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Periodic simple groups of finitary linear transformations

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In Memory of Dick and Brian

Abstract

A group is locally finite if every finite subset generates a finite subgroup. A group of linear transformations is finitary if each element minus the identity is an endomorphism of finite rank. The classification and structure theory for locally finite simple groups splits naturally into two cases—those groups that can be faithfully represented as groups of finitary linear transformations and those groups that are not finitary linear. This paper completes the finitary case. We classify up to isomorphism those infinite, locally finite, simple groups that are finitary linear but not linear.

1. Introduction

A group G is *locally finite* if every finite subset S is contained in a finite subgroup of G. That is, every finite S generates a finite subgroup $\langle S\rangle$.

This paper presents one step in the classification of those locally finite groups that are simple. We shall be particularly interested in locally finite simple groups that have faithful representations as finitary linear groups—the finitary locally finite simple groups.

Let V be a left vector space over the field K. (For us fields will always be commutative.) Thus $\mathrm{End}_K(V)$ acts on the right with group of units $\mathrm{GL}_K(V)$. The element $g \in \mathrm{GL}_K(V)$ is *finitary* if $V(g-1) = [V, g]$ has finite K-dimension. This dimension is the *degree of g on V*, $\deg_V g = \dim_K [V, g]$. Equivalently, g is finitary on V if and only if $\dim_K V/C_V(g)$ is finite, where $C_V(g) = \ker(g-1)$. In this case $\dim_K V/C_V(g) = \deg_V g$.

The invertible finitary linear transformations of V form a normal subgroup of $\mathrm{GL}_K(V)$ that is denoted $\mathrm{FGL}_K(V)$, the *finitary general linear group*. A

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group G is *finitary linear* (sometimes shortened to *finitary*) if it has a faithful representation $\varphi\colon G \longrightarrow \mathrm{FGL}_K(V)$, for some vector space V over the field K.

A group G is *linear* if it has a faithful representation $\varphi\colon G \longrightarrow \mathrm{GL}_n(K)$ $(= \mathrm{GL}_K(K^n))$, for some integer n and some field K. Clearly a finite group is linear and a linear group is finitary, but the reverse implications are not valid in general.

This paper contains a proof of the following theorem.

(1.1) THEOREM. *A locally finite simple group that has a faithful representation as a finitary linear group is isomorphic to one of*:

(1) *a linear group in finite dimension*;

(2) *an alternating group* $\mathrm{Alt}(\Omega)$ *with* Ω *infinite*;

(3) *a finitary symplectic group* $\mathrm{FSp}_K(V,s)$;

(4) *a finitary special unitary group* $\mathrm{FSU}_K(V,u)$;

(5) *a finitary orthogonal group* $\mathrm{F}\Omega_K(V,q)$;

(6) *a finitary special linear group* $\mathrm{FSL}_K(V,W,m)$.

Here K is a (possibly finite) subfield of $\overline{\mathbb{F}}_p$, the algebraic closure of the prime subfield $\mathbb{F}_p$. The forms s, u, and q are nondegenerate on the infinite dimensional K-space V; and m is a nondegenerate pairing of the infinite dimensional K-spaces V and W. Conversely, each group in (2)–(6) *is locally finite, simple, and finitary but not linear in finite dimension.*

The classification theory for locally finite simple groups progresses in natural steps:

(i) Classification of finite simple groups;

(ii) Classification of nonfinite, linear locally finite simple groups;

(iii) Classification of nonlinear, finitary locally finite simple groups;

(iv) Description of nonfinitary locally finite simple groups.

The resolution of (i) is the well-known classification of finite simple groups (CFSG); see [11]. Less well-known is the full classification up to isomorphism of the groups in (ii):

(1.2) THEOREM (BBHST: Belyaev, Borovik, Hartley, Shute, and Thomas [4], [6], [18], [43]). *Each locally finite simple group that is not finite but has a faithful representation as a linear group in finite dimension over a field is isomorphic to a Lie type group $\Phi(K)$, where K is an infinite, locally finite field, that is, an infinite subfield of $\overline{\mathbb{F}}_p$, for some prime p.*

The present Theorem 1.1 resolves the third step, providing the classification up to isomorphism of all groups as in (iii). (An earlier discussion can be found in [15].)

The original proofs of the BBHST Theorem 1.2 appealed to CFSG, but the theorem of Larsen and Pink [26] now renders the BBHST theorem independent of CFSG. Our proof of Theorem 1.1 does not depend upon BBHST, but it does depend upon a weak version of CFSG (Theorem 5.1 below). The nature of that dependence is discussed more fully in Section 5. In particular it is conceivable that the necessary results of Section 5 have geometric, classification-free proofs.

Every group is the union of its finitely generated subgroups. Therefore every locally finite group is the union of its finite subgroups. This simple observation is the starting point for our proof of Theorem 1.1. After this introduction, the second section of the paper discusses the tools—sectional covers and ultraproducts—used to make the observation precise and useful. Sectional covers allow us to approximate our groups locally by finite simple groups. These can then be pasted together effectively via ultraproducts.

The third section on examples describes the conclusions to the theorem and some of their properties. Pairings of vector spaces and their isometry groups are discussed in some detail, since this material is not familiar to many but is crucial for the definition and identification of the examples. The fourth section gives needed results, several from the literature, on the representations of finite groups, particularly discussion and characterization of the natural representations of finite alternating and classical groups. This section includes Jordan's Theorem 4.2, which states that a finite primitive permutation group generated by elements that move only a small number of letters is alternating or symmetric. The material of Section 5 could be placed in the previous section since it is largely about representations of finite groups. Indeed its main result is a version of Jordan's Theorem valid for all finite linear groups, not just permutation groups. We have chosen to isolate this section since its Theorem 5.1 of Jordan type constitutes the weak version of the classification of finite simple groups that we use in proving Theorem 1.1.

The proof of the theorem begins in earnest in Section 6, where the cases are identified. In Theorem 6.5 an arbitrary nonlinear locally finite simple group that is finitary is seen to bear a strong resemblance either to an alternating group or to a finitary classical group. The alternating case is then resolved in Section 7 and the classical case in Section 8.

Although a classification of locally finite simple groups under (iv) up to isomorphism is not possible, Meierfrankenfeld [30] has shown that a great deal of useful structural information can be obtained and then applied. The finitary classification is important here, since Meierfrankenfeld's structural results depend critically, via Corollary 2.13 below, on the impossibility of finitary representation under (iv) .

Wehrfritz [44] has proved that Theorem 1.1 with K allowed to be an arbitrary division ring can be reduced to the case of K a field. Theorem 1.1 also has applications outside of the realm of pure group theory. Finitary groups can be thought of as those that are "nearly trivial" on the associated module. An application in this context can be found in work of Passman on group rings [32], [33].

A *periodic* group is one in which all elements have finite order. The first published result on locally finite groups was:

(1.3) THEOREM (Schur [38]). *A periodic linear group is locally finite.*

An easy consequence [13], [35] is

(1.4) THEOREM. *A periodic finitary linear group is locally finite.*

Therefore the groups of the title are classified by Theorem 1.1.

Our basic references for group theory are [1], [10] and [25] for locally finite groups. For basic geometry, see [3], [42]. For more detailed discussion of finitary groups, locally finite simple groups, and their classification, see the articles [15], [17], [30], [36] in the proceedings of the Istanbul NATO Advanced Institute.

2. Tools

We have already remarked that every locally finite group is the union of its finite subgroups. In this section we formalize and refine this observation in several ways. For further discussion on several of the topics in this section, see [25, Chaps. 1§§A,L, 4§A] and [15, Appendix].

2.1. *Systems and covers.* We say that the set I is *directed* by the partial order $\preceq$ if, for every pair i, j of elements of I, there is a $k \in I$ with $i \preceq k \succeq j$. An important example of a directed set is the set of all finite subsets of a given G, ordered by containment.

Just as we can reconstruct a set from the set of its finite subsets, we wish to reconstruct a more structured object G from a large enough collection $\mathcal{G}$ of its subobjects. We say that the direct ordering $(I, \preceq)$ on the index set I is *compatible* with $\mathcal{G} = \{\, G_i \mid i \in I \,\}$ if $G_i \le G_j$ whenever $i \preceq j$. (We write $A \le B$ and $B \ge A$ when we mean that A is a subobject of B.) Then, for each pair $i, j \in I$, there is a $k \in I$ with $G_i \le G_k \ge G_j$ as I is directed. If additionally $G = \bigcup_{i \in I} G_i$ then $\mathcal{G}$ is called a *directed system in* G with respect to the directed set $(I, \preceq)$. For us the canonical example of a directed system is the set of all finitely generated subgroups of a group—in particular, the set of all finite subgroups of a locally finite group—with respect to containment.

A *local system* $\{\, G_i \mid i \in I \,\}$ in G (here typically a group, field, or vector space) is a set of $G_i \le G$ with the properties

(a) $G = \bigcup_{i \in I} G_i$ and

(b) for every $i, j \in I$ there is a k with $G_i \le G_k \ge G_j$.

Therefore a local system is a directed system in G with respect to any direct ordering of its index set that is compatible. In this situation G is not only the union of the G_i, it is actually (isomorphic to) the direct limit $\varinjlim_{(I,\preceq)} G_i$ of the G_i with respect to containment. (For a formal discussion of direct limits, see [19, §2.5].) If G is a group then a local system is also called a *subgroup cover*.

A group G is *quasisimple* if it is perfect ($G = G'$, the derived subgroup) and $G/Z(G)$ is simple.

(2.1) LEMMA. *Let the group G have a subgroup cover $\{\, G_i \,|\, i \in I \,\}$ that consists of quasisimple groups. Then G itself is quasisimple. Indeed G is simple if and only if, for every $g \in G$, there is some i with $g \in G_i \setminus Z(G_i)$.*

Proof. We must prove that G is perfect and $G/Z(G)$ is simple. For any element $g \in G$, there is an $i \in I$ with $g \in G_i = G_i' \le G'$; so G is perfect. In particular $Z(G/Z(G)) = 1$, so we now assume that $Z(G) = 1$ and aim to prove that G is simple. The group G is simple if and only if $h \in \langle g^G \rangle$ for all pairs $g, h \in G$ of nonidentity elements. As g is not central in G, there are $i, j \in I$ with $g \in G_i \setminus Z(G_i)$ and $h \in G_j$. Then there is a $k \in I$ with $\langle G_i, G_j \rangle \le G_k$, hence $g \in G_k \setminus Z(G_k)$ and $h \in G_k$. As G_k is quasisimple, $h \in G_k = \langle g^{G_k} \rangle \le \langle g^G \rangle$ as desired.

A *section* of the group X is a quotient of a subgroup. That is, for a subgroup $A \le X$ and normal subgroup B of A, the group A/B is a section of X. We often write the section A/B as an ordered pair (A, B), keeping track of the subgroups involved, not just the isomorphism type of the quotient A/B.

In the group G consider the set of pairs $\mathcal{S} = \{\, (G_i, N_i) \,|\, i \in I \,\}$ with each (G_i, N_i) a section of G. Give I an ordering such that

$$i \prec j \quad \Longrightarrow \quad G_i < G_j \text{ and } G_i \cap N_j = 1\,.$$

If $(I, \preceq)$ is a directed set and $\{\, G_i \setminus N_i \,|\, i \in I \,\}$ is a directed system in $G \setminus 1$ with respect to $(I, \preceq)$, then $\mathcal{S}$ is called a *sectional cover* of G. That is, $\mathcal{S} = \{\, (G_i, N_i) \,|\, i \in I \,\}$ is a sectional cover of G precisely when it satisfies:

(c) $G = \bigcup_{i \in I} G_i$ and

(d) for every $i, j \in I$ there is a $k \in I$ with $G_i \le G_k \ge G_j$ and $G_i \cap N_k = 1 = G_j \cap N_k$.

If $\{\, (G_i, N_i) \,|\, i \in I \,\}$ is a sectional cover, then $\{\, G_i \,|\, i \in I \,\}$ is a subgroup cover. Conversely, if $\{\, G_i \,|\, i \in I \,\}$ is a subgroup cover, then $\{\, (G_i, 1) \,|\, i \in I \,\}$ is a sectional cover.

A sectional cover $\mathcal{S} = \{\, (G_i, N_i) \,|\, i \in I \,\}$ is said to have property $\mathcal{P}$ if each section G_i/N_i has property $\mathcal{P}$. In particular $\mathcal{S}$ is a *finite* sectional cover precisely when each G_i/N_i is finite, and $\mathcal{S}$ is a *finite simple* sectional cover precisely when each G_i/N_i is a finite simple group.

We then have:

(2.2) Lemma. *Let* $\mathcal{S} = \{\,(G_i, N_i) \mid i \in I\,\}$ *be a collection of sections from the group* G. *The following are equivalent*:

(1) $\mathcal{S}$ *is a finite sectional cover of* G;

(2) G *is locally finite, and* $\mathcal{S}$ *satisfies*:

(c′) $G = \bigcup_{i \in I} G_i$, *with each* G_i *finite, and*

(d′) *for every* $i \in I$ *there is a* $k \in I$ *with* $G_i \le G_k$ *and* $G_i \cap N_k = 1$;

(3) G *is locally finite, and* $\mathcal{S}$ *satisfies*:

(c″) *each* G_i *is finite, and*

(d″) *for every finite* $A \le G$ *there is a* $k \in I$ *with* $A \le G_k$ *and* $A \cap N_k = 1$.

The modern approach to locally finite simple groups began with Otto Kegel's fundamental observation:

(2.3) Theorem (Kegel). *Every locally finite simple group has a finite simple sectional cover.*

There are numerous proofs. See Kegel's original paper [24] and also [15, Prop. 3.2], [30, Lemma 2.15], or [34, Th. 1].

Kegel's result provides the critical fact that every locally finite simple group can be papered over with its finite simple sections, leaving no seams showing. Finite simple sectional covers $\{\,(G_i, N_i) \mid i \in I\,\}$ are therefore called *Kegel covers*. The subgroups N_i are the *Kegel kernels*, while the simple quotients G_i/N_i are the *Kegel quotients* or *Kegel factors*. (The converse of the theorem does not hold. That is, a locally finite group with a Kegel cover need not be simple; see [25, Remark, p. 116].)

It is easy to see that, for a locally finite simple group G with the finite quasisimple sectional cover $\mathcal{Q} = \{\,(H_i, O_i) \mid i \in I\,\}$, the set $\{\,(H_i, Z_i) \mid i \in I\,\}$ is a Kegel cover, where Z_i is the preimage of $Z(H_i/O_i)$ in H_i. Accordingly, we call such $\mathcal{Q}$ a *quasisimple Kegel cover*.

An infinite locally finite simple group G will have many Kegel covers. Theorem 1.1 is proved by finding particularly nice Kegel covers and then using them to construct the geometry for G. An important tool for taking a Kegel cover and pruning it down to a more useful one is the following:

(2.4) Lemma (coloring argument). *Let* G *be a locally finite group, and suppose that the pairs of the finite sectional cover* $\mathcal{S} = \{\,(G_i, N_i) \mid i \in I\,\}$ *are colored with a finite set* $1, \dots, n$ *of colors. Then* $\mathcal{S}$ *contains a monochromatic* *subcover. That is, if $\mathcal{S}_j$ is the set of pairs from $\mathcal{S}$ with color j, for $1 \le j \le n$, then there is a color j for which $\mathcal{S}_j$ is itself a sectional cover of G.*

Proof. Otherwise, for each j, there is a finite subgroup A_j of G that is not covered by any section colored by j. The subgroup $A = \langle A_1, \ldots, A_j, \ldots, A_n \rangle$ is therefore not covered by a section with any of the colors $1, 2, \ldots, n$. As A is generated by a finite number of finite groups, it is finite itself. Therefore some section of $\mathcal{S}$ covers A, a contradiction which proves the lemma.

As an easy application we have

(2.5) Corollary. *Let G be a locally finite group with sectional cover $\mathcal{S} = \{\, (G_i, N_i) \,|\, i \in I \,\}$. For the finite subgroup $A \le G$, let*

$$\mathcal{S}_A = \{\, (G_i, N_i) \,|\, i \in I,\ A \le G_i, A \cap N_i = 1 \,\}\,.$$

Then $\mathcal{S}_A$ is also a sectional cover of G.

We can also use simplicity to trade one Kegel cover for another.

(2.6) Lemma. *Let $\{\, G_i \,|\, i \in I \,\}$ be a directed system of subgroups of G with respect to the directed set $(I, \preceq)$. For each $i \in I$, let H_i be a normal subgroup of G_i with the additional property that $H_i \le H_j$ whenever $i \preceq j$. Then $\{\, H_i \,|\, i \in I \,\}$ is a directed system in H with respect to $(I, \preceq)$, where $H = \bigcup_{i \in I} H_i = \varinjlim_{(I,\preceq)} H_i$ is the direct limit of the H_i and is normal in G.*

In particular, if G is simple and some H_i is nontrivial, then $H = G$. Assume additionally that $\{\, (G_i, N_i) \,|\, i \in I \,\}$ is a Kegel cover of simple G, and set $O_i = H_i \cap N_i$ for $i \in I$. Then there is a subset I_0 of I with $\{\, (H_i, O_i) \,|\, i \in I_0 \,\}$ a Kegel cover of G whose collection of Kegel quotients is contained in that of the original cover.

Proof. As the G_i are directed by $(I, \preceq)$, so are the normal subgroups H_i. Therefore their direct limit H is normal in G.

Assume now that G is simple and that H_0 is nontrivial. Let

$$I_0 = \{\, i \in I \,|\, G_0 \le G_i,\ G_0 \cap N_i = 1 \,\}.$$

By Corollary 2.5 $\{\, (G_i, N_i) \,|\, i \in I_0 \,\}$ is a Kegel cover. For $i \in I_0$,

$$H_i/O_i = H_i/H_i \cap N_i \simeq H_i N_i/N_i = G_i/N_i\,.$$

If G_i/N_i covers G_j, then H_i/O_i covers H_j; so $\{\, (H_i, O_i) \,|\, i \in I_0 \,\}$ is a Kegel cover as described.

One case of interest sets $H_i = G_i^{(\infty)}$, the last term in the derived series of G_i. If locally finite G is nonabelian and simple, then the lemma provides a Kegel cover $\{\, (H_i, O_i) \,|\, i \in I_0 \,\}$ with each H_i perfect. In particular a locally finite simple group that is locally solvable must be abelian hence cyclic.

Let

$$\mathcal{K}^* = \{ (G_i, N_i) \mid i \in I \}$$

be a Kegel cover of the locally finite simple group G. We know that, for many subsets I^0 of I, the set

$$\mathcal{K}^0 = \{ (G_i, N_i) \mid i \in I^0 \}$$

is actually a Kegel subcover, perhaps by Lemma 2.4. Equally well, for any nonidentity finite subset S of G, by Lemma 2.6 there is a subset I^1 of I for which the set

$$\mathcal{K}^1 = \{ (G_i^1 = \langle (S \cap G_i)^{G_i} \rangle, N_i^1 = G_i^1 \cap N_i) \mid i \in I^1 \}$$

is also a Kegel cover. We call any Kegel cover $\mathcal{K}$, got by a succession of these operations from $\mathcal{K}^*$, an *abbreviation* of $\mathcal{K}^*$. An abbreviation of $\mathcal{K}^*$ is indexed by a subset I^∞ of I; and, for each $i \in I^\infty$, the Kegel quotient is the same as that for $\mathcal{K}^*$.

Additionally, we say that one quasisimple Kegel cover is an abbreviation of another if the associated Kegel cover of the first is an abbreviation of that for the second.

2.2. *Ultraproducts and representation.* Let I be any nonempty set. A *filter* $\mathcal{F}$ on I is a set of subsets of I that satisfies two axioms:

(a) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;

(b) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

The set of all subsets of I is the *trivial* filter. If the set I is infinite, then the *cofinite* filter, consisting all subsets of I with finite complement, is nontrivial.

If the filter $\mathcal{F}$ on I contains A and B with $A \cap B = \emptyset$, then it is trivial. A filter that instead satisfies:

(c) for all $A \subseteq I$, $A \in \mathcal{F}$ if and only if $I \setminus A \notin \mathcal{F}$

is a maximal nontrivial filter and is called an *ultrafilter*. A degenerate example is the *principal* ultrafilter $\mathcal{F}_x$, composed of all subsets containing the element $x \in I$. A nontrivial filter is principal if and only if it contains a set with exactly one element.

The union of an ascending chain of nontrivial filters on I is itself a nontrivial filter, so that by Zorn's lemma every nontrivial filter is contained in an ultrafilter. In particular, for infinite I there are nonprincipal ultrafilters containing the cofinite filter.

Compare the following with Lemma 2.4 and Corollary 2.5.

(2.7) Lemma. *Let $\mathcal{F}$ be a filter on I.*

(1) *If $\mathcal{F}$ is an ultrafilter and $A \in \mathcal{F}$, then for any finite coloring of A there is exactly one color class that belongs to $\mathcal{F}$.*

(2) *For* $A \in \mathcal{F}$, *put* $\mathcal{F}_A = \{B \in \mathcal{F} \mid B \subseteq A\}$. *Then* $\mathcal{F}_A$ *is a filter on* A, *and if* $\mathcal{F}$ *is an ultrafilter then so is* $\mathcal{F}_A$.

Proof. For (1), consider first a 2-coloring $A = A_1 \cup A_2$. If both $I \setminus A_1$ and $I \setminus A_2$ were in $\mathcal{F}$ then $(I \setminus A_1) \cap (I \setminus A_2) = I \setminus A$ would be as well, which is not the case. Thus by axiom (c) (applied twice) exactly one of the disjoint sets A_1 and A_2 belongs to $\mathcal{F}$. Part (1) then follows by induction.

For (2), axioms (a) and (b) for $\mathcal{F}_A$ come from the same axioms for $\mathcal{F}$. Axiom (c) for A is the 2-coloring case of (1).

If $(I, \preceq)$ is a directed set, define

$$\mathcal{F}(i) = \{a \in I \mid i \preceq a\}\,.$$

The filter *generated* by the directed set $(I, \preceq)$ is then

$$\mathcal{F}_{(I,\preceq)} = \{A \mid A \supseteq \mathcal{F}(i),\ \text{for some } i \in I\}\,.$$

This filter is nonprincipal precisely when $(I, \preceq)$ has no maximum element.

The ultraproduct construction starts with a collection of sets (structures) $\mathcal{G} = \{\, G_i \mid i \in I \,\}$. If $\mathcal{F}$ is any ultrafilter on the index set I, then the *ultraproduct* $\prod_{\mathcal{F}} G_i$ is defined as the Cartesian product $\prod_{i \in I} G_i$ modulo the equivalence relation

$$(x_i)_{i\in I} \sim_{\mathcal{F}} (y_i)_{i \in I} \qquad \Longleftrightarrow \qquad \{\, i \in I \mid x_i = y_i \,\} \in \mathcal{F}\,.$$

The ultraproduct provides a formal and logical method for pasting together local information that is putatively related. Ultraproducts share many properties with their coordinate structures. Ultraproducts of groups are groups, and (more surprisingly) ultraproducts of fields are fields. Ultraproducts commute with regular products. If we are given coordinate maps $\alpha_i \colon G_i \longrightarrow H_i$, then there is a naturally defined ultraproduct map

$$\alpha_{\mathcal{F}} = \textstyle\prod_{\mathcal{F}} \alpha_i \colon\ \prod_{\mathcal{F}} G_i \longrightarrow \prod_{\mathcal{F}} H_i\,.$$

Therefore we can carry actions over to ultraproducts. In particular, ultraproducts of vector spaces are vector spaces. (See [15, Appendix] for more.)

Certain ultraproducts may be thought of as enveloping directed systems and direct limits.

(2.8) PROPOSITION. *Let* $\mathcal{G} = \{\, G_i \mid i \in I \,\}$ *be a directed system in* G *with respect to the directed set* $(I, \preceq)$. *Let* $\mathcal{F}$ *be an ultrafilter containing* $\mathcal{F}_{(I,\preceq)}$.

Consider the map

$$\Gamma \colon G \longrightarrow \prod_{i \in I} G_i \quad \textit{given by} \quad g \mapsto (g_i)_{i \in I}\ ,$$

where, for $g \in G$,

$$\begin{aligned} g_k &= g \quad \textit{if } g \in G_k \\ &= \ ? \quad \textit{otherwise}\,. \end{aligned}$$

(By ? we mean any arbitrary member of G_k. When the G_i have algebraic structure, it is convenient but not necessary to choose the neutral element.) Then Γ induces an isomorphism $\Gamma_{\mathcal{F}}$ of G into the ultraproduct $\prod_{\mathcal{F}} G_i$.

Proof. See [15, Th. C.1].

We often identify G with its image in an ultraproduct as in the proposition. One difficulty with this construction is that the ultraproduct may be a great deal larger than G. In particular, if the G_i are finite and G is countably infinite, then $\prod_{\mathcal{F}} G_i$ is uncountable.

The next lemma is a permutation version of the representation theoretic Theorem 2.10(2) below, and its proof is typical of ultraproduct arguments.

For a permutation $g \in \mathrm{Sym}(\Omega)$, the *support* of g, denoted $[\Omega, g]$, is the set $\{\, \omega \in \Omega \,|\, \omega.g \neq \omega \,\}$ of letters in Ω moved by g. The *degree of g in Ω*, $\deg_\Omega g$, is then the cardinality of the support, $|[\Omega, g]|$.

(2.9) Lemma. *Let the group G have the subgroup cover $\{\, G_i \,|\, i \in I \,\}$, and let $(I, \preceq)$ be a compatible directed order of the index set I with $\mathcal{F}$ an ultrafilter containing $\mathcal{F}_{(I,\preceq)}$. For each $i \in I$, let Ω_i be a permutation space for G_i. Suppose that g is a fixed but arbitrary nonidentity element of G.*

If, for each $i \in I$ with $g \in G_i$, the degree $\deg_{\Omega_i} g$ is at most k, then in the action of $G \leq \prod_{\mathcal{F}} G_i$ on $\Omega = \prod_{\mathcal{F}} \Omega_i$ we have $\deg_\Omega g \leq k$. The element g is in the kernel of the action on Ω if and only if $\{\, i \in I \,|\, g \in G_i \cap \ker(\Omega_i) \,\} \in \mathcal{F}$.

Proof. For each i, give the points ω of Ω_i that are moved by g distinct colors from $1, \ldots, k$ (possible, by hypothesis). In each Ω_i, color ω with color 0 if $\omega.g = \omega$. (If $g \notin G_i$ then by convention $\omega.g = \omega$ for all $\omega \in \Omega_i$, and so all points of Ω_i are colored with 0. This amounts to choosing $g_i = 1$ in the embedding of G in $\prod_{\mathcal{F}} G_i$.)

Consider arbitrary $o = (\omega_i)_{i \in I}$, representing the point $o_{\mathcal{F}}$ of Ω. The coordinate entries of o are $(k+1)$-colored from $\{0, \ldots, k\}$. As $\mathcal{F}$ is an ultrafilter, by Lemma 2.7(1), exactly one of the monochromatic coordinate subsets for o belongs to $\mathcal{F}$. We then color the point $o_{\mathcal{F}}$ of Ω with the corresponding color j. (This is well-defined: if $o' = (\omega'_i)_{i \in I}$ also represents $o_{\mathcal{F}}$, then $\{\, i \in I \,|\, \omega_i = \omega'_i$ has color $j \,\} \in \mathcal{F}$.)

For a given color j not 0, there is either one point of Ω colored j or no point colored j, depending upon whether or not $\{i \in I \mid$ a unique $\omega \in \Omega_i$ has color $j\,\}$ belongs to $\mathcal{F}$.

If $o_{\mathcal{F}}$ receives the color j, then $I_o = \{\, i \in I \,|\, \omega_i$ has color $j \,\}$ is in $\mathcal{F}$. If $j = 0$ then $C_o(g) = \{\, i \in I \,|\, \omega_i = \omega_i.g \,\}$ is equal to I_o, and $o_{\mathcal{F}}$ is fixed by g. If $j > 0$ then $C_o(g)$ is within $I \backslash I_o$ and so is not in $\mathcal{F}$. That is, $o_{\mathcal{F}} \neq (o.g)_{\mathcal{F}} = o_{\mathcal{F}}.g$. We conclude that, in its action on Ω, the element g moves at most k points, namely those colored other than with 0.

The set $\{ i \in I \mid g \in G_i \}$ contains members of $\mathcal{F}_{(I,\preceq)}$, and so its complement $J = \{ i \in I \mid g \notin G_i \}$ is not in $\mathcal{F}$. Certainly if $\ker_I(g) = \{ i \in I \mid g \in G_i \cap \ker(\Omega_i) \}$ belongs to $\mathcal{F}$, then $g \in \ker(\Omega)$.

Suppose now that $\ker_I(g) \notin \mathcal{F}$. There exist elements o whose only coordinates colored 0 are those of $J \cup \ker_I(g)$. As $J \notin \mathcal{F}$, we must have $I \setminus \{J \cup \ker_I(g)\} \in \mathcal{F}$ by Lemma 2.7(1). Therefore such elements o are not colored 0. Hence $o_{\mathcal{F}}.g \neq o_{\mathcal{F}}$, and $g \notin \ker(\Omega)$.

In our applications we need to work with projective representations—homomorphisms into projective groups $\mathrm{PGL}_F(U)$—since the natural representations of the classical simple groups are projective representations. We define projective representation in a different but equivalent form. The map $\varphi\colon G \longrightarrow \mathrm{GL}_F(U)$ with associated cocycle $c\colon G \times G \longrightarrow F$ is a *projective representation* provided, for all $g, h \in G$,

$$\varphi(g)\varphi(h) = c(g,h)\varphi(gh)\,.$$

Thus a projective representation whose cocycle is identically 1 is a representation in the usual sense. As a consequence of this definition, the cocycle c is characterized by the property:

$$c(g,h)c(gh,k) = c(g,hk)c(h,k),\ \text{for all } g,h,k \in G\,.$$

The *kernel* of the projective representation φ is

$$\ker(\varphi) = \{g \in G \mid \varphi(g) \text{ is scalar on } U\}\,,$$

and φ is *nontrivial* if $\ker(\varphi) \neq G$.

For a linear transformation $g \in \mathrm{GL}_F(U)$ and $W \leq U$, we set $[W,g] = W(g-1)$; and, for $G \subseteq \mathrm{GL}_F(U)$, we set $[W,G] = \sum_{g\in G}[W,g]$. The *degree of G on U*, $\deg_U G$, is then the dimension $\dim_F [U,G]$. We define iterated commutators via $[W,G,H] = [[W,G],H]$ for $G, H \subseteq \mathrm{GL}_F(U)$.

(2.10) THEOREM ([15, App. §§B, C, D]). *Let the group G have the subgroup cover $\{ G_i \mid i \in I \}$, and let $(I, \preceq)$ be a compatible directed order of the index set I with $\mathcal{F}$ an ultrafilter containing $\mathcal{F}_{(I,\preceq)}$. For each $i \in I$ let $(\varphi_i, c_i)\colon G_i \longrightarrow \mathrm{GL}_{F_i}(U_i)$ be a projective representation. Then $(\Phi_{\mathcal{F}}, c_{\mathcal{F}})\colon G \longrightarrow \mathrm{GL}_F(U)$ is a projective representation, where $c_{\mathcal{F}} = \prod_{\mathcal{F}} c_i$, $F = \prod_{\mathcal{F}} F_i$, $U = \prod_{\mathcal{F}} U_i$, and $\Phi_{\mathcal{F}} = (\prod_{\mathcal{F}} \varphi_i)|_G$. If $\{ i \in I \mid \mathrm{char}\, F_i = p \} \in \mathcal{F}$, then $\mathrm{char}\, F = p$. The element $g \in G$ is in $\ker(\Phi_{\mathcal{F}})$ if and only if $\{ i \in I \mid g \in G_i \cap \ker(\varphi_i) \} \in \mathcal{F}$.*

(1) (Mal'cev's Theorem) *If, for each $i \in I$, the dimension $\dim_{F_i} U_i$ is at most k, then $\dim_F U$ is at most k.*

(2) *If, for some $g \in G$ and each $i \in I$ with $g \in G_i$, the degree of g on U_i, $\deg_{U_i} \varphi_i(g)$, is at most k, then the degree of g on U, $\deg_U \Phi_{\mathcal{F}}(g)$, is at most k.*

(3) *If each* U_i *has a* $\varphi_i(G_i)$*-invariant* (*nondegenerate, nonsingular*) *form of type* Cl, *then on* V *there is a* $\Phi_{\mathcal{F}}(G)$*-invariant* (*nondegenerate, nonsingular*) *form of type* Cl. (*See Sections* 3.2 *and* 3.3 *for the appropriate definitions.*)

Theorem 2.10(1) is Mal'cev's famous Representation Theorem (see [25, 1.L.6]).

Theorem 2.10(2) is of greatest import to us here. A version of this first appeared as [13, Th. (3.3)]. We present this and two further versions as corollaries.

Consider the subset $B \neq 1$ of the group A. The *degree of* B *in* A, $\deg_A B$, is the minimum of $\deg_U \varphi(B)$ over all projective representations $\varphi\colon A \longrightarrow \mathrm{GL}_F(U)$ with $b \notin \ker\varphi$, for all $1 \neq b \in B$. The *degree of* A, $\deg A$, is then $\deg_A A$.

If $\mathcal{S} = \{\, (G_i, N_i) \mid i \in I\,\}$ is a sectional cover of the group G, then the degrees of $\mathcal{S}$ are the degrees of the various quotients $Q_i = G_i/N_i$. For $g \in G$, the degrees of g in $\mathcal{S}$ are the degrees $\deg_{Q_i} gN_i$, for those $i \in I$ with $g \in G_i$.

(2.11) Corollary ([13, Th. (3.3)]). *A locally finite simple group* G *that has a sectional cover in which the degrees of the element* $g \neq 1$ *are bounded has a faithful representation as a finitary linear group.*

If $Q = G/N$ is an alternating group $\mathrm{Alt}(\Omega)$, then the *natural degree* of g in Q is $\deg_\Omega gN$. If Q is a classical group on F^n, then the *natural degree* of g in Q is the minimum of $\deg_{F^n} \varphi(gN)$ over all nontrivial projective representations $\varphi\colon Q \longrightarrow \mathrm{GL}_n(F)$.

(2.12) Corollary ([15, Cor. 3.13]). *A locally finite simple group* G *that has a sectional cover composed of alternating or classical groups in which the natural degrees of the element* $g \neq 1$ *are bounded has a faithful representation as a finitary linear group.*

(2.13) Corollary. *For the nonfinitary locally finite simple group* G, *in every sectional cover* $\mathcal{S}$ *the degree of every element* $g \neq 1$ *is unbounded. In particular, the degrees of* $\mathcal{S}$ *are unbounded.*

From the point of view of classification theory for locally finite simple groups, the present paper completes the classification of all finitary examples; so to go further we would only need to consider nonfinitary groups. In that case the corollaries, together with the classification of finite simple groups, imply that Kegel covers are essentially composed of alternating and classical groups of unbounded degree in which every nonidentity element has unbounded (natural) degree. The attendant stretching of elements and groups can be put to good use; see [30].

3. The examples

3.1. *Alternating groups.* For any permutation group $G \leq \mathrm{Sym}(\Omega)$ and any field K, the vector space $K\Omega = \{\, \sum_{\omega \in \Omega} a_\omega \omega \mid a_\omega \in K \,\}$ has a natural structure as a KG-module given by

$$\Big(\sum_{\omega \in \Omega} a_\omega \omega\Big).g = \sum_{\omega \in \Omega} a_\omega (\omega.g)\,.$$

The *augmentation submodule* $[K\Omega, G] = \sum_{g \in G} K\Omega(g-1)$ has codimension t, where t is the number of orbits of G on Ω.

For the element $g \in \mathrm{Sym}(\Omega)$, we have defined previously the degree of g on $K\Omega$, $\deg_{K\Omega} g = \dim_K [V, g]$, and the degree of g on Ω, $\deg_\Omega g = |[\Omega, g]|$, where $[\Omega, g] = \{\, \omega \in \Omega \mid \omega.g \neq \omega \,\}$, the support of g. For nonidentity g these two degrees are not equal; indeed,

$$\deg_{K\Omega} g = \deg_\Omega g - t\,,$$

where t is the number of orbits of g on $[\Omega, g]$ (the number of nontrivial orbits of g on Ω). Therefore

$$\deg_{K\Omega} g \leq \deg_\Omega g \leq 2\deg_{K\Omega} g\,.$$

In particular, $\deg_\Omega g$ is finite if and only if $\deg_{K\Omega} g$ is finite; and, over a collection of permutation spaces Ω, $\deg_\Omega g$ is bounded if and only if $\deg_{K\Omega} g$ is bounded. (Compare with Lemma 2.9 and Theorem 2.10(2).)

For any set Ω, the *finitary symmetric group* $\mathrm{FSym}(\Omega)$ consists of all permutations g of Ω whose support is finite, the elements of finite degree. If the set Ω is finite then $\mathrm{FSym}(\Omega)$ is just the full symmetric group $\mathrm{Sym}(\Omega)$, but for infinite Ω the finitary group $\mathrm{FSym}(\Omega)$ is a proper normal subgroup of $\mathrm{Sym}(\Omega)$.

$\mathrm{FSym}(\Omega)$ is generated by its 2-cycles. The alternating group, $\mathrm{Alt}(\Omega)$, is then the subgroup of all even finitary permutations (products of an even number of 2-cycles). For $|\Omega| > 1$, it is a normal subgroup of index 2 in $\mathrm{FSym}(\Omega)$. Indeed, for $|\Omega| > 4$, $\mathrm{Alt}(\Omega)$ is the unique minimal normal subgroup of $\mathrm{Sym}(\Omega)$.

As discussed previously, the infinite set Ω has a directed system consisting of its finite subsets Δ (with respect to containment). This means that $\mathrm{FSym}(\Omega)$ has a subgroup cover consisting of its finite symmetric subgroups $\mathrm{Sym}(\Delta)$ (identified with the pointwise stabilizer of $\Omega \setminus \Delta$ in $\mathrm{Sym}(\Omega)$), and $\mathrm{Alt}(\Omega)$ has the subgroup cover of its finite simple subgroups $\mathrm{Alt}(\Delta)$ [1, (15.16)]. Therefore, by Lemma 2.2, $\mathrm{FSym}(\Omega)$ and $\mathrm{Alt}(\Omega)$ are locally finite; and, by Lemma 2.1, infinite $\mathrm{Alt}(\Omega)$ is simple.

If G is an alternating or (finitary) symmetric group on Ω (with $|\Omega| > 3$), then a *natural* module for G is the nontrivial irreducible factor in the permutation module $K\Omega$, for any field K. If Ω is infinite or $\mathrm{char}\, K$ does not divide $|\Omega|$, this is the augmentation module $[K\Omega, G]$. Otherwise it is the quotient of

$[K\Omega, G]$ by the submodule of constant vectors $K(1, 1, \ldots, 1)$. The degree inequalities above imply that both modules $K\Omega$ and $[K\Omega, G]$ are finitary. Indeed, for $|\Omega| > 2$,

$$\mathrm{FSym}(\Omega) = \mathrm{Sym}(\Omega) \cap \mathrm{FGL}_K(K\Omega) = \mathrm{Sym}(\Omega) \cap \mathrm{FGL}_K([K\Omega, G])\,.$$

On the other hand, for infinite Ω the alternating group $\mathrm{Alt}(\Omega)$ is not linear of finite degree by

(3.1) Proposition ([13, (4.4)]). *If* $H \le \mathrm{GL}_m(K)$ *with* $H/M \simeq \mathrm{Alt}(n)$ *for* $n \ge 16$, *then* $m \ge n - 2$.

In partial summary, we have

(3.2) Theorem. *Let* Ω *be infinite. The group* $\mathrm{Alt}(\Omega)$ *is a locally finite simple group. Over any field* K, *the permutation module* $K\Omega$ *and the natural, irreducible augmentation submodule of codimension* 1 *give faithful and finitary representations.* $\mathrm{Alt}(\Omega)$ *is not linear in finite dimension.*

In [13] it was proved that any faithful, finitary representation of infinite $\mathrm{Alt}(\Omega)$ on V has augmentation module $[V, \mathrm{Alt}(\Omega)]$ equal to a direct sum of irreducible natural modules.

3.2. *Pairings and forms.* Let K be a division ring, $V = {}_KV$, a left K-space, and $W = W_K$, a right K-space. Following Baer [3, pp. 34–36], a *pairing* of V and W is a bilinear map $m\colon V \times W \longrightarrow K$. That is,

(a) we have

(i) $m(u + v, w) = m(u, w) + m(v, w)$ and
(ii) $m(u, w + y) = m(u, w) + m(u, y)$,
for all $u, v \in V$ and $w, y \in W$; and

(b) $m(av, wb) = am(v, w)b$, for all $v \in V$, $w \in W$, and $a, b \in K$.

The canonical example $m = m^{\mathrm{can}}$ lets $W = V^*$, the dual of V, and sets $m^{\mathrm{can}}(v, \lambda) = v\lambda$, for all $v \in V$ and $\lambda \in V^*$.

Let U be a subspace of V and Y a subspace of W. Then we set

$$U^\perp = \{\, w \in W \mid m(u, w) = 0, \text{ for all } u \in U \,\} \text{ and}$$
$${}^\perp Y = \{\, v \in V \mid m(v, y) = 0, \text{ for all } y \in Y \,\}\,.$$

The pairing $m\colon V \times W \longrightarrow K$ is *nondegenerate* if the *radicals* $\mathrm{Rad}(W, m) = V^\perp$ and $\mathrm{Rad}(V, m) = {}^\perp W$ are both 0. If $U \le V$ and $Y \le W$ with $m|_{U\times Y}$ identically 0, then we call the pair (U, Y) *(iv) totally isotropic.*

The following are elementary:

(3.3) LEMMA. *The pairing* $m\colon V\times W \longrightarrow K$ *is nondegenerate if and only if the map* $w \mapsto m(\cdot, w)$ *is an injection of* W *into* V^* *and the map* $v \mapsto m(v, \cdot)$ *is an injection of* V *into* W^*.

(3.4) LEMMA. *Let* $m\colon V \times W \longrightarrow K$ *be a nondegenerate pairing. Let finite dimensional* $U \leq V$ *and finite dimensional* $Y \leq W$.

(1) *The codimension of* $U^{\perp}$ *in* W *equals the dimension of* U, *and* ${}^{\perp}(U^{\perp}) = U$.

(2) *The codimension of* ${}^{\perp}Y$ *in* V *equals the dimension of* Y, *and* $({}^{\perp}Y)^{\perp} = Y$.

(3) $m|_{U\times Y}$ *is nondegenerate if and only if* $\dim_K U = \dim_K Y$, $V = U \oplus {}^{\perp}Y$, *and* $W = Y \oplus U^{\perp}$.

In particular, for the finite dimensional space $V = U$ there is an essentially unique nondegenerate pairing, the canonical one, m^{can} with $W = Y = V^*$. This is not the case for infinite dimensional V. Let $\mathcal{B} = \{\, v_i \mid i \in I \,\}$ be a K-basis for V. For each $i \in I$ we have the element $v_i^* \in V^*$ given by

$$v_i.v_i^* = 1 \text{ and } v_j.v_i^* = 0, \text{ for } j \neq i\,.$$

The set $\{\, v_i^* \mid i \in I \,\}$ is "dual" to $\mathcal{B}$ and linearly independent (although it is a basis of V^* if and only if $\dim_K V$ is finite). Let $V^{\mathcal{B}}$ be the subspace of V^* spanned by the v_i^*. Then the restriction of the canonical pairing, $m^{\mathcal{B}} = m^{\mathrm{can}}|_{V\times V^{\mathcal{B}}}$, is a nondegenerate pairing of V and $V^{\mathcal{B}}$. For $\dim_K V$ infinite,

$$\dim_K V^* = |K|^{\dim_K V} > \dim_K V = \dim_K V^{\mathcal{B}}$$

(see [7, Lemma 5.1]); and the two nondegenerate pairings m^{can} and $m^{\mathcal{B}}$ of V are different in an essential way.

We shall need the following.

(3.5) LEMMA. *Let* $m\colon V \times W \longrightarrow K$ *be a nondegenerate pairing. Let finite dimensional* $U_0 \leq V$ *and finite dimensional* $Y_0 \leq W$. *Then there are* U *and* Y *with* $U_0 \leq U \leq V$, $Y_0 \leq Y \leq W$, $m|_{U\times Y}$ *nondegenerate, and* $\dim_K U = \dim_K Y \leq 2\max(\dim_K U_0, \dim_K Y_0)$.

Proof. We may assume that $\dim_K U_0 = \dim_K Y_0 = d$, say.

Let U_1 be a complement to ${}^{\perp}Y_0 + U_0$ in V:

$$V = ({}^{\perp}Y_0 + U_0) \oplus U_1 = {}^{\perp}Y_0 + (U_0 \oplus U_1)\,.$$

Set $U = U_0 \oplus U_1$ with $\dim_K U = k \leq 2d$. Now

$$0 = V^{\perp} = ({}^{\perp}Y_0)^{\perp} \cap U^{\perp} = Y_0 \cap U^{\perp}$$

by Lemma 3.4(2), so there is a Y with $Y_0 \leq Y \leq W$ and

$$W = Y \oplus U^{\perp}.$$

Here $U^{\perp}$ has codimension k in W; hence $\dim_K Y = \dim_K U = k \leq 2d$. As before

$$0 = {}^{\perp}W = {}^{\perp}Y \cap {}^{\perp}(U^{\perp}) = {}^{\perp}Y \cap U.$$

Since ${}^{\perp}Y$ has codimension k and U has dimension k,

$$V = {}^{\perp}Y \oplus U.$$

Therefore $m|_{U \times Y}$ is nondegenerate by Lemma 3.4(3).

We next wish to study self-pairings of the left K-space V. To make sense of this, we must give V the structure of a right K-space. When σ is an anti-isomorphism of K, V can be viewed as a right K-space V^{σ} whose addition is that of V but with scalar multiplication given by

$$b.v = v.b^{\sigma},$$

for all $v \in V$ and $b \in K$. The same equality allows us to associate with each right K-space V a left K-space $V^{\sigma^{-1}}$ (so that $(V^{\sigma})^{\sigma^{-1}} = V$). The identity map is an anti-isomorphism precisely when K is a field. The associated right (respectively, left) K-space V^1 is the *transpose* of the left (respectively, right) K-space V.

A self-pairing for V is then a pairing of V and V^{σ} (for some anti-isomorphism σ of K) and so can be thought of as a map $m\colon {}_KV \times {}_KV \longrightarrow K$ that is biadditive (as in (a) above) and satisfies the law

(b′) $m(av, bw) = a\,m(v,w)\,b^{\sigma}$, for all $v, w \in V$ and $a, b \in K$.

A map $m\colon V \times V \longrightarrow K$ with (a) and (b′) is usually called a *σ-sesquilinear form* on V. In particular, the classical reflexive sesquilinear forms can be discussed in this framework.

The σ-sesquilinear form is *reflexive* provided ${}^{\perp}U = U^{\perp}$, for all $U \subseteq V$. The three cases we study are the *classical sesquilinear forms*:

(1) s is a *symplectic form* on V if $s(x,x) = 0$, for all $x \in V$, so that $s(x,y) = -s(y,x)$, for all $x, y \in V$, with $\sigma = 1$.

(2) u is a *unitary form* on V if $u(x,y) = u(y,x)^{\sigma}$, for all $x, y \in V$, where σ has order 2.

(3) b is an *orthogonal form* on V if $b(x,y) = b(y,x)$, for all $x, y \in V$, with $\sigma = 1$. (Note that in characteristic 2 a symplectic form is a special type of orthogonal form.)

For a sesquilinear form f on V, a subspace U of V is *totally isotropic* if the pair (U, U) is totally isotropic. The subspace U is *nondegenerate* if the radicals $U \cap U^{\perp}$ and $U \cap {}^{\perp}U$ are both 0, that is, if the restriction of m to $U \times U$ is nondegenerate. For the reflexive form f, the *radical* of U is $\mathrm{Rad}(U, f) = U \cap U^{\perp}$.

Related to Lemma 3.5 is the well-known

(3.6) LEMMA. *Let f be a nondegenerate classical sesquilinear form on the K-space V. Let finite dimensional $U_0 \leq V$. Then there is a nondegenerate U with $U_0 \leq U \leq V$ and $\dim_K U \leq 2 \dim_K U_0$.*

A *quadratic form* $q\colon V \longrightarrow K$ on the (left) vector space V over the field K is a map that satisfies

(c) $q(av) = a^2 q(v)$, for all $a \in K$ and $v \in V$;

(d) $b(u, v) = q(u + v) - q(u) - q(v)$ is an orthogonal form on V.

In characteristic other than 2 we have $q(v) = b(v, v)/2$, and conversely $q(v) = b(v, v)/2$ gives a quadratic form associated with orthogonal b. Therefore in this case quadratic forms and orthogonal forms are essentially equivalent. When $\mathrm{char}\, K = 2$ the orthogonal form b associated with the quadratic form f is in fact symplectic, but a given symplectic form may have many associated quadratic forms.

If q is a quadratic form on V, then the subspace U is *totally singular* if the restriction of q to U is identically 0. A totally singular subspace for q must be totally isotropic for the associated orthogonal form b, but in characteristic 2 totally isotropic subspaces need not be totally singular.

We continue to call q nondegenerate when $\mathrm{Rad}(V, q) = \mathrm{Rad}(V, b) = V^{\perp} = 0$. We also say that q is *nonsingular* when its *singular radical*

$$\mathrm{SRad}(V, q) = \{\, v \in \mathrm{Rad}(V, q) \mid q(v) = 0 \,\}$$

is 0. If q is nondegenerate, then it is nonsingular. If $\mathrm{char}\, K \neq 2$ the converse is true, but if $\mathrm{char}\, K = 2$ this need not be the case.

Let K be a field of characteristic 2, and further assume that K is perfect. (That is, the Frobenius endomorphism $\varphi\colon a \mapsto a^2$ is an automorphism. This is certainly the case when K is finite, locally finite, or algebraically closed.) The restriction of q to $\mathrm{Rad}(V, q)$ then satisfies

$$0 = b(u, v) = q(u + v) - q(u) - q(v) \quad \text{and} \quad q(av) = a^2 q(v)\,,$$

for all $u, v \in \mathrm{Rad}(V, q)$ and $a \in K$. Therefore q is a φ-semilinear map from $\mathrm{Rad}(V, q)$ to K. The kernel of this map is $\mathrm{SRad}(V, q)$, which thus is a subspace of codimension at most 1 in $\mathrm{Rad}(V, q)$.

In the interest of uniformity, we shall refer to each of the various pairings and forms discussed above as a *form of type* Cl, for an appropriate

$\mathrm{Cl} \in \{\mathrm{GL}, \mathrm{SL}, \mathrm{Sp}, \mathrm{GU}, \mathrm{SU}, \mathrm{GO}, \Omega\}$. (The labels actually refer to the associated classical isometry groups. See Section 3.3 below.) Specifically, the form f is of type Sp if it is symplectic. The form f is of type GU or SU if it is a unitary σ-sesquilinear form. By a form of type GO or Ω, we shall mean a quadratic or orthogonal form (as determined by the context). If f is a pairing of some V and W, then f is a form of type GL or SL. For $\mathrm{Cl} \in \{\mathrm{Sp}, \mathrm{GU}, \mathrm{SU}, \mathrm{GO}, \Omega\}$, a form f of type Cl (but not quadratic) can be viewed either as a classical σ-sesquilinear form $f\colon V \times V \longrightarrow K$ or as a pairing $f\colon V \times V^\sigma \longrightarrow K$. Similarly, if V and W are both left spaces over the field K, then by a form f of type GL or SL on $V \times W$ we mean a pairing $f\colon V \times W^1 \longrightarrow K$ of V with the transpose of W.

Furthermore, when we say that f is a *form of type* Cl *with respect to* σ, we mean that either $\mathrm{Cl} \in \{\mathrm{GU}, \mathrm{SU}\}$ and f is a unitary σ-sesquilinear form with σ an order 2 automorphism of the associated field or $\mathrm{Cl} \notin \{\mathrm{GU}, \mathrm{SU}\}$ and σ is the identity automorphism of the field.

3.3. *Classical isometry groups.* If V is a left or right K-space, then $\mathrm{GL}_K(V)$ is the group of all invertible K-linear transformations. We also use $\mathrm{GL}(V_K)$ for a right K-space V and $\mathrm{GL}({}_KV)$ for a left space. The *finitary general linear group* $\mathrm{FGL}_K(V)$ is the corresponding group of invertible finitary linear transformations. If K is a field, then the determinant homomorphism $\det\colon \mathrm{FGL}_K(V) \longrightarrow K$, given by $\det(g) = \det(g|_{[V,g]})$, has kernel the *finitary special linear group* $\mathrm{FSL}_K(V)$. As is usual, we write $\mathrm{SL}_K(V)$ in place of $\mathrm{FSL}_K(V)$ when V has finite dimension over the field K.

As $\mathrm{GL}({}_KV)$ acts on V on the right and $\mathrm{GL}(W_K)$ acts on W on the left, the pair $a = (g,h) \in \mathrm{GL}({}_KV) \times \mathrm{GL}(W_K)$ acts on $V \times W$ on the right by

$$(v,w).a = (v,w).(g,h) = (v.g, h.w)\,,$$

for all $(v,w) \in V \times W$. (We also write $v.a$ for $v.g$ and $a.w$ for $h.w$.) We then have

$$(v,w)(g_1,h_1)(g_2,h_2) = (v.g_1, h_1.w)(g_2,h_2) = (v.g_1g_2, h_2h_1.w)\,.$$

Thus multiplication in the group $\mathrm{GL}({}_KV) \times \mathrm{GL}(W_K)$ is, for us, given by $(g_1,h_1)(g_2,h_2) = (g_1g_2, h_2h_1)$.

An *isometry* of the pairing $m\colon V \times W \longrightarrow K$ is an element $a = (g,h)$ of $\mathrm{GL}({}_KV) \times \mathrm{GL}(W_K)$ with

$$m(v,w) = m(v.g, h.w) = m(v.a, a.w)\,,$$

for all $(v,w) \in V \times W$. The subgroup of $\mathrm{GL}({}_KV) \times \mathrm{GL}(W_K)$ consisting of all isometries of m will be denoted $\mathrm{GL}_K(V,W,m)$. If G is a subgroup of $\mathrm{GL}_K(V,W,m)$, then we say that m is *G-invariant*.

We shall be concerned primarily with nondegenerate pairings. In these cases, by Lemma 3.3 we may view W as a subspace of V^* or V as a subspace of W^* and m as a restriction of m^{can}. Each element $g \in \mathrm{GL}_K(V)$ acts naturally on V^* via

$$v(g\mu) = (vg)\mu\,,$$

for all $v \in V$ and $\mu \in V^*$; hence $(g, g^{-1}) \in \mathrm{GL}_K(V, V^*, m^{\mathrm{can}})$.

(3.7) LEMMA. *Let* $m\colon V \times W \longrightarrow K$ *be a pairing, and let* $A \le \mathrm{GL}_K(V, W, m)$.

(1) *With a slight abuse of notation,*

$$C_{W/V^\perp}(A) = \bigcap_{(g,h)\in A} C_{W/V^\perp}(h) = \sum_{(g,h)\in A} (V(g-1))^\perp = [V, A]^\perp$$

and

$$C_{V/{}^\perp W}(A) = \bigcap_{(g,h)\in A} C_{V/{}^\perp W}(g) = \sum_{(g,h)\in A} {}^\perp((h-1)W) = {}^\perp[A, W]\,.$$

(2) *If the restriction of* m *to* $[V, A] \times [A, W]$ *is trivial, then*

$$[[V, A], A] = [V, A, A] \le {}^\perp W$$

and

$$[A, [A, W]] = [A, A, W] \le V^\perp\,.$$

Proof. (1) For all $v \in V$, fixed $w \in W$, and $a = (g, h) \in A$,

$$\begin{aligned} m(v(g-1), w) &= m(vg, w) - m(v, w) \\ &= m(vg, w) - m(vg, hw) = m(vg, (1-h)w)\,. \end{aligned}$$

Therefore $w \in V(g-1)^\perp = [V, a]^\perp$ if and only if $w + V^\perp \in C_{W/V^\perp}(h) = C_{W/V^\perp}(a)$.

(2) By (1) and assumption, $C_{W/V^\perp}(A) = [V, A]^\perp \ge [A, W]$.

(3.8) PROPOSITION. *Let* $m\colon V \times W \longrightarrow K$ *be a nondegenerate pairing, and let* $a = (g, h) \in \mathrm{GL}_K(V, W, m)$.

(1) $g = 1$ *if and only if* $h = 1$.

(2) $g \in \mathrm{FGL}_K(V)$ *if and only if* $h \in \mathrm{FGL}_K(W)$. *In this case* $\deg_V g = \deg_W h$ *(written as* $\deg_{V\times W} a$*).*

(3) *For* K *a field and* $\dim_K V = \dim_K W$ *finite,* $g \in \mathrm{SL}_K(V)$ *if and only if* $h \in \mathrm{SL}_K(W)$.

(4) *For* K *a field,* $g \in \mathrm{FSL}_K(V)$ *if and only if* $h \in \mathrm{FSL}_K(W)$.

Proof. Part (1) is an immediate consequence of Lemma 3.7(1).

For (2) assume that $\deg_V g$ is finite. Then

$$\begin{aligned}\deg_V g &= \dim_K V(g-1) = \operatorname{codim}_K V(g-1)^{\perp} \\ &= \operatorname{codim}_K C_W(h) = \dim_K (h-1)W = \deg_W h\,,\end{aligned}$$

as desired.

By Lemma 3.3, for (3) we can identify W with V^*, so that the elements of $\mathrm{GL}_K(V,W,m)$ have the form (g,g^{-1}), as g runs over $\mathrm{GL}_K(V)$. Since $(\det g)^{-1} = \det g^{-1}$, (3) follows.

Part (4) is then a consequence of (2), (3), and Lemma 3.5.

Let $\mathrm{FGL}_K(V,W,m)$ consist of those elements $(g,h) \in \mathrm{GL}_K(V,W,m)$ with $g \in \mathrm{FGL}_K(V)$ and $h \in \mathrm{FGL}_K(W)$. Similarly, for a field K, let $\mathrm{FSL}_K(V,W,m)$ consist of those elements $(g,h) \in \mathrm{GL}_K(V,W,m)$ with $g \in \mathrm{FSL}_K(V)$ and $h \in \mathrm{FSL}_K(W)$. For finite dimensional V and W over a field K, $\mathrm{SL}_K(V,W,m)$ will be the subgroup of all $(g,h) \in \mathrm{GL}_K(V,W,m)$ with $g \in \mathrm{SL}_K(V)$ and $h \in \mathrm{SL}_K(W)$.

By Proposition 3.8(1), for a nondegenerate pairing m, restriction to the first coordinate, $(g,h) \mapsto (g,h)|_V = g$, gives an isomorphism of $\mathrm{GL}_K(V,W,m)$ with a subgroup of $\mathrm{GL}_K(V)$. Similarly, $(g,h) \mapsto (g,h)|_W = h$ is an anti-isomorphism of $\mathrm{GL}_K(V,W,m)$ into $\mathrm{GL}_K(W)$. In particular,

(3.9) Corollary. (1) *For K a division ring,*

$$\mathrm{GL}_K(V,V^*,m^{\mathrm{can}}) \simeq \mathrm{GL}_K(V,V^*,m^{\mathrm{can}})|_V = \mathrm{GL}_K(V)$$

and

$$\mathrm{FGL}_K(V,V^*,m^{\mathrm{can}}) \simeq \mathrm{FGL}_K(V,V^*,m^{\mathrm{can}})|_V = \mathrm{FGL}_K(V)\,.$$

(2) *For K a field,*

$$\mathrm{FSL}_K(V,V^*,m^{\mathrm{can}}) \simeq \mathrm{FSL}_K(V,V^*,m^{\mathrm{can}})|_V = \mathrm{FSL}_K(V)\,.$$

(3.10) Corollary. *Let $m\colon U \times Y \longrightarrow K$ be a nondegenerate pairing with U or Y finite dimensional over the division ring K.*

(1) *We have*

$$\mathrm{GL}_K(U,Y,m) \simeq \mathrm{GL}_K(U,Y,m)|_U = \mathrm{GL}({}_K U) = \mathrm{GL}_K(U)$$

and

$$\mathrm{GL}_K(U,Y,m) \simeq \mathrm{GL}_K(U,Y,m)|_Y = \mathrm{GL}(Y_K) = \mathrm{GL}_K(Y)\,.$$

(2) *For K a field,*

$$\mathrm{SL}_K(U,Y,m) \simeq \mathrm{SL}_K(U,Y,m)|_U = \mathrm{SL}_K(U)$$

and

$$\mathrm{SL}_K(U,Y,m) \simeq \mathrm{SL}_K(U,Y,m)|_Y = \mathrm{SL}_K(Y)\,.$$

(3.11) THEOREM. *Let K be a field and U a K-space of finite dimension at least 3. Then $\mathrm{SL}_K(U,Y,m)$ is quasisimple if and only if m is nondegenerate. In this case $\mathrm{SL}_K(U,Y,m) = \mathrm{GL}_K(U,Y,m)'$.*

Proof. This follows from [42, Th. 4.4].

If σ is an anti-isomorphism of K and $g \in \mathrm{GL}_K(V)$, then we define an associated $g^\sigma \in \mathrm{GL}_K(V^\sigma)$ acting on the left:

$$g^\sigma.v = v.g \quad \text{or, equivalently,} \quad v.g^{\sigma^{-1}} = g.v\,.$$

For a basis $\{\, e_i \,|\, i \in I \,\}$ of V, if we have $e_i.g = \sum_{j\in I} g_{ij} e_j$ then $g^\sigma.e_i = \sum_{j\in I} e_j.g^\sigma_{ij}$; so the matrix representing g^σ in this basis is the transpose-σ-conjugate of that representing g. In the special case of a field K and the identity anti-isomorphism $\sigma = 1$, the element g^1 acts on the transpose space V^1 as the transpose of g. When $g \in \mathrm{GL}_K(V)$ acts on V on the left via transposes, we have $[g^1, V^1] = [V, g]$; so we write $[g, V] = [V, g]$. We further this by setting $[A, V] = [V, A]$ for all $A \subseteq \mathrm{GL}_K(V)$ when K is a field.

An *isometry* of the σ-sesquilinear form $f\colon V \times V \longrightarrow K$ is a $g \in \mathrm{GL}_K(V)$ with

$$f(u,v) = f(ug, vg)\,,$$

for all $u, v \in V$. In terms of the associated pairing $m\colon V \times V^\sigma \longrightarrow K$, we have

$$m(u,v) = f(u,v) = f(ug,vg) = m(ug, g^\sigma v).$$

Therefore g is an isometry of f if and only if $(g, g^\sigma) \in \mathrm{GL}_K(V, V^\sigma, m)$.

For finite dimensional V and nondegenerate m, we can identify V^σ with V^*, in which case $(g, g^{-1}) \in \mathrm{GL}_K(V, V^\sigma, m)$. By Proposition 3.8(1) we conclude in this case that $g^{-1} = g^\sigma$. We have recovered the familiar matrix identity $gg^\sigma = 1$.

An *isometry* of the quadratic form $q\colon V \longrightarrow K$ is a $g \in \mathrm{GL}_K(V)$ with

$$q(v) = q(vg)\,,$$

for all $v \in V$. Isometries of q are also isometries of the associated orthogonal form b.

The full isometry group of a form f of type $\mathrm{Cl} \in \{\mathrm{Sp}, \mathrm{GU}, \mathrm{GO}\}$ on the K-space V is written $\mathrm{Cl}_K(V,f)$. The corresponding finitary isometry group is then $\mathrm{FCl}_K(V,f) = \mathrm{FGL}_K(V) \cap \mathrm{Cl}_K(V,f)$. When K is a field we have $\mathrm{FSp}_K(V,f) \le \mathrm{FSL}_K(V)$. We set $\mathrm{FSU}_K(V,f) = \mathrm{FSL}_K(V) \cap \mathrm{GU}_K(V,f)$ and $\mathrm{F\Omega}_K(V,f) = \mathrm{FGO}_K(V,f)'$ (often proper in $\mathrm{FSL}_K(V) \cap \mathrm{GO}_K(V,f)$; see [42, 11.44, 11.51]). As usual, when V has finite dimension over the field K we write $\mathrm{SU}_K(V,f)$ in place of $\mathrm{FSU}_K(V,f)$ and $\Omega_K(V,f)$ in place of $\mathrm{F\Omega}_K(V,f)$. The groups $\mathrm{Cl}_K(V,f)$ for $\mathrm{Cl} \in \{\mathrm{GL}, \mathrm{SL}, \mathrm{Sp}, \mathrm{GU}, \mathrm{SU}, \mathrm{GO}, \Omega\}$ are the *classical groups.*

Another common piece of notation for the finite classical groups is $\mathrm{Cl}_n(q)$ for $\mathrm{Cl}_{\mathbb{F}_q}(\mathbb{F}_q^n)$ (so, for instance, $\mathrm{SL}_n(q) = \mathrm{SL}_{\mathbb{F}_q}(\mathbb{F}_q^n)$). This notation presupposes a nondegenerate or nonsingular form. If $\mathrm{Cl} \notin \{\mathrm{GO}, \Omega\}$, then a nondegenerate form of type Cl on $\mathbb{F}_q^n$ is essentially unique, and the isometry group is uniquely determined up to isomorphism by the parameters Cl, n, q. If $\mathrm{Cl} \in \{\mathrm{GO}, \Omega\}$ then there are at most two essentially distinct nonsingular quadratic forms on $\mathbb{F}_q^n$, so there are at most two distinct isometry groups. (See [42, pp. 138-9] for a precise discussion.)

One often writes $\mathrm{PCl}_K(V, f)$ for the group induced by $\mathrm{Cl}_K(V, f)$ on the projective space $\mathbb{P}V$. For nondegenerate forms the kernel will consist of scalars. The finite groups $\mathrm{PCl}_n(q)$ are typically the simple quotients of the quasisimple groups $\mathrm{Cl}_n(q)$. (See Theorems 3.11 and 3.13.) Nevertheless, the projective groups appear rarely in the present work, because a nonidentity scalar acting on an infinite dimensional space is not a finitary transformation.

Let f be a classical σ-sesquilinear form of type $\mathrm{Cl} \in \{\mathrm{Sp}, \mathrm{GU}, \mathrm{SU}\}$. We have seen above that $g \in \mathrm{Cl}_K(V, f)$ if and only if $(g, g^\sigma) \in \mathrm{GL}_K(V, V^\sigma, f)$. We set

$$\mathrm{Cl}_K(V, V^\sigma, f) = \{\, (g, g^\sigma) \mid g \in \mathrm{Cl}_K(V, f) \,\} \leq \mathrm{GL}_K(V, V^\sigma, f)\,.$$

The corresponding finitary group is

$$\mathrm{FCl}_K(V, V^\sigma, f) = \{\, (g, g^\sigma) \mid g \in \mathrm{FCl}_K(V, f) \,\} \leq \mathrm{FGL}_K(V, V^\sigma, f)\,.$$

Similarly for the quadratic form f on the K-space V over the field K and $\mathrm{Cl} \in \{\mathrm{GO}, \Omega\}$, we set

$$\mathrm{Cl}_K(V, V^1, f) = \{\, (g, g^1) \mid g \in \mathrm{Cl}_K(V, f) \,\} \leq \mathrm{GL}_K(V, V^1, b)$$

and

$$\mathrm{FCl}_K(V, V^1, f) = \{\, (g, g^1) \mid g \in \mathrm{FCl}_K(V, f) \,\} \leq \mathrm{FGL}_K(V, V^1, b)\,,$$

where b is the orthogonal form associated with the quadratic form f. In all cases we have $\mathrm{Cl}_K(V, f) = \mathrm{Cl}_K(V, V^\sigma, f)|_V$ and $\mathrm{FCl}_K(V, f) = \mathrm{FCl}_K(V, V^\sigma, f)|_V$. (Compare Corollary 3.9.) The various groups $\mathrm{Cl}_K(V, W, f)$ (including SL and *GL*) are the *classical isometry groups*. We sometimes blur the distinction between a classical isometry group and the corresponding classical group.

If G is a subgroup of $\mathrm{Cl}_K(V, W, f)$ or the corresponding classical group, then we say that f is a *G-invariant form* of type Cl.

(3.12) PROPOSITION. (1) *Assume V is a vector space over the perfect field K in characteristic 2 and that the quadratic form q is degenerate but nonsingular on finite dimensional $V = K^n$. Then $n = 2m + 1$ is odd, and $R = \mathrm{Rad}(V, b)$ has dimension 1. The associated form b is symplectic and induces a nondegenerate symplectic form $\tilde{b}$ on $\tilde{V} = V/R$. Furthermore*

$$\Omega_K(V, q) \simeq \mathrm{Sp}_K(\tilde{V}, \tilde{b})\,.$$

(2) *For K a finite field of characteristic* 2 *and a nondegenerate symplectic form s on $\tilde{V} = K^{2m}$, there is a nonsingular quadratic form q on $V = K^{2m+1}$ with $R = \mathrm{Rad}(V, b)$ of dimension* 1*, $V/R = \tilde{V}$, and $s = \tilde{b}$. Furthermore*

$$\mathrm{Sp}_K(\tilde{V}, s) \simeq \Omega_K(V, q)\,.$$

Proof. See Taylor [42, Th. 11.9].

Therefore the isometry group of a nondegenerate symplectic form over a finite field of characteristic 2 can be thought of as the isometry group of a degenerate, nonsingular quadratic form over the same field.

(3.13) Theorem. *Let V have finite dimension at least* 6 *over the finite field K. Then $\mathrm{Sp}_K(V, s)$, $\mathrm{SU}_K(V, u)$, and $\Omega_K(V, q)$* (*respectively*) *are quasisimple if and only if s and u are nondegenerate and q is nonsingular* (*respectively*).

Proof. See Taylor [42, Ths. 8.8, 10.23, 11.48].

(3.14) Proposition. *Let* $\mathrm{Cl} \in \{\mathrm{Gl}, \mathrm{SL}, \mathrm{Sp}, \mathrm{GU}, \mathrm{SU}, \mathrm{GO}, \Omega\}$*, and let $m\colon V \times W \longrightarrow K$ be a nondegenerate form of type* Cl*. The group $\mathrm{FCl}_K(V, W, m)$ has a subgroup cover consisting of those subgroups*

$$G_{U,Y} \simeq \mathrm{Cl}_K(U, Y, m|_{U\times Y})$$

with U finite dimensional in V, Y finite dimensional in W, and $m|_{U\times Y}$ nondegenerate. Here the element (g, h) of $G_{U,Y}$ corresponding to the element $(g_0, h_0) \in \mathrm{Cl}_K(U, Y, m|_{U\times Y})$ acts on $V = U \oplus {}^{\perp}Y$ via $g|_U = g_0$ and ${}^{\perp}Y.(g-1) = 0$ and acts on $W = Y \oplus U^{\perp}$ via $h|_Y = h_0$ and $(h-1).U^{\perp} = 0$.

If $W = V^{\sigma}$ for m a nondegenerate form of type $\mathrm{Cl}\,(\neq \mathrm{SL}, \mathrm{GL})$ *on V with respect to σ, then this remains true with $Y = U^{\sigma}$ additionally.*

Proof. The subgroups $G_{U,Y}$ are certainly in $\mathrm{FCl}_K(V, W, m)$ and are directed by containment. Each $(g, h) \in \mathrm{FCl}_K(V, W, m)$ is in some $G_{U,Y}$ by Lemma 3.5 with $U_0 = V(g-1)$ and $Y_0 = (h-1)W$. If we have a quadratic or classical σ-sesquilinear form on V, we instead use Lemma 3.6 with $U_0 = V(g-1)$.

(3.15) Theorem. *For V and W of dimension at least* 6 *over the locally finite field K and nondegenerate* (*or nonsingular*) *f of type* $\mathrm{Cl} \in \{\mathrm{SL}, \mathrm{Sp}, \mathrm{SU}, \Omega\}$*, the finitary group $\mathrm{FCl}_K(V, W, f)$ is locally finite and quasisimple. Indeed if V and W are infinite dimensional, then $\mathrm{FCl}_K(V, W, f)$ is simple and is not linear in finite dimension.*

Proof. First consider the finite dimensional case. By Proposition 3.12, the result for nonsingular f follows from the nondegenerate case.

Let S be a finite subset of $\mathrm{Cl}_K(V, f) \simeq \mathrm{Cl}_K(V, W, f) = \mathrm{FCl}_K(V, W, f)$. Choose a basis $\mathcal{B} = \{\, e_i \mid i \in I \,\}$ for V. Let the set S' consist of all the $f(e_i, e_j)$

(or $f(e_i)$ and $f(e_i+e_j)$), for $i,j\in I$, and all entries of the $|S|$ matrices that describe the action of members of S on $\mathcal{B}$. Then S' is a finite subset of K and so lies in a finite subfield K_S of K on which the associated automorphism is nontrivial in the case $\mathrm{Cl}=\mathrm{SU}$. Let V_S be the K_S-span of $\mathcal{B}$. Then V_S is a $K_S\langle S\rangle$-submodule of $V=K\otimes_{K_S}V_S$. Therefore $S\subseteq \mathrm{Cl}_{K_S}(V_S,f|_{V_S})$, a finite quasisimple subgroup of $\mathrm{Cl}_K(V,f)$ by Theorems 3.11 and 3.13. As this was true for any finite subset S, we have proved that $\mathrm{Cl}_K(V,f)$ has a finite quasisimple subgroup cover. As the cover is finite, $\mathrm{Cl}_K(V,f)$ is locally finite. Since the cover is quasisimple, $\mathrm{Cl}_K(V,f)$ is quasisimple by Lemma 2.1.

Now we consider the case of infinite dimensional V and W. Again the result for nonsingular f follows from the nondegenerate case and so we assume f to be nondegenerate.

By Proposition 3.14 and the finite dimensional case, $\mathrm{FCl}_K(V,W,f)$ has a locally finite, quasisimple subgroup cover. Therefore by Lemma 2.1, G itself is locally finite and quasisimple. Central elements are scalar by Schur's lemma and Proposition 3.14, but the identity is the only finitary scalar on an infinite dimensional space. Therefore quasisimple $\mathrm{FCl}_K(V,W,f)$ is simple. By Proposition 3.14, $\mathrm{FCl}_K(V,W,f)$ has alternating sections of arbitrarily large degree. Therefore by Proposition 3.1 it is not linear of any finite degree.

3.4. *ℓ-root elements.* The finitary symmetric group is generated by its 2-cycles, and the alternating group is essentially defined as the group generated by all 3-cycles. The classical groups also have special generating elements of small degree called root elements—the transvections (of degree 1) and the orthogonal Siegel elements (of degree 2).

By Proposition 3.8(2), the element $t=(g,h)$ of $\mathrm{FCl}_K(V,W,f)$ has

$$\dim_K V(g-1)=\dim_K (h-1)W=\deg_{V\times W} t=\ell\,,$$

say. In this case we call t an *ℓ-root element* provided that the restriction of f to the commutator of t is trivial. That is, $(V(g-1),(h-1)W)$ is totally isotropic when f is not a quadratic form and $V(g-1)$ is totally singular when f is a quadratic form. The identity is the only 0-root element.

(3.16) Lemma. *Let $t\in \mathrm{FCl}_K(V,W,f)$ with $\deg_{V\times W} t=\ell$.*

(1) *Assume that f is nondegenerate and that f is not a quadratic form when $\mathrm{char}\,K=2$. Then t is an ℓ-root element if and only if $(t-1)^2=0$. (That is, $V(g-1)^2=0$ and $(h-1)^2W=0$.)*

(2) *Assume that f is a nonsingular quadratic form. Then t is an ℓ-root element if and only if $(t-1)^2=0$ and $v\in v^\perp(t-1)$ for all $v\in V(t-1)$ if and only if $(t-1)^2=0$ and $v\in v^\perp(t-1)$ for a spanning set of $v\in V(t-1)$.*

(3) *If t is an ℓ-root element, then $t\in \mathrm{FSL}_K(V,W,f)$.*

Proof. $[V, t]$ is the image of $t - 1$ and $C_V(t)$ is the kernel of $t - 1$, so (1) follows directly from Lemma 3.7.

For (2), Lemma 3.7 still applies to say that $V(t - 1)$ is totally isotropic if and only if $V(t - 1)^2 \leq V^\perp$, the only singular vector of this radical being 0. Let b be the orthogonal form associated with f. For $v = x(t - 1)$, we have

$$\begin{aligned} f(v) &= f(xt - x) \\ &= f(xt) + f(-x) + b(xt, -x) \\ &= f(x) + f(-x) + b(x + v, -x) \\ &= f(x) + f(-x) + b(x, -x) + b(v, -x) \\ &= 0 + b(v, -x) = b(v, -x)\,. \end{aligned}$$

Thus $v \in V(t - 1)$ is singular if and only if $v \in v^\perp(t - 1)$. As the singular vectors of the totally isotropic $V(t-1)$ form a K-subspace, we need only check a spanning set to see if $V(t - 1)$ is totally singular. This completes (2).

(3) holds as $(t - 1)^2 = 0$.

Given an ℓ-root element t, its associated *ℓ-root subgroup* is the subgroup consisting of all elements t_0 with $V(t - 1) \geq V(t_0 - 1)$ and $(t - 1)W \geq (t_0-1)W$. (All such t_0 will themselves be ℓ-root elements for possibly smaller ℓ.) A $\mathrm{Cl}_K(V, W, f)$ conjugate of an ℓ-root element is an ℓ-root element, and a $\mathrm{Cl}_K(V, W, f)$ conjugate of an ℓ-root subgroup is an ℓ-root subgroup.

We are interested in the cases $\ell = 1$ or 2.

An element $t \in \mathrm{GL}_K(V)$ with $\deg_V t = 1$ and $(t-1)^2 = 0$ is a *transvection*. Every transvection t has the form $t(v, \lambda)$, with action on $x \in V$ given by

$$x.t(v, \lambda) = x + x\lambda.v\,,$$

for some nonzero $v \in V$ and $\lambda \in V^*$ with $v\lambda = 0$. The 1-space $\langle v\rangle = Kv \leq V$ is called the *center* of $t(\lambda, v)$ while the 1-space $\langle \lambda\rangle \leq V^*$ is its *axis*. A transvection on V also is a transvection on V^*, the action on V^* given by

$$t(v, \lambda).\mu = \mu + \lambda.v\mu\,.$$

Thus the transvections of $\mathrm{GL}_K(V)$ are exactly those elements giving 1-root elements of $\mathrm{GL}_K(V, V^*, m^{\mathrm{can}})$.

(3.17) THEOREM. *Let K be a field.*

(1) *Let $m\colon V \times W \longrightarrow K$ be a nondegenerate pairing. The 1-root elements of $\mathrm{SL}_K(V, W, m)$ are the transvections $t(v, w)$, which are given by*

$$x.t(v, w) = x + m(x, w)v, \quad \textit{for} \quad x \in V,$$

and

$$t(v, w).y = y + w\, m(v, y), \quad \textit{for} \quad y \in W,$$

for any nonzero $v \in V$ *and* $w \in W$ *with* $m(v, w) = 0$. *The* 1*-root subgroup containing the transvection* $t(v, w)$ *is* $T_{v,w} = \{\, t(bv, w) \mid b \in K \,\} \simeq (K, +)$.

(2) *Let* s *be a nondegenerate symplectic form on the space* V *over* K.

(a) *The transvections of* $\mathrm{Sp}_K(V, s)$ *are the* 1*-root elements of* $\mathrm{Sp}_K(V, s)$.

(b) *The* 1*-root elements of* $\mathrm{Sp}_K(V, s)$ *are the symplectic transvections* $t(av, v)$, *which are given by*

$$x.t(av, v) = x + s(x, v).av\,,$$

for any nonzero $v \in V$ *and nonzero* $a \in K$. *The* 1*-root subgroup containing the symplectic transvection* $t(av, v)$ *is* $T_{v,v} = \{\, t(bv, v) \mid b \in K \,\} \simeq (K, +)$.

(3) *Let* u *be a nondegenerate* σ*-sesquilinear unitary form on the space* V *over* K.

(a) *The transvections of* $\mathrm{SU}_K(V, u)$ *are the* 1*-root elements of* $\mathrm{SU}_K(V, u)$.

(b) *The* 1*-root elements of* $\mathrm{SU}_K(V, u)$ *are the unitary transvections* $t(av, v)$, *which are given by*

$$x.t(av, v) = x + u(x, v).av\,,$$

for any nonzero $v \in V$ *with* $u(v, v) = 0$ *and any nonzero* a *from the subgroup* $K_\sigma = \{\, c \in K \mid c + c^\sigma = 0 \,\}$. *The* 1*-root subgroup containing the unitary transvection* $t(av, v)$ *is* $T_{v,v} = \{\, t(bv, v) \mid b \in K_\sigma \,\} \simeq (K_\sigma, +)$.

(4) *Let* q *be a nonsingular quadratic form on the space* V *over* K *with associated orthogonal form* b.

(a) $\Omega_K(V, q)$ *contains no transvections.*

(b) *The* 2*-root elements of* $\Omega_K(V, q)$ *are the Siegel elements, which are given by*

$$x.r(v, w) = x + b(x, v)w - b(x, w)v\,,$$

for $v, w \in V$ *with* $U = \langle v, w \rangle$ *any totally singular* 2*-space. The* 2*-root subgroup containing the Siegel element* $r(v, w)$ *is* $R_U = \{\, r(av, w) \mid a \in K \,\} \simeq (K, +)$.

Proof. The structure of the ℓ-root elements follows from relatively easy calculations. The corresponding ℓ-root subgroups are then apparent. For (4a), any transvections of $\mathrm{GO}_K(V, q)$ are not in the derived group $\Omega_K(V, q)$; see [42, Th. 11.43].

Note that for consistency we set

$$t(v, 0) = 1 = t(0, w) \quad \text{and} \quad r(v, 0) = 1 = r(0, w)\,.$$

(3.18) Lemma. *Let* K *be a field and* $T_{v,w}$ *a transvection subgroup of the group* $\mathrm{SL}_K(V, W, f)$.

(1) *If* $1 \neq T \leq T_{v',w'}$ *with* $f(v,w') \neq 0 \neq f(v',w)$, *then*

$$\langle T, T_{v,w} \rangle = \mathrm{SL}_K(\langle v, v' \rangle, \langle w, w' \rangle, f|_{\langle v,v' \rangle \times \langle w,w' \rangle}) \simeq \mathrm{SL}_2(K)\,.$$

(2) *If further* $x \in W$ *with* $f(v,x) = 0 = f(v',x)$, *then* $T_{v',w'}$ *and* $T_{v',w'+x}$ *are conjugate transvection subgroups of* $\langle T, T_{v,w}, T_{v,w+x} \rangle \simeq K^2 \colon \mathrm{SL}_2(K)$, *where* K^2 *is a natural module for* $\mathrm{SL}_2(K)$.

Proof. This is well-known and an elementary calculation.

Let V be a K-space and W a subspace of V^*. The groups

$$\mathrm{T}_K(W,V) = \langle t(v,\lambda) \,|\, v \in V, \lambda \in W,\ v.\lambda = 0 \rangle \leq \mathrm{GL}_K(V)$$

were introduced in [7]. By Proposition 3.14 we have

(3.19) Proposition. *Let* K *be a field. For* $W \leq V^*$ *with* $m = m^{\mathrm{can}}|_{V \times W}$ *nondegenerate, we have* $\mathrm{T}_K(W,V) = \mathrm{FSL}_K(V,W,m)|_V \simeq \mathrm{FSL}_K(V,W,m)$.

4. Representations of finite groups

4.1. *Unique nontrivial composition factors.*

(4.1) Proposition (Meierfrankenfeld [15, 4.13]). *Let finite* $B = O^p(B)$ *act on the finite dimensional* F*-vector space* U *in characteristic* p *with a unique nontrivial composition factor. Assume also that* $B = B_0 O_p(B)$ *for* $B_0 \leq B$ *implies* $B = B_0$. *Then* $[B, O_p(B)] \leq C_B(U)$.

Proof. We proceed by induction on $\dim_F U$. Set $Q = [B, O_p(B)]$.

First assume that $[U,Q]$ is not trivial as an FB-module. Then since $B = O^p(B)$, the unique nontrivial composition factor is in $[U,Q] = [U,B]$. As Q itself is unipotent, we also have $[U,Q] < U$.

Let Y be a B-invariant hyperplane of U that contains $[U,Q] = [U,B]$. By induction we have $Y \leq C_U(Q)$. In particular, the action of Q on U is quadratic: $[U,Q,Q] = 0$. Choose $x \in U \setminus Y$, so that $U = Fx \oplus Y$ and $[U,Q] = [Fx,Q]$. By quadratic action the set $W = \{[x,q] \,|\, q \in Q\}$ is an $\mathbb{F}_p Q$-submodule of U. Indeed it is an $\mathbb{F}_p B$-submodule since $[U,B] \leq C_U(Q)$, so that, for $q \in Q$ and $b \in B$,

$$[x,q]^b = [x^b, q^b] = [x + [x,b], q^b] = [x, q^b] + [[x,b], q^b] = [x, q^b] \in W\,.$$

Consider now the FB-module $\bar{U} = U/C_U(B)$. The image of $[U,B]$ is an irreducible FB-submodule $\bar{T}$ with $\bar{T} = F\bar{W}$ because $[U,B] = [Fx,Q]$. As an $\mathbb{F}_p B$-module (of possibly infinite dimension), $\bar{T}$ has a nonzero irreducible submodule $\bar{W}_0$ within finite $\bar{W}$. Thus $\bar{T} = F\bar{W} = F\bar{W}_0$ is a sum of $\mathbb{F}_p B$-irreducibles and so is completely reducible. Therefore $\bar{W}$ is complemented in $\bar{T}$; there is an $\mathbb{F}_p B$-submodule $\bar{Z}$ of $\bar{T}$ with $\bar{T} = \bar{W} \oplus \bar{Z}$.

For an arbitrary $b \in B$, we have

$$\bar{x}^b = \bar{x} + [\bar{x}, b] = \bar{x} + (\bar{w} + \bar{z}) = \bar{x} + [\bar{x}, q] + \bar{z} = \bar{x}^q + \bar{z}\,,$$

where $\bar{z}$ is in $\bar{Z}$ and $\bar{w}$ is in $\bar{W}$, so that $\bar{w} = [\bar{x}, q]$, for some $q \in Q$. Therefore $(\bar{x} + \bar{Z})^b = (\bar{x} + \bar{Z})^q$, and generally $(\bar{x} + \bar{Z})^B = (\bar{x} + \bar{Z})^Q$. By a Frattini argument ([1, (6.3)]) $B = QN_B(\bar{x} + \bar{Z})$, so by assumption $B = N_B(\bar{x} + \bar{Z})$. That is, for each $b \in B$, we have $[\bar{x}, b] \in \bar{Z}$. In particular for each $q \in Q$ this gives $[\bar{x}, q] \in \bar{Z}$, but already $[\bar{x}, q] \in \bar{W}$. Therefore $[\bar{x}, q] \in \bar{W} \cap \bar{Z} = \bar{0}$. We conclude that $[\bar{x}, Q] = \bar{0}$, which is not true since $\bar{W}$ is nonzero. The contradiction shows that this case cannot occur, and therefore $[U, Q]$ must be trivial as an FB-module.

Dually, $U/C_U(Q)$ is a trivial B-module. Therefore

$$[U, Q, B] = [B, U, Q] = 0\,,$$

whence $[Q, B, U] = 0$ by the Three Subgroups Lemma [1, (8.7)]. As $B = O^p(B)$, we have $[Q, B] = [O_p(B), B, B] = [O_p(B), B] = Q$. Therefore $0 = [Q, B, U] = [Q, U]$; that is, Q is trivial on U, as required.

4.2. *Representation of finite alternating groups.* In the next result, the function $c(d)$ is a fixed nondecreasing function, defined on the positive integers. While it is possible to give a precise function, this is not necessary for us. The existence of such a function is good enough.

(4.2) Theorem (Jordan's Theorem [21]). *Let finite $H \leq \mathrm{Sym}(\Omega)$ be primitive on Ω. There is a nondecreasing function $c(d)$ such that, if H contains a nontrivial element of degree at most d on Ω with $|\Omega| > c(d)$, then H is* $\mathrm{Alt}(\Omega)$ *or* $\mathrm{Sym}(\Omega)$.

This can be found in [10, Th. 3.3D] with a short and elementary proof that uses a function $c(d)$ exponential in $d \log d$. A more sophisticated but still elementary argument of Babai [2, Th. 0.3] shows that Jordan's Theorem remains valid with $c(d) = 4d^2$ (see [10, Ths. 5.3A, 5.4A]).

(4.3) Proposition. *Let finite H be contained in $\mathrm{GL}_F(U)$ with H irreducible but not primitive on finite dimensional U. Let $\Omega = \{\, U_k \mid 1 \leq k \leq m \,\}$ be the blocks of a system of imprimitivity that maximizes $\dim_F U_k = e$. With the function $c(d)$ of Theorem* 4.2, *we have*:

If H is generated by a set D of elements of degree at most d on U with $\dim_F U > d\,c(2d)$, then in its action on Ω the group H induces $\mathrm{Alt}(\Omega)$ *or* $\mathrm{Sym}(\Omega)$ *with $|\Omega| \geq (\dim_F U)/d$. The kernel of this action is a subgroup of $\prod_\Omega \mathrm{GL}_e(F)$. Each element $g \in D$ permutes Ω with degree at most $\lfloor 2d/e \rfloor$. In particular $e \leq d$.*

Proof. See [15, Prop. 3.1], [36, Th. 9.1], [37].

The group $\bar{H}$ induced on Ω is transitive and generated by the elements of $\bar{D}$. As the U_i are maximal blocks, this action is actually primitive. For $g \in D$, if $U_i^g \neq U_i$ then $[U_i, g] = U_i(g-1)$ has dimension $e \leq d$. Therefore $d \geq e\bar{d}/2$, where $\bar{d}$ is the degree of $\bar{g}$ on Ω. Hence $\bar{d} \leq \lfloor 2d/e \rfloor$ while $|\Omega| = (\dim_F U)/e \geq (\dim_F U)/d$ (as claimed).

We have

$$|\Omega| \geq (\dim_F U)/d > d\,c(2d)/d = c(2d) \geq c(\lfloor 2d/e \rfloor) \geq c(\bar{d})\,;$$

so by Jordan's Theorem 4.2, $\bar{H}$ is $\mathrm{Alt}(\Omega)$, or $\mathrm{Sym}(\Omega)$ as desired. The kernel of the action is contained in $\prod_k \mathrm{GL}_F(U_k)$, as described.

(4.4) Proposition. *Let $G \simeq \mathrm{Alt}(\Omega)$ for finite $|\Omega| \geq 5$. If V is a natural KG-module and E is a subfield of K, then within V there is a natural EG-submodule Y with $V = K \otimes_E Y$. As an EG-module, V is completely reducible with all irreducible submodules equal to $c \otimes Y$, for some $c \in K$.*

Proof. This is well-known. Matrices for the representation of G on its permutation module $K\Omega$ all have entries 0 or 1. The module V is then isomorphic to $[K\Omega, G]/[K\Omega, G] \cap K(1,1,1,\ldots,1)$. As the augmentation module $[K\Omega, G]$ consists of those vectors with coefficient sum 0, the representing matrices all have entries from the prime subfield. Thus natural modules are realized over the prime subfield and are absolutely irreducible, and the rest follows.

4.3. *Representation of finite classical groups.* We present several results about the uniform behavior of finite classical groups with sufficiently large degree. The actual lower bound on the degrees (8, in fact) is not crucial, only the fact that some such bound exists.

(4.5) Proposition. *The Schur multiplier of the classical group $\mathrm{Cl}_n(p^a)$, for $n > 8$, has trivial p-part.*

Proof. See [11, p. 302].

(4.6) Proposition. *Let H be a finite perfect group with $H/O_p(H) \simeq \mathrm{Cl}_n(p^a)$ with $n > 8$. Furthermore, let H act faithfully on the finite dimensional F-vector space U in characteristic p with a unique nontrivial composition factor. Then H splits over $O_p(H)$.*

Proof. Let B be a minimal supplement to $O_p(H)$ in H. In particular B is perfect and satisfies all the hypotheses of Proposition 4.1. Therefore B intersects $O_p(H)$ only in a central p-subgroup. By Proposition 4.5, the Schur multiplier of the classical group $H/O_p(H)$ in characteristic p has trivial p-part, so this intersection is trivial. Therefore the extension is split by B.

Let G be a nontrivial quotient of the quasisimple classical group $\mathrm{Cl}_n(q)$. A *natural* module for G is the module $\mathbb{F}_q^n$ for its defining irreducible projective representation or its dual module (or any twist of these modules via field automorphisms). A *nearly natural* module is a natural module tensored up to a (possibly) larger field.

If $G = \mathrm{Sp}_{2m}(q)$ with q even, then an *orthogonal module* for G is the module $\mathbb{F}_q^{2m+1}$ on which G acts as the orthogonal group $\Omega_{2m+1}(q)$ (or any twist via field automorphisms). An orthogonal module for G is therefore a reducible but indecomposable extension of a trivial module $\mathbb{F}_q$ by a natural symplectic module for G; see Proposition 3.12. In this case, a *nearly orthogonal* module is an orthogonal module again tensored up to a (possibly) larger field. (Note that in this case, natural and nearly natural modules still have dimension $2m$.)

A module for $\mathrm{Cl}_n(q)$ that is either nearly natural or nearly orthogonal is *nearly nonsingular*. For a discussion of the structure of nearly nonsingular modules, see Proposition 4.11 below.

The tranpose of a natural right G-module is a natural left G-module and so forth.

(4.7) PROPOSITION. *Let C be a finite quasisimple classical group $\mathrm{Cl}_n(q)$ with $n > 6$, and let U be an extension of a trivial FC-module Z by a nearly natural FC-module Y. Then either*

(1) *$U = Z \oplus [U, C]$ with $[U, C]$ nearly natural, or*

(2) *for $n = 2m$, $C \simeq \mathrm{Sp}_{2m}(q)$ with q even, $Z \cap [U, C]$ has dimension 1, and $[U, C]$ is a nearly orthogonal module for $C \simeq \Omega_{2m+1}(q)$.*

In particular, $[U, C]$ is nearly nonsingular.

Proof. The dual of a nearly natural module is also nearly natural, and nearly orthogonal modules have the stated structure. Therefore the proposition is equivalent to the cohomological statement that $H^1(C, Y) \simeq F \otimes H^1(C, \mathbb{F}_q^n)$ is 0 but for the exceptional case (2) where it has dimension 1. As such, the result is a compendium of results by many people and is reasonably well-known; see [20], [23, Th. 2.14], and [28, §1].

(4.8) THEOREM. *Let S be a finite quasisimple classical group $\mathrm{Cl}_n(q)$ with $n > 8$. If $S \leq \mathrm{GL}_F(U)$ with $\dim_F U \leq n$, then U is a nearly nonsingular module for S. In particular, nearly natural modules are absolutely irreducible.*

Proof. For irreducible U, this is a reasonably well-known consequence of Steinberg's representation theory for Lie type groups in natural characteristic [41]. In particular it is contained in [27, Th. 1.1]. The full result then follows from Proposition 4.7.

(4.9) Proposition. *For* $H \le \mathrm{Cl}_F(U,f)$ *with either* $\mathrm{Cl} \in \{\mathrm{Sp}, \mathrm{GU}\}$ *and* f *nondegenerate or* $\mathrm{Cl} = \mathrm{GO}$ *and* f *nonsingular*,

$$\mathrm{Rad}([U,H], f|_{[U,H]}) = [U,H] \cap C_U(H)\,.$$

Proof. By definition $\mathrm{Rad}([U,H], f|_{[U,H]}) = [U,H] \cap [U,H]^\perp$, so this follows directly from Lemma 3.7(1) except when $(\mathrm{Cl}, \mathrm{char}\, F) = (\mathrm{GO}, 2)$. Now assume we are in that case. From Lemma 3.7(1) we still get $C_U(H) \le C_{U/U^\perp}(H) = [U,H]^\perp$, so it suffices to prove $R = \mathrm{Rad}([U,H], f|_{[U,H]}) \le C_U(H)$.

For $u \in R \le R^\perp$ we also have $u.h \in R^h = R$ for all $h \in H$. Therefore

$$\begin{aligned} f([u,h]) &= f(u(h-1)) \\ &= f(u.h) + f(-u) + b(u.h, -u) \\ &= f(u) + f(u) + 0 = 0\,, \end{aligned}$$

where b is the symplectic form associated with the quadratic form f.

By Lemma 3.7(2), $[u,h]$ is in $\mathrm{Rad}(U,f)$, and by the previous paragraph it is singular; so $[u,h] \in \mathrm{SRad}(U,f) = 0$. Therefore $u \in C_U(h)$, as desired.

Motivated by this proposition, we define the radical of H in U as

$$\mathrm{R}_U(H) = [U,H] \cap C_U(H)\,,$$

for $H \le \mathrm{GL}_F(U)$.

We also want a version of the singular radical relative to subgroups H. For $H \le \mathrm{Cl}_F(U,f)$ with $\mathrm{Cl} \in \{\mathrm{Sp}, \mathrm{GU}\}$ and f nondegenerate, we define

$$\mathrm{S}_U(H) = \mathrm{R}_U(H) = \mathrm{Rad}([U,H], f|_{[U,H]})\,.$$

Similarly, for $H \le \mathrm{GO}_F(U,f)$ and f nonsingular, we define

$$\mathrm{S}_U(H) = \mathrm{SRad}([U,H], f|_{[U,H]})\,,$$

of codimension at most 1 in $\mathrm{R}_U(H)$ for perfect F. Finally, for a subgroup $H \le \mathrm{GL}_F(U^+, U^-, f)$ with f nondegenerate, set

$$\mathrm{S}_{U^\varepsilon}(H) = \mathrm{R}_{U^\varepsilon}(H) = \mathrm{Rad}([U^\varepsilon, H], f|_{[U^+,H]\times[U^-,H]})\,,$$

again by Lemma 3.7(1).

In all cases, we then set

$$U_H^\varepsilon = [U^\varepsilon, H]/\mathrm{S}_{U^\varepsilon}(H)\,,$$

a section of U^ε $(= U)$ which contains all nontrivial H-composition factors of U^ε.

For $\mathrm{Cl} \ne \mathrm{GO}$ the classical sesquilinear form f induces an H-invariant nondegenerate form $f_H \colon U_H^+ \times U_H^- \longrightarrow F$ of the same type Cl as f. Similarly for $\mathrm{Cl} = \mathrm{GO}$, the quadratic form f induces an H-invariant nonsingular quadratic form $f_H \colon U_H \longrightarrow F$.

(4.10) PROPOSITION. *Let finite quasisimple* $C = \mathrm{Cl}_F(W^+, W^-, f)$, *with* $\dim_F W^\varepsilon > 6$, *be a subgroup of the finite quasisimple group* $\mathrm{Cl}_E(U^+, U^-, e)$ *of the same classical type* Cl.

Let $p = \mathrm{char}\, E = \mathrm{char}\, F$. *Assume that, for* $Q = \langle C^Q \rangle = O_p(Q)C$, *there is a unique nontrivial* Q*-composition factor in* U^ε *and it is nearly natural for* C. *Assume further that* $(\mathrm{Cl}, p) \neq (\mathrm{Sp}, 2)$. *Then*

(1) $[U^\varepsilon, Q] = \mathrm{S}_{U^\varepsilon}(Q) \oplus [U^\varepsilon, C]$ *with* $[U^\varepsilon, C]$ *nearly nonsingular for* C.

(2) Q *is quasisimple if and only if* $\mathrm{S}_{U^+}(Q) = \mathrm{S}_{U^-}(Q) = 0$.

(3) *If* $(\mathrm{Cl}, p) = (\Omega, 2)$ *with* $C \simeq \mathrm{Sp}(F^{2m}, b) \simeq \Omega(F^{2m+1}, q)$, *then* $[U^\varepsilon, C]$ *is nearly orthogonal for* C. *Furthermore, the restriction of* e *to* $[U^\varepsilon, C]$ *is nonsingular but degenerate.*

Proof. First consider (3). By Proposition 4.7, either $[U^\varepsilon, C]$ is of dimension $2m+1$ and is nearly orthogonal for $C \simeq \Omega(F^{2m+1}, q)$ or it is of dimension $2m$ and is nearly natural for $C \simeq \mathrm{Sp}(F^{2m}, b)$. The second case cannot happen by Theorem 3.17(4a). In the first case, $[U^\varepsilon, C]$ is indecomposable with a trivial submodule Z of dimension 1 and the quotient $[U^\varepsilon, C]/Z$ is irreducible and nearly natural (symplectic) for C. The subspace $[U^\varepsilon, C]$ cannot be totally isotropic for e by Lemma 3.7 and Proposition 4.9, so it has radical $Z = [U^\varepsilon, C] \cap [U^\varepsilon, C]^\perp = \mathrm{R}_{U^\varepsilon}(C)$. If Z were singular for e, then e would induce a nondegenerate C-invariant quadratic form on the quotient $[U^\varepsilon, C]/Z$. This would again contradict Theorem 3.17(4a).

We next consider (1). Note that $[U^\varepsilon, Q, O_p(Q)] \leq \mathrm{S}_{U^\varepsilon}(Q)$. Let Z^ε be a maximal proper EQ-submodule of $[U^\varepsilon, Q]$. As $Q = \langle C^Q \rangle$ is perfect, $[U^\varepsilon, Q]/Z^\varepsilon$ is nearly natural and

$$Z^\varepsilon = C_{[U^\varepsilon, Q]}(Q) = [U^\varepsilon, Q] \cap C_U(Q) = \mathrm{R}_{U^\varepsilon}(Q)\,.$$

Proposition 4.7 and (3) then give (1). Together with Theorems 3.11 and 3.13, this gives (2).

(4.11) PROPOSITION. *Let finite quasisimple* $Q \simeq \mathrm{Cl}_F(W^+, W^-, f)$, *with* $\dim_F W^\varepsilon > 6$, *be a subgroup of the group* $\mathrm{Cl}_E(U^+, U^-, e)$ *of the same classical type* Cl *over the field* E. *If* $\mathrm{Cl} \neq \mathrm{SL}$, *let* σ *be the automorphism of* E *for which* $U^\varepsilon = (U^{-\varepsilon})^\sigma$. *Assume that each* U^ε *is a nearly nonsingular module for* Q.

The field E *has a unique subfield* K *isomorphic to* F. *Let* τ *be the restriction of* σ *to* K. *Let* V^ε *be a minimal nontrivial* KQ*-submodule of* U^ε (*with* $V^\varepsilon = (V^{-\varepsilon})^\tau$ *if* $\mathrm{Cl} \neq \mathrm{SL}$). *Then there is a constant* $\kappa = \kappa^\sigma \in E$ *with* $Q = \mathrm{Cl}_K(V^+, V^-, \kappa\, e|_{V^\pm})$. *Here* $\kappa\, e|_{V^\pm}$ *is the appropriate restriction* $e|_{V^+ \times V^-}$ *or* $e|_{V^+} = e|_{V^-}$ *multiplied by the scalar* κ. *The form* $\kappa\, e|_{V^\pm}$ *is of type* Cl *with respect to* τ *and is nondegenerate on* $V^+ \times V^-$ *if* $\mathrm{Cl} \neq \Omega$ *and nonsingular on* $V^\varepsilon = (V^{-\varepsilon})^1$ *if* $\mathrm{Cl} = \Omega$.

The K-spaces V^ε are uniquely determined up to multiplication by scalars from E. The constant κ with $Q = \mathrm{Cl}_K(V^+, V^-, \kappa\, e|_{V^\pm})$ is then uniquely determined up to multiplication by an element of K fixed by τ.

Proof. As F is finite and U^ε is a nearly nonsingular EQ-module, E indeed has a unique subfield K isomorphic to F and K is invariant under σ.

We first discuss the situation where U^ε is nearly natural for Q. Then $U^\varepsilon \simeq E \otimes_K V^\varepsilon$, where V^ε is an irreducible, natural KQ-module. (This is an abuse of notation. In fact $U^+ \simeq E \otimes_K V^+$ and $U^- \simeq V^- \otimes_K E$.) Thus U^ε is completely reducible as a KQ-module, and every irreducible KQ-submodule (and quotient) is isomorphic to V^ε. As V^ε is an absolutely irreducible KQ-module (by Theorem 4.8), U^ε is an absolutely irreducible EQ-module. Thus by Schur's Lemma any two KQ-irreducible submodules of U^ε differ by a scalar of E.

Assume that $\mathrm{Cl} = \mathrm{SL}$. Let V^ε be an irreducible KQ-submodule of U^ε, and let $t \in Q$ be a transvection on V^+ with center Kx. Then t is also a transvection on V^- with center, say, x_0K. As t remains a transvection on U^ε, it is a 1-root element of $\mathrm{SL}_E(U^+, U^-, e)$ by Theorem 3.17(1). In particular $e(x, x_0) = 0$.

The Q-stabilizer of Kx has two orbits on the 1-spaces of V^-. For zK in the orbit of x_0K, we have $e(x, z) = 0$. As $x \notin {}^\perp U^-$, we have $e(x, z) \neq 0$ when zK is from the other orbit. Choose such a z, and set $\kappa = e(x, z)^{-1}$.

We have $\kappa\, e(x, x_0) = 0$ and $\kappa\, e(x, z) = 1$; hence $\kappa\, e(x, v) \in K$ for all $v \in V^-$. As Q is transitive on the 1-spaces of V^+, we conclude that $\kappa\, e(u, v) \in K$ for all $(u, v) \in V^+ \times V^-$. Therefore $\kappa\, e|_{V^\pm} \colon V^+ \times V^- \longrightarrow K$ is a Q-invariant pairing that is nondegenerate, since $\kappa\, e(x, y) = 1$ and Q is irreducible on V^ε. After comparing orders, we have that quasisimple Q is equal to $\mathrm{SL}_K(V^+, V^-, \kappa\, e|_{V^\pm})$. By Lemma 3.3, the map $v \mapsto \kappa\, e(\cdot, v)$ gives a KQ-isomorphism of V^- and $(V^+)^*$. Therefore by Schur's Lemma, given V^ε, a constant κ affording this equality is uniquely determined up to multiplication by a scalar from K.

Next assume that $\mathrm{Cl} \neq \mathrm{SL}$ with the form e of type Cl being nondegenerate on U $(= U^+ = (U^-)^\sigma)$, a nearly natural module for Q. Let $V = V^+$ be an irreducible KQ-submodule of U and put $V^- = (V^+)^\tau = V^\tau$. Let b be the σ-sesquilinear form on U associated with e (equal to e if $\mathrm{Cl} \neq \Omega$).

As $Q \simeq \mathrm{Cl}_F(W^+, W^-, f)$, there is a nondegenerate form f_K on V that is of type Cl—quadratic or classical ρ-sesquilinear for some automorphism ρ of K—with $Q = \mathrm{Cl}_K(V, V^\rho, f_K)$. Let $b_K = f_K$ except if $\mathrm{Cl} = \Omega$ where b_K will be the orthogonal form associated with the quadratic form f_K.

By Lemma 3.16 the ℓ-root elements of $Q = \mathrm{Cl}_K(V, V^\rho, f_K)$ are also ℓ-root elements of $\mathrm{Cl}_E(U^+, U^-, e)$. Considering the case $\ell = 1, 2$ and Theorem 3.17, we learn that V is the K-span of its vectors that are isotropic (and even singular when $\mathrm{Cl} = \Omega$) for both e and f_K. Let Kx and Ky be two isotropic (even singular) 1-spaces in V, chosen so that $b(x, y) \neq 0$. We also have $b_K(x,y) \neq 0$ (again, by consideration of ℓ-root elements for $\ell = 1,2$). The K-subspace $H = Kx \oplus Ky$ is hyperbolic for f_K.

Let $b' \colon U \times U \longrightarrow E$ be given by $b'(u,v) = b(x,y)^{-1} b(u,v)$, so that b' is a nondegenerate σ-sesquilinear form on U but not necessarily classical. Thus

$$b'(x,x) = b'(y,y) = 0 \quad \text{and} \quad b'(x,y) = b'(y,x) = 1\,,$$

and the range of b' on H is in K. As Q is transitive on 2-spaces that are hyperbolic for f_K, the range of b' on all V is in K. That is, $b'|_{V\times V} = b'_K$ is a nondegenerate τ-sesquilinear form on the K-space V.

By Lemma 3.3 the map $v \mapsto b_K(\cdot, v)$ gives a KQ-isomorphism of V^ρ and V^*, while $v \mapsto b'_K(\cdot,v)$ gives a KQ-isomorphism of V^τ and V^*. We conclude that $\rho = \tau$ (by [41] or direct calculation). As Q is absolutely irreducible on V^* (by Theorem 4.8), the forms b_K and b'_K differ by a scalar. That is, for all $u, v \in V$,

$$b_K(u,v) = k b'_K(u,v) = \kappa\, b(u,v)$$

for some $k \in K$ and $\kappa = k b(x,y)^{-1} \in E$. If $\sigma = 1$, then certainly $\kappa^\sigma = \kappa$. When $\sigma \neq 1$, we can find $v \in V$ with $b(v,v) \neq 0$. Then

$$\begin{aligned}\kappa &= b_K(v,v) b(v,v)^{-1} = b_K(v,v)^\tau (b(v,v)^\sigma)^{-1}\\ &= (b_K(v,v) b(v,v)^{-1})^\sigma = \kappa^\sigma\,.\end{aligned}$$

If $\mathrm{Cl} \neq \Omega$ then $f_K = b_K$ and $e|_{V^\pm} = b|_{V^\pm}$, so that $Q \le \mathrm{Cl}_K(V^+, V^-, \kappa\, e|_{V^\pm})$ with $\kappa = \kappa^\sigma$. Equality follows by order considerations. Schur's Lemma again guarantees that κ is unique up to multiplication by a member of K (necessarily fixed by τ as we have classical sesquilinear forms).

If $\mathrm{Cl} = \Omega$ then we know that singular vectors for f_K span V and U and are also singular for e and $\kappa\, e|_{V^\pm}$. Additionally, we have from above that $b_K = \kappa\, b|_{V^\pm}$, with κ unique up to a scalar from K. Any quadratic form is uniquely determined by its values at a basis and its associated orthogonal form. Therefore $f_K = \kappa\, e|_{V^\pm}$ and $Q \le \mathrm{Cl}_K(V^+, V^-, \kappa\, e|_{V^\pm})$ with equality following from order considerations.

This concludes discussion of nearly natural U^ε. The remaining possibility is that $U (= U^+ = (U^-)^1)$ is nearly orthogonal for Q with $(\mathrm{Cl}, \mathrm{char}\, K) = (\Omega, 2)$, which we now assume. The space $Z = \mathrm{Rad}(U, e)$ is nonsingular of dimension 1 by Proposition 4.10(3). As before, let b be the symplectic form on U associated with e.

The previous remarks apply to $\tilde{U} = U/Z$ (with $\mathrm{Cl} = \mathrm{Sp}$). In particular, a minimal nontrivial KQ-submodule V of U must map to an irreducible submodule $\tilde{V}$ of $\tilde{U}$. Let V_0 be the KQ-submodule of U that is the full preimage of $\tilde{V}$. By Proposition 4.7 the K-space $V = [V_0, Q]$ is either a natural KQ-module or an orthogonal KQ-module. In the first case $Y = EV$ ($\simeq E \otimes_K V$) would be a completely reducible EQ-submodule of U with all composition factors natural and $\tilde{Y} = \tilde{U}$. That would go against the indecomposability of U. Therefore V is an orthogonal module for Q. Two such differ by a scalar from E, since the centralizer of Q in $\mathrm{End}_E(U)$ still consists of the scalars.

By earlier remarks $Q = \mathrm{Sp}_K(\tilde{V}, \tilde{V}^1, \kappa\, \tilde{b}|_{\tilde{V}^\pm})$, where $\tilde{b}$ is the symplectic form induced on $\tilde{U}$ by b and the constant $\kappa \in E$ is uniquely determined up to multiplication by a member of K. The map $\kappa\, b|_{V^\pm}$ is then a Q-invariant symplectic form on the K-space V with radical $Z \cap V$ of K-dimension 1 (nonsingular for e). As before, consideration of 2-root elements shows that V is the K-span of its vectors that are singular for e. Therefore $\kappa\, e|_{V^\pm}$ is a nonsingular quadratic form on V with associated symplectic form $\kappa\, b|_{V^\pm}$, and $Q \leq \Omega_K(V, V^1, \kappa\, e|_{V^\pm})$. Comparing orders again, we find equality and so complete the proof of Proposition 4.11.

(4.12) Proposition. (1) *Let H be an irreducible subgroup of the finite classical group $\mathrm{Cl}_K(V)$ with $\mathrm{Cl} \in \{\mathrm{SL}, \mathrm{Sp}, \mathrm{SU}\}$. Assume that H contains a 1-root (transvection) subgroup and a subgroup $C \simeq \mathrm{Cl}_F(U)$ with $F \leq K$ and $[V, C]$ a nearly natural module for C with $\dim_F U + 2 \geq \dim_K V \geq 7$. Then $H = \mathrm{Cl}_K(V)$.*

(2) *Let H be a subgroup of the finite orthogonal group $\Omega_K(V, f)$ for f nonsingular with H irreducible on $V/\mathrm{Rad}(V, f)$. Assume that H contains a 2-root (Siegel) subgroup and a quasisimple subgroup $C \simeq \Omega_F(U, f_U)$ with $F \leq K$ and $[V, C]$ a nearly nonsingular module for C with $[V, C] \geq \mathrm{Rad}(V, f)$ and $2\dim_F U > \dim_K V > \dim_F U + 8$. Then $H = \Omega_K(V, f)$.*

Proof. This can be proved by direct calculation, but it is immediate from results on groups generated by transvections and long root elements; see in particular [22], [29]. It could also be deduced from Corollary 5.4 below.

5. A classification result

In this section we present the weak version of the classification of finite simple groups (CFSG) used in proving our main theorem, Theorem 1.1. Indeed the result can be split into two parts, Corollary 5.3 and Corollary 5.4, both of which are highly geometric in flavor. Corollary 5.4 is certainly open to proof without CFSG, so that a classification-free proof of Corollary 5.3 should render the results of this paper independent of CFSG.

We begin with an abbreviated version of [16, Th. 4]:

(5.1) Theorem (Hall, Liebeck, Seitz [16]). *Let F be an algebraically closed field of characteristic p (possibly $p = 0$), and let U be a vector space of finite dimension $n > 1$ over F. Suppose that H is a finite primitive subgroup of $\mathrm{GL}_F(U)$ and is generated by elements of degree less than $\sqrt{n}/12$. Then one of the following holds*:

(1) *H is* $\mathrm{Alt}(m)$ *or* $\mathrm{Sym}(m)$, *and U is a natural FH-module*; *or*

(2) *$F^*(H)$ is a classical group over a finite field of characteristic p with natural module of dimension n.*

Here $F^*(H)$ is the self-centralizing characteristic subgroup of H generated by all subnormal nilpotent or quasisimple subgroups [1]. The proof of the theorem in [16] makes use of CFSG.

Guralnick and Saxl [12] have extended the theorem, in particular improving the bound $\sqrt{n}/12$ to $\sqrt{n}/2$ (still using CFSG).

The version of the theorem that we use is:

(5.2) Theorem. *Let finite H be contained in $\mathrm{GL}_F(U)$, for F algebraically closed, with H primitive on finite dimensional U. There is a nondecreasing function $k(d)$ such that, if H is generated by elements of degree at most d on U with $\dim_F U > k(d)$, then either*

(1) *H is* $\mathrm{Alt}(\Delta)$ *or* $\mathrm{Sym}(\Delta)$, *and U is a natural module*; *or*

(2) *$F^*(H)$ is a quasisimple classical group in the same characteristic as F, and U is a nearly natural module for $F^*(H)$.*

Proof. If we set $k(d) = 144d^2$, then the theorem follows from Theorems 4.8 and 5.1.

A version of Theorem 5.2 for imprimitive groups appeared earlier as Proposition 4.3. Its proof did not depend upon CFSG.

There is some virtue in splitting Theorem 5.2 into two parts.

(5.3) Corollary. *Let finite H be contained in $\mathrm{GL}_F(U)$, for F algebraically closed, with H primitive on finite dimensional U. There is a nondecreasing function $k(d)$ such that, if H is generated by elements of degree at most d on U with $\dim_F U > k(d)$, then H is generated by elements of degree at most* 2 *on U.*

(5.4) Corollary. *Let finite H be contained in $\mathrm{GL}_F(U)$, for F algebraically closed, with H primitive on finite dimensional U. There is a constant $k(2)$ such that, if H is generated by elements of degree at most* 2 *on U with $\dim_F U > k(2)$, then either*:

(1) *H is* $\mathrm{Alt}(\Delta)$ *or* $\mathrm{Sym}(\Delta)$, *and U is a natural module*; *or*

(2) *$F^*(H)$ is a quasisimple classical group in the same characteristic as F, and U is a nearly natural module for $F^*(H)$.*

The point here is that most (and perhaps all) of Corollary 5.4 has already been proved without CFSG. See, for instance, [22], [29], [40].

The results of Guralnick and Saxl [12] imply that Theorem 5.2 and Corollary 5.3 hold with $k(d) = 4d^2$ for $d \geq 3$. Additionally they prove that Theorem 5.2 and Corollary 5.4 are valid with $k(2) = 10$ and that this is best possible. Their work makes use of CFSG.

6. The division of cases in Theorem 1.1

We have now assembled everything needed to prove Theorem 1.1. The converse is immediate from Theorems 3.2 and 3.15:

(6.1) THEOREM. *Each group in Theorem* 1.1(2–6) *is locally finite*, *simple*, *and finitary but not linear in finite dimension.*

To attack the direct part of Theorem 1.1, we let G be a locally finite simple group that has a faithful representation as a finitary linear group in characteristic p (possibly 0) but has no faithful characteristic p representation as a linear group in finite dimension. In this section we show that G resembles either an alternating group or a classical group in positive characteristic p.

Choose an arbitrary but fixed nonidentity element g of G and let $D = g^G$. Considering all faithful characteristic p representations $\varphi\colon G \longrightarrow \mathrm{FGL}_F(X)$, we let d be the minimum value of the degree $\dim_F [X, \varphi(g)]$.

Let φ be one such minimizing representation. We identify G with its image under φ, so that $G \leq \mathrm{FGL}_F(X)$ and $d = \dim_F [X, g]$. Let $\hat{F}$ be an algebraic closure of F.

Let $\mathcal{O} = \{\, (G_i, N_i) \mid i \in I \,\}$ be a Kegel cover of G. Passing to an abbreviation if necessary, we may assume that each $G_i = \langle G_i \cap D \rangle$ with $g \in G_i \setminus N_i$ by Lemmas 2.4 and 2.6.

(6.2) PROPOSITION. *Let f be a constant. Let I_f be the set of all $i \in I$ for which*

(a) *G_i has a unique composition factor X_i in X in which g acts nontrivially*;

(b) *X_i is an absolutely irreducible G_i-module*;

(c) $\dim_F X_i > f$; *and*

(d) $\dim_F [X_i, g] = d$.

Then $\mathcal{O}_f = \{\, (G_i, N_i) \mid i \in I_f \,\}$ is a Kegel cover for G.

Proof. We color $\mathcal{O}$ with three colors—SMALL-d, SMALL-X, and BIG.

If G_i has at least two composition factors in X on which g is nontrivial, give (G_i, N_i) color SMALL-d. Equally well give (G_i, N_i) color SMALL-d if G_i has a unique composition factor X_i in X on which g acts, but X_i is not absolutely irreducible or $\dim_F [X_i, g] < d$.

If G_i has a unique composition factor X_i in X on which g acts, X_i is absolutely irreducible, and $\dim_F [X_i, g] = d$ but $\dim_F X_i \le f$, then give (G_i, N_i) color SMALL-X.

Finally, if G_i has a unique nontrivial composition factor X_i in X on which g acts, X_i is absolutely irreducible, $\dim_F [X_i, g] = d$, and $\dim_F X_i > f$, then (G_i, N_i) gets color BIG.

We wish to show that those pairs colored BIG form a Kegel cover, so that by Lemma 2.4 we need only show that the other two color classes are not Kegel covers.

If SMALL-X colors a cover, then by Mal'cev's Theorem 2.10(1), G is linear of degree at most f in characteristic p, against our overall assumption.

Now suppose that those pairs colored SMALL-d form a Kegel cover. For each such pair (G_i, N_i), let X_i be a G_i-composition factor in X on which g acts nontrivially. Let W_i be an irreducible $\hat{F}G_i$-submodule of $\hat{F} \otimes_F X_i$. Consider the representation $\varphi_i \colon G_i \longrightarrow \mathrm{GL}_{\hat{F}}(W_i)$.

If X_i is absolutely irreducible and the unique composition factor for G_i on which g is nontrivial, then $0 < \dim_{\hat{F}} [W_i, \varphi_i(g)] = \dim_F [X_i, g] < d$ as we are in color class SMALL-d. Otherwise X_i is not absolutely irreducible or is absolutely irreducible but not the only FG_i-composition factor in X on which g acts nontrivially. In either case $0 < \dim_F [W_i, \varphi_i(g)] < d$ since $\dim_F [X, g] = d$ and W_i is not the only $\hat{F}G_i$-factor in $\hat{F} \otimes_F X$ on which g acts nontrivially. Therefore by Theorem 2.10(2), when we represent G on a characteristic p ultraproduct W_∞ of the W_i, we have $0 < \dim [W_\infty, \varphi_\infty(g)] < d$. As this degree is nonzero and G is simple, the representation φ_∞ is faithful. This contradicts our choice of d as the minimum degree for g in any faithful characteristic p finitary representation of G.

Choose an $f > \max(d\, c(2d), k(d), d^2 + 15d)$ (the functions c, k being those of Theorem 4.2 and Theorem 5.2). We consider the Kegel cover $\mathcal{O}_f$. Write $I_f = I_{\mathrm{prim}} \cup I_{\mathrm{imprim}}$ where $i \in I_{\mathrm{prim}}$ if $\hat{X}_i = \hat{F} \otimes_F X_i$ is primitive for G_i and $i \in I_{\mathrm{imprim}}$ otherwise. By coloring (Lemma 2.4), at least one of $\mathcal{O}_{\mathrm{prim}} = \{\, (G_i, N_i) \mid i \in I_{\mathrm{prim}} \,\}$ or $\mathcal{O}_{\mathrm{imprim}} = \{\, (G_i, N_i) \mid i \in I_{\mathrm{imprim}} \,\}$ is a Kegel cover.

(6.3) PROPOSITION. *For each* $i \in I_{\mathrm{prim}}$, *either*

(1) $G_i/\ker_{G_i}(X_i)$ *is* $\mathrm{Alt}(\Delta_i)$ *or* $\mathrm{Sym}(\Delta_i)$ *and* X_i *is a natural module*; *or*

(2) $F^*(G_i/\ker_{G_i}(X_i))$ *is a quasisimple classical group in characteristic* p (*which thus is positive*) *and* X_i *is a nearly natural module for* $F^*(G_i/\ker_{G_i}(X_i))$.

Proof. By Theorem 5.2 the induced group $G_i/\ker_{G_i}(X_i) = G_i/\ker_{G_i}(\hat{X}_i)$ is as in (1) or (2). Let M_i be $G_i/\ker_{G_i}(X_i)$ under (1) and $F^*(G_i/\ker_{G_i}(X_i))$ under (2). Theorem 5.2 also says that $\hat{X}_i$ is, respectively, a natural or nearly natural $\hat{F}M_i$-module. We must still identify the FM_i-module X_i.

In case (1), by Proposition 4.4 the module X_i is natural for $G_i/\ker_{G_i}(X_i)$, isomorphic to $\mathrm{Alt}(\Delta_i)$ or $\mathrm{Sym}(\Delta_i)$.

In case (2), let E be $\mathbb{F}_{q_i}$ where $M_i = F^*(G_i/\ker_{G_i}(X_i)) \simeq \mathrm{Cl}_{n_i}(q_i)$. By Theorem 4.8, the nearly natural $\hat{F}M_i$-module $\hat{X}_i$ is completely reducible as an EM_i-module and irreducible submodules differ only by scalar multiplication by a member of $\hat{F}$. So we can choose Y_i to be an irreducible natural EM_i-module within the submodule X_i. Then $X_i = F \otimes_E Y_i$ is a nearly natural FM_i-module, as desired.

(6.4) Proposition. *Suppose $\mathcal{O}_{\mathrm{imprim}}$ is a Kegel cover, and choose (G_0, N_0) in $\mathcal{O}_{\mathrm{imprim}}$. Let J be the set of all $i \in I_{\mathrm{imprim}}$ with $G_0 \le G_i$ and $G_0 \cap N_i = 1$. Then $\mathcal{P}_{\mathrm{imprim}} = \{\, (G_j, N_j) \,|\, j \in J \,\}$ is a Kegel cover. For each $j \in J$, $G_j/N_j \simeq \mathrm{Alt}(\Delta_j)$ with g acting as a nontrivial permutation of degree at most $2d$ on Δ_j (and $|\Delta_j| > f/d$).*

Proof. By Corollary 2.5, $\mathcal{P}_{\mathrm{imprim}}$ is a Kegel cover. We must identify the Kegel quotients G_j/N_j for $j \in J$. Recall that $\ker_{G_i}(X_i) = \ker_{G_i}(\hat{X}_i)$.

For $i \in I_{\mathrm{imprim}}$ the only G_i-composition factor in X on which g acts nontrivially is X_i, so $\langle g^{G_i} \rangle \cap \ker(X_i)$ is unipotent. As $g \in G_i \setminus N_i$, simple G_i/N_i is a composition factor of $G_i/\ker_{G_i}(X_i)$. By Proposition 4.3 the group G_i, in its action on a maximal block system Δ_i in $\hat{X}_i$, induces $\mathrm{Alt}(\Delta_i)$ or $\mathrm{Sym}(\Delta_i)$ with kernel B_i inducing on X_i a subgroup of $\prod_{\Delta_i} \mathrm{GL}_d(\hat{F})$ (as the dimension e of a block is at most d). Furthermore, the degree of each member of $D \cap G_i$ on Δ_i is at most $\lfloor 2d/e \rfloor \le 2d$ and

$$|\Delta_i| = (\dim_F X_i)/e > f/d \ge d + 15 \ge 16\,.$$

We claim that, for each $j \in J$, we have $g \notin B_j$. Assume that this is not true for j. Then simple G_j/N_j is a section of $\mathrm{GL}_d(\hat{F})$ containing a subgroup G_0 with $G_0/B_0 \simeq \mathrm{Alt}(\Delta_0)$, for $|\Delta_0| \ge 16$. By Proposition 3.1 this implies that $d \ge |\Delta_0| - 2$. As $|\Delta_0| > f/d$, we conclude that $d^2 + 2d > f$, against our choice of f. Thus for all $j \in J$ we have $g \notin B_j$, and g is nontrivial on Δ_j.

Now G_j/N_j is a composition factor of G_j/B_j, which is itself isomorphic to $\mathrm{Sym}(\Delta_j)$ or $\mathrm{Alt}(\Delta_j)$. Therefore $G_j/N_j \simeq \mathrm{Alt}(\Delta_j)$, completing the proof of the proposition.

(6.5) Theorem. *Let G be a locally finite simple group that has a faithful characteristic p representation as a finitary linear group but has no faithful characteristic p representation as a linear group in finite dimension. Take g to be an arbitrary but fixed nontrivial element of G.*

Then any Kegel cover $\mathcal{O}$ for G can be abbreviated to a quasisimple Kegel cover $\mathcal{K} = \{\, (H_i, O_i) \,|\, i \in I \,\}$ such that one of two cases holds:

(1) (Alternating Case) *There is an infinite set Δ such that $G \le \mathrm{Alt}(\Delta)$. There is a constant δ such that, for each $i \in I$, $g \in H_i = \langle g^G \cap H_i \rangle$ and H_i/O_i*

is an alternating group $\mathrm{Alt}(\Delta_i)$ *with* g *acting nontrivially but having degree at most* δ *in its action on* Δ_i. *The set* $\{\,|\Delta_i|\,|\, i \in I\,\}$ *is unbounded.*

(2) (Classical Case) *There are a perfect field* L *of characteristic* $p > 0$, L-*spaces* X^ε (*for* $\varepsilon = \pm$), *and a classical type* $\mathrm{Cl} \in \{\mathrm{SL}, \mathrm{Sp}, \mathrm{SU}, \Omega\}$ (*with* $(\mathrm{Cl}, p) \neq (\mathrm{Sp}, 2)$) *such that* $G \leq \mathrm{FCl}_L(X^+, X^-, f)$, *where* f *is a nondegenerate form of type* Cl *on* $X^+ \times X^-$ *or a nonsingular quadratic form on* $X^\varepsilon = (X^{-\varepsilon})^1$. *Each* O_i *is unipotent and each* H_i *is perfect with* $H_i/O_i \simeq \mathrm{Cl}_{n_i}(p^{a_i})$. *The section* $X^\varepsilon_{H_i} = [X^\varepsilon, H_i]/\mathrm{S}_{X^\varepsilon}(H_i)$ *is a nearly nonsingular module for* H_i/O_i. *The set* $\{\, n_i \,|\, i \in I\,\}$ *is unbounded.*

Proof. Adopting all the notation of this section, we may assume that $\mathcal{O} = \mathcal{O}_f = \{\,(G_i, N_i) \,|\, i \in I\,\}$ and write

$$\mathcal{O} = \mathcal{O}_{\mathrm{imprim}} \cup \mathcal{O}_1 \cup \mathcal{O}_2\,,$$

where $\mathcal{O}_k = \{\,(G_i, N_i) \,|\, i \in I_k\,\}$ (for $k = 1, 2$) consists of those $(G_i, N_i) \in \mathcal{O}_{\mathrm{prim}}$ that come under Proposition 6.3(k). One of these three subsets of $\mathcal{O}$ is a Kegel cover.

If $\mathcal{O}_{\mathrm{imprim}}$ is a Kegel cover, then so is $\mathcal{P}_{\mathrm{imprim}} = \{\,(G_j, N_j) \,|\, j \in J\,\}$ of Proposition 6.4. For each $j \in J$, set $H_j = \langle g^{G_j}\rangle$ and $O_j = H_j \cap N_j$. By that proposition and Lemma 2.6, $\mathcal{K} = \{\,(H_j, O_j) \,|\, j \in J\,\}$ is a Kegel cover as in the alternating case of the theorem. (As G is infinite, $|\Delta_j|$ is unbounded.)

If $\mathcal{O}_{\mathrm{prim}} = \mathcal{O}_1 \cup \mathcal{O}_2$ is a Kegel cover, then for all $i \in I_{\mathrm{prim}}$ set $A_i = \langle g^{G_i}\rangle$, $M_i = A_i \cap N_i$, and $P_i = \ker_{A_i}(X_i)$. By Lemma 2.6 the abbreviation $\{\,(A_i, M_i) \,|\, i \in I\,\}$ of $\mathcal{O}$ is a Kegel cover with $A_i/M_i \simeq G_i/N_i$. Therefore $\{\,(A_i, P_i) \,|\, i \in I\,\}$ is a sectional cover of G with $A_i/P_i \simeq A_i \ker_{G_i}(X_i)/\ker_{G_i}(X_i)$, a normal subgroup of $G_i/\ker_{G_i}(X_i)$. As the generating set g^{G_i} of A_i acts trivially on all composition factors other than X_i, the kernel P_i is unipotent. If P_i were not contained in M_i, then the simple group $A_i/M_i = P_iM_i/M_i$ would be isomorphic to the unipotent group $P_i/P_i \cap M_i$, which is not the case. Therefore $M_i \geq P_i$ and

$$A_i/M_i \simeq A_i/P_i \Big/ M_i/P_i$$

is simple.

If $\mathcal{O}_1$ is a Kegel cover, then by Proposition 6.3(1) we have that $P_i = M_i$ and $A_i/P_i \simeq \mathrm{Alt}(\Delta_i)$. The module X_i is natural for A_i/P_i with g of degree d, so the permutation degree of g on Δ_i is at most $2d$. Thus, with $H_i = A_i$ and $O_i = P_i$, $\mathcal{K} = \{\,(H_i, O_i) \,|\, i \in I_1\,\}$ is a Kegel cover as in the alternating case of the theorem. (Again $|\Delta_i|$ is unbounded.)

Assume now that $\mathcal{O}_{\mathrm{imprim}} \cup \mathcal{O}_1$ is a Kegel cover. Therefore, as in the alternating case, we have an alternating Kegel cover $\mathcal{K} = \{\,(H_i, O_i) \,|\, i \in I\,\}$ in which the permutation degree of the element g is nonzero but bounded by some δ. Order the index set I by

$$i \prec j \iff H_i < H_j \text{ and } H_i \cap O_j = 1\,,$$

for all (H_i, O_i) and (H_j, O_j) from the Kegel cover $\mathcal{K}$. Let $\mathcal{F}$ be an ultrafilter on I containing $\mathcal{F}_{(I,\preceq)}$, and let Δ be the ultraproduct over $\mathcal{F}$ of the sets Δ_i. Then by Lemma 2.9 there is a homomorphism of G into $\mathrm{Sym}(\Delta)$ with the image of g of nonzero but finite degree and so a nonidentity in $\mathrm{FSym}(\Omega)$. Since G is infinite and simple, G is embedded in $\mathrm{Alt}(\Omega)$. As $\mathcal{O}_{\mathrm{imprim}} \cup \mathcal{O}_1$ is an abbreviation of $\mathcal{O}$, we have the alternating case of the theorem in full.

We assume for the balance of the proof that G does not come under the alternating case of the theorem. In particular $\mathcal{O}_2$ and $\{\, (A_i, M_i) \mid i \in I_2 \,\}$ are Kegel covers which are abbreviations of $\mathcal{O}$. By Proposition 6.3(2) we have, for all $i \in I_2$, that $M_i/P_i = Z(A_i/P_i)$ and A_i/P_i is a quasisimple classical group with nearly natural module X_i.

For each $i \in I_2$, let

$$A_i/P_i \simeq \mathrm{Cl}_{F_i}(X_i^+, X_i^-, f_i) \simeq \mathrm{Cl}_{n_i}(p^{a_i}),$$

where $\mathrm{char}\, F_i = \mathrm{char}\, F = p$ and f_i is a nondegenerate form of type Cl on $X_i^+ \times X_i^-$ with respect to the automorphism σ_i of F_i or a nonsingular quadratic form on $X_i^\varepsilon = (X_i^{-\varepsilon})^1$ (where we set $\sigma_i = 1$). By coloring, $\{\, (A_i, M_i) \mid i \in I_2 \,\}$ has a subcover $\{\, (A_j, M_j) \mid j \in J \,\}$ in which each Cl is of a fixed classical type from $\{\mathrm{SL}, \mathrm{Sp}, \mathrm{SU}, \Omega\}$. By Proposition 3.12 finite symplectic groups in characteristic 2 can be viewed as orthogonal groups. Thus we may additionally assume that $(\mathrm{Cl}, p) \neq (\mathrm{Sp}, 2)$, the corresponding groups being included under $(\mathrm{Cl}, p) = (\Omega, 2)$.

Order J by

$$i \prec j \iff A_i < A_j \text{ and } A_i \cap P_j = 1\,.$$

Let $\mathcal{F}$ be an ultrafilter on J containing $\mathcal{F}_{(J,\preceq)}$. Set

$$L = \prod_{\mathcal{F}} F_j\,, \quad X^\varepsilon = \prod_{\mathcal{F}} X_j^\varepsilon\,, \quad f = \prod_{\mathcal{F}} f_j\,, \quad \sigma = \prod_{\mathcal{F}} \sigma_j\,,$$

so that X^ε (for $\varepsilon = \pm$) is a vector space over the field L of characteristic p, and f is a form of type Cl on $X^+ \times X^-$ with respect to σ or a quadratic form on $X^\varepsilon = (X^{-\varepsilon})^1$. As in Theorem 2.10(3), such a pairing is nondegenerate, because each f_j is, and such a quadratic form is nonsingular. The field L is an ultraproducts of perfect fields of characteristic p, so L itself is perfect of characteristic p.

As in Theorem 2.10(2-3), G acts as a subgroup of $\mathrm{Cl}_L(X^+, X^-, f)$ with the element g having degree d. Identify G with its image in $\mathrm{Cl}_L(X^+, X^-, f)$. The group G is simple and g has finite degree, so that in fact, $G \leq \mathrm{FCl}_L(X^+, X^-, f)$. As G is not linear in characteristic p, X^ε is infinite dimensional. In particular, the n_j $(= \dim_{F_j} X_j)$ are unbounded.

Propositions 6.2–6.4 now apply to the representation of G on X^ε. As we are not in the alternating case, there is a subset J' of J such that

$$\{\, (A_j, P_j) \mid j \in J' \,\}$$

is a quasisimple Kegel cover with each P_j unipotent. Furthermore, for each $j \in J'$, the unique A_j-composition factor in X^ε on which g acts nontrivially is a nearly natural module for the finite classical group A_j/P_j. Therefore the module $[X^\varepsilon, A_j]/\mathrm{S}_{X^\varepsilon}(A_j)$ is a nearly nonsingular module for A_j/P_j by Propositions 4.7 and 4.9.

For each $j \in J'$, set $H_j = A_j^{(\infty)}$ and $O_j = H_j \cap P_j$. By Lemma 2.6, $\{\,(H_j, O_j) \mid j \in J'\,\}$ is a perfect, quasisimple Kegel cover of G in which $H_j/O_j \simeq A_j/P_j$. Choose $0 \in J'$ and let $J'' = \{\, j \in J' \mid H_0 \le H_j,\ H_0 \cap O_j = 1\,\}$. As O_j and P_j are unipotent while H_0 and H_j are perfect, we have $H_j = \langle H_0^{H_j}\rangle = \langle H_0^{A_j}\rangle$ for each $j \in J''$. By Lemma 2.6 the abbreviation $\mathcal{K} = \{\,(H_j, O_j) \mid j \in J''\,\}$ of $\mathcal{O}$ is a quasisimple Kegel cover, and it has all properties required under the classical case of the theorem. (Note that g might not be contained in H_j.)

7. The alternating case

In this section we prove

(7.1) THEOREM ([13, Th. 5.2]). *Let G be a locally finite simple group with a faithful representation as a finitary linear group. Assume that G has a Kegel cover $\mathcal{S} = \{\,(G_i, N_i) \mid i \in I\,\}$ with all Kegel quotients G_i/N_i alternating. Then there is a set Ω with $G \simeq \mathrm{Alt}(\Omega)$.*

(7.2) THEOREM. *Let G be a locally finite simple group with a faithful representation as a finitary linear group as in the Alternating Case*, Theorem 6.5(1). *Then there is a set Ω with $G \simeq \mathrm{Alt}(\Omega)$.*

The set of Kegel quotients for an abbreviation of a Kegel cover $\mathcal{S}$ is contained in the set of Kegel quotients for $\mathcal{S}$. Therefore by Theorem 6.5 a group G as in Theorem 7.1 must come under the Alternating Case, and Theorem 7.1 follows directly from Theorem 7.2.

In the Alternating Case, Theorem 6.5(1), the infinite locally finite group G is a subgroup of $\mathrm{Alt}(\Delta)$ for some Δ. Theorem 7.2 is thus a consequence of the known and elementary

(7.3) THEOREM ([31], [39]). *An infinite, simple subgroup of* $\mathrm{Alt}(\Delta)$ *is isomorphic to* $\mathrm{Alt}(\Omega)$, *for some Ω.*

Proof. We sketch a proof using our methods. Note that for this result we appeal to Jordan's Theorem 4.2 but not to CFSG.

Let G be an infinite simple finitary permutation group as in the theorem. Let g be a nontrivial element of G. Among all embeddings $G \le \mathrm{Alt}(\Omega)$, choose one in which the degree k of g on Ω is minimal. We may assume that G is transitive on Ω. Let $\mathcal{L} = \{ (G_j, N_j) \,|\, j \in J \}$ be a Kegel cover for G with $g \in G_j \setminus N_j$ and $G_j = \langle g^G \cap G_j \rangle$, for all $j \in J$.

Let $\mathcal{L}_{\mathrm{trans}}$ consist of those pairs (G_j, N_j) from $\mathcal{L}$ with $\langle g^{G_j} \rangle$ having a single orbit Ω_j on Ω in which g acts nontrivially, that orbit having $|\Omega_j| > \max(k\,c(k), k^2)$. If $\mathcal{L}_{\mathrm{trans}}$ is not a Kegel cover then, by Lemma 2.9 and a coloring argument as in Proposition 6.2, we can realize G via an ultraproduct as a finitary permutation group in which the degree of g is smaller than that on Ω, against assumption.

Therefore $\mathcal{L}_{\mathrm{trans}}$ is a Kegel cover. Replacing G_j by $\langle g^{G_j} \rangle$, we may assume that, for each (G_j, N_j) from $\mathcal{L}_{\mathrm{trans}}$, the G_j-orbit Ω_j is the support of G_j in Ω. Let $\mathcal{L}_{\mathrm{prim}}$ consist of those (G_j, N_j) from $\mathcal{L}_{\mathrm{trans}}$ with G_j primitive on Ω_j. If $\mathcal{L}_{\mathrm{prim}}$ is not a Kegel cover then its complement $\mathcal{L}_{\mathrm{imprim}}$ in $\mathcal{L}_{\mathrm{trans}}$ is. In that case, by Lemma 2.9 and an argument similar to that of Proposition 6.4, we can again realize G via an ultraproduct as a finitary permutation group in which the degree of g is smaller than that on Ω, against assumption. Therefore $\mathcal{L}_{\mathrm{prim}}$ is a Kegel cover.

By Jordan's Theorem 4.2 each G_j from the Kegel cover $\mathcal{L}_{\mathrm{prim}}$ induces either $\mathrm{Sym}(\Omega_j)$ or $\mathrm{Alt}(\Omega_j)$ on its support Ω_j. As $G \leq \mathrm{Alt}(\Omega)$, in fact, $G_j = \mathrm{Alt}(\Omega_j)$ and $N_j = 1$.

Let x be any element of $\mathrm{Alt}(\Omega)$. Then the support of x is finite and so contained in some Ω_j, since G is transitive. Therefore $x \in G_j \leq G$. This reveals G as $\mathrm{Alt}(\Omega)$.

Remarks. (1) A similar argument proves that any infinite simple section of $\mathrm{Alt}(\Delta)$ is isomorphic to $\mathrm{Alt}(\Omega)$, for some Ω, a result due to Segal [39].

(2) A consequence of these arguments is that every Kegel cover of an alternating group $\mathrm{Alt}(\Omega)$ has an abbreviation that is natural; that is, it equals $\{ (\mathrm{Alt}(\Omega_i), 1) \,|\, i \in I \}$, where Ω is the direct limit of its finite subsets Ω_i, for $i \in I$. This can be proved directly. See [15, Prop. 3.6] and [13, Th. 8.1].

(3) The paper [13] contains two other characterizations of alternating groups: the infinite locally finite simple group G is alternating either if it is finitary in characteristic 0 ([13, Th. 1]) or if it has an alternating Kegel cover in which the degree of some nonidentity element is bounded ([13, Th. 5.1]). As with Theorem 7.1 above, both of these can be proved by reduction to Theorem 7.2. Indeed, for a group to be in the Classical Case, Theorem 6.5(2), all finitary representations are in positive characteristic p; so a finitary group in characteristic 0 must belong to the Alternating Case, and hence be alternating by Theorem 7.2. (Alternating groups are finitary in all characteristics, including 0.) A locally finite simple group G with a nonidentity element whose degree is bounded in a Kegel cover $\mathcal{K}$ is finitary by Corollary 2.11. If all the Kegel quotients of $\mathcal{K}$ are alternating, then the same is true of any abbreviation of $\mathcal{K}$. Such a finitary G cannot then come under the Classical Case, Theorem 6.5(2), and so must be an alternating group by Theorem 7.2.

8. The classical case

In this section we handle the direct part of Theorem 1.1 in the classical case of Theorem 6.5(2). As in Theorem 6.5(2), let the locally finite simple group G be contained in $\mathrm{FCl}_L(X^+, X^-, f)$ with $\mathcal{K}$ a perfect, quasisimple Kegel cover of type Cl for $(\mathrm{Cl}, p) \neq (\mathrm{Sp}, 2)$, where $p\,(> 0)$ is the characteristic of the perfect field L. For Cl $=$ SU let σ be the associated automorphism of L of order 2, and in all other cases let $\sigma = 1$.

Recall that, for $A \leq G$, we write $X^\varepsilon_A = [X^\varepsilon, A]/\mathrm{S}_{X^\varepsilon}(A)$, a section of X^ε covering all nontrivial A-composition factors in X^ε.

Set

$$\mathcal{P} = \{\, P \mid P = P',\ P/O_p(P) \simeq \mathrm{Cl}_{K_P}(W^+_P, W^-_P, f_P) \simeq \mathrm{Cl}_{n_P}(K_P),\ n_P > 10\,,$$
$$O_p(P) \leq \ker X^\varepsilon_P,\ P \text{ finite with } X^\varepsilon_P \text{ nearly nonsingular for } P/O_p(P)\}.$$

As $(\mathrm{Cl}, p) \neq (\mathrm{Sp}, 2)$ and by Propositions 4.7, 4.10, and 4.11, we can take $K_P \leq L$ and view W^ε_P as a $K_P P$-submodule of $X^\varepsilon_P\ (= L \otimes_{K_P} W^\varepsilon_P)$. Also there is a constant κ_P with f_P equal to the form induced on W^ε_P by $\kappa_P f$, the form f_P being of type Cl with respect to $\sigma_P = \sigma|_{K_P}$.

Note that $H \in \mathcal{P}$ for all $(H, O) \in \mathcal{K}$. We are particularly interested in the quasisimple members of $\mathcal{P}$:

$$\mathcal{Q} = \{\, Q \mid Q = Q' \simeq \mathrm{Cl}_{K_Q}(W^+_Q, W^-_Q, f_Q) \simeq \mathrm{Cl}_{n_Q}(K_Q),\ n_Q > 10\,,$$
$$Q \text{ finite with } X^\varepsilon_Q = [X^\varepsilon, Q] \text{ nearly nonsingular for } Q\}.$$

(8.1) Proposition. *For all $P \in \mathcal{P}$, P splits over $O_p(P)$. Each complement Q to $O_p(P)$ in P belongs to $\mathcal{Q}$ with $n_Q = \dim_L X^\varepsilon_Q = \dim_L X^\varepsilon_P$. In particular ℓ-root subgroups of Q, thought of as $\mathrm{Cl}_{K_Q}(W^+_Q, W^-_Q, f_Q)$, are contained in ℓ-root subgroups of $\mathrm{FCl}_L(X^+, X^-, f)$.*

Proof. As X^ε_P contains all nontrivial compositions factors in X^ε for P, we have $\ker_P X^\varepsilon_P = O_p(P)$. Since P is finite, there is in X^ε a P-invariant finite dimensional subspace U with $X^\varepsilon = U \oplus U'$, for some $U' \leq C_{X^\varepsilon}(P)$. Now perfect P is faithful in its action on U, so the extension splits by Proposition 4.6.

Each complement then belongs to $\mathcal{Q}$ by Proposition 4.7.

As mentioned above, we can take $Q = \mathrm{Cl}_{K_Q}(W^+_Q, W^-_Q, f_Q) \in \mathcal{Q}$, for $K_Q \leq L$ and W^ε_Q a $K_Q Q$-submodule of $X^\varepsilon_Q = L \otimes_{K_Q} W^\varepsilon_Q$. The form f_Q is then the appropriate restriction of $\kappa_Q f$ for some constant κ_Q. Since ℓ-root elements are defined in terms of commutator dimensions and the triviality of the form on these commutators, the element $t \in Q$ is an ℓ-root element of $\mathrm{Cl}_{K_Q}(W^+_Q, W^-_Q, f_Q)$ if and only if it is an ℓ-root element of $G \leq \mathrm{FCl}_L(X^+, X^-, f)$. Indeed the ℓ-root subgroup of Q determined by t is within the ℓ-root subgroup of $\mathrm{FCl}_L(X^+, X^-, f)$ determined by t.

Remark. At this point the classification could be finished by appealing to work on groups generated by 1- and 2-root elements, particularly [8], [9]. We choose not to do that since these results are difficult (appealing in part to Timmesfeld's deep work on abstract root subgroups) and such an approach would also require special handling of certain cases (in particular $L = \mathbb{F}_2$). We have enough information to identify G directly without increasing the length of the argument significantly.

(8.2) PROPOSITION. *G has the quasisimple subgroup cover $\mathcal{Q}$ of type* Cl, *and $[X^\varepsilon, Q]$ is a nearly nonsingular module for each $Q \in \mathcal{Q}$.*

Proof. $\mathcal{C} = \{ (P, O_p(P)) \mid P \in \mathcal{P} \}$ is a quasisimple Kegel cover since it contains $\mathcal{K}$. Therefore it is enough to show that, for a fixed but arbitrary $H \in \mathcal{P}$, there is a $Q \in \mathcal{Q}$ with $H \le Q$.

For each $P \in \mathcal{P}$ set

$$s_P = \dim_L \mathrm{S}_{X^+}(P) + \dim_L \mathrm{S}_{X^-}(P)\,.$$

Let $P \in \mathcal{P}_H = \{ N \mid (N, O_p(N)) \in \mathcal{C}_H \}$. If $s_P \neq 0$ then we shall find a $Q \in \mathcal{P}_P$ with $s_Q < s_P$. Replacing P with Q and continuing in this manner, we ultimately reach a $Q \in \mathcal{P}_H$ with $s_Q = 0$. Proposition 4.10 then says that Q is quasisimple with $[X^\varepsilon, Q]$ nearly nonsingular. That is, $H \le Q \in \mathcal{Q}$, which proves the proposition.

Assume now that $s_P \neq 0$. Let $(N, O_p(N)) \in \mathcal{P}_P \subseteq \mathcal{P}_H$, and let N_0 be a complement to $O_p(N)$ in N. We may assume $n_N > n_P + 8$. We write $\bar{N}$ for $N/O_p(N)$, $\bar{W}^\varepsilon$ for W^ε_N, and $\bar{K}$ for K_N. As $P \le N$, X^ε_P is a section of X^ε_N, both being nearly nonsingular. Since $X^\varepsilon_N = L \otimes_{K_N} W^\varepsilon_N$, the module $\bar{W}^\varepsilon_{\bar{P}} = [\bar{W}^\varepsilon, \bar{P}]/\mathrm{S}_{\bar{W}^\varepsilon}(\bar{P})$ is then a nearly nonsingular $\bar{P}/O_p(\bar{P})$-section in $\bar{W}^\varepsilon = W^\varepsilon_N$.

First assume that $\mathrm{Cl} \in \{\mathrm{SL}, \mathrm{Sp}, \mathrm{SU}\}$. By Proposition 8.1 we may choose a complement $P_0 \in \mathcal{Q}$ to $O_p(P)$ in P and let T be a 1-root subgroup of P_0, which is contained in a 1-root subgroup of $\mathrm{FCl}_L(X^+, X^-, f)$. Recall that T is isomorphic to $(K_P, +)$ if $\mathrm{Cl} \neq \mathrm{SU}$ and to $((K_P)_{\sigma_P}, +)$ if $\mathrm{Cl} = \mathrm{SU}$. Since P is perfect and $O_p(P)$ is not central (by Proposition 4.5), we have $[O_p(P), P_0] \neq 1$; therefore we can choose an $a \in O_p(P)$ with $T^a \neq T$. Set $P_1 = \langle P_0, T^a \rangle \simeq \bar{P}_1$. Thus $[\bar{W}^\varepsilon, \bar{P}_1] = \mathrm{S}_{\bar{W}^\varepsilon}(\bar{P}_1) \oplus \bar{W}^\varepsilon_{\bar{P}_0}$ with $\dim_{\bar{K}} \mathrm{S}_{\bar{W}^\varepsilon}(\bar{P}_1) \le 1$ by Proposition 4.10(1). There are two cases:

$$\dim_{\bar{K}} \mathrm{S}_{\bar{W}^+}(\bar{P}_1) + \dim_{\bar{K}} \mathrm{S}_{\bar{W}^-}(\bar{P}_1) \quad = \quad 1 \text{ or } 2\,.$$

We begin with the first case, which can only occur when $\mathrm{Cl} = \mathrm{SL}$. By symmetry we may assume that $\mathrm{S}_{\bar{W}^+}(\bar{P}_1)$ has dimension 1 and $\mathrm{S}_{\bar{W}^-}(\bar{P}_1) = \bar{0}$. Choose $s \in [X^+, P_1]$ ($\le [X^+, N]$) so that its image $\bar{s}$ in $X^+_N = [X^+, N]/\mathrm{S}_{X^+}(N)$ has $\mathrm{S}_{\bar{W}^+}(\bar{P}_1) = \bar{K}\bar{s}$. (Recall that $\bar{W}^+ = W^+_N$ is contained in X^+_N.)

Let the K_P-transvection (1-root) subgroup T of P_0 be contained in the L-transvection subgroup T_{v_T, w_T} of $\mathrm{Cl}_L(X^+, X^-, f)$. In N_0, let $T_0 = N_0 \cap$

$T_{v_N,w_N} = N \cap T_{v_N,w_N}$ be a K_N-transvection subgroup of N_0 (and N) chosen with $f(v_T, w_N) \neq 0 \neq f(v_N, w_T)$ (possible as $[\bar{W}^\varepsilon, \bar{P}]$ has large codimension in $\bar{W}^\varepsilon$). Then by Lemma 3.18(1), $\langle T_0, T\rangle \simeq \mathrm{SL}_2(\bar{K})$, and $T_1 = N \cap T_{v_T,w_T}$ is a K_N-transvection subgroup of N containing T. Next in N_0 find a K_N-transvection subgroup $T_2 = N \cap T_{v_N,w_N+x}$ where $f(v_T, x) = 0\,(= f(v_N, x))$, $f(s, x) \neq 0$, and $x \notin [X^-, P_1]$. By Lemma 3.18(2),

$$\langle \bar{T}_0, \bar{T}_1, \bar{T}_2\rangle \simeq \langle T_0, T_1, T_2\rangle \simeq K_N^2 \colon \mathrm{SL}_2(K_N).$$

Set $A = T_{v_T,w_T+x} \cap \langle T_0, T_1, T_2\rangle = T_{v_T,w_T+x} \cap N$, a K_N-transvection subgroup.

The group $\bar{Q}_1 = \langle \bar{A}, \bar{P}_1\rangle \leq \bar{N}$ has $[\bar{W}^+, \bar{Q}_1] = [\bar{W}^+, \bar{P}_1]$ of dimension n_P+1 and $[\bar{W}^-, \bar{Q}_1] = [\bar{W}^-, \bar{P}_1] \oplus \bar{K}\bar{x}$, also of dimension $n_P + 1$. The subgroup $\bar{P}_1$ is uniserial with only two composition factors on $[\bar{W}^+, \bar{Q}_1]$, one a submodule spanned by $\bar{s}$. As $f(s, w_T + x) \neq 0$, $\bar{A}$ moves $\bar{K}\bar{s}$; so $\bar{Q}_1$ acts irreducibly on $[\bar{W}^+, \bar{Q}_1]$, hence also on $[\bar{W}^-, \bar{Q}_1]$. By Proposition 4.12(1), $\bar{Q}_1 \simeq \mathrm{SL}_{n_P+1}(\bar{K})$ with $\mathrm{S}_{\bar{W}^+}(\bar{Q}_1) = \bar{0} = \mathrm{S}_{\bar{W}^-}(\bar{Q}_1)$.

For $Q = \langle A, P\rangle \in \mathcal{P}_P$, we have $[X^+, Q] = [X^+, A] + [X^+, P] = [X^+, P]$ and $[X^-, Q] = [X^-, A] + [X^-, P]$; so $\dim_L [X^-, Q] \leq 1 + \dim_L [X^-, P]$. Also

$$\dim_L X_Q^\varepsilon = n_Q = 1 + n_P = 1 + \dim_L X_P^\varepsilon\,.$$

Therefore

$$\begin{aligned}
s_P - s_Q &= \Big(\dim_L \mathrm{S}_{X^+}(P) + \dim_L \mathrm{S}_{X^-}(P)\Big)\\
&\qquad -\Big(\dim_L \mathrm{S}_{X^+}(Q) + \dim_L \mathrm{S}_{X^-}(Q)\Big)\\
&= \Big(\dim_L [X^+, P] - \dim_L X_P^+ + \dim_L [X^-, P] - \dim_L X_P^-\Big)\\
&\qquad -\Big(\dim_L [X^+, Q] - \dim_L X_Q^+ + \dim_L [X^-, Q] - \dim_L X_Q^-\Big)\\
&\geq \Big(\dim_L [X^+, P] - \dim_L X_P^+ + \dim_L [X^-, P] - \dim_L X_P^-\Big)\\
&\qquad -\Big(\dim_L [X^+, P] - (1 + \dim_L X_P^+)\\
&\qquad\quad + (1 + \dim_L [X^-, P]) - (1 + \dim_L X_P^-)\Big) \geq 1\,,
\end{aligned}$$

as desired.

We now move to the second case for $\mathrm{Cl} \in \{\mathrm{SL}, \mathrm{Sp}, \mathrm{SU}\}$, where each singular radical $\mathrm{S}_{\bar{W}^\varepsilon}(\bar{P}_1)$ has dimension 1 and is spanned, say, by $\bar{s}^\varepsilon$. Choose vectors $\bar{w}^\varepsilon \in \bar{W}^\varepsilon \setminus [\bar{W}^\varepsilon, \bar{P}_1]$ with $f_N(\bar{w}^+, \bar{w}^-) = \bar{0}$ but $f_N(\bar{w}^+, \bar{s}^-) \neq \bar{0}$ and $f_N(\bar{s}^+, \bar{w}^-) \neq \bar{0}$. (If we are not in the case $\mathrm{Cl} = \mathrm{SL}$, then we require that $\bar{w}^+ = \bar{w}^-$.) Let $\bar{A}$ be the 1-root subgroup $T_{\bar{w}^+,\bar{w}^-}$ of $\bar{N}$, and set $\bar{Q}_1 = \langle \bar{A}, \bar{P}_1\rangle$.

Then $\bar{Q}_1$ acts on $[\bar{W}^\varepsilon, \bar{Q}_1]$ of dimension $n_P + 2$, being spanned by $[\bar{W}^\varepsilon, \bar{P}_1]$ and $\bar{w}^\varepsilon$. The group $\bar{P}_1$ is uniserial on $[\bar{W}^\varepsilon, \bar{P}_1]$ with only two composition factors, one a submodule spanned by $\bar{s}^\varepsilon$.

We first claim that the form induced by f_N on $[\bar{W}^+,\bar{Q}_1]\times[\bar{W}^-,\bar{Q}_1]$ is nondegenerate. (That is, $\mathrm{S}_{\bar{W}^\varepsilon}(\bar{Q}_1)=\bar{0}$.) Suppose $\bar{0}\neq\bar{x}^+\in{}^\perp[\bar{W}^-,\bar{Q}_1]\cap[\bar{W}^+,\bar{Q}_1]$. As $f_N(\bar{s}^+,\bar{w}^-)\neq 0$, ${}^\perp[\bar{W}^-,\bar{Q}_1]$ meets $[\bar{W}^+,\bar{P}_1]$ trivially. Therefore $[\bar{W}^+,\bar{Q}_1]=[\bar{W}^+,\bar{P}_1]\oplus\bar{K}\bar{x}^+$. But then $\bar{s}^-\in[\bar{W}^+,\bar{Q}_1]^\perp$, which is not the case. Thus ${}^\perp[\bar{W}^-,\bar{Q}_1]\cap[\bar{W}^+,\bar{Q}_1]=\bar{0}$. Similarly $[\bar{W}^-,\bar{Q}_1]\cap[\bar{W}^+,\bar{Q}_1]^\perp=\bar{0}$, so f_N is nondegenerate as claimed.

Next we prove that $\bar{Q}_1$ is irreducible on $[\bar{W}^\varepsilon,\bar{Q}_1]$. If $\bar{M}^+$ is a $\bar{Q}_1$-submodule of $[\bar{W}^+,\bar{Q}_1]$, then $\bar{M}^-=(\bar{M}^+)^\perp\cap[\bar{W}^-,\bar{Q}_1]$ is a $\bar{Q}_1$-submodule of $[\bar{W}^-,\bar{Q}_1]$. If k is the dimension of $[\bar{W}^\varepsilon,\bar{Q}_1]$ then $\dim\bar{M}^++\dim\bar{M}^-=k$, so at least one $\bar{M}^\varepsilon$ has dimension greater than or equal to $k/2$. We can assume, without loss, that $\dim\bar{M}^+\geq k/2$; so, in particular, $\bar{M}^+\cap[\bar{W}^+,\bar{P}_1]$ is not confined to $\mathrm{S}_{\bar{W}^+}(\bar{Q}_1)$. Therefore $\bar{M}^+$ contains $[\bar{W}^+,\bar{P}_1]$, which has codimension 1 in $[\bar{W}^+,\bar{Q}_1]$. As $f_N(\bar{s}^+,\bar{w}^-)\neq 0$, $\bar{A}$ moves $\bar{s}^+\in\bar{M}^+$, hence $[\bar{W},\bar{A}]=\bar{K}\bar{w}^+\leq\bar{M}^+$. But now

$$[\bar{W}^+,\bar{Q}_1]\geq\bar{M}^+\geq[\bar{W}^+,\bar{P}_1]\oplus\bar{K}\bar{w}^+=[\bar{W}^+,\bar{Q}_1]\,.$$

Therefore $\bar{Q}_1$ is irreducible on $[\bar{W}^+,\bar{Q}_1]$ and so also on $[\bar{W}^-,\bar{Q}_1]$ by nondegeneracy. By Proposition 4.12(1), $\bar{Q}_1\simeq\mathrm{Cl}_{n_P+2}(\bar{K})$ with $\mathrm{S}_{\bar{W}^+}(\bar{Q}_1)=\bar{0}=\mathrm{S}_{\bar{W}^-}(\bar{Q}_1)$.

Again we lift $\bar{A}$ to a 1-root subgroup A of N and set $Q=\langle A,P\rangle\in\mathcal{P}_P$. We have $[X^\varepsilon,Q]=[X^\varepsilon,A]+[X^\varepsilon,P]$, so that $\dim_L[X^\varepsilon,Q]\leq 1+\dim_L[X^\varepsilon,P]$. Also

$$\dim_L X^\varepsilon_Q=n_Q=2+n_P=2+\dim_L X^\varepsilon_P\,.$$

Calculating as above, we find

$$\begin{aligned}
s_P-s_Q=&\Big(\dim_L\mathrm{S}_{X^+}(P)+\dim_L\mathrm{S}_{X^-}(P)\Big)\\
&-\Big(\dim_L\mathrm{S}_{X^+}(Q)+\dim_L\mathrm{S}_{X^-}(Q)\Big)\\
\geq&\Big(\dim_L[X^+,P]-\dim_L X^+_P+\dim_L[X^-,P]-\dim_L X^-_P\Big)\\
&-\Big((1+\dim_L[X^+,P])-(2+\dim_L X^+_P)\\
&\quad+(1+\dim_L[X^-,P])-(2+\dim_L X^-_P)\Big)\geq 2\,.
\end{aligned}$$

Now we may assume that $\mathrm{Cl}=\Omega$. By Proposition 8.1 we may choose a complement P_0 to $O_p(P)$ in P. Let T be a 2-root (Siegel) subgroup of P_0, and $a\in O_p(P)$ with $T^a\neq T$. Set $P_1=\langle P_0,T^a\rangle\simeq\bar{P}_1$. Thus $[\bar{W}^\varepsilon,\bar{P}_1]=\mathrm{S}_{\bar{W}^\varepsilon}(\bar{P}_1)\oplus\bar{W}^\varepsilon_{\bar{P}_0}$ with $0<\dim_{\bar{K}}\mathrm{S}_{\bar{W}^\varepsilon}(\bar{P}_1)=d\leq 2$. As $\mathrm{Cl}=\Omega$, we have $\bar{W}^\varepsilon=(\bar{W}^{-\varepsilon})^1$ and $X^\varepsilon=(X^{-\varepsilon})^1$. Dropping the exponents, we do our calculations in $\bar{W}=\bar{W}^+$.

Let $\bar{W}_0$ be the nonsingular (but possibly degenerate) space $[\bar{W},\bar{P}_0]$, and let the radical of $\bar{W}_0$ be $\bar{Z}$, which is either $\bar{0}$ or nonsingular of dimension 1. Choose $\bar{U}_0\leq\bar{W}_0^\perp$ to be an 8-dimensional, nondegenerate subspace of $\bar{W}$ with $\mathrm{S}_{\bar{W}}(\bar{P}_1)=\bar{U}_0\cap[\bar{W},\bar{P}_1]$.

Let $\bar{U}_1$ be a nonsingular complement to $\mathrm{S}_{\bar{W}}(\bar{P}_1)$ in $\bar{U}_0$, and choose a totally singular subspace $\bar{W}_s \leq \bar{W}_0$ with $\dim_{\bar{K}} \bar{W}_s = \dim_{\bar{K}} \bar{U}_1 = 8-d$. We then choose a subspace $\bar{Y}$ of dimension $8-d$ on the diagonal of $\bar{W}_s \oplus \bar{U}_1$. Therefore $\bar{Y}$ is disjoint from $\bar{W}_s$ and $\bar{U}_1$ and is isometric to $\bar{U}_1$. Thus we have nonsingular

$$\bar{W}_0 \oplus \bar{U}_0 = [\bar{W}, \bar{P}_1] \oplus \bar{U}_1 = [\bar{W}, \bar{P}_1] \oplus \bar{Y} = \bar{U}\,,$$

say, of dimension $n_P + 8$. The radical of $\bar{U}$ is equal to $\bar{Z}$, the radical of $\bar{W}_0$.

Let $\bar{A}_0$ be the largest subgroup of $\bar{N}$ that is trivial on $\bar{Y}^{\perp}$, so that quasisimple $\bar{A} = \bar{A}_0'$ acts irreducibly on nonsingular $\bar{Y} = [\bar{U}, \bar{A}] = [\bar{W}, \bar{A}]$ modulo its radical, which is either $\bar{0}$ or nonsingular of dimension 1. Set $\bar{Q}_1 = \langle \bar{A}, \bar{P}_1 \rangle$.

We claim that $\bar{Q}_1$ is irreducible on $\bar{U} = [\bar{W}, \bar{Q}_1]$ modulo the radical $\bar{Z}$. Let $\bar{M}$ ($> \bar{Z}$) be a nontrivial $\bar{Q}_1$-submodule of $\bar{U}$. Either $\bar{M}$ or $\bar{M}^{\perp} \cap \bar{U}$ has dimension at least half that of $\bar{U}$. Replacing $\bar{M}$ by $\bar{M}^{\perp}$ if necessary, we have $\dim_{\bar{K}} \bar{M} \geq \lceil (n_P+8)/2\,\rceil > 9$. Any $\bar{P}_0$-invariant subspace of $\bar{U}$ either contains $\bar{W}_0$ of dimension n_P or is contained in $\bar{W}_0^{\perp} \cap \bar{U} = \bar{Z} \oplus \bar{U}_0$ of dimension at most 9. Thus $\bar{M}$ contains $\bar{W}_0$; indeed, then $\bar{P}_1$-invariant $\bar{M}$ contains $[\bar{W}, \bar{P}_1]$. As $\bar{U} = [\bar{W}, \bar{P}_1] \oplus [\bar{W}, \bar{A}]$, we now consider $[\bar{M}, \bar{A}] \leq [\bar{W}, \bar{A}] = \bar{Y}$. By design $[[\bar{W}, \bar{P}_1], \bar{A}] \neq \bar{0}$, so we are done except possibly when $\operatorname{char} \bar{K} = 2$ and $[\bar{M}, \bar{A}]$ is the radical of $\bar{Y}$, nonsingular and of dimension 1. The quasisimple group $\bar{A}$ is generated by its 2-root elements $\bar{t}$, but for such an element $[\bar{M}, \bar{t}] \leq [\bar{M}, \bar{A}] \cap [\bar{W}, \bar{t}] = \bar{0}$. In that case all 2-root elements of $\bar{A}$ would fix $\bar{M}$, giving $[\bar{M}, \bar{A}] = \bar{0}$ which is not the case. We conclude that $\bar{Q}_1$ is irreducible on $\bar{U}$; so, by Proposition 4.12(2), $\bar{Q}_1 \simeq \Omega_{\bar{K}}(\bar{U}, f_N|_{\bar{U}})$ is quasisimple with $\mathrm{S}_{\bar{W}}(\bar{Q}_1) = \bar{0}$.

Using Proposition 8.1 we lift $\bar{A}$ to a quasisimple subgroup A of N_0 with $[X^{\varepsilon}, A]$ nearly nonsingular (as $[X^{\varepsilon}, N_0]$ is). We set $Q = \langle A, P \rangle \in \mathcal{P}_P$. Again we have $[X^{\varepsilon}, Q] = [X^{\varepsilon}, A] + [X^{\varepsilon}, P]$, so that $\dim_L [X^{\varepsilon}, Q] \leq (8-d) + \dim_L [X^{\varepsilon}, P]$. Also

$$\dim_L X_Q^{\varepsilon} = n_Q = 8 + n_P = 8 + \dim_L X_P^{\varepsilon}\,.$$

Therefore

$$\begin{aligned} s_P - s_Q &= 2(\dim_L \mathrm{S}_{X^+}(P) - \dim_L \mathrm{S}_{X^+}(Q)) \\ &= 2\Big(\dim_L [X^+, P] - \dim_L X_P^+ \\ &\qquad -(\dim_L [X^+, Q] - \dim_L X_Q^+)\Big) \\ &\geq 2\Big(\dim_L [X^+, P] - \dim_L X_P^+ \\ &\qquad -(8 - d + \dim_L [X^+, P] - (8 + \dim_L X_P^+))\Big) \geq 2d\,, \end{aligned}$$

as desired.

This completes the proof of the proposition.

(8.3) PROPOSITION. *Let* $\mathcal{Q}$ *be the quasisimple cover for* G *from Proposition* 8.2, *and assume that* $\mathcal{Q}$ *has type* Ω *in characteristic* 2. *Then either* $\mathcal{Q}_{\text{orth}} = \{ Q \in \mathcal{Q} \,|\, [X^\varepsilon, Q] \text{ is nondegenerate} \}$ *or* $\mathcal{Q}_{\text{symp}} = \{ Q \in \mathcal{Q} \,|\, \mathrm{Rad}([X^\varepsilon, Q], f) = \mathrm{Rad}([X^\varepsilon, G], f) \neq 0 \}$ *is a subcover.*

Proof. Let $Q \in \mathcal{Q}$. Color the member P of the cover $\mathcal{Q}_Q$ green if $0 \neq \mathrm{Rad}([X^\varepsilon, Q], f) \leq \mathrm{Rad}([X^\varepsilon, P], f)$ and white otherwise. If the set $\mathcal{G}_Q$ of those P that are colored green is a cover, then

$$0 \neq \mathrm{Rad}([X^\varepsilon, Q], f) \leq \varinjlim_{P \in \mathcal{G}_Q} \mathrm{Rad}([X^\varepsilon, P], f) = \mathrm{Rad}([X^\varepsilon, G], f)\,.$$

Because always $\dim_L \mathrm{Rad}([X^\varepsilon, P], f) \leq 1$, the subcover $\mathcal{G}_Q$ is contained in the subcover $\mathcal{Q}_{\text{symp}}$.

Now we can assume that for no Q of $\mathcal{Q}$ is $\mathcal{G}_Q$ a cover. Thus for each Q the set $\mathcal{W}_Q$ of white $P \in \mathcal{Q}_Q$ is a cover. Within $P \in \mathcal{W}_Q$, we can find a nondegenerate quasisimple subgroup $P_Q \in \mathcal{Q}_{\text{orth}}$ containing Q. The set $\{ P_Q \,|\, Q \in \mathcal{Q} \}$ is then a cover contained in the subcover $\mathcal{Q}_{\text{orth}}$.

We now let Y^ε be the L-space $[X^\varepsilon, G]/\mathrm{Rad}([X^\varepsilon, G], f)$.

If $(\mathrm{Cl}, p) \neq (\Omega, 2)$, then by Proposition 8.2 we have $Y^\varepsilon = [X^\varepsilon, G]$ and e, the restriction of f to $Y^+ \times Y^-$ (or $Y^\varepsilon = (Y^{-\varepsilon})^1$ if $\mathrm{Cl} = \Omega$), is nondegenerate of type Cl. The quasisimple cover $\mathcal{R} = \mathcal{Q}$ has $[Y^\varepsilon, Q]$ nearly natural for each $Q \in \mathcal{R}$.

If $(\mathrm{Cl}, p) = (\Omega, 2)$, then by Proposition 8.3 either $\mathcal{Q}_{\text{orth}}$ or $\mathcal{Q}_{\text{symp}}$ is a quasisimple cover. If $\mathcal{Q}_{\text{orth}}$ is a cover, then $Y^\varepsilon = [X^\varepsilon, G] = \langle [X^\varepsilon, Q] \,|\, Q \in \mathcal{Q}_{\text{orth}} \rangle$ is nondegenerate for the restriction of the quadratic form f, while if $\mathcal{Q}_{\text{symp}}$ is a cover, then $[X^\varepsilon, G] = \langle [X^\varepsilon, Q] \,|\, Q \in \mathcal{Q}_{\text{symp}} \rangle$ is nonsingular but degenerate for f. In particular, exactly one of $\mathcal{Q}_{\text{orth}}$ and $\mathcal{Q}_{\text{symp}}$ is a cover; call it $\mathcal{R}$.

If $\mathcal{R} = \mathcal{Q}_{\text{orth}}$, then $e = f|_{Y^\varepsilon}$ is a nondegenerate quadratic form on Y^ε, and $[Y^\varepsilon, Q]$ is nearly natural for each $Q \in \mathcal{R}$, an orthogonal quasisimple cover.

If $\mathcal{R} = \mathcal{Q}_{\text{symp}}$ is a cover, then $\mathrm{Rad}([X^\varepsilon, G], f)$ is nonsingular and of dimension 1 over the perfect field L. The quotient space $Y^\varepsilon = [X^\varepsilon, G]/\mathrm{Rad}([X^\varepsilon, G], f)$ is nondegenerate for the symplectic form e on Y^ε induced by f (or, more precisely, by the symplectic form b associated with f). For each $Q \in \mathcal{R}$, the LQ-module $[Y^\varepsilon, Q]$ is a nearly natural module for the symplectic group Q.

Thus, in all cases, $G \leq \mathrm{FCl}_L(Y^+, Y^-, e)$ with $[Y^\varepsilon, G] = Y^\varepsilon$, where e is a nondegenerate form of type Cl and $\mathcal{R}$ is a quasisimple cover with $[Y^\varepsilon, Q]$ nearly natural for each $Q \in \mathcal{R}$, also of type Cl. (Note that we may be in the situation, for $p = 2$, where $G \leq \mathrm{F\Omega}_L(X^+, X^-, f)$ whereas $G \leq \mathrm{FSp}_L(Y^+, Y^-, e)$; the type Cl may have changed from Ω to Sp.)

If e is a classical σ-sesquilinear form and K is a finite subfield of L, then we view σ as an automorphism of K when more properly we should talk of the restriction of σ to K. (If $\mathrm{Cl} \neq \mathrm{SU}$, then we set $\sigma = 1$.)

(8.4) Lemma. (1) *For* $Q \in \mathcal{R}$, *there are a subfield* $K_Q \le L$, K_Q*-subspaces* $V_Q^\varepsilon \le [Y^\varepsilon, Q]$, *and a form* e_Q *of type* Cl (*with respect to* σ) *such that* $[Y^\varepsilon, Q] = L \otimes_{K_Q} V_Q^\varepsilon$ *and* $Q = \mathrm{Cl}_{K_Q}(V_Q^+, V_Q^-, e_Q)$ (*where* Q *acts trivially on* $(V_Q^+)^\perp = [Y^+, Q]^\perp$ *and* ${}^\perp(V_Q^-) = {}^\perp[Y^-, Q]$). *The field* K_Q *is unique and the* K_Q*-spaces* V_Q^+, *and* V_Q^- *are uniquely determined up to scalar multiplication by elements of* L. *The form* e_Q *is unique up to scalar multiplication by a constant* $c = c^\sigma \in K_Q$ *and is equal to* $\kappa_Q\, e|_{V_Q^\pm}$ *for some constant* $\kappa_Q = \kappa_Q^\sigma \in L$. (*Here by* $e|_{V_Q^\pm}$ *we mean the appropriate restriction* $e|_{V_Q^+ \times V_Q^-}$ *or* $e|_{V_Q^+} = e|_{V_Q^-}$. *Note that* e_Q *depends not just on the subgroup* Q *but also on the particular choice of modules* V_Q^ε.)

(2) *Let* $Q = \mathrm{Cl}_{K_Q}(V_Q^+, V_Q^-, e_Q)$ *as in* (1), *and let* $P \in \mathcal{R}$ *with* $P \ge Q$. *Then there are a unique subfield* K_P *of* L *and unique* K_P*-subspaces* V_P^ε *of* $[Y^\varepsilon, P]$ *with* $V_P^\varepsilon \ge V_Q^\varepsilon$ *and* $P = \mathrm{Cl}_{K_P}(V_P^+, V_P^-, e_P)$.

Proof. (1) As $Q \in \mathcal{Q}$, we have $Q \simeq \mathrm{Cl}_{K_Q}(W_Q^+, W_Q^-, f_Q)$ with $[Y^\varepsilon, Q]$ nearly nonsingular and indeed nearly natural for Q. By Proposition 4.10(3), Q acts trivially on $[Y^+, Q]^\perp$ and ${}^\perp[Y^-, Q]$. Part (1) then follows from Proposition 4.11, where we have identified K_Q with its isomorphic image in L.

(2) As $P \ge Q = \mathrm{Cl}_{K_Q}(V_Q^+, V_Q^-, e_Q)$, the module $[Y^\varepsilon, Q]$, nearly natural for Q, is contained in $[Y^\varepsilon, P]$, nearly natural for P. By (1), there are a subfield K_P of L, K_P-subspaces $U_P^\varepsilon \le [Y^\varepsilon, P]$, and a form d_P with $P = \mathrm{Cl}_{K_P}(U_P^+, U_P^-, d_P)$. Now $[U_P^\varepsilon, Q]$ has dimension at most $\dim_L [Y^\varepsilon, Q] = \dim_{K_Q} V_Q^\varepsilon$. Therefore by Theorem 4.8, $[U_P^\varepsilon, Q]$ is nearly natural for Q. Applying Proposition 4.11 next to $Q \le P = \mathrm{Cl}_{K_P}(U_P^+, U_P^-, d_P)$, we learn that $K_Q \le K_P$ and that there are K_Q subspaces $U_Q^\varepsilon \le U_P^\varepsilon$ and a form d_Q with $Q = \mathrm{Cl}_{K_Q}(U_Q^+, U_Q^-, d_Q)$. By (1), there are scalars $a_\varepsilon \in L$ with $V_Q^+ = a_+ U_Q^+$ and $V_Q^- = U_Q^- a_-$. Set $V_P^+ = a_+ U_P^+$ and $V_P^- = U_P^- a_-$. Then $P = \mathrm{Cl}_{K_P}(V_P^+, V_P^-, e_P)$ with $V_P^\varepsilon \ge V_Q^\varepsilon$, as desired. By (1) a second pair of subspaces V_P^ε having these properties would be scalar multiples of the first. As distinct scalar multiples intersect trivially, the V_P^ε are unique. This gives (2).

Choose $R \in \mathcal{R}$ with $\dim_L [Y^\varepsilon, R] > 8$. Let

$$\mathcal{S} = \mathcal{R}_R = \{\, Q \in \mathcal{R} \mid Q \ge R \,\},$$

a quasisimple subgroup cover. By Lemma 8.4(1) there are a subfield K_R of L, K_R-subspaces $V_R^\varepsilon \le Y^\varepsilon$, and a form e_R with $R = \mathrm{Cl}_{K_R}(V_R^+, V_R^-, e_R)$. Furthermore there is a constant $\kappa_R = \kappa_R^\sigma$ with $e_R = \kappa_R\, e|_{V_R^\pm}$. Let $e^Y = \kappa_R\, e$, so that e^Y is a form of type Cl (with respect to σ) for which $G \le \mathrm{FCl}_L(Y^+, Y^-, e^Y)$.

For each $P \in \mathcal{S}$, let $K_P \ge K_R$ and $V_P^\varepsilon \ge V_R^\varepsilon$ be uniquely determined as in Lemma 8.4(2), so that $P = \mathrm{Cl}_{K_P}(V_P^+, V_P^-, e_P)$ with $e_P = \kappa_P\, e|_{V_P^\pm}$ and $\kappa_P^\sigma = \kappa_P$ by Lemma 8.4(1).

(8.5) LEMMA. (1) *The forms* e_P *can be chosen so that, for all* $P \in \mathcal{S}$, $P = \mathrm{Cl}_{K_P}(V_P^+, V_P^-, e_P)$ *with* $e_P = e^Y|_{V_P^\pm}$.

(2) *For* $Q, P \in \mathcal{S}$ *with* $Q \le P$, *we have* $K_Q \le K_P$ *and* $V_Q^\varepsilon \le V_P^\varepsilon$. *Furthermore, with the choice of forms as in* (1), $e_Q = e_P|_{V_Q^\pm}$.

Proof. (1) As $K_R \le K_P$, we have the constant $c_P = \kappa_R \kappa_P^{-1} = c_P^\sigma \in K_P$, and $P = \mathrm{Cl}_{K_P}(V_P^+, V_P^-, c_P e_P)$. Here

$$c_P e_P = \kappa_R \kappa_P^{-1}\, e_P = \kappa_R \kappa_P^{-1}(\kappa_P\, e|_{V_P^\pm}) = \kappa_R\, e|_{V_P^\pm} = e^Y|_{V_P^\pm}\,.$$

Therefore, if necessary replacing e_P by $c_P e_P$, we have (1).

(2) $K_Q \le K_P$ is immediate. Also $V_Q^\varepsilon \cap V_P^\varepsilon \ge V_R^\varepsilon$, so uniqueness as in Lemma 8.4(2) forces $V_Q^\varepsilon \le V_P^\varepsilon$. Furthermore $e_Q = e^Y|_{V_Q^\pm} = e^Y|_{V_P^\pm}|_{V_Q^\pm} = e_P|_{V_Q^\pm}$.

Set

$$K = \bigcup_{Q \in \mathcal{S}} K_Q \subseteq L$$

and

$$V^\varepsilon = \bigcup_{Q \in \mathcal{S}} V_Q^\varepsilon \subseteq Y^\varepsilon\,.$$

$\mathcal{S}$ is a local system for G; so that, by Lemma 8.5(2), $\{\, K_Q \,|\, Q \in \mathcal{S}\,\}$ is a local system for the field K and $\{\, V_Q^\varepsilon \,|\, Q \in \mathcal{S}\,\}$ is a local system for the K-space V^ε. In particular K is a locally finite field. Let $e^V = e^Y|_{V^\pm}$, the restriction of e^Y to $V^+ \times V^-$ (when $\mathrm{Cl} \ne \Omega$) or to $V^\varepsilon = (V^{-\varepsilon})^1$ (when $\mathrm{Cl} = \Omega$). Then e^V is a G-invariant form of type Cl with respect to σ. Indeed by Lemma 8.5(2), $e^V = \varinjlim_{Q \in \mathcal{S}} e_Q$, the direct limit of the forms e_Q, chosen according to Lemma 8.5(1). With this choice we then have $Q = \mathrm{Cl}_{K_Q}(V_Q^+, V_Q^-, e^V|_{V_Q^\pm})$ for each $Q \in \mathcal{S}$.

Theorem 1.1 is now a consequence of Theorems 6.1, 6.5, 7.2, and

(8.6) THEOREM. *With the notation of this section,* e^V *is nondegenerate and* $G = \mathrm{FCl}_K(V^+, V^-, e^V)$.

Proof. The form e^Y is nondegenerate and $Y^\varepsilon = L \otimes_K V^\varepsilon$, and so e^V is nondegenerate.

By construction, $G \le \mathrm{FCl}_K(V^+, V^-, e^V)$. We must prove the containment in the other direction. Consider an arbitrary $h \in \mathrm{FCl}_K(V^+, V^-, e^V)$. Since K is a locally finite field, $\mathrm{FCl}_K(V^+, V^-, e^V)$ is a locally finite group by Theorem 3.15.

By Lemmas 3.5 (when $\mathrm{Cl} = \mathrm{SL}$) and 3.6 (when $\mathrm{Cl} \ne \mathrm{SL}$), there exist appropriate finite dimensional K-subspaces U_h^ε of V^ε with $e^V|_{U_h^\pm}$ nondegenerate and $[V^\varepsilon, h] \le U_h^\varepsilon$. As $\{\, V_Q^\varepsilon \,|\, Q \in \mathcal{S}\,\}$ is a local system for V^ε, we can choose

$Q \in \mathcal{S}$ with $Q = \mathrm{Cl}_{K_Q}(V_Q^+, V_Q^-, e^V|_{V_Q^\pm})$ and $U_h^\varepsilon \le [V^\varepsilon, Q] = U_Q^\varepsilon = K \otimes_{K_Q} V_Q^\varepsilon$. Then $\langle h, Q\rangle$ is a finite subgroup of quasisimple $\mathrm{Cl}_K(U_Q^+, U_Q^-, e^V|_{U_Q^\pm})$. Therefore there is a finite field K_h with $K_Q \le K_h \le K$ and $\langle h, Q\rangle \le \mathrm{Cl}_{K_h}(V_h^+, V_h^-, e^V|_{V_h^\pm})$ where $V_h^\varepsilon = K_h \otimes_{K_Q} V_Q^\varepsilon$.

As $\{\, K_P \,|\, Q \le P \in \mathcal{S}\,\}$ is a local system for the field K, there is a $P \in \mathcal{S}$ with $K_h \le K_P$ and $Q \le P$, hence $V_h^\varepsilon \le V_P^\varepsilon$. But then $h \in \mathrm{Cl}_{K_P}(V_P^+, V_P^-, e^V|_{V_P^\pm}) = P \le G$. That is, $h \in G$ as desired, proving the theorem.

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