problems.

Problem 6.2. Do the CAP_n properties stabilize at some integer, i.e., is there a $p \in \mathbb{N}$ depending on Y (or perhaps only on $\dim Y$) such that $\mathrm{CAP}_p \implies \mathrm{CAP}_n$ for all $n > p$? Does this hold for $p = \dim Y$?

Problem 6.3. Let B be a closed ball in $\mathbb{C}^p$ for some $p \geq 2$. Does $\mathbb{C}^p \backslash B$ satisfy CAP (and hence the Oka property)? Does $\mathbb{C}^p \backslash B$ admit any nontrivial sprays?

The same problem makes sense for every compact convex set $B \subset \mathbb{C}^p$. What makes this problem particularly intriguing is the absence of any obvious obstruction; indeed, $\mathbb{C}^p \backslash B$ is a union of Fatou-Bieberbach domains [42].

Problem 6.4 (Gromov [28, p. 881, 3.4.(D)]). Suppose that every holomorphic map from a ball $B \subset \mathbb{C}^n$ to Y (for any $n \in \mathbb{N}$) can be approximated by entire maps $\mathbb{C}^n \to Y$. Does Y enjoy the Oka property?

Problem 6.5. Let $\pi \colon Y \to Y_0$ be a holomorphic fiber bundle. Does the Oka property of Y imply the Oka property of the base Y_0 and of the fiber?

Remark 6.6 (Mappings from Stein spaces). Although we have stated our results only for mappings from *Stein manifolds*, it is not difficult to see that the CAP property of a complex manifold Y implies the Oka property for maps $X \to Y$ also when X is a (reduced, finite dimensional) *Stein space* with singularities. To this end we choose a stratification $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ by a finite descending sequence of closed Stein subspaces $X_j \subset X$ such that $\dim X_0 = 0$ and $X_j \backslash X_{j-1}$ is nonsingular for $j = 1, 2, \ldots, n$. Assuming that our map $X \to Y$ has already been made holomorphic on X_{j-1}, the methods in this paper (and in the sequel [19] where the interpolation is explained more carefully) allow us to make it holomorphic on X_j by a homotopy that is fixed on X_{j-1}. A more precise outline of this proof (in the context of stratified submersions with sprays) is found in Section 7 of [23].

INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, UNIVERSITY OF LJUBLJANA, LJUBLJANA, SLOVENIA
E-mail address: franc.forstneric@fmf.uni-lj.si

REFERENCES

[1]   E. ANDERSÉN, Volume-preserving automorphisms of $\mathbb{C}^n$, *Complex Variables* **14** (1990), 223–235.

[2]   E. ANDERSÉN and L. LEMPERT, On the group of holomorphic automorphisms of $\mathbb{C}^n$, *Invent. Math.* **110** (1992), 371–388.

[3]   W. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact Complex Surfaces*, 2nd ed., Springer-Verlag, New York, 2004.

[4]   R. Brody, Compact manifolds and hyperbolicity, *Trans. Amer. Math. Soc.* **235** (1978), 213–219.

[5]   G. Buzzard and S. S. Y. Lu, Algebraic surfaces holomorphically dominable by $\mathbb{C}^2$, *Invent. Math.* **139** (2000), 617–659.

[6]   J. Carlson and P. Griffiths, A defect relation for equidimensional holomorphic mappings between algebraic varieties, *Ann. of Math.* **95** (1972), 557–584.

[7]   H. Cartan, Espaces fibrés analytiques, *Symposium Internat. de Topologia Algebraica*, Mexico, 97–121 (1958) (Also in H. Cartan, *Oeuvres*, Vol. 2, Springer-Verlag, New York, 1979).

[8]   D. Eisenman, Intrinsic measures on complex manifolds and holomorphic mappings, *Mem. Amer. Math. Soc.* **96**, A. M. S., Providence, R.I., 1970.

[9]   Y. Eliashberg and M. Gromov, Embeddings of Stein manifolds, *Ann. of Math.* **136** (1992), 123–135.

[10] J. E. Fornæss, Embedding strictly pseudoconvex domains in convex domains, *Amer. J. Math.* **98** (1976), 529–569.

[11] O. Forster, Plongements des variétés de Stein. *Comment. Math. Helv.* **45** (1970), 170–184.

[12] F. Forstnerič, Interpolation by holomorphic automorphisms and embeddings in $\mathbb{C}^n$, *J. Geom. Anal.* **9** (1999), 93–118.

[13] ———, The Oka principle for sections of subelliptic submersions, *Math. Z.* **241** (2002), 527–551.

[14] ———, The Oka principle for multivalued sections of ramified mappings, *Forum Math.* **15** (2003), 309–328.

[15] ———, The homotopy principle in complex analysis: A survey, in *Explorations in Complex and Riemannian Geometry*: *A Volume Dedicated to Robert E. Greene* (J. Bland, K.-T. Kim, and S. G. Krantz, eds.), *Contemporary Math.* **32**, 73–99, A. M. S., Providence, R.I., 2003.

[16] ———, Noncritical holomorphic functions on Stein manifolds, *Acta Math.* **191** (2003), 143–189.

[17] ———, Holomorphic submersions from Stein manifolds, *Ann. Inst. Fourier* **54** (2004), 1913–1942.

[18] ———, Holomorphic flexibility properties of complex manifolds, *Amer. J. Math.* **28** (2006), 239–270.

[19] ———, Extending holomorphic mappings from subvarieties in Stein manifolds, *Ann. Inst. Fourier* **55** (2005), 733–751.

[20] F. Forstnerič and C. Laurent-Thiébaut, Stein compacts in Levi-flat hypersurfaces, *Trans. Amer. Math. Soc.*, to appear; arXiv: math.CV/0410386.

[21] F. Forstnerič and J. Prezelj, Oka's principle for holomorphic fiber bundles with sprays. *Math. Ann.* **317** (2000), 117–154.

[22] ———, Oka's principle for holomorphic submersions with sprays, *Math. Ann.* **322** (2002), 633–666.

[23] ———, Extending holomorphic sections from complex subvarieties, *Math. Z.* **236** (2001), 43–68.

[24] H. Grauert, Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen, *Math. Ann.* **133** (1957), 139–159.

[25] ———, Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen, *Math. Ann.* **133** (1957), 450–472.

[26] H. Grauert, Analytische Faserungen über holomorph-vollständigen Räumen, *Math. Ann.* **135** (1958), 263–273.

[27] M. Gromov, Stable maps of foliations into manifolds, *Izv. Akad. Nauk, S.S.S.R.* **33** (1969), 707–734.

[28] ———, Oka's principle for holomorphic sections of elliptic bundles, *J. Amer. Math. Soc.* **2** (1989), 851–897.

[29] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[30] G. M. Henkin and E. M. Chirka, Boundary properties of holomorphic functions of several complex variables, in *Current Problems in Mathematics* **4**, 12–142 (Russian) (English transl. in *Soviet Math. J.* **5** (1976), 612–687).

[31] G. M. Henkin and J. Leiterer, The Oka-Grauert principle without induction over the basis dimension, *Math. Ann.* **311** (1998), 71–93.

[32] ———, *Andreotti-Grauert Theory by Integral Formulas*, *Progr. Math.* **74**, Birkhäuser, Boston, 1988.

[33] L. Hörmander, *Notions of Convexity*, Birkhäuser, Boston, 1994.

[34] S. Kobayashi, Intrinsic distances, measures and geometric function theory, *Bull. Amer. Math. Soc.* **82** (1976), 357–416.

[35] S. Kobayashi and T. Ochiai, Meromorphic mappings onto compact complex spaces of general type, *Invent. Math.* **31** (1975), 7–16.

[36] K. Kodaira, Pluricanonical systems on algebraic surfaces of general type, *J. Math. Soc. Japan* **20** (1968), 170–192.

[37] F. Lárusson, Model structures and the Oka principle, *J. Pure Appl. Algebra* **192** (2004), 203–223.

[38] ———, Mapping cylinders and the Oka principle, *Indiana Univ. Math. J.* **54** (2005), 1145–1159.

[39] J. Leiterer, Holomorphic vector bundles and the Oka-Grauert principle, *Encyclopedia of Mathematical Sciences* **10**, 63–103, *Several Complex Variables* IV, Springer-Verlag, New York, 1989.

[40] K. Oka, Sur les fonctions des plusieurs variables. III: Deuxième problème de Cousin, *J. Sc. Hiroshima Univ.* **9** (1939), 7–19.

[41] A. Phillips, Submersions of open manifolds, *Topology* **6** (1967), 170–206.

[42] J.-P. Rosay and W. Rudin, Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, *Trans. Amer. Math. Soc.* **310** (1988), 47–86.

[43] J. Schürmann, Embeddings of Stein spaces into affine spaces of minimal dimension, *Math. Ann.* **307** (1997), 381–399.

[44] J.-T. Siu, Every Stein subvariety admits a Stein neighborhood, *Invent. Math.* **38** (1976), 89–100.

[45] G. W. Whitehead, *Elements of Homotopy Theory. Grad. Texts in Math.* **61**, Springer-Verlag, New York, 1978.