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The resolution of the Nirenberg-Treves conjecture

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Abstract

We give a proof of the Nirenberg-Treves conjecture: that local solvability of principal-type pseudo-differential operators is equivalent to condition (Ψ). This condition rules out sign changes from $-$ to $+$ of the imaginary part of the principal symbol along the oriented bicharacteristics of the real part. We obtain local solvability by proving a localizable *a priori* estimate for the adjoint operator with a loss of two derivatives (compared with the elliptic case).

The proof involves a new metric in the Weyl (or Beals-Fefferman) calculus which makes it possible to reduce to the case when the gradient of the imaginary part is nonvanishing, so that the zeroes form a smooth submanifold. The estimate uses a new type of weight, which measures the changes of the distance to the zeroes of the imaginary part along the bicharacteristics of the real part between the minima of the curvature of the zeroes. By using condition (Ψ) and the weight, we can construct a multiplier giving the estimate.

1. Introduction

In this paper we shall study the question of local solvability of a classical pseudo-differential operator $P \in \Psi^m_{cl}(M)$ on a C^∞ manifold M. Thus, we assume that the symbol of P is an asymptotic sum of homogeneous terms, and that $p = \sigma(P)$ is the homogeneous principal symbol of P. We shall also assume that P is of principal type, which means that the Hamilton vector field H_p and the radial vector field are linearly independent when $p = 0$; thus $dp \neq 0$ when $p = 0$.

Local solvability of P at a compact set $K \subseteq M$ means that the equation

$$Pu = v \tag{1.1}$$

has a local solution $u \in \mathcal{D}'(M)$ in a neighborhood of K for any $v \in C^\infty(M)$ in a set of finite codimension. We can also define microlocal solvability at any compactly based cone $K \subset T^*M$, see [9, Def. 26.4.3]. Hans Lewy's famous counterexample [19] from 1957 showed that not all smooth linear differential operators are solvable. It was conjectured by Nirenberg and Treves [21] in 1970 that local solvability of principal type pseudo-differential operators is equivalent to condition (Ψ), which means that

$$\operatorname{Im}(ap) \text{ does not change sign from } - \text{ to } + \quad \text{along the oriented bicharacteristics of } \operatorname{Re}(ap) \tag{1.2}$$

for any $0 \neq a \in C^\infty(T^*M)$. The oriented bicharacteristics are the positive flow-outs of the Hamilton vector field $H_{\operatorname{Re}(ap)} \neq 0$ on $\operatorname{Re}(ap) = 0$ (also called semi-bicharacteristics). Condition (1.2) is invariant under multiplication of p with nonvanishing factors, and conjugation of P with elliptic Fourier integral operators; see [9, Lemma 26.4.10]. Thus, it suffices to check (1.2) for some $a \in C^\infty(T^*M)$ such that $H_{\operatorname{Re}(ap)} \neq 0$.

The necessity of (Ψ) for local solvability of pseudo-differential operators was proved by Moyer [20] in 1978 for the two dimensional case, and by Hörmander [8] in 1981 for the general case. In the analytic category, the sufficiency of condition (Ψ) for solvability of microdifferential operators acting on microfunctions was proved by Trépreau [22] in 1984 (see also [10, Ch. VII]). The sufficiency of condition (Ψ) for solvability of pseudo-differential operators in two dimensions was proved by Lerner [13] in 1988, leaving the higher dimensional case open.

For differential operators, condition (Ψ) is equivalent to condition (P), which rules out any sign changes of $\operatorname{Im}(ap)$ along the bicharacteristics of $\operatorname{Re}(ap)$ for nonvanishing $a \in C^\infty(T^*M)$. The sufficiency of (P) for local solvability of pseudo-differential operators was proved in 1970 by Nirenberg and Treves [21] in the case when the principal symbol is real analytic. Beals and Fefferman [1] proved the general case in 1973, by using a new calculus that was later developed by Hörmander into the Weyl calculus.

In all these solvability results, one obtains *a priori* estimates for the adjoint operator with loss of one derivative (compared with the elliptic case). In 1994 Lerner [14] constructed counterexamples to the sufficiency of (Ψ) for local solvability with loss of one derivative in dimensions greater than two, raising doubts on whether the condition really was sufficient for solvability. But it was proved in 1996 by the author [4] that Lerner's counterexamples are locally solvable with loss of at most two derivatives (compared with the elliptic case). There are other results giving local solvability with loss of one derivative under conditions stronger than (Ψ), see [5], [11], [15] and [17].

In this paper we shall prove local and microlocal solvability of principal type pseudo-differential operators satisfying condition (Ψ); this resolves the Nirenberg-Treves conjecture. To get local solvability at a point x_0 we shall also assume a strong form of the nontrapping condition at x_0:

$$p = 0 \implies \partial_\xi p \neq 0. \tag{1.3}$$

This means that all semi-bicharacteristics are transversal to the fiber $T^*_{x_0}M$, which originally was the condition for the principal type of Nirenberg and Treves [21]. Microlocally, we can always obtain (1.3) after a canonical transformation.

THEOREM 1.1. *If $P \in \Psi^m_{cl}(M)$ is of principal type and satisfies condition (Ψ) given by (1.2) microlocally near $(x_0, \xi_0) \in T^*M$, then*

$$\|u\| \le C(\|P^*u\|_{(2-m)} + \|Ru\| + \|u\|_{(-1)}), \qquad u \in C_0^\infty(M). \tag{1.4}$$

Here $R \in \Psi^1_{1,0}(M)$ such that $(x_0, \xi_0) \notin \mathrm{WF}\, R$, which gives microlocal solvability of P at (x_0, ξ_0) with a loss of at most two derivatives. If P satisfies conditions (Ψ) and (1.3) locally near $x_0 \in M$, then (1.4) holds with $x \ne x_0$ in $\mathrm{WF}\, R$, which gives local solvability of P at x_0 with a loss of two derivatives.

Thus, we lose at most two derivatives in the estimate of the adjoint, which is one more compared to the condition (P) case.

Most of the earlier results on local solvability have relied on finding a factorization of the imaginary part of the principal symbol; see for example [5] and [17]. We have not been able to find a factorization in terms of sufficiently good symbol classes in order to prove local solvability. The best result seems to be given by Lerner [16], who obtained a factorization showing that every first order principal type pseudo-differential operator satisfying condition (Ψ) is a sum of a solvable operator and an L^2-bounded operator. But the bounded perturbation has a very bad symbol, and the solvable operator is solvable with a loss of more than one derivative, so that this does not imply solvability.

This paper is a shortened and simplified version of [6], and the plan is as follows. In Section 2 we reduce the proof of Theorem 1.1 to an estimate for a microlocal normal form for the adjoint operator $P^* = D_t + iF(t, x, D_x)$. Here F has real principal symbol $f \in C^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$, and P_0 satisfies the corresponding condition $(\overline{\Psi})$: $t \mapsto f(t, x, \xi)$ does not change sign from $+$ to $-$ with increasing t for any (x, ξ). In Corollary 2.7 we shall for any $T > 0$ prove the estimate

$$\|u\|^2 \le T\, \mathrm{Im}\,(P^*u, B_T u) + C\|\langle D_x\rangle^{-1} u\|^2 \tag{1.5}$$

for $u \in \mathcal{S}(\mathbf{R}^{n+1})$ having support where $|t| \le T$. Here $\|u\|$ is the L^2 norm on $\mathbf{R}^{n+1}$, (u, v) the corresponding sesquilinear inner product, $\langle D_x\rangle = 1 + |D_x|$ and $B_T(t, x, D_x) \in \Psi^1_{1/2,1/2}(\mathbf{R}^n)$ is symmetric, with symbol having homogeneous gradient

$$\nabla B_T = (\partial_x B_T, |\xi|\partial_\xi B_T) \in S^1_{1/2,1/2}(\mathbf{R}^n).$$

This gives local solvability by the Cauchy-Schwarz inequality after microlocalization. Since $\mathrm{Re}\, P^* = D_t$ is solvable and $\nabla B_T \in S^1_{1/2,1/2}(\mathbf{R}^n)$, the estimate (1.5) is localizable and independent of lower order terms in the expansion of F (see Lemma 2.6). Clearly, the estimate (1.5) follows if we have suitable lower bounds on $2\operatorname{Im}(B_T P^*) = \partial_t B_T + 2\operatorname{Re}(B_T F)$.

Let $g_{1,0}(dx, d\xi) = |dx|^2 + |d\xi|^2/|\xi|^2$ be the homogeneous metric and $g_{1/2,1/2} = |\xi| g_{1,0}$. The symbol B_T of the multiplier is essentially a lower order perturbation of the signed $g_{1/2,1/2}$ distance δ_0 to the sign changes of f in $T^*\mathbf{R}^n$ for fixed t. Then $\delta_0 f \ge 0$ and we find from condition $(\overline{\Psi})$ that $\partial_t \delta_0 \ge 0$.

In Section 3 we shall make a second microlocalization with a new metric $G_1 \cong H_1 g_{1/2,1/2}$, where $c|\xi|^{-1} \le H_1 \le 1$ so that $cg_{1,0} \le G_1 \le g_{1/2,1/2}$ (see Definition 3.4). This metric has the property that if $H_1 \ll 1$ at $f^{-1}(0)$, then $|\nabla f| \ne 0$ and $f^{-1}(0)$ is a C^∞ surface with curvature bounded by $CH_1^{1/2}$. The implicit function theorem then gives $f = \alpha\delta_0$ where $|\partial_{x,\xi}\delta_0| \ne 0$, $\alpha \ne 0$, and these factors are in suitable symbol classes in the Weyl calculus by Proposition 3.9.

In Section 5 we introduce the weight, which for fixed (x, ξ) is defined by

(1.6)
$$m_1(t_0) = \inf_{t_1 \le t_0 \le t_2} \left\{ \delta_0(t_2) - \delta_0(t_1) + \max(H_1^{1/2}(t_1)\langle\delta_0(t_1)\rangle, H_1^{1/2}(t_2)\langle\delta_0(t_2)\rangle) \right\}$$

where $\langle\delta_0\rangle = 1 + |\delta_0|$ (see Definition 5.1). This is a weight for the metric $g_{1/2,1/2}$ by Proposition 5.4, such that $c|\xi|^{-1/2} \le m_1 \le 1$. The weight m_1 essentially measures how much the signed distance δ_0 changes between the minima of $H_1^{1/2}$. From (1.6) we immediately obtain the convexity property of $t \mapsto m_1(t, x, \xi)$ given by Proposition 5.7:

$$\sup_I m_1 \le |\Delta_I \delta_0| + 2\sup_{\partial I} m_1, \qquad I = [a, b] \times (x, \xi)$$

where $|\Delta_I \delta_0| = |\delta_0(b, x, \xi) - \delta_0(a, x, \xi)|$ is the variation of δ_0 on I. This makes it possible to add a perturbation ϱ_T so that $|\varrho_T| \le m_1$ and

$$\partial_t(\delta_0 + \varrho_T) \ge m_1/2T \qquad \text{in } |t| \le T$$

by Proposition 5.8. Using the Wick quantization $B_T = (\delta_0 + \varrho_T)^{\mathrm{Wick}}$ in Section 6 we obtain that positive symbols give positive operators, and

$$\partial_t B_T \ge m_1^{\mathrm{Wick}}/2T \ge c|D_x|^{-1/2}/2T \qquad \text{in } |t| \le T.$$

Now if $m_1 \ll 1$ at (t_0, x_0, ξ_0), then we obtain that $|\delta_0| \ll H_1^{-1/2}$ and $H_1^{1/2} \ll 1$ at both (t_1, x_0, ξ_0) and (t_2, x_0, ξ_0) for some $t_1 \le t_0 \le t_2$. We also find that

$$\Delta_I \delta_0 = \mathcal{O}(m_1(t_0, x, \xi)), \qquad I = [t_1, t_2] \times (x_0, \xi_0)$$

and because of condition $(\overline{\Psi})$ the sign changes of $(x, \xi) \mapsto f(t_0, x, \xi)$ are located in the set where $\delta_0(t_1, x, \xi)\delta_0(t_2, x, \xi) \le 0$. This makes it possible to estimate $\nabla^2 f$ in terms of m_1 (see Proposition 5.5), and we obtain the lower bound: $\operatorname{Re}(B_T F) \ge -C_0 m_1^{\mathrm{Wick}}$ in Section 7. By replacing B_T with $|D_x|^{1/2} B_T$ we obtain for small enough T the estimate (1.5) and the Nirenberg-Treves conjecture.

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2. The multiplier estimate

In this section we shall microlocalize and reduce the proof of Theorem 1.1 to the semiclassical multiplier estimate of Proposition 2.5 for a microlocal normal form of the adjoint operator. We shall consider operators

$$P_0 = D_t + iF(t, x, D_x) \tag{2.1}$$

where $F \in C^\infty(\mathbf{R}, \Psi^1_{1,0}(\mathbf{R}^n))$ has real principal symbol $\sigma(F) = f$. In the following, we shall assume that P_0 satisfies condition $(\overline{\Psi})$:

$$f(t, x, \xi) > 0 \quad \text{and } s > t \implies f(s, x, \xi) \ge 0 \tag{2.2}$$

for any t, $s \in \mathbf{R}$ and $(x, \xi) \in T^*\mathbf{R}^n$. This means that the adjoint P_0^* satisfies condition (Ψ). Observe that if $\chi \ge 0$ then χf also satisfies (2.2), thus the condition can be localized.

Remark 2.1. We shall also consider symbols $f \in L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$, that is, $f(t, x, \xi) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ is bounded in $S^1_{1,0}(\mathbf{R}^n)$ for almost all t. Then we say that P_0 satisfies condition $(\overline{\Psi})$ if for every (x, ξ), condition (2.2) holds for almost all s, $t \in \mathbf{R}$. Since $(x, \xi) \mapsto f(t, x, \xi)$ is continuous for almost all t it suffices to check (2.2) for (x, ξ) in a countable dense subset of $T^*\mathbf{R}^n$. Then we find that f has a representative satisfying (2.2) for any t, s and (x, ξ) after putting $f(t, x, \xi) \equiv 0$ for t in a countable union of null sets.

In order to prove Theorem 1.1 we shall make a second microlocalization using the specialized symbol classes of the Weyl calculus, and the Weyl quantization of symbols $a \in \mathcal{S}'(T^*\mathbf{R}^n)$ defined by:

$$(a^w u, v) = (2\pi)^{-n} \iint \exp\left(i\langle x - y, \xi\rangle\right) a\left(\tfrac{x+y}{2}, \xi\right) u(y)\overline{v}(x)\, dxdyd\xi, \qquad u, v \in \mathcal{S}(\mathbf{R}^n).$$

Observe that $\operatorname{Re} a^w = (\operatorname{Re} a)^w$ is the symmetric part and $i \operatorname{Im} a^w = (i \operatorname{Im} a)^w$ the antisymmetric part of the operator a^w. Also, if $a \in S^m_{1,0}(\mathbf{R}^n)$ then $a^w(x, D_x) = a(x, D_x)$ modulo $\Psi^{m-1}_{1,0}(\mathbf{R}^n)$ by [9, Th. 18.5.10].

We recall the definitions of the Weyl calculus: let g_w be a Riemannean metric on $T^*\mathbf{R}^n$, $w = (x, \xi)$, then we say that g is slowly varying if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies $g_w \cong g_{w_0}$; i.e., $1/C \le g_w / g_{w_0} \le C$. Let σ be the standard symplectic form on $T^*\mathbf{R}^n$, and let $g^\sigma(w) \ge g(w)$ be the dual metric of $w \mapsto g(\sigma(w))$. We say that g is σ temperate if it is slowly varying and

$$g_w \le C g_{w_0}(1 + g^\sigma_w(w - w_0))^N, \qquad w,\, w_0 \in T^*\mathbf{R}^n.$$

A positive real-valued function $m(w)$ on $T^*\mathbf{R}^n$ is g continuous if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies $m(w) \cong m(w_0)$. We say that m is σ, g temperate if it is g continuous and

$$m(w) \le C m(w_0)(1 + g^\sigma_w(w - w_0))^N, \qquad w,\, w_0 \in T^*\mathbf{R}^n.$$

If m is σ, g temperate, then m is a weight for g and we can define the symbol classes: $a \in S(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and

(2.3)
$$|a|^g_j(w) = \sup_{T_i \ne 0} \frac{|a^{(j)}(w, T_1, \dots, T_j)|}{\prod_1^j g_w(T_i)^{1/2}} \le C_j m(w), \qquad w \in T^*\mathbf{R}^n \qquad \text{for } j \ge 0,$$

which gives the seminorms of $S(m, g)$. If $a \in S(m, g)$ then we say that the corresponding Weyl operator $a^w \in \operatorname{Op} S(m, g)$. For more on the Weyl calculus, see [9, §18.5].

Definition 2.2. Let m be a weight for the metric g. Then $a \in S^+(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and $|a|^g_j \le C_j m$ for $j \ge 1$.

Observe that by Taylor's formula we find that

$$\begin{aligned} |a(w) - a(w_0)| &\le C_1 \sup_{\theta \in [0,1]} g_{w_\theta}(w - w_0)^{1/2} m(w_\theta) \\ &\le C_N m(w_0)(1 + g^\sigma_{w_0}(w - w_0))^N \end{aligned}$$

where $w_\theta = \theta w + (1 - \theta) w_0$, which implies that $m + |a|$ is a weight for g. Clearly, $a \in S(m + |a|, g)$, so the operator a^w is well-defined.

LEMMA 2.3. *Assume that m_j is a weight for $g_j = h_j g^\sharp \le g^\sharp = (g^\sharp)^\sigma$ and $a_j \in S^+(m_j, g_j)$, $j = 1, 2$. Let $g = g_1 + g_2$ and $h^2 = \sup g_1/g_2^\sigma = \sup g_2/g_1^\sigma = h_1 h_2$, then*

$$a_1^w a_2^w - (a_1 a_2)^w \in \operatorname{Op} S(m_1 m_2 h, g), \tag{2.4}$$

and we have the usual expansion of (2.4) *with terms in $S(m_1 m_2 h^k, g)$, $k \ge 1$.*

This result is well known, but for completeness we give a proof.

Proof. As shown after Definition 2.2 we have that $m_j + |a_j|$ is a weight for g_j and $a_j \in S(m_j + |a_j|, g_j)$, $j = 1, 2$. Thus

$$a_1^w a_2^w \in \operatorname{Op} S((m_1 + |a_1|)(m_2 + |a_2|), g)$$

is given by Proposition 18.5.5 in [9]. We find that $a_1^w a_2^w - (a_1a_2)^w = a^w$ with

$$a(w) = E(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2})a_1(w_1)a_2(w_2)\big|_{w_1=w_2=w}$$

where $E(z) = (e^z-1)/z = \int_0^1 e^{\theta z}\, d\theta$. We have that $\sigma(D_{w_1}, D_{w_2})a_1(w_1)a_2(w_2) \in S(M,G)$ where

$$M(w_1, w_2) = m_1(w_1)m_2(w_2)h_1^{1/2}(w_1)h_2^{1/2}(w_2)$$

and $G_{w_1,w_2}(z_1, z_2) = g_{1,w_1}(z_1) + g_{2,w_2}(z_2)$. Now the proof of Theorem 18.5.5 in [9] works when $\sigma(D_{w_1}, D_{w_2})$ is replaced by $\theta\sigma(D_{w_1}, D_{w_2})$, uniformly in $0 \le \theta \le 1$. By integrating over $\theta \in [0,1]$ we obtain that $a(w)$ has an asymptotic expansion in $S(m_1m_2h^k, g)$, which proves the lemma. □

Remark 2.4. The conclusions of Lemma 2.3 also hold if a_1 has values in $\mathcal{L}(B_1, B_2)$ and a_2 in B_1 where B_1 and B_2 are Banach spaces (see §18.6 in [9]).

For example, if $\{ a_j \}_j \in S(m_1, g_1)$ with values in ℓ^2, and $b_j \in S(m_2, g_2)$ uniformly in j, then $\left\{ a_j^w b_j^w \right\}_j \in \mathrm{Op}(m_1m_2, g)$ with values in ℓ^2. In the proof of Theorem 1.1 we shall microlocalize near (x_0, ξ_0) and put $h^{-1} = \langle\xi_0\rangle = 1 + |\xi_0|$. Then after a symplectic dilation: $(x, \xi) \mapsto (h^{-1/2}x, h^{1/2}\xi)$, we find that $S^k_{1,0} = S(h^{-k}, hg^\sharp)$ and $S^k_{1/2,1/2} = S(h^{-k}, g^\sharp)$, $(g^\sharp)^\sigma = g^\sharp$, $k \in \mathbf{R}$. Therefore, we shall prove a semiclassical estimate for a microlocal normal form of the operator.

Let $\|u\|$ be the L^2 norm on $\mathbf{R}^{n+1}$, and (u, v) the corresponding sesquilinear inner product. As before, we say that $f \in L^\infty(\mathbf{R}, S(m, g))$ if $f(t, x, \xi)$ is measurable and bounded in $S(m, g)$ for almost all t. The following is the main estimate that we shall prove.

Proposition 2.5. *Assume that $P_0 = D_t + if^w(t, x, D_x)$, with real $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfying condition $(\overline{\Psi})$ given by (2.2); here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. Then there exists $T_0 > 0$ and real-valued symbols $b_T(t, x, \xi) \in L^\infty(\mathbf{R}, S(h^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp))$ uniformly for $0 < T \le T_0$, so that*

$$h^{1/2}\|u\|^2 \le T\,\mathrm{Im}\,(P_0u, b_T^w u) \tag{2.5}$$

for $u(t, x) \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^n)$ having support where $|t| \le T$. The constant T_0 and the seminorms of b_T only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

It follows from the proof (see the end of Section 7) that $|b_T| \le CH_1^{-1/2}$, where H_1 is a weight for $g^\sharp$ such that $h \le H_1 \le 1$, and $G_1 = H_1g^\sharp$ is σ temperate (see Proposition 6.3 and Definition 3.4).

There are two difficulties present in estimates of the type (2.5). The first is that b_T is not C^∞ in the t variables. Therefore one has to be careful not to involve b_T^w in the calculus with symbols in all the variables. We shall avoid this problem by using tensor products of operators and the Cauchy-Schwarz inequality. The second difficulty lies in the fact that $|b_T| \gg h^{1/2}$, so it is not obvious that lower order terms and cut-off errors can be controlled.

LEMMA 2.6. *The estimate* (2.5) *can be perturbed with terms in* $L^\infty(\mathbf{R}, S(1, hg^\sharp))$ *in the symbol of* P_0 *for small enough* T, *by changing* b_T (*satisfying the same conditions*). *Thus it can be microlocalized*: *if* $\phi(w) \in S(1, hg^\sharp)$ *is real-valued and independent of* t, *then*

$$\operatorname{Im}(P_0\phi^w u, b_T^w \phi^w u) \le \operatorname{Im}(P_0 u, \phi^w b_T^w \phi^w u) + Ch^{1/2}\|u\|^2 \tag{2.6}$$

where $\phi^w b_T^w \phi^w$ *satisfies the same conditions as* b_T^w.

Proof. It is clear that the estimate (2.5) can be perturbed with terms in $L^\infty(\mathbf{R}, S(h, hg^\sharp))$ in the symbol expansion of P_0 for small enough T. Now, we can also perturb with symmetric terms $r^w \in L^\infty(\mathbf{R}, \operatorname{Op} S(1, hg^\sharp))$. In fact, if $r \in S(1, hg^\sharp)$ is real and $b \in S^+(1, g^\sharp)$ is real modulo $S(h^{1/2}, g^\sharp)$, then

$$|\operatorname{Im}(r^w u, b^w u)| \le |([(\operatorname{Re} b)^w, r^w]u, u)|/2 + |(r^w u, (\operatorname{Im} b)^w u)| \le Ch^{1/2}\|u\|^2, \tag{2.7}$$

since $[(\operatorname{Re} b)^w, r^w] \in \operatorname{Op} S(h^{1/2}, g^\sharp)$ by Lemma 2.3. Now assume $P_1 = P_0 + r^w(t, x, D_x)$ with complex-valued $r \in L^\infty(\mathbf{R}, S(1, hg^\sharp))$, and let

$$\begin{aligned} E(t, x, \xi) &= \exp\left(-\int_0^t \operatorname{Im} r(s, x, \xi)\, ds\right) \\ &\in C(\mathbf{R}, S(1, hg^\sharp) \textstyle\bigcap S^+(T, hg^\sharp)), \qquad |t| \le T \end{aligned}$$

since $\partial_w E = -E\int_0^t \operatorname{Im} \partial_w r\, ds$. Then E is real and we have by Lemma 2.3 that

$$E^w (E^{-1})^w = 1 = (E^{-1})^w E^w \qquad \text{modulo } \operatorname{Op} S(T^2 h, hg^\sharp)$$

uniformly when $|t| \le T$. Thus, for small enough T we obtain that $\|u\| \cong \|E^w u\|$. We also find that

$$(E^{-1})^w P_0 E^w = P_0 + i \operatorname{Im} r^w + (E^{-1}\{f, E\})^w = P_1$$

modulo $L^\infty(\mathbf{R}, \operatorname{Op} S(h, hg^\sharp))$ and symmetric terms in $L^\infty(\mathbf{R}, \operatorname{Op} S(1, hg^\sharp))$. Thus we obtain the estimate with P_0 replaced with P_1 by substituting $E^w u$ in (2.5) and using (2.7) to perturb with symmetric terms in $L^\infty(\mathbf{R}, \operatorname{Op} S(1, hg^\sharp))$. We find that b_T^w is replaced with $B_T^w = E^w b_T^w E^w$ which is symmetric, satisfying the same conditions as b_T^w by Lemma 2.3, since $E \in S(1, hg^\sharp)$ is real so that $B_T = b_T E^2$ modulo $S(h, g^\sharp)$ for almost all t.

If $\phi(w) \in S(1, hg^\sharp)$ then we find that $[P_0, \phi^w] = \{f, \phi\}^w$ modulo $L^\infty(\mathbf{R}, \operatorname{Op} S(h, hg^\sharp))$ where $\{f, \phi\} \in L^\infty(\mathbf{R}, S(1, hg^\sharp))$ is real-valued. By using (2.7) with $r^w = \{f, \phi\}^w$ and $b^w = b_T^w \phi^w$, we obtain (2.6) since $b_T^w \phi^w \in \operatorname{Op} S^+(1, g^\sharp)$ is symmetric modulo $\operatorname{Op} S(h^{1/2}, g^\sharp)$ for almost all t by Lemma 2.3. We find that $\phi^w b_T^w \phi^w$ is symmetric, and as before $\phi^w b_T^w \phi^w = (b_T \phi^2)^w$ modulo $L^\infty(\mathbf{R}, \operatorname{Op} S(h, g^\sharp))$, which satisfies the same conditions as b_T^w. □

Next, we shall prove an estimate for the microlocal normal form of the adjoint operator.

COROLLARY 2.7. *Assume that* $P_0 = D_t + iF^w(t,x,D_x)$, *with* $F^w \in L^\infty(\mathbf{R}, \Psi^1_{1,0}(\mathbf{R}^n))$ *having real principal symbol* f *satisfying condition* $(\overline{\Psi})$ *given by* (2.2). *Then there exists* $T_0 > 0$ *and real-valued symbols* $b_T(t,x,\xi) \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$ *with homogeneous gradient*

$$\nabla b_T = (\partial_x b_T, |\xi|\partial_\xi b_T) \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$$

uniformly for $0 < T \le T_0$, *such that*

$$\|u\|^2 \le T\,\mathrm{Im}\,(P_0 u, b_T^w u) + C_0\|\langle D_x\rangle^{-1}u\|^2 \tag{2.8}$$

for $u \in \mathcal{S}(\mathbf{R}^{n+1})$ *having support where* $|t| \le T$. *The constants* T_0, C_0 *and the seminorms of* b_T *only depend on the seminorms of* F *in* $L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$.

Since $\nabla b_T \in L^\infty(\mathbf{R}, S^1_{1/2,1/2})$ we find that the commutators of b_T^w with operators in $L^\infty(\mathbf{R}, \Psi^0_{1,0})$ are in $L^\infty(\mathbf{R}, \Psi^0_{1/2,1/2})$. This will make it possible to localize the estimate.

Proof of Corollary 2.7. Choose real symbols $\{\,\phi_j(x,\xi)\,\}_j$, $\{\,\psi_j(x,\xi)\,\}_j$ and $\{\,\Psi_j(x,\xi)\,\}_j \in S^0_{1,0}(\mathbf{R}^n)$ having values in ℓ^2, such that $\sum_j \phi_j^2 = 1$, $\psi_j\phi_j = \phi_j$, $\Psi_j\psi_j = \psi_j$ and $\psi_j \ge 0$. We may assume that the supports are small enough so that $\langle\xi\rangle \cong \langle\xi_j\rangle$ in $\mathrm{supp}\,\Psi_j$ for some ξ_j. Then, after doing a symplectic dilation $(y,\eta) = (x\langle\xi_j\rangle^{1/2}, \xi/\langle\xi_j\rangle^{1/2})$ we obtain that $S^m_{1,0}(\mathbf{R}^n) = S(h_j^{-m}, h_j g^\sharp)$ and $S^m_{1/2,1/2}(\mathbf{R}^n) = S(h_j^{-m}, g^\sharp)$ in $\mathrm{supp}\,\Psi_j$, $m \in \mathbf{R}$, where $h_j = \langle\xi_j\rangle^{-1} \le 1$ and $g^\sharp(dy,d\eta) = |dy|^2 + |d\eta|^2$.

By using the calculus in the y variables we find $\phi_j^w P_0 = \phi_j^w P_{0j}$ modulo $\mathrm{Op}\,S(h_j, h_j g^\sharp)$, where

$$P_{0j} = D_t + i(\psi_j F)^w(t,y,D_y) = D_t + if_j^w(t,y,D_y) + r_j^w(t,y,D_y)$$

with

$$f_j = \psi_j f \in L^\infty(\mathbf{R}, S(h_j^{-1}, h_j g^\sharp))$$

satisfying (2.2), and $r_j \in L^\infty(\mathbf{R}, S(1, h_j g^\sharp))$ uniformly in j. Then, by using Proposition 2.5 and Lemma 2.6 for P_{0j}, we obtain real-valued symbols $b_{j,T}(t,y,\eta) \in L^\infty(\mathbf{R}, S(h_j^{-1/2}, g^\sharp) \bigcap S^+(1,g^\sharp))$ uniformly for $0 < T \ll 1$, such that

$$\|\phi_j^w u\|^2 \le T(h_j^{-1/2}\,\mathrm{Im}\,\big(P_0 u, \phi_j^w b_{j,T}^w \phi_j^w u\big) + C_0\|u\|^2) \qquad \forall\, j \tag{2.9}$$

for $u(t,y) \in \mathcal{S}(\mathbf{R} \times \mathbf{R}^n)$ having support where $|t| \le T$. Here and in the following, the constants are independent of T.

By substituting $\Psi_j^w u$ in (2.9) and summing up we obtain

$$\|u\|^2 \le T(\mathrm{Im}\,(P_0 u, b_T^w u) + C_1\|u\|^2) + C_2\|\langle D_x\rangle^{-1}u\|^2 \tag{2.10}$$

for $u(t,y) \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^n)$ having support where $|t| \le T$. Here

$$b_T^w = \sum_j h_j^{-1/2}\Psi_j^w\phi_j^w b_{j,T}^w\phi_j^w\Psi_j^w \in L^\infty(\mathbf{R},\Psi^1_{1/2,1/2})$$

is symmetric. In fact, $\sum_j \phi_j^2 = 1$ so that $\sum_j \phi_j^w\phi_j^w = 1$ modulo $\Psi^{-1}(\mathbf{R}^n)$, and since $\phi_j\Psi_j = \phi_j$ we have $\left\{\phi_j^w[F^w,\Psi_j^w]\right\}_j \in \Psi^{-1}_{1,0}(\mathbf{R}^n)$ with values in ℓ^2 for almost all t. We find the homogeneous gradient $\nabla b_T \in S^1_{1/2,1/2}$ since $b_T = \sum_j h_j^{-1/2}b_{j,T}\phi_j^2 \in S^1_{1/2,1/2}$ modulo $S^0_{1/2,1/2}$, where $\phi_j \in S(1,h_jg^\sharp)$ and $b_{j,T} \in S^+(1,g^\sharp)$ for almost all t. For small enough T we obtain (2.8) and the corollary. □

Proof that Corollary 2.7 *gives Theorem* 1.1. We shall prove that there exist ϕ and $\psi \in S^0_{1,0}(T^*M)$ such that $\phi = 1$ in a conical neighborhood of (x_0,ξ_0), $\psi = 1$ on $\operatorname{supp}\phi$, and for any $T > 0$ there exists $R_T \in S^1_{1,0}(M)$ with the property that $\mathrm{WF}\, R_T^w \bigcap T^*_{x_0}M = \emptyset$ and

(2.11)
$$\|\phi^w u\| \le C_1\left(\|\psi^w P^* u\|_{(2-m)} + T\|u\|\right) + \|R_T^w u\| + C_0\|u\|_{(-1)}, \quad u \in C_0^\infty(M).$$

Here $\|u\|_{(s)}$ is the Sobolev norm and the constants are independent of T. Then for small enough T we obtain (1.4) and microlocal solvability, since $(x_0,\xi_0) \notin \mathrm{WF}(1-\phi)^w$. In the case that P satisfies condition (Ψ) and $\partial_\xi p \ne 0$ near x_0 we may choose finitely many $\phi_j \in S^0_{1,0}(M)$ such that $\sum \phi_j \ge 1$ near x_0 and $\|\phi_j^w u\|$ can be estimated by the right-hand side of (2.11) for some suitable ψ and R_T. By elliptic regularity, we then obtain the estimate (1.4) for small enough T.

By multiplying with an elliptic pseudo-differential operator, we may assume that $m = 1$. Let $p = \sigma(P)$, then it is clear that it suffices to consider $w_0 = (x_0,\xi_0) \in p^{-1}(0)$; otherwise $P^* \in \Psi^1_{cl}(M)$ is elliptic near w_0 and we easily obtain the estimate (2.11). It is clear that we may assume that $\partial_\xi \operatorname{Re} p(w_0) \ne 0$, in the microlocal case after a conical transformation. Then, we may use Darboux' theorem and the Malgrange preparation theorem to obtain microlocal coordinates $(t,y;\tau,\eta) \in T^*\mathbf{R}^{n+1}$ so that $w_0 = (0,0;0,\eta_0)$, $t = 0$ on $T^*_{x_0}M$ and $\overline{p} = q(\tau + if)$ in a conical neighborhood of w_0, where $f \in C^\infty(\mathbf{R}, S^1_{1,0})$ is real and homogeneous satisfying condition (2.2), and $0 \ne q \in S^0_{1,0}$; see Theorem 21.3.6 in [9]. By conjugation with elliptic Fourier integral operators and using the Malgrange preparation theorem successively on lower order terms, we obtain that

$$P^* = Q^w(D_t + i\,(\chi F)^w) + R^w \tag{2.12}$$

microlocally in a conical neighborhood Γ of w_0 (see the proof of Theorem 26.4.7′ in [9]). Here $Q \in S^0_{1,0}(\mathbf{R}^{n+1})$ and $R \in S^1_{1,0}(\mathbf{R}^{n+1})$, such that Q^w has principal symbol $q \ne 0$ in Γ and $\Gamma \bigcap \mathrm{WF}\, R^w = \emptyset$. Moreover, $\chi(\tau,\eta) \in S^0_{1,0}(\mathbf{R}^{n+1})$ is equal to 1 in Γ, $|\tau| \le C|\eta|$ in $\operatorname{supp}\chi(\tau,\eta)$, and $F^w \in C^\infty(\mathbf{R},\Psi^1_{1,0}(\mathbf{R}^n))$ has real principal symbol f satisfying (2.2). By cutting off in the t variable we may assume that $f \in L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$. We shall choose ϕ and ψ so that $\operatorname{supp}\phi \subset \operatorname{supp}\psi \subset \Gamma$ and

$$\phi(t, y; \tau, \eta) = \chi_0(t, \tau, \eta)\phi_0(y, \eta)$$

where $\chi_0(t, \tau, \eta) \in S^0_{1,0}(\mathbf{R}^{n+1})$, $\phi_0(y, \eta) \in S^0_{1,0}(\mathbf{R}^n)$, $t \neq 0$ in $\operatorname{supp}\partial_t\chi_0$, $|\tau| \le C|\eta|$ in $\operatorname{supp}\chi_0$ and $|\tau| \cong |\eta|$ in $\operatorname{supp}\partial_{\tau,\eta}\chi_0$.

Since $q \neq 0$ and $R = 0$ on $\operatorname{supp}\psi$ it is no restriction to assume that $Q \equiv 1$ and $R \equiv 0$ when proving the estimate (2.11). Now, by Theorem 18.1.35 in [9] we may compose $C^\infty(\mathbf{R}, \Psi^m_{1,0}(\mathbf{R}^n))$ with operators in $\Psi^k_{1,0}(\mathbf{R}^{n+1})$ having symbols vanishing when $|\tau| \ge c(1 + |\eta|)$, and we obtain the usual asymptotic expansion in $\Psi^{m+k-j}_{1,0}(\mathbf{R}^{n+1})$ for $j \ge 0$. Since $|\tau| \le C|\eta|$ in $\operatorname{supp}\phi$ and $\chi = 1$ on $\operatorname{supp}\psi$, it thus suffices to prove (2.11) for $P^* = P_0 = D_t + iF^w$.

By using Corollary 2.7 on $\phi^w u$, we obtain that

(2.13)
$$\|\phi^w u\|^2 \le T\left(\operatorname{Im}\left(\phi^w P_0 u, b^w_T\phi^w u\right) + \operatorname{Im}\left([P_0, \phi^w]u, b^w_T\phi^w u\right)\right) + C_0\|u\|^2_{(-1)}$$

where $b^w_T \in L^\infty(\mathbf{R}, \Psi^1_{1/2,1/2}(\mathbf{R}^n))$ is symmetric with $\nabla b_T \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$. We find $[P_0, \phi^w] = -i\partial_t\phi^w + \{\, f, \phi\,\}^w \in \Psi^0_{1,0}(\mathbf{R}^{n+1})$ modulo $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$ by Theorem 18.1.35 in [9]. We have that

(2.14)
$$|\,(v, b^w_T u)\,| = |\,\left(\langle D_y\rangle v, \langle D_y\rangle^{-1} b^w_T u\right)| \le C(\|v\|^2_{(1)} + \|u\|^2) \qquad \forall\ u,\ v \in \mathcal{S}(\mathbf{R}^n)$$

since $\|\langle D_y\rangle v\| \le \|v\|_{(1)}$ and $\langle D_y\rangle^{-1} b^w_T \in L^\infty(\mathbf{R}, \Psi^0_{1/2,1/2}(\mathbf{R}^n))$, $\langle D_y\rangle = 1 + |D_y|$. Now $\phi^w = \phi^w\psi^w$ modulo $\Psi^{-2}_{1,0}(\mathbf{R}^{n+1})$. Thus we find from (2.14) that

(2.15)
$$|\,(\phi^w P_0 u, b^w_T\phi^w u)\,| \le C(\|\phi^w P_0 u\|^2_{(1)} + \|\phi^w u\|^2) \le C'(\|\psi^w P_0 u\|^2_{(1)} + \|u\|^2).$$

We also have to estimate the commutator term $\operatorname{Im}([P_0, \phi^w]u, b^w_T\phi^w u)$ in (2.13). Since $\phi = \chi_0\phi_0$ we find that

$$\{\, f, \phi\,\} = \phi_0\,\{\, f, \chi_0\,\} + \chi_0\,\{\, f, \phi_0\,\}\,,$$

where $\phi_0\,\{\, f, \chi_0\,\} = R_0 \in S^0_{1,0}(\mathbf{R}^{n+1})$ is supported when $|\tau| \cong |\eta|$ and $\psi = 1$. Now $(\tau + if)^{-1} \in S^{-1}_{1,0}(\mathbf{R}^{n+1})$ when $|\tau| \cong |\eta|$, thus by [9, Th. 18.1.35] we find that $R^w_0 = A^w_1\psi^w P_0$ modulo $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$ where $A_1 = R_0(\tau + if)^{-1} \in S^{-1}_{1,0}(\mathbf{R}^{n+1})$. As before, we find from (2.14) that

(2.16)
$$|\,(R^w_0 u, b^w_T\phi^w u)\,| \le C(\|R^w_0 u\|^2_{(1)} + \|\phi^w u\|^2) \le C_0(\|\psi^w P_0 u\|^2 + \|u\|^2)$$

and $|\,(\partial_t\phi^w u, b^w_T\phi^w u)\,| \le \|R^w_1 u\|^2 + C\|u\|^2$ by (2.14), where $R^w_1 = \langle D_y\rangle\partial_t\phi^w \in \Psi^1_{1,0}(\mathbf{R}^{n+1})$; thus $t \neq 0$ in $\operatorname{WF} R^w_1$.

It remains to estimate the term $\operatorname{Im}((\{ f, \phi_0 \} \chi_0)^w u, b_T^w \phi^w u)$, where $(\{ f, \phi_0 \} \chi_0)^w = \{ f, \phi_0 \}^w \chi_0^w$ and $\phi^w = \phi_0^w \chi_0^w$ modulo $\Psi_{1,0}^{-1}(\mathbf{R}^{n+1})$. By (2.14) we find $|(R^w u, b_T^w v)| \le C(\|u\|^2 + \|v\|^2)$ for $R \in S_{1,0}^{-1}(\mathbf{R}^{n+1})$, thus we find

$$|\operatorname{Im}((\{ f, \phi_0 \} \chi_0)^w u, b_T^w \phi^w u)| \le |\operatorname{Im}(\{ f, \phi_0 \}^w \chi_0^w u, b_T^w \phi_0^w \chi_0^w u)| + C\|u\|^2.$$

The calculus gives $b_T^w \phi_0^w = (b_T \phi_0)^w$ and

$$2i \operatorname{Im}((b_T \phi_0)^w \{ f, \phi_0 \}^w) = \{ b_T \phi_0, \{ f, \phi_0 \} \}^w = 0$$

modulo $L^\infty(\mathbf{R}, \Psi^0_{1/2,1/2}(\mathbf{R}^n))$ since $\nabla(b_T \phi_0) \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$. We obtain

$$|\operatorname{Im}(\{ f, \phi_0 \}^w \chi_0^w u, b_T^w \phi_0^w \chi_0^w u)| \le C\|\chi_0^w u\|^2 \le C'\|u\|^2 \tag{2.17}$$

and the estimate (2.11), which completes the proof of Theorem 1.1. □

It remains to prove Proposition 2.5, which will be done at the end of Section 7. The proof involves the construction of a multiplier b_T^w, and it will occupy most of the remaining part of the paper. In the following, we let $\|u\|(t)$ be the L^2 norm of $x \mapsto u(t,x)$ in $\mathbf{R}^n$ for fixed t, and $(u,v)(t)$ the corresponding sesquilinear inner product. Let $\mathcal{B} = \mathcal{B}(L^2(\mathbf{R}^n))$ be the set of bounded operators $L^2(\mathbf{R}^n) \mapsto L^2(\mathbf{R}^n)$. We shall use operators which depend measurably on t.

Definition 2.8. We say that $t \mapsto A(t)$ is *weakly measurable* if $A(t) \in \mathcal{B}$ for all t and $t \mapsto A(t)u$ is *weakly measurable* for every $u \in L^2(\mathbf{R}^n)$, i.e., $t \mapsto (A(t)u, v)$ is measurable for any u, $v \in L^2(\mathbf{R}^n)$. We say that $A(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}, \mathcal{B})$ if $t \mapsto A(t)$ is *weakly measurable* and *locally bounded* in $\mathcal{B}$.

If $A(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}, \mathcal{B})$, then we find that the function $t \mapsto (A(t)u, v) \in L^\infty_{\mathrm{loc}}(\mathbf{R})$ has weak derivative $\frac{d}{dt}(Au, v) \in \mathcal{D}'(\mathbf{R})$ for any u, $v \in \mathcal{S}(\mathbf{R}^n)$ given by

$$\tfrac{d}{dt}(Au, v)(\phi) = -\int (A(t)u, v)\, \phi'(t)\, dt, \qquad \phi(t) \in C_0^\infty(\mathbf{R}).$$

If $u(t)$, $v(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}, L^2(\mathbf{R}^n))$ and $A(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}, \mathcal{B})$, then we find $t \mapsto (A(t)u(t), v(t)) \in L^\infty_{\mathrm{loc}}(\mathbf{R})$ is measurable. We shall use the following multiplier estimate (see also [13] and [15] for similar estimates):

PROPOSITION 2.9. *Let $P_0 = D_t + iF(t)$ with $F(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}, \mathcal{B})$. Assume that $B(t) = B^*(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}, \mathcal{B})$, such that*

$$\tfrac{d}{dt}(Bu, u) + 2\operatorname{Re}(Bu, Fu) \ge (mu, u) \quad \text{in } \mathcal{D}'(I) \qquad \forall\, u \in \mathcal{S}(\mathbf{R}^n) \tag{2.18}$$

where $m(t) = m^*(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}, \mathcal{B})$ and $I \subseteq \mathbf{R}$ is open. Then

$$\int (mu, u)\, dt \le 2 \int \operatorname{Im}(Pu, Bu)\, dt \tag{2.19}$$

for $u \in C_0^1(I, \mathcal{S}(\mathbf{R}^n))$.

Proof. Since $B(t) \in L^\infty_{\rm loc}(\mathbf{R}, \mathcal{B})$, we may for u, $v \in \mathcal{S}(\mathbf{R}^n)$ define the regularization

$$(B_\varepsilon(t)u, v) = \varepsilon^{-1} \int (B(s)u, v)\, \phi((t-s)/\varepsilon)\, ds = (Bu, v)\, (\phi_{\varepsilon,t}), \qquad \varepsilon > 0,$$

where $\phi_{\varepsilon,t}(s) = \varepsilon^{-1}\phi((t-s)/\varepsilon)$ with $0 \le \phi \in C_0^\infty(\mathbf{R})$ satisfying $\int \phi(t)\, dt = 1$. Then $t \mapsto (B_\varepsilon(t)u, v)$ is in $C^\infty(\mathbf{R})$ with derivative equal to $\frac{d}{dt}\, (Bu, v)\, (\phi_{\varepsilon,t}) = -\,(Bu, v)\, (\phi'_{\varepsilon,t})$. Let I_0 be an open interval such that $I_0 \Subset I$. Then for small enough $\varepsilon > 0$ and $t \in I_0$ we find from condition (2.18) that

$$\frac{d}{dt}\, (B_\varepsilon(t)u, u) + 2\,\mathrm{Re}\,(Bu, Fu)\, (\phi_{\varepsilon,t}) \ge (mu, u)\, (\phi_{\varepsilon,t}), \qquad u \in \mathcal{S}(\mathbf{R}^n). \tag{2.20}$$

In fact, $\phi_{\varepsilon,t} \ge 0$ and $\operatorname{supp} \phi_{\varepsilon,t} \in C_0^\infty(I)$ for small enough ε when $t \in I_0$.

Now for $u(t) \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n))$ and $\varepsilon > 0$ we define

$$M_{\varepsilon,u}(t) = (B_\varepsilon(t)u(t), u(t)) = \varepsilon^{-1} \int (B(s)u(t), u(t))\, \phi((t-s)/\varepsilon)\, ds. \tag{2.21}$$

For small enough ε we obtain $M_{\varepsilon,u}(t) \in C_0^1(I_0)$, with derivative

$$\frac{d}{dt} M_{\varepsilon,u} = \big((\tfrac{d}{dt} B_\varepsilon)u, u\big) + 2\,\mathrm{Re}\,(B_\varepsilon u, \partial_t u)$$

since $B(t) \in L^\infty_{\rm loc}(\mathbf{R}, \mathcal{B})$. By integrating with respect to t, we obtain the vanishing average

$$0 = \int \frac{d}{dt} M_{\varepsilon,u}(t)\, dt = \int \big((\tfrac{d}{dt} B_\varepsilon)u, u\big)\, dt + \int 2\,\mathrm{Re}\,(B_\varepsilon u, \partial_t u)\, dt \tag{2.22}$$

when $u \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n))$. We obtain from (2.20) and (2.22) that

$$0 \ge \iint \big((m(s)u(t), u(t)) + 2\,\mathrm{Re}\,(B(s)u(t), \partial_t u(t) - F(s)u(t)) \big) \phi((t-s)/\varepsilon)\, ds dt.$$

By letting $\varepsilon \to 0$, we find by dominated convergence that

$$0 \ge \int (m(t)u(t), u(t)) + 2\,\mathrm{Re}\,(B(t)u(t), \partial_t u(t) - F(t)u(t))\, dt$$

since $u \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n))$ and $m(t)$, $B(t)$, $F(t) \in L^\infty_{\rm loc}(\mathbf{R}, \mathcal{B})$. Here $\partial_t u - Fu = iPu$ and $2\,\mathrm{Re}\,(Bu, iPu) = -2\,\mathrm{Im}\,(Pu, Bu)$; thus we obtain (2.19) for $u \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n))$. Since I_0 is an arbitrary open subinterval with compact closure in I, this completes the proof of the proposition. □

3. The symbol classes

In this section we shall define the symbol classes to be used. Assume that $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfies (2.2). Here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. The results are uniform in the usual sense; they only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$. Let

$$X_+(t) = \{\, w \in T^*\mathbf{R}^n : \exists\, s \le t,\ f(s, w) > 0 \,\}, \tag{3.1}$$

$$X_-(t) = \{\, w \in T^*\mathbf{R}^n : \exists\, s \ge t,\ f(s, w) < 0 \,\}. \tag{3.2}$$

Clearly, $X_\pm(t)$ are open in $T^*\mathbf{R}^n$, $X_+(s) \subseteq X_+(t)$ and $X_-(s) \supseteq X_-(t)$ when $s \le t$. By condition $(\overline{\Psi})$ we obtain that $X_-(t) \bigcap X_+(t) = \emptyset$ and $\pm f(t,w) \ge 0$ when $w \in X_\pm(t)$, $\forall\, t$. Let $X_0(t) = T^*\mathbf{R}^n \setminus (X_+(t) \bigcup X_-(t))$ which is closed in $T^*\mathbf{R}^n$. By the definition of $X_\pm(t)$ we have $f(t,w) = 0$ when $w \in X_0(t)$. Let

$$d_0(t_0, w_0) = \inf \left\{ g^\sharp(w_0 - z)^{1/2} : \ z \in X_0(t_0) \right\} \tag{3.3}$$

be the $g^\sharp$ distance in $T^*\mathbf{R}^n$ to $X_0(t_0)$ for fixed t_0. It is equal to $+\infty$ in the case that $X_0(t_0) = \emptyset$.

Definition 3.1. We define the signed distance function $\delta_0(t,w)$ by

$$\delta_0 = \operatorname{sgn}(f) \min(d_0, h^{-1/2}), \tag{3.4}$$

where d_0 is given by (3.3) and

$$\operatorname{sgn}(f)(t,w) = \begin{cases} 1, & w \in X_\pm(t) \\ 0, & w \in X_0(t) \end{cases} \tag{3.5}$$

so that $\operatorname{sgn}(f) f \ge 0$.

Definition 3.2. We say that $w \mapsto a(w)$ is *Lipschitz continuous* on $T^*\mathbf{R}^n$ with respect to the metric $g^\sharp$ if

$$\sup_{w \ne z \in T^*\mathbf{R}^n} |a(w) - a(z)| / g^\sharp(w - z)^{1/2} = \|a\|_{Lip} < \infty$$

where $\|a\|_{Lip}$ is the Lipschitz constant of a.

PROPOSITION 3.3. *The signed distance function $w \mapsto \delta_0(t,w)$ given by Definition* 3.1 *is Lipschitz continuous with respect to the metric $g^\sharp$ with Lipschitz constant equal to* 1, *for all t. We also find that $t \mapsto \delta_0(t,w)$ is nondecreasing,* $0 \le \delta_0 f$, $|\delta_0| \le h^{-1/2}$ *and* $|\delta_0| = d_0$ *when* $|\delta_0| < h^{-1/2}$.

Proof. Clearly, it suffices to show the Lipschitz continuity of $w \mapsto \delta_0(t,w)$ on $\complement X_\pm(t)$, and thus of $w \mapsto d_0(t,w)$ when $d_0 < \infty$. In fact, if $w_1 \in X_-(t)$ and $w_2 \in X_+(t)$ then we can find $w_0 \in X_0(t)$ on the line connecting w_1 and w_2. By using the Lipschitz continuity of d_0 and the triangle inequality we then find that

$$|\delta_0(t,w_2) - \delta_0(t,w_1)| \le |w_2 - w_0| + |w_0 - w_1| = |w_2 - w_1|.$$

The triangle inequality also shows that $w \mapsto g^\sharp(w-z)^{1/2}$ is Lipschitz continuous with Lipschitz constant equal to 1. By taking the infimum over z we find that $w \mapsto d_0(t,w)$ is Lipschitz continuous when $d_0 < \infty$, which gives the Lipschitz continuity of $w \mapsto \delta_0(t,w)$.

Clearly $\delta_0 f \ge 0$, and by the definition $|\delta_0| = \min(d_0, h^{-1/2}) \le h^{-1/2}$ so that $|\delta_0| = d_0$ when $|\delta_0| < h^{-1/2}$. Since $X_+(t)$ is nondecreasing and $X_-(t)$ is nonincreasing when t increases, we find that $t \mapsto \delta_0(t,w)$ is nondecreasing. $\square$

In the following, we shall treat t as a parameter which we shall suppress, and we shall denote $f' = \partial_w f$ and $f'' = \partial_w^2 f$. Also, in the following, assume that we have choosen $g^\sharp$ orthonormal coordinates so that $g^\sharp(w) = |w|^2$.

Definition 3.4. Let

$$H_1^{-1/2} = 1 + |\delta_0| + \frac{|f'|}{|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}} \tag{3.6}$$

and $G_1 = H_1 g^\sharp$.

Remark 3.5. We have that

$$1 \le H_1^{-1/2} \le 1 + |\delta_0| + h^{-1/4}|f'|^{1/2} \le Ch^{-1/2} \tag{3.7}$$

since $|f'| \le C_1 h^{-1/2}$ and $|\delta_0| \le h^{-1/2}$. Moreover,

$$|f'| \le H_1^{-1/2}(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2})$$

so that by the Cauchy-Schwarz inequality,

$$|f'| \le 2|f''|H_1^{-1/2} + 3h^{1/2}H_1^{-1} \le C_2 H_1^{-1/2}. \tag{3.8}$$

Definition 3.6. Let

$$M = |f| + |f'|H_1^{-1/2} + |f''|H_1^{-1} + h^{1/2}H_1^{-3/2}; \tag{3.9}$$

then $h^{1/2} \le M \le C_3 h^{-1}$.

Proposition 3.7. *We find that G_1 is σ temperate, such that $G_1 = H_1^2 G_1^\sigma$ and*

$$H_1(w) \le C_0 H_1(w_0)(1 + H_1(w) g^\sharp(w - w_0)). \tag{3.10}$$

Also, M is a weight for G_1 such that $f \in S(M, G_1)$ and

$$M(w) \le C_1 M(w_0)(1 + H_1(w_0) g^\sharp(w - w_0))^{3/2}. \tag{3.11}$$

In the case when $1 + |\delta_0(w_0)| \le H_1^{-1/2}(w_0)/2$, we have $|f'(w_0)| \ge h^{1/2}$,

$$|f^{(k)}(w_0)| \le C_k |f'(w_0)| H_1^{\frac{k-1}{2}}(w_0), \qquad k \ge 1, \tag{3.12}$$

and $1/C \le |f'(w)|/|f'(w_0| \le C$ when $|w - w_0| \le cH_1^{-1/2}(w_0)$ for some $c > 0$.

Since $G_1 \le g^\sharp \le G_1^\sigma$ we find that the conditions (3.10) and (3.11) are stronger than the property of being σ temperate (in fact, strongly σ temperate in the sense of [2, Def. 7.1]). When $1 + |\delta_0| < H_1^{-1/2}/2$ we find that $f' \in S(|f'|, G_1)$, $f^{-1}(0)$ is a C^∞ hypersurface, and then $H_1^{1/2}$ gives an upper bound on the curvature of $f^{-1}(0)$ by (3.12). Proposition 3.8 shows that (3.12) also holds for $k = 0$ when $1 + |\delta_0| \ll H_1^{-1/2}$.

Proof. If $H_1(w_0)g^\sharp(w-w_0) \ge c > 0$ then we immediately obtain (3.10) with $C_0 = c^{-1}$. Thus, in order to prove (3.10), it suffices to prove that $H_1(w) \le C_0 H_1(w_0)$ when $H_1(w_0)g^\sharp(w-w_0) \ll 1$, i.e., that G_1 is slowly varying.

First we consider the case $1 + |\delta_0(w_0)| \ge H_1^{-1/2}(w_0)/2$. Then we find by the uniform Lipschitz continuity of $w \mapsto |\delta_0(w)|$ that

$$H_1^{-1/2}(w) \ge 1 + |\delta_0(w)| \ge 1 + |\delta_0(w_0)| - H_1^{-1/2}(w_0)/6 \ge H_1^{-1/2}(w_0)/3$$

when $|w - w_0| \le H_1^{-1/2}(w_0)/6$, which gives the slow variation in this case with $C_0 = 9$.

In the case $1 + |\delta_0(w_0)| \le H_1^{-1/2}(w_0)/2$ we have that $H_1^{1/2}(w_0) \le 1/2$ and

$$|f''(w_0)| + h^{1/4}|f'(w_0)|^{1/2} + h^{1/2} \le 2H_1^{1/2}(w_0)|f'(w_0)| \le |f'(w_0)|. \tag{3.13}$$

Let $H_1 = H_1(w_0)$ and $F(z) = f'(w_0 + zH_1^{-1/2})/|f'(w_0)| \in C^\infty$. Then we find $|F(0)| = 1$, $|F'(0)| \le 2$ and $|F''(z)| \le C$, $\forall\, z$, since $h^{1/2} \le 4H_1|f'(w_0)|$ by (3.13). Taylor's formula gives that $1/C_1 \le |F(z)| \le C_1$ and $|F'(z)| \le C_2$ when $|z| \le \varepsilon$ is sufficiently small, depending on the seminorms of f. Thus when $|w - w_0| \le \varepsilon H_1^{-1/2}$ for $\varepsilon \ll 1$, we have $1/C_1 \le |f'(w)|/|f'(w_0| \le C_1$ and $|f''(w)| \le C_2 H_1^{1/2}|f'(w_0|$; thus (3.13) gives

$$H_1^{1/2}(w) \le |f''(w)||f'(w)|^{-1} + h^{1/4}|f'(w)|^{-1/2} + h^{1/2}|f'(w)|^{-1} \le C_3 H_1^{1/2}$$

and the slow variation. Observe that (3.12) follows from (3.13) for $k = 2$. When $k \ge 3$ we have

$$|f^{(k)}(w_0)| \le C_k h^{\frac{k-2}{2}} \le 4C_k C^{k-3}|f'(w_0)|H_1^{\frac{k-1}{2}},$$

since $h^{1/2} \le 4H_1|f'(w_0)|$ by (3.13) and $h^{(k-3)/2} \le C^{k-3}H_1^{(k-3)/2}$ by (3.7).

Next, we shall prove that M is a weight for G_1. By Taylor's formula,

$$|f^{(k)}(w)| \le C_4 \sum_{j=0}^{2-k} |f^{(k+j)}(w_0)||w-w_0|^j + C_4 h^{1/2}|w-w_0|^{3-k}, \qquad 0 \le k \le 2, \tag{3.14}$$

thus

$$M(w) \le C_5 \sum_{k=0}^{2} |f^{(k)}(w_0)|(|w-w_0| + H_1^{-1/2}(w))^k + C_5 h^{1/2}(|w-w_0| + H_1^{-1/2}(w))^3.$$

By interchanging w and w_0 in (3.10) we find

$$H_1^{-1/2}(w) + |w - w_0| \le C_0(H_1^{-1/2}(w_0) + |w - w_0|).$$

Thus

$$\begin{aligned}M(w) &\le C_6 \sum_{k=0}^{2} |f^{(k)}(w_0)| H_1^{-k/2}(w_0)(1 + H_1^{1/2}(w_0)|w - w_0|)^k \\ &\quad + C_6 h^{1/2} H_1^{-3/2}(w_0)(1 + H_1^{1/2}(w_0)|w - w_0|)^3 \\ &\le C_6 M(w_0)(1 + H_1^{1/2}(w_0)|w - w_0|)^3\end{aligned}$$

which gives (3.11). It is clear from the definition of M that $|f^{(k)}| \le M H_1^{k/2}$ when $k \le 2$, and when $k \ge 3$ we have $|f^{(k)}| \le C_k h^{\frac{k-2}{2}} \le C_k C^{k-3} M H_1^{\frac{k}{2}}$ since $h^{1/2} \le M H_1^{3/2}$ and $h^{(k-3)/2} \le C^{k-3} H_1^{(k-3)/2}$. This completes the proof of Proposition 3.7. ☐

Note that $f \in S(M, H_1 g^\sharp)$ for any choice of $H_1 \ge h$ in Definition 3.6. We shall compare our metric with the Beals-Fefferman metric $G = Hg^\sharp$ for f on $T^*\mathbf{R}^n$, where

$$H^{-1} = 1 + |f| + |f'|^2 \le Ch^{-1}. \tag{3.15}$$

This metric is σ temperate on $T^*\mathbf{R}^n$, $\sup G/G^\sigma = H^2 \le 1$ and $f \in S(H^{-1}, G)$ (see for example the proof of Lemma 26.10.2 in [9]).

Proposition 3.8. *We have $H^{-1} \le CH_1^{-1}$ and $M \le CH_1^{-1}$, which implies that $f \in S(H_1^{-1}, G_1)$ and*

$$1/C \le M/(|f''|H_1^{-1} + h^{1/2}H_1^{-3/2}) \le C. \tag{3.16}$$

When $|\delta_0| \le \kappa_0 H_1^{-1/2}$ and $H_1^{1/2} \le \kappa_0$ for $0 < \kappa_0$ sufficiently small, then

$$1/C_1 \le M/|f'|H_1^{-1/2} \le C_1. \tag{3.17}$$

Thus, we find that the metric G_1 gives a coarser localization than the Beals-Fefferman metric G and smaller cut-off errors.

Proof. First note that by the Cauchy-Schwarz inequality

$$M = |f| + |f'|H_1^{-1/2} + |f''|H_1^{-1} + h^{1/2}H_1^{-3/2} \le C(H^{-1} + H_1^{-1}).$$

Thus, $M \le CH_1^{-1}$ if $H^{-1} \le CH_1^{-1}$. Observe that we only have to do this when $|\delta_0| \ll H^{-1/2}$, since otherwise $H^{-1/2} \le C|\delta_0| \le CH_1^{-1/2}$.

If $|\delta_0(w_0)| \le \kappa H^{-1/2}(w_0) \le C\kappa h^{-1/2}$ and $C\kappa < 1$, then there exists $w \in f^{-1}(0)$ such that $|w - w_0| = |\delta_0(w_0)|$. Since $f(w) = 0$, Taylor's formula gives that

$$|f(w_0)| \le |f'(w_0||\delta_0(w_0)| + |f''(w_0)||\delta_0(w_0)|^2/2 + Ch^{1/2}|\delta_0(w_0)|^3. \tag{3.18}$$

We find from (3.18) and (3.15) that $|f(w_0)| \le C_0\kappa H^{-1}(w_0)$ when $|\delta_0(w_0)| \le \kappa H^{-1/2}(w_0)$. When $C_0\kappa < 1$ we obtain

$$H^{-1}(w_0) \le (1 - C\kappa)^{-1}(1 + |f'(w_0)|^2) \le C' H_1^{-1}(w_0)$$

by (3.8).

Observe that when $|\delta_0| \cong h^{-1/2}$ we have $H_1^{-1/2} \cong h^{-1/2}$, which gives $M \cong h^{-1}$ and proves (3.16) in this case. If $|\delta_0(w_0)| < h^{-1/2}$, then as before there exists $w \in f^{-1}(0)$ such that $|w - w_0| = |\delta_0(w_0)| \le H_1^{-1/2}(w_0)$. We obtain from (3.18) and (3.8) that

$$M \le C\big(|f'|H_1^{-1/2} + |f''|H_1^{-1} + h^{1/2}H_1^{-3/2}\big) \le C'\left(|f''|H_1^{-1} + h^{1/2}H_1^{-3/2}\right) \quad \text{at } w_0,$$

which gives (3.16). If $|\delta_0| \ll H_1^{-1/2} \le Ch^{-1/2}$ and $H_1^{1/2} \ll 1$, then we obtain by (3.18) and (3.12) that

$$M \le C\big(|f'|H_1^{-1/2} + |f''|H_1^{-1} + h^{1/2}H_1^{-3/2}\big) \le C'|f'|H_1^{-1/2} \qquad \text{at } w_0.$$

This gives (3.17) and completes the proof of the proposition. □

Proposition 3.9. *Let $H_1^{-1/2}$ be given by Definition 3.4 for $f \in S(h^{-1}, hg^\sharp)$. There exists $\kappa_1 > 0$ so that if $\langle\delta_0\rangle = 1 + |\delta_0| \le \kappa_1 H_1^{-1/2}$ then*

$$f = \alpha_0\delta_0 \tag{3.19}$$

where $\kappa_1 M H^{1/2} \le \alpha_0 \in S(MH_1^{1/2}, G_1)$, which implies that $\delta_0 = f/\alpha_0 \in S(H_1^{-1/2}, G_1)$.

Proof. We choose $g^\sharp$ orthonormal coordinates so that $w_0 = 0$, put $H_1^{1/2} = H_1^{1/2}(0)$ and $M = M(0)$. Let $\kappa_0 > 0$ be given by Proposition 3.8; then if $\kappa_1 \le \kappa_0$ we find $|f'(0)| \cong MH_1^{1/2}$. Next, we change coordinates, letting $w = H_1^{-1/2}z$ and

$$F(z) = H_1^{1/2} f(H_1^{-1/2}z)/|f'(0)| \cong f(H_1^{-1/2}z)/M \in C^\infty.$$

Now $\delta_1(z) = H_1^{1/2}\delta_0(H_1^{-1/2}z)$ is the signed distance to $F^{-1}(0)$ in the z coordinates. We have $|F(0)| \le C_0$, $|F'(0)| = 1$, $|F''(0)| \le C_2$ and $|F^{(3)}(z)| \le C_3$, for all z. It is no restriction to assume that $\partial_{z'}F(0) = 0$, and then $|\partial_{z_1}F(z)| \ge c > 0$ in a fixed neighborhood of the origin. If $|\delta_1(0)| = |\delta_0(0)H_1^{1/2}| \le \kappa_1 \ll 1$ then $F^{-1}(0)$ is a C^∞ manifold in this neighborhood, $\delta_1(z)$ is uniformly C^∞ and $\partial_{z_1}\delta_1(z) \ge c_0 > 0$ in a fixed neighborhood of the origin. By choosing $(F(z), z')$ as local coordinates and using Taylor's formula we find that $\delta_1(z) = \alpha_1(z)F(z)$, where $0 < c_1 \le \alpha_1 \in C^\infty$ in a fixed neighborhood of the origin. Thus, we obtain the proposition with $\alpha_0(w) = |f'(0)|/\alpha_1(H_1^{1/2}w) \in S(MH_1^{1/2}, G_1)$. □

The denominator $D = |f''| + h^{1/4}|f'|^{1/2} + h^{1/2}$ in the definition of $H_1^{-1/2}$ may seem strange, but it has the following explanation which we owe to Nicolas Lerner [18].

Remark 3.10. If $f \in S(h^{-1}, hg^\sharp)$ we find that $F = h^{-1/2}f' \in S(h^{-1}, hg^\sharp)$. The Beals-Fefferman metric for F is $G_2 = H_2 g^\sharp$ where $H_2^{-1} = 1 + |F'|^2 + |F| = 1 + h^{-1}|f''|^2 + h^{-1/2}|f'|$. Thus, we obtain that $D = |f''| + h^{1/4}|f'|^{1/2} + h^{1/2} \cong H_2^{-1/2}h^{1/2}$ and

$$H_1^{-1/2} \cong 1 + |\delta_0| + |F|H_2^{1/2} \le CH_2^{-1/2} \qquad \text{when } |\delta_0| \le CH_2^{-1/2} \tag{3.20}$$

which gives that $H_2^{-1/2} \cong H_1^{-1/2} + |F'|$ when $|\delta_0| \le CH_2^{-1/2}$ (or else $H_1^{-1/2} \cong |\delta_0| \ge CH_2^{-1/2}$). We find that $|f''| \le C(h^{1/4}|f'|^{1/2} + h^{1/2})$ if and only if $H_2^{-1/2} \cong 1 + |F|^{1/2}$. Thus G_1 is equivalent to the Beals-Fefferman metric G_2 for $F = h^{-1/2}f'$ in a G_2 neighborhood of $f^{-1}(0)$ if and only if

$$|f''| \le C(h^{1/4}|f'|^{1/2} + h^{1/2}).$$

In fact, the condition $|f''| \le C(h^{1/4}|f'|^{1/2} + h^{1/2})$ means that $H_2^{-1/2} \cong 1 + h^{-1/4}|f'|^{1/2} = 1 + |F|^{1/2}$. Now the Cauchy-Schwarz inequality gives that

$$1 + |F|^{1/2} \le 1 + \varepsilon H_2^{-1/2} + C_\varepsilon |F| H_2^{1/2}.$$

Thus, $H_1^{-1/2} \cong H_2^{-1/2}$ when $|\delta_0| \le CH_2^{-1/2}$. Observe that we can define the metric G_2 with h replaced by any constant H_0 such that $ch \le H_0 \le CH_1$, since $H_0^{-1/2}f' \in S(H_0^{-1}, H_0 g^\sharp)$ by (3.8) (see Remark 5.6).

4. Properties of the symbol

In this section we shall study the properties of the symbol near the sign changes. We start with a one dimensional result.

LEMMA 4.1. *Assume that* $f(t) \in C^3(\mathbf{R})$ *such that* $\|f^{(3)}\|_\infty = \sup_t |f^{(3)}(t)|$ *is bounded. If*

$$\operatorname{sgn}(t)f(t) \ge 0 \qquad \text{when } \varrho_0 \le |t| \le \varrho_1 \tag{4.1}$$

for $\varrho_1 \ge 3\varrho_0 > 0$, *then*

$$|f(0)| \le \frac{3}{2}\left(\varrho_0 f'(0) + \varrho_0^3 \|f^{(3)}\|_\infty/2\right), \tag{4.2}$$

$$|f''(0)| \le f'(0)/\varrho_0 + 7\varrho_0\|f^{(3)}\|_\infty/6. \tag{4.3}$$

Proof. By Taylor's formula,

$$0 \le \operatorname{sgn}(t)f(t) = |t|f'(0) + \operatorname{sgn}(t)(f(0) + f''(0)t^2/2) + R(t), \qquad \varrho_0 \le |t| \le \varrho_1,$$

where $|R(t)| \le \|f^{(3)}\|_\infty |t|^3/6$. This gives

$$\big|f(0) + t^2 f''(0)/2\big| \le f'(0)|t| + \|f^{(3)}\|_\infty |t|^3/6 \tag{4.4}$$

for any $|t| \in [\varrho_0, \varrho_1]$. By choosing $|t| = \varrho_0$ and $|t| = 3\varrho_0$, we obtain that

$$4\varrho_0^2 |f''(0)| \le 4f'(0)\varrho_0 + 28\|f^{(3)}\|_\infty \varrho_0^3/6$$

which gives (4.3). By letting $|t| = \varrho_0$ in (4.4) and substituting (4.3), we obtain (4.2). □

Proposition 4.2. *Let $f(w) \in C^\infty(T^*\mathbf{R}^n)$ such that $\|f^{(3)}\|_\infty < \infty$. Assume that there exists $0 < \varepsilon \le r/5$ such that*

$$\operatorname{sgn}(w_1) f(w) \ge 0 \quad \text{when } |w_1| \ge \varepsilon + |w'|^2/r \text{ and } |w| \le r \tag{4.5}$$

where $w = (w_1, w')$. Then

$$|f''(0)| \le 33(|\partial_{w_1} f(0)|/\varrho + \varrho\|f^{(3)}\|_\infty) \tag{4.6}$$

for any $\varepsilon \le \varrho \le r/\sqrt{10}$.

Proof. We shall consider the function $t \mapsto f(t, w')$ which satisfies (4.1) for fixed w' with

$$\varepsilon + |w'|^2/r = \varrho_0(w') \le |t| \le \varrho_1 \equiv 3r/\sqrt{10}$$

and $|w'| \le r/\sqrt{10}$ which we assume in what follows. In fact, then $t^2 + |w'|^2 \le r^2$ and $3\varrho_0(w') \le 9r/10 \le 3r/\sqrt{10} = \varrho_1$. We obtain from (4.2) and (4.3) that

$$|f(0, w')| \le \frac{3}{2}\partial_{w_1} f(0, w')\varrho + 3\varrho^3 \|f^{(3)}\|_\infty/4, \tag{4.7}$$

$$|\partial_{w_1}^2 f(0, w')| \le \partial_{w_1} f(0, w')/\varrho + 7\varrho\|f^{(3)}\|_\infty/6 \tag{4.8}$$

for $\varepsilon + |w'|^2/r \le \varrho \le r/\sqrt{10}$ and $|w'| \le r/\sqrt{10}$. By letting $w' = 0$ in (4.8) we find that

$$|\partial_{w_1}^2 f(0)| \le \partial_{w_1} f(0)/\varrho + 7\varrho\|f^{(3)}\|_\infty/6 \tag{4.9}$$

for $\varepsilon \le \varrho \le r/\sqrt{10}$. By letting $\varrho = \varrho_0(w')$ in (4.7) and dividing by $3\varrho_0(w')/2$, we obtain

$$0 \le \partial_{w_1} f(0, w') + 2\|f^{(3)}\|_\infty |w'|^2 \tag{4.10}$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$ since then $\varrho_0(w') \le \varepsilon + |w'| \le 2|w'|$. By using Taylor's formula for $w' \mapsto \partial_{w_1} f(0, w')$ in (4.10), we find that

$$0 \le \partial_{w_1} f(0) + \langle w', \partial_{w'}(\partial_{w_1} f)(0)\rangle + \frac{5}{2}\|f^{(3)}\|_\infty |w'|^2$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$. Thus, by optimizing over fixed $|w'|$, we obtain

(4.11)
$$|w'||\partial_{w'}(\partial_{w_1} f)(0)| \le \partial_{w_1} f(0) + \frac{5}{2}\|f^{(3)}\|_\infty |w'|^2 \qquad \text{when } \varepsilon \le |w'| \le r/\sqrt{10}.$$

By again putting $\varrho = \varrho_0(w')$ in (4.7), using Taylor's formula for $w' \mapsto \partial_{w_1} f(0, w')$ but this time substituting (4.11), we obtain

(4.12)
$$|f(0,w')| \le 6\partial_{w_1} f(0)|w'| + 15\|f^{(3)}\|_\infty |w'|^3 \qquad \text{when } \varepsilon \le |w'| \le r/\sqrt{10}.$$

We may also estimate the even terms in Taylor's formula by (4.12):

$$\begin{aligned}|f(0) + \langle \partial^2_{w'} f(0)w', w'\rangle/2| &\le \frac{1}{2}|f(0,w') + f(0,-w')| + \|f^{(3)}\|_\infty |w'|^3/6 \\ &\le 6\partial_{w_1} f(0)|w'| + \frac{91}{6}\|f^{(3)}\|_\infty |w'|^3\end{aligned}$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$. Thus, by using (4.7) with $\varrho = \varepsilon$ and $w' = 0$ to estimate $|f(0)|$ and optimizing over fixed $|w'|$, we obtain that

$$|\partial^2_{w'} f(0)||w'|^2/2 \le \frac{15}{2}|\partial_{w_1} f(0)||w'| + 16\|f^{(3)}\|_\infty |w'|^3 \tag{4.13}$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$. Thus we obtain (4.6) by taking $\varepsilon \le |w'| = \varrho \le r/\sqrt{10}$ in (4.9)–(4.13). ☐

As before, if $f \in C^\infty(\mathbf{R}^n)$ then we define the *signed distance function* of f as $\delta = \operatorname{sgn}(f)d$ where d is the Euclidean distance to $f^{-1}(0)$.

Proposition 4.3. *Let $f_j(w) \in C^\infty(\mathbf{R}^n)$, $j = 1, 2$, such that $f_1(w) > 0 \implies f_2(w) \ge 0$. Let $\delta_j(w)$ be the signed distance functions of $f_j(w)$, for $j = 1, 2$. There exists $c_0 > 0$, such that if $|f_j'(w_0)| \ge 1$, $|\delta_j(w_0)| \le c_0$ for $j = 1, 2$, and*

$$|\delta_1(w_0) - \delta_2(w_0)| = \varepsilon, \tag{4.14}$$

then there exist $g^\sharp$ orthonormal coordinates $w = (w_1, w')$ so that $w_0 = (x_1, 0)$ with $x_1 = \delta_1(w_0)$ and

$$\operatorname{sgn}(w_1)f_j(w) \ge 0 \quad \text{when } |w_1| \ge (\varepsilon + |w'|^2)/c_0 \text{ and } |w| \le c_0, \tag{4.15}$$

$$|\delta_2(w) - \delta_1(w)| \le (\varepsilon + |w - w_0|^2)/c_0 \qquad \text{when } |w| \le c_0. \tag{4.16}$$

The constant c_0 only depends on the seminorms of f_1 and f_2 in a fixed neighborhood of w_0.

Proof. Observe that the conditions get stronger and the conclusions weaker when c_0 decreases. Assume that f_1 and f_2 are uniformly bounded in C^∞ near w_0. We find that $|f_j'(w)| > 0$ for $|w - w_0| \le c_1 \ll 1$; thus $f_j^{-1}(0)$ is a C^∞ hypersurface in $|w - w_0| \le c_1$ when $|\delta_j(w_0)| \le c_0 \ll 1$, $j = 1, 2$. By decreasing c_0 we obtain (as in the proof of Proposition 3.9) that there exists $c_2 > 0$ so that $w \mapsto \delta_j(w) \in C^\infty(\mathbf{R}^n)$ uniformly in $|w - w_0| \le c_2$,

$j = 1, 2$. We may also choose $z_0 \in f_1^{-1}(0)$ so that $|\delta_1(w_0)| = |w_0 - z_0|$, and then choose $g^\sharp$ orthonormal coordinates so that $z_0 = 0$, $w_0 = (\delta_1(w_0), 0)$ and $\partial_{w'}\delta_1(0) = \partial_{w'}\delta_1(w_0) = 0$, $w = (w_1, w')$. If $c_0 \le c_2/3$ we find that $\delta_j \in C^\infty$ in $|w| \le c_3 = 2c_2/3$. Since $\operatorname{sgn}(f_1(w_0)) = \operatorname{sgn}(\delta_1(w_0))$ we find that $\partial_{w_1} f_1(0) > 0$.

Now, $|\partial_w^2 \delta_j(w)| \le C_0$ for $|w| \le c_3$, $j = 1, 2$, and $\Delta(w) = \delta_2(w) - \delta_1(w) \ge 0$ by the sign condition. By [9, Lemma 7.7.2] and (4.14), we obtain that $|\partial_w \Delta(w)|^2 \le C_1 \Delta(w) \le C_1 \varepsilon$ when $w = w_0$. This gives

$$
\begin{aligned}
|\Delta(w)| &\le |\Delta(w_0)| + |\partial_w \Delta(w_0)||w - w_0| + C_2 |w - w_0|^2 \\
&\le C_3(\varepsilon + |w - w_0|^2) \qquad \text{for } |w| \le c_3,
\end{aligned}
\tag{4.17}
$$

which proves (4.16). Since $|\partial_{w'}\delta_1(w_0)| = 0$ we find that

$$
|\partial_{w'}\delta_2(w)| \le C_4(\sqrt{\varepsilon} + |w - w_0|) \ll 1
$$

when $|w - w_0| \ll 1$ and $\varepsilon \le 2c_0 \ll 1$. Now $f_2(\overline{w}) = 0$ for some $|\overline{w}| \le 2c_0$. Thus for $c_0 \ll 1$ we obtain $|\partial_{w'}\delta_2(\overline{w})| \ll 1$, which gives that $|\partial_{w_1} f_2(\overline{w})| \ge c_4 |\partial_w f_2(\overline{w})| \ge c_4^2 > 0$ for some $c_4 > 0$. Since $\operatorname{sgn}(f_2(w_1, 0)) = 1$ when $w_1 > 0$, we obtain that $\partial_{w_1} f_2(w) \ge c_5 |\partial_w f_2(w)| \ge c_5^2$ when $|w| \le c_5$ for some $c_5 > 0$.

By using the implicit function theorem, we obtain $b_j(w') \in C^\infty(\mathbf{R}^{n-1})$, so that $\pm f_j(w) > 0$ if and only if $\pm(w_1 - b_j(w')) \ge 0$ when $|w| \le c_6$, $j = 1, 2$. Since $f_1(0) = |\partial_{w'} f_1(0)| = 0$ we obtain that $b_1(0) = |b_1'(0)| = 0$. This gives $|b_1(w')| \le C_5 |w'|^2$ and proves the positive part of (4.15) by the sign condition. Observe that the sign condition is equivalent to $f_2(w) < 0 \implies f_1(w) \le 0$, which gives $b_1(w') \ge b_2(w')$. Now $|\delta_2(w_0)| \le |\delta_1(w_0)| + \varepsilon$, thus we find $-\varepsilon \le b_2(\overline{w}') \le b_1(\overline{w}')$ for some $|\overline{w}'| \le C\sqrt{\varepsilon}$. This gives $b_2(\overline{w}') \le C_5 C^2 \varepsilon$ and $|b_1'(\overline{w}')| \le C_6\sqrt{\varepsilon}$, and we obtain as before that $|b_1'(\overline{w}') - b_2'(\overline{w}')| \le C_7\sqrt{\varepsilon}$. As in (4.17), we obtain

$$
|b_2(w')| \le C_8(\varepsilon + |w' - \overline{w}'|^2) \le C_9(\varepsilon + |w'|^2)
$$

which proves the negative part of (4.15) and the proposition. □

5. The weight function

In this section, we shall define the weight m_ϱ to be used; for technical reasons it will depend on a parameter $0 < \varrho \le 1$. Let $\delta_0(t, w)$ and $H_1^{-1/2}(t, w)$ be given by Definitions 3.1 and 3.4 for $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfying condition $(\overline{\Psi})$ given by (2.2). The weight m_ϱ will essentially measure how much $t \mapsto \delta_0(t, w)$ changes between the minima of $t \mapsto H_1^{1/2}(t, w)\langle \delta_0(t, w) \rangle$, which will give restrictions on the sign changes of the symbol. As before, we assume that we have chosen $g^\sharp$ orthonormal coordinates so that $g^\sharp(w) = |w|^2$, and the results will only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

Definition 5.1. For $0 < \varrho \le 1$ and $(t,w) \in \mathbf{R} \times T^*\mathbf{R}^n$ we let $m_\varrho = \min(\mathcal{M}_\varrho, \varrho^2)$ with

$$\mathcal{M}_\varrho(t,w) = \inf_{t_1 \le t \le t_2} \Big\{ \varrho^2 |\delta_0(t_1,w) - \delta_0(t_2,w)| + \max\big(H_1^{1/2}(t_1,w)\langle \varrho\delta_0(t_1,w)\rangle, H_1^{1/2}(t_2,w)\langle \varrho\delta_0(t_2,w)\rangle \big) \Big\} \tag{5.1}$$

where $\langle \delta_0 \rangle = 1 + |\delta_0|$.

Remark 5.2. When $t \mapsto \delta_0(t,w)$ is constant for fixed w, we find that $t \mapsto m_1(t,w)$ is equal to the largest quasi-convex minorant of $t \mapsto H_1^{1/2}(t,w)\langle \delta_0(t,w)\rangle$; i.e., $\sup_I m_1 = \sup_{\partial I} m_1$ for compact intervals $I \subset \mathbf{R}$; see [10, Def. 1.6.3].

We shall use the parameter ϱ to obtain suitable norms in Section 6, but this is just a technicality: all m_ϱ are equivalent according to the following proposition.

Proposition 5.3. *We have* $m_\varrho \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$,

$$\min(ch^{1/2}, \varrho^2) \le m_\varrho \le \min(H_1^{1/2}\langle \varrho\delta_0\rangle, \varrho^2) \le \varrho^2 \tag{5.2}$$

where $c^{-1} = C$ *is given by* (3.7), *and*

$$\varrho_1^2/\varrho_2^2 \le m_{\varrho_1}/m_{\varrho_2} \le 1 \tag{5.3}$$

when $0 < \varrho_1 \le \varrho_2 \le 1$. *If* $m_\varrho(t_0,w_0) < \varrho^2$, *then there exist* $t_1 \le t_0 \le t_2$ *so that* $H_0^{1/2} = \max(H_1^{1/2}(t_1,w_0),\, H_1^{1/2}(t_2,w_0)) < 2m_\varrho(t_0,w_0)$ *satisfies*

$$H_0^{1/2} < 4m_\varrho(t_0,w_0)/\langle \varrho\delta_0(t_j,w_0)\rangle \qquad \text{for } j = 0,\, 1,\, 2, \tag{5.4}$$

this implies that $H_0^{1/2} < 4H_1^{1/2}(t_0,w_0)$ *by* (5.2). *When* $m_\varrho(t_0,w_0) < \varrho^2 \ll 1$, $g^\sharp$ *orthonormal coordinates may be chosen so that* $w_0 = (x_1, 0)$, $|x_1| < |\delta_0(t_0,w_0)| + 1 < 4\varrho H_0^{-1/2}$, *and*

$$\operatorname{sgn}(w_1) f(t_0,w) \ge 0 \quad \text{when } |w_1| \ge (1 + H_0^{1/2}|w'|^2)/c_0, \tag{5.5}$$

$$|\delta_0(t_1,w) - \delta_0(t_2,w)| \le (\varrho^{-2} m_\varrho(t_0,w_0) + H_0^{1/2}|w - w_0|^2)/c_0 \tag{5.6}$$

when $|w| \le c_0 H_0^{-1/2}$ *for some constant* c_0, *which only depends on the seminorms of* f.

Observe that condition (5.5) is not empty when ϱ is sufficiently small since $H_0^{1/2} < 4\varrho^2$.

Proof. We obtain the first statement and (5.2) by taking the infimum, since $ch^{1/2} \le \mathcal{M}_\varrho \le H_1^{1/2}\langle \varrho\delta_0\rangle$ by (3.7). Next, we put

$$F_\varrho(s,t,w) = \varrho^2|\delta_0(s,w) - \delta_0(t,w)| + \max(H_1^{1/2}(s,w)\langle \varrho\delta_0(s,w)\rangle, H_1^{1/2}(t,w)\langle \varrho\delta_0(t,w)\rangle).$$

Then we have $F_{\varrho_1} \le F_{\varrho_2}$ and $\varrho_1^2 F_{\varrho_2} \le \varrho_2^2 F_{\varrho_1}$ when $\varrho_1 \le \varrho_2$. Since these estimates are preserved when taking the infimum, we obtain (5.3).

Next assume that $m_\varrho(t_0, w_0) < \varrho^2$; then $m_\varrho(t_0, w_0) = \mathcal{M}_\varrho(t_0, w_0)$. By approximating the infimum, we may choose $t_1 \le t_0 \le t_2$ so that $F_\varrho(t_1, t_2, w_0) < m_\varrho(t_0, w_0) + ch^{1/2}$, which gives

$$|\delta_0(t_1, w_0) - \delta_0(t_2, w_0)| < \varrho^{-2} m_\varrho(t_0, w_0) < 1 \qquad \text{and} \tag{5.7}$$

$$H_1^{1/2}(t_j, w_0)\langle \varrho\delta_0(t_j, w_0)\rangle < 2m_\varrho(t_0, w_0) < 2\varrho^2 \qquad \text{for } j = 1 \text{ and } 2. \tag{5.8}$$

We obtain that $H_0^{1/2} = \max(H_1^{1/2}(t_1, w_0),\, H_1^{1/2}(t_2, w_0)) < 2m_\varrho(t_0, w_0) < 2\varrho^2$ and

$$1/2 < \langle \varrho\delta_0(t_j, w_0)\rangle / \langle \varrho\delta_0(t_k, w_0)\rangle < 2 \qquad \text{when } j,\, k = 0,\, 1,\, 2 \tag{5.9}$$

by the monotonicity of $t \mapsto \delta_0(t, w_0)$; thus (5.8) gives (5.4). We obtain from (5.4) that

$$1 + |\delta_0(t_j, w_0)| < 4\varrho H_0^{-1/2} \qquad \text{when } j = 0,\ 1,\ 2. \tag{5.10}$$

Next, choose $g^\sharp$ orthonormal coordinates so that $w_0 = 0$. Since $H_1^{1/2}(t_j, 0) < 2\varrho^2$ and $|\delta_0(t_j, 0)| < 2\varrho H_1^{-1/2}(t_j, 0)$ by (5.8), we find from Proposition 3.7 for $\varrho \ll 1$ that

$$h^{1/2} \le |\partial_w f(t_j, 0)| \cong |\partial_w f(t_j, w)|$$

when

$$|w| \le cH_0^{-1/2} \le cH_1^{-1/2}(t_j, 0), \qquad j = 1, 2.$$

Now $f(t_j, \widetilde{w}_j) = 0$ for some $|\widetilde{w}_j| < 4\varrho H_0^{-1/2}$ by (5.10) when $\varrho \ll 1$ and $j = 1,\ 2$. Thus, when $4\varrho \le c$ we obtain that

$$|f(t_j, w)| \le C|\partial_w f(t_j, 0)| H_0^{-1/2} \qquad \text{when } |w| < cH_0^{-1/2}$$

and then (3.12) gives $f(t_j, w) \in S(|\partial_w f(t_j, 0)| H_0^{-1/2}, H_0 g^\sharp)$ since $H_1^{1/2}(t_j, 0) \le H_0^{1/2}$, $j = 1,\ 2$. Choosing coordinates $z = H_0^{1/2} w$, we shall use Proposition 4.3 with

$$f_j(z) = H_0^{1/2} f(t_j, H_0^{-1/2} z)/|\partial_w f(t_j, 0)| \in C^\infty \qquad \text{for } j = 1,\ 2.$$

Let $\delta_j(z) = H_0^{1/2}\delta_0(t_j, H_0^{-1/2} z)$ be signed distance functions to $f_j^{-1}(0)$; then $|f_j'(0)| = 1$, $|\delta_j(0)| \le 4\varrho$ and

$$|\delta_1(0) - \delta_2(0)| = \varepsilon \le H_0^{1/2} m_\varrho(t_0, 0)/\varrho^2$$

by (5.7). Thus, for sufficiently small ϱ we may use Proposition 4.3 to obtain $g^\sharp$ orthogonal coordinates so that $w_0 = (x_1, 0)$ where

$$|x_1| = |\delta_0(t_1, 0)| < |\delta_0(t_0, 0)| + 1 < 4\varrho H_0^{-1/2}$$

by (5.10). We then obtain (5.5) and (5.6) from (4.15) and (4.16) for some $c_0 > 0$. □

PROPOSITION 5.4. *There exists* $C > 0$ *such that*

$$m_\varrho(t_0, w) \le C m_\varrho(t_0, w_0)(1 + \varrho^2 g^\sharp(w - w_0)) \tag{5.11}$$

uniformly when $0 < \varrho \le 1$, *thus* m_ϱ *is a weight for* $g_\varrho = \varrho^2 g^\sharp$ *uniformly in* ϱ.

Proof. The weights m_ϱ are equivalent when $\varrho \ge \varrho_0 > 0$ by (5.3), so it suffices to consider the case when $\varrho \le \varrho_0 \ll 1$. In fact, if (5.11) holds for m_{ϱ_0} then it holds for m_ϱ when $\varrho_0 \le \varrho \le 1$, with C replaced by C/ϱ_0^2. Since $m_\varrho \le \varrho^2$ we only have to consider the case when

$$m_\varrho(t_0, w_0) < \varrho^2. \tag{5.12}$$

Now, for fixed w_0 and ϱ it suffices to prove (5.11) when $|w - w_0| \le \varrho/m_\varrho$, where $m_\varrho = m_\varrho(t_0, w_0) < \varrho^2$. In fact, when $|w - w_0| > \varrho/m_\varrho$ we obtain that

$$\varrho^2|w - w_0|^2 > \varrho^4/m_\varrho^2 > m_\varrho(t_0, w)/m_\varrho$$

by (5.12). In that case (5.11) is satisfied with $C = 1$; thus in the following we shall only consider w such that $|w - w_0| \le \varrho/m_\varrho$ for $m_\varrho < \varrho^2 \ll 1$. Then we may use Proposition 5.3 to obtain $t_1 \le t_0 \le t_2$ such that (5.6) and (5.4) hold with $H_0^{1/2} = \max(H_1^{1/2}(t_1, w_0),\ H_1^{1/2}(t_2, w_0)) < 2m_\varrho$. Thus

$$\begin{aligned} \varrho^2|\delta_0(t_1, w) - \delta_0(t_2, w)| &\le C_2(m_\varrho + H_0^{1/2}\varrho^2|w - w_0|^2) \\ &\le 2C_2 m_\varrho(1 + \varrho^2|w - w_0|^2) \end{aligned} \tag{5.13}$$

when $|w - w_0| \le \varrho/m_\varrho < 2\varrho H_0^{-1/2} \le c_0 H_0^{-1/2}$ for $\varrho \le c_0/2$. Now G_1 is slowly varying, uniformly in t. Thus we find for small enough $\varrho > 0$ that

$$H_1^{1/2}(t_j, w) \le C_3 H_1^{1/2}(t_j, w_0) \qquad \text{when } |w - w_0| \le 2\varrho H_0^{-1/2} \le 2\varrho H_1^{-1/2}(t_j, w_0)$$

for $j = 1,\ 2$. By the uniform Lipschitz continuity we find that

$$\langle \varrho\delta_0(t, w)\rangle \le \langle \varrho\delta_0(t, w_0)\rangle(1 + \varrho|w - w_0|) \tag{5.14}$$

which implies that

$$\begin{aligned} &H_1^{1/2}(t_j, w)\langle \varrho\delta_0(t_j, w)\rangle \\ &\qquad \le C_3 H_1^{1/2}(t_j, w_0)\langle \varrho\delta_0(t_j, w_0)\rangle(1 + \varrho|w - w_0|), \qquad j = 1,\ 2, \end{aligned} \tag{5.15}$$

when $|w - w_0| \le 2\varrho H_0^{-1/2}$. Now $H_1^{1/2}(t_j, w_0)\langle \varrho\delta_0(t_j, w_0)\rangle < 4m_\varrho$ by (5.4) for $j = 1,\ 2$. Thus, by using (5.13), (5.15) and taking the infimum we obtain

$$m_\varrho(t_0, w) \le C_4 m_\varrho(1 + \varrho^2|w - w_0|^2)$$

when $|w - w_0| \le \varrho/m_\varrho \le 2\varrho H_0^{-1/2}$ for $\varrho \le \varrho_0 \ll 1$. □

In Section 6, we shall choose a fixed $\varrho \ll 1$ in order to get suitable function spaces. In the following, we shall for simplicity only consider m_1, since all the m_ϱ are equivalent, this is really no restriction: the results also holds for any m_ϱ, but with constants depending on ϱ. The following result will be important for the proof of Proposition 2.5 in Section 7.

Proposition 5.5. *Let M be given by Definition 3.6 and m_1 by Definition 5.1. Then there exists $C_0 > 0$ such that*

$$MH_1^{3/2} \le C_0 m_1/\langle\delta_0\rangle. \tag{5.16}$$

Proof. We shall omit the dependence on t in the proof and put $m_1 = m_1(w_0)$. First we observe that if $m_1 \ge c > 0$, then $MH_1^{3/2}\langle\delta_0\rangle \le C \le Cm_1/c$ at w_0 since $\langle\delta_0\rangle \le H_1^{-1/2}$ and $M \le CH_1^{-1}$ by Proposition 3.8.

Thus, we only have to consider the case $m_1 < \varrho^2$ at w_0 for some $\varrho > 0$ to be chosen later. Since $m_\varrho \le m_1$ we may use Proposition 5.3 for $\varrho \ll 1$ to choose $g^\sharp$ orthonormal coordinates so that $|w_0| \le |\delta_0(w_0)| + 1 \le 4\varrho H_0^{-1/2}$ and f satisfies (5.5) with

$$ch^{1/2} \le H_0^{1/2} < 4m_\varrho(w_0)/\langle\varrho\delta_0(w_0)\rangle \le 4\varrho^2 \tag{5.17}$$

by (5.4). Observe that $H_0^{1/2} \le 4H_1^{1/2}(w_0)$ by (5.2). Thus it suffices to prove the estimate

$$MH_1^{3/2} \le CH_0^{1/2} \qquad \text{at } w_0$$

for this choice of ϱ. Since $ch^{1/2} \le H_0^{1/2}$, we find from Proposition 3.8 that this is equivalent to

$$|f''|H_1^{1/2} \le CH_0^{1/2} \tag{5.18}$$

at w_0. Now it actually suffices to prove (5.18) at $w = 0$. In fact, (3.10) gives $H_1^{1/2}(w_0) \le CH_1^{1/2}(0)$ since $|w_0| \le |\delta_0(w_0)| + 1 \le H_1^{-1/2}(w_0)$. Thus Taylor's formula gives

$$\begin{aligned}|f''(w_0)|H_1^{1/2}(w_0) &\le \left(|f''(0)| + C_3h^{1/2}|w_0|\right) H_1^{1/2}(w_0)\\ &\le C_1(|f''(0)|H_1^{1/2}(0) + h^{1/2})\end{aligned}$$

since $|f^{(3)}| \le C_3h^{1/2}$. By Definition 3.4 we find that

$$\begin{aligned}H_1^{-1/2} &\ge 1 + |f'|/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2})\\ &\ge (|f'| + |f''| + h^{1/2})/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}).\end{aligned}$$

Thus (5.18) follows if we prove

$$|f''|(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}) \le C\left(|f'| + |f''| + h^{1/2}\right) H_0^{1/2} \quad \text{at } 0. \tag{5.19}$$

Since $ch^{1/2} \le H_0^{1/2}$ we obtain (5.19) by the Cauchy-Schwarz inequality, if we prove that

$$|f''(0)| \le C(H_0^{1/4}|f'(0)|^{1/2} + h^{1/2}). \tag{5.20}$$

Because of (5.5) we can use Proposition 4.2 on $F(z) = H_0 f(H_0^{-1/2} z)$ with $r = c_0$ and $\varepsilon = H_0^{1/2}/c_0 \le 4\varrho^2/c_0 \le c_0/5$ by (5.17) when $\varrho \le c_0/\sqrt{20}$. Observe that $|F'(0)| \le C_0$ since $H_0^{1/2} \le 4H_1^{1/2}(w_0) \le 4CH_1^{1/2}(0)$. We obtain from Proposition 4.2 that

$$|F''(0)| \le C_1 \left(|F'(0)|/\varrho_0 + H_0^{-1/2} h^{1/2} \varrho_0 \right), \qquad H_0^{1/2}/c_0 \le \varrho_0 \le c_0/\sqrt{10}$$

since $\|F^{(3)}\|_\infty \le C_3 H_0^{-1/2} h^{1/2}$. By choosing $\varrho_0 = \lambda |F'(0)|^{1/2} + H_0^{1/2}/c_0 \le c_0/\sqrt{10}$ for $\lambda = c_0(\sqrt{10}-2)/10\sqrt{C_0}$, we obtain that $|F''(0)| \le C'(|F'(0)|^{1/2} + h^{1/2})$ since $H_0^{-1/2} \le Ch^{-1/2}$. Now $F' = H_0^{1/2} f'$ and $F'' = f''$; thus we obtain (5.20) for this choice of ϱ, which completes the proof of the proposition. $\square$

If $m_1 \cong 1$ then we find that the estimate (5.16) is trivial, and when $m_1 \ll 1$ we have the following interpretation of (5.20).

Remark 5.6. If $|f'| \le CH_1^{-1/2} \le C_0 H_0^{-1/2} \le C_1 h^{-1/2}$ we find that $F = H_0^{-1/2} f' \in S(H_0^{-1}, H_0 g^\sharp)$. If we take the corresponding Beals-Fefferman metric $G_3 = H_3 g^\sharp$ for F, $H_3^{-1} \cong 1 + H_0^{-1}|f''|^2 + H_0^{-1/2}|f'|$ (see Remark 3.10), then (5.20) means that $H_3^{-1} \cong 1 + H_0^{-1/2}|f'|$ in a G_3 neighborhood of 0. By replacing $h^{1/2}$ by $H_0^{1/2}$ in the definition of $H_1^{-1/2}$, we find that (5.20) means that G_1 is equivalent to G_3, in a G_3 neighborhood of $f^{-1}(0)$ by Remark 3.10.

Next, we shall prove a convexity property of $t \mapsto m_1(t, w)$, which will be essential for the proof.

PROPOSITION 5.7. *Let m_1 be given by Definition* 5.1. *Then*

(5.21)
$$\sup_{t_1 \le t \le t_2} m_1(t, w) \le \delta_0(t_2, w) - \delta_0(t_1, w) + m_1(t_1, w) + m_1(t_2, w) \qquad \forall\, w.$$

Proof. Since $t \mapsto \delta_0(t, w)$ is monotone, we find that

$$\inf_{\pm(t-t_0)\ge 0} \left(|\delta_0(t, w) - \delta_0(t_0, w)| + H_1^{1/2}(t, w)\langle \delta_0(t, w) \rangle \right) \le m_1(t_0, w). \tag{5.22}$$

Let $t \in [t_1, t_2]$; then by using (5.22) for $t_0 = t_1$, t_2, and taking the infima, we obtain that

$$\begin{aligned} m_1(t,w) \le \inf_{r\le t_1<t_2\le s} \Big(&\delta_0(s,w) - \delta_0(r,w) \\ &+ H_1^{1/2}(s,w)\langle\delta_0(s,w)\rangle + H_1^{1/2}(r,w)\langle\delta_0(r,w)\rangle\Big) \\ \le\, &\delta_0(t_2,w) - \delta_0(t_1,w) + m_1(t_1,w) + m_1(t_2,w) \end{aligned}$$

which gives (5.21) after we take the supremum. □

Next, we shall construct the pseudo-sign $B = \delta_0 + \varrho_0$, which we shall use in Section 7 to prove Proposition 2.5 with the multiplier $b^w = B^{\mathrm{Wick}}$.

Proposition 5.8. *Assume that* δ_0 *is given by Definition* 3.1 *and* m_1 *is given by Definition* 5.1. *Then for* $T > 0$ *there exists real-valued* $\varrho_T(t,w) \in L^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$ *with the property that*

$$|\varrho_T| \le m_1 \tag{5.23}$$

$$\partial_t(\delta_0 + \varrho_T) \ge m_1/2T \qquad \text{in } \mathcal{D}'(\mathbf{R}) \tag{5.24}$$

when $|t| < T$.

Proof. (We owe this argument to Lars Hörmander [12].) Let

(5.25)
$$\varrho_T(t,w) = \sup_{-T\le s\le t}\Big(\delta_0(s,w) - \delta_0(t,w) + \frac{1}{2T}\int_s^t m_1(r,w)\,dr - m_1(s,w)\Big)$$

for $|t| \le T$, then

$$\begin{aligned} \delta_0(t,w) + \varrho_T(t,w) = \sup_{-T\le s\le t}&\Big(\delta_0(s,w) - \frac{1}{2T}\int_0^s m_1(r,w)\,dr - m_1(s,w)\Big) \\ &+\frac{1}{2T}\int_0^t m_1(r,w)\,dr \end{aligned}$$

which immediately gives (5.24) since the supremum is nondecreasing. We find from Proposition 5.7 that

$$\begin{aligned} &\delta_0(s,w) - \delta_0(t,w) \\ &\qquad + \frac{1}{2T}\int_s^t m_1(r,w)\,dr - m_1(s,w) \le m_1(t,w) \qquad -T \le s \le t \le T. \end{aligned}$$

By taking the supremum, we obtain that $-m_1(t,w) \le \varrho_T(t,w) \le m_1(t,w)$ when $|t| \le T$, which proves the result. □

6. The Wick quantization

In order to define the multiplier we shall use the Wick quantization, and also define the function spaces to be used, following [2]. As before, we shall assume that $g^\sharp = (g^\sharp)^\sigma$ and that the coordinates are chosen so that $g^\sharp(w) = |w|^2$. For $a \in L^\infty(T^*\mathbf{R}^n)$ we define the Wick quantization:

$$a^{\mathrm{Wick}}(x, D_x)u(x) = \int_{T^*\mathbf{R}^n} a(y,\eta)\Sigma^w_{y,\eta}(x, D_x)u(x)\, dyd\eta, \qquad u \in \mathcal{S}(\mathbf{R}^n)$$

using the projections $\Sigma^w_{y,\eta}(x, D_x)$ with the Weyl symbol

$$\Sigma_{y,\eta}(x,\xi) = \pi^{-n} \exp(-g^\sharp(x-y, \xi-\eta))$$

(see [5, App. B] or [15, §4]). We find that $a^{\mathrm{Wick}}\colon \mathcal{S}(\mathbf{R}^n) \mapsto \mathcal{S}'(\mathbf{R}^n)$ is symmetric on $\mathcal{S}(\mathbf{R}^n)$ if a is real-valued,

$$a \ge 0 \implies \left(a^{\mathrm{Wick}}(x, D_x)u, u\right) \ge 0, \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{6.1}$$

and $\|a^{\mathrm{Wick}}(x, D_x)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le \|a\|_{L^\infty(T^*\mathbf{R}^n)}$, which is the main advantage with the Wick quantization (see [15, Prop. 4.2]). Now if $a_t(x,\xi) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ depends on a parameter t, then we find that

$$\int_{\mathbf{R}} \left(a_t^{\mathrm{Wick}}u, u\right) \phi(t)\, dt = \left(A_\phi^{\mathrm{Wick}}u, u\right), \qquad u \in \mathcal{S}(\mathbf{R}^n), \tag{6.2}$$

where $A_\phi(x,\xi) = \int_{\mathbf{R}} a_t(x,\xi)\phi(t)\, dt$. We obtain from the definition that $a^{\mathrm{Wick}} = a_0^w$ where

$$a_0(w) = \pi^{-n} \int_{T^*\mathbf{R}^n} a(z) \exp(-|w-z|^2)\, dz \tag{6.3}$$

is the Gaussian regularization. Thus Wick operators with real symbols have real Weyl symbols.

In the following, we shall assume that $G = Hg^\sharp \le g^\sharp$ is a slowly varying metric satisfying

$$H(w) \le C_0 H(w_0)(1 + |w - w_0|)^{N_0} \tag{6.4}$$

and m is a weight for G satisfying (6.4) with H replaced by m. This means that G and m are strongly σ temperate in the sense of [2, Def. 7.1]. Recall the symbol class $S^+(1, g^\sharp)$ given by Definition 2.2.

Proposition 6.1. *Assume that $a \in L^\infty(T^*\mathbf{R}^n)$ such that $|a| \le m$; then $a^{\mathrm{Wick}} = a_0^w$ where $a_0 \in S(m, g^\sharp)$ is given by (6.3). If $a \ge m$ then $a_0 \ge c_0 m$ for a fixed constant $c_0 > 0$, and if $a \in S(m, G)$, then $a_0 = a$ modulo symbols in $S(mH, G)$. If $|a| \le m$ and $a = 0$ in a fixed G ball with center w, then $a \in S(mH^N, G)$ at w for any N. If $\partial_w a \in L^\infty(T^*\mathbf{R}^n)$ then $a_0 \in S^+(1, g^\sharp)$.*

By localization we find, for example, that if $|a| \le m$ and $a \in S(m,G)$ in a G neighborhood of w_0, then $a_0 = a$ modulo $S(mH,G)$ in a smaller G neighborhood of w_0. The results are well known, but for completeness we give a proof. Observe that the results are uniform in the metrics and weights.

Proof. Since a is measurable satisfying $|a| \le m$, where

$$m(z) \le C_0 m(w)(1+|z-w|)^{N_0}$$

by (6.4), we find that $a^{\mathrm{Wick}} = a_0^w$ where $a_0 = \mathcal{O}(m)$ is given by (6.3). By differentiating on the exponential factor, we find $a_0 \in S(m, g^\sharp)$, and similarly we find that $a_0 \ge m/C$ if $a \ge m$.

If $a = 0$ in a G ball of radius $\varepsilon > 0$ and center at w, then we can write

$$\pi^n a_0(w) = \int_{|z-w|\ge \varepsilon H^{-1/2}(w)} a(z)\exp(-|w-z|^2)\,dz = \mathcal{O}(m(w)H^N(w))$$

for any N even after repeated differentiation. If $a \in S(m,G)$ then Taylor's formula gives

$$a_0(w) = a(w) + \pi^{-n}\int_0^1\int_{T^*\mathbf{R}^n}(1-\theta)\langle a''(w+\theta z)z, z\rangle e^{-|z|^2}\,dz d\theta$$

where $a'' \in S(mH,G)$ since $G = Hg^\sharp$. Because $m(w+\theta z) \le C_0 m(w)(1+|z|)^{N_0}$ and $H(w+\theta z) \le C_0 H(w)(1+|z|)^{N_0}$ when $|\theta| \le 1$, we find that $a_0(w) = a(w)$ modulo $S(mH,G)$. Since $\partial_w a_0(w) = \pi^{-n}\int_{T^*\mathbf{R}^n}\partial_w a(z)\exp(-|w-z|^2)\,dz$, we obtain the proposition. □

LEMMA 6.2. *If $a(t,w)$ and $\mu(t,w) \in L^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$ and $\partial_t a(t,w) \ge \mu(t,w)$ in $\mathcal{D}'(\mathbf{R})$ for almost all $w \in T^*\mathbf{R}^n$, then $\big(\partial_t(a^{\mathrm{Wick}})u,u\big) \ge \big(\mu^{\mathrm{Wick}}u,u\big)$ in $\mathcal{D}'(\mathbf{R})$ when $u \in \mathcal{S}(\mathbf{R}^n)$.*

Proof. The condition means that $-\int a(t,w)\phi'(t)\,dt \ge \int \mu(t,w)\phi(t)\,dt$ for all $0 \le \phi \in C_0^\infty(\mathbf{R})$ and almost all $w \in T^*\mathbf{R}^n$, which by (6.1) and (6.2) gives

$$-\int \Big(a^{\mathrm{Wick}}(t,x,D_x)u,u\Big)\,\phi'(t)\,dt \ge \int\Big(\mu^{\mathrm{Wick}}(t,x,D_x)u,u\Big)\,\phi(t)\,dt \qquad 0 \le \phi \in C_0^\infty(\mathbf{R})$$

for $u \in \mathcal{S}(\mathbf{R}^n)$. □

We shall compute the Weyl symbol for the Wick operator $(\delta_0 + \varrho_T)^{\mathrm{Wick}}$, where ϱ_T is as given by Proposition 5.8. In the following we shall suppress the t variable.

PROPOSITION 6.3. *Let $B = \delta_0 + \varrho_0$, where δ_0 is given by Definition 3.1 and ϱ_0 is real-valued satisfying $|\varrho_0| \le m_1$, with m_1 given by Definition 5.1. Then*

$$B^{\mathrm{Wick}} = b^w$$

where $b = \delta_1 + \varrho_1$ *is real-valued,* $\delta_1 \in S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$*, and* $\varrho_1 \in S(m_1, g^\sharp)$*. Also, there exists* $\kappa_2 > 0$ *with the following properties: If* $\langle \delta_0 \rangle \le \kappa_2 H_1^{-1/2}$ *then* $\delta_1 = \delta_0 + \varrho_2 \in S(H_1^{-1/2}, G_1)$ *with* $\varrho_2(w) \in S(H_1^{1/2}, G_1)$*. For any* $\lambda > 0$*, there exists* $c_\lambda > 0$ *such that* $|\delta_0| \ge \lambda H_1^{-1/2}$ *and* $H_1^{1/2} \le c_\lambda$ *gives*

$$|b| \ge \kappa_2 \lambda H_1^{-1/2}. \tag{6.5}$$

Proof. Let $\delta_0^{\mathrm{Wick}} = \delta_1^w$ and $\varrho_0^{\mathrm{Wick}} = \varrho_1^w$. Since $|\delta_0| \le H_1^{-1/2}$, $|\varrho_0| \le m_1$ and the symbols are real-valued, we obtain from Proposition 6.1 that $\delta_1 \in S(H_1^{-1/2}, g^\sharp)$ and $\varrho_1 \in S(m_1, g^\sharp)$ are real-valued. Since $|\delta_0'| \le 1$ almost everywhere, we find that $\delta_1 \in S^+(1, g^\sharp)$ by Proposition 6.1.

If $\langle \delta_0 \rangle \le \kappa H_1^{-1/2}$ at w_0 for sufficiently small $\kappa > 0$, then we find by Lipschitz continuity and slow variation that $\langle \delta_0 \rangle \le C_0 \kappa H_1^{-1/2}$ in a fixed G_1 neighborhood ω_κ of w_0 (depending on κ). Then we find that $\delta_0 \in S(H_1^{-1/2}, G_1)$ in ω_κ by Proposition 3.9, which implies that $\delta_1 - \delta_0 \in S(H_1^{1/2}, G_1)$ at w_0 by Proposition 6.1 after localization.

When $|\delta_0| \ge \lambda H_1^{-1/2} \ge \lambda > 0$ at w_0, then by Lipschitz continuity and slow variation we find that $|\delta_0| \ge \lambda H_1^{-1/2}/C_0$ in a G_1 neighborhood ω_λ of w_0 (depending on λ). Since $|\varrho_0| \le H_1^{1/2} \langle \delta_0 \rangle \le C_\lambda H_1^{1/2} |\delta_0|$ in ω_λ by (5.2), we find by the slow variation that

$$|\delta_0 + \varrho_0| \ge |\delta_0|/2 \ge \lambda H_1^{-1/2}/2C_0 \qquad \text{in } \omega_\lambda$$

when $H_1^{1/2}(w_0) \ll 1$. Proposition 6.1 then gives after localization that

$$|b| \ge c_0 \lambda H_1^{-1/2}/2C_0 - C_\lambda H_1^{1/2} \ge c_0 \lambda H_1^{-1/2}/3C_0 \qquad \text{at } w_0$$

when $H_1^{1/2}(w_0) \le c_\lambda \ll 1$, which completes the proof. □

Let m_ϱ be given by Definition 5.1; then m_ϱ is a weight for $g_\varrho = \varrho^2 g^\sharp$ uniformly in $0 < \varrho \le 1$ according to Proposition 5.4. We are going to use the symbol classes $S(m_\varrho^k, g_\varrho)$, $k \in \mathbf{R}$. Observe that $S(m_\varrho^k, g_\varrho) = S(m_1^k, g^\sharp)$ for all $0 < \varrho \le 1$ (but not uniformly), since $g_\varrho \cong g^\sharp$ and $m_\varrho \cong m_1$ by Proposition 5.3.

Definition 6.4. Let $H(m_1^k, g^\sharp)$, be the Hilbert space given by [2, Def. 4.1] so that

$$u \in H(m_1^k, g^\sharp) \iff a^w u \in L^2 \qquad \forall\, a \in S(m_1^k, g^\sharp), \qquad k \in \mathbf{R}. \tag{6.6}$$

We let $\|u\|_k$ be the norm of $H(m_1^k, g^\sharp)$.

This Hilbert space has the following properties: $\mathcal{S}$ is dense in $H(m_1^k, g^\sharp)$, the dual of $H(m_1^k, g^\sharp)$ is naturally identified with $H(m_1^{-k}, g^\sharp)$, and if $u \in$

$H(m_1^k, g^\sharp)$ then $u = a_0^w v$ for some $v \in L^2(\mathbf{R}^n)$ and $a_0 \in S(m_1^{-k}, g^\sharp)$ (see [2, Cor. 6.7]). It follows that $a^w \in \operatorname{Op} S(m_1^k, g^\sharp)$ is bounded:

$$u \in H(m_1^j, g^\sharp) \mapsto a^w u \in H(m_1^{j-k}, g^\sharp) \tag{6.7}$$

with bound only depending on the seminorms of a.

Now m_ϱ is not necessarily a symbol, but by (5.11) we can define the equivalent weight

$$\widetilde{m}_\varrho(t,w) = \sum_j \phi_{j,\varrho}(w) m_\varrho(t, w_j) \in S(m_\varrho, g_\varrho), \tag{6.8}$$

by using a partition of unity $\{\, \phi_{j,\varrho} \,\} = \{\, \phi_j(\varrho w) \,\} \in S(1, g_\varrho)$ uniformly in ϱ. Then $\widetilde{m}_\varrho \cong m_\varrho$, and we let $\mu_\varrho^w = \widetilde{m}_\varrho^{\mathrm{Wick}}$, i.e.,

$$\mu_\varrho(t,w) = \pi^{-n} \int_{T^*\mathbf{R}^n} \widetilde{m}_\varrho(t, w-z) \exp(-|z|^2)\, dz. \tag{6.9}$$

Since m_ϱ satisfies (5.11) and $m_\varrho \cong \widetilde{m}_\varrho \in S(m_\varrho, g_\varrho)$ uniformly in ϱ, we find by using Proposition 6.1 (with $G = g_\varrho$) that $m_\varrho / c \le \mu_\varrho \in L^\infty(\mathbf{R}, S(m_\varrho, g_\varrho))$ uniformly for $0 < \varrho \le 1$ and some $c > 0$.

The following proposition shows that the topology in $H(m_1^{1/2}, g^\sharp)$ can be defined by the operator μ_1^w.

PROPOSITION 6.5. *Assume that $\mu_1 \in L^\infty(\mathbf{R}, S(m_1, g^\sharp))$ such that $\mu_1^w = \widetilde{m}_1^{\mathrm{Wick}}$ with $\widetilde{m}_1 \in L^\infty(\mathbf{R}, S(m_1, g^\sharp))$ given by* (6.8). *Then there exist positive constants c_1, c_2 and C_0 such that*

$$c_1 h^{1/2} \|u\|^2 \le c_2 \|u\|_{1/2}^2 \le (\mu_1^w u, u) \le C_0 \|u\|_{1/2}^2, \qquad u \in \mathcal{S}(\mathbf{R}^n). \tag{6.10}$$

The constants only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

Proof. Let $a_\varrho = \mu_\varrho^{-1/2} \in S(m_\varrho^{-1/2}, g_\varrho)$ with $0 < \varrho \le 1$ to be chosen later. Since $g_\varrho = \varrho^2 g^\sharp$ is uniformly σ temperate, $g_\varrho / g_\varrho^\sigma = \varrho^4$, m_ϱ is uniformly σ, g_ϱ temperate, and $\mu_\varrho^{\pm 1/2} \in S(m_\varrho^{\pm 1/2}, g_\varrho)$ uniformly, the calculus gives that $(a_\varrho^{-1})^w a_\varrho^w = 1 + r_\varrho^w$ where $r_\varrho / \varrho^2 \in S(1, g^\sharp)$ uniformly for $0 < \varrho \le 1$. Similarly, we find that $a_\varrho^w \mu_\varrho^w a_\varrho^w = 1 + s_\varrho^w$ where $s_\varrho / \varrho^2 \in S(1, g^\sharp)$ uniformly. We obtain that the L^2 operator norms

$$\|r_\varrho^w\|_{\mathcal{L}(L^2)} + \|s_\varrho^w\|_{\mathcal{L}(L^2)} \le C_0 \varrho^2 \le 1/2$$

for sufficiently small ϱ. By fixing such a value of ϱ we find that $1/2 \le a_\varrho^w \mu_\varrho^w a_\varrho^w \le 2$ and

$$\|u\|_0 \le 2 \|(a_\varrho^{-1})^w a_\varrho^w u\|_0 \le C_1 \|a_\varrho^w u\|_{1/2} \le C_2 \|u\|_0. \tag{6.11}$$

Thus $u \mapsto a_\varrho^w u$ is a homeomorphism between L^2 and $H(m_1^{1/2}, g^\sharp)$. Since the constant metric $g^\sharp$ is trivially strongly σ temperate in the sense of [2, Def. 7.1] and $a_\varrho \in S(m_1^{-1/2}, g^\sharp)$, we find from [2, Cor. 7.7] that there exists $b_\varrho \in S(m_1^{1/2}, g^\sharp)$ such that $a_\varrho^w b_\varrho^w = b_\varrho^w a_\varrho^w = 1$. Then we obtain that

$$\|u\|_{1/2} = \|a_\varrho^w b_\varrho^w u\|_{1/2} \le C_3 \|b_\varrho^w u\|_0 \le C_4 \|u\|_{1/2}$$

and

$$\frac{1}{2}\|b_\varrho^w u\|_0^2 \le \left(\mu_\varrho^w a_\varrho^w b_\varrho^w u, a_\varrho^w b_\varrho^w u\right) = \left(\mu_\varrho^w u, u\right) \le 2\|b_\varrho^w u\|_0^2$$

which gives $\|b_\varrho^w u\|_0^2 \cong \left(\mu_\varrho^w u, u\right)$. Since $ch^{1/2} \le m_1 \cong \widetilde{m}_\varrho$ and $\mu_\varrho^w = \widetilde{m}_\varrho^{\mathrm{Wick}}$ we find $c_1 h^{1/2}\|u\|^2 \le \left(\mu_\varrho^w u, u\right)$, which completes the proof of the proposition. □

7. The lower bounds

In this section we shall obtain a proof of Proposition 2.5 by giving lower bounds on $\operatorname{Re} b_T^w f^w$, where $b_T^w = B_T^{\mathrm{Wick}}$ is given by Proposition 6.3. In the following, we shall omit the t variable and assume the coordinates chosen so that $g^\sharp(w) = |w|^2$. The results will hold for almost all $|t| \le T$ and only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

Proposition 7.1. *Assume that* $b = \delta_1 + \varrho_1$ *is as given by Proposition* 6.3. *Then*

$$\operatorname{Re}(b^w f^w u, u) \ge (C^w u, u) \qquad \forall\ u \in \mathcal{S}(\mathbf{R}^n) \tag{7.1}$$

where $C \in S(m_1, g^\sharp)$.

Proof. We shall localize in $T^*\mathbf{R}^n$ with respect to the metric $G_1 = H_1 g^\sharp$, and estimate the localized operators. We shall use the neighborhoods

$$\omega_{w_0}(\varepsilon) = \left\{ w :\ |w - w_0| < \varepsilon H_1^{-1/2}(w_0) \right\} \qquad \text{for } w_0 \in T^*\mathbf{R}^n. \tag{7.2}$$

We may in the following assume that ε is small enough so that $w \mapsto H_1(w)$ and $w \mapsto M(w)$ only vary with a fixed factor in $\omega_{w_0}(\varepsilon)$. Then by the uniform Lipschitz continuity of $w \mapsto \delta_0(w)$ we can find $\kappa_0 > 0$ with the following property: for $0 < \kappa \le \kappa_0$ there exist positive constants c_κ and ε_κ so that for any $w_0 \in T^*\mathbf{R}^n$,

$$|\delta_0(w)| \le \kappa H_1^{-1/2}(w), \qquad w \in \omega_{w_0}(\varepsilon_\kappa) \qquad \text{or} \tag{7.3}$$

$$|\delta_0(w)| \ge c_\kappa H_1^{-1/2}(w), \qquad w \in \omega_{w_0}(\varepsilon_\kappa). \tag{7.4}$$

In fact, we have by the Lipschitz continuity that $|\delta_0(w) - \delta_0(w_0)| \le \varepsilon_\kappa H_1^{-1/2}(w_0)$ when $w \in \omega_{w_0}(\varepsilon_\kappa)$. Thus, if $\varepsilon_\kappa \ll \kappa$, then (7.3) holds when $|\delta_0(w_0)| \ll \kappa H_1^{-1/2}(w_0)$ and (7.4) holds when $|\delta_0(w_0)| \ge c\kappa H_1^{-1/2}(w_0)$.

By shrinking κ_0 we may assume that $M \cong |f'| H_1^{-1/2}$ when $|\delta_0| \le \kappa_0 H_1^{-1/2}$ and $H_1^{1/2} \le \kappa_0$ according to Proposition 3.8. Let κ_1 be given by Proposition 3.9, κ_2 by Proposition 6.3, and let ε_κ and c_κ be given by (7.3)–(7.4) for

$\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. Using Proposition 6.3 with $\lambda = c_\kappa$ gives $\kappa_3 > 0$ such that

$$|b| \ge \kappa_2 c_\kappa H_1^{-1/2} \qquad \text{in } \omega_{w_0}(\varepsilon_\kappa) \tag{7.5}$$

if $H_1^{1/2} \le \kappa_3$ and (7.4) holds in $\omega_{w_0}(\varepsilon_\kappa)$.

Choose real symbols $\{\, \psi_j(w)\,\}_j$, $\{\, \Psi_j(w)\,\}_j$ and $\{\, \Phi_j(w)\,\}_j \in S(1, G_1)$ with values in ℓ^2, such that $\sum_k \psi_j^2 \equiv 1$, $\psi_j \Psi_j = \psi_j$, $\Psi_j \Phi_j = \Psi_j$, $\Psi_j = \phi_j^2 \ge 0$ for some $\{\, \phi_j(w)\,\}_j \in S(1, G_1)$ with values in ℓ^2 so that

$$\operatorname{supp} \phi_j \subseteq \omega_j = \omega_{w_j}(\varepsilon_\kappa).$$

Since $b \in S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$ we find that

$$A_j = \Psi_j b f \in S(M H_1^{-1/2}, g^\sharp) \bigcap S^+(M, g^\sharp) \qquad \text{uniformly in } j.$$

We have $\sum_j \psi_j^2 A_j = \sum_j \psi_j^2 \Psi_j b f = bf$, and we shall show that

$$\operatorname{Re}(b^w f^w) = (bf)^w = \sum_j \psi_j^w A_j^w \psi_j^w \qquad \text{modulo } \operatorname{Op} S(m_1, g^\sharp). \tag{7.6}$$

In order to estimate these localized operators, we shall use the following:

Lemma 7.2. *Let $b = \delta_1 + \varrho_1$ be as given by Proposition* 6.3, *and let $\Psi_j = \phi_j^2$ with $\phi_j \in S(1, G_1)$ uniformly with $\operatorname{supp} \phi_j \subseteq \omega_{w_j}(\varepsilon_\kappa)$ so that* (7.3) *or* (7.4) *holds for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. If $A_j = \Psi_j b f$ then there exists $C_j \in S(m_1, g^\sharp)$ uniformly, such that*

$$\left(A_j^w u, u\right) \ge \left(C_j^w u, u\right), \qquad u \in \mathcal{S}(\mathbf{R}^n). \tag{7.7}$$

We postpone the proof of Lemma 7.2 until later. We obtain from (7.6) that

$$\operatorname{Re}\,(b^w f^w u, u) \ge \sum_j \left(\psi_j^w C_j^w \psi_j^w u, u\right) + (R^w u, u)\,, \qquad u \in \mathcal{S}(\mathbf{R}^n)$$

where $\sum_j \psi_j^w C_j^w \psi_j^w$ and $R^w \in \operatorname{Op} S(m_1, g^\sharp)$, which gives Proposition 7.1.

It remains to prove (7.6). Proposition 5.5 gives that

$$M H_1^{3/2} \langle \delta_0 \rangle \le C m_1; \tag{7.8}$$

thus we may ignore terms in $\operatorname{Op} S(M H_1^{3/2} \langle \delta_0 \rangle, g^\sharp)$. Observe that since $b \in S(H_1^{-1/2}, g^\sharp)$ and $A_k \in S(M H_1^{-1/2}, g^\sharp)$ we find that the symbols of $b^w f^w$ and $\sum_k \psi_k^w A_k^w \psi_k^w$ have expansions in $S(M H_1^{j/2}, g^\sharp)$. Thus, we only have to compute the first terms in these expansions. Also observe that in the domains ω_j where $H_1^{1/2} \ge c > 0$, we find from Remark 2.4 that the symbols of $\sum_k \psi_k^w A_k^w \psi_k^w$ and $b^w f^w$ are in $S(M H_1^{3/2}, g^\sharp)$ giving the result in this case. Thus, in the following, we shall assume that $H_1^{1/2} \ll 1$, and we shall consider the neighborhoods where (7.3) or (7.4) hold.

If (7.4) holds then we find that $\langle \delta_0 \rangle \cong H_1^{-1/2}$ so that $S(MH_1, g^\sharp) \subseteq S(m_1, g^\sharp)$ in ω_j by (7.8). Since $b \in S^+(1, g^\sharp)$ we find from Lemma 2.3 that the symbol of $b^w f^w$ is equal to $bf + \frac{1}{2i}\{b, f\}$ modulo $S(MH_1, g^\sharp)$. Thus, we find that the symbol of $\mathrm{Re}(b^w f^w)$ is equal to bf modulo $S(m_1, g^\sharp)$ in ω_j. Similarly, since $\psi_k^w A_k^w \psi_k^w$ is symmetric, $\{\psi_k\}_k \in S(1, G_1)$ has values in ℓ^2 and $A_j \in S^+(M, g^\sharp)$ uniformly, we find from Remark 2.4 that the symbol of $\sum_k \psi_k^w A_k^w \psi_k^w$ is equal to $\sum_k \psi_k^2 A_k = bf$ modulo $S(MH_1, g^\sharp) \subseteq S(m_1, g^\sharp)$ in ω_j, which proves the result in this case.

Next, we consider the case when (7.3) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H_1^{1/2} \le \kappa_2/2$ in ω_j. Then $\langle \delta_0 \rangle \le \kappa_2 H_1^{-1/2}$ so $b = \delta_1 + \varrho_1 \in S(H_1^{-1/2}, G_1) + S(m_1, g^\sharp)$ in ω_j by Proposition 6.3. By taking the symmetric part of $b^w f^w = \delta_1^w f^w + \varrho_1^w f^w$ we obtain from Lemma 2.3 that the symbol of $\mathrm{Re}(b^w f^w - (bf)^w)$ is in $S(MH_1^{3/2}, G_1) + S(MH_1 m_1, g^\sharp) \subseteq S(m_1, g^\sharp)$ in ω_j since $M \le CH_1^{-1}$. Similarly, since $A_j \in S(MH_1^{-1/2}, G_1) + S(Mm_1, g^\sharp)$ uniformly, we find from Remark 2.4 that the symbol of $\sum_k \psi_k^w A_k^w \psi_k^w$ is equal to bf modulo $S(m_1, g^\sharp)$ in ω_j, which proves (7.6) and Proposition 7.1. $\square$

In order to simplify the computations of the symbols, we shall use the following result.

LEMMA 7.3. *Assume that M_1 is a weight for $G_1 = H_1 g^\sharp$, m_1 is a weight for $g^\sharp$, $p_1 \in S(M_1, G_1)$ and $p_2 \in S(m_1, g^\sharp)$. Then $p_1^w p_2^w$ and $p_2^w p_1^w$ have symbols which have expansions with terms in $S(M_1 m_1 H_1^{j/2}, g^\sharp)$, $j \ge 0$. Let $p_3^w = p_2^w p_2^w \in \mathrm{Op}\, S(m_1^2, g^\sharp)$ then*

$$(p_1 + p_2)^w (p_1 + p_2)^w = (p_1^2 + 2p_1 p_2 + p_3)^w \tag{7.9}$$

modulo $\mathrm{Op}\, S(M_1^2 H_1^2, G_1) + \mathrm{Op}\, S(M_1 m_1 H_1, g^\sharp)$. *If $p = p_1 p_2$, then we find $p^w p^w = (p_1^2 p_3)^w$ modulo $\mathrm{Op}\, S(M_1^2 m_1^2 H_1^{1/2}, g^\sharp)$.*

Proof. Since $g^\sharp / G_1^\sigma = H_1$, we obtain the expansions of $p_1^w p_2^w$ and $p_2^w p_1^w$ from Lemma 2.3. We also find that $p_1^w p_2^w = (p_1 p_2 + \frac{1}{2i}\{p_1, p_2\})^w$ and $p_2^w p_1^w = (p_2 p_1 - \frac{1}{2i}\{p_1, p_2\})^w$ modulo $S(M_1 m_1 H_1, g^\sharp)$. Since $p_1^w p_1^w = (p_1^2)^w$ modulo $\mathrm{Op}\, S(M_1^2 H_1^2, G_1)$ we obtain (7.9). Similarly, $p^w p^w = p_1^w p_2^w p_1^w p_2^w = p_1^w p_1^w p_3^w = (p_1^2 p_3)^w$ modulo $\mathrm{Op}\, S(M_1^2 m_1^2 H_1^{1/2}, g^\sharp)$ by the expansion. $\square$

Proof of Lemma 7.2. As before we are going to consider the cases when $H_1^{1/2} \cong 1$ or $H_1^{1/2} \ll 1$, and when (7.3) or (7.4) holds in $\omega_j = \omega_{w_j}(\varepsilon_\kappa)$ for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. When $H_1^{1/2} \ge c > 0$ we find that $A_j \in S(MH_1^{3/2}, g^\sharp) \subseteq S(m_1, g^\sharp)$ uniformly by (7.8) which gives the lemma with $C_j = A_j$ in this case.

Thus, we may assume that

$$H_1^{1/2} \le \kappa_4 = \min(\kappa_0, \kappa_1, \kappa_2, \kappa_3)/2 \qquad \text{in } \omega_j \tag{7.10}$$

so that (7.5) follows from (7.4).

Next, we consider the case when (7.3) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H_1^{1/2} \le \kappa_4 \le \kappa$ in ω_j. Then $\langle\delta_0\rangle \le 2\kappa H_1^{-1/2}$ so we obtain from Proposition 3.8 that $M \cong |f'| H_1^{-1/2}$ in ω_j. Similarly we find from Proposition 6.3 that $b = \delta_0 + \varrho_1 + \varrho_2 = \delta_0 + \varrho_3$ in ω_j, where $\varrho_1 \in S(m_1, g^\sharp)$ and $\varrho_2 \in S(H_1^{1/2}, G_1)$; thus $\varrho_3 \in S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp)$ by (5.2). Also, we find from Proposition 3.9 that $f = \alpha_0\delta_0$, where $\kappa_1 M H_1^{1/2} \le \alpha_0 \in S(MH_1^{1/2}, G_1)$ and $\delta_0 \in S(H_1^{-1/2}, G_1)$ in ω_j. Since $\Psi_j = \phi_j^2$, we find that

$$A_j = \Psi_j b f = \phi_j^2 \alpha_0\big(\delta_0^2 + \varrho_3\delta_0\big)$$

is real, and we shall construct an approximate square root γ_j^w so that

$$A_j^w = \gamma_j^w\gamma_j^w \ge 0 \qquad \text{modulo } \operatorname{Op} S(m_1, g^\sharp). \tag{7.11}$$

In the following, we shall suppress the index j, and let $\phi(w) = \phi_j(w)$ and $\gamma(w) = \gamma_j(w)$. By taking real-valued $\gamma(w) = \phi(w)(\mu_1\delta_0 + \mu_0)$, we see in the first approximation that $\mu_1 = \sqrt{\alpha_0} \in S(M^{1/2}H_1^{1/4}, G_1)$ and $\mu_0 = \varrho_3\sqrt{\alpha_0}/2 \in S(M^{1/2}H_1^{3/4}\langle\delta_0\rangle, g^\sharp)$. Then Lemma 7.3 gives

$$\gamma^w\gamma^w = \big(\phi^2(\mu_1^2\delta_0^2 + 2\mu_1\mu_0\delta_0)\big)^w + (\phi\mu_0)^w(\phi\mu_0)^w \tag{7.12}$$

modulo $\operatorname{Op} S(MH_1^{3/2}\langle\delta_0\rangle, g^\sharp) \subseteq \operatorname{Op} S(m_1, g^\sharp)$ by (7.8). By Lemma 7.3 we also have

$$(\phi\mu_0)^w(\phi\mu_0)^w = (\phi^2\nu_0)^w \qquad \text{mod } \operatorname{Op} S(MH_1^2\langle\delta_0\rangle^2, g^\sharp) \subseteq \operatorname{Op} S(m_1, g^\sharp) \tag{7.13}$$

where $\nu_0^w = \mu_0^w\mu_0^w \in \operatorname{Op} S(MH_1^{3/2}\langle\delta_0\rangle^2, g^\sharp)$ is symmetric. Observe that adding terms in $S(M^{1/2}H_1^{5/4}\langle\delta_0\rangle, g^\sharp)$ to μ_0 only give terms in $S(m_1, g^\sharp)$ in (7.13). By using $\chi(\delta_0) \in S(1, g^\sharp)$ for $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi(t) = 1$ for $|t| \le c$ we find that

$$\nu_0 = (1 - \chi(\delta_0))\nu_0 = \nu_1\delta_0 \qquad \text{modulo } S(MH_1^{3/2}, g^\sharp), \tag{7.14}$$

where $\nu_1 = (1 - \chi(\delta_0))\nu_0/\delta_0 \in S(MH_1^{3/2}\langle\delta_0\rangle, g^\sharp)$. By using (7.12)–(7.14) we obtain (7.11) if

$$\mu_1^2\delta_0^2 + (2\mu_1\mu_0 + \nu_1)\delta_0 = \alpha_0\delta_0^2 + \alpha_0\varrho_3\delta_0 \qquad \text{modulo } S(m_1, g^\sharp) \tag{7.15}$$

in ω_j. Subtracting $\nu_1/2\mu_1 \in S(M^{1/2}H_1^{5/4}\langle\delta_0\rangle, g^\sharp)$ from μ_0 does not change (7.13); thus we obtain (7.11) and the lemma in this case.

Finally, we consider the case when $H_1^{1/2} \le \kappa_4$ and (7.4) holds in ω_j. We shall use the uniform Fefferman-Phong estimate for $\Psi_j|f|$. Since $|\delta_0(w)| \ge$

$c_\kappa H_1^{-1/2}(w)$, we find $\langle\delta_0\rangle \cong H_1^{-1/2}$ in ω_j. Thus, we may ignore terms in $S(MH_1, g^\sharp) \subseteq S(MH_1^{3/2}\langle\delta_0\rangle, g^\sharp)$ supported in ω_j by (7.8). Since $H_1^{1/2} \le \kappa_4 \le \kappa_3/2$ and $|\delta_0| \ge c_\kappa H_1^{-1/2}$ in ω_j, we find from (7.5) that $|b| \ge \kappa_2 c_\kappa H_1^{-1/2}$ in ω_j. Since $b \in S^+(1, g^\sharp)$, we find by the chain rule that

$$|b|^\lambda \in S(H_1^{-\lambda/2}, g^\sharp) \bigcap S^+(H_1^{(1-\lambda)/2}, g^\sharp) \qquad \text{in} \;\; \omega_j \quad \forall\, \lambda.$$

In fact, we have $\partial_w |b|^\lambda = \operatorname{sgn}(b)\lambda|b|^{\lambda-1}\partial_w b \in S(H_1^{(1-\lambda)/2}, g^\sharp)$ in ω_j since $\partial_w b \in S(1, g^\sharp)$. Let $\Phi_j \in S(1, G_1)$ uniformly such that $\Psi_j\Phi_j = \Psi_j$ and $\operatorname{supp}\Phi_j \subseteq \omega_j$ as in the proof of Proposition 7.1. Since $\Phi_j \in S(1, G_1)$ we obtain that

$$\beta_j = \Phi_j |b|^{1/2} \in S(H_1^{-1/4}, g^\sharp) \bigcap S^+(H_1^{1/4}, g^\sharp). \tag{7.16}$$

Letting $0 \le a_j = \Psi_j|f| \in S(M, G_1)$, we find that $A_j = \Psi_j b f = a_j \beta_j^2$ since $\Psi_j\Phi_j = \Psi_j$. In order to estimate A_j^w we shall use the following lemma.

LEMMA 7.4. *Let $a \in S(M, G_1)$ and $\beta \in S(H_1^{-1/4}, g^\sharp) \bigcap S^+(H_1^{1/4}, g^\sharp)$ be real-valued symbols. Then there exists a real-valued symbol $r \in S(H_1^{1/2}, g^\sharp)$ such that*

$$\beta^w a^w \beta^w = \big(a(\beta^2 + r)\big)^w \tag{7.17}$$

modulo $\operatorname{Op} S(MH_1, g^\sharp)$.

Thus, we find that

$$A_j^w = \beta_j^w a_j^w \beta_j^w - (a_j r_j)^w \qquad \text{modulo } \operatorname{Op} S(MH_1, g^\sharp) \tag{7.18}$$

where $r_j \in S(H_1^{1/2}, g^\sharp)$ is real. Now we take the real symbol $\gamma_j = \Phi_j r_j |b|^{-1/2}/2 \in S(H_1^{3/4}, g^\sharp)$ and define

$$\lambda_j = \beta_j + \gamma_j \in S(H_1^{-1/4}, g^\sharp) \bigcap S^+(H_1^{1/4}, g^\sharp). \tag{7.19}$$

Then $2a_j\beta_j\gamma_j = a_j r_j$, and we shall show that

$$\lambda_j^w a_j^w \lambda_j^w = A_j^w \qquad \text{modulo } \operatorname{Op} S(MH_1, g^\sharp). \tag{7.20}$$

We obtain from Lemma 2.3 that that $a_j^w \gamma_j^w = (a_j\gamma_j)^w \in \operatorname{Op} S(MH_1^{3/4}, g^\sharp)$ modulo $\operatorname{Op} S(MH_1^{5/4}, g^\sharp)$. By Lemma 2.3,

$$2\beta_j^w a_j^w \gamma_j^w = 2\beta_j^w (a_j\gamma_j)^w = 2(\beta_j a_j \gamma_j)^w = (a_j r_j)^w$$

modulo $\operatorname{Op} S(MH_1, g^\sharp)$. Because $\gamma_j^w a_j^w \gamma_j^w \in \operatorname{Op} S(MH_1^{3/2}, g^\sharp)$ we obtain (7.20) from (7.18). By multiplying with $\Phi_j^w \in \operatorname{Op} S(1, G_1)$ we find from Lemma 2.3 that

$$\Phi_j^w \lambda_j^w a_j^w \lambda_j^w \Phi_j^w = A_j^w \qquad \text{modulo } \operatorname{Op} S(m_1, g^\sharp) \tag{7.21}$$

since $A_j^w = \Phi_j^w A_j^w \Phi_j^w$ modulo $\operatorname{Op} S(m_1, g^\sharp)$.

Because $0 \le a_j \in S(M(w_j), H_1(w_j)g^\sharp)$, the uniform Fefferman-Phong estimate (see [9, Lemma 18.6.10]) gives a constant $C_0 > 0$ so that

$$\big(a_j^w u, u\big) \ge -C_0 M(w_j) H_1^2(w_j) \|u\|^2 \qquad \forall\ u \in \mathcal{S}(\mathbf{R}^n).$$

Since $\lambda_j \in S(H_1^{-1/4}, g^\sharp)$ and $\Phi_j \in S(1, G_1)$ are real-valued this gives

$$\big(\Phi_j^w \lambda_j^w a_j^w \lambda_j^w \Phi_j^w u, u\big) \ge -C_0 M(w_j) H_1^2(w_j) \|\lambda_j^w \Phi_j^w u\|^2 = \big(c_j^w u, u\big)$$

where $c_j^w = -C_0 M(w_j) H_1^2(w_j) \Phi_j^w \lambda_j^w \lambda_j^w \Phi_j^w \in \operatorname{Op} S(M H_1^{3/2}, g^\sharp)$ uniformly in j. By (7.21) this completes the proof of Lemma 7.2. □

Proof of Lemma 7.4. We have that $\beta^w a^w \beta^w$ is symmetric since a and β are real. Thus

$$\beta^w a^w \beta^w = \operatorname{Re}\left([\beta^w, a^w]\beta^w + a^w B^w\right) = \frac{1}{2}\left[[\beta^w, a^w], \beta^w\right] + \frac{1}{2}\left(a^w B^w + B^w a^w\right)$$

where $B^w = \beta^w \beta^w \in \operatorname{Op} S(H_1^{-1/2}, g^\sharp)$ is symmetric. From Lemma 2.3

$$B = \beta^2 + r$$

with real $r \in S(H_1^{1/2}, g^\sharp)$ and $\beta^2 \in S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$, since $\partial \beta^2 = 2\beta \partial \beta$ where $\partial\beta \in S(H_1^{1/4}, g^\sharp)$. Since $a \in S(M, G_1)$ and $B \in S^+(1, g^\sharp)$, we find from Lemma 2.3 that

$$\frac{1}{2}\left(a^w B^w + B^w a^w\right) = (aB)^w = (a(\beta^2 + r))^w$$

modulo $\operatorname{Op} S(M H_1, g^\sharp)$. Lemma 2.3 also gives $[\beta^w, a^w] \in \operatorname{Op} S(M H_1^{3/4}, g^\sharp)$ and then $[[\beta^w, a^w], \beta^w] \in \operatorname{Op} S(M H_1, g^\sharp)$, which completes the proof of the lemma. □

We shall finish the paper by giving a proof of Proposition 2.5.

Proof of Proposition 2.5. We have assumed that $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfies condition $(\overline{\Psi})$ given by (2.2). Let $B_T = \delta_0 + \varrho_T$, where $\delta_0 + \varrho_T$ is the pseudo-sign for f given by Proposition 5.8 for $0 < T \le 1$, so that $|\varrho_T| \le m_1$ and

$$\partial_t(\delta_0 + \varrho_T) \ge m_1/2T \qquad \text{in } \mathcal{D}'\big(]{-T}, T[\big). \tag{7.22}$$

Putting $B_T \equiv 0$ when $|t| > T$, we find that $B_T^{\mathrm{Wick}} = b_T^w$ where

$$b_T(t, w) \in L^\infty(\mathbf{R}, S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp))$$

uniformly by Proposition 6.3. Let $cm_1 \le \widetilde{m}_1 \in S(m_1, g^\sharp)$ be given by (6.8) and let $\mu_1 \in L^\infty(\mathbf{R}, S(m_1, g^\sharp))$ be defined by (6.9) so that $\mu_1^w = \widetilde{m}_1^{\mathrm{Wick}}$. By Lemma 6.2 and (7.22),

$$T\partial_t\left(b_T^w u, u\right) = T\left((\partial_t B_T)^{\mathrm{Wick}} u, u\right) \ge C_0\left(\mu_1^w u, u\right) \qquad \text{in } \mathcal{D}'\big(]{-T}, T[\big) \tag{7.23}$$

when $u \in \mathcal{S}(\mathbf{R}^n)$. We obtain from Proposition 6.5 that there exist positive constants c_1 and c_2 so that

$$(\mu_1^w u, u) \geq c_2 \|u\|_{1/2}^2 \geq c_1 h^{1/2} \|u\|^2, \qquad u \in \mathcal{S}(\mathbf{R}^n). \tag{7.24}$$

Here $\|u\|_{1/2}$ is the norm of the Hilbert space $H(m_1^{1/2}, g^\sharp)$ given by Definition 6.4. By Proposition 7.1, we find for almost all $t \in [-T, T]$ that

(7.25)
$$\operatorname{Re}\left((B_T^{\mathrm{Wick}} f^w)\big|_t u, u \right) = \operatorname{Re}\left((b_T^w f^w)\big|_t u, u \right) \geq (C^w(t)u, u)\,, \quad u \in \mathcal{S}(\mathbf{R}^n)$$

with $C(t) \in S(m_1, g^\sharp)$ uniformly. We obtain from (6.7), (7.24) and duality that there exists a positive constant c_3 such that

$$|\,(C^w(t)u, u)\,| \leq \|u\|_{1/2} \|C^w(t)u\|_{-1/2} \leq c_3 \|u\|_{1/2}^2 \leq c_3\,(\mu_1^w u, u)\,/c_2 \tag{7.26}$$

for $u \in \mathcal{S}(\mathbf{R}^n)$ and $|t| \leq T$. We obtain from (7.23)–(7.26) the estimate

$$(\partial_t b_T^w u, u) + 2\operatorname{Re}(f^w u, b_T^w u) \geq (C_0/T - 2c_3/c_2)\,(\mu_1^w u, u) \qquad \text{in } \mathcal{D}'\big(]-T, T[\big)$$

for $u \in \mathcal{S}(\mathbf{R}^n)$. By using Proposition 2.9 with $P_0 = D_t + i f^w(t, x, D_x)$, $B = b_T^w$ and $m = C_0 \mu_1^w / 2T$ we obtain that

$$c_1 h^{1/2} \int \|u\|^2 \, dt \leq \int (\mu_1^w u, u)\; dt \leq \frac{4T}{C_0} \int \operatorname{Im}(P_0 u, b_T^w u)\; dt$$

if $u \in \mathcal{S}(\mathbf{R} \times \mathbf{R}^n)$ has support where $|t| < T \leq c_2 C_0 / 4c_3$. Replacing b_T^w with $4b_T^w / C_0 c_1$ we get a proof of Proposition 2.5, which completes the proof of the Nirenberg-Treves conjecture. □

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