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Bilipschitz maps, analytic capacity, and the Cauchy integral

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Abstract

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a bilipschitz map. We prove that if $E \subset \mathbb{C}$ is compact, and $\gamma(E)$, $\alpha(E)$ stand for its analytic and continuous analytic capacity respectively, then $C^{-1}\gamma(E) \leq \gamma(\varphi(E)) \leq C\gamma(E)$ and $C^{-1}\alpha(E) \leq \alpha(\varphi(E)) \leq C\alpha(E)$, where C depends only on the bilipschitz constant of φ. Further, we show that if μ is a Radon measure on $\mathbb{C}$ and the Cauchy transform is bounded on $L^2(\mu)$, then the Cauchy transform is also bounded on $L^2(\varphi_\sharp\mu)$, where $\varphi_\sharp\mu$ is the image measure of μ by φ. To obtain these results, we estimate the curvature of $\varphi_\sharp\mu$ by means of a corona type decomposition.

1. Introduction

A compact set $E \subset \mathbb{C}$ is said to be removable for bounded analytic functions if for any open set Ω containing E, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. In order to study removability, in the 1940's Ahlfors [Ah] introduced the notion of analytic capacity. The *analytic capacity* of a compact set $E \subset \mathbb{C}$ is

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

In [Ah], Ahlfors proved that E is removable for bounded analytic functions if and only if $\gamma(E) = 0$.

Painlevé's problem consists of characterizing removable singularities for bounded analytic functions in a metric/geometric way. By Ahlfors' result this is equivalent to describing compact sets with positive analytic capacity in metric/geometric terms.

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Vitushkin in the 1950's and 1960's showed that analytic capacity plays a central role in problems of uniform rational approximation on compact sets of the complex plane. Further, he introduced the continuous analytic capacity α, defined as

$$\alpha(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all *continuous* functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ which are analytic on $\mathbb{C} \setminus E$, and uniformly bounded by 1 on $\mathbb{C}$. Many results obtained by Vitushkin in connection with uniform rational approximation are stated in terms of α and γ. See [Vi], for example.

Until quite recently it was not known if removability is preserved by an affine map such as $\varphi(x, y) = (x, 2y)$ (with $x, y \in \mathbb{R}$). From the results of [To3] (see Theorem A below) it easily follows that this is true even for $\mathcal{C}^{1+\varepsilon}$ diffeomorphisms. In the present paper we show that this also holds for bilipschitz maps. Remember that a map $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ is bilipschitz if it is bijective and there exists some constant $L > 0$ such that

$$L^{-1}|z - w| \leq |\varphi(z) - \varphi(w)| \leq L\,|z - w|$$

for all $z, w \in \mathbb{C}$. The precise result that we will prove is the following.

THEOREM 1.1. *Let $E \subset \mathbb{C}$ be a compact set and $\varphi : \mathbb{C} \to \mathbb{C}$ a bilipschitz map. There exists a positive constants C depending only on φ such that*

$$C^{-1}\gamma(E) \leq \gamma(\varphi(E)) \leq C\gamma(E) \tag{1.1}$$

and

$$C^{-1}\alpha(E) \leq \alpha(\varphi(E)) \leq C\alpha(E). \tag{1.2}$$

As far as we know, the question on the behaviour of analytic capacity under bilipschitz maps was first raised by Verdera [Ve1, p.435]. See also [Pa, p.113] for a more recent reference to the problem.

At first glance, the results stated in Theorem 1.1 may seem surprising, since f and $f \circ \varphi$ are rarely both analytic simultaneously. However, by the results of G. David [Da1], it turns out that if E is compact with finite length (i.e. $\mathcal{H}^1(E) < \infty$, where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure), then $\gamma(E) > 0$ if and only if $\gamma(\varphi(E)) > 0$. Moreover, Garnett and Verdera [GV] proved recently that $\gamma(E)$ and $\gamma(\varphi(E))$ are comparable for a large class of Cantor sets E which may have non σ-finite length.

Let us remark that the assumption that φ is bilipschitz in Theorem 1.1 is necessary for (1.1) or (1.2) to hold. The precise statement reads as follows.

PROPOSITION 1.2. *Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ be a homeomorphism such that either* (1.1) *holds for all compact sets $E \subset \mathbb{C}$, or* (1.2) *holds for all compact sets $E \subset \mathbb{C}$* (*in both cases with C independent of E*)*. Then φ is bilipschitz.*

We introduce now some additional notation. A positive Radon measure μ is said to have linear growth if there exists some constant C such that $\mu(B(x,r)) \leq Cr$ for all $x \in \mathbb{C}$, $r > 0$. The linear density of μ at $x \in \mathbb{C}$ is (if it exists)

$$\Theta_\mu(x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{r}.$$

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $R(x,y,z) = \infty$, $c(x,y,z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we set $c(x,y,z) = 0$. For a positive Radon measure μ, we define the *curvature of* μ as

$$c^2(\mu) = \iiint c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z). \tag{1.3}$$

The notion of curvature of measures was introduced by Melnikov [Me] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible for the recent advances in connection with analytic capacity.

Given a complex Radon measure ν on $\mathbb{C}$, the *Cauchy transform* of ν is

$$\mathcal{C}\nu(z) = \int \frac{1}{\xi - z}\, d\nu(\xi).$$

This definition does not make sense, in general, for $z \in \operatorname{supp}(\nu)$, although one can easily see that the integral above is convergent at a.e. $z \in \mathbb{C}$ (with respect to Lebesgue measure). This is the reason why one considers the ε-*truncated Cauchy transform* of ν, which is defined as

$$\mathcal{C}_\varepsilon\nu(z) = \int_{|\xi - z|>\varepsilon} \frac{1}{\xi - z}\, d\nu(\xi),$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$. Given a μ-measurable function f on $\mathbb{C}$ (where μ is some fixed positive Radon measure on $\mathbb{C}$), we write $\mathcal{C}_\mu f \equiv \mathcal{C}(f\,d\mu)$ and $\mathcal{C}_{\mu,\varepsilon} f \equiv \mathcal{C}_\varepsilon(f\,d\mu)$ for any $\varepsilon > 0$. It is said that the Cauchy transform is bounded on $L^2(\mu)$ if the operators $\mathcal{C}_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

The relationship between the Cauchy transform and curvature of measures was found by Melnikov and Verdera [MV]. They proved that if μ has linear growth, then

$$\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} = \frac{1}{6}c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})), \tag{1.4}$$

where $c^2_\varepsilon(\mu)$ is an ε-truncated version of $c^2(\mu)$ (defined as in the right-hand side of (1.3), but with the triple integral over $\{x,y,z \in \mathbb{C} : |x-y|, |y-z|, |x-z| > \varepsilon\}$).

Moreover, there is also a strong connection (see [Pa]) between the notion of curvature of measures and the β's from Jones' travelling salesman theorem [Jo]. The relationship with Favard length is an open problem (see Section 6 of the excellent survey paper [Matt], for example).

The proof of Theorem 1.1, as well as the one of the result of Garnett and Verdera [GV], use the following characterization of analytic capacity in terms of curvature of measures obtained recently by the author.

THEOREM A ([To3]). *For any compact $E \subset \mathbb{C}$,*

$$\gamma(E) \simeq \sup \mu(E),$$

where the supremum is taken over all Borel measures μ supported on E such that $\mu(B(x,r)) \leq r$ for all $x \in E$, $r > 0$ and $c^2(\mu) \leq \mu(E)$.

The notation $A \simeq B$ in the theorem means that there exists an absolute constant $C > 0$ such that $C^{-1}A \leq B \leq CA$.

The corresponding result for α is the following.

THEOREM B ([To4]). *For any compact $E \subset \mathbb{C}$,*

$$\alpha(E) \simeq \sup \mu(E),$$

where the supremum is taken over the Borel measures μ supported on E such that $\Theta_\mu(x) = 0$ for all $x \in E$, $\mu(B(x,r)) \leq r$ for all $x \in E$, $r > 0$, and $c^2(\mu) \leq \mu(E)$.

Although the notion of curvature of a measure has a definite geometric flavor, it is not clear if the characterizations of γ and α in Theorems A and B can be considered as purely metric/geometric. Nevertheless, Theorem 1.1 asserts that γ and α have a metric nature, in a sense.

Theorem 1.1 is a direct consequence of the next result and of Theorems A and B.

THEOREM 1.3. *Let μ be a Radon measure supported on a compact $E \subset \mathbb{C}$, such that $\mu(B(x,r)) \leq r$ for all $x \in E$, $r > 0$ and $c^2(\mu) < \infty$. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a bilipschitz mapping. There exists a positive constant C depending only on φ such that*

$$c^2(\varphi_\sharp \mu) \leq C\big(\mu(E) + c^2(\mu)\big).$$

In the inequality above, $\varphi_\sharp\mu$ stands for the image measure of μ by φ. That is to say, $\varphi_\sharp\mu(A) = \mu(\varphi^{-1}(A))$ for $A \subset \mathbb{C}$.

We will prove Theorem 1.3 using a corona type decomposition, analogous to the one used by David and Semmes in [DS1] and [DS2] for AD regular sets (i.e. for sets E such that $\mathcal{H}^1(E \cap B(x,r)) \simeq r$ for all $x \in E$, $r > 0$). The ideas go back to Carleson's corona construction. See [AHMTT] for a recent survey on similar techniques. In our situation, the measures μ that we will consider do not satisfy any doubling or homogeneity condition. This fact is responsible for most of the technical difficulties that appear in the proof of Theorem 1.3.

By the relationship (1.4) between curvature and the Cauchy integral, the results in [To1] (or in [NTV]), and Theorem 1.3, we also deduce the next result.

THEOREM 1.4. *Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ be a bilipschitz map and μ a Radon measure on $\mathbb{C}$ without atoms. Set $\sigma = \varphi_\sharp \mu$. If $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$, then $\mathcal{C}_\sigma$ is bounded on $L^2(\sigma)$.*

Notice that the theorem by Coifman, McIntosh and Meyer [CMM] concerning the L^2 boundedness of the Cauchy transform on Lipschitz graphs (with respect to arc length measure) can be considered as a particular case of Theorem 1.4. Indeed, if $x \mapsto A(x)$ defines a Lipschitz graph on $\mathbb{C}$, then the map $\varphi(x, y) = (x, y + A(x))$ is bilipschitz. Since φ sends the real line to the Lipschitz graph defined by A and the Cauchy transform is bounded on $L^2(dx)$ on the real line (because it coincides with the Hilbert transform), from Theorem 1.4 we infer that it is also bounded on the Lipschitz graph.

The plan of the paper is the following. In Section 2 we prove (the easy) Proposition 1.2 and introduce additional notation and definitions. The rest of the paper is devoted to the proof of Theorem 1.3, which we have split into two main lemmas. The first one, Main Lemma 3.1, deals with the construction of a suitable corona type decomposition of E, and it is proved in Sections 3–7. The second one, Main Lemma 8.1, is proved in Section 8, and it shows how one can estimate the curvature of a measure by means of a corona type decomposition. So the proof of Theorem 1.3 works as follows. In Main Lemma 3.1 we construct a corona type decomposition of E, which is stable under bilipschitz maps. That is to say, φ sends the corona decomposition of E (perhaps we should say of the pair (E, μ)) to another corona decomposition of $\varphi(E)$ (i.e. of the pair $(\varphi(E), \varphi_\sharp \mu)$). Then, Main Lemma 8.1 yields the required estimates for $c^2(\varphi_\sharp \mu)$.

2. Preliminaries

2.1. *Proof of Proposition* 1.2. Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ be a homeomorphism and suppose that $\gamma(\varphi(E)) \simeq \gamma(E)$ for all compact sets $E \subset \mathbb{C}$. Given $x, y \in \mathbb{C}$, consider the segment $E = [x, y]$. Then $\varphi(E)$ is a curve and its analytic capacity is comparable to its diameter. Thus,

$$|\varphi(x) - \varphi(y)| \leq \operatorname{diam}(\varphi(E)) \simeq \gamma(\varphi(E)) \simeq \gamma(E) \simeq |x - y|.$$

The converse inequality, $|x - y| \lesssim |\varphi(x) - \varphi(y)|$, follows by application of the previous argument to φ^{-1}.

If instead of $\gamma(\varphi(E)) \simeq \gamma(E)$ we assume now that with $\alpha(\varphi(E)) \simeq \alpha(E)$ for all compact sets E, a similar argument works. For example, given $x, y \in \mathbb{C}$, one can take E to be the closed ball centered at x with radius $2|x-y|$, and then one can argue as above. □

2.2. *Two remarks.* There are bijections $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\gamma(\varphi(E)) \simeq \gamma(E)$ and $\alpha(\varphi(E)) \simeq \alpha(E)$, for any compact $E \subset \mathbb{C}$, which are not homeomorphisms. For example, set $\varphi(z) = z$ if $\mathrm{Re}(z) \geq 0$ and $\varphi(z) = z + i$ if $\mathrm{Re}(z) < 0$. Using the semiadditivity of γ and α one easily sees that $\gamma(\varphi(E)) \simeq \gamma(E)$ and $\alpha(\varphi(E)) \simeq \alpha(E)$.

If the map $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ is assumed to be only Lipschitz, then none of the inequalities $\gamma(\varphi(E)) \gtrsim \gamma(E)$ or $\gamma(\varphi(E)) \lesssim \gamma(E)$ holds, in general. To check this, for the first inequality consider a constant map and E arbitrary with $\gamma(E) > 0$. For the second inequality, one only has to take into account that there are purely unrectifiable sets with finite length which project orthogonally onto a segment (with positive length) in some direction.

2.3. *Additional notation and definitions.* An Ahlfors-David regular curve (or AD regular curve) is a curve Γ such that $\mathcal{H}^1(\Gamma \cap B(x,r)) \leq C_3 r$ for all $x \in \Gamma$, $r > 0$, and some fixed $C_3 > 0$. If we want to specify the constant C_3, we will say that Γ is "C_3-AD regular".

In connection with the definition of $c^2(\mu)$, we also set

$$c_\mu^2(x) = \iint c(x,y,z)^2 \, d\mu(y) d\mu(z).$$

Thus, $c^2(\mu) = \int c_\mu^2(x) \, d\mu(x)$. If $A \subset \mathbb{C}$ is μ-measurable,

$$c_\mu^2(x,y,A) = \int_A c(x,y,z)^2 d\mu(z), \qquad x, y \in \mathbb{C},$$

and, if $A, B, C \subset \mathbb{C}$ are μ-measurable,

$$c_\mu^2(x,A,B) = \int_A \int_B c(x,y,z)^2 d\mu(y) d\mu(z), \qquad x \in \mathbb{C},$$

and

$$c_\mu^2(A,B,C) = \int_A \int_B \int_C c(x,y,z)^2 d\mu(x) d\mu(y) d\mu(z).$$

The curvature operator K_μ is

$$K_\mu(f)(x) = \int k_\mu(x,y) f(y) d\mu(y), \qquad x \in \mathbb{C}, f \in L^1_{\mathrm{loc}}(\mu),$$

where $k_\mu(x,y)$ is the kernel

$$k_\mu(x,y) = \int c(x,y,z)^2 d\mu(z) = c_\mu^2(x,y,\mathbb{C}), \qquad x, y \in \mathbb{C}.$$

For $j \in \mathbb{Z}$, the truncated operators $K_{\mu,j}$, $j \in \mathbb{Z}$, are defined as

$$K_{\mu,j} f(x) = \int_{|x-y|>2^{-j}} k_\mu(x,y)\, f(y)\, d\mu(y), \qquad x \in \mathbb{C}, f \in L^1_{\mathrm{loc}}(\mu).$$

In this paper, by a square we mean a square with sides parallel to the axes. Moreover, we assume the squares to be half closed - half open. The side length of a square Q is denoted by $\ell(Q)$. Given a square Q and $a > 0$, aQ denotes the square concentric with Q with side length $a\ell(Q)$. The average (linear) density of a Radon measure μ on Q is

$$\theta_\mu(Q) := \frac{\mu(Q)}{\ell(Q)}. \tag{2.1}$$

A square $Q \subset \mathbb{C}$ is called 4-dyadic if it is of the form $[j2^{-n}, (j+4)2^{-n}) \times [k2^{-n}, (k+4)2^{-n})$, with $j, k, n \in \mathbb{Z}$. So a 4-dyadic square with side length $4 \cdot 2^{-n}$ is made up of 16 dyadic squares with side length 2^{-n}. We will work quite often with 4-dyadic squares. All our arguments would also work with other variants of this type of square, such as squares $5Q$ with Q dyadic, say. However, our choice of 4-dyadic squares has some advantages. For example, if Q is 4-dyadic, $\frac{1}{2}Q$ is another square made up of 4 dyadic squares, and some calculations may be a little simpler.

Given a square Q (which may be nondyadic) with side length 2^{-n}, we denote $J(Q) := n$. Given $a, b > 1$, we say that Q is (a,b)-doubling if $\mu(aQ) \leq b\mu(Q)$. If we do not want to specify the constant b, we say that Q is a-doubling.

Remark 2.1. If $b > a^2$, then it easily follows that for μ-a.e. $x \in \mathbb{C}$ there exists a sequence of (a,b)-doubling squares $\{Q_n\}_n$ centered at x with $\ell(Q_n) \to 0$ (and with $\ell(Q_n) = 2^{-k_n}$ for some $k_n \in \mathbb{Z}$ if necessary).

As usual, in this paper the letter 'C' stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as C_1, retain their value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. So $A \simeq B$ is equivalent to $A \lesssim B \lesssim A$.

3. The corona decomposition

This section deals with the corona construction. In the next lemma we will introduce a family $\mathrm{Top}(E)$ of 4-dyadic squares (the top squares) satisfying some precise properties. Given any square $Q \in \mathrm{Top}(E)$, we denote by $\mathrm{Stop}(Q)$ the subfamily of the squares $P \in \mathrm{Top}(E)$ satisfying

(a) $P \cap 3Q \neq \varnothing$,

(b) $\ell(P) \leq \frac{1}{8}\ell(Q)$,

(c) P is maximal, in the sense that there does not exist another square $P' \in \mathrm{Top}(E)$ satisfying (a) and (b) which contains P.

We also denote by $Z(\mu)$ the set of points $x \in \mathbb{C}$ such that there does not exist a sequence of $(70, 5000)$-doubling squares $\{Q_n\}_n$ centered at x with $\ell(Q_n) \to 0$ as $n \to \infty$, so that moreover $\ell(Q_n) = 2^{-k_n}$ for some $k_n \in \mathbb{Z}$. By the preceding remark we have $\mu(Z(\mu)) = 0$.

The set of good points for Q is defined as

$$G(Q) := 3Q \cap \mathrm{supp}(\mu) \setminus \Big[Z(\mu) \cup \bigcup_{P \in \mathrm{Stop}(Q)} P \Big].$$

Given two squares $Q \subset R$, we set

$$\delta_\mu(Q, R) := \int_{R_Q \setminus Q} \frac{1}{|y - x_Q|}\, d\mu(y),$$

where x_Q stands for the center of Q, and R_Q is the smallest square concentric with Q that contains R.

Main Lemma 3.1 (The corona decomposition). *Let μ be a Radon measure supported on $E \subset \mathbb{C}$ such that $\mu(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$ and $c^2(\mu) < \infty$. There exists a family* $\mathrm{Top}(E)$ *of* 4*-dyadic* $(16, 5000)$*-doubling squares* (*called* top squares) *which satisfy the packing condition*

$$\sum_{Q \in \mathrm{Top}(E)} \theta_\mu(Q)^2 \mu(Q) \leq C\big(\mu(E) + c^2(\mu)\big), \tag{3.1}$$

*and such that for each square $Q \in \mathrm{Top}(E)$ there exists a C_3-*AD *regular curve Γ_Q such that*:

(a) $G(Q) \subset \Gamma_Q$.

(b) *For each $P \in \mathrm{Stop}(Q)$ there exists some square $\widetilde{P}$ containing P such that $\delta_\mu(P, \widetilde{P}) \leq C\theta_\mu(Q)$ and $\widetilde{P} \cap \Gamma_Q \neq \varnothing$.*

(c) *If P is a square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \varnothing$ or there is another square $P' \in \mathrm{Stop}(Q)$ such that $P \cap P' \neq \varnothing$ and $\ell(P') \leq \ell(P)$, then $\mu(P) \leq C\,\theta_\mu(Q)\,\ell(P)$.*

Moreover, $\mathrm{Top}(E)$ *contains some* 4*-dyadic square R_0 such that $E \subset R_0$.*

Notice that the AD regularity constant of the curves Γ_Q in the lemma is uniformly bounded above by the constant C_3.

In Subsections 3.1, 3.2 and 3.3 we explain how the 4-dyadic squares in $\mathrm{Top}(E)$ are chosen. Section 4 deals with the construction of the curves Γ_Q. The packing condition (3.1) is proved in Sections 5–7

The squares in $\mathrm{Top}(E)$ are obtained by stopping-time arguments. The first step consists of choosing a family $\mathrm{Top}_0(E)$ which is a kind of pre-selection of the 4-dyadic squares which are candidates to be in $\mathrm{Top}(E)$. In the second step, some unnecessary squares in $\mathrm{Top}_0(E)$ are eliminated. The remaining family of squares is $\mathrm{Top}(E)$.

3.1. *Pre-selection of the top squares.* To prove the Main Lemma 3.1, we will assume that E is contained in a dyadic square with side length comparable to $\mathrm{diam}(E)$. It easy to check that the lemma follows from this particular case.

All the squares in $\mathrm{Top}_0(E)$ will be chosen to be $(16, 5000)$-doubling. We define the family $\mathrm{Top}_0(E)$ by induction. Let R_0 be a 4-dyadic square with $\ell(R_0) \simeq \mathrm{diam}(E)$ such that E is contained in one of the four dyadic squares in $\frac{1}{2}R_0$ with side length $\ell(R_0)/4$. Then, we set $R_0 \in \mathrm{Top}_0(E)$. Suppose now that we have already decided that some squares belong to $\mathrm{Top}_0(E)$. If Q is one of them, then it generates a (finite or countable) family of "bad" $(16, 5000)$-doubling 4-dyadic squares, called $\mathrm{Bad}(Q)$. We will explain precisely below how this family is constructed. For the moment, let us say that if $P \in \mathrm{Bad}(Q)$, then $P \subset 4Q$ and $\ell(P) \leq \ell(Q)/8$. One should think that, in a sense, $\mathrm{supp}(\mu_{|3Q})$ can be well approximated by a "nice" curve Γ_Q up to the scale of the squares in $\mathrm{Bad}(Q)$. All the squares in $\mathrm{Bad}(Q)$ become also elements of the family $\mathrm{Top}_0(E)$.

In other words, we start the construction of $\mathrm{Top}_0(E)$ by R_0. The next squares that we choose as elements of $\mathrm{Top}_0(E)$ are the squares from the family $\mathrm{Bad}(R_0)$. And the following ones are those generated as bad squares of some square which is already in $\mathrm{Bad}(R_0)$, and so on. The family $\mathrm{Top}_0(E)$ is at most countable. Moreover, in this process of generation of squares of $\mathrm{Top}_0(E)$, *a priori*, it may happen that some bad square P is generated by two different squares $Q_1, Q_2 \in \mathrm{Top}_0(E)$ (i.e. $P \in \mathrm{Bad}(Q_1) \cap \mathrm{Bad}(Q_2)$). We do not care about this fact.

3.2. *The family* $\mathrm{Bad}(R)$. Let R be some fixed $(16, 5000)$-doubling 4-dyadic square. We will show now how we construct $\mathrm{Bad}(R)$. Roughly speaking, a square Q with center in $3R$ and $\ell(Q) \leq \ell(R)/32$ is not good (we prefer to reserve the terminology "bad" for the final choice) for the approximation of $\mu_{|3R}$ by an Ahlfors regular curve Γ_R if either:

(a) $\theta_\mu(Q) \gg \theta_\mu(R)$ (i.e. too high density), or

(b) $K_{\mu,J(Q)+10}\chi_E(x) - K_{\mu,J(R)-4}\chi_E(x)$ is too big for "many" points $x \in Q$ (i.e. too high curvature), or

(c) $\theta_\mu(Q) \ll \theta_\mu(R)$ (i.e. too low density).

A first attempt to construct $\mathrm{Bad}(R)$ might consist of choosing some kind of maximal family of squares satisfying (a), (b) or (c). However, we want the squares from $\mathrm{Bad}(R)$ to be doubling, and so the arguments for the construction will become somewhat more involved.

Let $A > 0$ be some big constant (to be chosen below, in Subsection 5.2), $\delta > 0$ be some small constant (which will be fixed in Section 7, depending on A, besides other things), and $\varepsilon_0 > 0$ be another small constant (to be chosen also in Section 7, depending on A and δ). Let Q be some $(70, 5000)$-*doubling* square centered at some point in $3R \cap \mathrm{supp}(\mu)$, with $\ell(Q) = 2^{-n}\ell(R)$, $n \geq 5$. We introduce the following notation:

(a) If $\theta_\mu(Q) \geq A\theta_\mu(R)$, then we write $Q \in \mathrm{HD}_{c,0}(R)$ (high density).

(b) If $Q \notin \mathrm{HD}_{c,0}(R)$ and

$$\mu\{x \in Q : K_{\mu,J(Q)+10}\chi_E(x) - K_{\mu,J(R)-4}\chi_E(x) \geq \varepsilon_0\theta_\mu(R)^2\} \geq \frac{1}{2}\,\mu(Q),$$

then $Q \in \mathrm{HC}_{c,0}(R)$ (high curvature).

(c) If $Q \notin \mathrm{HD}_{c,0}(R) \cup \mathrm{HC}_{c,0}(R)$ and there exists some square S_Q such that $Q \subset \frac{1}{100}S_Q$, with $\ell(S_Q) \leq \ell(R)/8$ and $\theta_\mu(S_Q) \leq \delta\,\theta_\mu(R)$, then we set $Q \in \mathrm{LD}_{c,0}(R)$ (low density).

The subindex c in $\mathrm{HD}_{c,0}$, $\mathrm{LD}_{c,0}$, and $\mathrm{HC}_{c,0}$ refers to the fact that these families contain squares whose *centers* belong to $\mathrm{supp}(\mu)$.

For each point $x \in 3R \cap \mathrm{supp}(\mu)$ which belongs to some square from $\mathrm{HD}_{c,0}(R) \cup \mathrm{HC}_{c,0}(R) \cup \mathrm{LD}_{c,0}(R)$ consider the largest square $Q_x \in \mathrm{HD}_{c,0}(R) \cup \mathrm{HC}_{c,0}(R) \cup \mathrm{LD}_{c,0}(R)$ which contains x. Let $\widehat{Q}_x$ be a 4-dyadic square with side length $4\ell(Q_x)$ such that $Q_x \subset \frac{1}{2}\widehat{Q}_x$. Now we apply Besicovitch's covering theorem to the family $\{\widehat{Q}_x\}_x$ (notice that this theorem can be applied because $x \in \frac{1}{2}\widehat{Q}_x$), and we obtain a family of 4-dyadic squares $\{\widehat{Q}_{x_i}\}_i$ with finite overlap such that the union of the squares from $\mathrm{HD}_{c,0}(R) \cup \mathrm{HC}_{c,0}(R) \cup \mathrm{LD}_{c,0}(R)$ is contained (as a set in $\mathbb{C}$) in $\bigcup_i \widehat{Q}_{x_i}$. We define

$$\mathrm{Bad}(R) := \{\widehat{Q}_{x_i}\}_i.$$

If $Q_{x_i} \in \mathrm{HD}_{c,0}(R)$, then we write $\widehat{Q}_{x_i} \in \mathrm{HD}_0(R)$, and analogously with $\mathrm{HC}_{c,0}(R)$, $\mathrm{LD}_{c,0}(R)$ and $\mathrm{HC}_0(R)$, $\mathrm{LD}_0(R)$.

Remark 3.2. The constants that we denote by C (with or without subindex) in the rest of the proof of Main Lemma 3.1 do not depend on A, δ, or ε_0, unless stated otherwise.

In the next two lemmas we show some properties fulfilled by the family $\mathrm{Bad}(R)$.

LEMMA 3.3. *Given $R \in \mathrm{Top}_0(E)$, the following properties hold for every $Q \in \mathrm{Bad}(R)$:*

(a) *Q is $(16, 5000)$-doubling and $\frac{1}{2}Q$ is $(32, 5000)$-doubling.*

(b) *If $Q \in \mathrm{HD}_0(R)$, then $\theta_\mu(Q) \gtrsim A\theta_\mu(R)$.*

(c) *If $Q \in \mathrm{HC}_0(R)$, then*

$$\mu\{x \in \frac{1}{2}Q : K_{\mu,J(Q)+12}\chi_E(x) - K_{\mu,J(R)-4}\chi_E(x) \geq C^{-1}\varepsilon_0\theta_\mu(R)^2\} \gtrsim \frac{1}{2}\mu(Q).$$

(d) *If $Q \in \mathrm{LD}_0(R)$, then there exists some square S_Q such that $Q \subset \frac{1}{20}S_Q$, $\ell(S_Q) \leq \ell(R)/8$, with $\theta_\mu(S_Q) \lesssim \delta\, \theta_\mu(R)$.*

Proof. The doubling properties of Q and $\frac{1}{2}Q$ follow easily. Let $x \in 3R \cap \operatorname{supp}(\mu)$ be such that $Q = \widehat{Q}_x$, by the notation above. Since $\frac{1}{2}\widehat{Q}_x \supset Q_x$, Q_x is $(70, 5000)$-doubling, and $70Q_x \supset 16\widehat{Q}_x$, we get

$$\mu(\widehat{Q}_x) \geq \mu(\tfrac{1}{2}\widehat{Q}_x) \geq \mu(Q_x) \geq \frac{1}{5000}\mu(70Q_x) \geq \frac{1}{5000}\mu(16\widehat{Q}_x).$$

The statements (b), (c) and (d) are a direct consequence of the definitions and of the fact that $\theta_\mu(Q_x) \simeq \theta_\mu(\widehat{Q}_x)$. □

LEMMA 3.4. *Given $R \in \mathrm{Top}_0(E)$, the following properties hold for every $Q \in \mathrm{Bad}(R)$:*

(a) *If P is a square such that $P \cap Q \neq \varnothing$ and $\ell(Q) \leq \ell(P) \leq \ell(R)$, then*

$$\mu(P) \leq C_4 A\, \theta_\mu(R)\, \ell(P).$$

(b) *If P is a square concentric with Q, $\ell(P) \leq \ell(R)/8$, and $\delta_\mu(Q, P) \geq C_5 A\theta_\mu(R)$ (where $C_5 > 0$ is big enough), then*

$$\mu(P) \geq \delta\, \theta_\mu(R)\, \ell(P) \tag{3.2}$$

and

$$K_{\mu,J(P)+12}\chi_E(x) - K_{\mu,J(R)-2}\chi_E(x) \lesssim A^2\theta_\mu(R)^2 \quad \textit{for all } x \in P. \tag{3.3}$$

Before proving the lemma we recall the following result, whose proof follows by standard arguments (see [To1, Lemma 2.4]).

LEMMA 3.5. *Let $x, y, z \in \mathbb{C}$ be three pairwise different points, and let $x' \in \mathbb{C}$ be such that $C_6^{-1}|x - y| \leq |x' - y| \leq C_6|x - y|$. Then,*

$$|c(x, y, z) - c(x', y, z)| \leq (4 + 2C_6)\, \frac{|x - x'|}{|x - y|\, |x - z|}.$$

Proof of Lemma 3.4. Let $x \in 3R \cap \operatorname{supp}(\mu)$ be such that $Q = \widehat{Q}_x$, by the notation above.

Let us prove (a). If $\ell(R)/8 < \ell(P) \leq \ell(R)$, then

$$\mu(P) \leq \mu(5R) \lesssim \mu(R) = \theta_\mu(R)\ell(R) \lesssim \theta_\mu(R)\ell(P).$$

If P is of the form $2^n Q_x$, $n \geq 1$, with $\ell(P) \leq \ell(R)/8$, and P is $(70, 5000)$-doubling, then

$$\mu(2^n Q_x) = \theta_\mu(2^n Q_x)\,\ell(2^n Q_x) \leq A\theta_\mu(R)\ell(2^n Q_x),$$

by the definition of Q_x. If P is of the same type but it is not doubling, then we take the smallest $(70, 5000)$-doubling square $\widetilde{P} := 2^m Q_x$ such that $P \subset \widetilde{P}$ and $\ell(P) \leq \ell(R)/8$, in case that it exists. If all the squares $2^m Q_x$ containing P with $\ell(2^m Q) \leq \ell(R)/8$ are non-$(70, 5000)$-doubling, we set $\widetilde{P} := 2^m Q_x$, with m such that $\ell(P) = \ell(R)/8$. In any case we have

$$\theta_\mu(P) \leq \theta_\mu(2P) \leq \theta_\mu(2^2 P) \leq \cdots \leq \theta_\mu(\widetilde{P}) \leq CA\theta_\mu(R).$$

The statement (a) for an arbitrary square P such that $P \cap \widehat{Q}_x \neq \varnothing$ and $\ell(\widehat{Q}_x) \leq \ell(P) \leq \ell(R)$ follows easily from the preceding instances.

Now we turn our attention to (b). Let $P \supset Q$ be a square concentric with Q such that $\ell(P) \leq \ell(R)/8$ and $\delta_\mu(Q, P) \geq C_5 A\theta_\mu(R)$. It is easy to check (by estimates analogous to the ones of [To2, Lemma 2.1]) that if C_5 is chosen big enough, then there exists some $(70, 5000)$-doubling square $2^n Q_x$ such that $2Q \subset 2^n Q_x \subset \frac{1}{100}P$. Then $2^n Q_x \notin \mathrm{LD}_{c,0}(R)$, and by construction $\theta_\mu(P) \geq \delta\theta_\mu(R)$.

On the other hand, also by construction, there exists some $y \in 2^n Q_x$ such that

$$\begin{aligned}
&K_{\mu,J(P)+12}\chi_E(y) - K_{\mu,J(R)-2}\chi_E(y)\\
&\qquad\qquad \leq\; K_{\mu,J(2^n Q_x)+12}\chi_E(y) - K_{\mu,J(R)-4}\chi_E(y) \;\leq\; \varepsilon_0\theta_\mu(R)^2.
\end{aligned}$$

Then, for any $x \in P$,

$$\begin{aligned}
(3.4)\quad & K_{\mu,J(P)+12}\chi_E(x) - K_{\mu,J(R)-2}\chi_E(x)\\
&= \iint_{2^{-12}\ell(P)<|x-t|\leq 4\ell(R)} c(x,t,z)^2\,d\mu(t)d\mu(z)\\
&\leq 2\iint_{2^{-12}\ell(P)<|x-t|\leq 4\ell(R)} c(y,t,z)^2\,d\mu(t)d\mu(z)\\
&\quad +2\iint_{2^{-12}\ell(P)<|x-t|\leq 4\ell(R)} \Big[c(x,t,z) - c(y,t,z)\Big]^2\,d\mu(t)d\mu(z)\\
&=: 2I_1 + 2I_2.
\end{aligned}$$

We have

$$I_1 = K_{\mu,J(P)+12}\chi_E(y) - K_{\mu,J(R)-2}\chi_E(y) \leq \varepsilon_0\theta_\mu(R)^2.$$

To estimate I_2 notice that by Lemma 3.5,

$$\Big[c(x,t,z) - c(y,t,z)\Big]^2 \lesssim \max\left(\frac{1}{\ell(P)^2}, \frac{\ell(2^n Q_x)^2}{|x-t|^2\,|x-z|^2}\right).$$

Integrating this inequality over $\{(t,z) \in \mathbb{C}^2 : 2^{-12}\ell(P) < |x-t| \leq 4\ell(R)\}$ (dividing $\mathbb{C}$ into annuli, for example), one easily gets

$$I_2 \leq \left(\sup_{\lambda>1} \frac{\mu(\lambda P \cap 16R)}{\ell(\lambda P)}\right)^2 \lesssim A^2\theta_\mu(R)^2.$$

Summing the estimates for I_1 and I_2, we see that (3.3) follows. □

3.3. *Elimination of unnecessary squares from* $\mathrm{Top}_0(E)$. Some of the bad squares generated by each square $R \in \mathrm{Top}_0(E)$ may not be contained in R. This fact may cause troubles when we try to prove a packing condition such as (3.1) (because of the possible superposition of squares coming from different R's in $\mathrm{Top}_0(E)$). The class $\mathrm{Top}(E)$ that we are going to construct will be a refined version of $\mathrm{Top}_0(E)$, where some unnecessary squares will be eliminated.

Let us introduce some notation first. We say that a square $Q \in \mathrm{Top}_0(E)$ is a descendant of another square $R \in \mathrm{Top}_0(E)$ if there is a chain $R = Q_1, Q_2, \ldots, Q_n = Q$, with $Q_i \in \mathrm{Top}_0(E)$ such that $Q_{i+1} \in \mathrm{Bad}(Q_i)$ for each i. Observe that, in principle, some square Q may be a descendant of more than one square R. Then, in the algorithm of elimination below, Q must be counted with multiplicity (so that Q is completely eliminated if it has been eliminated m times, where m is the multiplicity of Q, etc.)

Let us describe the algorithm for constructing $\mathrm{Top}(E)$. We have to decide when any square in $\mathrm{Top}_0(E)$ belongs to $\mathrm{Top}(E)$. We follow an induction procedure of elimination. A square in $\mathrm{Top}_0(E)$ that (during the algorithm we decide belongs to $\mathrm{Top}(E)$) is said to be "chosen for $\mathrm{Top}(E)$". If we decide that it will not belong to $\mathrm{Top}(E)$ (i.e. we eliminate it), we say that it is "unnecessary". If we have not decided already if it is either chosen for $\mathrm{Top}(E)$ or unnecessary, we say that it is "available". We start with all the squares in $\mathrm{Top}_0(E)$ being available, and at each step of the algorithm, some squares are chosen for $\mathrm{Top}(E)$ and others become unnecessary.

Let R_0 be the 4-dyadic square containing E defined at the beginning of Subsection 3.1. We start by choosing R_0 for $\mathrm{Top}(E)$. Let R_1 be (one of) the squares from $\mathrm{Bad}(R_0)$ with maximal side length. We choose R_1 for $\mathrm{Top}(E)$ too. Next, we choose for $\mathrm{Top}(E)$ (one of) the available square(s) $R_2 \in \mathrm{Top}_0(E)$ with maximal side length. At this point, some available squares in $\mathrm{Top}_0(E)$ may become unnecessary. First, these are the squares Q with $Q \in \mathrm{Bad}(R)$ for some $R \in \mathrm{Top}_0(E)$ such that $Q \subset R_2$ and $\ell(R_2) \leq \ell(R)/8$ (notice that this implies that either $R = R_0$ or $R = R_1$). Also, all the squares which are descendants of unnecessary squares become unnecessary.

Suppose now that we have chosen $R_0, R_1, \dots, R_{k-1}$ for $\mathrm{Top}(E)$, with $\ell(R_0) \geq \ell(R_1) \geq \cdots \geq \ell(R_{k-1})$, and that the available squares in $\mathrm{Top}_0(E)$ have side length $\leq \ell(R_{k-1})$. Let R_k be (one of) the *available* square(s) in $\mathrm{Top}_0(E)$ with maximal side length. We choose R_k for $\mathrm{Top}(E)$. The squares that become unnecessary are those available squares Q such that $Q \in \mathrm{Bad}(R)$ for some $R \in \mathrm{Top}_0(E)$ with $Q \subset R_k$ and $\ell(R_k) \leq \ell(R)/8$ (this implies that R coincides with one of the squares $R_1, \dots, R_{k-1}$). Again, all the squares which are descendants of unnecessary squares become unnecessary too.

It is easily seen that following this algorithm one will decide if any square in $\mathrm{Top}_0(E)$ is unnecessary or chosen for $\mathrm{Top}(E)$. Notice that the squares $Q \in \mathrm{Bad}(R)$ which are eliminated after choosing R_k are the ones such that R_k becomes an "intermediate" square (in a sense) between Q and its generator R (i.e. the square $R \in \mathrm{Top}_0(E)$ such that $Q \in \mathrm{Bad}(R)$), as well as descendants of already eliminated squares. Moreover, if a square Q has been eliminated but its generator R has been chosen for $\mathrm{Top}(E)$, it means that there is another chosen square $Q' \in \mathrm{Top}(E)$ which contains Q, with $\ell(Q') \leq \ell(R)/8$. Thus, if $R \in \mathrm{Top}(E)$, then

$$\bigcup_{Q \in \mathrm{Bad}(R)} Q \subset \bigcup Q', \tag{3.5}$$

where the union on the right side is over the squares $Q' \in \mathrm{Top}(E)$ such that there exists $Q \in \mathrm{Bad}(R)$ contained in Q' and $\ell(Q') \leq \ell(R)/8$.

Remember the definition of $\mathrm{Stop}(R)$, for $R \in \mathrm{Top}(E)$, given at the beginning of Section 3. Notice that $\mathrm{Bad}(R) \cap \mathrm{Top}(E) \subset \mathrm{Stop}(R)$. Of course, the opposite inclusion is false in general. By (3.5), we also have

$$\bigcup_{Q \in \mathrm{Bad}(R)} Q \subset \bigcup_{Q' \in \mathrm{Stop}(R)} Q'.$$

Given $R \in \mathrm{Top}(E)$ and $Q \in \mathrm{Stop}(R)$, we write $Q \in \mathrm{HD}(R)$ if there exists some $R' \in \mathrm{Top}(E)$ such that $Q \in \mathrm{Bad}(R') \cap \mathrm{HD}(R')$ analogously with $\mathrm{LD}(R)$ and $\mathrm{HC}(R)$.

Remark 3.6. Let us insist again on the following fact: for any $R \in \mathrm{Top}(E)$, if $Q \in \mathrm{Bad}(R)$, then either $Q \in \mathrm{Stop}(R)$ or otherwise there is some $Q' \in \mathrm{Stop}(R)$ such that $Q' \supset Q$.

Remark 3.7. Changing constants if necessary, the properties (a) and (b) of Lemma 3.4 also hold if instead of assuming $Q \in \mathrm{Bad}(R)$, we suppose that $Q \in \mathrm{Stop}(R)$. This is due to the fact that, with the new assumption $Q \in \mathrm{Stop}(R)$, the squares P considered in Lemma 3.4 (a), (b) will be, roughly speaking, a subset of the corresponding squares P with the assumption $Q \in \mathrm{Bad}(R)$, because of the preceding remark.

On the other hand, in principle, (b), (c) and (d) in Lemma 3.3 may fail. Nevertheless, we will see in Lemma 5.3 below that they still do hold in some special cases.

4. Construction of the curves Γ_R, $R \in \mathrm{Top}(E)$

4.1. *P. Jones' travelling salesman theorem.* To construct the curves Γ_R for $R \in \mathrm{Top}(E)$, we will apply P. Jones' techniques. Before stating the precise result that we will use, we need to introduce some notation. Given a set $K \subset \mathbb{C}$ and a square Q, let V_Q be an infinite strip (or line in the degenerate case) of smallest possible width which contains $K \cap 3Q$, and let $w(V_Q)$ denote the width of V_Q. Then we set

$$\beta_K(Q) = \frac{w(V_Q)}{\ell(Q)}.$$

We will use the following version of Jones' travelling salesman theorem [Jo]:

THEOREM C (P. Jones). *A set $K \subset \mathbb{C}$ is contained in an* AD *regular curve if and only if there exists some constant C_7 such that for every dyadic square Q*

$$\sum_{P \in \mathcal{D}, P \subset Q} \beta_K(P)^2 \ell(P) \leq C_7 \ell(Q). \tag{4.1}$$

The AD *regularity constant of the curve depends on C_7.*

Let us mention that in [MMV], using an L^2 version of Jones' theorem due to David and Semmes [DS2], the authors showed that if μ is a measure such that $\mu(B(x,r)) \simeq r$ and $c^2(\mu_{|B(x,r)}) \leq C\mu(B(x,r))$ for all $x \in \mathrm{supp}(\mu)$, $0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu))$, then $\mathrm{supp}(\mu)$ is contained in an AD regular curve. In our case, the measure $\mu_{|R}$ does not satisfy these conditions. However, in a sense, they do hold for "big" balls $B(x,r)$, at scales sufficiently above the stopping squares.

In order to apply Jones' result, we will construct a set K which approximates $\mathrm{supp}(\mu) \cap 3R$ at some level above the stopping squares and then, using Theorem C, we will show that there exists a curve Γ_R which contains K. We have not been able to extend the arguments in [MMV] to our situation. Instead, our approach is based on another idea of Jones which shows how one can estimate the β numbers of an AD regular set in terms of the curvature of a measure (see [Pa, Th. 38]).

4.2. *Balanced squares.* Before constructing the appropriate set K which approximates $\mathrm{supp}(\mu)$ we need to show the existence of some squares that will be called *balanced squares*, which will be essential for our calculations.

Given $a, b > 0$, we say that a square Q is balanced with parameters a, b (or (a, b)-balanced) with respect to μ if there exist two squares Q_1, $Q_2 \subset Q$ such that

- $\operatorname{dist}(Q_1, Q_2) \geq a\,\ell(Q)$,
- $\ell(Q_i) \leq \dfrac{a}{10}\ell(Q)$ for $i = 1, 2$, and
- $\mu(Q_i) \geq b\mu(Q)$ for $i = 1, 2$.

We write $Q \in \mathrm{Bal}_\mu(a, b)$.

The next lemmas deal with the existence of balanced squares.

Lemma 4.1. *Let $a = 1/40$. If $Q \notin \mathrm{Bal}_\mu(a, b)$, then there exists a square $P \subset Q$ with $\ell(P) \leq \ell(Q)/10$ such that $\mu(P) \geq (1 - 2 \cdot 10^5 b)\mu(Q)$.*

Proof. We set $N := 400$. We split Q into N^2 squares Q_k, $k = 1, \ldots, N^2$, of side length $\ell(Q)/N = a\ell(Q)/10$. We put

$$G = \{Q_k : 1 \leq k \leq N^2,\ \mu(Q_k) \geq b\mu(Q)\}.$$

Since $Q \notin \mathrm{Bal}_\mu(a, b)$, given any pair of squares Q_j, Q_k, $1 \leq j, k \leq N^2$ such that $\operatorname{dist}(Q_j, Q_k) \geq a\ell(Q)$, it turns out that one of the two squares, say Q_j, satisfies $\mu(Q_j) \leq b\mu(Q)$. Therefore, all the squares from G are contained in a ball B_0 of radius $2a\ell(Q)$, since

$$\Big(\frac{2a}{10} + a\Big) 2^{1/2} \ell(Q) \leq 2a\ell(Q).$$

Thus,

$$\sum_{k: Q_k \in G} \mu(Q_k) \leq \mu(B_0 \cap Q).$$

Also,

$$\sum_{k: Q_k \notin G} \mu(Q_k) \leq b \sum_{k: Q_k \notin G} \mu(Q) \leq bN^2 \mu(Q) = \frac{100b}{a^2}\, \mu(Q).$$

Then we have

$$\mu(Q) \leq \mu(B_0 \cap Q) + \frac{100b}{a^2}\, \mu(Q).$$

Thus,

$$\mu(B_0 \cap Q) \geq \Big(1 - \frac{100b}{a^2}\Big)\mu(Q) \geq (1 - 2 \cdot 10^5 b)\mu(Q).$$

Since the radius of B_0 equals $2a\ell(Q) = \ell(Q)/20$, there exists some square $P \subset Q$ with side length $\ell(Q)/10$ which contains $B_0 \cap Q$, and we are done. $\square$

LEMMA 4.2. *Let Q be a square such that $2Q \notin \mathrm{Bal}_\mu(1/40,\, b)$, and suppose that $\theta_\mu(2Q) \leq C_4 A\theta_\mu(R)$ and $\theta_\mu(\frac{1}{2}Q) \geq C_4^{-1}\delta\theta_\mu(R)$ (with C_4 given by Lemma 3.4 (a)). If $b \ll \delta/A$, then*

$$\mu(2Q \setminus Q) \leq \frac{1}{10}\,\mu(Q).$$

Proof. By Lemma 4.1 there exists a square $P \subset 2Q$ with $\ell(P) \leq \ell(2Q)/10$ such that $\mu(P) \geq (1 - 2\cdot 10^5 b)\mu(2Q)$. If $P \not\subset Q$, then $P \subset 2Q \setminus \frac{1}{2}Q$, and so

$$\mu(2Q \setminus \tfrac{1}{2}Q) \geq \mu(P) \geq (1 - 2\cdot 10^5 b)\mu(2Q).$$

Thus, $\mu(\frac{1}{2}Q) \leq 2{\cdot}10^5 b\mu(2Q)$. If $b \ll \delta/A$, then we derive $\theta_\mu(\frac{1}{2}Q) < C_4^{-2}\delta A^{-1}\theta_\mu(2Q)$, which is a contradiction. Therefore, $P \subset Q$, and then

$$\mu(2Q \setminus Q) \leq \mu(2Q \setminus P) \leq 2\cdot 10^5 b\mu(2Q) \leq \frac{1}{10}\,\mu(\tfrac{1}{2}Q),$$

since $b \ll \delta/A$. $\square$

Remark 4.3. From now on, we assume that $a = 1/40$, and moreover that $b = b(A,\delta) \ll \delta/A$, so that the preceding lemma holds. For these precise values of a and b we write $\mathrm{Bal}(\mu) := \mathrm{Bal}_\mu(a, b)$.

LEMMA 4.4. *Let Q be a square whose center lies on $3R$ and is such that $\ell(Q) \leq C_8^{-1}A^{-2}\ell(R)$ and*

$$\delta\theta_\mu(R) \lesssim \theta_\mu(2^n Q) \lesssim A\theta_\mu(R)$$

for all $n \geq 0$ with $\ell(2^n Q) \leq 8\ell(R)$. If C_8 is big enough, then there exists some square $\widehat{Q} \in \mathrm{Bal}(\mu)$ concentric with Q such that $2\ell(Q) \leq \ell(\widehat{Q}) \leq 8\ell(R)$ and also $\ell(\widehat{Q}) \leq C_9(A,\delta)\ell(Q)$.

Proof. If $2^n Q \notin \mathrm{Bal}(\mu)$ for $n = 1, \ldots, N$, with $\ell(2^N Q) \leq 8\ell(R)$, then

$$\mu(2^n Q \setminus 2^{n-1}Q) \leq \frac{1}{10}\mu(2^{n-1}Q) \qquad \text{for } n = 1, \ldots, N.$$

Thus, $\mu(2^n Q) \leq 1.1\,\mu(2^{n-1}Q)$ for $n = 1, \ldots, N$, and so

$$\begin{aligned}\mu(2^N Q) &\leq 1.1^N \mu(Q) \leq 2^{N/2}\mu(Q)\\ &\leq CA\theta_\mu(R)2^{N/2}\ell(Q) = CA\theta_\mu(R)2^{-N/2}\ell(2^N Q).\end{aligned}$$

Therefore,

$$\theta_\mu(2^N Q) \leq CA2^{-N/2}\theta_\mu(R). \tag{4.2}$$

Suppose that N is such that $\ell(2^N Q) = 8\ell(R)$. Then we have $\theta_\mu(R) \simeq \theta_\mu(2^N Q)$, and by (4.2) we get $CA2^{-N/2} \geq 1$, and so

$$\ell(R) = \frac{1}{8}\,\ell(2^N Q) \leq CA^2\ell(Q) =: \frac{C_8 A^2}{2}\,\ell(Q),$$

which contradicts the assumptions of the lemma.

We infer that there is a square $2^nQ \in \mathrm{Bal}(\mu)$ with $n \geq 1$ and $\ell(2^nQ) \leq 8\ell(R)$. Let $\widehat{Q}$ be the smallest one. Let N be such that $2^NQ = \frac{1}{2}\widehat{Q}$. From (4.2), since $\theta_\mu(2^NQ) \gtrsim \delta\theta_\mu(R)$, we deduce $CA\delta^{-1}2^{-N/2} \geq 1$, and then $\ell(\widehat{Q}) \leq CA^2\delta^{-2}\ell(Q)$. □

4.3. *Construction of K.* From Lemma 3.4 (b) it easily follows that for any square $Q \in \mathrm{Bad}(R)$ there exists another square $\widetilde{Q}$ concentric with Q satisfying $2\ell(Q) \leq \ell(\widetilde{Q}) \leq 8\ell(R)$ and $\delta_\mu(Q,\widetilde{Q}) \lesssim A\theta_\mu(R)$, such that

$$\text{if } n \geq -1 \text{ and } \ell(2^n\widetilde{Q}) \leq 8\ell(R), \text{ then} \quad \delta\theta_\mu(R) \lesssim \theta_\mu(2^n\widetilde{Q}) \lesssim A\theta_\mu(R) \tag{4.3}$$

and moreover

$$\begin{aligned}\iint_{\substack{y,z\in 3R\\ 2^{-12}\ell(\widetilde{Q})<|x-y|\leq 4\ell(R)}} & c(x,y,z)^2\, d\mu(y)d\mu(z) \\ & \leq\; K_{\mu,J(\widetilde{Q})+12}\chi_E(x) - K_{\mu,J(R)-2}\chi_E(x) \;\lesssim\; A^2\theta_\mu(R)^2\end{aligned} \tag{4.4}$$

for all $x \in \widetilde{Q}$.

As explained in Remark 3.7, the same holds if $Q \in \mathrm{Stop}(R)$ (instead of $\mathrm{Bad}(R)$). Further, we can take the squares $\widehat{Q}$ to be 4-dyadic (in this way, some of the calculations below will become simpler). That is to say, given $Q \in \mathrm{Stop}(R)$ there exists a 4-dyadic square $\widehat{Q}$ such that $Q \subset \frac{1}{2}\widehat{Q}$ (we cannot assume Q and $\widehat{Q}$ to be concentric) with $2\ell(Q) \leq \ell(\widetilde{Q}) \leq 8\ell(R)$ and $\delta_\mu(Q,\widetilde{Q}) \lesssim A\theta_\mu(R)$ which satisfies (4.3) and (4.4). We denote by $\mathrm{Qstp}(R)$ the family of squares $\widetilde{Q}$, with $Q \in \mathrm{Stop}(R)$, and we say that $\widetilde{Q}$ is a *quasi-stopping* square of R.

We intend to construct some set K containing $G(R)$ (remember that $G(R)$ is the set of good points of R) such that, besides other properties, for each $Q \in \mathrm{Qstp}(R)$ there exists some point $a_Q \in K$ with $\mathrm{dist}(a_Q, Q) \leq C\ell(Q)$. In the next subsection, we will show that K verifies (4.1), and thus K will be contained in an AD regular curve Γ_R. This curve will fulfill the required properties in Main Lemma 3.1.

In the next lemma we deal with the details of the construction of K and selection of the points a_Q, $Q \in \mathrm{Qstp}(R)$. Most difficulties are due to the fact that the squares $Q \in \mathrm{Qstp}(R)$ are not disjoint, in general.

LEMMA 4.5. *Let $\eta > 3$ be some fixed constant to be chosen below. For each $x \in 3R$,*

$$\ell_x := \inf_{Q\in\mathrm{Qstp}(R)} \Big(\ell(Q) + \frac{1}{40}\,\mathrm{dist}(x,Q),\ \frac{1}{40}\,\mathrm{dist}(x,G(R))\Big). \tag{4.5}$$

There exists a family of points $\{a_Q\}_{Q\in\mathrm{Qstp}(E)}$ such that if

$$K := G(R) \cup \{a_Q\}_{Q\in\mathrm{Qstp}(R)},$$

the following properties hold:

(a) *For each* $Q \in \mathrm{Qstp}(R)$, $\mathrm{dist}(a_Q, Q) \leq C\ell(Q)$.

(b) *There exists* $C > 0$ *such that for all* $x \in K$, $K \cap \bar{B}(x, C^{-1}\ell_x) = \{x\}$.

(c) *If* $x \in K$ *and* $\ell_x > 0$, *then*

$$c^2_{\mu_{|B(x,\eta\ell_x)\setminus B(x,\ell_x)}}(x) \leq \frac{C(A,\delta)}{\mu(B(x,\ell_x))} \iiint_{\substack{y,z,t\in B(x,C\eta\ell_x)\\ |y-z|\geq \ell_x}} c(y,z,t)^2 d\mu(y)d\mu(z)d\mu(t).$$

We note that the lemma is understood better if we think about the points in $G(R)$ as degenerate quasi-stopping squares with zero side length. In our construction some points a_Q may coincide for different squares $Q \in \mathrm{Qstp}(R)$.

We also remark that if (b) were not required in the lemma, then its proof would be much simpler.

Proof. First we explain the algorithm for assigning a point a_Q to each square $Q \in \mathrm{Qstp}(R)$. Finally we will show that (a), (b) and (c) hold for our construction.

Take a fixed square $Q \in \mathrm{Qstp}(R)$. Since ℓ_x is a continuous (and Lipschitz) function of x, there exists some $z_0 \in \overline{10Q}$ such that ℓ_x attains its minimum over $\overline{10Q}$ at z_0. If $\ell_{z_0} = 0$, $a_Q := z_0$.

Suppose now that $\ell_{z_0} > 0$. Assume first that there exists a sequence of squares $\{P_n\}_n \subset \mathrm{Qstp}(R)$ with $\ell(P_n) \to 0$ or points $p_n \in G(R)$ such that

$$\ell(P_n) + \frac{1}{40}\,\mathrm{dist}(z_0, P_n) \to \ell_{z_0} \tag{4.6}$$

(we identify a point p_n with a square P_n with $\ell(P_n) = 0$). Since $\ell_{z_0} \leq \ell(Q)$ (because if $x \in Q$, then $\ell_x \leq \ell(Q)$ and $\ell_{z_0} \leq \ell_x$), we may assume $P_n \subset B(z_0, 41\ell(Q)) \subset \overline{90Q}$. Thus, there exists some point $z_1 \in \overline{90Q}$ such that a subsequence $\{P_{n_k}\}_k$ accumulates on z_1. We set $a_Q := z_1$. Observe that $\ell_{a_Q} = 0$ in this case.

Assume now that $\ell_{z_0} > 0$ and that a sequence $\{P_n\}_n$ as above does not exist. This implies that the infimum which defines ℓ_{z_0} (in (4.5) replacing x by z_0) is attained over a subfamily of squares $P \in \mathrm{Qstp}(R)$ with $\ell(P) \geq \delta$, for some fixed $\delta > 0$, which further satisfy $\mathrm{dist}(P, Q) \leq 41\ell(Q)$ (because $\ell_{z_0} \leq \ell(Q)$). Then it turns out that such a subfamily of squares must be finite, because we are dealing with 4-dyadic squares. Thus the infimum in (4.5) (with x replaced by z_0) is indeed a minimum (attained by only a finite number of squares in $\mathrm{Qstp}(R)$). Among the squares where the minimum is attained, let P_Q be one with minimal side length.

Let us apply Vitali's covering theorem to the family of squares P_Q obtained above. Then there exists a countable (or finite) subfamily of pairwise disjoint squares P_i (which are of the type P_Q) such that

$$\bigcup P_Q \subset \bigcup_i 5P_i,$$

where the union on the left side is over all the squares P_Q which arise in the algorithm above when one considers all the squares $Q \in \mathrm{Qstp}(R)$.

Now, for each i we choose a "good" point $a_i \in \frac{1}{2}P_i \cap \operatorname{supp}(\mu)$, so that

$$\iint_{\substack{y,z\in 2\eta P_i\\ |a_i-y|\geq \ell(P_i)}} c(a_i,y,z)^2\,d\mu(y)d\mu(z) \leq \frac{1}{\mu(\frac{1}{2}P_i)} \iiint_{\substack{x\in\frac{1}{2}P_i\\ y,z\in 2\eta P_i\\ |x-y|\geq \ell(P_i)}} c(x,y,z)^2\,d\mu(x)d\mu(y)d\mu(z). \tag{4.7}$$

We claim that for each square $Q \in \mathrm{Qstp}(R)$ for which a_Q has not been chosen yet (which means $\ell_{z_0} > 0$ and there is no sequence $\{P_n\}_n \subset \mathrm{Qstp}(R)$ with $\ell(P_n) \to 0$ satisfying (4.6)), there exists some a_i such that $\operatorname{dist}(a_i, Q) \leq C\ell(Q)$. Then we set $a_Q := a_i$.

Before proving our claim, we show that if $Q \in \mathrm{Qstp}(R)$ is such that $Q \cap 5P_i \neq \varnothing$ for some i, then $\ell(Q) \geq \ell(P_i)$. Indeed, if $\ell(Q) \leq \ell(P_i)/2$ (remember that Q and P_i are 4-dyadic squares), then $Q \subset 10P_i$, and so it easily follows that for any $y \in \mathbb{C}$ we have

$$\ell(P_i) + \frac{1}{40}\operatorname{dist}(y,P_i) \geq \ell(Q) + \frac{1}{40}\operatorname{dist}(y,Q), \tag{4.8}$$

which is not possible because of our construction (remember that there exists some z_0 such that the infimum defining ℓ_{z_0} is attained by P_i).

Let us prove the claim now. By our construction, there exists some square P_Q with $\ell(P_Q) \leq \ell(Q)$ such that $\operatorname{dist}(P_Q, Q) \leq C\ell(Q)$. Let P_i be such that $5P_i \cap P_Q \neq \varnothing$. Since $\ell(P_Q) \geq \ell(P_i)$,

$$\begin{aligned}\operatorname{dist}(a_i,Q) &\leq \operatorname{dist}(a_i,P_Q) + 2^{1/2}\ell(P_Q) + \operatorname{dist}(P_Q,Q)\\ &\leq C\ell(P_Q) + 2^{1/2}\ell(P_Q) + C\ell(Q) \leq C\ell(Q).\end{aligned}$$

Let us consider the statement (b). If $\ell_x = 0$, there is nothing to prove. The only points such that $\ell_x > 0$ are the a_i's. Notice that $\ell_{a_i} \leq \ell(P_i)$ because $a_i \in P_i$. On the other hand, we also have $\ell_{a_i} \geq \frac{1}{40}\ell(P_i)$. Otherwise, it is easily seen that there exists some $P \in \mathrm{Qstp}(R)$ such that $P \cap 5P_i \neq \varnothing$ and $\ell(P) \leq \ell(P_i)/2$, which is not possible as shown above (in the paragraph of (4.8)). Thus (b) follows from the fact that $a_i \in \frac{1}{2}P_i$, the squares P_i are disjoint, $\ell(P_i) \simeq \ell_{a_i}$, etc.

Finally (c) follows easily from (4.7) and the fact that $B(a_i, \eta\ell_{a_i}) \subset 2\eta P_i \subset B(a_i, C\eta\ell_{a_i})$, for some $C > 0$. □

4.4. *Estimate of* $\sum_{P\in\mathcal{D},P\subset Q}\beta_K(P)^2\ell(P)$. We will need the following result.

Lemma 4.6. *There exists some* $\lambda > 4$ *depending on* A *and* δ *such that, given any* $Q \in \mathrm{Qstp}(R)$, *for each* $n \geq 1$ *with* $\lambda^{n+1}\ell(Q) \leq \ell(R)$ *there exist two squares* Q_n^a *and* Q_n^b *fulfilling the following properties*:

(a) $Q_n^a, Q_n^b \subset \frac{\lambda^{n+1}}{2} Q \setminus \lambda^n Q$,

(b) $\operatorname{dist}(Q_n^a, Q_n^b) \gtrsim \lambda^n \ell(Q)$,

(c) $\lambda^n \ell(Q) \lesssim \ell(Q_n^i) \lesssim \lambda^{n+1}\ell(Q)$, *for* $i = a, b$,

(d) $C(A,\delta)^{-1}\theta_\mu(R) \lesssim \theta_\mu(Q_n^i) \lesssim A\theta_\mu(R)$, *for* $i = a, b$.

Proof. The lemma follows easily from the existence of balanced squares (see Lemma 4.4). Indeed, if $Q \in \operatorname{Bal}(\mu)$, then there are two squares Q_1, $Q_2 \subset Q$ fulfilling the properties stated just above Lemma 4.1. Since $\operatorname{dist}(Q_1, Q_2) \geq \frac{1}{40}\ell(Q)$, one of the squares Q_i is contained in $Q \setminus 2^{-7}Q$. From this fact one can easily deduce the existence of some constant $\lambda_0 > 2$ (depending on $A, \delta, C_8, C_9, \ldots$) such that for each $n \geq 1$ with $\lambda_0^{n+1}\ell(Q) \leq \ell(R)$ there exists some square $P_n \subset \frac{1}{2}\lambda_0^{n+1} Q \setminus \lambda_0^n Q$ satisfying $\ell(P_n) \simeq \lambda_0^{n+1}\ell(Q)$ and $\theta_\mu(P_n) \geq C(A,\delta)^{-1}\theta_\mu(R)$. If we set $\lambda := \lambda_0^2$, $Q_n^a := P_{2n}$, and $Q_n^b := P_{2n+1}$, the lemma follows. □

Remark 4.7. The lemma above also holds for $x \in G(R)$ (interchanged with the square $Q \in \operatorname{Qstp}(R)$ in the lemma) and for x such that $\ell_x = 0$. That is to say, increasing λ if necessary, for each $n \geq 1$ we have squares $Q_n^a, Q_n^b \subset B(x, \frac{1}{2}\lambda^{-n}\ell(R)) \setminus B(x, \lambda^{-n-1}\ell(R))$ which satisfy properties analogous to (b), (c) and (d).

In the following lemma we show a version of (4.4) which involves the curvature $c^2(\mu)$ truncated by the function ℓ_x.

LEMMA 4.8. *Let* $C_{10} > 0$ *be a fixed constant. For all* $x \in 3R$,

$$\iint_{\substack{y,z\in 3R\\ |x-y|\geq C_{10}\ell_x}} c(x,y,z)^2\, d\mu(y)d\mu(z) \;\leq\; C_{11}A^2\theta_\mu(R)^2,$$

where C_{11} *depends on* C_{10}.

The proof of the preceding estimate follows easily from (3.3) and Lemma 3.5 (as in (3.4)). We will not go through the details.

Proof of (4.1). We follow quite closely Jones' ideas (see [Pa, pp. 39–44]).

It is enough to show that (4.1) holds for dyadic squares Q with $\ell(Q) \leq (C_{12}\lambda)^{-1}\ell(R)$ (with λ given by Lemma 4.6 and C_{12} to be fixed below). So we assume $\ell(Q) \leq (C_{12}\lambda)^{-1}\ell(R)$. Also, the sum (4.1) can be restricted to those squares $P \in \mathcal{D}$ such that $P \cap K \neq \varnothing$. That is, it suffices to prove that

$$\sum_{P\in\mathcal{D},\, P\subset Q,\, P\cap K\neq\varnothing} \beta_K(P)^2\ell(P) \leq C(A,\delta)\ell(Q). \tag{4.9}$$

The main step of the proof of (4.9) consists of estimating $\beta_K(P)$ for some P as in the sum above in terms of $c^2(\mu)$. By standard arguments, if $\beta(P) \neq 0$, there are three pairwise different points z_0, z_1, $z_2 \in K \cap 3P$ such that

$$\beta_K(P) \simeq \frac{w(z_0, z_1, z_2)}{\ell(P)}, \tag{4.10}$$

where $w(z_0, z_1, z_2)$ stands for the width of the thinnest infinite strip containing z_0, z_1, z_2. By (b) of Lemma 4.5, $\ell_{z_0} \leq C|z_0 - z_1| \leq C\ell(P)$. So either $\ell_{z_0} = 0$ or there is some $P_0 \in \mathrm{Qstp}(R)$ with $\ell(P_0) + \frac{1}{40}\mathrm{dist}(z_0, P_0) \leq C\ell(P)$. In any case, if C_{12} has been chosen big enough, there are "many" balanced squares $\widehat{P}$ which contain z_0, z_1, z_2 such that $\ell(P) \leq \ell(\widehat{P}) \leq C(A, \delta)\ell(P)$. Arguing as in Lemma 4.6 , we deduce that there exist two squares S_a, S_b contained in $C(A, \delta)P \setminus P$ satisfying the properties stated just above Lemma 4.1. For any $x_0 \in S_a$ and $y_0 \in S_b$,

$$w(z_0, z_1, z_2) \lesssim \mathrm{dist}(z_0, L_{x_0,y_0}) + \mathrm{dist}(z_1, L_{x_0,y_0}) + \mathrm{dist}(z_2, L_{x_0,y_0}),$$

as it is easy to check. Integrating over $x_0 \in S_a$ and $y_0 \in S_b$ we get

$$\begin{aligned} &w(z_0, z_1, z_2) \lesssim \frac{1}{\mu(S_a)\,\mu(S_b)} \\ &\times \iint_{\substack{x_0 \in S_a \\ y_0 \in S_b}} \big[\mathrm{dist}(z_0, L_{x_0,y_0}) + \mathrm{dist}(z_1, L_{x_0,y_0}) + \mathrm{dist}(z_2, L_{x_0,y_0})\big]\, d\mu(x_0) d\mu(y_0). \end{aligned}$$

Thus there exists some z_i, say z_0, such that

$$w(z_0, z_1, z_2) \lesssim \frac{1}{\mu(S_a)\,\mu(S_b)} \iint_{\substack{x_0 \in S_a \\ y_0 \in S_b}} \mathrm{dist}(z_i, L_{x_0,y_0})\, d\mu(x_0) d\mu(y_0). \tag{4.11}$$

From Lemma 4.6 and the subsequent remark we deduce that there are two families of squares $\{P_n^a\}_{n\geq 1}$, $\{P_n^b\}_{n\geq 1}$ which satisfy the following properties for any $n \geq 1$ such that $\lambda^{n+1} \leq \ell(P)/\ell_{z_0}$:

(a) $P_n^a, P_n^b \subset B(z_0, \frac{1}{2}\lambda^{-n}\ell(P)) \setminus B(z_0, \lambda^{-n-1}\ell(P))$,

(b) $\mathrm{dist}(P_n^a, P_n^b) \gtrsim \lambda^{-n-1}\ell(P)$,

(c) $\lambda^{-n-1}\ell(P) \lesssim \ell(P_n^i) \lesssim \lambda^{-n}\ell(P)$, for $i = a, b$,

(d) $C(A, \delta)^{-1}\theta_\mu(R) \lesssim \theta_\mu(P_n^i) \lesssim A\theta_\mu(R)$, for $i = a, b$,

with $\lambda > 4$. We also set $P_0^a := S_a$ and $P_0^b := S_b$.

Let N be the biggest positive integer such that $\lambda^{N+1} \leq \ell(P)/\ell_{z_0}$ (with $N = \infty$ if $\ell_{z_0} = 0$). We claim that for all points $x_n \in P_n^a$ and $y_n \in P_n^b$ we have

$$\mathrm{dist}(z_0, L_{x_0,y_0}) \leq C \sum_{n=0}^{N} \big[\mathrm{dist}(x_{n+1}, L_{x_n,y_n}) + \mathrm{dist}(y_{n+1}, L_{x_n,y_n})\big], \tag{4.12}$$

where $x_{N+1} = y_{N+1} = z_0$ if $N < \infty$, and C depends on A, δ, λ (like all the following constants denoted by C in the rest of the proof of (4.9)).

Assuming the claim for the moment, from (4.12) we get

$$\begin{aligned}\operatorname{dist}(z_0, L_{x_0,y_0}) &\lesssim \sum_{n=0}^{N}\Big[c(x_n,y_n,x_{n+1})\,|x_n - x_{n+1}|\,|y_n - x_{n+1}| \\ &\qquad + c(x_n,y_n,y_{n+1})\,|x_n - y_{n+1}|\,|y_n - y_{n+1}|\Big] \\ &\lesssim \sum_{n=0}^{N}\big[c(x_n,y_n,x_{n+1}) + c(x_n,y_n,y_{n+1})\big]\lambda^{-2n}\ell(P)^2.\end{aligned}$$

From (4.11), taking the μ-mean of the above inequality over $x_0 \in P_0^a = S_a$ and $y_0 \in P_0^b = S_b$, then over $x_1 \in P_1^a$ and $y_1 \in P_1^b$, over $x_2 \in P_2^a$ and $y_2 \in P_2^b$, and so on, we obtain

(4.13)

$$\begin{aligned}w(z_0,z_1,z_2) &\lesssim \ell(P)^2 \sum_{n=0}^{N-1}\lambda^{-2n}\left[\frac{A_n}{\mu(P_n^a)\mu(P_n^b)\mu(P_{n+1}^a)} + \frac{B_n}{\mu(P_n^a)\mu(P_n^b)\mu(P_{n+1}^b)}\right] \\ &\qquad + \ell_{z_0}^2 \iint_{\substack{x_N \in P_N^a \\ y_N \in P_N^b}} c(x_N,y_N,z_0)\,d\mu(x_N)d\mu(y_N),\end{aligned}$$

where

$$A_n := \iiint_{\substack{x_n \in P_n^a \\ y_n \in P_n^b \\ x_{n+1} \in P_{n+1}^a}} c(x_n,y_n,x_{n+1})\,d\mu(x_n)d\mu(y_n)d\mu(x_{n+1})$$

and

$$B_n := \iiint_{\substack{x_n \in P_n^a \\ y_n \in P_n^b \\ y_{n+1} \in P_{n+1}^b}} c(x_n,y_n,y_{n+1})\,d\mu(x_n)d\mu(y_n)d\mu(y_{n+1}).$$

Note that the last term in (4.13) (the one involving ℓ_{z_0}) only appears when $N < \infty$ (i.e. when $\ell_{z_0} > 0$). By Hölder inequality, the estimates for the squares P_n^i below (4.11), and Lemma 4.5 (c) (with η big enough), we get

$$\begin{aligned}w(z_0,z_1,z_2) &\lesssim \ell(P)^2 \sum_{n=0}^{N-1}\lambda^{-2n}\left[\frac{c_\mu^2(P_n^a,P_n^b,P_{n+1}^a)^{1/2}}{\big(\mu(P_n^a)\mu(P_n^b)\mu(P_{n+1}^a)\big)^{1/2}}\right. \\ &\qquad \left. + \frac{c_\mu^2(P_n^a,P_n^b,P_{n+1}^b)^{1/2}}{\big(\mu(P_n^a)\mu(P_n^b)\mu(P_{n+1}^b)\big)^{1/2}}\right] + \ell(P)^2\lambda^{-2N}\frac{c_\mu^2(z_0,P_N^a,P_N^b)^{1/2}}{\big(\mu(P_N^a)\mu(P_N^b)\big)^{1/2}} \\ &\lesssim \theta_\mu(R)^{-3/2}\ell(P)^{1/2}\left[\sum_{n=0}^{N-1}\lambda^{-n/2}c_\mu^2(P_n^a,P_n^b,P_{n+1}^a \cup P_{n+1}^b)^{1/2}\right. \\ &\qquad \left. + \lambda^{-N/2}\left(\iiint_{\substack{x,y,z \in B(z_0,C\eta\ell_{z_0}) \\ |y-z| \geq \ell_{z_0}}} c(x,y,z)^2\,d\mu(x)d\mu(y)d\mu(z)\right)^{1/2}\right].\end{aligned}$$

By (4.10) and Cauchy-Schwartz, we obtain

$$\beta_K(P)^2 \lesssim \frac{1}{\theta_\mu(R)^3\ell(P)}\Bigg[\sum_{n=0}^N \lambda^{-n/2} c_\mu^2(P_n^a, P_n^b, P_{n+1}^a \cup P_{n+1}^b) + \lambda^{-N/2} \iiint_{\substack{x,y,z\in B(z_0, C\eta\ell_{z_0})\\ |y-z|\geq \ell_{z_0}}} c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z)\Bigg].$$

Notice that for every $x_n \in P_n^a$, we have $\ell_{x_n} \lesssim \lambda^{-n}\ell(P)$, because $\ell_{z_0} \lesssim \lambda^{-n}\ell(P)$, $\operatorname{dist}(x_n, z_0) \simeq \lambda^{-n}\ell(P)$, and ℓ_x is a Lipschitz function of x. Analogously, if $y_n \in P_n^b$, then $\ell_{y_n} \lesssim \lambda^{-n}\ell(P)$. Moreover, since for each n there exists some dyadic square $S \subset P$ such that $P_n^a \cup P_n^b \cup P_{n+1}^a \cup P_{n+1}^b$ is contained in $3S$, with $\ell(S) \simeq \lambda^{-n}\ell(P)$, we infer that

$$\theta_\mu(R)^3\beta_K(P)^2\ell(P) \lesssim \sum_{S\subset P}\left(\frac{\ell(S)}{\ell(P)}\right)^{1/2} \iiint_{(x,y,z)\in S^*\cap R^\ell} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z),$$

where S^* is the set of $(\xi_1,\xi_2,\xi_3) \in (3S)^3$ such that $|\xi_2 - \xi_3| \geq C^{-1}\ell(S)$, and R^ℓ is the set of $(\xi_1,\xi_2,\xi_3) \in (3R)^3$ such that $|\xi_2 - \xi_3| \geq C^{-1}(\ell_{\xi_2} + \ell_{\xi_3})$.

Now, for a fixed dyadic square Q with $\ell(Q) \leq (C_{12}\lambda)^{-1}\ell(R)$, by Lemma 4.8, we get (with the sums over P and S only for dyadic squares)

$$\begin{aligned}
&\theta_\mu(R)^3 \sum_{P:P\subset Q} \beta_K(P)^2\ell(P) \\
&\lesssim \sum_{S:S\subset Q} \iiint_{(x,y,z)\in S^*\cap R^\ell} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z) \sum_{P:S\subset P\subset Q}\left(\frac{\ell(S)}{\ell(P)}\right)^{1/2} \\
&\lesssim \sum_{S:S\subset Q} \iiint_{(x,y,z)\in S^*\cap R^\ell} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z) \\
&\lesssim \iiint_{(x,y,z)\in (3Q)^3\cap R^\ell} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z) \\
&\lesssim \theta_\mu(R)^2\mu(3Q) \lesssim \theta_\mu(R)^3\ell(Q).
\end{aligned}$$

It only remains to prove (4.12). Indeed, suppose for simplicity that $N < \infty$. We set $a_{N+1} := z_0$, and for $0 \leq n \leq N$, let a_n be the orthogonal projection of a_{n+1} onto L_{x_n,y_n}. Then

$$\operatorname{dist}(z_0, L_{x_0,y_0}) \leq \sum_{n=0}^N \operatorname{dist}(a_n, a_{n+1}). \tag{4.14}$$

Let us check that $a_n \in B(z_0, \lambda^{-n}\ell(R))$ for $0 \leq n \leq N+1$. We argue by (backward) induction. The statement is clearly true for a_{N+1}. Suppose now that $a_{n+1} \in B(z_0, \lambda^{-n-1}\ell(R))$. By construction, we have $x_n, y_n \in B(z_0, \frac{1}{2}\lambda^{-n}\ell(R))$.

On the other hand, since we are assuming $\lambda > 4$, then $a_{n+1} \in B(z_0, \frac{1}{2}\lambda^{-n}\ell(R))$ too. Then, by elementary geometry,

$$a_n \in B(z_0, 2^{-1/2}\lambda^{-n}\ell(R)) \subset B(z_0, \lambda^{-n}\ell(R)).$$

Since a_{n+1}, x_{n+1}, $y_{n+1} \in B(z_0, \lambda^{-n-1}\ell(R))$, they are collinear, and

$$|x_{n+1} - y_{n+1}| \gtrsim \lambda^{-n-1}\ell(R),$$

we deduce

$$\operatorname{dist}(a_n, a_{n+1}) = \operatorname{dist}(a_{n+1}, L_{x_n,y_n}) \lesssim \operatorname{dist}(x_{n+1}, L_{x_n,y_n}) + \operatorname{dist}(y_{n+1}, L_{x_n,y_n}).$$

Thus (4.12) follows from (4.14). □

5. The packing condition for the top squares

5.1. *The family* $\operatorname{Stop}^{1/2}_{\max}(R)$. In order to prove the packing condition

$$\sum_{R\in\operatorname{Top}(E)} \theta_\mu(R)^2\mu(R) \le C(\mu(E) + c^2(\mu))$$

(with C depending on A, δ, $\varepsilon_0, \dots$), we need to introduce some auxiliary families of squares. Let $\operatorname{Stop}_{\max}(R)$ be the subfamily of those squares $Q \in \operatorname{Stop}(R)$ such that there does not exist another square $Q' \in \operatorname{Top}(E)$, with $Q' \cap 3R \neq \varnothing$ and $\ell(Q') \le \ell(R)/8$, such that $Q \in \operatorname{Stop}(Q')$. So $\operatorname{Stop}_{\max}(R)$ is a maximal subfamily of $\operatorname{Stop}(R)$ in a sense. Notice, in particular, that if $Q \in \operatorname{Stop}_{\max}(R)$, then $Q \notin \operatorname{Stop}(Q')$ for any $Q' \in \operatorname{Stop}(R)$. We also denote by $\operatorname{Stop}^{1/2}_{\max}(R)$ the subfamily of the squares $Q \in \operatorname{Stop}_{\max}(R)$ such that $4Q \cap \frac{1}{2}R \neq \varnothing$.

Lemma 5.1. *For every* $R \in \operatorname{Top}(E)$, *there is* $\sum_{Q\in\operatorname{Stop}(R)} \chi_{\frac{1}{2}Q} \le C$.

Proof. Suppose that $\frac{1}{2}Q\cap\frac{1}{2}Q' \neq \varnothing$ for $Q, Q' \in \operatorname{Stop}(R)$. If $\ell(Q') \le \ell(Q)/4$, then $Q' \subset Q$, which contradicts the definition of $\operatorname{Stop}(R)$. Thus, $\ell(Q') \ge \ell(Q)/2$, and in an analogous way, we have $\ell(Q) \ge \ell(Q')/2$. The lemma follows from the fact that there is a bounded number of 4-dyadic squares Q' such that $\ell(Q)/2 \le \ell(Q') \le 2\ell(Q)$ for Q fixed. □

Lemma 5.2. (a) *Suppose that* $R_1, R_2 \in \operatorname{Top}(E)$ *and* $Q \in \operatorname{Stop}^{1/2}_{\max}(R_1) \cap \operatorname{Stop}^{1/2}_{\max}(R_2)$. *Then,*

$$\ell(R_2)/4 \le \ell(R_1) \le 4\ell(R_2). \tag{5.1}$$

(b) *There exists an absolute constant* N_0 *such that for any* $Q \in \operatorname{Top}(E)$

$$\#\{R \in \operatorname{Top}(E) :\, Q \in \operatorname{Stop}^{1/2}_{\max}(R)\} \le N_0.$$

(c) *For* $R \in \mathrm{Top}(E)$,

$$\bigcup_{P\in \mathrm{Stop}(R)} P \subset \bigcup_{Q\in \mathrm{Stop}_{\max}(R)} 4Q,$$

and, more generally, if $P, R \in \mathrm{Top}(E)$ *are such that* $P \cap 3R \neq \varnothing$ *and* $\ell(P) \leq \ell(R)/8$, *then there exists some square* $Q \in \mathrm{Stop}_{\max}(R)$ *such that* $P \subset 4Q$.

(d) *For* $R \in \mathrm{Top}(E)$,

$$G^{1/2}(R) := \Big\{x \in \tfrac{1}{2}R \setminus \bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} 4Q\Big\}.$$

Then, for each $x \in E$,

$$\#\{R \in \mathrm{Top}(E) : x \in G^{1/2}(R)\} \leq N_1,$$

where N_1 *is an absolute constant.*

(e) *If* $R_1, R_2 \in \mathrm{Top}(E)$, $Q \in \mathrm{Stop}^{1/2}_{\max}(R_1) \cap \mathrm{Bad}(R_2)$, *then*

$$\ell(R_1)/4 \leq \ell(R_2) \leq 4\ell(R_1)$$

and $\theta_\mu(R_1) \simeq \theta_\mu(R_2)$.

Proof. First we show (a). If $Q \in \mathrm{Stop}^{1/2}_{\max}(R_1) \cap \mathrm{Stop}^{1/2}_{\max}(R_2)$, then

$$4Q \cap \tfrac{1}{2}R_1 \neq \varnothing \qquad \text{and} \qquad 4Q \cap \tfrac{1}{2}R_2 \neq \varnothing. \tag{5.2}$$

It is easily seen that if $\ell(R_2) \leq \ell(R_1)/4$, then (5.2) and the fact that $\ell(Q) \leq \min(\ell(R_1), \ell(R_2))/8$ imply that $R_2 \subset R_1$. As a consequence, if $\ell(R_2) \leq \ell(R_1)/8$, then $Q \subset R_2 \subset R_1$ and so, by definition, $Q \notin \mathrm{Stop}(R_1)$.

The same happens if we reverse the roles of R_1 and R_2, and so (5.1) holds.

It is easy to check that (b) follows from (a). This is left for the reader.

The statement (c) of the lemma follows from the fact that if there is a sequence of squares $Q_1, Q_2, \ldots, Q_n = P$ with $Q_1 \in \mathrm{Stop}_{\max}(R)$ and $Q_{j+1} \in \mathrm{Stop}(Q_j)$ for $j \geq 1$, then $Q_{j+1} \cap 3Q_j \neq \varnothing$ and $\ell(Q_{j+1}) \leq \ell(Q_j)/8$, and so

$$\begin{aligned}\mathrm{dist}_\infty(Q_1, Q_n) + \ell(Q_n) &\leq \sum_{j=1}^{n-1} \big[\mathrm{dist}_\infty(Q_j, Q_{j+1}) + \ell(Q_{j+1})\big] \\ &\leq \sum_{j=1}^{\infty} \big[8^{1-j} + 8^{-j}\big]\, \ell(Q_1) = \frac{9}{7}\,\ell(Q_1) \leq \frac{3}{2}\,\ell(Q_1),\end{aligned}$$

which implies that $Q_n \subset 4Q_1$ (dist_∞ stands for the distance induced by the norm $\|\cdot\|_\infty$).

Let us show (d) now. Let $R_1, R_2 \in \mathrm{Top}(E)$ be such that $x \in G^{1/2}(R_1) \cap G^{1/2}(R_2)$. If $\ell(R_2) \leq \ell(R_1)/8$, then by (c), R_2 is contained in $4Q$ for some

$Q \in \mathrm{Stop}_{\max}(R_1)$. Since $x \in \frac{1}{2}R_1 \cap \frac{1}{2}R_2$, we have $4Q \cap \frac{1}{2}R_1 \neq \varnothing$, and so $Q \in \mathrm{Stop}_{\max}^{1/2}(R_1)$, which is a contradiction. Therefore, $\ell(R_2) > \ell(R_1)/8$. The same inequality holds interchanging R_1 by R_2. Thus,

$$\ell(R_2)/4 \leq \ell(R_1) \leq 4\ell(R_2).$$

That is, all the squares $R \in \mathrm{Top}(E)$ such that $x \in G^{1/2}(R)$ have comparable sizes, which implies that the number of these squares R is bounded above by some absolute constant.

Finally we will prove (e). Suppose that $\ell(R_2) \leq \ell(R_1)/8$. From

$$4Q \cap \frac{1}{2}R_1 \neq \varnothing \qquad \text{and} \qquad \ell(Q) \leq \ell(R_2)/8, \tag{5.3}$$

we infer that $R_2 \subset 3R_1$, and so $R_2 \in \mathrm{Stop}(R_1)$. This implies that $Q \notin \mathrm{Stop}_{\max}(R_1)$, which is a contradiction. Thus, $\ell(R_2) \geq \ell(R_1)/4$.

The inequality $\ell(R_2) \leq 4\ell(R_1)$ also holds. Otherwise $\ell(R_1) \leq \ell(R_2)/8$ and $Q \subset R_1$ (by (5.3)) imply that $Q \notin \mathrm{Top}(E)$ (it should have been eliminated when constructing $\mathrm{Top}(E)$ from $\mathrm{Top}_0(E)$).

The comparability between $\theta_\mu(R_1)$ and $\theta_\mu(R_2)$ follows from $4R_1 \cap 4R_2 \neq \varnothing$ (since Q is contained in the intersection) and $1/4 \leq \ell(R_1)/\ell(R_2) \leq 4$. Indeed, one easily deduces that then $R_1 \subset 16R_2$ and $R_2 \subset 16R_1$, and by the doubling properties of R_1 and R_2, one gets $\theta_\mu(R_1) \simeq \theta_\mu(R_2)$. $\square$

The next result is a consequence of Lemma 3.3 and the statement (e) in the preceding lemma.

LEMMA 5.3. *If $R \in \mathrm{Top}(E)$ and $Q \in \mathrm{Stop}_{\max}^{1/2}(R)$, then*:

(a) *If $Q \in \mathrm{HD}(R)$, then $\theta_\mu(Q) \gtrsim A\,\theta_\mu(R)$.*

(b) *If $Q \in \mathrm{HC}(R)$, then*

$$\mu\big\{x \in \tfrac{1}{2}Q\colon K_{\mu,J(Q)+12}\chi_E(x) - K_{\mu,J(R)-6}\chi_E(x) \geq C^{-1}\varepsilon_0\theta_\mu(R)^2\big\} \geq C_{13}^{-1}\mu(Q). \tag{5.4}$$

(c) *If $Q \in \mathrm{LD}(R)$, then there exists some square S_Q such that $Q \subset \frac{1}{20}S_Q$, with $\ell(S_Q) \leq \ell(R)/2$, and $\theta_\mu(S_Q) \leq C_{14}\delta\theta(R)$. Also,*

$$\mu\big\{x \in \tfrac{1}{2}Q:\ K_{\mu,J(Q)+12}\chi_E(x) - K_{\mu,J(R)-2}\chi_E(x) \leq C\varepsilon_0\theta_\mu(R)^2\big\} \geq C_{15}^{-1}\mu(Q). \tag{5.5}$$

Proof. Let $R_2 \in \mathrm{Top}(E)$ be such that $Q \in \mathrm{Bad}(R_2)$.

If $Q \in \mathrm{HD}(R)$, then $Q \in \mathrm{HD}_0(R_2)$ by definition, and by (b) in Lemma 3.3 and (e) in the preceding lemma, $\theta_\mu(Q) \gtrsim A\theta_\mu(R_2) \simeq A\theta_\mu(R)$.

If $Q \in \mathrm{HC}(R)$, then $Q \in \mathrm{HC}_0(R_2)$. Inequality (5.4) follows from Lemma 3.3 (c), and the fact that $\theta_\mu(R) \simeq \theta_\mu(R_2)$ and $|J(R) - J(R_2)| \leq 2$ by (e) in the preceding lemma.

The statement (c) also follows easily from Lemma 5.2 (e) and the definition of $\mathrm{LD}_0(R)$ and Lemma 3.3 (d). $\square$

Notice that, since $\frac{1}{2}R$ is doubling,

$$\mu(R) \lesssim \mu(\tfrac{1}{2}R) \le \mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} 4Q\Big) + \mu\Big(\tfrac{1}{2}R \setminus \bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} 4Q\Big). \tag{5.6}$$

We distinguish a special kind of square $R \in \mathrm{Top}(E)$. We set $R \in VC(E)$ (and we say that μ is *very concentrated* on R) if

$$\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HD}(R)} 4Q\Big) > \frac{1}{2}\,\mu(\tfrac{1}{2}R).$$

5.2. *Squares with μ very concentrated.* For $R \in VC(E)$, using the doubling properties of $\frac{1}{2}R$ and $Q \in \mathrm{Stop}^{1/2}_{\max}(R)$, we get

$$\begin{aligned}\mu(R) \lesssim \mu(\tfrac{1}{2}R) &\le 2\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HD}(R)} 4Q\Big)\\ &\lesssim \sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HD}(R)} \mu(Q),\end{aligned}$$

and since $\theta_\mu(Q) \gtrsim A\theta_\mu(R)$ for $Q \in \mathrm{Stop}^{1/2}_{\max}(R) \cap \mathrm{HD}(R)$,

$$\begin{aligned}\theta_\mu(R)^2\mu(R) &\lesssim \frac{1}{A^2} \sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HD}(R)} \theta_\mu(Q)^2\mu(Q)\\ &\lesssim \frac{1}{A^2} \sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} \theta_\mu(Q)^2\mu(Q).\end{aligned}$$

Then by (b) of Lemma 5.2,

$$\begin{aligned}\sum_{R\in \mathrm{Top}(E)\cap VC(E)} \theta_\mu(R)^2\mu(R) &\le \frac{C}{A^2} \sum_{R\in \mathrm{Top}(E)} \sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} \theta_\mu(Q)^2\mu(Q)\\ &\le \frac{C_{16}N_0}{A^2} \sum_{Q\in \mathrm{Top}(E)} \theta_\mu(Q)^2\mu(Q).\end{aligned}$$

If we choose A such that $C_{16}N_0/A^2 \le 1/2$, we deduce (see Remark 5.4 below)

$$\sum_{R\in \mathrm{Top}(E)} \theta_\mu(R)^2\mu(R) \le 2 \sum_{R\in \mathrm{Top}(E)\setminus VC(E)} \theta_\mu(R)^2\mu(R). \tag{5.7}$$

5.3. *Squares with μ not very concentrated.* If $R \notin VC(E)$, then

$$\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HD}(R)} 4Q\Big) \le \frac{1}{2}\,\mu(\tfrac{1}{2}R),$$

and by (5.6) we get

$$\mu(\tfrac{1}{2}R) \le 2\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\setminus \mathrm{HD}(R)} 4Q\Big) + 2\mu\Big(\tfrac{1}{2}R \setminus \bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} 4Q\Big).$$

Therefore,

$$\mu(R) \le C_{17}\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q\Big) + C_{17}\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HC}(R)} 4Q\Big) + C_{17}\mu\Big(\tfrac{1}{2}\,R \setminus \bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} 4Q\Big). \tag{5.8}$$

We will show in Section 7 below that

$$\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q\Big) \le \eta\mu(R), \tag{5.9}$$

with $\eta \le 1/(2C_{17})$, and with δ and ε_0 chosen appropriately. Thus,

$$\sum_{R\in \mathrm{Top}(E)\setminus VC(E)} \theta_\mu(R)^2\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q\Big) \le \eta \sum_{R\in \mathrm{Top}(E)\setminus VC(E)} \theta_\mu(R)^2\mu(R). \tag{5.10}$$

Also, in Section 6 we will prove that

$$\sum_{R\in \mathrm{Top}(E)} \sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HC}(R)} \theta_\mu(R)^2\mu(Q) \le C_{18}c^2(\mu), \tag{5.11}$$

with C_{18} possibly depending on A, δ, and ε_0.

Now we deal with the term

$$\sum_{R\in \mathrm{Top}(E)} \mu\Big(\tfrac{1}{2}\,R \setminus \bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} 4Q\Big) = \sum_{R\in \mathrm{Top}(E)} \mu(G^{1/2}(R)).$$

From (d) of Lemma 5.2 we deduce

$$\sum_{R\in \mathrm{Top}(E)} \mu(G^{1/2}(R)) \le N_1\mu(E).$$

Therefore,

$$\sum_{R\in \mathrm{Top}(E)} \theta_\mu(R)^2\mu\Big(\frac{1}{2}\,R \setminus \bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)} 4Q\Big) \le C_0^2 N_1\,\mu(E). \tag{5.12}$$

From (5.8), (5.10), (5.11) and (5.12), we get

$$\sum_{R\in \mathrm{Top}(E)\setminus VC(E)} \theta_\mu(R)^2\mu(R) \le C_{17}\eta \sum_{R\in \mathrm{Top}(E)\setminus VC(E)} \theta_\mu(R)^2\mu(R) + Cc^2(\mu) + C_{17}C_0^2N_1\,\mu(E).$$

Therefore, since $\eta < 1/(2C_{17})$,

$$\sum_{R\in \mathrm{Top}(E)\setminus VC(E)} \theta_\mu(R)^2\mu(R) \;\le\; C_{18}\big[c^2(\mu) + \mu(E)\big],$$

where C_{18} depends on A and ε_0 (see the remark below). So (3.1) follows from this estimate and (5.7).

Remark 5.4. The arguments above work if one assumes *a priori* that

$$\sum_{R\in \mathrm{Top}(E)} \theta_\mu(R)^2\mu(R) < \infty.$$

To circumvent this difficulty it is necessary to argue more carefully. For example, we can operate with finite subsets of $\mathrm{Top}(E)$. Let $\mathrm{Top}_n(E)$ be the subfamily of $\mathrm{Top}(E)$ of those squares with side length $\geq 2^{-n}$; and let $A_n(E)$ be the family

$$\bigl\{Q \in \mathrm{Top}(E)\setminus \mathrm{Top}_n(E) : \exists R\in \mathrm{Top}_n(E) \text{ such that } Q\in \mathrm{Stop}^{1/2}_{\max}(R)\bigr\}.$$

Then, it can be checked that a slight modification of the preceding estimates yields

$$\begin{aligned}\sum_{R\in \mathrm{Top}_n(E)} \theta_\mu(R)^2\mu(R) \lesssim \Bigl(\frac{1}{A^2}+\eta\Bigr) &\sum_{R\in \mathrm{Top}_n(E)} \theta_\mu(R)^2\mu(R)\\ &+\sum_{R\in A_n(E)} \theta_\mu(R)^2\mu(R) + c^2(\mu)+\mu(E).\end{aligned}$$

If we choose A big enough and η sufficiently small, we obtain

$$\sum_{R\in \mathrm{Top}_n(E)} \theta_\mu(R)^2\mu(R) \lesssim \sum_{R\in A_n(E)} \theta_\mu(R)^2\mu(R) + c^2(\mu)+\mu(E). \tag{5.13}$$

It can be shown that $\sum_{R\in A_n(E)}\chi_{\frac12 R}\leq C$ (this is left for the reader). Then

$$\sum_{R\in A_n(E)} \theta_\mu(R)^2\mu(R) \leq C_0^2 \sum_{R\in A_n(E)} \mu(\tfrac12 R) \lesssim C\mu(E),$$

and from (5.13) we deduce

$$\sum_{R\in \mathrm{Top}_n(E)} \theta_\mu(R)^2\mu(R) \lesssim c^2(\mu)+\mu(E),$$

uniformly on n.

6. Estimates for the high curvature squares

6.1. *The class* $\widehat{\mathrm{Top}}(E)$. To prove (5.11) it will be simpler to use dyadic squares than 4-dyadic squares.

LEMMA 6.1. *Let $R\in \mathrm{Top}(E)$ and $Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HC}(R)$. There exists a dyadic square $\widehat{Q}\subset \frac12 Q$, with $\ell(\widehat{Q})=\ell(Q)/4$, such that $\mu(\widehat{Q})\geq C_{19}^{-1}\mu(Q)$ and*

$$\int_{\widehat{Q}} \bigl(K_{\mu,J(\widehat{Q})+10}\chi_E - K_{\mu,J(R)-6}\chi_E\bigr)\,d\mu \gtrsim \varepsilon_0\theta_\mu(R)^2\mu(\widehat{Q}). \tag{6.1}$$

Proof. Let $P_1, \dots, P_4$ be the disjoint dyadic squares with side length $\ell(Q)/4$ such that $\frac{1}{2}Q = \bigcup_{1\le i\le 4} P_i$. Remember that

$$\mu\bigl\{x\in \tfrac{1}{2}Q\colon K_{\mu,J(Q)+12}\chi_E(x) - K_{\mu,J(R)-6}\chi_E(x) \ge C^{-1}\varepsilon_0\theta_\mu(R)^2\bigr\} \ge C_{13}^{-1}\mu(Q).$$

Let $\widehat{Q}$ be the square P_i such that

$$\mu\bigl\{x \in P_i :\ K_{\mu,J(Q)+12}\chi_E(x) - K_{\mu,J(R)-6}\chi_E(x) \ge C^{-1}\varepsilon_0\theta_\mu(R)^2\bigr\}$$

is maximal. Clearly, $\widehat{Q}$ satisfies (6.1), and $\mu(\widehat{Q}) \ge C_{13}^{-1}\mu(Q)/4$. □

For each square $Q \in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{HC}(R)$, with $R\in \mathrm{Top}(E)$, we choose a dyadic subsquare $\widehat{Q}$ of Q as in the lemma. In the following Subsections 6.2 and 6.3 we denote by $\widehat{\mathrm{Top}}(E)$ the class made up of all the chosen subsquares $\widehat{Q}$, and $\widehat{R}_0$ (which is the dyadic subsquare of $\frac{1}{2}R_0$ with side length $\ell(R_0)/4$ which contains E). Notice that it may happen that $\#\widehat{\mathrm{Top}}(E) < \#\mathrm{Top}(E)$, because not all the squares in $\mathrm{Top}(E)$ are high curvature squares.

6.2. *Decomposition of* $c^2(\mu)$. We denote the class of all dyadic squares contained in $\widehat{R}_0$ by Δ, and the class of dyadic squares contained in $\widehat{R}_0$ with side length 2^{-j}, by Δ_j.

Given $Q \in \widehat{\mathrm{Top}}(E)$, let $\widehat{G}(Q)$ be the set of points $x\in Q$ which do not belong to any square $P\in\widehat{\mathrm{Top}}(E)$, with $P\subsetneq Q$. Let us denote by $\mathrm{Term}(Q)$ the family of *maximal* dyadic squares $P\in\widehat{\mathrm{Top}}(E)$, with $P\subsetneq Q$. Finally, we let $\mathrm{Tree}(Q)$ be the class of dyadic squares contained in Q, different from Q, which contain either a point $x\in\widehat{G}(Q)$ or a square from $\mathrm{Term}(Q)$. The squares in $\mathrm{Term}(Q)$ are called *terminal* squares of the tree $\mathrm{Tree}(Q)$, for obvious reasons. Notice that we have

$$\Delta = \{\widehat{R}_0\} \cup \bigcup_{Q\in\widehat{\mathrm{Top}}(E)} \mathrm{Tree}(Q),$$

and that $\mathrm{Tree}(Q)\cap\mathrm{Tree}(R) = \varnothing$ if $Q\neq R$.

Given $Q \in \widehat{\mathrm{Top}}(E)$ with $Q\neq\widehat{R}_0$, we denote by $\mathrm{Root}(Q)$ the square R such that Q is a terminal square of $\mathrm{Tree}(R)$.

We split the curvature $c^2(\mu)$ as follows:

$$\begin{aligned} c^2(\mu) &\gtrsim \sum_j \sum_{Q\in\Delta_j} \int_Q (K_{\mu,j+10}\chi_E - K_{\mu,j-11}\chi_E)\,d\mu \\ &\gtrsim \sum_{R\in\widehat{\mathrm{Top}}(E)} \sum_{Q\in\mathrm{Tree}(R)} \int_Q (K_{\mu,J(Q)+10}\chi_E - K_{\mu,J(Q)-11}\chi_E)\,d\mu. \end{aligned}$$

Observe that if $P\in\mathrm{Term}(R)$ and $x\in P$, then

$$\begin{aligned} \sum_{Q\in\mathrm{Tree}(R)} &\chi_Q(x)\bigl(K_{\mu,J(Q)+10}\chi_E(x) - K_{\mu,J(Q)-11}\chi_E(x)\bigr) \\ &\ge K_{\mu,J(P)+10}\chi_E(x) - K_{\mu,J(R)-10}\chi_E(x). \end{aligned}$$

Therefore,

$$
\begin{aligned}
(6.2)\quad c^2(\mu) &\gtrsim \sum_{R\in\widehat{\operatorname{Top}}(E)} \sum_{Q\in\operatorname{Term}(R)} \int_Q \big(K_{\mu,J(Q)+10}\chi_E - K_{\mu,J(R)-10}\chi_E\big)\,d\mu \\
&= \sum_{Q\in\widehat{\operatorname{Top}}(E),Q\neq\widehat{R}_0} \int_Q \big(K_{\mu,J(Q)+10}\chi_E - K_{\mu,J(\operatorname{Root}(Q))-10}\chi_E\big)\,d\mu.
\end{aligned}
$$

6.3. *Proof of* (5.11). By Lemma 6.1 (we use the same notation as in the lemma), we have

$$
\begin{aligned}
&\sum_{R\in\operatorname{Top}(E)} \sum_{Q\in\operatorname{Stop}^{1/2}_{\max}(R)\cap\operatorname{HC}(R)} \theta_\mu(R)^2\mu(Q) \\
&\qquad\lesssim \sum_{R\in\operatorname{Top}(E)} \sum_{Q\in\operatorname{Stop}^{1/2}_{\max}(R)\cap\operatorname{HC}(R)} \theta_\mu(R)^2\mu(\widehat{Q}) \\
&\qquad\lesssim N_0 \sum_{P\in\widehat{\operatorname{Top}}(E),P\neq\widehat{R}_0} \theta_\mu(R_P)^2\mu(P),
\end{aligned}
$$

where N_0 is the constant appearing in (a) of Lemma 5.2, and $R_P\in\operatorname{Top}(E)$ is a square such that $P=\widehat{P}_1$ and $P_1\in\operatorname{Stop}^{1/2}_{\max}(R_P)$ for some P_1. The square R_P is not unique, but in any case remember that if $P_1\in\operatorname{Stop}^{1/2}_{\max}(R^1_P)\cap\operatorname{Stop}^{1/2}_{\max}(R^2_P)$, then $\theta_\mu(R^1_P)\simeq\theta_\mu(R^2_P)$. By (6.1),

$$
\begin{aligned}
(6.3)\quad &\sum_{R\in\operatorname{Top}(E)} \sum_{Q\in\operatorname{Stop}^{1/2}_{\max}(R)\cap\operatorname{HC}(R)} \theta_\mu(R)^2\mu(Q) \\
&\qquad\lesssim \varepsilon_0^{-1} \sum_{P\in\widehat{\operatorname{Top}}(E),P\neq\widehat{R}_0} \int_P \big(K_{\mu,J(P)+10}\chi_E - K_{\mu,J(R_P)-6}\chi_E\big)d\mu.
\end{aligned}
$$

For every $P\in\widehat{\operatorname{Top}}(E)$ different from R_0, we have $\ell(R_P)\leq 16\ell(\operatorname{Root}(P))$. This is clear if $\operatorname{Root}(P)=\widehat{R}_0$. For $\operatorname{Root}(P)\neq\widehat{R}_0$, let $P_1,R_1\in\operatorname{Top}(E)$ be such that $P=\widehat{P}_1$, $\operatorname{Root}(P)=\widehat{R}_1$. It is easily seen that $P_1\subsetneq R_1$. If $\ell(R_P)/16>\ell(\operatorname{Root}(P))=\ell(R_1)/4$, then $P_1\notin\operatorname{Stop}(R_P)$, by the definition of the family $\operatorname{Stop}(\cdot)$ (since $P_1\subset R_1$ and $\ell(R_1)\leq\ell(R_P)/8$). Thus,

$$
J(R_P)\geq J(\operatorname{Root}(P))-4,
$$

and so

$$
K_{\mu,J(P)+10}\chi_E - K_{\mu,J(R_P)-6}\chi_E \leq K_{\mu,J(P)+10}\chi_E - K_{\mu,J(\operatorname{Root}(P))-10}\chi_E.
$$

From (6.3), (6.2), and the preceding estimate we get

$$
\sum_{R\in\operatorname{Top}(E)} \sum_{Q\in\operatorname{Stop}^{1/2}_{\max}(R)\cap\operatorname{HC}(R)} \theta_\mu(R)^2\mu(Q) \lesssim \varepsilon_0^{-1}\,c^2(\mu). \qquad\square
$$

7. Estimates for the low density squares

To prove the packing condition (3.1) it remains to show that

$$\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q\Big) \leq \eta\mu(R), \tag{7.1}$$

with $\eta = 1/(2C_{17})$ (notice that η is an absolute constant and it does not depend either on A or δ).

7.1. *The big and small squares S_j, $j \in I_{\mathrm{LD}(R)}$.* For each $x \in 4Q$, $Q \in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)$, let S_x be a square such that $x \in \frac{1}{5}S_x$, $\theta_\mu(S_x) \leq C_{14}\delta\theta_\mu(R)$, and $\ell(S_x) = 2^{-m}\ell(R)$ with $m \geq 1$ (this square exists because of (c) in Lemma 5.3). Moreover, we assume that S_x has maximal side length among all the squares with these properties.

Let $\bigcup_{j\in I_{\mathrm{LD}(R)}} S_j$ be a Besicovitch covering of $\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q$, with $S_j := S_{x_j}$ as explained above.

LEMMA 7.1. *There exist $n_0 \geq 4$ and $C_{20} > 0$ such that if $\ell(S_j) \leq C_{20}^{-1}\ell(R)$ for $j \in I_{\mathrm{LD}(R)}$, then $\ell(64S_j) \leq \ell(2^{n_0}S_j) \leq \ell(R)$ and $\mu(2^{n_0}S_j) \geq 2\mu(S_j)$.*

Proof. Given n_0 such that $\ell(64S_j) \leq \ell(2^{n_0}S_j) \leq \ell(R)$, we have

$$\mu(2^{n_0}S_j) \gtrsim \delta\theta_\mu(R)\ell(2^{n_0}S_j) = 2^{n_0}\delta\theta_\mu(R)\ell(S_j) \gtrsim 2^{n_0}\mu(S_j).$$

Thus, for n_0 big enough, we have $\mu(2^{n_0}S_j) \geq 2\mu(S_j)$. So the lemma follows if C_{20} is big enough too (so that $\ell(2^{n_0}S_j) \leq \ell(R)$). □

To prove (7.1) we will distinguish two types of squares S_j. If S_j, $j \in I_{\mathrm{LD}(R)}$, satisfies

$$\ell(S_j) \geq \min\Big(C_8^{-1}A^{-2}, C_{20}^{-1}\Big)\,\ell(R), \tag{7.2}$$

where C_8 is as defined in Lemma 4.4, and C_{20} in Lemma 7.1, then we write $j \in I^b_{\mathrm{LD}(R)}$, and otherwise we set $j \in I^s_{\mathrm{LD}(R)}$ (the superindices "b" and "s" stand for "big" and "small" respectively).

Next we estimate the measure μ of the family of the big squares S_j:

LEMMA 7.2.

$$\mu\Big(\bigcup_{j\in I^b_{\mathrm{LD}(R)}} S_j\Big) \lesssim A^4\delta\mu(R).$$

Proof. We set $C_{21} := \max(C_8, C_{20})$. Then each square S_j, $j \in I^b_{\mathrm{LD}(R)}$, satisfies $\ell(S_j) \geq C_{21}^{-1}A^{-2}\ell(R)$. Since the family $\{S_j\}_{j\in I^b_{\mathrm{LD}(R)}}$ has finite superposition we have

$$\#I^b_{\mathrm{LD}(R)} \lesssim \Big(\frac{\ell(R)}{\inf_{j\in I^b_{\mathrm{LD}(R)}} \ell(S_j)}\Big)^2 \lesssim A^4.$$

Therefore,

$$\sum_{j\in I^b_{\mathrm{LD}(R)}} \mu(S_j) \lesssim \sum_{j\in I^b_{\mathrm{LD}(R)}} \delta\ell(S_j)\theta_\mu(R) \lesssim \delta\mu(R)\cdot \# I^b_{\mathrm{LD}(R)} \lesssim A^4\delta\mu(R). \qquad \square$$

7.2. *Estimates for the small squares* S_j, $j \in I^s_{\mathrm{LD}(R)}$. Now we turn our attention to the small squares S_j. For each $Q \in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)$, let W_Q be the set

$$\{x \in \tfrac{1}{2}Q \cap \mathrm{supp}(\mu) :\ K_{\mu,J(Q)+12}\chi_E(x) - K_{\mu,J(R)-2}\chi_E(x) \le C\varepsilon_0\theta_\mu(R)^2\}.$$

Remember that, by (5.5), $\mu(W_Q) \ge C^{-1}\mu(Q) \simeq \mu(4Q)$, since Q is 16-doubling.

For $j \in I_{\mathrm{LD}(R)}$, we denote

$$W_j := \bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} W_Q \cap S_j.$$

Remember that the family $\{S_j\}_{j\in I^s_{\mathrm{LD}(R)}}$ was obtained by an application of the Besicovitch covering theorem. So $\{S_j\}_{j\in I^s_{\mathrm{LD}(R)}}$ can be split into N_B subfamilies of pairwise disjoint squares S_j. Thus there exists a subfamily $\{S_j\}_{j\in I^{s,0}_{\mathrm{LD}(R)}}$, $I^{s,0}_{\mathrm{LD}(R)} \subset I^s_{\mathrm{LD}(R)}$, of pairwise disjoint squares such that

$$\mu\Bigl(\bigcup_{j\in I^{s,0}_{\mathrm{LD}(R)}} W_j\Bigr) \ge \frac{1}{N_B}\,\mu\Bigl(\bigcup_{j\in I^s_{\mathrm{LD}(R)}} W_j\Bigr).$$

We set

$$W := \bigcup_{j\in I_{\mathrm{LD}^{s,0}(R)}} W_j.$$

By the preceding lemma and since $\sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} \chi_{\frac{1}{2}Q} \le C$ (see Lemma 5.1), we get

$$\begin{aligned}
\mu\Bigl(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q\Bigr) &\le \sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} \mu(4Q)\\
&\lesssim \sum_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} \mu(W_Q)\\
&\lesssim \mu\Bigl(\bigcup_{j\in I^b_{\mathrm{LD}(R)}} W_j\Bigr) + \mu\Bigl(\bigcup_{j\in I^s_{\mathrm{LD}(R)}} W_j\Bigr)\\
&\lesssim A^4\delta\mu(R) + N_B \sum_{j\in I_{\mathrm{LD}^{s,0}(R)}} \mu(W_j).
\end{aligned} \tag{7.3}$$

Remark 7.3. Another useful property of our construction of the squares S_j is the following: *If* $Q \in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)$ *is such that* $Q\cap S_j \ne \varnothing$ *(for some* $j \in I_{\mathrm{LD}(R)}$*), then* $\ell(Q) \le \ell(S_j)$ *and* $Q \subset 3S_j$.

Indeed, suppose that $\ell(Q) > \ell(S_j)$. By Lemma 5.3 there exists some square S_Q such that $Q \subset \frac{1}{20}S_Q$ with $\theta_\mu(S_Q) \leq C_{14}\delta\theta_\mu(R)$, and $\ell(S_Q) = 2^{-m}\ell(R)$ with $m \geq 1$. Then we have $S_j \subset 3Q \subset \frac{1}{5}S_Q$, which is not possible, because of the choice of S_j with maximal size (besides other properties). The inclusion $Q \subset 3S_j$ is a direct consequence of the inequality $\ell(Q) \leq \ell(S_j)$ and the fact that $Q \cap S_j \neq \varnothing$.

Similar arguments show that, *if* $Q \in \mathrm{Stop}^{1/2}_{\max}(R) \cap \mathrm{LD}(R)$, *then*

$$\ell(Q) \leq \mathrm{dist}(Q, S_j) + \ell(S_j).$$

For $x \in W$, we set

$$\ell_x := 2^{-12} \inf\big\{\ell(Q) : \, Q \in \mathrm{Stop}^{1/2}_{\max}(R) \cap \mathrm{LD}(R), \, x \in Q\big\}.$$

Notice that, by the preceding remark, we have

$$\ell_x \leq 2^{-12}\big(\mathrm{dist}(x, S_j) + \ell(S_j)\big) \tag{7.4}$$

for each $j \in I_{\mathrm{LD}^{s,0}(R)}$. As a consequence, if $x \in S_j$, then $\ell_x \leq 2^{-12}\ell(S_j)$.

We consider the following truncated version of the curvature $c_\mu(x, 2R, 2R)$, for $x \in W$:

$$c^2_{tr,\mu}(x, 2R, 2R) := \iint_{\substack{y,z\in 2R\\ |x-y|>\ell_x}} c(x, y, z)^2 d\mu(y)d\mu(z).$$

The next lemma follows easily from our construction.

LEMMA 7.4. *For every* $x \in W$,

$$c^2_{tr,\mu}(x, 2R, 2R) \lesssim \varepsilon_0\theta_\mu(R)^2.$$

Proof. Let $Q \in \mathrm{Stop}^{1/2}_{\max}(R) \cap \mathrm{LD}(R)$ be such that $x \in W_Q$. By the definition of W_Q, we have

$$\begin{aligned} c^2_{tr,\mu}(x, 2R, 2R) &\leq \iint_{2^{-12}\ell(Q)<|x-y|\leq 4\ell(R)} c(x, y, z)^2 \, d\mu(y)d\mu(z) \\ &= K_{\mu,J(Q)+12}\chi_E(x) - K_{\mu,J(R)-2}\chi_E \lesssim \varepsilon_0\theta_\mu(R)^2. \qquad \square \end{aligned}$$

For each $j \in I^{s,0}_{\mathrm{LD}(R)}$, let L_j be a segment of length $\mathcal{H}^1(L_j) = \ell(S_j)/8$ contained in $\frac{1}{2}S_j$. The exact position and orientation of L_j will be fixed below. Let ν be the following measure

$$d\nu = \sum_{j\in I^{s,0}_{\mathrm{LD}(R)}} \frac{\mu(W_j)}{\mathcal{H}^1(L_j)} \, d\mathcal{H}^1_{|L_j}.$$

LEMMA 7.5. *The measure* ν *satisfies*

$$\nu(B(x, r)) \lesssim CA\theta_\mu(R)r \qquad \textit{for all } x \in \mathbb{C} \textit{ and } r > 0, \tag{7.5}$$

and if the position and orientation of each L_j are chosen appropriately, also

$$c^2(\nu) \leq C(A,\delta)\varepsilon_0^{1/50}\theta_\mu(R)^2\mu(R). \tag{7.6}$$

We defer the proof of this lemma until Subsection 7.4. For the moment, we only remark that it follows from the estimate of $c^2_{tr,\mu}(x,2R,2R)$ for $x \in W$ in Lemma 7.4, by comparison.

Now we recall David-Léger's theorem [Lé] (the quantitative version in [Lé, Prop. 1.2]).

THEOREM D. *For any $c_0 > 0$, there exists some $\varepsilon_L > 0$ such that if τ is a Radon measure whose support is contained in a square R and satisfies*:

(a) $\tau(R) \geq \ell(R)$,

(b) $\tau(B(x,r)) \leq c_0 r$ *for any* $x \in \mathbb{C}$, $r > 0$, *and*

(c) $c^2(\tau) \leq \varepsilon_L \ell(R)$,

then there exists a Lipschitz graph Γ with slope $\leq 1/10$ (with respect to the appropriate axes) such that $\tau(\Gamma) \geq \dfrac{99}{100}\,\tau(R)$.

7.3. *Proof of* (5.9). Suppose that

$$\mu\Big(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q\Big) > \eta\mu(R). \tag{7.7}$$

Remember that if δ is small enough, by (7.3),

$$\mu\Bigg(\bigcup_{Q\in \mathrm{Stop}^{1/2}_{\max}(R)\cap \mathrm{LD}(R)} 4Q\Bigg) \leq \frac{\eta}{2}\,\mu(R) + CN_B \sum_{j\in I_{\mathrm{LD}^{s,0}(R)}} \mu(W_j). \tag{7.8}$$

So we only have to estimate

$$\sum_{j\in I_{\mathrm{LD}^{s,0}(R)}} \mu(W_j) = \nu(R).$$

From the assumption (7.7) and inequality (7.8) we deduce

$$\nu(R) \geq C^{-1}N_B^{-1}\frac{\eta}{2}\,\mu(R) =: C_{22}^{-1}\eta\mu(R). \tag{7.9}$$

Considering the measure $\tau := \dfrac{C_{22}}{\eta\theta_\mu(R)}\,\nu$, we have

$$\tau(R) \geq \frac{C_{22}}{\eta\theta_\mu(R)}\,C_{22}^{-1}\eta\mu(R) = \ell(R).$$

On the other hand, any ball $B(x,r)$ satisfies

$$\tau(B(x,r)) \leq \frac{C_{22}}{\eta}\,CA\,r,$$

because of the estimate on the linear growth of ν in Lemma 7.5. Further, from the same lemma we also get the following estimate for $c^2(\tau)$:

$$\begin{aligned} c^2(\tau) &= \frac{C_{22}^3}{\eta^3 \theta_\mu(R)^3}\, c^2(\nu) \\ &\le \frac{C_{22}^3}{\eta^3 \theta_\mu(R)^3}\, C(A,\delta)\, \varepsilon_0^{1/50} \theta_\mu(R)^2\, \mu(R) = \frac{C_{22}^3 C(A,\delta)}{\eta^3}\, \varepsilon_0^{1/50}\, \ell(R). \end{aligned}$$

Therefore, by Theorem D, if ε_0 is small enough, there exists a Lipschitz graph Γ with slope $\le 1/10$ such that $\tau(\Gamma) \ge \frac{99}{100}\,\tau(R)$, which is equivalent to saying

$$\nu(\Gamma) \ge \frac{99}{100}\,\nu(R).$$

Let J be the subset of indices $j \in I^{s,0}_{\mathrm{LD}(R)}$ such that $L_j \cap \Gamma \neq \varnothing$. Notice that if $j \in J$, we have $\mathcal{H}^1(\Gamma \cap S_j) \ge \frac{1}{2}\,\ell(S_j)$ because L_j is contained in $\frac{1}{2}\,S_j$. Thus, since the squares S_j, $j \in J$, are disjoint, we have

$$\sum_{j\in J} \ell(S_j) \le 2\mathcal{H}^1(\Gamma) \le 10\ell(R)$$

(of course, "10" is not the best constant here). Then we obtain

$$\begin{aligned} \nu(R) &\le \frac{100}{99} \sum_{j\in J} \nu(\Gamma \cap L_j) \;\le\; \frac{100}{99} \sum_{j\in J} \nu(L_j) \;\le\; \frac{100}{99} \sum_{j\in J} \mu(S_j) \\ &\lesssim \delta\, \theta_\mu(R) \sum_{j\in J} \ell(S_j) \;\lesssim\; \delta\, \theta_\mu(R)\, \ell(R) \;=\; \delta\mu(R). \end{aligned}$$

Thus, if δ has been chosen small enough, we get a contradiction to (7.9). $\square$

7.4. *Proof of Lemma* 7.5. To simplify notation, in this subsection we write $J_0 := I^{s,0}_{\mathrm{LD}(R)}$.

The linear growth condition (7.5) follows easily from the fact that if $x \in S_j$, $j \in J_0$, then $\mu(B(x,r)) \lesssim A\theta_\mu(R) r$ for $r \ge \ell(S_j)$, and also

$$\frac{\mu(W_j)}{\mathcal{H}^1(L_j)} \lesssim \theta_\mu(S_j) \lesssim \theta_\mu(R).$$

The details are left for the reader.

The proof of (7.6) is more delicate. If, instead of choosing an appropriate orientation for each segment L_j, we assume all the L_j's to be parallel to the x axis, say, then instead of (7.6) we would get an estimate such as

$$c^2(\nu) \le C_{23}(A,\delta) \theta_\mu(R)^2 \mu(R),$$

where $C_{23}(A,\delta)$ is a large constant. Unfortunately this estimate is not enough for our purposes, because for the application of Léger's theorem to the measure τ in Subsection 7.3, we need $C_{23}(A,\delta) \le \varepsilon_L$.

The position and orientation of each segment L_j, $j \in J_0$, will be fixed with the help of a balanced square $\widehat{S}_j$ concentric with S_j, with $S_j \subset \widehat{S}_j \subset C(A,\delta)S_j$. We will show that W_j is contained in a thin strip V_j associated to $\widehat{S}_j$. The segment L_j will be a segment parallel to the strip V_j, with length $\ell(S_j)/8$, so that the middle point of L_j coincides with some point in $W_j \cap \frac{1}{5}S_j$.

7.4.1. *Preliminary lemmas.* For each $j \in J_0$, by Lemmas 4.4 and 7.1 there exists a square $\widehat{S}_j$ concentric with S_j satisfying

$$64\ell(S_j) \leq \ell(\widehat{S}_j) \leq \min\bigl(8\ell(R), C(A,\delta)\bigr)\,\ell(S_j),$$

such that $\widehat{S}_j \in \mathrm{Bal}(\mu)$ and $\mu(\widehat{S}_j \setminus S_j) \geq \frac{1}{2}\,\mu(\widehat{S}_j)$.

LEMMA 7.6. *For each $j \in J_0$ there exist two squares Q_j^1, $Q_j^2 \subset \widehat{S}_j$ and an infinite strip V_j of width $\leq \varepsilon_0^{1/6}\ell(S_j)$ which contains $10\varepsilon_0^{-1/50}\widehat{S}_j \cap W$ such that*

(a) $\mathrm{dist}(Q_j^1, Q_j^2) \geq a\ell(\widehat{S}_j)$,

(b) $\ell(Q_j^i) \leq \dfrac{a}{10}\,\ell(\widehat{S}_j)$ *for $i = 1, 2$, and*

(c) $\mu(Q_j^i \cap V_j) \geq \dfrac{b}{2}\,\mu(\widehat{S}_j)$ *for $i = 1, 2$,*

when ε_0 is small enough.

The constants a and b which appear in the lemma are the ones in Remark 4.3.

Proof. Since $\widehat{S}_j \in \mathrm{Bal}(\mu)$, there are squares $Q_j^1, Q_j^2 \subset \widehat{S}_j$ satisfying the properties (a) and (b) and such that $\mu(Q_j^i) \geq b\mu(\widehat{S}_j)$, $i = 1, 2$. In order to show the existence of the strip V_j, we will first prove that most of $\mathrm{supp}(\mu) \cap \widehat{S}_j$ is very close to some line.

Let $x_0 \in W \cap \frac{1}{5}S_j$ (x_0 exists because of the construction of S_j). By (7.4), for any $y \in \widehat{S}_j \setminus S_j$, we have $|y - x_0| > \ell_{x_0}$. Thus, by Lemma 7.4,

$$\iint_{\substack{y \in \widehat{S}_j \setminus S_j \\ z \in \widehat{S}_j}} c(x_0,y,z)^2\,d\mu(y)d\mu(z) \leq \iint_{\substack{y,z \in \widehat{S}_j \\ |x_0 - y| > \ell_{x_0}}} c(x_0,y,z)^2\,d\mu(y)d\mu(z) \lesssim \varepsilon_0\theta_\mu(R)^2.$$

Therefore, there exists some $y_0 \in \widehat{S}_j \setminus S_j$ such that

$$\int_{z \in \widehat{S}_j} c(x_0,y_0,z)^2\,d\mu(z) \lesssim \frac{\varepsilon_0\theta_\mu(R)^2}{\mu(\widehat{S}_j \setminus S_j)} \leq \frac{2\varepsilon_0\theta_\mu(R)^2}{\mu(\widehat{S}_j)}.$$

By Tchebychev, we obtain

$$
\begin{aligned}
&\mu\{z \in \widehat{S}_j : \operatorname{dist}(z, L_{x_0,y_0}) > \varepsilon_0^{1/4}\ell(\widehat{S}_j)\}\\
&\qquad \le \frac{1}{4\varepsilon_0^{1/2}} \int_{\widehat{S}_j} \left(\frac{2\operatorname{dist}(z, L_{x_0,y_0})}{\ell(\widehat{S}_j)}\right)^2 d\mu(z)\\
&\qquad \le \frac{\ell(\widehat{S}_j)^2}{\varepsilon_0^{1/2}} \int_{\widehat{S}_j} \left(\frac{2\operatorname{dist}(z, L_{x_0,y_0})}{|x_0 - z||z - y_0|}\right)^2 d\mu(z) \;=\; \frac{\ell(\widehat{S}_j)^2}{\varepsilon_0^{1/2}}\, c_\mu^2(x_0, y_0, \widehat{S}_j)\\
&\qquad \lesssim 2\varepsilon_0^{1/2} \left(\frac{\theta_\mu(R)}{\theta_\mu(\widehat{S}_j)}\right)^2 \mu(\widehat{S}_j) \;\lesssim\; \frac{\varepsilon_0^{1/2}}{\delta^2}\, \mu(\widehat{S}_j).
\end{aligned}
$$

Let $\widetilde{V}_j$ be the infinite strip with axis L_{x_0,y_0} and width $2\varepsilon_0^{1/4}\ell(\widehat{S}_j)$. If ε_0 is small enough, we infer that

$$
\mu(\widehat{S}_j \setminus \widetilde{V}_j) \le \frac{C\varepsilon_0^{1/2}}{\delta^2}\, \mu(\widehat{S}_j) \le \frac{1}{2}\mu(Q_j^i) \tag{7.10}
$$

for $i = 1, 2$, since $\mu(Q_j^i) \ge b\mu(\widehat{S}_j)$. Therefore, $\mu(Q_j^i \cap \widetilde{V}_j) \ge \frac{1}{2}\mu(Q_j^i)$ for each i. This will imply the statement (c) because we will construct V_j so that $V_j \supset \widetilde{V}_j$.

It remains to define V_j and to show that $10\varepsilon_0^{-1/50}\widehat{S}_j \cap W \subset V_j$. Take $y \in Q_j^1 \cap \widetilde{V}_j$ and $z \in Q_j^2 \cap \widetilde{V}_j$. Since $\operatorname{dist}(y, z) \ge a\ell(\widehat{S}_j)$ (with $a = 1/40$), the segment $L_{y,z} \cap 30\varepsilon_0^{-1/50}\widehat{S}_j$ is contained in some strip with the same axis as $\widetilde{V}_j$ and width $C\varepsilon_0^{1/4}\ell(30\varepsilon_0^{-1/50}\widehat{S}_j) \le \varepsilon_0^{1/5}\ell(\widehat{S}_j)/3$ (assuming ε_0 small enough).

Let V_j be the strip with the same axis as $\widetilde{V}_j$ and width $\varepsilon_0^{1/5}\ell(\widehat{S}_j)$ (which is $\le \varepsilon_0^{1/6}\ell(S_j)$ for ε_0 small). If $x \in 10\varepsilon_0^{-1/50}\widehat{S}_j \setminus V_j$, then $\operatorname{dist}(x, L_{y,z}) > \varepsilon_0^{1/5}\ell(\widehat{S}_j)/3$, and so

$$
c(x, y, z) \ge \frac{C^{-1}\varepsilon_0^{1/5}\ell(\widehat{S}_j)}{\ell\big(10\varepsilon_0^{-1/50}\widehat{S}_j\big)^2} = C^{-1}\varepsilon_0^{6/25}\ell(\widehat{S}_j)^{-1}.
$$

Thus,

$$
c_\mu^2(x, Q_j^1, Q_j^2) \ge C^{-1}\varepsilon_0^{12/25}\ell(\widehat{S}_j)^{-2}\, \mu(Q_j^1)\, \mu(Q_j^2) \ge C(A, \delta)^{-1}\varepsilon_0^{12/25}\theta_\mu(R)^2,
$$

which is larger than $C\varepsilon_0\theta_\mu(R)^2$ as ε_0 has been taken small enough. Further, from (7.4) it easily follows that $\ell_x \le 2^{-12}\big(\operatorname{dist}(x, \widehat{S}_j) + \ell(\widehat{S}_j)\big)$, and then either $\ell_x \le \operatorname{dist}(x, Q_j^1)$ or $\ell_x \le \operatorname{dist}(x, Q_j^2)$. As a consequence,

$$
c_{tr,\mu}^2(x, 2R, 2R) \ge c_\mu^2(x, Q_j^1, Q_j^2) > C\varepsilon_0\theta_\mu(R)^2,
$$

and so $x \notin W$. □

The *orientation of the segments* L_j, $j \in J_0$, which support ν is chosen so that each L_j is supported on the axis of V_j. Remember also that L_j has length $\ell(S_j)/8$. We assume that its middle point coincides with some point in $W \cap \frac{1}{5}S_j$ (for example, the point x_0 appearing in the proof of the preceding lemma). Notice that, in particular, we have $L_j \subset \frac{1}{2}S_j \cap V_j$.

We denote $\widehat{\widehat{S}}_j := \varepsilon_0^{-1/50}\widehat{S}_j$. Given two lines L and M, $\measuredangle(L,M)$ stands for the angle between L and M (it does not matter which one of the two possible angles because we will always deal with its sinus). Also, given $x,y,z\in\mathbb{C}$, we set $\measuredangle(x,y,z) := \measuredangle(L_{x,y},L_{y,z})$.

slowly, in a sense.

Lemma 7.7. *Let S_j, S_k, $j,k\in J_0$, be such that $3\widehat{\widehat{S}}_j\cap 3\widehat{\widehat{S}}_k\neq\varnothing$. Suppose that $\ell(\widehat{\widehat{S}}_j)\geq\ell(\widehat{\widehat{S}}_k)$. Then, either $\sin\measuredangle(L_j,L_k)\leq C\varepsilon_0^{1/6}$ or $\ell(S_k)\leq\varepsilon_0^{3/5}\ell(S_j)$. In any case, L_k is contained in a strip with the same axis as V_j and width $\varepsilon_0^{1/8}\ell(S_j)$.*

Proof. First we will show that either $\sin\measuredangle(L_j,L_k)\leq C\varepsilon^{1/6}$ or $\ell(S_k)\leq\varepsilon_0^{3/5}\ell(S_j)$.

From the assumptions in the lemma we deduce that $3\widehat{\widehat{S}}_k\subset 9\widehat{\widehat{S}}_j$. By construction there exists some $x\in W\cap L_k$. Consider the squares $Q_j^1,Q_j^2\subset\widehat{S}_j$ mentioned in Lemma 7.6. Suppose that Q_j^1 is the one which is farther from x, so that $\operatorname{dist}(x,Q_j^1)\geq C^{-1}\ell(\widehat{S}_j)$. Take also the square Q_k^i, $i=1$ or 2, which is farther from x. Suppose this is Q_k^1, and so $\operatorname{dist}(x,Q_k^1)\geq C^{-1}\ell(\widehat{S}_k)$. Take $y\in Q_j^1\cap V_j$ and $z\in Q_k^1\cap V_k$. Since $x,y\in V_j$, clearly we have

$$\sin\measuredangle(L_j,\overleftrightarrow{xy})\leq\frac{\text{width of } V_j}{|x-y|}\leq C\varepsilon^{1/6},$$

and, analogously, since $x\in V_k$, $\sin\measuredangle(L_k,\overleftrightarrow{xz})\leq C\varepsilon_0^{1/6}$. So we infer that

$$\sin\measuredangle(L_j,L_k)\leq C\sin\measuredangle(y,x,z)+C\varepsilon_0^{1/6}.$$

Therefore,

$$c(x,y,z)=\frac{2\sin\measuredangle(y,x,z)}{|y-z|}\geq C^{-1}\frac{\sin\measuredangle(L_j,L_k)-C\varepsilon_0^{1/6}}{\ell(\widehat{\widehat{S}}_j)}.$$

Thus,

$$\begin{aligned}
&\iint_{\substack{y\in Q_j^1\cap V_j\\ z\in Q_k^1\cap V_k}} c(x,y,z)^2\,d\mu(y)d\mu(z)\\
&\qquad\geq C^{-1}\frac{\left(\sin\measuredangle(L_j,L_k)-C\varepsilon_0^{1/6}\right)^2}{\ell(\widehat{\widehat{S}}_j)^2}\,\mu(Q_j^1\cap V_j)\,\mu(Q_k^1\cap V_k)\\
&\qquad\geq C(A,\delta)^{-1}\frac{\left(\sin\measuredangle(L_j,L_k)-C\varepsilon_0^{1/6}\right)^2}{\varepsilon_0^{-1/25}\ell(\widehat{S}_j)^2}\,\mu(\widehat{S}_j)\,\mu(\widehat{S}_k)\\
&\qquad\geq C(A,\delta)^{-1}\frac{\left(\sin\measuredangle(L_j,L_k)-C\varepsilon_0^{1/6}\right)^2}{\varepsilon_0^{-1/25}\ell(\widehat{S}_j)}\,\theta_\mu(R)^2\ell(\widehat{S}_k).
\end{aligned}$$

On the other hand, it is easily seen that

$$c_{tr,\mu}^2(x,2R,2R) \geq \iint_{\substack{y\in Q_j^1\cap V_j\\ z\in Q_k^1\cap V_k}} c(x,y,z)^2\, d\mu(y)d\mu(z).$$

Since $x\in W$, we have $c_{tr,\mu}^2(x,2R,2R)\leq \varepsilon_0\theta_\mu(R)^2$, and then we get

$$\ell(\widehat{S}_k)\big(\sin\measuredangle(L_j,L_k) - C_{24}\varepsilon_0^{1/6}\big)^2 \leq C(A,\delta)\varepsilon_0^{24/25}\ell(\widehat{S}_j).$$

So we deduce that either $\sin\measuredangle(L_j,L_k)\leq 2C_{24}\varepsilon_0^{1/6}$, or otherwise,

$$\ell(\widehat{S}_k)\leq C(A,\delta)\varepsilon_0^{24/25}\varepsilon_0^{-1/3}\ell(\widehat{S}_j).$$

Thus,

$$\ell(S_k)\leq C(A,\delta)\varepsilon_0^{47/75}\ell(\widehat{S}_j)\leq \varepsilon_0^{3/5}\ell(S_j),$$

assuming ε_0 small enough.

It remains to show that, in any case, L_k lies in a thin strip with the same axis as V_j. Remember that $x\in L_k\cap V_j$. If $\ell(S_k)\leq \varepsilon_0^{3/5}\ell(S_j)$, then S_k (and thus L_k) is contained in a strip with the same axis as V_j and width

$$\varepsilon_0^{1/6}\ell(S_j) + 2\varepsilon_0^{3/5}\ell(S_j)\leq \varepsilon_0^{1/8}\ell(S_j)$$

(for ε_0 small).

Supposing now that $\sin\measuredangle(L_j,L_k)\leq C\varepsilon_0^{1/6}$, we have

$$\ell(S_k)\leq \ell(\widehat{S}_k) = \varepsilon_0^{1/50}\ell(\widehat{\widehat{S}}_k)\leq \varepsilon_0^{1/50}\ell(\widehat{\widehat{S}}_j) = \ell(\widehat{S}_j)\leq C(A,\delta)\ell(S_j).$$

We deduce that L_k is also contained in a strip with the same axis as V_j and width

$$\varepsilon_0^{1/6}\ell(S_j) + 2\ell(S_k)\sin\measuredangle(L_j,L_k)\leq \varepsilon_0^{1/6}\ell(S_j) + C(A,\delta)\varepsilon_0^{1/6}\ell(S_j)\leq \varepsilon_0^{1/8}\ell(S_j),$$

for ε_0 sufficiently small again. □

LEMMA 7.8. *Given $j\in J_0$, let $x\in L_j$, $y,z\notin S_j$, $x_1\in Q_j^1\cap V_j$, and $x_2\in Q_j^2\cap V_j$. Then,*

$$c(x,y,z)\leq C(A,\delta)\left[c(x_1,y,z)+c(x_2,y,z)\right] + \frac{C\varepsilon_0^{1/6}\ell(S_j)}{|x-y||x-z|}.$$

Proof. Let x' be the orthogonal projection of x onto the line L_{x_1,x_2}. Since $x_1,x_2\in V_j$ and

$$|x_1-x_2|\geq \ell(\widehat{S}_j)/40 \gg \text{width of } V_j,$$

the segment $L_{x_1,x_2}\cap S_j$ is contained in CV_j, where CV_j stands for the strip with the same axis as V_j and width C times the one of V_j. Remember also that L_j is a segment supported on the axis of V_j. As a consequence,

$$|x-x'| = \operatorname{dist}(x,L_{x_1,x_2})\leq C \text{ width of } V_j\leq C\varepsilon_0^{1/6}\ell(S_j).$$

By Lemma 3.5,

$$c(x,y,z) \leq c(x',y,z) + \frac{C|x-x'|}{|x-y||x-z|}, \tag{7.11}$$

because x, x' are in S_j and far from ∂S_j, while $y, z \notin S_j$.

It can be shown that there exists some absolute constant C such that

$$\operatorname{dist}(x, L_{y,z}) \leq C\big(\operatorname{dist}(x_1, L_{y,z}) + \operatorname{dist}(x_2, L_{y,z})\big).$$

This follows easily from the fact that x', x_1, x_2 are collinear and $|x_1 - x_2| \geq C^{-1}|x' - x_1|$. Notice also that, for $i = 1, 2$,

$$|x_i - y| \leq C(A,\delta)|x' - y| \qquad \text{and} \qquad |x_i - z| \leq C(A,\delta)|x' - z|.$$

In fact, the constants $C(A,\delta)$ above depend on the ratio $\ell(\widehat{S}_j)/\ell(S_j)$. We get

$$\begin{aligned} c(x',y,z) &= \frac{2\operatorname{dist}(x, L_{y,z})}{|x-y||x-z|} \leq \frac{C(A,\delta)\operatorname{dist}(x_1, L_{y,z})}{|x_1-y||x_1-z|} + \frac{C(A,\delta)\operatorname{dist}(x_2, L_{y,z})}{|x_2-y||x_2-z|} \\ &= C(A,\delta)\,\big[c(x_1,y,z) + c(x_2,y,z)\big]. \end{aligned}$$

From this estimate, (7.11), and the fact that $|x - x'| \lesssim \varepsilon_0^{1/6}\ell(S_j)$, the lemma follows. □

Lemma 7.9. *Let $M_{tr,\mu}$ be the following* (*truncated*) *maximal operator*

$$M_{tr,\mu}f(x) = \sup_{\substack{Q: x \in \frac{1}{2}Q \\ \ell(Q) > 8\ell(S_x)}} \frac{1}{\mu(Q)} \int_{Q \cap 2R} |f|\, d\mu, \qquad \text{for } x \in \bigcup_{j \in J_0} S_j,$$

where S_x is the square S_j, $j \in J_0$, which contains x. Then, $M_{tr,\mu}$ is bounded from $L^2(\mu)$ into $L^2(\nu)$, with norm depending on A and δ.

Notice that the notation "S_x" was also used at the beginning of Subsection 7.1, but with a different meaning.

Proof. We immediately check that $M_{tr,\mu}$ is bounded from $L^\infty(\mu)$ into $L^\infty(\nu)$. So, by interpolation it is enough to show that it is also bounded from $L^1(\mu)$ into $L^{1,\infty}(\nu)$. Take a fixed $\lambda > 0$. If $M_{tr,\mu}f(x) > \lambda$ for some $x \in \operatorname{supp}(\nu)$, there is a square Q_x such that $x \in \frac{1}{2}Q_x$, $\ell(Q_x) > 8\ell(S_x)$, and $\int_{Q_x} |f| d\mu/\mu(Q_x) > \lambda$. By the Besicovitch covering theorem, there exists a family of points $\{x_i\}_i \subset \operatorname{supp}(\nu)$ so that the family of squares $\{Q_{x_i}\}_i$ has finite overlap and $\{x : M_{tr,\mu}f(x) > \lambda\} \subset \bigcup_i Q_{x_i}$. Since $\ell(Q_{x_i}) > 8\ell(S_{x_i})$ and $x_i \in \frac{1}{2}Q_{x_i}$, it is easy to check that there exists a square P concentric with S_{x_i} and with side length $\ell(P) = \ell(Q_{x_i})/2$ such that $S_{x_i} \subset P \subset Q_{x_i}$. Then,

$$\nu(Q_{x_i}) \leq CA\theta_\mu(R)\ell(Q_{x_i}) = CA\theta_\mu(R)\ell(P) \leq CA\delta^{-1}\mu(P) \leq C(A,\delta)\mu(Q_{x_i}).$$

Thus,

$$\begin{aligned}\nu\{x : M_{tr,\mu}f(x) > \lambda\} &\le \sum_i \nu(Q_{x_i}) \le C(A,\delta)\sum_i \mu(Q_{x_i})\\ &\le \frac{C(A,\delta)}{\lambda}\sum_i \int_{Q_{x_i}} |f|\,d\mu \le \frac{C(A,\delta)}{\lambda}\int |f|\,d\mu. \qquad \square\end{aligned}$$

7.4.2. *Proof of* (7.6). As in the preceding lemma, for $x \in \bigcup_{j\in J_0} S_j$, we denote by S_x be the square S_j, $j \in J_0$, which contains x. Analogously, $\widehat{S}_x$, $\widehat{\widehat{S}}_x$, Q_x^1, Q_x^2, and V_x stand for $\widehat{S}_j$, $\widehat{\widehat{S}}_j$, Q_j^1, Q_j^2, and V_j respectively.

We denote

$$\begin{aligned}
&F_1 := \big\{(x,y,z) \in (\textstyle\bigcup_{j\in J_0} S_j)^3 :\ S_x = S_y \neq S_z\big\},\\
&F_2 := \big\{(x,y,z) \in (\textstyle\bigcup_{j\in J_0} S_j)^3 :\ S_x \neq S_y \neq S_z \neq S_x\big\},\\
&F_3 := \big\{(x,y,z) \in F_2 :\ 3\widehat{\widehat{S}}_y \cap 3\widehat{\widehat{S}}_z = \varnothing\big\},\\
&F_4 := \big\{(x,y,z) \in F_2 :\ 3\widehat{\widehat{S}}_x \cap 3\widehat{\widehat{S}}_y \neq \varnothing,\ 3\widehat{\widehat{S}}_x \cap 3\widehat{\widehat{S}}_z \neq \varnothing,\ 3\widehat{\widehat{S}}_y \cap 3\widehat{\widehat{S}}_z \neq \varnothing\big\}.
\end{aligned}$$

Since $c^2(\nu_{|S_j}) = 0$ for all $j \in J_0$,

(7.12)
$$\begin{aligned}
c^2(\nu) &= \iiint_{\left(\bigcup_{j\in J_0} S_j\right)^3} c(x,y,z)^2\,d\nu(x)d\nu(y)d\nu(z)\\
&= 3\iiint_{F_1}\cdots + \iiint_{F_2}\cdots \ \le\ 3\iiint_{F_1}\cdots + 3\iiint_{F_3}\cdots + \iiint_{F_4}\cdots\\
&=: 3I_1 + 3I_3 + I_4.
\end{aligned}$$

• *Estimates for* I_3. If $y' \in S_y \cap W$ and $z' \in S_z \cap W$, by Lemma 3.5

$$\begin{aligned}
c(x,y,z) &\le c(x,y',z') + \frac{C\ell(S_y)}{|y-x||y-z|} + \frac{C\ell(S_z)}{|z-x||z-y|}\\
&=: c(x,y',z') + C\big[T_y(x,y,z) + T_z(x,y,z)\big].
\end{aligned} \tag{7.13}$$

Then it easily follows that

$$\begin{aligned}
I_3 \le\ & 2\iiint_{\substack{(x,y,z)\in F_2\\ y,z\in W\\ 3\widehat{\widehat{S}}_y\cap 3\widehat{\widehat{S}}_z=\varnothing}} c(x,y,z)^2\,d\nu(x)d\mu(y)d\mu(z)\\
&+C\iiint_{\substack{(x,y,z)\in F_2\\ 3\widehat{\widehat{S}}_y\cap 3\widehat{\widehat{S}}_z=\varnothing}} \big[T_y(x,y,z)^2 + T_z(x,y,z)^2\big]\,d\nu(x)d\nu(y)d\nu(z)\\
=:\ & 2I_{3,1} + C\,I_{3,2}.
\end{aligned} \tag{7.14}$$

Although it is not written explicitly, all the integrals above are restricted to $(2R)^3$ (and the same for the rest of the proof of (7.6)).

First, dealing with the term $I_{3,2}$, we have

$$\begin{aligned}\iiint_{\substack{(x,y,z)\in F_2\\ 3\widehat{\widehat{S}}_y\cap 3\widehat{\widehat{S}}_z=\varnothing}} T_y(x,y,z)^2\,d\nu(x)d\nu(y)d\nu(z) \;\leq\; \iiint_{\substack{|y-x|>\ell(S_y)/2\\ |y-z|>\ell(\widehat{\widehat{S}}_y)}}\cdots\\ \leq \int\Big(\int_{|y-x|>\ell(S_y)/2}\frac{\ell(S_y)}{|y-x|^2}\,d\nu(x)\Big)\Big(\int_{|y-z|>\ell(\widehat{\widehat{S}}_y)}\frac{\ell(S_y)}{|y-z|^2}\,d\nu(z)\Big)d\nu(y)\\ \leq C\Big(A\theta_\mu(R)\Big)\left(A\theta_\mu(R)\frac{\ell(S_y)}{\ell(\widehat{\widehat{S}}_y)}\right)\nu(\mathbb{C}) \leq CA^2\theta_\mu(R)^2\,\varepsilon_0^{1/50}\,\mu(R).\end{aligned}$$

We have analogous estimates for the integral of $T_z(\cdots)^2$. Thus,

$$I_{3,2} \leq CA^2\theta_\mu(R)^2\,\varepsilon_0^{1/50}\,\mu(R).$$

Now we consider the term $I_{3,1}$ in (7.14). By Lemma 7.8, for all $x_1 \in Q_x^1\cap V_x$ and $x_2 \in Q_x^2\cap V_x$ we have

$$c(x,y,z) \leq C(A,\delta)\,\big[c(x_1,y,z)+c(x_2,y,z)\big] + \frac{C\varepsilon_0^{1/6}\ell(S_x)}{|x-y||x-z|}.$$

Integrating over $x_1 \in Q_x^1\cap V_x$ and over $x_2 \in Q_x^2\cap V_x$ with respect to μ, we obtain

$$\begin{aligned}c(x,y,z) \leq &\frac{C(A,\delta)}{\mu(\widehat{S}_x)}\Big(\int_{x_1\in V_x\cap Q_x^1} c(x_1,y,z)d\mu(x_1) + \int_{x_2\in V_x\cap Q_x^2} c(x_2,y,z)d\mu(x_2)\Big)\\ &+C\varepsilon_0^{1/6}T_x(x,y,z),\end{aligned}$$

where $T_x(x,y,z) := \ell(S_x)/\big(|x-y||x-z|\big)$. Therefore,

$$\begin{aligned}c(x,y,z) &\leq \frac{C(A,\delta)}{\mu(\widehat{S}_x)}\int_{\widehat{S}_x} c(w,y,z)\,d\mu(w) + C\varepsilon_0^{1/6}T_x(x,y,z)\\ &\leq C(A,\delta)\,M_{tr,\mu}\big[c(\cdot,y,z)\big](x) + C\varepsilon_0^{1/6}T_x(x,y,z).\end{aligned}$$

Thus,

$$\begin{aligned}I_{3,1} \leq C(A,\delta)&\iiint_{3\widehat{\widehat{S}}_y\cap 3\widehat{\widehat{S}}_z=\varnothing} M_{tr,\mu}\big[c(\cdot,y,z)\big](x)^2\,d\nu(x)d\mu(y)d\mu(z)\\ &+ C\varepsilon_0^{1/3}\iiint_{|x-y|,|x-z|\geq\ell(S_x)/2} T_x(x,y,z)^2\,d\nu(x)d\mu(y)d\mu(z).\end{aligned}$$

The last integral on the right side is estimated as follows:

$$\begin{aligned}\iiint_{|x-y|,|x-z|\geq\ell(S_x)/2} &T_x(x,y,z)^2\,d\nu(x)d\mu(y)d\mu(z)\\ &\leq \int\Big(\int_{|x-y|>\ell(S_x)/2}\frac{1}{|x-y|^2}\,d\mu(y)\Big)^2 d\nu(x) \leq CA^2\theta_\mu(R)^2\mu(R).\end{aligned}$$

On the other hand, by Lemma 7.9 we know that $M_{tr,\mu}$ is bounded from $L^2(\mu)$ into $L^2(\nu)$. So we have

$$\begin{aligned}I_{3,1} \le C(A,\delta) \iiint_{3\widehat{\widehat{S}}_y \cap 3\widehat{\widehat{S}}_z = \varnothing} c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z)\\ + C\varepsilon_0^{1/3} A^2 \theta_\mu(R)^2 \mu(R).\end{aligned}$$

It is easy to check that

$$\begin{aligned}\iiint_{3\widehat{\widehat{S}}_y \cap 3\widehat{\widehat{S}}_z = \varnothing} c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z) &\le \int c^2_{tr,\mu}(y,2R,2R)\, d\mu(y)\\ &\le C\varepsilon_0 \theta_\mu(R)^2 \mu(2R).\end{aligned}$$

From the preceding estimates for $I_{3,1}$ and $I_{3,2}$, we get

$$I_3 \le C(A,\delta)\varepsilon_0^{1/50} \theta_\mu(R)^2 \mu(R).$$

- *Estimates for I_4.* By Fubini, we have

$$I_4 \le 3 \iiint_{\substack{(x,y,z)\in F_4\\ \ell(\widehat{\widehat{S}}_x)\ge \ell(\widehat{\widehat{S}}_y)\ge \ell(\widehat{\widehat{S}}_z)}} c(x,y,z)^2\, d\nu(x)d\nu(y)d\nu(z) =: 3I_4'.$$

Now we split I_4' as follows:

$$\begin{aligned}I_4' &= \Bigg(\iiint_{\substack{(x,y,z)\in F_4\\ \ell(\widehat{\widehat{S}}_x)\ge \ell(\widehat{\widehat{S}}_y)\ge \ell(\widehat{\widehat{S}}_z)\\ \ell(S_x)\ge \ell(\widehat{\widehat{S}}_y)}} + \iiint_{\substack{(x,y,z)\in F_4\\ \ell(\widehat{\widehat{S}}_x)\ge \ell(\widehat{\widehat{S}}_y)\ge \ell(\widehat{\widehat{S}}_z)\\ \ell(S_x)< \ell(\widehat{\widehat{S}}_y)}} \Bigg) c(x,y,z)^2\, d\nu(x)d\nu(y)d\nu(z)\\ &=: I_{4,1} + I_{4,2}.\end{aligned}$$

First we will study $I_{4,1}$. For (x,y,z) in the domain of integration of $I_{4,1}$ we have

$$|x-y| \ge \frac{\ell(S_x)}{4} \ge \frac{\ell(\widehat{\widehat{S}}_y)}{4} \qquad \text{and} \qquad |x-z| \ge \frac{\ell(S_x)}{4} \ge \frac{\ell(\widehat{\widehat{S}}_y)}{4} \ge \frac{\ell(\widehat{\widehat{S}}_z)}{4}.$$

The estimates for $I_{4,1}$ are similar to the ones for I_3. Indeed, consider $y' \in S_y \cap W$ and $z' \in S_z \cap W$, so that (7.13) also holds in this case. Instead of (7.14) now we get

(7.15)

$$\begin{aligned}I_{4,1} \le 2 \iiint_{\substack{(x,y,z)\in F_4\\ y,z\in W}} c(x,y,z)^2\, d\nu(x)d\mu(y)d\mu(z)\\ + C \iiint_{\substack{(x,y,z)\in F_4\\ |x-y|\ge \ell(\widehat{\widehat{S}}_y)/4\\ |x-z|\ge \ell(\widehat{\widehat{S}}_z)/4}} \left[T_y(x,y,z)^2 + T_z(x,y,z)^2\right] d\nu(x)d\mu(y)d\mu(z).\end{aligned}$$

We have

$$\iiint_{\substack{(x,y,z)\in F_4\\ |x-y|\geq \ell(\widehat{\widehat{S}}_y)/4\\ |x-z|\geq \ell(\widehat{\widehat{S}}_z)/4}} T_y(x,y,z)^2\, d\nu(x)d\mu(y)d\mu(z) \;\leq\; \iiint_{\substack{|y-x|>\ell(\widehat{\widehat{S}}_y)/4\\ |y-z|>\ell(S_y)/4}}\cdots$$

$$\leq \int\Big(\int_{|y-x|>\ell(S_y)/4}\frac{\ell(S_y)}{|y-x|^2}\,d\nu(x)\Big)\Big(\int_{|y-z|>\ell(\widehat{\widehat{S}}_y)/4}\frac{\ell(S_y)}{|y-z|^2}\,d\mu(z)\Big)d\mu(y)$$

$$\leq C\Big(A\theta_\mu(R)\Big)\left(A\theta_\mu(R)\frac{\ell(S_y)}{\ell(\widehat{\widehat{S}}_y)}\right)\mu(R) \leq CA^2\theta_\mu(R)^2\,\varepsilon_0^{1/50}\,\mu(R).$$

Analogous estimates hold for the integral of $T_z(\cdots)^2$ since in the domain of integration we have $|x-z|\geq \ell(\widehat{\widehat{S}}_z)/4$.

To estimate the integral

$$\iiint_{\substack{(x,y,z)\in F_4\\ y,z\in W}} c(x,y,z)^2\, d\nu(x)d\mu(y)d\mu(z)$$

in (7.15), the same arguments used for $I_{3,1}$ work in this case. Only some minor changes which are left for the reader are required.

Now we deal with $I_{4,2}$. Take (x,y,z) in the domain of integration of $I_{4,2}$. Since $\ell(\widehat{\widehat{S}}_x)\geq\ell(\widehat{\widehat{S}}_y)\geq\ell(\widehat{\widehat{S}}_z)$, $3\widehat{\widehat{S}}_x\cap 3\widehat{\widehat{S}}_y\neq\varnothing$, and $3\widehat{\widehat{S}}_x\cap 3\widehat{\widehat{S}}_z\neq\varnothing$, we have $3\widehat{\widehat{S}}_y, 3\widehat{\widehat{S}}_z\subset 9\widehat{\widehat{S}}_x$. Remember that, by Lemma 7.7, the segments L_y, L_z (and thus y and z) are contained in a strip with the same axis as V_x and width $\varepsilon_0^{1/8}\ell(S_x)$. Therefore,

$$\sin\measuredangle(L_x,\overleftrightarrow{xy}) \leq \frac{\varepsilon_0^{1/8}\ell(S_x)}{|x-y|} \leq \frac{\varepsilon_0^{1/8}\ell(S_x)}{C^{-1}\ell(S_x)} \leq C\varepsilon_0^{1/8},$$

and, in the same way, $\sin\measuredangle(L_x,\overleftrightarrow{xz})\leq C\varepsilon_0^{1/8}$. Thus, we get

$$\sin\measuredangle(y,x,z) \leq C\big(\sin\measuredangle(L_x,\overleftrightarrow{xy}) + \sin\measuredangle(L_x,\overleftrightarrow{xz})\big) \leq C\varepsilon_0^{1/8}.$$

Therefore, since $\ell(S_x)<\ell(\widehat{\widehat{S}}_y)$,

$$c(x,y,z) = \frac{2\sin\measuredangle(y,x,z)}{|y-z|} \leq \frac{C\varepsilon_0^{1/8}}{\ell(S_y)} \leq \frac{C(A,\delta)\,\varepsilon_0^{1/8}}{\varepsilon_0^{1/50}\ell(\widehat{\widehat{S}}_y)} \leq \frac{C(A,\delta)\,\varepsilon_0^{21/200}}{\ell(S_x)},$$

and so

$$I_{4,2} \leq \int_{x\in 2R}\left(\iint_{y,z\in 9\widehat{\widehat{S}}_x}\frac{C(A,\delta)\,\varepsilon_0^{21/100}}{\ell(S_x)^2}\,d\mu(y)d\mu(z)\right)d\mu(x)$$
$$\leq C(A,\delta)\varepsilon_0^{21/100}\varepsilon^{-1/25}\theta_\mu(R)^2\mu(R) = C(A,\delta)\varepsilon_0^{17/100}\theta_\mu(R)^2\mu(R).$$

Gathering the estimates for $I_{4,1}$ and $I_{4,2}$, we obtain

$$I_4 \leq C(A,\delta)\,(\varepsilon_0^{1/50}+\varepsilon_0^{17/100})\,\theta_\mu(R)^2\mu(R) \leq C(A,\delta)\varepsilon_0^{1/50}\theta_\mu(R)^2\mu(R).$$

• *Estimates for* I_1. We have

$$I_1 = \iiint_{\substack{x\in\operatorname{supp}(\nu)\\ y\in L_x\\ z\notin\widehat{\widehat{S}}_x}} c(x,y,z)^2\,d\nu(x)d\nu(y)d\nu(z) + \iiint_{\substack{x\in\operatorname{supp}(\nu)\\ y\in L_x\\ z\in\widehat{\widehat{S}}_x\setminus S_x}} c(x,y,z)^2\,d\nu(x)d\nu(y)d\nu(z) \;=:\; I_{1,1}+I_{1,2}.$$

The term $I_{1,1}$ is estimated as follows:

$$\begin{aligned} I_{1,1} &\le \iiint_{\substack{x\in\operatorname{supp}(\nu)\\ y\in L_x\\ |x-z|>\ell(\widehat{\widehat{S}}_x)/4}} \frac{1}{|x-z|^2}\,d\nu(x)d\nu(y)d\nu(z)\\ &\le \int_{x\in\operatorname{supp}(\nu)} \frac{CA\theta_\mu(R)\nu(L_x)}{\ell(\widehat{\widehat{S}}_x)}\,d\nu(x) \;\le\; CA\theta_\mu(R)^2\varepsilon_0^{1/50}\mu(R), \end{aligned}$$

since $\nu(L_x)/\ell(\widehat{\widehat{S}}_x)\le C\theta_\mu(R)\ell(S_x)/\ell(\widehat{\widehat{S}}_x)\le C\theta_\mu(R)\varepsilon^{1/50}$ and $\nu(\mathbb{C})\le C\mu(R)$.

Finally we turn our attention to $I_{1,2}$. Consider (x,y,z) in the domain of integration of $I_{1,2}$. Clearly, in this case we have $\widehat{\widehat{S}}_x\cap\widehat{\widehat{S}}_z\neq\varnothing$. If $\ell(\widehat{\widehat{S}}_z)\le\ell(\widehat{\widehat{S}}_x)$, by Lemma 7.7, z is contained in a strip with the same axis as V_x and width $\varepsilon_0^{1/8}\ell(S_x)$.

Suppose now that $\ell(\widehat{\widehat{S}}_z)>\ell(\widehat{\widehat{S}}_x)$. Then, again by Lemma 7.7, either $\sin\measuredangle(L_x,L_z)\le C\varepsilon_0^{1/6}$ or $\ell(S_x)\le\varepsilon_0^{3/5}\ell(S_z)$. However, the latter inequality cannot hold because it implies

$$|x-z|\ge\ell(S_z)/4\ge\varepsilon_0^{-3/5}\ell(S_x)/4\gg\ell(\widehat{\widehat{S}}_x),$$

and so $z\notin\widehat{\widehat{S}}_x$. Then the condition $\sin\measuredangle(L_x,L_z)\le C\varepsilon_0^{1/6}$ holds. Further, we have $\ell(S_z)\le 2\ell(\widehat{\widehat{S}}_x)$ because $z\in\widehat{\widehat{S}}_x$. As a consequence, we easily infer that z lies in a thin strip with the same axis as V_x and width

$$\varepsilon_0^{1/6}\ell(S_x)+C\varepsilon_0^{1/6}\ell(\widehat{\widehat{S}}_x)\le C(A,\delta)\varepsilon_0^{1/6-1/50}\ell(S_x)\le\varepsilon_0^{2/15}\ell(S_x),$$

for ε_0 small enough.

So in any case z is contained in the strip with the same axis as V_x and width $\varepsilon_0^{2/15}\ell(S_x)$. As a consequence, we deduce

$$\sin\measuredangle(x,y,z)\lesssim\frac{\varepsilon_0^{2/15}\ell(S_x)}{|y-z|}\lesssim\varepsilon_0^{2/15}.$$

Thus, $c(x,y,z)\lesssim\varepsilon_0^{2/15}/|x-z|\lesssim\varepsilon_0^{2/15}/\ell(S_x)$, and so

$$\begin{aligned} I_{1,2} &\lesssim \int_{x\in\operatorname{supp}(\nu)} \frac{A\varepsilon_0^{4/15}}{\ell(S_x)^2}\,\theta_\mu(R)^2\ell(\widehat{\widehat{S}}_x)\ell(S_x)\,d\nu(x)\\ &\le C(A,\delta)\varepsilon_0^{4/15-1/50}\,\theta_\mu(R)^2\mu(R) \;=\; C(A,\delta)\varepsilon_0^{37/150}\,\theta_\mu(R)^2\mu(R). \end{aligned}$$

• *End of the proof.* By the estimates obtained for I_1, I_3 and I_4, we get

$$c^2(\nu) \leq C(A,\delta)\theta_\mu(R)^2\varepsilon_0^{1/50}\mu(R). \qquad \square$$

8. The curvature of $\varphi_\sharp\mu$

In this section we denote $\sigma := \varphi_\sharp\mu$ and $F := \varphi(E)$.

Given a square Q, we say that $\varphi(Q)$ is a φ-square. If x_Q is the center of Q, then we call $\varphi(x_Q)$ the center of $\varphi(Q)$. We also set $\ell(\varphi(Q)) := \ell(Q)$. Since φ is bilipschitz, we have $\ell(\varphi(Q)) \simeq \operatorname{diam}(\varphi(Q))$. We will often use the letters P, Q, R to denote φ-squares too. If Q is a dyadic (or 4-dyadic) square, we say that $\varphi(Q)$ is a dyadic (or 4-dyadic) φ-square.

If $Q = \varphi(Q_0)$ is a φ-square, we let $\lambda Q = \varphi(\lambda Q_0)$, for $\lambda > 0$. Then Q is λ-doubling if $\sigma(\lambda Q) \leq C\sigma(Q)$ for some $C \geq 1$. We also set

$$\theta_\sigma(Q) := \frac{2^{1/2}\sigma(Q)}{\operatorname{diam}(Q)}$$

(the number $2^{1/2}$ is due to aesthetic reasons; if φ is the identity, then the definition coincides with (2.1)) and if R is another φ-square which contains Q, we put

$$\delta_\sigma(Q,R) := \int_{R_Q\setminus Q} \frac{1}{|y - x_Q|}\, d\sigma(y),$$

where x_Q stands for the center of Q and R_Q is the smallest φ-square concentric with Q that contains R.

Given a family $\operatorname{Top}(F)$ of 4-dyadic φ-squares and a fixed $Q \in \operatorname{Top}(F)$, we denote by $\operatorname{Stop}(Q)$ the subfamily of φ-squares which satisfy the properties (a), (b), (c) stated at the beginning of Section 3 (with squares replaced by φ-squares). The set $G(Q)$ is also defined as in Section 3, with φ-squares instead of squares.

MAIN LEMMA 8.1. *Let σ be a Radon measure supported on a compact $F \subset \mathbb{C}$. Suppose that $\sigma(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$. Let $\operatorname{Top}(F)$ be a family of 4-dyadic 16-doubling φ-squares (called top φ-squares) which contains some 4-dyadic φ-square R_0 such that $F \subset R_0$, and such that for each $Q \in \operatorname{Top}(F)$ there exists a C_{25}-AD regular curve Γ_Q satisfying*:

(a) *σ-almost every point in $G(Q)$ belongs to Γ_Q.*

(b) *For each $P \in \operatorname{Stop}(Q)$ there exists some φ-square $\widetilde{P}$ containing P such that $\delta_\sigma(P,\widetilde{P}) \leq C\theta_\sigma(Q)$ and $\widetilde{P} \cap \Gamma_Q \neq \varnothing$.*

(c) *If P is a φ-square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \varnothing$ or there is another φ-square $P' \in \operatorname{Stop}(Q)$ such that $P \cap P' \neq \varnothing$ and $\ell(P') \leq \ell(P)$, then $\sigma(P) \leq C\,\theta_\sigma(Q)\,\ell(P)$.*

Then,

$$c^2(\sigma) \leq C \sum_{Q \in \mathrm{Top}(F)} \theta_\sigma(Q)^2 \sigma(Q).$$

We will prove this lemma in Subsections 8.2–8.8.

8.1. *Proof of Theorem* 1.3. This is an easy consequence of Main Lemmas 3.1 and 8.1. Indeed, if $c^2(\mu) < \infty$, then we have the corona decomposition given by Main Lemma 3.1. Applying the bilipschitz map φ, we obtain another corona decomposition for $F = \varphi(E)$ like the one required in Main Lemma 8.1. In particular, notice that φ sends AD regular curves to AD regular curves, and also if Q, R are squares such that $Q \subset R$, then

$$\begin{aligned}
\delta_\sigma(\varphi(Q),\, \varphi(R)) &= \int_{\varphi(R_Q)\setminus\varphi(Q)} \frac{1}{|y - x_{\varphi(Q)}|}\, d\sigma(y) \\
&= \int_{R_Q\setminus Q} \frac{1}{|\varphi(y) - \varphi(x_Q)|}\, d\mu(y) \\
&\leq \int_{R_Q\setminus Q} \frac{C}{|y - x_Q|}\, d\mu(y) = C\delta_\mu(Q, R),
\end{aligned}$$

with C depending on φ. So, by Main Lemma 8.1, $c^2(\sigma) \lesssim \big(\mu(E) + c^2(\mu)\big)$. $\square$

8.2. *Decomposition of* $c^2(\sigma)$. We start the proof of Main Lemma 8.1. Observe that

$$c^2(\sigma) \leq 3 \iiint_{|x-y|\geq|x-z|,|y-z|} c(x,y,z)^2\, d\sigma(x) d\sigma(y) d\sigma(z).$$

We now introduce a variant of the curvature operator K_σ. Consider the kernel

$$\widehat{k}_\sigma(x,y) = \int_{z:|x-y|\geq|x-z|,|y-z|} c(x,y,z)^2\, d\sigma(z),$$

and set

$$\widehat{K}_\sigma f(x) = \int \widehat{k}_\sigma(x,y)\, f(y)\, d\sigma(y), \qquad x \in \mathbb{C}, f \in L^1_{\mathrm{loc}}(\sigma)$$

(compare with the definition of $k_\mu(x,y)$ and K_μ in Section 2). We have

$$\int_F \widehat{K}_\sigma \chi_F(x)\, d\sigma(x) \leq c^2(\sigma) \leq 3 \int_F \widehat{K}_\sigma \chi_F(x)\, d\sigma(x).$$

The truncated operator $\widehat{K}_{\sigma,j}$, $j \in \mathbb{Z}$, is

$$\widehat{K}_{\sigma,j} f(x) = \int_{|x-y|>L^{-1}2^{-j}} \widehat{k}_\sigma(x,y)\, f(y)\, d\sigma(y), \qquad x \in \mathbb{C}, f \in L^1_{\mathrm{loc}}(\sigma),$$

where L is the bilipschitz constant of φ. We say that a φ-square $Q \in \mathrm{Top}(F)$ is a descendant of another φ-square $R \in \mathrm{Top}(F)$ if there is a chain $R =$

$Q_1, Q_2, \dots, Q_n = Q$, with $Q_i \in \mathrm{Top}(F)$ such that $Q_{i+1} \in \mathrm{Stop}(Q_i)$ for each i. Only the φ-squares from $\mathrm{Top}(F)$ which are descendants of R_0 will be relevant to estimate $c^2(\sigma)$. So we assume that all the φ-squares in $\mathrm{Top}(F)$ are of this type.

To decompose $c^2(\sigma)$, we prefer to use dyadic φ-squares instead of 4-dyadic φ-squares. A φ-square Q belongs to the family $\mathrm{Top}_{\mathrm{dy}}(F)$ if there exists some $R \in \mathrm{Top}(F)$ such that Q is one of the 16 dyadic φ-squares contained in R with side length $\ell(R)/4$.

Note that if $Q \in \mathrm{Top}_{\mathrm{dy}}(F)$, then Q is contained in a 4-dyadic φ-square R such that $Q \subset R \subset 7Q \subset 3R$. Moreover, since each 4-dyadic φ-square $R \in \mathrm{Top}(F)$ is made up of 16 dyadic φ-squares $Q \in \mathrm{Top}_{\mathrm{dy}}(F)$, we get (using the doubling properties of the φ-squares in $\mathrm{Top}(F)$)

$$\sum_{Q\in\mathrm{Top}_{\mathrm{dy}}(F)} \theta_\sigma(7Q)^2\sigma(7Q) \lesssim \sum_{R\in\mathrm{Top}(F)} \theta_\sigma(R)^2\sigma(R). \tag{8.1}$$

Given $Q \in \mathrm{Top}(F)$ or $Q \in \mathrm{Top}_{\mathrm{dy}}(F)$, we denote by $\mathrm{Term}(Q)$ the family of *maximal* dyadic (and thus disjoint) φ-squares $P \in \mathrm{Top}_{\mathrm{dy}}(F)$, with $P \subsetneq Q$. Finally, we let $\mathrm{Tree}(Q)$ be the class of dyadic φ-squares contained in Q, different from Q, which are not proper φ-subsquares of any $P \in \mathrm{Term}(Q)$.

We denote by $\varphi\Delta$ the class of dyadic φ-squares contained in R_0, and by $\varphi\Delta_j$ those φ-squares in $\varphi\Delta$ with side length 2^{-j}. We have

$$\varphi\Delta = \big\{Q \in \varphi\Delta : \ell(Q) \geq \ell(R_0)/4\big\} \cup \bigcup_{Q\in\mathrm{Top}_{\mathrm{dy}}(F)} \mathrm{Tree}(Q).$$

Observe that the φ-squares $Q \in \varphi\Delta$ such that $\ell(Q) \geq \ell(R_0)/4$ are the only φ-squares in $\mathrm{Top}_{\mathrm{dy}}(F)$ which may not belong to any $\mathrm{Tree}(R)$, $R \in \mathrm{Top}_{\mathrm{dy}}(F)$. Notice also that $\mathrm{Tree}(Q) \cap \mathrm{Tree}(R) = \varnothing$ if $Q \neq R$.

We split the curvature $c^2(\sigma)$ as follows:

$$\begin{aligned} c^2(\sigma) &\simeq \sum_j \sum_{Q\in\varphi\Delta_j} \int_Q (\widehat{K}_{\sigma,j+1}\chi_F - \widehat{K}_{\sigma,j}\chi_F)\, d\sigma + \int_F \widehat{K}_{\sigma,J(R_0)+2}\chi_F\, d\sigma \\ &= \sum_{R\in\mathrm{Top}_{\mathrm{dy}}(F)} \sum_{Q\in\mathrm{Tree}(R)} \int_Q (\widehat{K}_{\sigma,J(Q)+1}\chi_F - \widehat{K}_{\sigma,J(Q)}\chi_F)\, d\sigma \\ &\quad + \int_F \widehat{K}_{\sigma,J(R_0)+2}\chi_F\, d\sigma, \end{aligned}$$

where $J(Q)$ stands for the integer j such that $Q \in \varphi\Delta_j$. Since

$$\int_F \widehat{K}_{\sigma,J(R_0)+2}\chi_F\, d\sigma \lesssim \theta_\sigma(R_0)^2\sigma(F),$$

to prove Main Lemma 8.1 it is enough to show that

$$\sum_{Q\in\mathrm{Tree}(R)} \int_Q (\widehat{K}_{\sigma,J(Q)+1}\chi_F - \widehat{K}_{\sigma,J(Q)}\chi_F)\, d\sigma \lesssim \theta_\sigma(7R)^2\sigma(7R), \tag{8.2}$$

for every $R \in \mathrm{Top}_{\mathrm{dy}}(F)$, by (8.1).

8.3. *Regularization of the stopping φ-squares.* Given a fixed $R \in \mathrm{Top}_{\mathrm{dy}}(F)$, let R_1 be a 4-dyadic φ-square $R_1 \in \mathrm{Top}(F)$ such that $R \subset R_1 \subset 7R$ (it does not matter which R_1 if it is not unique). Let $\Gamma_R := \Gamma_{R_1}$ be the AD regular curve satisfying (a) and (b) in Main Lemma 8.1.

It seems that after defining $\mathrm{Top}_{\mathrm{dy}}(F)$ we should introduce the family $\mathrm{Stop}_{\mathrm{dy}}(R)$ analogously. However, for technical reasons, it is better to introduce a regularized version of $\mathrm{Stop}_{\mathrm{dy}}(R)$ (it does not matter what $\mathrm{Stop}_{\mathrm{dy}}(R)$ means precisely), that we will denote by $\mathrm{Reg}_{\mathrm{dy}}(R)$. First we set

$$d_R(x) := \inf_{Q\in\mathrm{Stop}(R_1)} \big\{\mathrm{dist}(x,Q) + \ell(Q),\ \mathrm{dist}(x, G(R_1))\big\}.$$

For each $x \in 3R \cap \mathrm{supp}(\sigma) \setminus [G(R_1) \cup Z(\sigma)]$ (recall that $Z(\sigma)$ is a set of zero σ-measure, defined similarly to $Z(\mu)$ at the beginning of Section 3), let Q_x be a dyadic φ-square containing x such that

$$\frac{d_R(x)}{20L} < \ell(Q_x) \le \frac{d_R(x)}{10L}. \tag{8.3}$$

Remember that L is the bilipschitz constant of φ. Then, $\mathrm{Reg}_{\mathrm{dy}}(R)$ is a maximal (and thus disjoint) subfamily of $\{Q_x\}_{x\in 3R\cap\mathrm{supp}(\sigma)\setminus[G(R_1)\cup Z(\sigma)]}$.

Lemma 8.2. (a) *If $P, Q \in \mathrm{Reg}_{\mathrm{dy}}(R)$ and $2P \cap 2Q \ne \varnothing$, then $\ell(Q)/2 \le \ell(P) \le 2\ell(Q)$.*

(b) *If $Q \in \mathrm{Reg}_{\mathrm{dy}}(R)$ and $x \in Q$, $r \ge \ell(Q)$, then $\sigma(B(x,r)\cap 4R) \le C\theta_\sigma(R_1)r$.*

(c) *For each $Q \in \mathrm{Reg}_{\mathrm{dy}}(R)$, there exist some φ-square $\widetilde{Q}$ which contains Q such that $\delta_\sigma(Q,\widetilde{Q}) \le C\theta_\sigma(R_1)$ and $\frac{1}{2}\widetilde{Q} \cap \Gamma_R \ne \varnothing$.*

Proof. (a) Consider $P, Q \in \mathrm{Reg}_{\mathrm{dy}}(R)$ such that $2P \cap 2Q \ne \varnothing$. By construction, there exist some $x \in P$ and some φ-square $P_0 \in \mathrm{Stop}(R_1)$ or point $P_0 \in G(R_1)$ (for convenience, in this proof we identify points in $G(R_1)$ with stopping squares in $\mathrm{Stop}(R_1)$ with zero side length) such that $\ell(P) \ge d_R(x)/20L$ and

$$\mathrm{dist}(x,P_0) + \ell(P_0) \le 1.1 d_R(x) \le 22L\ell(P).$$

Thus, for any $y \in Q$,

$$\begin{aligned}\mathrm{dist}(y,P_0) + \ell(P_0) &\le \mathrm{diam}(2Q) + \mathrm{diam}(2P) + \mathrm{dist}(x,P_0) + \ell(P_0)\\ &\le 3L\ell(Q) + 3L\ell(P) + 22L\ell(P),\end{aligned}$$

since $\operatorname{diam}(2Q) \le L \operatorname{diam}(\varphi^{-1}(2Q)) = L 8^{1/2}\ell(Q) \le 3L\ell(Q)$. So $d_R(y) \le 3L\ell(Q) + 25L\ell(P)$ for all $y \in Q$. Therefore,

$$\ell(Q) \le \frac{1}{10L}(3L\ell(Q) + 25L\ell(P)),$$

which yields $\ell(Q) \le \frac{25}{7}\ell(P) < 4\ell(P)$. This implies $\ell(Q) \le 2\ell(P)$, because P and Q are φ-dyadic squares.

The inequality $\ell(P) \le 2\ell(Q)$ is proved in an analogous way.

(b) Take now $Q \in \mathrm{Reg}_{\mathrm{dy}}(R)$ and $x \in Q$, $r \ge \ell(Q)$. There exists some $y \in Q$ and some φ-square $P_0 \in \mathrm{Stop}(R_1)$ such that $d_R(y)/20L < \ell(Q) \le d_R(y)/10$ and

$$\operatorname{dist}(y, P_0) + \ell(P_0) \le 1.1 d_R(y) \le 22L\ell(Q).$$

Thus $B(x,r)$ is contained in some φ-square of the form $\frac{Cr}{\ell(P_0)}P_0$, with $\frac{Cr}{\ell(P_0)} \ge 1$ and C depending on L. Then,

$$\sigma(B(x,r) \cap 4R) \le \sigma\big(\tfrac{Cr}{\ell(P_0)}P_0 \cap 4R\big) \le C\theta_\sigma(R_1)r.$$

(c) We continue with the same notation as in (b). Let $\widetilde{P}_0$ be a φ-square containing P_0 such that $\delta_\sigma(P_0, \widetilde{P}_0) \le C\theta_\sigma(R_1)$ and $\frac{1}{2}\widetilde{P}_0 \cap \Gamma_R \ne \varnothing$ (given by (c) of Main Lemma 8.1). It is easily checked that there exists some absolute constant $C_{26} \ge 1$ such that $C_{26}\widetilde{P}_0$ contains Q. We set $\widetilde{Q} := C_{26}\widetilde{P}_0$. $\square$

Given $R \in \mathrm{Top}_{\mathrm{dy}}(F)$, we denote by $\mathrm{Tree}^{\mathrm{Reg}}(R)$ the tree of dyadic φ-squares whose top φ-square is R and whose terminal φ-squares are the φ-squares $Q_i \in \mathrm{Reg}_{\mathrm{dy}}(R)$ which are contained in R (this is the same definition as the one for $\mathrm{Tree}(R)$ in Subsection 8.2, but with $\mathrm{Term}(R)$ replaced by $\mathrm{Reg}_{\mathrm{dy}}(R)$).

LEMMA 8.3. *Given any $R \in \mathrm{Top}_{\mathrm{dy}}(F)$, if $Q \in \mathrm{Tree}(R)$ and $\sigma(Q) > 0$, then $Q \in \mathrm{Tree}^{\mathrm{Reg}}(R)$.*

Roughly speaking, the lemma asserts that if we do not care about squares with vanishing σ-measure, then $\mathrm{Tree}(R) \subset \mathrm{Tree}^{\mathrm{Reg}}(R)$, and so we always stop later in $\mathrm{Tree}^{\mathrm{Reg}}(R)$ than in $\mathrm{Tree}(R)$.

Proof. Let R_1 be the 4-dyadic φ-square $R_1 \in \mathrm{Top}(F)$ such that $R \subset R_1 \subset 7R$ is as in the definition of $\mathrm{Reg}_{\mathrm{dy}}(R)$.

Let $Q_0 \in \mathrm{Tree}(R)$ be such that $\sigma(Q_0) > 0$. To see that $Q_0 \in \mathrm{Tree}^{\mathrm{Reg}}(R)$, it is enough to show that Q_0 is not contained in any square Q_x like the ones appearing in (8.3), with $x \in 3R \cap \mathrm{supp}(\sigma) \setminus [G(R_1) \cup Z(\sigma)]$. Suppose that this is not the case, so that $Q_0 \subset Q_x$ for some Q_x as above. Since

$$R \cap \mathrm{supp}(\sigma) \subset \bigcup_{P \in \mathrm{Stop}(R_1)} P \cup G(R_1) \cup Z(\sigma)$$

and $\sigma(Q_0) \neq 0$, by the definition of $\mathrm{Tree}(R)$, either there exists some square $P \in \mathrm{Stop}(R_1)$ such that one of the 16 dyadic squares which form P (which is 4-dyadic) is contained in P, or there exists some $y_0 \in G(R_1) \cap Q_0$. In any case, we deduce (identifying y_0 with a square P with $\ell(P) = 0$ in the latter case) that for any $y \in Q_x$

$$d_R(y) \leq \ell(P) + 2^{1/2} L\ell(Q_x) \leq (4 + 2^{1/2} L)\, \ell(Q_x).$$

In particular, this holds for $x = y$, and so

$$\ell(Q_x) \leq \frac{1}{10L}\, d_R(x) \leq \frac{4 + 2^{1/2}}{10} \ell(Q_x) \leq \frac{3}{5} \ell(Q_x).$$

Thus $\ell(Q_x) = 0$, which is a contradiction. □

8.4. *Construction of the approximating measure on* Γ_R. In this subsection we denote $\mathrm{Reg}_{\mathrm{dy}}(R) =: \{Q_i\}_{i\geq 1}$. For each i, let $\widetilde{Q}_i$ be a φ-square containing P_i such that $\delta_\sigma(Q_i, \widetilde{Q}_i) \leq C\theta_\sigma(R_1) \simeq \theta_\sigma(7R)$ and $\frac{1}{2}\widetilde{Q}_i \cap \Gamma_R \neq \varnothing$. We may also suppose that $\mathrm{diam}(\Gamma_R) \geq 10\ell(R)$, since we can always extend Γ_R if necessary.

LEMMA 8.4. *For each $i \geq 1$ there exists some function $g_i \geq 0$ supported on $\Gamma_R \cap \widetilde{Q}_i$ such that*

$$\int_{\Gamma_R} g_i \, d\mathcal{H}^1 = \sigma(Q_i), \tag{8.4}$$

$$\sum_i g_i \lesssim \theta_\sigma(R_1), \tag{8.5}$$

and

$$\|g_i\|_\infty \ell(\widetilde{Q}_i) \lesssim \sigma(Q_i). \tag{8.6}$$

Proof. The arguments are inspired by the Calderón-Zygmund decomposition of [To2, Lemma 7.3].

We assume first that the family $\mathrm{Reg}_{\mathrm{dy}}(R) = \{Q_i\}_i$ is finite. We also suppose that $\ell(\widetilde{Q}_i) \leq \ell(\widetilde{Q}_{i+1})$ for all i. The functions g_i that we will construct will be of the form $g_i = \alpha_i \chi_{A_i}$, with $\alpha_i \geq 0$ and $A_i \subset \widetilde{Q}_i$. We set $\alpha_1 := \sigma(Q_1)/\mathcal{H}^1(\widetilde{Q}_1 \cap \Gamma_R)$ and $A_1 := \widetilde{Q}_1 \cap \Gamma_R$, so that $\int_{\Gamma_R} g_1 d\mathcal{H}^1 = \sigma(Q_1)$. Notice by the way that $\|g_1\|_\infty \leq C\sigma(Q_1)/\ell(\widetilde{Q}_1) \leq C\theta_\sigma(R_1)$.

To define g_k, $k \geq 2$, we argue by induction. Suppose that $g_1, \ldots, g_{k-1}$ have been constructed, satisfy (8.4) and $\sum_{i=1}^{k-1} g_i \leq B\theta_\sigma(R_1)$, where B is some constant which will be chosen below. Let $\widetilde{Q}_{s_1}, \ldots, \widetilde{Q}_{s_m}$ be the subfamily of $\widetilde{Q}_1, \ldots, \widetilde{Q}_{k-1}$ such that $\widetilde{Q}_{s_j} \cap \widetilde{Q}_k \neq \varnothing$. Since $\ell(\widetilde{Q}_{s_j}) \leq \ell(\widetilde{Q}_k)$ (because of the nondecreasing sizes of the $\widetilde{Q}_i$'s), we have $\widetilde{Q}_{s_j} \subset 3\widetilde{Q}_k$. Using (8.4) for $i = s_j$, we get

$$\sum_j \int_{\Gamma_R} g_{s_j}\, d\mathcal{H}^1 \le \sum_j \sigma(Q_{s_j})$$
$$\le \sigma(3\widetilde{Q}_k) \le C\theta_\sigma(R_1)\ell(\widetilde{Q}_k) \le C_{27}\theta_\sigma(R_1)\mathcal{H}^1(\Gamma_R \cap \widetilde{Q}_k).$$

Therefore,

$$\mathcal{H}^1\Big(\Gamma_R \cap \Big\{\sum_j g_{s_j} > 2C_{27}\theta_\sigma(R_1)\Big\}\Big) \le \frac{1}{2}\mathcal{H}^1(\Gamma_R \cap \widetilde{Q}_k).$$

So we set

$$A_k := \Gamma_R \cap \widetilde{Q}_k \cap \Big\{\sum_j g_{s_j} \le 2C_{27}\theta_\sigma(R_1)\Big\},$$

and then $\mathcal{H}^1(A_k) \ge \mathcal{H}^1(\Gamma_R \cap \widetilde{Q}_k)/2$. Also, we put $\alpha_k := \dfrac{\sigma(Q_k)}{\mathcal{H}^1(A_k)}$, so that $\int_{\Gamma_R} g_k\, d\mathcal{H}^1 = \sigma(Q_k)$. Then,

$$\alpha_k \le \frac{2\sigma(Q_k)}{\mathcal{H}^1(\Gamma_R \cap \widetilde{Q}_k)} \le \frac{C\sigma(Q_k)}{\ell(\widetilde{Q}_k)} \le C_{28}\theta_\sigma(R_1). \tag{8.7}$$

Thus,

$$g_k + \sum_j g_{s_j} \le (2C_{27} + C_{28})\theta_\sigma(R_1).$$

We choose $B := 2C_{27} + C_{28}$ and (8.5) follows. Notice that (8.6) is proved in (8.7).

Suppose now that $\{Q_i\}_i$ is not finite. For each fixed N we consider a family of squares $\{Q_i\}_{1\le i\le N}$. As above, we construct functions $g_1^N, \dots, g_N^N$ with $\operatorname{supp}(g_i^N) \subset \widetilde{Q}_i \cap \Gamma_R$ satisfying

$$\int_{\Gamma_R} g_i^N d\mathcal{H}^1 = \sigma(Q_i), \qquad \sum_{i=1}^N g_i^N \le B\theta_\sigma(R_1), \quad \text{and} \quad \|g_i^N\|_\infty \ell(\widetilde{Q}_i) \le C\sigma(Q_i).$$

Then there is a subsequence $\{g_1^k\}_{k\in I_1}$ which is convergent in the weak $*$ topology of $L^\infty(\mathcal{H}^1_{\Gamma_R})$ to some function $g_1 \in L^\infty(\mathcal{H}^1_{\Gamma_R})$. Now we take another subsequence $\{g_2^k\}_{k\in I_2}$, $I_2 \subset I_1$, convergent in the weak $*$ topology of $L^\infty(\mathcal{H}^1_{\Gamma_R})$ to another function $g_2 \in L^\infty(\mathcal{H}^1_{\Gamma_R})$, etc. We have $\operatorname{supp}(g_i) \in \widetilde{Q}_i$. Further, (8.4), (8.5) and (8.6) also hold, because of the weak $*$ convergence. □

8.5. *A symmetrization lemma.* Recall that by Lemma 8.3, $\operatorname{Tree}(R) \subset \operatorname{Tree}^{\mathrm{Reg}}(R)$. As a consequence,

$$\sum_{Q\in \operatorname{Tree}(R)} \int_Q (\widehat{K}_{\sigma,J(Q)+1}\chi_F - \widehat{K}_{\sigma,J(Q)}\chi_F)\, d\sigma \le \sum_{Q\in \operatorname{Tree}^{\mathrm{Reg}}(R)} \int_Q \cdots\ d\sigma.$$

Observe also that if $x \in Q_i$, then

$$\begin{aligned}(8.8)\qquad \sum_{Q\in\mathrm{Tree}^{\mathrm{Reg}}(R)} &\chi_Q(x)\big(\widehat{K}_{\sigma,J(Q)+1}\chi_F(x)-\widehat{K}_{\sigma,J(Q)}\chi_F(x)\big)\\ &= \widehat{K}_{\sigma,J(Q_i)+1}\chi_F(x)-\widehat{K}_{\sigma,J(R)+1}\chi_F(x)\\ &= \iint_{\substack{\frac{1}{2L}\ell(Q_i)<|x-y|\le\frac{1}{2L}\ell(R)\\ |x-z|,|y-z|\le|x-y|}} c(x,y,z)^2d\sigma(y)d\sigma(z)\\ &\le \iint_{\substack{y,z\in 2R\\ |x-y|>\frac{1}{2L}\ell(Q_i),\\ |x-z|,|y-z|\le|x-y|}} c(x,y,z)^2d\sigma(y)d\sigma(z).\end{aligned}$$

In the last inequality we took into account that if $x\in R$ and $|x-y|,|x-z|\le \ell(R)/(2L)$, then $y,z\in 2R$. Analogously, if $x\in R\setminus\bigcup_i Q_i$, we get

$$(8.9)\qquad \sum_{Q\in\mathrm{Tree}^{\mathrm{Reg}}(R)}\chi_Q(x)\big(\widehat{K}_{\sigma,J(Q)+1}\chi_F(x)-\widehat{K}_{\sigma,J(Q)}\chi_F(x)\big)\le c^2_{\sigma|2R}(x).$$

The lack of symmetry with respect to x,y,z in the truncation of the integrals that appear in (8.8) might cause some difficulties in our estimates. This question is solved in the next lemma.

For any $y\in 3R$, we denote $\ell_y:=\ell(Q_i)$ if $y\in Q_i$, and $\ell_y:=0$ if $y\in 3R\setminus\bigcup_i Q_i$.

Lemma 8.5. (a) *If $|x-y|\ge C_{29}^{-1}\ell_x$, then $|x-y|\ge C_{30}^{-1}\ell_y$, with C_{30} depending only on C_{29} and L.*

(b) *There exists a sufficiently small constant $\varepsilon>0$ such that*

$$\begin{aligned}(8.10)\qquad &\iiint_{\substack{x,y,z\in 2R\\ |x-y|>\frac{1}{2L}\ell_x\\ |x-z|,|y-z|\le|x-y|}} c(x,y,z)^2d\sigma(x)d\sigma(y)d\sigma(z)\\ &\le C\theta_\sigma(R_1)^2\sigma(R_1)+\iiint_{\substack{x,y,z\in 2R\\ |x-y|\ge\varepsilon(\ell_x+\ell_y)\\ |x-z|\ge\varepsilon(\ell_x+\ell_z)\\ |y-z|\ge\varepsilon(\ell_y+\ell_z)}} c(x,y,z)^2d\sigma(x)d\sigma(y)d\sigma(z).\end{aligned}$$

Proof. First we show (a). Suppose $\ell_x\neq 0$, $\ell_y\neq 0$. Take $Q_i,Q_j\in\mathrm{Reg}_{\mathrm{dy}}(R)$ such that $x\in Q_i$, and $y\in Q_j$. So $\ell_x=\ell(Q_i)$ and $\ell_y=\ell(Q_j)$. If $|x-y|\le \ell(Q_j)/(2L)$, then $x\in 2Q_j$. Thus $Q_i\cap 2Q_j\neq\varnothing$, and then $\ell(Q_i)\ge\ell(Q_j)/2$, which yields

$$|x-y|\ge C_{29}^{-1}\ell(Q_i)\ge\frac{C_{29}^{-1}}{2}\ell(Q_j).$$

So in any case we have

$$|x-y|\ge\min\Big(\frac{1}{2L},\frac{C_{29}^{-1}}{2}\Big)\ell(Q_j).$$

If $\ell_x=0$ or $\ell_y=0$ the arguments above also work, with the convention $Q_i\equiv\{x\}$ or $Q_j\equiv\{y\}$.

Let us prove (b) now. We put

$$\begin{aligned}\iiint_{\substack{x,y,z\in 2R\\ |x-y|>\frac{1}{2L}\ell_x\\ |x-z|,|y-z|\leq|x-y|}} & c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z)\\ &= \iiint_{\substack{x,y,z\in 2R\\ |x-y|>\frac{1}{2L}\ell_x\\ |x-y|\geq|x-z|\geq|y-z|}} \cdots + \iiint_{\substack{x,y,z\in 2R\\ |x-y|>\frac{1}{2L}\ell_x\\ |x-y|\geq|y-z|>|x-z|}} \cdots =: A+B.\end{aligned}$$

First we deal with the term A. By (a) we deduce that if $|x-y|\geq\frac{1}{2L}\ell_x$, then $|x-y|\geq C^{-1}\ell_y$, and so $|x-y|\geq\varepsilon(\ell_x+\ell_y)$. If moreover $|x-y|\geq|x-z|\geq|y-z|$, then $|x-z|\geq\frac{1}{2}|x-y|\geq\frac{1}{4L}\ell_x$. Thus, $|x-z|\geq C^{-1}\ell_z$ by (a), and so $|x-z|\geq\varepsilon(\ell_x+\ell_z)$. We obtain

$$\begin{aligned}A &\leq \iiint_{\substack{x,y,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)}} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z)\\ &= \iiint_{\substack{x,y,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|>\ell_y}} \cdots + \iiint_{\substack{x,y,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|\leq\ell_y}} \cdots =: A_1+A_2.\end{aligned}$$

To estimate A_1 we apply (a) again. Indeed, if $|y-z|>\ell_y$, then $|y-z|\geq C^{-1}\ell_z$, and we get $|y-z|\geq\varepsilon(\ell_y+\ell_z)$. Therefore,

$$A_1 \leq \iiint_{\substack{x,y,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|\geq\varepsilon(\ell_y+\ell_z)}} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z).$$

Now we deal with A_2. For each $y\in 2R$ we have

$$\begin{aligned}\iint_{\substack{x,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|\leq\ell_y}} c(x,y,z)^2 d\sigma(x)d\sigma(z) &\leq \iint_{\substack{x,z\in 2R\\ |x-y|\geq\varepsilon\ell_y\\ |y-z|\leq\ell_y}} \frac{C}{|x-y|^2} d\sigma(x)d\sigma(z)\\ &\leq C\sigma(B(y,\ell_y))\int_{|x-y|\geq\varepsilon\ell_y} \frac{1}{|x-y|^2}d\sigma(x) \leq C\varepsilon^{-1}\theta_\sigma(R_1)^2.\end{aligned}$$

Therefore,

$$A_2 \leq C\varepsilon^{-1}\theta_\sigma(R_1)^2\sigma(R_1).$$

Thus, A is bounded above by the right-hand side of (8.10).

The term B is estimated similarly to A. We will not go through the details. Then we obtain

$$\begin{aligned}\iiint_{\substack{x,y,z\in 2R\\ |x-y|>\frac{1}{2L}\ell_x\\ |x-z|,|y-z|\leq|x-y|}} & c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z)\\ &\leq C\theta_\sigma(R_1)^2\sigma(R_1)+2\iiint_{\substack{x,y,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|\geq\varepsilon(\ell_y+\ell_z)}} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z),\end{aligned}$$

The reader may check that the '2' preceding the integral on the right-hand side can be eliminated if one argues a little more carefully (although this fact will be not needed for the estimates below). □

We denote

$$c^2_\ell(\sigma_{|2R}) := \iiint_{\substack{x,y,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|\geq\varepsilon(\ell_y+\ell_z)}} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z).$$

We also set

$$c^2_{\ell,\sigma_{|2R}}(x) := \iint_{\substack{y,z\in 2R\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|\geq\varepsilon(\ell_y+\ell_z)}} c(x,y,z)^2 d\sigma(y)d\sigma(z),$$

and

$$c^2_{\ell,\sigma}(A,B,C) := \iiint_{\substack{x\in A,\, y\in B,\, z\in C\\ |x-y|\geq\varepsilon(\ell_x+\ell_y)\\ |x-z|\geq\varepsilon(\ell_x+\ell_z)\\ |y-z|\geq\varepsilon(\ell_y+\ell_z)}} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z).$$

Notice that $c^2_{\ell,\sigma}(A,B,C)$ is symmetric with respect to A, B, C.

By (8.8), (8.9) and Lemma 8.5, to prove (8.2) it is enough to show that

$$c^2_\ell(\sigma_{|2R}) \leq C\theta_\sigma(R_1)^2\sigma(R_1). \tag{8.11}$$

We set

$$G_R := 2R\setminus\bigcup_i Q_i.$$

Observe that σ-almost all G_R are contained in Γ_R, by (a) of Lemma 8.1.

We split $c^2_\ell(\sigma_{|2R})$ as follows:

$$\begin{aligned} c^2_\ell(\sigma_{|2R}) = {} & c^2_{\ell,\sigma}\big(\cup_i Q_i,\cup_i Q_i,\cup_i Q_i\big)+3c^2_{\ell,\sigma}\big(\cup_i Q_i,\cup_i Q_i,G_R\big)\\ & +3c^2_{\ell,\sigma}\big(\cup_i Q_i,G_R,G_R\big)+c^2_\sigma\big(G_R,G_R,G_R\big).\end{aligned} \tag{8.12}$$

8.6. *Estimate of* $c^2_\sigma(G_R,G_R,G_R)$. The measure $\sigma_{|G_R}$ coincides with $f\,d\mathcal{H}^1_{\Gamma_R}$, where f is some function such that $\|f\|_\infty \leq C\theta_\sigma(R_1)$. Since the

Cauchy transform is bounded on $L^2(\mathcal{H}^1_{\Gamma_R})$ (with $\|\mathcal{C}\|_{L^2(\mathcal{H}^1_{\Gamma_R}),L^2(\mathcal{H}^1_{\Gamma_R})}$ bounded above by some absolute constant), we have

$$\begin{aligned} c^2_\sigma\big(G_R, G_R, G_R\big) &\leq \|f\|^3_\infty\, c^2(\mathcal{H}^1_{\Gamma_R}) \lesssim \theta_\sigma(R_1)^3\mathcal{H}^1(\Gamma_R) \\ &\lesssim \theta_\sigma(R_1)^3 \mathrm{diam}(R_1) = C\theta_\sigma(R_1)^2\sigma(R_1). \end{aligned}$$

8.7. *Estimate of* $c^2_{\ell,\sigma}\big(\cup_i Q_i, \cup_i Q_i, \cup_i Q_i\big)$. We set

$$c^2_{\ell,\sigma}\big(\cup_i Q_i, \cup_j Q_j, \cup_k Q_k\big) = \sum_{i,j,k} c^2_{\ell,\sigma}(Q_i, Q_j, Q_k). \tag{8.13}$$

Now we put

$$\begin{aligned} Q_i &= \Big[Q_i \cap (6\widetilde{Q}_j \cup 6\widetilde{Q}_k)\Big] \cup \Big[Q_i \setminus (6\widetilde{Q}_j \cup 6\widetilde{Q}_k)\Big], \\ Q_j &= \Big[Q_j \cap (6\widetilde{Q}_i \cup 6\widetilde{Q}_k)\Big] \cup \Big[Q_j \setminus (6\widetilde{Q}_i \cup 6\widetilde{Q}_k)\Big], \\ Q_k &= \Big[Q_k \cap (6\widetilde{Q}_i \cup 6\widetilde{Q}_j)\Big] \cup \Big[Q_k \setminus (6\widetilde{Q}_i \cup 6\widetilde{Q}_j)\Big]. \end{aligned}$$

We replace Q_i, Q_j, and Q_k in (8.13) by the right-hand side of the identities above, and we get

$$\begin{aligned} &c^2_{\ell,\sigma}\big(\cup_i Q_i, \cup_j Q_j, \cup_k Q_k\big) \\ &\quad \leq \sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i \setminus (6\widetilde{Q}_j \cup 6\widetilde{Q}_k),\ Q_j \setminus (6\widetilde{Q}_i \cup 6\widetilde{Q}_k),\ Q_k \setminus (6\widetilde{Q}_i \cup 6\widetilde{Q}_j)\big) \\ &\qquad + \sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i \cap (6\widetilde{Q}_j \cup 6\widetilde{Q}_k),\ Q_j,\ Q_k\big) \\ &\qquad + \sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i,\ Q_j \cap (6\widetilde{Q}_i \cup 6\widetilde{Q}_k),\ Q_k\big) \\ &\qquad + \sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i,\ Q_j,\ Q_k \cap (6\widetilde{Q}_i \cup 6\widetilde{Q}_j)\big) \\ &\quad =: U + V_1 + V_2 + V_3. \end{aligned}$$

First we estimate V_1, V_2, V_3. By symmetry, $V_1 = V_2 = V_3$, and also

$$\begin{aligned} V_1 &\leq \sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i \cap 6\widetilde{Q}_j,\ Q_j,\ Q_k\big) + \sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i \cap 6\widetilde{Q}_k,\ Q_j,\ Q_k\big) \\ &= 2\sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i \cap 6\widetilde{Q}_j,\ Q_j,\ Q_k\big) \\ &\leq 2\sum_j c^2_{\ell,\sigma}(Q_j,\ 6\widetilde{Q}_j,\ 2R) \\ &= 2\sum_j c^2_{\ell,\sigma}(Q_j,\ 6\widetilde{Q}_j,\ 6\widetilde{Q}_j) + 2\sum_j c^2_{\ell,\sigma}(Q_j,\ 6\widetilde{Q}_j,\ 2R \setminus 6\widetilde{Q}_j) \\ &=: V_{1,1} + V_{1,2}. \end{aligned}$$

Now we deal with $V_{1,1}$:

$$\begin{aligned}
c^2_{\ell,\sigma}\big(Q_j, 6\widetilde{Q}_j, 6\widetilde{Q}_j\big) &\leq 2 \iiint_{\substack{x\in Q_j,\, y,z\in 6\widetilde{Q}_j\\ |x-z|\geq|x-y|\geq\varepsilon\ell(Q_j)}} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z)\\
&\lesssim \int_{x\in Q_j}\int_{\substack{y\in 6\widetilde{Q}_j\\ |x-y|\geq\varepsilon\ell(Q_j)}} \bigg(\int_{\substack{z\in 6\widetilde{Q}_j\\ |x-z|\geq|x-y|}} \frac{1}{|x-z|^2}\, d\sigma(z)\bigg) d\sigma(y)d\sigma(x)\\
&\lesssim \int_{x\in Q_j}\int_{\substack{y\in 6\widetilde{Q}_j\\ |x-y|\geq\varepsilon\ell(Q_j)}} \frac{\theta_\sigma(R_1)}{|x-y|} d\sigma(y)d\sigma(x)\\
&\lesssim \int_{x\in Q_j} \theta_\sigma(R_1)^2 d\sigma(x) = \theta_\sigma(R_1)^2\sigma(Q_j).
\end{aligned}$$

Thus, $V_{1,1} \lesssim \theta_\sigma(R_1)^2\sigma(R_1)$.

The term $V_{1,2}$ is estimated likewise:

$$\begin{aligned}
&c^2_{\ell,\sigma}\big(Q_j, 6\widetilde{Q}_j, 2R\setminus 6\widetilde{Q}_j\big)\\
&\quad\lesssim \int_{x\in Q_j}\int_{\substack{y\in 6\widetilde{Q}_j\\ |x-y|\geq\varepsilon\ell(Q_j)}} \bigg(\int_{\substack{z\in 2R\\ |x-z|\geq C^{-1}\ell(\widetilde{Q}_j)}} \frac{1}{|x-z|^2}\, d\sigma(z)\bigg) d\sigma(y)d\sigma(x)\\
&\quad\lesssim \int_{x\in Q_j}\int_{\substack{y\in 6\widetilde{Q}_j\\ |x-y|\geq\varepsilon\ell(Q_j)}} \frac{\theta_\sigma(R_1)}{|x-y|} d\sigma(y)d\sigma(x)\\
&\quad\lesssim \theta_\sigma(R_1)^2\sigma(Q_j),
\end{aligned}$$

and so $V_{1,2} \leq C\theta_\sigma(R_1)^2\sigma(R_1)$.

It only remains to estimate U. Notice that if

$$Q_i\setminus 6\widetilde{Q}_j \neq \varnothing \qquad \text{and} \qquad Q_j\setminus 6\widetilde{Q}_i \neq \varnothing, \tag{8.14}$$

then $2\widetilde{Q}_i\cap 2\widetilde{Q}_j = \varnothing$. Otherwise, $2\widetilde{Q}_i\cap 2\widetilde{Q}_j \neq \varnothing$ implies that either $Q_i \subset 2\widetilde{Q}_i \subset 6\widetilde{Q}_j$ or $Q_j \subset 2\widetilde{Q}_j \subset 6\widetilde{Q}_i$, which contradicts (8.14). Thus,

$$\begin{aligned}
U &= \sum_{i,j,k} c^2_{\ell,\sigma}\big(Q_i\setminus(6\widetilde{Q}_j\cup 6\widetilde{Q}_k),\ Q_j\setminus(6\widetilde{Q}_i\cup 6\widetilde{Q}_k),\ Q_k\setminus(6\widetilde{Q}_i\cup 6\widetilde{Q}_j)\big)\\
&\leq \sum_{\substack{i,j,k:\, 2\widetilde{Q}_i\cap 2\widetilde{Q}_j=\varnothing,\\ 2\widetilde{Q}_i\cap 2\widetilde{Q}_k=\varnothing,\\ 2\widetilde{Q}_j\cap 2\widetilde{Q}_k=\varnothing}} c^2_\sigma(Q_i, Q_j, Q_k).
\end{aligned}$$

Next we wish to compare $c^2_\sigma(Q_i, Q_j, Q_k)$ (for Q_i, Q_j, Q_k as in the last sum) with the curvature $c^2_{\mathcal{H}^1_{\Gamma_R}}(g_i, g_j, g_k)$, where g_i, g_j, g_k are the bounded functions constructed in Lemma 8.4, which are supported on $\widetilde{Q}_i, \widetilde{Q}_j, \widetilde{Q}_k$ respectively. We set

$$c^2_{\mathcal{H}^1_{\Gamma_R}}(g_i, g_j, g_k) := \iiint_{\Gamma_R^3} c(x,y,z)^2\, g_i(x)\, g_j(y)\, g_k(z)\, d\mathcal{H}^1(x)d\mathcal{H}^1(y)d\mathcal{H}^1(z).$$

If $x, x' \in \widetilde{Q}_i$, $y, y' \in \widetilde{Q}_j$ and $z, z' \in \widetilde{Q}_k$, then

$$c(x,y,z)^2 \leq 2c(x',y',z')^2 + \frac{C\ell(\widetilde{Q}_i)^2}{|x-y|^2|x-z|^2} + \frac{C\ell(\widetilde{Q}_j)^2}{|y-x|^2|y-z|^2} + \frac{C\ell(\widetilde{Q}_k)^2}{|z-x|^2|z-y|^2},$$

by Lemma 3.5. If we integrate $x \in Q_i$, $y \in Q_j$, and $z \in Q_k$ with respect to σ, and also $x' \in \widetilde{Q}_j$ with respect to the measure $g_i\, d\mathcal{H}^1_{\Gamma_R}$, $y' \in \widetilde{Q}_j$ with respect to $g_j\, d\mathcal{H}^1_{\Gamma_R}$, and $z' \in \widetilde{Q}_k$ with respect to $g_k\, d\mathcal{H}^1_{\Gamma_R}$, then we get

$$\begin{aligned} c^2_\sigma(Q_i,Q_j,Q_k) \ \leq\ & 2c^2_{\mathcal{H}^1_{\Gamma_R}}(g_i,g_j,g_k) \\ & + \iiint_{\substack{x\in Q_i\\ y\in Q_j\\ z\in Q_k}} \frac{C\ell(\widetilde{Q}_i)^2}{|x-y|^2|x-z|^2}\, d\sigma(x) d\sigma(y)\, d\sigma(z) \\ & + \iiint_{\substack{x\in Q_i\\ y\in Q_j\\ z\in Q_k}} \frac{C\ell(\widetilde{Q}_j)^2}{|y-x|^2|y-z|^2}\, d\sigma(x)\, d\sigma(y)\, d\sigma(z) \\ & + \iiint_{\substack{x\in Q_i\\ y\in Q_j\\ z\in Q_k}} \frac{C\ell(\widetilde{Q}_k)^2}{|z-x|^2|z-y|^2}\, d\sigma(x)\, d\sigma(y)\, d\sigma(z). \end{aligned}$$

Therefore, by symmetry,

$$U \leq 2\sum_{i,j,k} c^2_{\mathcal{H}^1_{\Gamma_R}}(g_i,g_j,g_k) + 3\sum_i \iiint_{\substack{x\in Q_i\\ |x-y|>C^{-1}\ell(\widetilde{Q}_i)\\ |x-z|>C^{-1}\ell(\widetilde{Q}_i)}} \frac{C\ell(\widetilde{Q}_i)^2}{|y-x|^2|y-z|^2}\, d\sigma(x)\, d\sigma(y)\, d\sigma(z). \tag{8.15}$$

By Lemma 8.4, $g := \sum_i g_i \lesssim \theta_\sigma(R_1)$, and then

$$\sum_{i,j,k} c^2_{\mathcal{H}^1_{\Gamma_R}}(g_i,g_j,g_k) = c^2(g\, d\mathcal{H}^1_{\Gamma_R}) \lesssim \theta_\sigma(R_1)^3 c^2(\mathcal{H}^1_{\Gamma_R}) \lesssim \theta_\sigma(R_1)^3 \mathcal{H}^1(\Gamma_R) \lesssim \theta_\sigma(R_1)^2 \sigma(R_1). \tag{8.16}$$

The integral in (8.15) is estimated as follows:

$$\begin{aligned} \iiint_{\substack{x\in Q_i\\ |x-y|>C^{-1}\ell(\widetilde{Q}_i)\\ |x-z|>C^{-1}\ell(\widetilde{Q}_i)}} & \frac{\ell(\widetilde{Q}_i)^2}{|y-x|^2|y-z|^2}\, d\sigma(x)\, d\sigma(y)\, d\sigma(z) \\ & = \ell(\widetilde{Q}_i)^2 \int_{x\in Q_i} \left(\int_{|x-y|>C^{-1}\ell(\widetilde{Q}_i)} \frac{1}{|y-x|^2}\, d\sigma(y) \right)^2 d\sigma(x) \\ & \lesssim \ell(\widetilde{Q}_i)^2 \int_{x\in Q_i} \frac{\theta_\sigma(R_1)^2}{\ell(\widetilde{Q}_i)^2}\, d\sigma(x) = \theta_\sigma(R_1)^2 \sigma(Q_i). \end{aligned}$$

From (8.15), (8.16) and the preceding estimate we get

$$U \lesssim \theta_\sigma(R_1)^2\sigma(R_1).$$

We are done.

8.8. *Estimates of* $c^2_{\ell,\sigma}\big(\cup_i Q_i, \cup_i Q_i, G_R\big)$ *and* $c^2_{\ell,\sigma}\big(\cup_i Q_i, G_R, G_R\big)$. We leave these estimates for the reader. The arguments are similar to the ones used for $c^2_{\ell,\sigma}\big(\cup_i Q_i, \cup_i Q_i, \cup_i Q_i\big)$. In fact, notice that if by convention one allows the squares Q_i to be points, then $\big(\cup_i Q_i\big) \times \big(\cup_i Q_i\big) \times G_R$ and $\big(\cup_i Q_i\big) \times G_R \times G_R$ are subsets of $\big(\cup_i Q_i\big)^3$.

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