Annals of Mathematics, **163** (2006), 347–368

Asymptotic faithfulness of the quantum $\mathrm{SU}(n)$ representations of the mapping class groups

By Jørgen Ellegaard Andersen*

Abstract

We prove that the sequence of projective quantum $\mathrm{SU}(n)$ representations of the mapping class group of a closed oriented surface, obtained from the projective flat $\mathrm{SU}(n)$-Verlinde bundles over Teichmüller space, is asymptotically faithful. That is, the intersection over all levels of the kernels of these representations is trivial, whenever the genus is at least 3. For the genus 2 case, this intersection is exactly the order 2 subgroup, generated by the hyper-elliptic involution, in the case of even degree and $n = 2$. Otherwise the intersection is also trivial in the genus 2 case.

1. Introduction

In this paper we shall study the finite dimensional *quantum* $\mathrm{SU}(n)$ *representations* of the mapping class group of a genus g surface. These form the only rigorously constructed part of the gauge-theoretic approach to topological quantum field theories in dimension 3, which Witten proposed in his seminal paper [W1]. We discovered the *asymptotic faithfulness* property for these representations by studying this approach, which we will now briefly describe, leaving further details to Sections 2 and 3 and the references given there.

Let Σ be a closed oriented surface of genus $g \geq 2$ and p a point on Σ. Fix $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{\mathrm{SU}(n)}$ in the center of $\mathrm{SU}(n)$. Let M be the moduli space of flat $\mathrm{SU}(n)$-connections on $\Sigma - p$ with holonomy d around p.

By applying geometric quantization to the moduli space M one gets a certain finite rank vector bundle over Teichmüller space $\mathcal{T}$, which we will call the *Verlinde* bundle $\mathcal{V}_k$ at level k, where k is any positive integer. The fiber of this bundle over a point $\sigma \in \mathcal{T}$ is $\mathcal{V}_{k,\sigma} = H^0(M_\sigma, \mathcal{L}_\sigma^k)$, where M_σ is M equipped with a complex structure induced from σ and $\mathcal{L}_\sigma$ is an ample generator of the Picard group of M_σ.

*This research was conducted for the Clay Mathematics Institute at University of California, Berkeley.

The main result pertaining to this bundle $\mathcal{V}_k$ is that its projectivization $\mathbb{P}(\mathcal{V}_k)$ supports a natural flat connection. This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. Now, since there is an action of the mapping class group Γ of Σ on $\mathcal{V}_k$ covering its action on $\mathcal{T}$, which preserves the flat connection in $\mathbb{P}(\mathcal{V}_k)$, we get for each k a finite dimensional projective representation, say $\rho_k^{n,d}$, of Γ, namely on the covariant constant sections of $\mathbb{P}(\mathcal{V}_k)$ over $\mathcal{T}$. This sequence of projective representations $\rho_k^{n,d}$, $k \in \mathbb{N}_+$, is the *quantum* $\mathrm{SU}(n)$ *representation* of the mapping class group Γ.

For each given (n, d, k), we cannot expect $\rho_k^{n,d}$ to be faithful. However, V. Turaev conjectured a decade ago (see e.g. [T]) that there should be no nontrivial element of the mapping class group representing trivially under $\rho_k^{n,d}$ for all k, keeping (n, d) fixed. We call this property *asymptotic faithfulness* of the quantum $\mathrm{SU}(n)$ representations $\rho_k^{n,d}$. In this paper we prove Turaev's conjecture:

THEOREM 1. *Assume that n and d are coprime or that $(n, d) = (2, 0)$ when $g = 2$. Then,*

$$\bigcap_{k=1}^{\infty} \ker(\rho_k^{n,d}) = \begin{cases} \{1, H\} & g = 2, \ \ n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise,} \end{cases}$$

where H is the hyperelliptic involution.

This theorem states that for any element ϕ of the mapping class group Γ, which is not the identity element (and not the hyperelliptic involution in genus 2), there is an integer k such that $\rho_k^{n,d}(\phi)$ is not a multiple of the identity. We will suppress the superscript on the quantum representations and simply write $\rho_k = \rho_k^{n,d}$ throughout the rest of the paper.

Our key idea in the proof of Theorem 1 is the use of the endomorphism bundle $\mathrm{End}(\mathcal{V}_k)$ and the construction of sections of this bundle via *Toeplitz operators* associated to smooth functions on the moduli space M. By showing that these sections are *asymptotically* flat sections of $\mathrm{End}(\mathcal{V}_k)$ (see Theorem 6 for the precise statement), we prove that any element in the above intersection of kernels acts trivially on the smooth functions on M, hence acts by the identity on M (see the proof of Theorem 7). Theorem 1 now follows directly from knowing which elements of the mapping class group act trivially on the moduli space M.

The assumptions on the pair (n, d) in Theorem 1 exactly pick out the cases where the moduli space M is smooth. This means we can apply the work of Bordemann, Meinrenken and Schlichenmaier on Toeplitz operators on smooth Kähler manifolds, in particular their formula for the asymptotics in k of the operator norm of Toeplitz operators and the asymptotic expansion of the product of two Toeplitz operators. Using these results we establish that the Toeplitz operator sections are asymptotically flat with respect to Hitchin's connection.

In the remaining cases, where M is singular, we also have a proof of asymptotic faithfulness, where we use the desingularization of the moduli space, but this argument is technically quite a bit more involved. However, together with Michael Christ we have in [AC] extended some of the results of Bordemann, Meinrenken and Schlichenmaier and Karabegov and Schlichenmaier to the case of singular varieties. In [A3] the argument of the present paper will be repeated in the noncoprime case, where we make use of the results of [AC] to show that Theorem 1 holds in general without the coprime assumption.

The abelian case, i.e. the case where $\mathrm{SU}(n)$ is replaced by $\mathrm{U}(1)$, was considered in [A2], before we considered the case discussed in this paper. In this case, with the use of theta-functions, explicit expressions for the Toeplitz operators associated to holonomy functions can be obtained. From these expressions it follows that the Toeplitz operators are not covariant constant even in this much simpler case (although the relevant connection is the one induced from the L_2-projection as shown by Ramadas in [R1]). However, they are asymptotically covariant constant; in fact we find explicit perturbations to all orders in k, which in this case, we argue, can be summed and used to create actual covariant constant sections of the endomorphism bundle. The result as far as the mapping class group goes, is that the intersection of the kernels over all k, in that case, is the Torelli group.

Returning to the non-abelian case at hand, we know by the work of Laszlo [La], that $\mathbb{P}(\mathcal{V}_k)$ with its flat connection is isomorphic to the projectivization of the bundle of conformal blocks for $\mathrm{sl}(n, \mathbb{C})$ with its flat connection over $\mathcal{T}$ as constructed by Tsuchiya, Ueno and Yamada [TUY]. This means that the quantum $\mathrm{SU}(n)$ representations ρ_k is the same sequence of representations as the one arising from the space of conformal blocks for $\mathrm{sl}(n, \mathbb{C})$.

By the work of Reshetikhin-Turaev, Topological Quantum Field Theories have been constructed in dimension 3 from the quantum group $U_q\mathrm{sl}(n, \mathbb{C})$ (see [RT1], [RT2] and [T]) or alternatively from the Kauffman bracket and the Homfly-polynomial by Blanchet, Habegger, Masbaum and Vogel (see [BHMV1], [BHMV2] and [B1]).

In ongoing work of Ueno joint with this author (see [AU1], [AU2] and [AU3]), we are in the process of establishing that the TUY construction of the bundle of conformal blocks over Teichmüller space for $\mathrm{sl}(n, \mathbb{C})$ gives a modular functor, which in turn gives a TQFT, which is isomorphic to the $U_q\mathrm{sl}(n, \mathbb{C})$-Reshetikhin-Turaev TQFT. A corollary of this will be that the quantum $\mathrm{SU}(n)$ representations are isomorphic to the ones that are part of the $U_q\mathrm{sl}(n, \mathbb{C})$-Reshetikhin-Turaev TQFT. Since it is well known that the Reshetikhin-Turaev TQFT is unitary one will get unitarity of the quantum $\mathrm{SU}(n)$ representations from this. We note that unitarity is not clear from the geometric construction of the quantum $\mathrm{SU}(n)$ representations. If the quantum $\mathrm{SU}(n)$ representations ρ_k are unitary, then we have for all $\phi \in \Gamma$ that

$$|\operatorname{Tr}(\rho_k(\phi))| \leq \dim \rho_k. \tag{1}$$

Assuming unitarity Theorem 1 implies the following:

COROLLARY 1. *Assume that n and d are coprime or that $(n, d) = (2, 0)$ when $g = 2$. Then equality holds in* (1) *for all k, if and only if*

$$\phi \in \begin{cases} \{1, H\} & g = 2, \;\; n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise.} \end{cases}$$

Furthermore, one will get that the norm of the Reshetikhin-Turaev quantum invariant for all k and $n = 2$ ($n = 3$ in the genus 2 case) can separate the mapping torus of the identity from the rest of the mapping tori as a purely TQFT consequence of Corollary 1.

In this paper we have initiated the program of using of the theory of Toeplitz operators on the moduli spaces in the study of TQFT's. The main insight behind the program is the relation among these Toeplitz operators and Hitchin's connection asymptotically in the quantum level k. Here we have presented the initial application of this program, namely the establishment of the asymptotic faithfulness property for the quantum representations of the mapping class groups. However this program can also be used to study other asymptotic properties of these TQFT's. In particular we have used them to establish that the quantum invariants for closed 3-manifolds have asymptotic expansions in k. Topological consequences of this are that certain classical topological properties are determined by the quantum invariants, resulting in interesting topological conclusions, including very strong knot theoretical corollaries. Writeup of these further developments is in progress.

It is also an interesting problem to understand how the Toeplitz operator constructions used in this paper are related to the deformation quantization of the moduli spaces described in [AMR1] and [AMR2]. In the abelian case, the resulting Berezin-Toeplitz deformation quantization was explicitly described in [A2] and it turns out to be equivalent to the one constructed in [AMR2].

This paper is organized as follows. In Section 2 we give the basic setup of applying geometric quantization to the moduli space to construct the Verlinde bundle over Teichmüller space. In Section 3 we review the construction of the connection in the Verlinde bundle. We end that section by stating the properties of the moduli space and the Verlinde bundle. There are only a few elementary properties about the moduli space, Teichmüller space and the general form of the connection in the Verlinde bundle really needed. In Section 4 we review the general results about Toeplitz operators on smooth compact Kähler manifolds used in the following Section 5, where we prove that the

Toeplitz operators for smooth functions on the moduli space give asymptotically flat sections of the endomorphism bundle of the Verlinde bundle. Finally, in Section 6 we prove the asymptotic faithfulness (Theorem 1 above).

After the completion of this work Freedman and Walker, together with Wang, found a proof of the asymptotic faithfulness property for the SU(2)-BHMV-representations which uses BHMV-technology. Their paper has already appeared [FWW] (see also [M2] for a discussion). As alluded to before, we are working jointly with K. Ueno to establish that these representations are equivalent to our sequence $\rho_k^{2,0}$. However, as long as this has not been established, our result is logically independent of theirs.

For the SU(2)-BHMV-representations it is already known by the work of Roberts [Ro], that they are irreducible for $k+2$ prime and that they have infinite image by the work of Masbaum [M1], except for a few low values of k.

We would like to thank Nigel Hitchin, Bill Goldman and Gregor Masbaum for valuable discussion. Further thanks are due to the Clay Mathematical Institute for their financial support and to the University of California, Berkeley for their hospitality, during the period when this work was completed.

2. The gauge theory construction of the Verlinde bundle

Let us now very briefly recall the construction of the Verlinde bundle. Only the details needed in this paper will be given. We refer to [H] for further details. As in the introduction we let Σ be a closed oriented surface of genus $g \geq 2$ and $p \in \Sigma$. Let P be a principal $\mathrm{SU}(n)$-bundle over Σ. Clearly, all such P are trivializable. As above let $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{\mathrm{SU}(n)}$. Throughout the rest of this paper we will assume that n and d are coprime, although in the case $g = 2$ we also allow $(n, d) = (2, 0)$. Let M be the moduli space of flat $\mathrm{SU}(n)$-connections in $P|_{\Sigma - p}$ with holonomy d around p. We can identify

$$M = \mathrm{Hom}_d(\tilde{\pi}_1(\Sigma), \mathrm{SU}(n))/\mathrm{SU}(n).$$

Here $\tilde{\pi}_1(\Sigma)$ is the universal central extension

$$0 \to \mathbb{Z} \to \tilde{\pi}_1(\Sigma) \to \pi_1(\Sigma) \to 1$$

as discussed in [H] and in [AB] and Hom_d means the space of homomorphisms from $\tilde{\pi}_1(\Sigma)$ to $\mathrm{SU}(n)$ which send the image of $1 \in \mathbb{Z}$ in $\tilde{\pi}_1(\Sigma)$ to d (see [H]). When n and d are coprime, M is a compact smooth manifold of dimension $m = (n^2 - 1)(g - 1)$. In general, when n and d are not coprime M is not smooth, except in the case where $g = 2$, $n = 2$ and $d = 0$. In this case M is in fact diffeomorphic to $\mathbb{CP}^3$. There is a natural homomorphism from the mapping class group to the outer automorphisms of $\tilde{\pi}_1(\Sigma)$; hence Γ acts on M.

We choose an invariant bilinear form $\{\cdot,\cdot\}$ on $\mathfrak{g} = \mathrm{Lie}(\mathrm{SU}(n))$, normalized such that $-\frac{1}{6}\{\vartheta \wedge [\vartheta \wedge \vartheta]\}$ is a generator of the image of the integer cohomology in the real cohomology in degree 3 of $\mathrm{SU}(n)$, where ϑ is the $\mathfrak{g}$-valued Maurer-Cartan 1-form on $\mathrm{SU}(n)$.

This bilinear form induces a symplectic form on M. In fact

$$T_{[A]}M \cong H^1(\Sigma, d_A),$$

where A is any flat connection in P representing a point in M and d_A is the induced covariant derivative in the associated adjoint bundle. Using this identification, the symplectic form on M is:

$$\omega(\varphi_1, \varphi_2) = \int_\Sigma \{\varphi_1 \wedge \varphi_2\},$$

where φ_i are d_A-closed 1-forms on Σ with values in $\operatorname{ad} P$. See e.g. [H] for further details on this. The natural action of Γ on M is symplectic.

Let $\mathcal{L}$ be the Hermitian line bundle over M and ∇ the compatible connection in $\mathcal{L}$ constructed by Freed [Fr]. This is the content of Corollary 5.22, Proposition 5.24 and equation (5.26) in [Fr] (see also the work of Ramadas, Singer and Weitsman [RSW]). By Proposition 5.27 in [Fr], the curvature of ∇ is $\frac{\sqrt{-1}}{2\pi}\omega$. We will also use the notation ∇ for the induced connection in $\mathcal{L}^k$, where k is any integer.

By an almost identical construction, we can lift the action of Γ on M to act on $\mathcal{L}$ such that the Hermitian connection is preserved (see e.g. [A1]). In fact, since $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(M, \mathbb{Z}) = 0$, it is clear that the action of Γ leaves the isomorphism class of $(\mathcal{L}, \nabla)$ invariant, thus alone from this one can conclude that a central extension of Γ acts on $(\mathcal{L}, \nabla)$ covering the Γ action on M. This is actually all we need in this paper, since we are only interested in the projectivized action.

Let now $\sigma \in \mathcal{T}$ be a complex structure on Σ. Let us review how σ induces a complex structure on M which is compatible with the symplectic structure on this moduli space. The complex structure σ induces a $*$-operator on 1-forms on Σ and via Hodge theory we get that

$$H^1(\Sigma, d_A) \cong \ker(d_A + *d_A*).$$

On this kernel, consisting of the harmonic 1-forms with values in $\operatorname{ad} P$, the $*$-operator acts and its square is -1; hence we get an almost complex structure on M by letting $I = I_\sigma = -*$. From a classical result by Narasimhan and Seshadri (see [NS1]), this actually makes M a smooth Kähler manifold, which as such, we denote M_σ. By using the $(0,1)$ part of ∇ in $\mathcal{L}$, we get an induced holomorphic structure in the bundle $\mathcal{L}$. The resulting holomorphic line bundle will be denoted $\mathcal{L}_\sigma$. See also [H] for further details on this.

From a more algebraic geometric point of view, we consider the moduli space of S-equivalence classes of semi-stable bundles of rank n and determinant isomorphic to the line bundle $\mathcal{O}(d[p])$. By using Mumford's geometric invariant theory, Narasimhan and Seshadri (see [NS2]) showed that this moduli space is a smooth complex algebraic projective variety which is isomorphic as a Kähler manifold to M_σ. Referring to [DN] we recall that

THEOREM 2 (Drezet & Narasimhan). *The Picard group of M_σ is generated by the holomorphic line bundle $\mathcal{L}_\sigma$ over M_σ constructed above*:

$$\mathrm{Pic}(M_\sigma) = \langle \mathcal{L}_\sigma \rangle.$$

Definition 1. The *Verlinde bundle* $\mathcal{V}_k$ over Teichmüller space is by definition the bundle whose fiber over $\sigma \in \mathcal{T}$ is $H^0(M_\sigma, \mathcal{L}_\sigma^k)$, where k is a positive integer.

3. The projectively flat connection

In this section we will review Axelrod, Della Pietra and Witten's and Hitchin's construction of the projective flat connection over Teichmüller space in the Verlinde bundle. We refer to [H] and [ADW] for further details.

Let $\mathcal{H}_k$ be the trivial $C^\infty(M, \mathcal{L}^k)$-bundle over $\mathcal{T}$ which contains $\mathcal{V}_k$, the Verlinde sub-bundle. If we have a smooth one-parameter family of complex structures σ_t on Σ, then that induces a smooth one-parameter family of complex structures on M, say I_t. In particular we get $\sigma_t' \in T_{\sigma_t}(\mathcal{T})$, which gives an $I_t' \in H^1(M_{\sigma_t}, T)$ (here T refers to the holomorphic tangent bundle of M_{σ_t}).

Suppose s_t is a corresponding smooth one-parameter family in $C^\infty(M, \mathcal{L}^k)$ such that $s_t \in H^0(M_{\sigma_t}, \mathcal{L}^k_{\sigma_t})$. By differentiating the equation

$$(1 + \sqrt{-1} I_t)\nabla s_t = 0,$$

we see that

$$\sqrt{-1} I_t' \nabla s + (1 + \sqrt{-1} I_t)\nabla s_t' = 0.$$

Hence, if we have an operator

$$u(v) : C^\infty(M, \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k)$$

for all real tangent vectors to Teichmüller space $v \in T(\mathcal{T})$, varying smoothly with respect to v, and satisfying

$$\sqrt{-1} I_t' \nabla^{1,0} s_t + \nabla^{0,1} u(\sigma_t')(s_t) = 0,$$

for all smooth curves σ_t in $\mathcal{T}$, then we get a connection induced in $\mathcal{V}_k$ by letting

$$\hat{\nabla}_v = \hat{\nabla}^t_v - u(v), \tag{2}$$

for all $v \in T(\mathcal{T})$, where $\hat{\nabla}^t$ is the trivial connection in $\mathcal{H}_k$.

In order to specify the particular u we are interested in, we use the symplectic structure on $\omega \in \Omega^{1,1}(M_\sigma)$ to define the tensor $G = G(v) \in \Omega^0(M_\sigma, S^2(T))$ by

$$v[I_\sigma] = G(v)\omega,$$

where $v[I_\sigma]$ means the derivative in the direction of $v \in T_\sigma(\mathcal{T})$ of the complex structure I_σ on M. Following Hitchin, we give an explicit formula for G in terms of $v \in T_\sigma(\mathcal{T})$:

The holomorphic tangent space to Teichmüller space at $\sigma \in \mathcal{T}$ is given by

$$T_\sigma^{1,0}(\mathcal{T}) \cong H^1(\Sigma_\sigma, K^{-1}).$$

Furthermore, the holomorphic co-tangent space to the moduli space of semi-stable bundles at the equivalence class of a stable bundle E is given by

$$T^*_{[E]}M_\sigma \cong H^0(\Sigma_\sigma, \mathrm{End}_0(E) \otimes K).$$

Thinking of $G(v) \in \Omega^0(M_\sigma, S^2(T))$ as a quadratic function on $T^* = T^*M_\sigma$, we have that

$$G(v)(\alpha, \alpha) = \int_\Sigma \mathrm{Tr}(\alpha^2) v^{(1,0)}$$

where $v^{(1,0)}$ is the image of v under the projection onto $T^{1,0}(\mathcal{T})$. From this formula it is clear that $G(v) \in H^0(M_\sigma, S^2(T))$ and that $\hat{\nabla}$ agrees with $\hat{\nabla}^t$ along the anti-holomorphic directions $T^{0,1}(\mathcal{T})$. From Proposition (4.4) in [H] we have that this map $v \mapsto G(v)$ from $T_\sigma(\mathcal{T})$ to $H^0(M_\sigma, S^2(T))$ is an isomorphism.

The particular $u(v)$ we are interested in is $u_{G(v)}$, where

$$u_G(s) = \frac{1}{2(k+n)}(\Delta_G - 2\nabla_{G\partial F} + \sqrt{-1}kf_G)s. \tag{3}$$

The leading order term Δ_G is the 2[nd] order operator given by

$$\begin{aligned}\Delta_G : C^\infty(M, \mathcal{L}^k) &\xrightarrow{\nabla^{1,0}} C^\infty(M, T^* \otimes \mathcal{L}^k) \xrightarrow{G} C^\infty(M, T \otimes \mathcal{L}^k) \\ &\xrightarrow{\nabla^{1,0}\otimes 1 + 1 \otimes \nabla^{1,0}} C^\infty(M, T^* \otimes T \otimes \mathcal{L}^k) \xrightarrow{\mathrm{Tr}} C^\infty(M, \mathcal{L}^k),\end{aligned}$$

where we have used the Chern connection in T on the Kähler manifold (M_σ, ω).

The function $F = F_\sigma$ is the Ricci potential uniquely determined as the real function with zero average over M, which satisfies the following equation

$$\mathrm{Ric}_\sigma = 2n\omega + 2\sqrt{-1}\partial\bar{\partial}F_\sigma. \tag{4}$$

We usually drop the subscript σ and think of F as a smooth map from $\mathcal{T}$ to $C^\infty(M)$.

The complex vector field $G\partial F \in C^\infty(M_\sigma, T)$ is simply just the contraction of G with $\partial F \in C^\infty(M_\sigma, T^*)$.

The function $f_G \in C^\infty(M)$ is defined by

$$f_G = -\sqrt{-1}v[F],$$

where v is determined by $G = G(v)$ and $v[F]$ means the derivative of F in the direction of v. We refer to [ADW] for this formula for f_G.

Theorem 3 (Axelrod, Della Pietra & Witten; Hitchin). *The expression* (2) *above defines a connection in the bundle* $\mathcal{V}_k$, *which induces a flat connection in* $\mathbb{P}(\mathcal{V}_k)$.

Faltings has established this theorem in the case where one replaces $\mathrm{SU}(n)$ with a general semisimple Lie group (see [Fal]).

We remark about genus 2, that [ADW] covers this case, but [H] excludes it; however, the work of Van Geemen and De Jong [vGdJ] extends Hitchin's approach to the genus 2 case.

As discussed in the introduction, we see by Laszlo's theorem that this particular connection is the relevant one to study.

It will be essential for us to consider the induced flat connection $\hat{\nabla}^e$ in the endomorphism bundle $\mathrm{End}(\mathcal{V}_k)$. Suppose Φ is a section of $\mathrm{End}(\mathcal{V}_k)$. Then for all sections s of $\mathcal{V}_k$ and all $v \in T(\mathcal{T})$ we have that

$$(\hat{\nabla}^e_v \Phi)(s) = \hat{\nabla}_v \Phi(s) - \Phi(\hat{\nabla}_v(s)).$$

Assume now that we have extended Φ to a section of $\mathrm{Hom}(\mathcal{H}_k, \mathcal{V}_k)$ over $\mathcal{T}$. Then

$$\hat{\nabla}^e_v \Phi = \hat{\nabla}^{e,t}_v \Phi - [\Phi, u(v)] \tag{5}$$

where $\hat{\nabla}^{e,t}$ is the trivial connection in the trivial bundle $\mathrm{End}(\mathcal{H}_k)$ over $\mathcal{T}$.

Let us end this section by summarizing the properties we use about the moduli space in Section 5 to prove Theorem 6, which in turn implies Theorem 1:

The moduli space M is a finite dimensional smooth compact manifold with a symplectic structure ω, a Hermitian line bundle $\mathcal{L}$ and a compatible connection ∇, whose curvature is $\frac{\sqrt{-1}}{2\pi}\omega$. Teichmüller space $\mathcal{T}$ is a smooth connected finite dimensional manifold, which smoothly parametrizes Kähler structures I_σ, $\sigma \in \mathcal{T}$, on (M, ω). For any positive integer k, we have inside the trivial bundle $\mathcal{H}_k = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ the finite dimensional subbundle $\mathcal{V}_k$, given by

$$\mathcal{V}_k(\sigma) = H^0(M_\sigma, \mathcal{L}^k_\sigma)$$

for $\sigma \in \mathcal{T}$. We have a connection in $\mathcal{V}_k$ given by

$$\hat{\nabla}_v = \hat{\nabla}^t_v - u(v)$$

where $\hat{\nabla}^t_v$ is the trivial connection in $\mathcal{H}_k$ and $u(v)$ is the second order differential operator $u_{G(v)}$ given in (3). All we will need about the operator $\Delta_G - 2\nabla_{G\partial F}$ is that there is some finite set of vector fields $X_r(v), Y_r(v), Z(v) \in C^\infty(M_\sigma, T)$, $r = 1, \dots, R$ (where $v \in T_\sigma(\mathcal{T})$), all varying smoothly[1] with $v \in T(\mathcal{T})$, such

[1]This makes sense when we consider the holomorphic tangent bundle T of M_σ inside the complexified real tangent bundle $TM \otimes \mathbb{C}$ of M.

that

$$\Delta_{G(v)} - 2\nabla_{G(v)\partial F} = \sum_{r=1}^{R} \nabla_{X_r(v)} \nabla_{Y_r(v)} + \nabla_{Z(v)}. \tag{6}$$

This follows immediately from the definition of $\Delta_{G(v)}$. From this we have the expression

$$u(v) = \frac{1}{2(k+n)} \left(\sum_{r=1}^{R} \nabla_{X_r(v)} \nabla_{Y_r(v)} + \nabla_{Z(v)} + nv[F] \right) - \frac{1}{2} v[F]. \tag{7}$$

All we need to use about $F : \mathcal{T} \to C^\infty(M)$ is that it is a smooth function, such that F_σ is real-valued on M for all $\sigma \in \mathcal{T}$.

4. Toeplitz operators on compact Kähler manifolds

In this section (N^{2m}, ω) will denote a compact Kähler manifold on which we have a holomorphic line bundle L admitting a compatible Hermitian connection whose curvature is $\frac{\sqrt{-1}}{2\pi}\omega$. On $C^\infty(N, L^k)$ we have the L_2-inner product:

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_N (s_1, s_2) \omega^m$$

where $s_1, s_2 \in C^\infty(N, L^k)$ and $(\cdot,\cdot)$ is the fiberwise Hermitian structure in L^k. This L_2-inner product gives the orthogonal projection

$$\pi : C^\infty(N, L^k) \to H^0(N, L^k).$$

For each $f \in C^\infty(N)$ consider the associated *Toeplitz operator* $T_f^{(k)}$ given as the composition of the multiplication operator $M_f : H^0(N, L^k) \to C^\infty(N, L^k)$ with the orthogonal projection $\pi : C^\infty(N, L^k) \to H^0(N, L^k)$, so that

$$T_f^{(k)}(s) = \pi(fs).$$

Since the multiplication operator is a zero-order differential operator, $T_f^{(k)}$ is a zero-order Toeplitz operator. Sometimes we will suppress the superscript (k) and just write $T_f = T_f^{(k)}$.

Let us here give an explicit formula for π: Let $h_{ij} = \langle s_i, s_j \rangle$, where s_i is a basis of $H^0(N, L^k)$. Let h_{ij}^{-1} be the inverse matrix of h_{ij}. Then

$$\pi(s) = \sum_{i,j} \langle s, s_i \rangle h_{ij}^{-1} s_j. \tag{8}$$

This formula will be useful when we have to compute the derivative of π along a one-parameter curve of complex structures on the moduli space.

Suppose we have a smooth section $X \in C^\infty(N,TN)$ of the holomorphic tangent bundle of N. We then claim that the operator $\pi\nabla_X$ is a zero-order Toeplitz operator. Suppose $s_1 \in C^\infty(N,L^k)$ and $s_2 \in H^0(N,L^k)$; then

$$X(s_1,s_2) = (\nabla_X s_1, s_2).$$

Now, calculating the Lie derivative along X of $(s_1,s_2)\omega^m$ and using the above, one obtains after integration that

$$\langle \nabla_X s_1, s_2\rangle = -\langle \Lambda d(i_X\omega)s_1, s_2\rangle,$$

where Λ denotes contraction with ω. Thus

$$\pi\nabla_X = T^{(k)}_{f_X}, \tag{9}$$

as operators from $C^\infty(N,L^k)$ to $H^0(N,L^k)$, where $f_X = -\Lambda d(i_X\omega)$. Iterating this, we find for all $X_1, X_2 \in C^\infty(TN)$ that

$$\pi\nabla_{X_1}\nabla_{X_2} = T^{(k)}_{f_{X_2}f_{X_1} - X_2(f_{X_1})} \tag{10}$$

again as operators from $C^\infty(N,L^k)$ to $H^0(N,L^k)$.

For $X \in C^\infty(N,TN)$, the complex conjugate vector field $\bar{X} \in C^\infty(N,\bar{T}N)$ is a section of the antiholomorphic tangent bundle, and for $s_1, s_2 \in C^\infty(N,L^k)$, we have that

$$\bar{X}(s_1,s_2) = (\nabla_{\bar{X}} s_1, s_2) + (s_1, \nabla_X s_2).$$

Computing the Lie derivative along $\bar{X}$ of $(s_1,s_2)\omega^m$ and integrating, we get that

$$\langle \nabla_{\bar{X}} s_1, s_2\rangle + \langle (\nabla_X)^* s_1, s_2\rangle = \langle \Lambda d(i_{\bar{X}}\omega)s_1, s_2\rangle.$$

Hence we see that

$$(\nabla_X)^* = -\left(\nabla_{\bar{X}} + f_{\bar{X}}\right)$$

as operators on $C^\infty(N,L^k)$. In particular, we see that

$$\pi(\nabla_X)^*\pi = -T_{f_{\bar{X}}}|_{H^0(N,L^k)} : H^0(N,L^k) \to H^0(N,L^k). \tag{11}$$

For two smooth sections X_1, X_2 of the holomorphic tangent bundle TN and a smooth function $h \in C^\infty(N)$, we deduce from the formula for $(\nabla_X)^*$ that

$$\begin{aligned} \pi(\nabla_{X_1})^*(\nabla_{X_2})^* h\pi = {} & \pi\bar{X}_1\bar{X}_2(h)\pi \\ & + \pi f_{\bar{X}_1}\bar{X}_2(h)\pi + \pi f_{\bar{X}_2}\bar{X}_1(h)\pi \\ & + \pi\bar{X}_1(f_{\bar{X}_2})h\pi + \pi f_{\bar{X}_1}f_{\bar{X}_2}h\pi \end{aligned} \tag{12}$$

as operators on $H^0(N,L^k)$.

We need the following theorems on Toeplitz operators. The first is due to Bordemann, Meinrenken and Schlichenmaier (see [BMS]). The L_2-inner product on $C^\infty(N,L^k)$ induces an inner product on $H^0(N,L^k)$, which in turn induces the operator norm $\|\cdot\|$ on $\mathrm{End}(H^0(N,L^k))$.

Theorem 4 (Bordemann, Meinrenken and Schlichenmaier). *For any* $f \in C^\infty(N)$,

$$\lim_{k\to\infty} \|T_f^{(k)}\| = \sup_{x\in N} |f(x)|.$$

Since the association of the sequence of Toeplitz operators T_f^k, $k \in \mathbb{Z}_+$, is linear in f, we see from this theorem, that this association is faithful.

Theorem 5 (Schlichenmaier). *For any pair of smooth functions* $f_1, f_2 \in C^\infty(N)$, *there is an asymptotic expansion*

$$T_{f_1}^{(k)} T_{f_2}^{(k)} \sim \sum_{l=0}^{\infty} T_{c_l(f_1,f_2)}^{(k)} k^{-l},$$

where $c_l(f_1, f_2) \in C^\infty(N)$ *are uniquely determined since* $\sim$ *means the following*: *For all* $L \in \mathbb{Z}_+$,

$$\|T_{f_1}^{(k)} T_{f_2}^{(k)} - \sum_{l=0}^{L} T_{c_l(f_1,f_2)}^{(k)} k^{-l}\| = O(k^{-(L+1)}). \tag{13}$$

Moreover, $c_0(f_1, f_2) = f_1 f_2$.

This theorem was proved in [Sch] and is published in [Sch1] and [Sch2], where it is also proved that the formal generating series for the $c_l(f_1, f_2)$'s gives a formal deformation quantization of the Poisson structure on N induced from ω. By examining the proof in [Sch] (or in [Sch1] and [Sch2]) of this theorem, one observes that for continuous families of functions, the estimates in Theorem 5 are uniform over compact parameter spaces.

5. Toeplitz operators on moduli space and the projective flat connection

Let $f \in C^\infty(M)$ be a smooth function on the moduli space. We consider $T_f^{(k)}$ as a section of the endomorphism bundle $\mathrm{End}(\mathcal{V}_k)$. The flat connection $\hat\nabla$ in the projective bundle $\mathbb{P}(\mathcal{V}_k)$ induces the flat connection $\hat\nabla^e$ in the endomorphism bundle $\mathrm{End}(\mathcal{V}_k)$ as described in Section 3. We shall now establish that the sections $T_f^{(k)}$ are in a certain sense asymptotically flat by proving the following theorem.

Theorem 6. *Let* σ_0 *and* σ_1 *be two points in Teichmüller space and* P_{σ_0,σ_1} *be the parallel transport in the flat bundle* $\mathrm{End}(\mathcal{V}_k)$ *from* σ_0 *to* σ_1. *Then*

$$\|P_{\sigma_0,\sigma_1} T_{f,\sigma_0}^{(k)} - T_{f,\sigma_1}^{(k)}\| = O(k^{-1}),$$

where $\|\cdot\|$ *is the operator norm on* $H^0(M_{\sigma_1}, \mathcal{L}_{\sigma_1}^k)$.

In the proof of this theorem we will make use of the following Hermitian structure on $\mathcal{H}_k$:

$$\langle s_1, s_2\rangle_F = \frac{1}{m!}\int_M (s_1, s_2)e^{-F}\omega^m, \tag{14}$$

where we recall that $F = F_\sigma$ is the Ricci potential, which is a real smooth function on M_σ for each $\sigma \in \mathcal{T}$ determined by equation (4). In Lemma 1 below we will see that $\langle\cdot,\cdot\rangle_F$ is uniformly equivalent to the constant L_2-Hermitian structure on $\mathcal{H}_k$, when both are restricted to $\mathcal{V}_k$ over any compact subset of $\mathcal{T}$. The constant L_2-Hermitian structure on $\mathcal{H}_k$ is not asymptotically flat with respect to $\hat{\nabla}$, but Proposition 2 below shows that the Hermitian structure $\langle\cdot,\cdot\rangle_F$ restricted to $\mathcal{V}_k$ is asymptotically flat with respect to $\hat{\nabla}$. It therefore induces a Hermitian structure on $\mathcal{V}_k^*\otimes\mathcal{V}_k$ (which we also denote $\langle\cdot,\cdot\rangle_F$), which is asymptotically flat with respect to $\hat{\nabla}^e$.

This suggests that one consider the smooth function

$$t \mapsto |P_{\sigma_0,\sigma_t}T^{(k)}_{f,\sigma_0} - T^{(k)}_{f,\sigma_t}|_F$$

and establishes an $O(k^{-1})$ estimate for its derivative uniformly over J, since by Lemma 1 and the first inequality in (17) below, $O(k^{-1})$ control on

$$|P_{\sigma_0,\sigma_1}T^{(k)}_{f,\sigma_0} - T^{(k)}_{f,\sigma_1}|_F$$

implies Theorem 6. An $O(k^{-1})$ estimate on $|\hat{\nabla}^e_{\sigma_t'}T^{(k)}_{f,\sigma_t}|_F$ uniformly over J would imply this estimate; however we are only able to establish that $\|\hat{\nabla}^e_{\sigma_t'}T^{(k)}_{f,\sigma_t}\|$ is $O(k^{-1})$ uniformly over the interval, which is considerably weaker because of the $\left(\frac{m}{2}\right)^{\text{th}}$ power of k in the second inequality in (17) below.

We shall therefore perturb the function f by adding on sufficiently many terms of the form $(h_l)_t k^{-l}$ ($l = 1, \dots, r > m/2$), where $(h_l)_t \in C^\infty(M)$, $t \in J$, so as to obtain a smooth one-parameter family of functions

$$(f_r)_t = f + \sum_{l=1}^{r}(h_l)_t k^{-l}.$$

The $(h_l)_t$'s are determined inductively in l such that $\|\hat{\nabla}^e_{\sigma_t'}T^{(k)}_{(f_r)_t,\sigma_t}\|$ is $O(k^{-r-1})$ uniformly over J. This is the content of Proposition 1 below. That estimate on the covariant derivative of the Toeplitz operator $T^{(k)}_{(f_r)_t,\sigma_t}$ will allow us to prove that $\|P_{\sigma_0,\sigma_1}T^{(k)}_{(f_r)_0,\sigma_0} - T^{(k)}_{(f_r)_1,\sigma_1}\|$ is $O(k^{-1})$ by analyzing the derivative of

$$t \mapsto |P_{\sigma_0,\sigma_t}T^{(k)}_{(f_r)_0,\sigma_0} - T^{(k)}_{(f_r)_t,\sigma_t}|_F^2.$$

Since we can arrange that $(f_r)_0 = f$ and since $\|T^{(k)}_{(f_r)_1,\sigma_1} - T^{(k)}_{f,\sigma_1}\|$ is $O(k^{-1})$, this will allow us to prove Theorem 6.

First however, we need to establish a useful formula for the derivative of the orthogonal projection π along the curve σ_t. To this end, consider a basis of covariant constant sections[2] $s_i = (s_i)_t$, $i = 1, \dots, \operatorname{Rank} \mathcal{V}_k$, of $\mathcal{V}_k$ over the curve σ_t:

$$s_i' = u_G(s_i), \qquad i = 1, \dots, \operatorname{Rank} \mathcal{V}_k.$$

Recall formula (8) for the projection $\pi : C^\infty(M, \mathcal{L}^k) \to H^0(M_{\sigma_t}, \mathcal{L}^k_{\sigma_t})$ and compute the derivative along σ_t: For any fixed $s \in C^\infty(M, \mathcal{L}^k)$, we have that

$$\pi'(s) = \sum_{i,j} \langle s, s_i' \rangle h_{ij}^{-1} s_j + \sum_{i,j} \langle s, s_i \rangle (h_{ij}^{-1})' s_j + \sum_{i,j} \langle s, s_i \rangle h_{ij}^{-1} s_j'.$$

An easy computation gives that

$$(h_{ij}^{-1})' = -\sum_{l,r} h_{il}^{-1} (\langle s_l', s_r \rangle + \langle s_l, s_r' \rangle) h_{rj}^{-1},$$

so that

$$\begin{aligned} \pi\pi'(s) &= \sum_{i,j} \langle u_G^* s, s_i \rangle h_{ij}^{-1} s_j - \sum_{i,l,m,j} \langle s, s_i \rangle h_{il}^{-1} \langle s_l, s_m' \rangle h_{mj}^{-1} s_j \\ &= \pi u_G^*(s) - \sum_{m,j} \langle \pi s, s_m' \rangle h_{mj}^{-1} s_j = \pi u_G^*(s) - \pi u_G^* \pi(s). \end{aligned}$$

Hence we conclude that

$$\pi\pi' = \pi u_G^* - \pi u_G^* \pi. \tag{15}$$

Having derived the formula for the derivative of π, we now proceed to construct the needed perturbation of f. Let r be a nonnegative integer and let

$$(f_r)_t = \sum_{i=0}^{r} (h_i)_t k^{-i},$$

where the $(h_i)_t$, $t \in J$, for now are arbitrary, smooth, one-parameter families of smooth functions on M; however, we will fix $(h_0)_t = f$ for all $t \in J$. We have that $T_{f_r}^{(k)}$ is a section of $\operatorname{End}(\mathcal{V}_k)$ over the curve σ_t. According to formula (5),

$$\hat{\nabla}^e_{\sigma'}(T_{f_r}) = (T_{f_r})' - [u_G, T_{f_r}] = \pi f_r' + \pi' f_r - [u_G, \pi f_r].$$

Since $\hat{\nabla}^e_{\sigma'}(T_{f_r})$ is a section of $\operatorname{End}(\mathcal{V}_k)$, we have of course that

$$\pi \hat{\nabla}^e_{\sigma'}(T_{f_r}) \pi = \hat{\nabla}^e_{\sigma'}(T_{f_r}) \pi : H^0(M_{\sigma_t}, \mathcal{L}^k_{\sigma_t}) \to H^0(M_{\sigma_t}, \mathcal{L}^k_{\sigma_t}).$$

[2] We will from now on mostly suppress the subscript t on various quantities defined along the curve σ_t.

Proposition 1. *Given f and a nonnegative integer r, there exist unique smooth one-parameter families of functions $(h_i)_t \in C^\infty(M)$, $i = 1, \ldots, r$ and $t \in J$ such that*

$$\sup_{t\in J} \|\hat{\nabla}^e_{\sigma'}(T_{f_r})\| = O(k^{-r-1}), \tag{16}$$

and $(h_i)_0 = 0$, $i = 1, \ldots, r$.

Proof. For any choice of h_i's, $i = 1, \ldots, r$, we compute using (15) that

$$\begin{aligned}\pi\hat{\nabla}^e_{\sigma'}(T_{f_r})\pi = &\sum_{i=1}^{r} \pi h_i' \pi k^{-i}\\ &+ \sum_{i=0}^{r} (\pi u_G^* h_i \pi - \pi u_G^* \pi h_i \pi) k^{-i}\\ &- \sum_{i=0}^{r} (\pi u_G \pi h_i \pi - \pi h_i u_G \pi) k^{-i}.\end{aligned}$$

Using (6) together with (9) and (10) we define the function $H_G \in C^\infty(M)$ independent of k, as follows:

$$\pi(\Delta_G - 2\nabla_{G\partial F}) = \pi H_G.$$

We also define $H_i \in C^\infty(M)$ which depend on h_i and G but are independent of k, by

$$\pi h_i(\Delta_G - 2\nabla_{G\partial F}) = \pi H_i.$$

Using (11) and (12) we further define the function $H_G^* \in C^\infty(M)$ independent of k, as follows:

$$\pi(\Delta_G^* - 2(\nabla_{G\partial F})^*)\pi = \pi H_G^* \pi.$$

And we define $H_i^* \in C^\infty(M)$ which depend on h_i and G but are independent of k, by

$$\pi(\Delta_G^* - 2(\nabla_{G\partial F})^*) h_i \pi = \pi H_i^* \pi.$$

Then we have that

$$\begin{aligned}\pi\hat{\nabla}^e_{\sigma'}(T_{f_r})\pi = &\sum_{i=1}^{r} \pi h_i' \pi k^{-i}\\ &- \sum_{i=0}^{r} \frac{1}{2(k+n)} (\pi H_G^* \pi h_i \pi - \pi H_i^* \pi) k^{-i}\\ &- \sum_{i=0}^{r} \frac{1}{2(k+n)} (\pi H_G \pi h_i \pi - \pi H_i \pi) k^{-i}\\ &+ \sum_{i=0}^{r} \frac{k}{2(k+n)} \sqrt{-1} (\pi \overline{f}_G \pi h_i \pi - \pi \overline{f}_G h_i \pi) k^{-i}\\ &- \sum_{i=0}^{r} \frac{k}{2(k+n)} \sqrt{-1} (\pi f_G \pi h_i \pi - \pi h_i f_G \pi) k^{-i}.\end{aligned}$$

Because of (13) and Theorem 4, we see for any r, that condition (16) is equivalent to

$$\begin{aligned}\sup_{t\in J}\|\sum_{i=1}^{r}\pi h_i'\pi k^{-i}\\ &-\frac{1}{2k}\sum_{i=0}^{m}\sum_{j=0}^{r}(-1)^j n^j(\sum_{l=0}^{r}\pi c_l(H_G^*,h_i)\pi k^{-l}-\pi H_i^*\pi)k^{-i-j}\\ &-\frac{1}{2k}\sum_{i=0}^{r}\sum_{j=0}^{r}(-1)^j n^j(\sum_{l=0}^{r}\pi c_l(H_G,h_i)\pi k^{-l}-\pi H_i\pi)k^{-i-j}\\ &+\frac{\sqrt{-1}}{2k}\sum_{i=0}^{r}\sum_{j=0}^{r}\sum_{l=1}^{r}(-1)^j n^j\pi c_l(\overline{f}_G,h_i)\pi k^{-l-i-j+1}\\ &-\frac{\sqrt{-1}}{2k}\sum_{i=0}^{r}\sum_{j=0}^{r}\sum_{l=1}^{r}(-1)^j n^j\pi c_l(f_G,h_i)\pi k^{-l-i-j+1}\|=O(k^{-r-1}).\end{aligned}$$

Note that the expression inside the norm is a polynomial in k^{-1}, say $\sum_{i=1}^{N_r}\pi(C_i)_t\pi k^{-i}$, where each coefficient $\pi(C_i)_t\pi$ is the Toeplitz operator associated to a smooth one-parameter family $(C_i)_t\in C^\infty(M)$, $t\in J$. We also note that the functions C_i, $i=1,\dots,r$ do not depend on r and in the same range for i they are all of the following form:

$$C_i=h_i'-D_i$$

where $(D_i)_t\in C^\infty(M)$, $t\in J$, is a linear combination of

$$H_j^*,H_j,c_l(H_G^*,h_j),c_l(H_G,h_j),c_l(\bar f_G,h_j),c_l(f_G,h_j)\in C^\infty(J,C^\infty(M)),$$

for $j=0,\dots,i-1$ and $l=0,\dots,i$; e.g. for $i=1$

$$D_1=-\frac{1}{2}(H_G^*h_0-H_0^*-H_Gh_0-H_0-\sqrt{-1}c_1(\bar f_G,h_0)+\sqrt{-1}c_1(f_G,h_0)).$$

Now observe by Theorem 4 that condition (16) is equivalent to $C_i=0$ for $i=1,\dots r$. From this it follows that we can inductively uniquely determine the functions h_i. First of all, since the coefficient of k^0 is zero, we see that (16) holds for $r=0$. Now assume that we have determined $h_1,\dots h_{r-1}$ uniquely such that $(h_i)_0=0$ and

$$h_i'=D_i,$$

for $i=1,\dots,r-1$. By the above we then observe that condition (16) holds if and only if $C_r=0$, which means if and only if

$$h_r'=D_r.$$

But since we require $(h_r)_0=0$, we must have that

$$(h_r)_t=\int_0^t(D_r)_s ds. \qquad \square$$

Lemma 1. *The Hermitian structure on $\mathcal{H}_k$*

$$\langle s_1, s_2\rangle_F = \frac{1}{m!}\int_M (s_1,s_2)e^{-F}\omega^m$$

and the constant L_2-Hermitian structure on $\mathcal{H}_k$

$$\langle s_1, s_2\rangle = \frac{1}{m!}\int_M (s_1,s_2)\omega^m$$

are equivalent uniformly in k when restricted to $\mathcal{V}_k$ over any compact subset K of $\mathcal{T}$.

Proof. We clearly have that

$$|s|_F^2 \leq \|T^{(k)}_{e^{-F}}\|\ |s|^2,$$

so that by Theorem 4, there exists a constant C (depending on K) such that

$$|s|_F \leq C|s|,$$

for all k. Conversely, we have that

$$\begin{aligned}|s|^2 &= \langle \pi e^{\frac{1}{2}F}e^{-\frac{1}{2}F}\pi s, s\rangle\\ &\leq |\langle(\pi e^{\frac{1}{2}F}e^{-\frac{1}{2}F}\pi - \pi e^{\frac{1}{2}F}\pi e^{-\frac{1}{2}F}\pi)s, s\rangle| + |\langle \pi e^{\frac{1}{2}F}\pi e^{-\frac{1}{2}F}\pi s, s\rangle|\\ &\leq \|\pi e^{\frac{1}{2}F}e^{-\frac{1}{2}F}\pi - \pi e^{\frac{1}{2}F}\pi e^{-\frac{1}{2}F}\pi\|\ |s|^2 + \|\pi e^{\frac{1}{2}F}\pi\|\ |\pi e^{-\frac{1}{2}F}\pi s|\ |s|.\end{aligned}$$

By Theorems 4 and 5 we see there exist constants C' and C'' (again depending on K) such that

$$|s| \leq \frac{C'}{k}|s| + C''|s|_F.$$

But then we have for all sufficiently large k that

$$|s| \leq 2C''|s|_F.$$

Hence we have established the claimed equivalence. □

Along any smooth one-parameter family of complex structures σ_t,

$$\frac{d}{dt}\langle s_1,s_2\rangle_F = \langle \hat{\nabla}^t_{\sigma_t'}s_1, s_2\rangle_F + \langle s_1, \hat{\nabla}^t_{\sigma_t'}s_2\rangle_F - \langle \frac{\partial F}{\partial t}s_1, s_2\rangle_F.$$

So, if we let

$$E(s) = \frac{d}{dt}|s|_F^2 - \langle \hat{\nabla}_{\sigma_t'}s, s\rangle_F - \langle s, \hat{\nabla}_{\sigma_t'}s\rangle_F,$$

recalling that $\sqrt{-1}f_G = \frac{\partial F}{\partial t}$, $\hat{\nabla}_v = \hat{\nabla}^t_v - u(v)$ and formula (7), we have for all sections s of $\mathcal{V}_k$ that

$$\begin{aligned}E(s) = \frac{1}{2(k+n)}\Big(&\langle \pi e^{-F}(\Delta_G - 2\nabla_{G\partial F} - \sqrt{-1}nf_G)s, s\rangle\\ &+ \langle s, \pi e^{-F}(\Delta_G - 2\nabla_{G\partial F} - \sqrt{-1}nf_G)s\rangle\Big).\end{aligned}$$

Hence by combining Theorem 4, (6), (9) and (10) we have proved

PROPOSITION 2. *The Hermitian structure* (14) *is asymptotically flat with respect to the connections* $\hat{\nabla}$, *i.e. for any compact subset* K *of* $\mathcal{T}$, *there exists a constant* C *such that for all sections* s *of* $\mathcal{V}_k$ *over* K,

$$|E(s)| \leq \frac{C}{k+n}|s|_F^2$$

over K.

We note that this proposition implies the same proposition for sections of $\operatorname{End}(\mathcal{V}_k)$ with respect to the induced Hermitian structure on $\operatorname{End}(\mathcal{V}_k) = \mathcal{V}_k^* \otimes \mathcal{V}_k$, which we also denote $\langle \cdot, \cdot \rangle_F$. We denote the analogous quantity of E for the endomorphism bundle by E_e.

Proof of Theorem 6. Let σ_t, $t \in J$ be a smooth one-parameter family of complex structures such that σ_t is a curve in $\mathcal{T}$ between the two points in question. By Lemma 1 the Hermitian structures $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_F$ on $\mathcal{V}_k$ are equivalent uniformly in k over compact subsets of $\mathcal{T}$. Then there exist constants C_1 and C_2, such that we have the following inequalities over the image of σ_t in $\mathcal{T}$ for the operator norm $\|\cdot\|$ and the norm $|\cdot|_F$ on $\operatorname{End}(\mathcal{V}_k)$:

$$\|\cdot\| \leq C_1 |\cdot|_F \leq C_2 \sqrt{P_{g,n}(k)} \|\cdot\|, \tag{17}$$

where $P_{g,n}(k)$ is the rank of $\mathcal{V}_k$ given by the Verlinde formula. By the Riemann-Roch theorem this is a polynomial in k of degree m.

Because of these inequalities, we choose an integer r bigger than $m/2$ and let f_r be as provided by Proposition 1 and we define $n_k : J \to [0, \infty)$ by

$$n_k(t) = |\Theta_k(t)|_F^2$$

where

$$\Theta_k(t) : \mathcal{V}_{k,\sigma_t} \to \mathcal{V}_{k,\sigma_t}$$

is given by

$$\Theta_k(t) = P_{\sigma_0,\sigma_t} T^{(k)}_{(f_r)_0,\sigma_0} - T^{(k)}_{(f_r)_t,\sigma_t}.$$

The functions n_k are differentiable in t and we compute that

$$\begin{aligned}
\frac{dn_k}{dt} &= \langle \hat{\nabla}^e_{\sigma_t'}(\Theta_k(t)), \Theta_k(t) \rangle_F + \langle \Theta_k(t), \hat{\nabla}^e_{\sigma_t'}(\Theta_k(t)) \rangle_F + E_e(\Theta_k(t)) \\
&= -\langle \hat{\nabla}^e_{\sigma_t'} T^{(k)}_{(f_r)_t,\sigma_t}, \Theta_k(t) \rangle_F - \langle \Theta_k(t), \hat{\nabla}^e_{\sigma_t'} T^{(k)}_{(f_r)_t,\sigma_t} \rangle_F + E_e(\Theta_k(t)).
\end{aligned}$$

Using the above, we get the following estimate

$$\begin{aligned}
\left|\frac{dn_k}{dt}\right| &\leq 2 |\hat{\nabla}^e_{\sigma_t'} T^{(k)}_{(f_r)_t,\sigma_t}|_F |\Theta_k(t)|_F + |E_e(\Theta_k(t))| \\
&\leq 2C_2 \sqrt{P_{g,n}(k)} \|\hat{\nabla}^e_{\sigma_t'} T^{(k)}_{(f_r)_t,\sigma_t}\| n_k^{1/2} + |E_e(\Theta_k(t))|.
\end{aligned}$$

Consequently we can apply Propositions 1 and 2 to obtain a constant C such that

$$|\frac{dn_k}{dt}| \leq \frac{C}{k}(n_k^{1/2} + n_k).$$

This estimate implies that

$$n_k(t) \leq (\exp(\frac{Ct}{2k}) - 1)^2.$$

But by (17) we get that

$$\|P_{\sigma_0,\sigma_1} T^{(k)}_{(f_r)_0,\sigma_0} - T^{(k)}_{(f_r)_1,\sigma_1}\| = \|\Theta_k(1)\| \leq C_1 n_k(1)^{1/2}.$$

The theorem then follows from these two estimates, since $(f_r)_0 = f$ and

$$\|T^{(k)}_{(f_r)_1,\sigma_1} - T^{(k)}_{f,\sigma_1}\| = O(k^{-1}). \qquad \square$$

6. Asymptotic faithfulness

Recall that the flat connection in the bundle $\mathbb{P}(\mathcal{V}_k)$ gives the projective representation of the mapping class group

$$\rho_k : \Gamma \to \operatorname{Aut}(\mathbb{P}(V_k)),$$

where $\mathbb{P}(V_k)$ = covariant constant sections of $\mathbb{P}(\mathcal{V}_k)$ over Teichmüller space.

THEOREM 7. *For any $\phi \in \Gamma$,*

$$\phi \in \bigcap_{k=1}^{\infty} \ker \rho_k$$

if and only if ϕ induces the identity on M.

Proof. Suppose we have a $\phi \in \Gamma$. Then ϕ induces a symplectomorphism of M which we also just denote ϕ and we get the following commutative diagram for any $f \in C^\infty(M)$

$$\begin{array}{ccccc}
H^0(M_\sigma, \mathcal{L}^k_\sigma) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}^k_{\phi(\sigma)}) & \xrightarrow{P_{\phi(\sigma),\sigma}} & H^0(M_\sigma, \mathcal{L}^k_\sigma) \\
\Big\downarrow{\scriptstyle T^{(k)}_{f,\sigma}} & & \Big\downarrow{\scriptstyle T^{(k)}_{f\circ\phi,\phi(\sigma)}} & & \Big\downarrow{\scriptstyle P_{\phi(\sigma),\sigma} T^{(k)}_{f\circ\phi,\phi(\sigma)}} \\
H^0(M_\sigma, \mathcal{L}^k_\sigma) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}^k_{\phi(\sigma)}) & \xrightarrow{P_{\phi(\sigma),\sigma}} & H^0(M_\sigma, \mathcal{L}^k_\sigma),
\end{array}$$

where $P_{\phi(\sigma),\sigma} : H^0(M_{\phi(\sigma)}, \mathcal{L}^k_{\phi(\sigma)}) \to H^0(M_\sigma, \mathcal{L}^k_\sigma)$ on the horizontal arrows refer to parallel transport in the Verlinde bundle itself, whereas $P_{\phi(\sigma),\sigma}$ refers to the parallel transport in the endomorphism bundle $\operatorname{End}(\mathcal{V}_k)$ in the last vertical arrow. Suppose now $\phi \in \bigcap_{k=1}^{\infty} \ker \rho_k$; then $P_{\phi(\sigma),\sigma} \circ \phi^* = \rho_k(\phi) \in \mathbb{C}\,\mathrm{Id}$ and we get that $T^{(k)}_{f,\sigma} = P_{\phi(\sigma),\sigma} T^{(k)}_{f\circ\phi,\phi(\sigma)}$. By Theorem 6,

$$\begin{aligned}\lim_{k\to\infty} \|T^{(k)}_{f-f\circ\phi,\sigma}\| &= \lim_{k\to\infty} \|T^{(k)}_{f,\sigma} - T^{(k)}_{f\circ\phi,\sigma}\| \\ &= \lim_{k\to\infty} \|P_{\phi(\sigma),\sigma} T^{(k)}_{f\circ\phi,\phi(\sigma)} - T^{(k)}_{f\circ\phi,\sigma}\| = 0.\end{aligned}$$

By Bordemann, Meinrenken and Schlichenmaier's Theorem 4, we must have that $f = f \circ \phi$. Since this holds for any $f \in C^\infty(M)$, we must have that ϕ acts by the identity on M. □

Proof of Theorem 1. Our main Theorem 1 now follows directly from Theorem 7, since it is known that the only element of Γ, which acts by the identity on the moduli space M is the identity, if $g > 2$. If $g = 2$, $n = 2$ and d is even, it is contained in the sub-group generated by the hyper-elliptic involution, else it is also the identity.

A way to see this using the moduli space of flat $\mathrm{SL}(n,\mathbb{C})$-connections goes as follows: M is a component of the real slice in $\mathcal{M}$, the moduli space of flat $\mathrm{SL}(n,C)$-connections on $\Sigma - p$, whose holonomy around p has trace $n\exp(2\pi\sqrt{-1}\frac{d}{n})$. Hence if ϕ acts by the identity on M, it will also act by the identity on an open neighbourhood of M in $\mathcal{M}$, since it acts holomorphically on $\mathcal{M}$. But since $\mathcal{M}$ is connected, ϕ must act by the identity on the entire $\mathrm{SL}(n,\mathbb{C})$-moduli space $\mathcal{M}$. Now the generalized Teichmüller space $\tilde{\mathcal{T}}_p$ of $\Sigma - p$ is also included in $\mathcal{M}$, hence we get that ϕ acts by the identity on $\tilde{\mathcal{T}}_p$. But then the statement about ϕ follows by classical theory of the action of Γ on $\tilde{\mathcal{T}}_p$. □

Institut for Matematiske Fag, Aarhus Universitet, Aarhus, Denmark
E-mail address: andersen@imf.au.dk
URL: http://home.imf.au.dk.andersen/

References

[A1] J. E. Andersen, Jones-Witten theory and the Thurston boundary of Teichmüller space, University of Oxford D. Philos. thesis (1992), 129pp.

[A2] ———, Deformation quantization and geometric quantification of abelian moduli space, *Comm. Math. Phys.* **255** (2005), 727–745.

[A3] ———, Asymptotic faithfulness of the quantum $\mathrm{SU}(n)$ representations of the mapping class groups in the singular case, in preparation.

[AC] J. E. Andersen and M. Christ, Asymptotic expansion of the Szegö kernel on singular algebraic varieties, in preparation.

[AMR1] J. E. Andersen, J. Mattes, and N. Reshetikhin, The Poisson structure on the moduli space of flat connections and chord diagrams, *Topology* **35** (1996), 1069–1083.

[AMR2] ———, Quantization of the algebra of chord diagrams, *Math. Proc. Cambridge Philos. Soc.* **124** (1998), 451–467.

[AU1] J. E. Andersen and K. Ueno, Abelian conformal field theories and determinant bundles, preprint; math.QA/0304135.

[AU2] ———, Construction of modular functors from conformal field theories, preprint; math.DG/0306235.

[AU3] ———, Construction of the Reshetikhin-Turaev TQFT from conformal field theory, in preparation.

[At] M. Atiyah, The Jones-Witten invariants of knots, *Sém. Bourbaki*, Vol. 1989/90, *Astérisque* No. **189-190** (1990), Exp. No. 715, 7–16.

[AB] M. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, *Philos. Trans. Royal Soc. London* **A308** (1982), 523–615.

[ADW] S. Axelrod, S. Della Pietra, and E. Witten, Geometric quantization of Chern-Simons gauge theory, *J. Differential Geom.* **33** (1991), 787–902.

[B1] C. Blanchet, Hecke algebras, modular categories and 3-manifolds quantum invariants, *Topology* **39** (2000), 193–223.

[BHMV1] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, Three-manifold invariants derived from the Kauffman bracket, *Topology* **31** (1992), 685–699.

[BHMV2] ———, Topological quantum field theories derived from the Kauffman bracket, *Topology* **34** (1995), 883–927.

[BMS] M. Bordeman, E. Meinrenken, and M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and $gl(N), N \to \infty$ limit, *Comm. Math. Phys.* **165** (1994), 281–296.

[DN] J.-M. Drezet and M. S. Narasimhan, Groupe de Picard des variétés de modules de fibres semi-stables sur les courbes algébriques, *Invent. Math.* **97** (1989), 53–94.

[Fal] G. Faltings, Stable G-bundles and projective connections, *J. Algebraic Geom.* **2** (1993), 507–568.

[Fr] D. S. Freed, Classical Chern-Simons theory, Part 1, *Adv. Math.* **113** (1995), 237–303.

[FWW] M. H. Freedman, K. Walker, and Z. Wang, Quantum SU(2) faithfully detects mapping class groups modulo center, *Geom. Topol.* **6** (2002), 523–539.

[vGdJ] B. Van Geemen and A. J. Jong, On Hitchin's connection, *J. Amer. Math. Soc.* **11** (1998), 189–228.

[H] N. Hitchin, Flat connections and geometric quantization, *Comm. Math. Phys.* **131** (1990), 347–380.

[KS] A. V. Karabegov and M. Schlichenmaier, Identification of Berezin-Toeplitz deformation quantization, *J. Reine Angew. Math.* **540** (2001), 49–76.

[La] Y. Laszlo, Hitchin's and WZW connections are the same, *J. Differential Geom.* **49** (1998), 547–576.

[M1] G. Masbaum, An element of infinite order in TQFT-representations of mapping class groups, in *Low-dimensional Topology* (Funchal, 1998), 137–139, *Contemp. Math.* **233**, A. M. S., Providence, RI, 1999.

[M2] ———, Quantum representations of mapping class groups, in *Groupes et Géométrie* (Journée annuelle 2003 de la SMF), 19–36, 2003.

[NS1] M. S. Narasimhan and C. S. Seshadri, Holomorphic vector bundles on a compact Riemann surface, *Math. Ann.* **155** (1964), 69–80.

[NS2] ———, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–567.

[R1] T. R. Ramadas, Chern-Simons gauge theory and projectively flat vector bundles on M_g, *Comm. Math. Phys.* **128** (1990), 421–426.

[RSW] T. R. Ramadas, I. M. Singer, and J. Weitsman, Some comments on Chern-Simons gauge theory, *Comm. Math. Phys.* **126** (1989), 409–420.

[RT1] N. Reshetikhin and V. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Comm. Math. Phys.* **127** (1990), 1–26.

[RT2] ———, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–597.

[Ro] J. Roberts, Irreducibility of some quantum representations of mapping class groups, *J. Knot Theory and its Ramifications* **10** (2001), 763–767.

[Sch] M. Schlichenmaier, Zwei Anwendungen algebraisch-geometrischer Methoden in der theoretischen Physik: Berezin-Toeplitz-Quantisierung und globale Algebren der zweidimensionalen konformen Feldtheorie, Habilitation thesis, Universitaet Mannheim, 1996.

[Sch1] M. Schlichenmaier, Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization, in *Conférence Moshé Flato* 1999), Vol. II (Dijon), 289–306, *Math. Phys. Stud.* **22**, Kluwer Acad. Publ., Dordrecht (2000).

[Sch2] ———, Berezin-Toeplitz quantization and Berezin transform, in *Long Time Behaviour of Classical and Quantum Systems* (Bologna, 1999), *Ser. Concr. Appl. Math.* **1**, World Sci. Publ. Co., River Edge, NJ (2001), 271–287.

[TUY] A. Tsuchiya, K. Ueno, and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmetries, *Adv. Studies in Pure Math.* **19** (1989), 459–566.

[T] V. G. Turaev, *Quantum Invariants of Knots and* 3*-manifolds*, *de Gruyter Studies in Math.* **18**, Walter de Gruyter & Co., Berlin, 1994.

[W1] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), 351–398.

(Received March 1, 2004)