Annals of Mathematics, **163** (2006), 369–379

A refined version of the Siegel-Shidlovskii theorem

By F. Beukers

Abstract

Using Y. André's result on differential equations satisfied by E-functions, we derive an improved version of the Siegel-Shidlovskii theorem. It gives a complete characterisation of algebraic relations over the algebraic numbers between values of E-functions at any nonzero algebraic point.

1. Introduction

In this paper we consider E-functions. An entire function $f(z)$ is called an E-function if it has a power series expansion of the form

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

where

(1) $a_k \in \overline{\mathbb{Q}}$ for all k.

(2) $h(a_0, a_1, \dots, a_k) = O(k)$ for all k where h denotes the log of the absolute height.

(3) f satisfies a linear differential equation $Ly = 0$ with coefficients in $\overline{\mathbb{Q}}[z]$.

The linear differential equation $Ly = 0$ of minimal order which is satisfied by f is called the *minimal differential equation* of f.

Furthermore, in all of our consideration we take a fixed embedding $\overline{\mathbb{Q}} \to \mathbb{C}$.

Siegel first introduced E-functions around 1929 in his work on transcendence of values of Bessel-functions and related functions. Actually, Siegel's definition was slightly more general in that condition (3) reads $h(a_0, a_1, \dots, a_k) = o(k \log k)$. But until now no E-functions in Siegel's original definition are known which fail to satisfy condition (2) above. Around 1955 Shidlovski managed to remove Siegel's technical normality conditions and we now have the following theorem (see [Sh, Ch. 4.4], [FN, Th. 5.23]).

THEOREM 1.1 (Siegel-Shidlovskii, 1956). *Let $f_1, \ldots, f_n$ be a set of E-functions which satisfy the system of first order equations*

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

where A is an $n \times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$. Denote the common denominator of the entries of A by $T(z)$. Then, for any $\xi \in \overline{\mathbb{Q}}$ such that $\xi T(\xi) \neq 0$,

$$\deg \operatorname{tr}_{\overline{\mathbb{Q}}}(f_1(\xi), \ldots, f_n(\xi)) = \deg \operatorname{tr}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_n(z)).$$

In [B1] Daniel Bertrand gives an alternative proof of the Siegel-Shidlovskii theorem using Laurent's determinants.

Of course the Siegel-Shidlovskii theorem suggests strongly that all relations between values of E-functions at algebraic points arise by specialisation of polynomial relations over $\overline{\mathbb{Q}}(z)$. Using the techniques of Siegel and Shidlovskii this turns out to be true up to a finite exceptional set of algebraic points.

THEOREM 1.2 (Nesterenko-Shidlovskii, 1996). *There exists a finite set S such that for all $\xi \in \overline{\mathbb{Q}}$, $\xi \notin S$ the following holds. For any homogeneous polynomial relation $P(f_1(\xi), \ldots, f_n(\xi)) = 0$ with $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ there exists $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_n]$, homogeneous in X_i, such that $Q(z, f_1(z), \ldots, f_n(z)) \equiv 0$ and $P(X_1, \ldots, X_n) = Q(\xi, X_1, \ldots, X_n)$.*

In the statement of the theorem one can drop the word 'homogeneous' if one wants, simply by considering the set of E-functions $1, f_1(z), \ldots, f_n(z)$ instead. Loosely speaking, for almost all $\xi \in \overline{\mathbb{Q}}$, polynomial relations between the values of f_i at $z = \xi$ arise by specialisation of polynomial relations between the $f_i(z)$ over $\overline{\mathbb{Q}}(z)$.

In [NS] it is also remarked that the exceptional set S can be computed in principle. Although Theorem 1.2 is not stated explicitly in [NS], it is immediate from Theorem 1 and Lemmas 1, 2 in [NS].

Around 1997 Y. André (see [A1] and Theorem 2.1 below) discovered that the nature of differential equations satisfied by E-functions is very simple. Their only nontrivial singularities are at $0, \infty$. Even more astounding is that this observation allowed André to prove transcendence statements, as illustrated in Theorem 2.2. In particular André managed to give a completely new proof of the Siegel-Shidlovskii theorem using his discovery. In order to achieve this, a defect relation for linear equations with irregular singularities had to be invoked. For a survey one can consult [A2] or, more detailed, [B2].

However, it turns out that even more is possible. Theorem 2.1 allows us to prove the following theorem.

THEOREM 1.3. *Theorem 1.2 holds for any $\xi \in \overline{\mathbb{Q}}$ with $\xi T(\xi) \neq 0$.*

The proof of this theorem will be given in Section 3, after the necessary preparations. In particular we will use some very basic facts about differential Galois groups of systems of differential equations. All that we require is contained in Section 1.4 of the book [PS].

One particular consequence of Theorem 1.3 is the solution of Conjecture A in [NS]. As pointed out by the referee this conjecture was already alluded to in S. Lang's book on transcendental numbers; see [L, p. 100]

Corollary 1.4. *Let assumptions be as in Theorem* 1.1. *Suppose that* $f_1(z),\dots,f_n(z)$ *are linearly independent over* $\overline{\mathbb{Q}}(z)$. *Then for any* $\xi\in\overline{\mathbb{Q}}$, *with* $\xi T(\xi)\ne 0$, *the numbers* $f_1(\xi),\dots,f_n(\xi)$ *are* $\overline{\mathbb{Q}}$*-linear independent.*

A question that remains is about the nature of relations between values of E-functions at singular points $\ne 0$. The best known example is $f(z)=(z-1)e^z$. Its minimal differential equation has a singularity at $z=1$ and it vanishes at $z=1$, even though $f(z)$ is transcendental over $\overline{\mathbb{Q}}(z)$. Of course the vanishing of $f(z)$ at $z=1$ arises in a trivial way and one would probably agree that it is better to look at e^z itself. It turns out that all relations between values of E-functions at singularities $\ne 0$ arise in a similar trivial fashion. This is a consequence of the following theorem.

Theorem 1.5. *Let* $f_1,\dots,f_n$ *be as above and suppose they are* $\overline{\mathbb{Q}}(z)$*-linear independent. Then there exist* E*-functions* $e_1(z),\dots,e_n(z)$ *and an* $n\times n$*-matrix* M *with entries in* $\overline{\mathbb{Q}}[z]$ *such that*

$$\begin{pmatrix} f_1(z)\\ \vdots \\ f_n(z)\end{pmatrix} = M\begin{pmatrix} e_1(z)\\ \vdots \\ e_n(z)\end{pmatrix}$$

and where $(e_1(z),\dots,e_n(z))$ *is the vector solution of a system of* n *homogeneous first order equations with coefficients in* $\overline{\mathbb{Q}}[z,1/z]$.

Acknowledgement. At this point I would like to express my gratitude to Daniel Bertrand who critically read and commented on a first draft of this paper. His remarks were invaluable to me. I would also like to thank the anonymous referee who did a lot to improve the paper with a number of expert comments.

2. André's theorem and first consequences

Everything we deduce in this paper hinges on the following beautiful theorem plus corollary by Yves André.

Theorem 2.1 (Y. André). *Let f be an E-function and let $Ly = 0$ be its minimal differential equation. Then at every point $z \neq 0, \infty$ the equation has a basis of holomorphic solutions.*

All results that follow now, depend on a limited version of Theorem 2.1 where the E-function has rational coefficients. Although the following theorem occurs in [A1] we want to give a proof of it to make this paper self-contained to the extent only of accepting Theorem 2.1.

Corollary 2.2 (Y. André). *Let f be an E-function with rational coefficients and let $Ly = 0$ be its minimal differential equation. Suppose $f(1) = 0$. Then all solutions of $Ly = 0$ vanish at $z = 1$ and consequently $z = 1$ is an apparent singularity of $Ly = 0$.*

Proof. Suppose

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

Let $g(z) = f(z)/(1-z)$. Note that $g(z)$ is also holomorphic in $\mathbb{C}$. Moreover, $g(z)$ is again an E-function. Write

$$g(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$$

where

$$\frac{b_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!}.$$

Since $f(1) = 0$ we see that

$$\frac{b_n}{n!} = -\sum_{k=n+1}^{\infty} \frac{a_k}{k!}.$$

Since f is an E-function there exist $B, C > 0$ such that $|a_k| \leq B \cdot C^k$. Hence

$$\begin{aligned} |b_n| &\leq Bn! \left| \sum_{k=n}^{\infty} \frac{C^k}{k!} \right| \\ &\leq Bn! \frac{C^n}{n!} \left(1 + \frac{C}{1!} + \frac{C^2}{2!} + \cdots \right) \\ &\leq Be^C \cdot C^n. \end{aligned}$$

Furthermore, the common denominator of $b_0, \ldots, b_n$ is bounded above by the common denominator of $a_0, a_1, \ldots, a_n$, hence bounded by $B_1 \cdot C_1^n$ for some $B_1, C_1 > 0$. This shows that $f(z)/(z-1)$ is an E-function. The minimal differential operator which annihilates $g(z)$ is simply $L \circ (z-1)$. From André's

Theorem 2.1 it follows that the kernel of $(z-1)^{-1} \circ L \circ (z-1)$ around $z=1$ is spanned by holomorphic functions. Hence the kernel of L is spanned by holomorphic solutions times $z-1$. In other words, all solutions of $Ly=0$ vanish at $z=1$ and therefore $z=1$ is an apparent singularity. $\square$

LEMMA 2.3. *Let f be an E-function with minimal differential equation $Ly=0$ of order n. Let G be its differential Galois group and let G^o be the connected component of the identity in G. Let V be the vector space spanned by all images of $f(z)$ under G^o. Then V is the complete solution space of $Ly=0$.*

Proof. The fixed field of G^o is an algebraic Galois extension K of $\overline{\mathbb{Q}}(z)$ with Galois group G/G^o. Suppose that V has dimension m. Then f satisfies a linear differential equation with coefficients in K of order m. In particular we have a relation

$$f^{(m)} + p_{m-1}(z) f^{(m-1)} + \cdots + p_1(z) f' + p_0(z) f = 0 \tag{1}$$

for some $p_i \in K$. We subject this relation to analytic continuation. Since f is an entire function, it has trivial monodromy. By choosing suitable paths we obtain the conjugate relations

$$f^{(m)} + \sigma(p_{m-1}) f^{(m-1)} + \cdots + \sigma(p_1) f' + \sigma(p_0) f = 0$$

for all $\sigma \in G/G^o$. Taking the sum over all these relations gives us a nontrivial differential equation for f of order m over $\overline{\mathbb{Q}}(z)$. From the minimality of $Ly=0$ we now conclude that $m=n$; i.e., the dimension of V is n. $\square$

Actually it follows from Theorem 2.1 that the fixed field of G^o is of the form $K=\overline{\mathbb{Q}}(z^{1/r})$ for some positive integer r. But we do not need that in our proof. Lemma 2.3 also follows from a lemma of O. Gabber which states that the monodromy group surjects onto G/G^o. See [PS, p. 282].

The following lemma is a straightforward consequence of the general theory of algebraic groups.

LEMMA 2.4. *Let $G_1, \ldots, G_r$ be linear algebraic groups and denote by G_i^o their components of the identity. Let $H \subset G_1 \times G_2 \times \cdots \times G_r$ be an algebraic subgroup such that the natural projection $\pi_i : H \to G_i$ is surjective for every i. Let H^o be the connected component of the identity in H. Then the natural projections $\pi_i : H^o \to G_i^o$ are surjective.*

Now we prove a generalisation of André's Corollary 2.2 to general nonzero algebraic points.

THEOREM 2.5. *Let f be an E-function with minimal differential equation $Ly=0$ of order n. Suppose that $\xi \in \overline{\mathbb{Q}}^*$ and $f(\xi)=0$. Then all solutions of Ly vanish at $z=\xi$. In particular, $Ly=0$ has an apparent singularity at $z=\xi$.*

Proof. By replacing $f(z)$ by $f(\xi z)$ if necessary, we can assume that f vanishes at $z=1$. Let $f^{\sigma_1}(z),\dots,f^{\sigma_r}(z)$ be the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of $f(z)$ where we take $f^{\sigma_1}(z)=f(z)$. Let $L^{\sigma_i}y=0$ be the σ_i-conjugate of $Ly=0$. Note that this is the minimal differential equation satisfied by $f^{\sigma_i}(z)$. Let G_i be the differential Galois group and G_i^o the connected component of the identity. By Lemma 2.3 the images of $f^{\sigma_i}(z)$ under G_i^o span the complete solution space of $L^{\sigma_i}y=0$.

The product $F(z)=\prod_{i=1}^r f^{\sigma_i}(z)$ is an E-function having rational coefficients. Let $\mathcal{L}y=0$ be its minimal differential equation. Furthermore, $F(1)=0$. Hence, from André's Theorem 2.2 it follows that all solutions of $\mathcal{L}y=0$ vanish at $z=1$.

Let H be the differential Galois group of the differential compositum of the Picard-Vessiot extensions corresponding to $L^{\sigma_i}y=0$. Note that the image of $F(z)$ under any $h\in H$ is again a solution of $\mathcal{L}y=0$. In particular this image also vanishes at $z=1$.

Furthermore, H is an algebraic subgroup of $G_1\times G_2\times\cdots\times G_r$ such that the natural projections $\pi_i: H\to G_i$ are surjective. Let H^o be the connected component of the identity of H. Then, by Lemma 2.4, the projections π_i : $H^o\to G_i^o$ are surjective.

Let V_i be the solution space of local solutions at $z=1$ of $L^{\sigma_i}y=0$. In view of Theorem 2.1 all these solutions are holomorphic at $z=1$. Let W_i be the linear subspace of solutions vanishing at $z=1$. The group H^o acts linearly on each space V_i. Let $v_i\in V_i$ be the vector corresponding to the solution $f^{\sigma_i}(z)$. Define $H_i=\{h\in H^o|\pi_i(h)v_i\in W_i\}$. Then H_i is a Zariski closed subset of H^o. Furthermore, because all solutions of $\mathcal{L}y=0$ vanish at $z=1$, we have that $H^o=\cup_{i=1}^r H_i$. Since H^o is connected this implies that $H_i=H^o$ for at least one i. Hence $\pi_i(H_i)=\pi_i(H^o)=G_i^o$ and we see that $gv_i\in W_i$ for all $g\in G_i^o$. We conclude that $W_i=V_i$. In other words, all local solutions of $L^{\sigma_i}y=0$ around $z=1$ vanish in $z=1$. By conjugation we now see that the same is true for $Ly=0$. □

3. Independence results

We now consider a set of E-functions $f_1,\dots,f_n$ which satisfy a system of homogeneous first order equations

$$y'=Ay$$

where y is a vector of unknown functions $(y_1,\dots,y_n)^t$ and A an $n\times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$. The common denominator of these entries is denoted by $T(z)$.

LEMMA 3.1. *Let us assume that the $\overline{\mathbb{Q}}(z)$-rank of $f_1, \ldots, f_n$ is m. Then the $\overline{\mathbb{Q}}[z]$-relations bewteen $f_1, \ldots, f_n$ have a basis*

(1)
$$C_{i,1}(z)f_1(z) + C_{i,2}(z)f_2(z) + \cdots + C_{i,n}(z)f_n(z) \equiv 0, \quad i = 1, 2, \ldots, n-m,$$

such that for any $\xi \in \overline{\mathbb{Q}}$ the matrix

$$\begin{pmatrix} C_{11}(\xi) & C_{12}(\xi) & \ldots & C_{1n}(\xi) \\ \vdots & \vdots & & \vdots \\ C_{n-m,1}(\xi) & C_{n-m,2}(\xi) & \ldots & C_{n-m,n}(\xi) \end{pmatrix}$$

has rank precisely $n - m$.

Proof. The $\overline{\mathbb{Q}}(z)$-dimension of all relations is $n - m$. Choose an independent set of $n - m$ relations of the form (1) (without the extra specialisation condition).

Denote the greatest common divisor of the determinants of all $(n-m) \times (n-m)$ submatrices of $(C_{ij}(z))$ by $D(z)$. Suppose that $D(\xi) = 0$ for some ξ. Then the matrix $(C_{ij}(\xi))$ has linearly dependent rows. By taking $\overline{\mathbb{Q}}$-linear relations between the rows, if necessary, we can assume that $C_{1j}(\xi) = 0$ for $j = 1, \ldots, n$. Hence all $C_{1j}(z)$ are divisible by $z - \xi$ and the polynomials $C_{1j}(z)/(z - \xi)$ are the coefficients of another $\overline{\mathbb{Q}}(z)$-linear relation. Replace the first relation by this new relation. The new greatest divisor of all $(n-m) \times (n-m)$-determinants is now $D(z)/(z - \xi)$. By repeating this argument we can find an independent set of $n - m$ relations of the form (1) whose associated $D(z)$ is a nonzero constant.

But now it is not hard to see that (1) is a $\overline{\mathbb{Q}}[z]$-basis of all $\overline{\mathbb{Q}}[z]$-relations. Furthermore, $D(\xi) \neq 0$ for all ξ (because $D(z)$ is constant), so all specialisations have maximal rank. □

THEOREM 3.2. *Let $f_1, \ldots, f_n$ be a vector solution of the system*

$$y' = Ay$$

consisting of E-functions. Let $T(z)$ be the common denominator of the entries in A. Then, for any $\xi \in \overline{\mathbb{Q}}$, $\xi T(\xi) \neq 0$, any $\overline{\mathbb{Q}}$-linear relation between $f_1(\xi), \ldots, f_n(\xi)$ arises by specialisation of a $\overline{\mathbb{Q}}(z)$-linear relation.

Proof. Suppose there exists a $\overline{\mathbb{Q}}$-linear relation

$$\alpha_1 f_1(\xi) + \alpha_2 f_2(\xi) + \cdots + \alpha_n f_n(\xi) = 0$$

which does not come from specialisation of a $\overline{\mathbb{Q}}(z)$-linear relation at $z = \xi$. Consider the function

$$F(z) = A_1(z)f_1(z) + A_2(z)f_2(z) + \cdots + A_n(z)f_n(z)$$

where $A_i(z) \in \overline{\mathbb{Q}}[z]$ to be specified later. Let $Ly = 0$ be the minimal differential equation satisfied by F. Suppose that the $\overline{\mathbb{Q}}(z)$-rank of $f_1, \ldots, f_n$ is m. Then the order of $Ly = 0$ is at most m.

We now show how to choose $A_1(z), \ldots, A_n(z)$ such that

(i) $A_i(\xi) = \alpha_i$ for $i = 1, 2, \ldots, n$.

(ii) The order of $Ly = 0$ is m.

(iii) ξ is a regular point of $Ly = 0$.

By using the system $y' = Ay$ recursively we can find $A_i^j(z) \in \overline{\mathbb{Q}}(z)$ such that

$$F^{(j)}(z) = \sum_{i=1}^{n} A_i^j(z) f_i(z).$$

In addition we fix a $\overline{\mathbb{Q}}(z)$-basis of linear relations

$$C_{i,1}(z)f_1(z) + \cdots + C_{i,n}f_n(z) \equiv 0, \quad i = 1, \ldots, n-m,$$

with polynomial coefficients $C_{ij}(z)$ such that the $(n-m) \times n$-matrix of values $C_{ij}(\xi)$ has maximal rank $n-m$. This is possible in view of Lemma 3.1. Consider the $(n+1) \times n$-matrix

$$\mathcal{M} = \begin{pmatrix} C_{11}(z) & \ldots & C_{1n}(z) \\ \vdots & & \vdots \\ C_{n-m,1}(z) & \ldots & C_{n-m,n}(z) \\ A_1(z) & \ldots & A_n(z) \\ \vdots & & \vdots \\ A_1^m(z) & \ldots & A_n^m(z) \end{pmatrix}$$

where $A_i^0(z) = A_i(z)$. We denote the submatrix obtained from $\mathcal{M}$ by deleting the row with A_i^j $(i = 1, \ldots, n)$ by $\mathcal{M}_j$. There exists a $\overline{\mathbb{Q}}(z)$-linear relation between the rows of $\mathcal{M}$ which explains why the minimal equation $Ly = 0$ of F satisfies a differential equation of order $\leq m$. Observe that if the order is $< m$, then there exists a nontrivial $\overline{\mathbb{Q}}(z)$-linear relation between the rows $(A_1^j, \ldots, A_n^j(z))$ $(j = 0, \ldots, m-1)$ which gives a vanishing relation between $f-1, \ldots, f_n$. Hence $\det(\mathcal{M}_m) = 0$. So, if the rank of $\mathcal{M}_m$ equals n, the order of $Ly = 0$ should be m. In that case the minimal differential equation for F is given by

$$\Delta_m F^{(m)} + \cdots + \Delta_1 F' + \Delta_0 F = 0$$

where $\Delta_j = (-1)^j \det(\mathcal{M}_j)$.

By induction it is not hard to show that $A_i^0(z) = A_i(z)$ and

$$A_i^j(z) = A_i^{(j)} + P_{ij}(A_1, \ldots, A_n, \ldots, A_1^{(j-1)}, \ldots, A_n^{(j-1)})$$

where

$$P_{ij} \in \overline{\mathbb{Q}}[z, 1/T(z)][X_{10}, \ldots, X_{n0}, \ldots, X_{1,j-1}, \ldots, X_{n,j-1}]$$

are linear forms with coefficients in $\overline{\mathbb{Q}}[z, 1/T(z)]$. In what follows we choose the $A_i(z)$ and their derivatives in such a way that $\det(\mathcal{M}_m)$ does not vanish in the point ξ. The choice of $A_i(\xi)$ is fixed by taking $A_i(\xi) = \alpha_i$. Since the relation $\sum_{i=1}^n \alpha_i f_i(\xi) = 0$ does not come from specialisation, the rows of values $(C_{i1}(\xi), \ldots, C_{in}(\xi))$ for $i = 1, \ldots, n-m$ and $(\alpha_1, \ldots, \alpha_n)$ have maximal rank $n - m + 1$. We can now choose the derivatives $A_i^{(j)}$ recursively with respect to j such that $\det(\mathcal{M}_m)(\xi) \neq 0$. With this choice we note that conditions (i), (ii), (iii) are satisfied.

On the other hand, $F(\xi) = 0$, so it follows from Theorem 2.5 that ξ is a singularity of $Ly = 0$. This contradicts condition (iii). □

Proof of Theorem 1.3. Consider the vector of E-functions given by the monomials $\mathbf{f}(z)^{\mathbf{i}} := f_1(z)^{i_1} \cdots f_n(z)^{i_n}$, $i_1 + \cdots + i_n = N$ of degree N in $f_1(z), \ldots, f_n(z)$. This vector again satisfies a system of linear first order equations with singularities in the set $T(z) = 0$. So we now apply Theorem 3.2 to the set of E-functions $\mathbf{f}(z)^{\mathbf{i}}$. The relation $P(f_1(\xi), \ldots, f_n(\xi))$ is now a $\overline{\mathbb{Q}}$-linear relation between the values $\mathbf{f}(\xi)^{\mathbf{i}}$. Hence, by Theorem 3.2, there is a $\overline{\mathbb{Q}}[z]$-linear relation between the $\mathbf{f}(z)^{\mathbf{i}}$ which specialises to the linear relation between the values at $z = \xi$. This proves our theorem. □

4. Removal of nonzero singularities

In this section we prove Theorem 1.5. For this we require the following proposition.

PROPOSITION 4.1. *Let f be an E-function and $\xi \in \mathbb{Q}^*$ such that $f(\xi) = 0$. Then $f(z)/(z - \xi)$ is again an E-function.*

Proof. By replacing $f(z)$ by $f(\xi z)$ if necessary, we can restrict our attention to $\xi = 1$. Write down a basis of local solutions of $Ly = 0$ around $z = 1$. Since f vanishes at $z = 1$, Theorem 2.5 implies that all solutions of $Ly = 0$ vanish at $z = 1$. But then, by conjugation, this holds for the solutions around $z = 1$ of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates $L^\sigma y = 0$ as well. In particular, the conjugate E-function $f^\sigma(z)$ vanishes at $z = 1$ for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Taking up the notations of the proof of Theorem 2.2 we see that

$$\frac{b_n^\sigma}{n!} = -\sum_{k=n+1}^{\infty} \frac{a_k^\sigma}{k!}$$

for every σ. We can now bound $|b_n^\sigma|$ exponentially in n for every σ. Since the coefficients of an E-function lie in a finite extension of $\mathbb{Q}$, only finitely many conjugates are involved. So we get our desired bound $h(b_0, \ldots, b_n) = O(n)$. □

Proof of Theorem 1.5. Denote the column vector $(f_1(z),\ldots,f_n(z))^t$ by $\mathbf{f}(z)$. Let

$$\mathbf{y}'(z)=A(z)\mathbf{y}(z)$$

be the system of equations satisfied by $\mathbf{f}$ and let G be its differential Galois group. Because the $f_i(z)$ are $\overline{\mathbb{Q}}(z)$-linear independent, the images of $\mathbf{f}$ under G span the complete solution set of $\mathbf{y}'=A\mathbf{y}$. So the images under G give us a fundamental solution set $\mathcal{F}$ of our system. We assume that the first column is $\mathbf{f}(z)$ itself. Since the $f_i(z)$ are E-functions, it follows from Theorem 2.1 that the entries of $\mathcal{F}$ are holomorphic at every point $\neq 0$. Consequently, the determinant $W(z)=\det(\mathcal{F})$ is holomorphic outside 0. Since $W(z)$ satisfies $W'(z)=\mathrm{Trace}(A)W(z)$, we see that $W(\alpha)=0$ implies that α is a singularity of our system. In particular, $\alpha\in\overline{\mathbb{Q}}$. Let k be the highest order with which α occurs as a pole in A. Write $\tilde{A}(z)=(z-\alpha)^kA(z)$. Then it follows from specialisation at $z=\alpha$ of $(z-\alpha)^k\mathbf{f}'(z)=\tilde{A}(z)\mathbf{f}(z)$ that there is a nontrivial vanishing relation between the components of $\mathbf{f}(\alpha)$. By choosing a suitable $M\in\mathrm{GL}(n,\overline{\mathbb{Q}})$ we can see to it that $M\mathbf{f}(z)$ is a vector of E-functions, of which the first component vanishes at α. But then, by Theorem 2.5, the whole first row of $M\mathcal{F}(z)$ which, by construction, is composed of images under G of its first element, vanishes at $z=\alpha$. Hence we can write $M\mathcal{F}(z)=D\mathcal{F}_1$ where

$$D=\begin{pmatrix} z-\alpha & 0 & \ldots & 0\\ 0 & 1 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \ldots & 1\end{pmatrix}$$

and $\mathcal{F}_1$ has entries holomorphic around $z=\alpha$. Thanks to Proposition 4.1, the entries of the first column in $\mathcal{F}_1$ are again E-functions. Moreover, $\mathcal{F}_1$ satisfies the new system of equations

$$\mathcal{F}_1'=(D^{-1}MAM^{-1}D-D^{-1}D')\mathcal{F}_1.$$

Notice that the order of vanishing of $W_1(z)=\det(\mathcal{F}_1)$ at $z=\alpha$ is one lower than the vanishing order of $W(z)$. We repeat our argument when $W_1(\alpha)=0$. By using this reduction procedure to all zeros of $W(z)$ we end up with an $n\times n$-matrix B, with entries in $\overline{\mathbb{Q}}[z]$, and an $n\times n$-matrix of functions $\mathcal{E}$ such that $\mathcal{F}=B\mathcal{E}$, the first column of $\mathcal{E}$ consists of E-functions and $\det(\mathcal{E})$ is nowhere vanishing in $\mathbb{C}^*$. As a result we have $\mathcal{E}'(z)=A_E(z)\mathcal{E}(z)$ where $A_E(z)$ is an $n\times n$-matrix with entries in $\overline{\mathbb{Q}}[z,1/z]$. ∎

Universiteit Utrecht, Utrecht, The Netherlands
E-mail address: F.Beukers@math.uu.nl
URL: http://www.math.uu.nl/people/beukers/

References

[A1] Y. André, Séries Gevrey de type arithmétique I, II, *Ann. of Math.* **151** (2000), 705–740, 741–756.

[A2] ———, Arithmetic Gevrey series and transcendence, a survey, *J. Théorie des Nombres de Bordeaux* **15** (2003), 1–10.

[B1] D. Bertrand, Le théorème de Siegel-Shidlovskii revisité, prépublication de L'Institut de Maths, Jussieu 370 (2004),

[B2] ———, On André's proof of the Siegel-Shidlovskii theorem, in *Colloq. Franco-Japonais: Théorie des Nombres Transcendants* (Tokyo 1998), 51–63, *Sem. Math. Sci.* **27**, Keio Univ., Yokohama, 1999.

[FN] N. I. Fel'dman and Yu. V. Nesterenko, *Transcendental Numbers*, *Encyclopedia of Mathematical Sciences* (A. N. Parshin, I. R. Shafarevich, eds.), Vol. 44 (*Number Theory* IV, translated by N. Koblitz), Springer-Verlag, New York, 1998.

[L] S. Lang, *Transcendental Numbers*, Addison-Wesley Publ. Co., Reading, Mass., 1966.

[NS] Yu. V. Nesterenko and A. B. Shidlovskii, On the linear independence of values of E-functions, *Mat. Sb.* **187** (1996), 93–108, translated in *Sb. Math.* **187** (1996), 1197–1211.

[PS] M. van der Put and M. Singer, *Galois Theory of Differential Equations*, Vol. 328, Grundlehen, Springer-Verlag, New York, 2003.

[Sh] A. B. Shidlovskii, *Transcendental Numbers*, *W. de Gruyter Studies in Math.* **12**, Walter de Gruyter & Co., Berlin, 1989.

(Received August 9, 2004)