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Logarithmic singularity of the Szegö kernel and a global invariant of strictly pseudoconvex domains

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1. Introduction

This paper is a continuation of Fefferman's program [7] for studying the geometry and analysis of strictly pseudoconvex domains. The key idea of the program is to consider the Bergman and Szegö kernels of the domains as analogs of the heat kernel of Riemannian manifolds. In Riemannian (or conformal) geometry, the coefficients of the asymptotic expansion of the heat kernel can be expressed in terms of the curvature of the metric; by integrating the coefficients one obtains index theorems in various settings. For the Bergman and Szegö kernels, there has been much progress made on the description of their asymptotic expansions based on invariant theory ([7], [1], [15]); we now seek for invariants that arise from the integral of the coefficients of the expansions.

We here prove that the integral of the coefficient of the logarithmic singularity of the Szegö kernel gives a biholomorphic invariant of a domain Ω, or a CR invariant of the boundary $\partial\Omega$, and moreover that the invariant is unchanged under perturbations of the domain (Theorem 1). We also show that the same invariant appears as the coefficient of the logarithmic term of the volume expansion of the domain with respect to the Bergman volume element (Theorem 2). This second result is an analogue of the derivation of a conformal invariant from the volume expansion of conformally compact Einstein manifolds which arises in the AdS/CFT correspondence — see [10] for a discussion and references.

The proofs of these results are based on Kashiwara's microlocal analysis of the Bergman kernel in [17], where he showed that the reproducing property of the Bergman kernel on holomorphic functions can be "quantized" to a reproducing property of the microdifferential operators (i.e., classical analytic pseudodifferential operators). This provides a system of microdifferential equations that characterizes the singularity of the Bergman kernel (which can be formulated as a microfunction) up to a constant multiple; such an argument

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can be equally applied to the Szegö kernel. These systems of equations are used to overcome one of the main difficulties, when we consider the analogy to the heat kernel, that the Bergman and Szegö kernels are not defined as solutions to differential equations.

Let Ω be a relatively compact, smoothly bounded, strictly pseudoconvex domain in a complex manifold M. We take a pseudo Hermitian structure θ, or a contact form, of $\partial\Omega$ and define a surface element $d\sigma = \theta \wedge (d\theta)^{n-1}$. Then we may define the Hardy space $\mathcal{A}(\partial\Omega, d\sigma)$ consisting of the boundary values of holomorphic functions on Ω that are L^2 in the norm $\|f\|^2 = \int_{\partial\Omega} |f|^2 d\sigma$. The Szegö kernel $S_\theta(z, \overline{w})$ is defined as the reproducing kernel of $\mathcal{A}(\partial\Omega, d\sigma)$, which can be extended to a holomorphic function of $(z, \overline{w}) \in \Omega \times \overline{\Omega}$ and has a singularity along the boundary diagonal. If we take a smooth defining function ρ of the domain, which is positive in Ω and $d\rho \neq 0$ on $\partial\Omega$, then (by [6] and [2]) we can expand the singularity as

$$S_\theta(z, \overline{z}) = \varphi_\theta(z)\rho(z)^{-n} + \psi_\theta(z) \log \rho(z), \tag{1.1}$$

where φ_θ and ψ_θ are functions on Ω that are smooth up to the boundary. Note that $\psi_\theta|_{\partial\Omega}$ is independent of the choice of ρ and is shown to gives a local invariant of the pseudo Hermitian structure θ.

Theorem 1. (i) *The integral*

$$L(\partial\Omega, \theta) = \int_{\partial\Omega} \psi_\theta \, \theta \wedge (d\theta)^{n-1}$$

is independent of the choice of a pseudo Hermitian structure θ of $\partial\Omega$. Thus $L(\partial\Omega) = L(\partial\Omega, \theta)$.

(ii) *Let $\{\Omega_t\}_{t \in \mathbb{R}}$ be a C^∞ family of strictly pseudoconvex domains in M. Then $L(\partial\Omega_t)$ is independent of t.*

In case $n = 2$, we have shown in [13] that

$$\psi_\theta|_{\partial\Omega} = \frac{1}{24\pi^2}(\Delta_b R - 2 \operatorname{Im} A_{11,}{}^{11}),$$

where Δ_b is the sub-Laplacian, R and $A_{11,}{}^{11}$ are respectively the scalar curvature and the second covariant derivative of the torsion of the Tanaka-Webster connection for θ. Thus the integrand $\psi_\theta \, \theta \wedge d\theta$ is nontrivial and does depend on θ, but it also turns out that $L(\partial\Omega) = 0$ by Stokes' theorem. For higher dimensions, we can still give examples of $(\partial\Omega, \theta)$ for which $\psi_\theta|_{\partial\Omega} \not\equiv 0$. However, the evaluation of the integral is not easy and, so far, we can only give examples with trivial $L(\partial\Omega)$ — see Proposition 3 below.

We were led to consider the integral of ψ_θ by the works of Branson-Ørstead [4] and Parker-Rosenberg [20] on the constructions of conformal invariants from the heat kernel $k_t(x, y)$ of the conformal Laplacian, and their CR analogue for CR invariant sub-Laplacian by Stanton [22]. For a conformal manifold of even dimension $2n$ (resp. CR manifold of dimension $2n-1$), the integral of the coefficient a_n of the asymptotic expansion $k_t(x,x) \sim t^{-n}\sum_{j=0}^{\infty} a_j(x)t^j$ is shown to be a conformal (resp. CR) invariant, while the integrand $a_n dv_g$ does depend on the choice of a scale $g \in [g]$ (resp. a contact form θ). This is a natural consequence of the variational formula for the kernel $k_t(x,y)$ under conformal scaling, which follows from the heat equation. Our Theorem 1 is also a consequence of a variational formula of the Szegö kernel, which is obtained as a part of a system of microdifferential equations for the family of Szegö kernels (Proposition 3.4).

We next express $L(\partial\Omega)$ in terms of the Bergman kernel. Take a C^∞ volume element dv on M. Then the Bergman kernel $B(z,\overline{w})$ is defined as the reproducing kernel of the Hilbert space $\mathcal{A}(\Omega, dv)$ of L^2 holomorphic functions on Ω with respect to dv. The volume of Ω with respect to the volume element $B(z,\overline{z})dv$ is infinite. We thus set $\Omega_\varepsilon = \{z \in \Omega : \rho(z) > \varepsilon\}$ and consider the asymptotic behavior of

$$\mathrm{Vol}(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} B(z,\overline{z})\, dv$$

as $\varepsilon \to +0$.

THEOREM 2. *For any volume element dv on M and any defining function ρ of Ω, the volume $\mathrm{Vol}(\Omega_\varepsilon)$ admits an expansion*

$$\mathrm{Vol}(\Omega_\varepsilon) = \sum_{j=0}^{n-1} C_j\, \varepsilon^{j-n} + L(\partial\Omega)\log\varepsilon + O(1), \tag{1.2}$$

where C_j are constants, $L(\partial\Omega)$ is the invariant given in Theorem 1 and $O(1)$ is a bounded term.

The volume expansion (1.2) can be compared with that of conformally compact Einstein manifolds ([12], [10]); there one considers a complete Einstein metric g_+ on the interior Ω of a compact manifold with boundary and a conformal structure $[g]$ on $\partial\Omega$, which is obtained as a scaling limit of g_+. For each choice of a preferred defining function ρ corresponding to a conformal scale, we can consider the volume expansion of the form (1.2) with respect to g_+. If $\dim_{\mathbb{R}} \partial\Omega$ is even, the coefficient of the logarithmic term is shown to be a conformal invariant of the boundary $\partial\Omega$. Moreover, it is shown in [11] and [8] that this conformal invariant can be expressed as the integral of Branson's Q-curvature [3], a local Riemannian invariant which naturally arises from conformally invariant differential operators. We can relate this result to ours via Fefferman's Lorentz conformal structure defined on an S^1-bundle over the CR manifold $\partial\Omega$. In case $n=2$, we have shown in [9] that $\psi_\theta|_{\partial\Omega}$ agrees with the Q-curvature of the Fefferman metric; while such a relation is not known for higher dimensions.

So far, we have only considered the coefficient $L(\partial\Omega)$ of the expansion (1.2). But other coefficients may have some geometric meaning if one chooses ρ properly; here we mention one example. Let $E \to X$ be a positive Hermitian line bundle over a compact complex manifold X of dimension $n-1$; then the unit tube in the dual bundle $\Omega = \{v \in E^* : |z| < 1\}$ is strictly pseudoconvex. We take $\rho = -\log|z|^2$ as a defining function of Ω and fix a volume element dv on E^* of the form $dv = i\,\partial\rho \wedge \overline{\partial}\rho \wedge \pi^* dv_X$, where $\pi^* dv_X$ is the pullback of a volume element dv_X on X.

Proposition 3. *Let $B(z, \overline{z})$ be the Bergman kernel of $\mathcal{A}(\Omega, dv)$. Then the volume of the domain $\Omega_\varepsilon = \{v \in E^* | \rho(z) > \varepsilon\}$ with respect to the volume element Bdv satisfies*

$$\mathrm{Vol}(\Omega_\varepsilon) = \int_0^\infty e^{-\varepsilon\, t} P(t)dt + f(\varepsilon). \tag{1.3}$$

Here $f(\varepsilon)$ is a C^∞ function defined near $\varepsilon = 0$ and $P(t)$ is the Hilbert polynomial of E, which is determined by the condition $P(m) = \dim H^0(M, E^{\otimes m})$ for $m \gg 0$.

This formula suggests a link between the expansion of $\mathrm{Vol}(\Omega_\varepsilon)$ and index theorems. But in this case the right-hand side of (1.3) does not contain a $\log\varepsilon$ term and hence $L(\partial\Omega) = 0$. (Note that dv is singular along the zero section, but we can modify it to a C^∞ volume element without changing (1.3).)

Finally, we should say again that we know no example of a domain with nontrivial $L(\partial\Omega)$ and need to ask the following:

Question. *Does there exist a strictly pseudoconvex domain Ω such that $L(\partial\Omega) \neq 0$?*

This paper is organized as follows. In Section 2, we formulate the Bergman and Szegö kernels as microfunctions. We here include a quick review of the theory of microfunctions in order for the readers to grasp the arguments of this paper even if they are unfamiliar with the subject. In Section 3 we recall Kashiwara's theorem on the microlocal characterization of the Bergman and Szegö kernels and derive a microdifferential relation between the two kernels and a first variational formula of the Szegö kernel. After these preparations, we give in Section 4 the proofs of the main theorems. Finally in Section 5, we prove Proposition 3 by relating $\mathrm{Vol}(\Omega_\varepsilon)$ to the trace of the operator with the kernel $B(\lambda z, \overline{w})$, $|\lambda| \leq 1$. This proof, suggested by the referee, utilizes essentially only the fact that dv is homogeneous of degree 0, and one can considerably weaken the assumption of the proposition — see Remark 5.1. We also derive here, by following Catlin [5] and Zelditch [24], an asymptotic relation between the fiber integral of Bdv and the Bergman kernel of $H^0(M, E^{\otimes m})$; this is a localization of (1.3).

I am very grateful to the referee for simplifying the proof of Proposition 3.

2. The Bergman and Szegö kernels as microfunctions

In this preliminary section, we explain how to formulate the theorems in terms of microfunctions, which are the main tools of this paper. We here recall all the definitions and results we use from the theory of microfunctions, with an intention to make this section introductory to the theory. A fundamental reference for this section is Sato-Kawai-Kashiwara [21], but a concise review of the theory by Kashiwara-Kawai [18] will be sufficient for understating the arguments of this paper. For comprehensive introductions to microfunctions and microdifferential operators, we refer to [19], [23] and [16].

2.1. *Singularity of the Bergman kernel.* We start by recalling the form of singularity of the Bergman kernel, which naturally lead us to the definition of homomorphic microfunctions.

Let Ω be a strictly pseudoconvex domain in a complex manifold M with real analytic boundary $\partial\Omega$. We denote by $M_{\mathbb{R}}$ the underlying real manifold and its complexification by $X = M \times \overline{M}$ with imbedding $\iota : M_{\mathbb{R}} \to X$, $\iota(z) = (z, \overline{z})$. We fix a real analytic volume element dv on M and define the Bergman kernel as the reproducing kernel of $\mathcal{A}(\Omega, dv) = L^2(\Omega, dv) \cap \mathcal{O}(\Omega)$, where $\mathcal{O}$ denotes the sheaf of holomorphic functions. Clearly we have $B(z, \overline{w}) \in \mathcal{O}(\Omega \times \overline{\Omega})$, while we can also show that $B(z, \overline{w})$ has singularity on the boundary diagonal. If we take a defining function $\rho(z, \overline{z})$ of $\partial\Omega$, then at each boundary point $p \in \partial\Omega$, we can write the singularity of $B(z, \overline{w})$ as

$$B(z, \overline{w}) = \varphi(z, \overline{w})\rho(z, \overline{w})^{-n-1} + \psi(z, \overline{w}) \log \rho(z, \overline{w}).$$

Here $\rho(z, \overline{w})$ is the complexification of $\rho(z, \overline{z})$ and $\varphi, \psi \in \mathcal{O}_{X,p}$, where p is identified with $\iota(p) \in X$. Moreover it is shown that this singularity is locally determined: if Ω and $\widetilde{\Omega}$ are strictly pseudoconvex domains that agree near a boundary point p, then $B_\Omega(z, \overline{w}) - B_{\widetilde{\Omega}}(z, \overline{w}) \in \mathcal{O}_{X,p}$. See [17] and Remark 3.2 below. Such an $\mathcal{O}_X$ modulo class plays an essential role in the study of the system of differential equations and is called a holomorphic microfunction, which we define below in a more general setting.

2.2. *Microfunctions*: *a quick review.* Microfunctions are the "singular parts" of holomorphic functions on wedges at the edges. To formulate them, we first introduce the notion of hyperfunctions, which are generalized functions obtained by the sum of "ideal boundary values" of holomorphic functions.

For an open set $V \subset \mathbb{R}^n$ and an open convex cone $\Gamma \subset \mathbb{R}^n$, we denote by $V + i\Gamma 0 \subset \mathbb{C}^n$ an open set that asymptotically agrees with the wedge $V + i\Gamma$ at the edge V. The space of *hyperfunctions* on V is defined as a vector space of formal sums of the form

$$f(x) = \sum_{j=1}^{m} F_j(x + i\Gamma_j 0), \tag{2.1}$$

where F_j is a holomorphic function on $V + i\Gamma_j 0$, that allow the reduction $F_j(x + i\Gamma_j 0) + F_k(x + i\Gamma_k 0) = F_{jk}(x + i\Gamma_{jk} 0)$, where $\Gamma_{jk} = \Gamma_j \cap \Gamma_k \neq \emptyset$ and $F_{jk} = F_j|_{\Gamma_{jk}} + F_k|_{\Gamma_{jk}}$, and its reverse conversion. We denote the sheaf of hyperfunctions by $\mathcal{B}$. Note that if each F_j is of polynomial growth in y at $y = 0$ (i.e., $|F_j(x+iy)| \le \text{const.}|y|^{-m}$), then $\sum_j \lim_{\Gamma_j \ni y \to 0} F_j(x+iy)$ converges to a distribution $\widetilde{f}(x)$ on V and such a hyperfunction $f(x)$ can be identified with the distribution $\widetilde{f}(x)$. When $n = 1$, we only have to consider two cones $\Gamma_\pm = \pm(0, \infty)$ and we simply write (2.1) as $f(x) = F_+(x + i0) + F_-(x - i0)$. For example, the delta function and the Heaviside function are given by

$$\delta(x) = (-2\pi i)^{-1}\big((x + i0)^{-1} - (x - i0)^{-1}\big)$$

and

$$H(x) = (-2\pi i)^{-1}\big(\log(x + i0) - \log(x - i0)\big),$$

where $\log z$ has slit along $(0, \infty)$.

We next define the singular part of hyperfunctions. We say that a hyperfunction $f(x)$ is *micro-analytic* at $(x_0; i\xi_0) \in iT^*\mathbb{R}^n \setminus \{0\}$ if $f(x)$ admits, near x_0, an expression of the form (2.1) such that $\langle \xi_0, y\rangle < 0$ for any $y \in \cup_j \Gamma_j$. The sheaf of *microfunctions* $\mathcal{C}$ is defined as a sheaf on $iT^*\mathbb{R}^n \setminus \{0\}$ with the stalk at $(x_0; i\xi_0)$ given by the quotient space

$$\mathcal{C}_{(x_0;i\xi_0)} = \mathcal{B}_{x_0}/\{f \in \mathcal{B}_{x_0} : f \text{ is micro-analytic at } (x_0; i\xi_0)\}.$$

Since the definition of $\mathcal{C}$ is given locally, we can also define the sheaf of microfunctions $\mathcal{C}_M$ on $iT^*M \setminus \{0\}$ for a real analytic manifold M.

We now introduce a subclass of microfunctions that contains the Bergman and Szegö kernels. Let $N \subset M$ be a real hypersurface with a real analytic defining function $\rho(x)$ and let Y be its complexification given by $\rho(z) = 0$ in X. Then, for each point $p \in N$, we consider a (multi-valued) holomorphic function of the form

$$u(z) = \varphi(z)\rho(z)^{-m} + \psi(z)\log\rho(z), \tag{2.2}$$

where $\varphi, \psi \in \mathcal{O}_{X,p}$ and m is a positive integer. A class modulo $\mathcal{O}_{X,p}$ of $u(z)$ is called a germ of a *holomorphic microfunction* at $(p; i\xi) \in iT^*_N M \setminus \{0\} = \{(z; \lambda d\rho(z)) \in T^*M : z \in N, \lambda \in \mathbb{R} \setminus \{0\}\}$, and we denote the sheaf of holomorphic microfunctions on $iT^*_N M \setminus \{0\}$ by $\mathcal{C}_{N|M}$. For a holomorphic microfunction u, we may assign a microfunction by taking the "boundary values" from $\pm \operatorname{Im}\rho(z) > 0$ with signature ± 1, respectively, as in the expression of $\delta(x)$ above, which corresponds to $(-2\pi i\, z)^{-1}$. Thus we may regard $\mathcal{C}_{N|M}$ as a subsheaf of $\mathcal{C}_M$ supported on $iT^*_N M \setminus \{0\}$. With respect to local coordinates (x', ρ) of M, each $u \in \mathcal{C}_{N|M}$ admits a unique expansion

$$u(x', \rho) = \sum_{j=k}^{-\infty} a_j(x')\Phi_j(\rho), \tag{2.3}$$

where $a_j(x')$ are real analytic functions and

$$\Phi_j(t) = \begin{cases} j!\, t^{-j-1} & \text{for } j \geq 0, \\ \frac{(-1)^j}{(-j-1)!}\, t^{-j-1} \log t & \text{for } j < 0. \end{cases}$$

If $u \neq 0$ we may choose k so that $a_k(x') \not\equiv 0$ and call k the *order of* u; moreover, if $a_k(x') \neq 0$ then we say that u is *nondegenerate* at $(x', 0) \in N$.

A differential operator

$$P(x, D_x) = \sum a_\alpha(x) D_x^\alpha, \quad \text{where} \quad D_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n},$$

with real analytic coefficients acts on microfunctions; it is given by the application of the complexified operator $P(z, D_z)$ to each $F_j(z)$ in the expression (2.1). Moreover, at $(p; i(1, 0, \dots, 0)) \in iT^*\mathbb{R}^n$, we can also define the inverse operator $D_{x_1}^{-1}$ of D_{x_1} by taking indefinite integrals of each F_j in z_1. The microdifferential operators are defined as a ring generated by these operators. A germ of a *microdifferential operator of order* m at $(x_0; i\xi_0) \in iT^*\mathbb{R}^n$ is a series of holomorphic functions $\{P_j(z, \zeta)\}_{j=m}^{-\infty}$ defined on a conic neighborhood U of $(x_0; i\xi_0)$ in $T^*\mathbb{C}^n$ satisfying the following conditions:

(1) $P_j(z, \lambda\zeta) = \lambda^j P(z, \zeta)$ for $\lambda \in \mathbb{C} \setminus \{0\}$;

(2) For each compact set $K \subset U$, there exists a constant $C_K > 0$ such that $\sup_K |P_{-j}(z, \zeta)| \leq j!\, C_K^j$ for any $j \in \mathbb{N} = \{0, 1, 2, \dots\}$.

The series $\{P_j\}$ is denoted by $P(x, D_x)$, and the formal series $P(z, \zeta) = \sum P_j(z, \zeta)$ is called the *total symbol*, while $\sigma_m(P) = P_m(z, \zeta)$ is called the *principal symbol*. The *product* and *adjoint* of microdifferential operators can be defined by the usual formulas of symbol calculus:

$$(PQ)(z, \zeta) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (D_\zeta^\alpha P(z, \zeta)) D_z^\alpha Q(z, \zeta),$$

$$P^*(z, \zeta) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_z^\alpha D_\zeta^\alpha P(z, -\zeta).$$

It is then shown that P is invertible on a neighborhood of $(x_0; i\xi_0)$ if and only if $\sigma_m(P)(x_0, i\xi_0) \neq 0$.

While these definitions based on the choice of coordinates, we can introduce a transformation law of microdifferential operators under coordinate changes and define the sheaf of the ring microdifferential operators $\mathcal{E}_M$ on iT^*M for real analytic manifolds M. It then turns out that the adjoint depends only on the choice of volume element $dx = dx_1 \wedge \cdots \wedge dx_n$.

The action of differential operators on microfunctions can be extended to the action of microdifferential operators so that $\mathcal{C}_M$ is a left $\mathcal{E}_M$-module. This is done by using the Laurent expansion of $P(z, \zeta)$ in ζ and then substituting D_z and $D_{z_1}^{-1}$, or by introducing a kernel function associated with the symbol (analogous to the distribution kernel of a pseudodifferential operator). Then

$\mathcal{C}_{N|M}$ becomes an $\mathcal{E}_M$-submodule of $\mathcal{C}_M$. We can also define the right action of $\mathcal{E}_M$ on $\mathcal{C}_M \otimes \pi^{-1} v_M$, where v_M is the sheaf of densities on M and $\pi\colon iT^*M \to M$ is the projection. It is given by $(udx)P = (P^*u)dx$, where the adjoint is taken with respect to dx (here P^* depends on dx, but $(P^*u)dx$ is determined by udx).

We also consider microdifferential operators with a real analytic parameter, that is, a $P = P(x, t, D_x, D_t) \in \mathcal{E}_{M\times\mathbb{R}}$ that commutes with t. This is equivalent to saying that the total symbol of P is independent of the dual variable of t; so we denote P by $P(x, t, D_x)$. Note that $P(x, t, D_x)$, when t is regarded as a parameter, acts on $\mathcal{C}_M \otimes \pi^{-1} v_M$ from the right.

2.3. *Microfunctions associated with domains.* Now we go back to our original setting where M is a complex manifold and $N = \partial\Omega$. We have already seen that the Bergman kernel determines a section of $\mathcal{C}_{\partial\Omega|M}$, which we call the *local Bergman kernel* $B(x)$. Here x indicates a variable on $M_{\mathbb{R}}$. Note that the local Bergman kernel is defined for a germ of strictly pseudoconvex hypersurfaces. Similarly, we can define the *local Szegö kernel*: if we fix a real analytic surface element $d\sigma$ on $\partial\Omega$ and define the Szegö kernel, then the coefficients of the expansion (1.1) are shown to be real analytic and to define a section $S(x)$ of $\mathcal{C}_{\partial\Omega|M}$; see Remark 3.2 below. We sometimes identify the surface element $d\sigma$ with the delta function $\delta(\rho(x))$, or $\delta(\rho(x))dv$, normalized by $d\rho \wedge d\sigma = dv$. Note that the microfunction $\delta(\rho(x))$ corresponds to the holomorphic microfunction $(-2\pi i \rho(z, \overline{w}))^{-1} \bmod \mathcal{O}_X$, which we denote by $\delta[\rho]$. Similarly, the Heaviside function $H(\rho(x))$ corresponds to a section $H[\rho]$ of $\mathcal{C}_{\partial\Omega|M}$, which is represented by $(-2\pi i)^{-1} \log \rho(z, \overline{w})$.

Our main object $\mathrm{Vol}(\Omega_\varepsilon)$ can also be seen as a holomorphic microfunction. In fact, since $u(\varepsilon) = \mathrm{Vol}(\Omega_\varepsilon)$ is a function of the form $u(\varepsilon) = \varphi(\varepsilon)\varepsilon^{-n} + \psi(\varepsilon)\log\varepsilon$, where φ and ψ are real analytic near 0, we may complexify $u(\varepsilon)$ and define a germ of a holomorphic microfunction $u(\widehat{\varepsilon}) \in \mathcal{C}_{\{0\}|\mathbb{R}}$ at $(0; i)$. Note that $\mathrm{Vol}(\Omega_\varepsilon) \in \mathcal{C}_{\{0\}|\mathbb{R}}$ is expressed as an integral of the local Bergman kernel:

$$\int B(x) H[\rho - \varepsilon](x) dv(x). \tag{2.4}$$

Here $H[\rho - \varepsilon](x)$ is a section of $\mathcal{C}_{\partial\widetilde{\Omega}|\widetilde{M}}$, where

$$\widetilde{\Omega} = \{(x, \varepsilon) \in \widetilde{M} = M \times \mathbb{R} : \rho(x) > \varepsilon\}.$$

See Remark 2.2 for the definition of this integral.

More generally, for a section $u(x, \varepsilon)$ of $\mathcal{C}_{\partial\widetilde{\Omega}|\widetilde{M}}$ defined globally in x for small ε and a global section $w(x)dx$ of $\mathcal{C}_{\partial\Omega|M} \otimes \pi^{-1} v_M$, we can define the integral of microfunction

$$\int u(x, \varepsilon) w(x) dx$$

at $(0;i) \in iT^*\mathbb{R}$, which takes values in $\mathcal{C}_{\{0\}|\mathbb{R}}$. For such an integral, we have a formula of integration by parts, which is clear from the definition of the action of microdifferential operators in terms of kernel functions [19].

LEMMA 2.1. *If $P(x, \varepsilon, D_x)$ is a microdifferential operator defined on a neighborhood of the support of $u(x, \varepsilon)$, then*

$$\int (Pu) w dx = \int u(w dx\, P). \tag{2.5}$$

Remark 2.2. We here recall the definition of the integral (2.4) and show that it agrees with $\mathrm{Vol}(\Omega_\varepsilon)$. For a general definition of the integral of microfunctions, we refer to [19]. Write $dv = \lambda d\rho \wedge d\sigma$ and complexify $\lambda(x', \rho)$ to $\lambda(x', \widetilde{\rho})$ for $\widetilde{\rho} \in \mathbb{C}$ near 0. Then, define a holomorphic function $f(\widetilde{\varepsilon})$ on $\mathrm{Im}\, \widetilde{\varepsilon} > 0$, $|\widetilde{\varepsilon}| \ll 1$, by the path integral

$$f(\widetilde{\varepsilon}) = \int_{\partial\Omega} \int_{\gamma_1} B(x', \widetilde{\rho}) \frac{1}{2\pi i} \log(\widetilde{\rho} - \widetilde{\varepsilon}) \lambda(x', \widetilde{\rho})\, d\widetilde{\rho} d\sigma(x'), \tag{2.6}$$

where γ_1 is a path connecting a and b, with $a < 0 < b$, such that the image is contained in $0 < \mathrm{Im}\, \widetilde{\rho} < \mathrm{Im}\, \widetilde{\varepsilon}$ except for both ends. Then (2.4) is given by $f(\varepsilon + i0) \in \mathcal{C}_{\mathbb{R},\,(0;i)}$, which is independent of the choice of a, b and γ. We now show $f(\varepsilon + i0) = \mathrm{Vol}(\Omega_\varepsilon)$ as a microfunction. For each $\widetilde{\varepsilon}$ with $\mathrm{Im}\, \widetilde{\varepsilon} > 0$, choose another path connecting b and a so that $\gamma_2\gamma_1$ is a closed path surrounding $\widetilde{\varepsilon}$ in the positive direction. Since the integral along γ_2 gives a function that can be analytically continued to 0, we may replace γ_1 in (2.6) by $\gamma_2\gamma_1$ without changing its $\mathcal{O}_{\mathbb{C},0}$ modulo class. Now restricting $\widetilde{\varepsilon}$ to the positive real axis, and letting the path $\gamma_2\gamma_1$ shrink to the line segment $[\varepsilon, b]$, we see that $f(\varepsilon)$ agrees with $\mathrm{Vol}(\Omega_\varepsilon)$ modulo analytic functions at 0.

2.4. *Quantized contact transformations.* We finally recall a property of holomorphic microfunctions that follows from the strictly pseudoconvexity of $\partial\Omega$. Let z be local holomorphic coordinates of M. Then we write $P(x, D_x) = P(z, D_z)$ (resp. $P(\overline{z}, D_{\overline{z}})$) if P commutes with $\overline{z}_j$ and $D_{\overline{z}_j}$ (resp. z_j and D_{z_j}). Similarly for $P(x, t, D_x, D_t) \in \mathcal{E}_{M\times\mathbb{R}}$ we write, e.g., $P(x, t, D_x, D_t) = P(z, t, D_z)$ if P commutes with $\overline{z}_j$, $D_{\overline{z}_j}$ and t. Clearly, the class of operators $P(z, D_z)$ and $P(\overline{z}, D_{\overline{z}})$ is determined by the complex structure of M.

LEMMA 2.3. *Let N be a strictly pseudoconvex hypersurface in M with a defining function ρ. Then for each section u of $\mathcal{C}_{N|M}$, there exists a unique microdifferential operator $R(z, D_z)$ such that $u = R(z, D_z)\delta[\rho]$. Moreover, u and R have the same order, and u is nondegenerate if and only if R is invertible.*

Note that the same lemma holds when $\delta[\rho]$ is replaced by $H[\rho]$, or more generally, by a nondegenerate section u of $\mathcal{C}_{N|M}$, except for the statement about the order.

The strictly pseudoconvexity of N implies that the projection $p_1 : T^*_Y X \subset T^*(M \times \overline{M}) \to T^*M$ is a local biholomorphic map, where Y is the complexification of N in X. The surjectivity and the injectivity of p_1 imply the existence and uniqueness of $R(z, D_z)$, respectively. If we apply the same argument for $\overline{z}$, we obtain a local biholomorphic map $p_2 : T^*_Y X \to T^*\overline{M}$ and a contact (or homogeneous symplectic) transformation $\phi(z,\zeta) = p_1 \circ p_2^{-1}(z, -\zeta)$. Then, for each nondegenerate u, the lemma above gives a map $\Phi : P(z, D_z) \mapsto Q(\overline{z}, D_{\overline{z}})$ such that $(P - Q^*)u = 0$, where the adjoint is taken with respect to $|dz|^2$. This is an isomorphism of the rings and satisfies $\sigma_m(\Phi(P)) \circ \phi = \sigma_m(P)$ if P has order m; hence Φ is called a *quantized contact transformation* with a *generating function* u. It is shown that a quantization of ϕ determines a generating function uniquely up to a constant multiple. Chapter 1 of [23] is a good reference for this subject.

3. Kashiwara's analysis of the kernel functions

In this section we recall Kashiwara's analysis of the Bergman kernel and its analogy to the Szegö kernel. Then we derive some microdifferential equations satisfied by these kernels.

3.1. *A relation between the local Bergman and Szegö kernels.* Under the formulation of the previous section, Kashiwara's theorem [17] for the Bergman kernel and its analogy to the Szegö kernel can be stated as follows:

THEOREM 3.1. (i) *The local Bergman kernel satisfies*

$$\big(P(z, D_z) - Q(\overline{z}, D_{\overline{z}})\big)B = 0$$

for any pair of microdifferential operators $P(z, D_z)$ and $Q(\overline{z}, D_{\overline{z}})$ such that

$$(H[\rho]dv)(P(z, D_z) - Q(\overline{z}, D_{\overline{z}})) = 0. \tag{3.1}$$

Moreover, the local Bergman kernel is uniquely determined by this property up to a constant multiple.

(ii) *The local Szegö kernel satisfies*

$$\big(P(z, D_z) - Q(\overline{z}, D_{\overline{z}})\big)S = 0$$

for any pair of microdifferential operators $P(z, D_z)$ and $Q(\overline{z}, D_{\overline{z}})$ such that

$$(\delta[\rho]dv)(P(z, D_z) - Q(\overline{z}, D_{\overline{z}})) = 0. \tag{3.2}$$

Moreover, the local Szegö kernel is uniquely determined by this property up to a constant multiple.

Remark 3.2. In [17], Kashiwara stated (i) and gave its heuristic proof, which can be equally applied to (ii). Also, as a premise for this theorem, he stated the real analyticity of the coefficients of the asymptotic expansion of the Bergman kernel, though the proof was not published. Now a proof of this theorem and claim, based on Kashiwara's lectures, is available in Kaneko's lecture notes [16]; the arguments there can also be applied to the case of the Szegö kernel.

Take holomorphic coordinates z and write $dv = \varphi|dz|^2$. Then (3.1) can be rewritten as

$$(P^* - Q^*)\varphi H[\rho] = 0,$$

where the adjoint is taken with respect to $|dz|^2$. It follows that the maps $P^*(z, D_z) \mapsto Q(\overline{z}, D_{\overline{z}})$ and $Q(\overline{z}, D_{\overline{z}}) \mapsto P^*(z, D_z)$ are the quantized contact transformations generated by $\varphi H[\rho]$ and B, respectively, and are the inverse of each other. Thus we can say that the theorem states the reproducing property of the kernel on microdifferential operators. In particular, we see that the uniqueness statement of the theorem follows from that of the generating function.

From Theorem 3.1, we can easily derive a microdifferential relation between the local Bergman and Szegö kernels.

Proposition 3.3. *Let $R(z, D_z)$ be the microdifferential operator such that $(H[\rho]dv)R = \delta[\rho]dv$. Then $RS = B$.*

Proof. We first show that $(P(z, D_z) - Q(\overline{z}, D_{\overline{z}}))RS = 0$ for any pair P and Q satisfying $(H[\rho]dv)(P - Q) = 0$. Noting that $Q(\overline{z}, D_{\overline{z}})$ commutes with $R(z, D_z)$, we see from $H[\rho]dv = (\delta[\rho]dv)R^{-1}$ that $(\delta[\rho]dv)(R^{-1}PR - Q) = 0$. Since $R^{-1}PR$ is an operator of a z-variable, Theorem 3.1 implies

$$(R^{-1}PR - Q)S = 0$$

and thus $(P - Q)RS = 0$. Now by the uniqueness statement of Theorem 3.1, we have $B = c\,RS$ for a constant c.

It remains to show that $c = 1$. This can be done by computing explicitly the leading term of these kernels. Take local coordinates $z = (z', z_n)$ such that the boundary $\partial\Omega$ is locally given by the defining function

$$\rho_0(z, \overline{z}) = z_n + \overline{z}_n - z' \cdot \overline{z}' + F(z, \overline{z}), \quad F = O(|z|^3).$$

Then write $\rho = e^{-f(z,\overline{z})}\rho_0$ and $dv = e^{g(z,\overline{z})}dV$, where dV is the standard volume element on $\mathbb{C}^n$. Since Ω is osculated at 0 to the third order by the Siegel domain, we see that

$$B = \frac{n!}{\pi^n} e^{-g}\rho_0^{-n-1}\big(1 + O(|z|)\big) \quad \text{and} \quad S = \frac{(n-1)!}{\pi^n} e^{-f-g}\rho_0^{-n}\big(1 + O(|z|)\big).$$

On the other hand, setting $f_0 = f(0,0)$ and $g_0 = g(0,0)$, we have $e^{f_0} D_{z_n} e^g H[\rho] = e^g \delta[\rho] + u$ for a degenerate germ $u \in \mathcal{C}_{\partial\Omega|M}$ at $(0; id\rho)$. Thus $R(z, D_z) = -e^{f_0} D_{z_n} + P(z, D_z)$, where P has order at most 1 and $\sigma_1(P)(z, \zeta)$ vanishes at $(0; id\rho)$. Using the expression of S above, we have

$$RS = -e^{f_0} D_{z_n} S + PS = \frac{n!}{\pi^n} e^{-g} \rho_0^{-n-1} \big(1 + O(|z|)\big),$$

which implies $c = 1$. $\square$

3.2. *Variational formula of the local Szegö kernel.* Let $\{\Omega_t\}_{t\in I}$ be a real analytic family of strictly pseudoconvex domains, where $I \subset \mathbb{R}$ is an open interval. Here a real analytic family means that $\widetilde{\Omega} = \{(x,t) \in \widetilde{M} = M \times I : x \in \Omega_t\}$ admits a real analytic defining function $\rho_t(x)$ such that $d_x \rho_t(x) \neq 0$ on $\partial\widetilde{\Omega}$. If we fix ρ_t, we can assign for each $\partial\Omega_t$ a surface element $d\sigma_t$ by $\delta[\rho_t] dv$. We here consider the microdifferential equations for the family of the local Szegö kernels of $(\partial\Omega_t, d\sigma_t)$.

Proposition 3.4. *There exists a section $S_t(x)$ of $\mathcal{C}_{\partial\widetilde{\Omega}|\widetilde{M}}$ such that, for each t, $S_t(x)$ gives the local Szegö kernel of $(\partial\Omega_t, d\sigma_t)$. Moreover, $S_t(x)$ satisfies*

$$\big(P(z,t,D_z) - Q(\overline{z}, D_{\overline{z}}, D_t)\big) S_t(x) = 0 \tag{3.3}$$

for any pair of microdifferential operators $P(z,t,D_z)$ and $Q(\overline{z}, D_{\overline{z}}, D_t)$ such that

$$(\delta[\rho_t] d\widetilde{v})\big(P(z,t,D_z) - Q(\overline{z}, D_{\overline{z}}, D_t)\big) = 0,$$

where $d\widetilde{v} = dv \wedge dt$ is a volume element on $\widetilde{M}$. In particular, if $R(z,t,D_z)$ satisfies $D_t \delta[\rho_t] dv = (\delta[\rho_t] dv) R(z,t,D_z)$, then

$$-D_t S_t = R(z,t,D_z) S_t. \tag{3.4}$$

An analogous proposition for the local Bergman kernel was given in [14], where we considered the family of local Bergman kernels $B_t(x)$ of $(\Omega_t, |dz|^2)$ for domains in $\mathbb{C}^n$ and obtained exactly the same statement for B_t with $H[\rho_t]|dz|^2$ in place of $\delta[\rho_t] dv$. In particular, we have

$$-D_t B_t = \widetilde{R}(z,t,D_z) B_t \tag{3.5}$$

for $\widetilde{R}$ satisfying $D_t H[\rho_t]|dz|^2 = (H[\rho_t]|dz|^2) \widetilde{R}(z,t,D_z)$. We here use Proposition 3.3 to translate this formula into the one for S_t.

Proof. First note that the ring of operators of the form $Q(\overline{z}, D_{\overline{z}}, D_t)$ is the ring generated by $\overline{z}_1, \ldots, \overline{z}_n, D_{\overline{z}_1}, \ldots, D_{\overline{z}_n}, D_t$, and hence it suffices to prove (3.3) when Q is one of these generators. For $\overline{z}_j$ and $D_{\overline{z}_j}$, this is clear from Theorem 3.1. To prove the case $Q = D_t$, take $A(z,t,D_z)$ such that

$\delta[\rho_t]dv = (H[\rho_t]dv)A$ and compute

$$\begin{aligned}(\delta[\rho_t]d\widetilde{v})D_t &= (H[\rho_t]d\widetilde{v})AD_t \\ &= (H[\rho_t]d\widetilde{v})[A, D_t] + (H[\rho_t]d\widetilde{v})D_tA \\ &= (H[\rho_t]d\widetilde{v})([A, D_t] - \widetilde{R}A) \\ &= (\delta[\rho_t]d\widetilde{v})A^{-1}([A, D_t] - \widetilde{R}A).\end{aligned}$$

Since $[t, [D_t, A]] = 0$, we have $R(z, t, D_z) = A^{-1}(\widetilde{R}A - [A, D_t])$. On the other hand, Proposition 3.3 implies $B_t = AS_t$ and thus

$$\begin{aligned}RS_t &= A^{-1}(\widetilde{R}A - [A, D_t])S_t \\ &= (A^{-1}\widetilde{R}A + A^{-1}D_tA - D_t)S_t \\ &= A^{-1}(\widetilde{R} + D_t)B_t - D_tS_t.\end{aligned}$$

Therefore, by (3.5), we get $RS_t = -D_tS_t$. □

4. Proofs of the main theorems

Now we are ready to prove the main theorems. We first note that the theorems can be reduced to the ones in the real analytic category by approximations. The key fact is that the asymptotic expansions up to each fixed order of the Bergman and Szegö kernels are determined by the finite jets of ρ, $d\sigma$ and dv at each boundary point. Thus, for a domain Ω with C^∞ defining function ρ and the contact form $\theta = i(\partial\rho - \overline{\partial}\rho)$ on $\partial\Omega$, by taking a series of real analytic functions ρ_j that converge to ρ in C^k-norm for any k, we may express $L(\partial\Omega, \theta)$ as the limit of $L(\partial\Omega_j, \theta_j)$, where $\Omega_j = \{\rho_j > 0\}$. To reduce Theorem 1 (i) to the real analytic case, we only have to take another sequence of real analytic contact forms $\{e^{f_j}\theta_j\}$ approximating a given contact form $e^f\theta$ so that $L(\partial\Omega_j, e^{f_j}\theta_j) = L(\partial\Omega_j, \theta_j)$ implies $L(\partial\Omega, e^f\theta) = L(\partial\Omega, \theta)$. Similar arguments of approximation can be applied to the other cases.

In the following we prove the theorems in the real analytic category.

4.1. *Proof of Theorem* 1. Taking a real analytic family of defining functions $\rho_t(x)$, we define $S_t(x)$ to be the local Szegö kernel for the surface element given by $\delta[\rho_t]dv$. Let $\widetilde{\Omega} = \{(x, \varepsilon, t) \in \widetilde{M} = M \times \mathbb{R}^2 : \rho_t(x) > \varepsilon\}$ so that $\delta[\rho_t - \varepsilon]$ defines a section of $\mathcal{C}_{\partial\widetilde{\Omega}|\widetilde{M}}$ and consider the integral

$$A(\varepsilon, t) = \int S_t(x)\delta[\rho_t - \varepsilon](x)dv(x),$$

which is well-defined as a germ of $\mathcal{C}_{\{0\}\times\mathbb{R}|\mathbb{R}^2}$ at $(0, 0; i(1, 0))$. Write

$$A(\varepsilon, t) = \varphi(\varepsilon, t)\varepsilon^{-n} + \psi(\varepsilon, t)\log\varepsilon$$

and set $L_t = \psi(0, t)$, which we call the coefficient of $\varepsilon^0 \log\varepsilon$. Then our goal is to prove the independence of L_t from t, because it contains the theorem: For the statement (i), we take $\rho_t = e^{tf}\rho$ so that $\delta[\rho_0]dv$ and $\delta[\rho_1]dv$ correspond to $\theta\wedge(d\theta)^{n-1}$ and $\widetilde{\theta}\wedge(d\widetilde{\theta})^{n-1}$ respectively; then $L_0 = L(\partial\Omega,\theta)$ and $L_1 = L(\partial\Omega,\widetilde{\theta})$ agree. For the statement (ii), we have $L_t = L(\partial\Omega_t)$, which is independent of t.

Take a microdifferential operator $R(z,t,D_z)$ such that

$$D_t\delta[\rho_t]dv = (\delta[\rho_t]dv)R.$$

Then Proposition 3.4 implies $-D_tS_t = RS_t$. Using this and (2.5), we have

$$\begin{aligned} D_tA(t,\varepsilon) &= D_t\int S_t\,\delta[\rho_t-\varepsilon]dv \\ &= \int (D_tS_t)\delta[\rho_t-\varepsilon] + S_tD_t\delta[\rho_t-\varepsilon]dv \\ &= \int (-RS_t)\delta[\rho_t-\varepsilon] + S_tD_t\delta[\rho_t-\varepsilon]dv \\ &= \int S_t\big(-(\delta[\rho_t-\varepsilon]dv)R + D_t\delta[\rho_t-\varepsilon]dv\big). \end{aligned}$$

Since $D_t\delta[\rho_t-\varepsilon]dv - (\delta[\rho_t-\varepsilon]dv)R$ vanishes at $\varepsilon=0$, we may take, by using Lemma 4.1 below, a section u of $\mathcal{C}_{\partial\widetilde{\Omega}|\widetilde{M}}$ such that

$$D_t\delta[\rho_t-\varepsilon]dv - (\delta[\rho_t-\varepsilon]dv)R = \varepsilon\,udv.$$

Hence we have

$$D_tA(t,\varepsilon) = \varepsilon\int S_t(x)\,u(x,t,\varepsilon)dv(x).$$

The integral on the right-hand side takes value in $\mathcal{C}_{\{0\}\times\mathbb{R}|\mathbb{R}^2}$; thus the right-hand side does not contain an $\varepsilon^0\log\varepsilon$ term. This implies $D_tL_t = 0$. $\square$

Lemma 4.1. *Let $\{\Omega_\varepsilon\}$ be a real analytic family of domains in M and $\rho(x,\varepsilon)$ be the defining function of $\widetilde{\Omega} = \{(x,\varepsilon)\in M\times\mathbb{R} : x\in\Omega_\varepsilon\}$ such that $d_x\rho(x,\varepsilon)\neq 0$ on the boundary. If $u(x,\varepsilon)\in\mathcal{C}_{\partial\widetilde{\Omega}|\widetilde{M}}$ satisfies $u(x,0)=0$ in $\mathcal{C}_{\partial\Omega_0|M}$, then there exists a germ $v\in\mathcal{C}_{\partial\widetilde{\Omega}|\widetilde{M}}$ such that $u=\varepsilon\,v$.*

Proof. Take coordinates (x',ρ,ε) for $\widetilde{M}$ and expand u as in (2.3) with coefficients $a_j(x',\varepsilon)$. Then $u(x',\rho,0)=0$ implies $a_j(x',0)=0$ so that $a_j=\varepsilon a_j'$ for real analytic functions $a_j'(x,\varepsilon)$. Thus we may set $v=\sum a_j'(x',\varepsilon)\Phi_j(\rho)$. $\square$

4.2. *Proof of Theorem* 2. Take a microdifferential operator $R(z,\varepsilon,D_z)$ such that

$$(H[\rho-\varepsilon]dv)R = \delta[\rho-\varepsilon]dv. \tag{4.1}$$

Then $(H[\rho]dv)R(z,0,D_z) = \delta[\rho]dv$ and hence $R(z,0,D_z)S = B$ by Proposition 3.3. So, applying Lemma 4.1 for $\Omega_\varepsilon = \Omega$, we have

$$R(z,\varepsilon,D_z)S(x) = B(x) + \varepsilon\,B'(x,\varepsilon), \tag{4.2}$$

where B' is a section of $\mathcal{C}_{\partial\Omega\times\mathbb{R}|M\times\mathbb{R}}$. Using (4.1) and (4.2), we compute

$$\begin{aligned}\int S\delta[\rho-\varepsilon]dv &= \int S\big((H[\rho-\varepsilon]dv)R\big)\\ &= \int (RS)H[\rho-\varepsilon]dv\\ &= \mathrm{Vol}(\Omega_\varepsilon) + \varepsilon\int B'H[\rho-\varepsilon]dv.\end{aligned}$$

Since the integral on the right-hand side takes value in $\mathcal{C}_{\{0\}|\mathbb{R}}$, its ε multiple cannot contain an $\varepsilon^0\log\varepsilon$ term. Therefore the coefficients of $\varepsilon^0\log\varepsilon$ of $\int S\,\delta[\rho-\varepsilon]dv$ and $\mathrm{Vol}(\Omega_\varepsilon)$ agree; the former gives $L(\partial\Omega)$ and the theorem follows. □

5. Proof of Proposition 3

Let $\mathcal{A}_m(\Omega)$ be the subspace of $\mathcal{A}(\Omega)=\mathcal{A}(\Omega,dv)$ consisting of homogeneous functions of degree m (i.e., $\varphi(\lambda z)=\lambda^m\varphi(z)$ for any $\lambda\in\mathbb{C}$ with $|\lambda|\le 1$). Then, except for the case $\mathcal{A}_0(\Omega)=\{0\}$, each $\mathcal{A}_m(\Omega)$ can be identified with $H^0(X,E^{\otimes m})$ and hence has finite dimension d_m. We take, for each m, an orthonormal basis $\varphi_{1,m},\dots,\varphi_{d_m,m}$ of $\mathcal{A}_m(\Omega)$ and form a complete orthonormal system $\{\varphi_{j,m}\}_{j,m}$ of $\mathcal{A}(\Omega)$, so that

$$B(z,\overline{w})=\sum_{j,m}\varphi_{j,m}(z)\overline{\varphi_{j,m}(w)}. \tag{5.1}$$

It is then clear that $B(\lambda z,\overline{w})=B(\sqrt{\lambda}\,z,\sqrt{\lambda}\,\overline{w})$ for $0<\lambda<1$ and thus, using the homogeneity of dv, we get

$$\mathrm{Vol}(\Omega_\varepsilon)=\int_\Omega B(\sqrt{\lambda}\,z,\sqrt{\lambda}\,\overline{z})dv=\int_\Omega B(\lambda z,\overline{z})dv,\quad \text{with } \lambda=e^{-\varepsilon}.$$

Again by (5.1), we see that the integral on the right-hand side is given by the power series $F(\lambda)=\sum_{m=1}^\infty d_m\lambda^m$. Since $d_m=P(m)$ for $m\gg 0$, we have, modulo holomorphic functions at $\varepsilon=0$,

$$F(e^{-\varepsilon})=P(-D_\varepsilon)((1-e^{-\varepsilon})^{-1})=P(-D_\varepsilon)(\varepsilon^{-1}).$$

The final formula is the Laplace transform of $P(t)$. □

Remark 5.1. In the proof above, we have only used the facts that Ω is bounded, that we have $\lambda\Omega\subset\Omega$ if $|\lambda|\le 1$, and that dv is homogeneous of degree 0. The result still holds if Ω is not pseudoconvex or smooth; ρ does not even need to be continuous.

With the assumption of strictly pseudoconvexity, we can further express the asymptotic expansion as $\varepsilon\to 0$ of the integration of $B(z,\overline{z})$ on each fiber $\Omega_\varepsilon(x)$ of $\pi\colon\Omega_\varepsilon\to X$ in terms of the Bergman kernel on the diagonal $B_m(x)$ of

$H^0(X, E^{\otimes m})$. The formulas (5.3) and (5.4) below can be seen as a localization of Proposition 3.

First note that the L^2-inner product on the space $H^0(X, E^{\otimes m})$, defined with respect to the fiber metric and a volume element dv_X on X, is the $(1/2\pi)$-multiple of that on the subspace $\mathcal{A}_m(\partial\Omega) \subset \mathcal{A}(\partial\Omega, d\sigma)$ consisting of functions of homogeneous degree m, where $d\sigma = i\partial\rho \wedge \pi^* dv_X$. Thus using the Szegö kernel of $\mathcal{A}(\partial\Omega)$, we may write the Bergman kernels $B_m(x)$ as Fourier series

$$B_m(x) = \int_0^{2\pi} e^{-im\phi} S(e^{i\phi}z, \overline{z}) d\phi, \quad z \in \partial\Omega \cap \pi^{-1}(x).$$

On the other hand, since $\int_{\partial\Omega_\varepsilon} f\,\overline{f} d\sigma = \int_{\Omega_\varepsilon} f\,\overline{Vf} dv$ for $f \in \mathcal{A}(\partial\Omega)$, where V is the vector field that generates the $\mathbb{C}$-action, we have $B(z, \overline{z}) = \overline{V} S(z, \overline{z})$ and thus

$$\int_{\Omega_\varepsilon(x)} B(z, \overline{z}) i\partial\rho \wedge \overline{\partial}\rho = \int_{\partial\Omega_\varepsilon(x)} S(z, \overline{z}) i\partial\rho.$$

Now we use the strictly pseudoconvexity of Ω and write $S(z, \overline{z})$ as the Laplace transform

$$S(z, \overline{z}) = \int_0^\infty e^{-t\rho} a(x, t) dt \tag{5.2}$$

of a classical symbol $a(x, t) \in S^n(X \times \mathbb{R}_+)$ with asymptotic expansion $a(x, t) \sim \sum_{j=n-1}^{-\infty} a_j(x) t^j$ at $t = \infty$ (this is another formulation of the expansion (2.3); cf. [2]). Then we have

$$\int_{\Omega_\varepsilon(x)} B(z, \overline{z}) i\partial\rho \wedge \overline{\partial}\rho = 2\pi \int_0^\infty e^{-\varepsilon t} a(x, t) dt \tag{5.3}$$

and

$$B_m(x) \sim 2\pi a(x, m) \quad \text{as } m \to \infty. \tag{5.4}$$

The latter is just an application of Fourier's inversion formula to the complexification of (5.2):

$$S(e^{i\phi}z, \overline{z}) = \int_0^\infty e^{it\phi} a(x, t) dt,$$

where we have used the fact that $\rho(e^{i\phi}z, \overline{z}) = -i\phi$ for small ϕ.

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