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Hypoellipticity and loss of derivatives

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(with an Appendix by Makhlouf Derridj and David S. Tartakoff)

Dedicated to Yum-Tong Siu for his 60^{th} *birthday.*

Abstract

Let $\{X_1, \dots, X_p\}$ be complex-valued vector fields in $\mathbb{R}^n$ and assume that they satisfy the bracket condition (i.e. that their Lie algebra spans all vector fields). Our object is to study the operator $E = \sum X_i^* X_i$, where X_i^* is the L_2 adjoint of X_i. A result of Hörmander is that when the X_i are real then E is hypoelliptic and furthemore it is subelliptic (the restriction of a destribution u to an open set U is "smoother" then the restriction of Eu to U). When the X_i are complex-valued if the bracket condition of order one is satisfied (i.e. if the $\{X_i, [X_i, X_j]\}$ span), then we prove that the operator E is still subelliptic. This is no longer true if brackets of higher order are needed to span. For each $k \geq 1$ we give an example of two complex-valued vector fields, X_1 and X_2, such that the bracket condition of order $k+1$ is satisfied and we prove that the operator $E = X_1^* X_1 + X_2^* X_2$ is hypoelliptic but that it is not subelliptic. In fact it "loses" k derivatives in the sense that, for each m, there exists a distribution u whose restriction to an open set U has the property that the $D^\alpha Eu$ are bounded on U whenever $|\alpha| \leq m$ and for some β, with $|\beta| = m - k + 1$, the restriction of $D^\beta u$ to U is not locally bounded.

1. Introduction

We will be concerned with local C^∞ hypoellipticity in the following sense. A linear differential operator operator E on $\mathbb{R}^n$ is hypoelliptic if, whenever u is a distribution such that the restriction of Eu to an open set $U \subset \mathbb{R}^n$ is in $C^\infty(U)$, then the restriction of u to U is also in $C^\infty(U)$. If E is hypoelliptic then it satisfies the following *a priori* estimates.

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(1) Given open sets U, U' in $\mathbb{R}^n$ such that $U \subset \bar{U} \subset U' \subset \mathbb{R}^n$, a nonnegative integer p, and a real number s_o, there exist an integer q and a constant $C = C(U, p, q, s_o)$ such that

$$\sum_{|\alpha|\leq p} sup_{\,x\in U}|D^\alpha u(x)| \leq C(\sum_{|\beta|\leq q} sup_{\,x\in U'}|D^\beta Eu(x)| + \|u\|_{-s_o}),$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

(2) Given $\varrho, \varrho' \in C_0^\infty(\mathbb{R}^n)$ such that $\varrho' = 1$ in a neighborhood of $\mathrm{supp}(\varrho)$, and $s_o, s_1 \in \mathbb{R}$, there exist $s_2 \in \mathbb{R}$ and a constant $C = C(\varrho, \varrho', s_1, s_2, s_0)$ such that

$$\|\varrho u\|_{s_1} \leq C(\|\varrho' Eu\|_{s_2} + \|u\|_{-s_o}),$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Assuming that E is hypoelliptic and that q is the smallest integer so that the first inequality above holds (for large s_o) then, if $q \leq p$, we say that E *gains* $p - q$ derivatives in the sup norms and if $q \geq p$, we say that E *loses* $q - p$ derivatives in the sup norms. Similarly, assuming that s_2 is the smallest real number so that the second inequality holds (for large s_o) then, if $s_2 \leq s_1$, we say that E *gains* $s_1 - s_2$ derivatives in the Sobolev norms and if $s_2 \geq s_1$, we say that E *loses* $s_2 - s_1$ derivatives in the Sobolev norms. In particular if E is of order m and if E is elliptic then E gains exactly m derivatives in the Sobolev norms and gains exactly $m - 1$ derivatives in the sup norms. Here we will present hypoelliptic operators E_k of order 2 which lose exactly $k - 1$ derivatives in the Sobolev norms and lose at least k derivatives in the sup norms.

Loss of derivatives presents a very major difficulty: namely, how to derive the *a priori* estimates? Such estimates depend on localizing the right-hand side and (because of the loss of derivatives) the errors that arise are apparently always larger then the terms one wishes to estimate. This difficulty is overcome here by the use of subelliptic multipliers in a microlocal setting. In this introduction I would like to indicate the ideas behind these methods, which were originally devised to study hypoellipticity with gain of derivatives. It should be remarked that that for global hypoellipticity the situation is entirely different; in that case loss of derivatives can occur and is well understood but, of course, the localization problems do not arise.

We will restrict ourselves to operators E of second order of the form

$$Eu = -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j},$$

where (a_{ij}) is a hermitian form with C^∞ complex-valued components. If at some point $P \in \mathbb{R}^n$ the form $(a_{ij}(P))$ has two nonzero eigenvalues of different signs then E is not hypoelliptic so that, without loss of generality, we will assume that $(a_{ij}) \geq 0$.

Definition 1. The operator E is subelliptic at $P \in \mathbb{R}^n$ if there exists a neighborhood U of P, a real number $\varepsilon > 0$, and a constant $C = (U, \varepsilon)$, such that

$$\|u\|_\varepsilon^2 \leq C(|(Eu, u)| + \|u\|^2),$$

for all $u \in C_0^\infty(U)$.

Here the Sobolev norm $\|u\|_s$ is defined by

$$\|u\|_s = \|\Lambda^s u\|,$$

and $\Lambda^s u$ is defined by its Fourier transform, which is

$$\widehat{\Lambda^s u}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi).$$

We will denote by $H^s(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm $\| \, \|_s$. If $U \subset \mathbb{R}^n$ is open, we denote by $H^s_{\text{loc}}(U)$ the set of all distributions on U such that $\zeta u \in H^s(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(U)$. The following result, which shows that subellipticity implies hypoellipticity with a gain of 2ε derivatives in Sobolev norms, is proved in [KN].

THEOREM. *Suppose that E is subelliptic at each $P \in U \subset \mathbb{R}^n$. Then E is hypoelliptic on U. More precisely, if $u \in H^{-s_\circ} \cap H^s_{\text{loc}}(U)$ and if $Eu \in H^s_{\text{loc}}(U)$, then $u \in H^{s+2\varepsilon}_{\text{loc}}(U)$.*

In [K1] and [K2] I introduced subelliptic multipliers in order to establish subelliptic estimates for the $\bar{\partial}$-Neumann problem. In the case of E, subelliptic multipliers are defined as follows.

Definition 2. A *subelliptic multiplier* for E at $P \in \mathbb{R}^n$ is a pseudodifferential operator A of order zero, defined on $C_0^\infty(U)$, where U is a neighborhood of P, such that there exist $\varepsilon > 0$, and a constant $C = C(\varepsilon, P, A)$, such that

$$\|Au\|_\varepsilon^2 \leq C(|(Eu, u)| + \|u\|^2),$$

for all $u \in C_0^\infty(U)$.

If A is a subelliptic multiplier and if A' is a pseudodifferential operator whose principal symbol equals the principal symbol of A then A' is also a subelliptic multiplier. The existence of subelliptic estimates can be deduced from the properties of the set symbols of subelliptic multipliers. In the case of the $\bar{\partial}$-Neumann problem this leads to the analysis of the condition of "D'Angelo finite type." Catlin and D'Angelo, in [C] and [D'A], showed that D'Angelo finite type is a necessary and sufficient condition for the subellipticity of the $\bar{\partial}$-Neumann problem. To illustrate some of these ideas, in the case of an operator E, we will recall Hörmander's theorem on the sum of squares of vector fields.

Let $\{X_1, \dots, X_m\}$ be vector fields on a neighborhood of the origin in $\mathbb{R}^n$.

Definition 3. The vectorfields $\{X_1, \dots, X_m\}$ satisfy the *bracket condition* at the origin if the Lie algebra generated by these vector fields evaluated at the origin is the tangent space.

In [Ho], Hörmander proved the following

THEOREM. *If the vectorfields $\{X_1, \dots, X_m\}$ are real and if they satisfy the* bracket condition *at the origin then the operator $E = \sum X_j^2$ is hypoelliptic in a neighborhood of the origin.*

The key point of the proof is to establish that for some neighborhoods of the origin U there exist $\varepsilon > 0$ and $C = C(\varepsilon, U)$ such that

$$\|u\|_\varepsilon^2 \le C\big(\sum \|X_j u\|^2 + \|u\|^2\big), \tag{1}$$

for all $u \in C_0^\infty(U)$. Here is a brief outline of the proof of estimate (1) using subelliptic multipliers. Note that

1. The operators $A_j = \Lambda^{-1} X_j$ are subelliptic multipliers with $\varepsilon = 1$, that is
$$\|A_j u\|_1^2 \le C\big(\sum \|X_j u\|^2 + \|u\|^2\big),$$
for all $u \in C_0^\infty(U)$.
2. If A is a subelliptic multiplier then $[X_j, A]$ is a subelliptic multiplier. (This is easily seen: we have $X_j^* = -X_j + a_j$ since X_j is real and
$$\begin{aligned}\|[X_j, A]u\|_{\frac{\varepsilon}{2}}^2 &\le |(X_j A u, R^\varepsilon u)| + |(A X_j u, R^\varepsilon u)| \\ &\le |(Au, \tilde{R}^\varepsilon u)| + O(\|u\|^2) + |(Au, R^\varepsilon X_j u)| + |(A X_j u, R^\varepsilon u)| \\ &\le C\left(\|Au\|_\varepsilon^2 + \sum \|X_j u\|^2 + \|u\|^2\right),\end{aligned}$$
where $R^\varepsilon = \Lambda^\varepsilon [X_j, A]$ and $\tilde{R}^\varepsilon = [X_j^*, R^\varepsilon]$ are pseudodifferential operators of order ε.)

Now using the bracket condition and the above we see that 1 is a subelliptic multiplier and hence the estimate (1) holds.

The more general case, where the a_{ij} are real but E cannot be expressed as a sum of squares (modulo L_2) has been analyzed by Oleinik and Radkevic (see [OR]). Their result can also be obtained by use of subelliptic multipliers and can then be connected to the geometric interpretation given by Fefferman and Phong in [FP]. The next question, which has been studied fairly extensively, is what happens when subellipticity fails and yet there is no loss. A striking example is the operator on $\mathbb{R}^2$ given by

$$E = -\frac{\partial^2}{\partial x^2} - a^2(x)\frac{\partial^2}{\partial y^2},$$

where $a(x) \geq 0$ when $x \neq 0$. This operator was studied by Fedii in [F], who showed that E is always hypoelliptic, no matter how fast $a(x)$ goes to zero as $x \to 0$. Kusuoka and Stroock (see [KS]) have shown that the operator on $\mathbb{R}^3$ given by

$$E = -\frac{\partial^2}{\partial x^2} - a^2(x)\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

where $a(x) \geq 0$ when $x \neq 0$, is hypoelliptic if and only if $\lim_{x\to 0} \log a(x) = 0$. Hypoellipticity when there is no loss but when the gain is smaller than in the subelliptic case has also been studied by Bell and Mohamed [BM], Christ [Ch1], and Morimoto [M]. Using subelliptic multipliers has provided new insights into these results (see [K4]); for example Fedii's result is proved when a^2 is replaced by a with the requirement that $a(x) > 0$ when $x \neq 0$. In the case of the $\bar\partial$-Neumann problem and of the operator $\Box_b$ on CR manifolds, subelliptic multipliers are used to established hypoellipticity in certain situations where there is no loss of derivatives in Sobolev norms but in which the gain is weaker than in the subelliptic case (see [K5]). Stein in [St] shows that the operator $\Box_b+\mu$ on the Heisenberg group $\mathcal{H} \subset \mathbb{C}^2$, with $\mu \neq 0$, is analytic hypoelliptic but does not gain or lose any derivatives. In his thesis Heller (see [He]), using the methods developed by Stein in [St], shows that the fourth order operator $\Box_b^2+X$ is analytic hypoelliptic and that it loses derivatives (here X denotes a "good" direction). In a recent work, C. Parenti and A. Parmeggiani studied classes of pseudodifferential operators with large losses of derivatives (see [PP1]).

The study of subelliptic multipliers has led to the concept of multiplier ideal sheaves (see [K2]). These have had many applications notably Nadel's work on Kähler-Einstein metrics (see [N]) and numerous applications to algebraic geometry. In algebraic geometry there are three areas in which multiplier ideals have made a decisive contribution: the Fujita conjecture, the effective Matsusaka big theorem, and invariance of plurigenera; see, for example, Siu's article [S]. Up to now the use of subelliptic multipliers to study the $\bar\partial$-Neumann problem and the laplacian $\Box_b$ has been limited to dealing with Sobolev norms, Siu has developed a program to use multipliers for the $\bar\partial$-Neumann problem to study Hölder estimates and to give an explicit construction of the critical varieties that control the D'Angelo type. His program leads to the study of the operator

$$E = \sum_1^m X_j^* X_j,$$

where the $\{X_1, \dots, X_m\}$ are *complex* vector fields satisfying the bracket condition. Thus Siu's program gives rise to the question of whether the above operator E is hypoelliptic and whether it satisfies the subelliptic estimate (1). These problems raised by Siu have motivated my work on this paper. At first I found that if the bracket condition involves only one bracket then (1) holds with $\varepsilon = \frac{1}{4}$ (if the X_j span without taking brackets then E is elliptic). Then I found a series of examples for which the bracket condition is satisfied with k brackets, $k > 1$, for which (1) does not hold. Surprisingly I found that the operators in these examples are hypoelliptic with a loss of $k-1$ derivatives in the Sobolev norms. The method of proof involves calculations with subelliptic multipliers and it seems very likely that it will be possible to treat the more general cases, that is when E given by complex vectorfields and, more generally, when (a_{ij}) is nonnegative hermitian, along the same lines.

The main results proved here are the following:

THEOREM A. *If $\{X_i, [X_i, X_j]\}$ span the complex tangent space at the origin then a subelliptic estimate is satisfied, with $\varepsilon = \frac{1}{2}$.*

THEOREM B. *For $k \geq 0$ there exist complex vector fields X_{1k} and X_2 on a neighborhood of the origin in $\mathbb{R}^3$ such that the two vectorfields $\{X_{1k}, X_2\}$ and their commutators of order $k+1$ span the complexified tangent space at the origin, and when $k > 0$ the subelliptic estimate* (1) *does not hold. Moreover, when $k > 1$, the operator $E_k = X_{1k}^* X_{1k} + X_2^* X_2$ loses k derivatives in the sup norms and $k-1$ derivatives in the Sobolev norms.*

Recently Christ (see [Ch2]) has shown that the operators $-\frac{\partial^2}{\partial s^2} + E_k$ on $\mathbb{R}^4$ are not hypoelliptic when $k > 0$.

THEOREM C. *If X_{1k} and X_2 are the vectorfields given in Theorem* B *then the operator $E_k = X_{1k}^* X_{1k} + X_2^* X_2$ is hypoelliptic. More precisely, if u is a distribution solution of $Eu = f$ with $u \in H^{-s_0}(\mathbb{R}^3)$ and if $U \subset \mathbb{R}^3$ is an open set such that $f \in H^{s_2}_{\mathrm{loc}}(U)$, then $u \in H^{s_2-k+1}_{\mathrm{loc}}(U)$.*

This paper originated with a problem posed by Yum-Tong Siu. The author wishes to thank Yum-Tong Siu and Michael Christ for fruitful discussions of the material presented here.

Remarks. In March 2005, after this paper had been accepted for publication, I circulated a preprint. Then M. Derridj and D. Tartakoff proved analytic hypoellipticity for the operators constructed here (see [DT]). The work of Derridj and Tartakoff used "balanced" cutoff functions to estimate the size of derivatives starting with the C^∞ local hypoellipticity proved here; then Bove, Derridj, Tartakoff, and I (see [BDKT]) proved C^∞ local hypoellipticity using the balanced cutoff functions, starting from the estimates for functions with compact support proved here. Also at this time, in [PP2], Parenti and Parmeggiani, following their work in [PP1], gave a different proof of hypoellipticity of the operators discussed here and in [Ch2].

2. Proof of Theorem A

The proof of Theorem A proceeds in the same way as given above in the outline of Hörmander's theorem. It works only when one bracket is involved because (unlike the real case) $\bar{X}_j$ is not in the span of the $\{X_1, \ldots, X_m\}$. The constant $\varepsilon = \frac{1}{2}$ is the largest possible, since (as proved in [Ho]) this is already so when the X_i are real.

First note that $\|X_i^* u\|^2_{-\frac{1}{2}} \leq \|X_i u\|^2 + C\|u\|^2$, since

$$\begin{aligned}\|X_i^* u\|^2_{-\frac{1}{2}} &= (X_i^* u, \Lambda^{-1} X_i^* u) = (X_i^* u, P^0 u) \\ &= (u, X_i P^0 u) = -(u, P^0 X_i u) + O(\|u\|^2);\end{aligned}$$

hence,

$$\|X_i^* u\|^2_{-\frac{1}{2}} \leq C\big(\sum \|X_k u\|^2 + \|u\|^2\big),$$

where $P^0 = \Lambda^{-1}\bar{X}_i$ is a pseudodifferential operator of order zero. Then we have

$$\begin{aligned}\|X_i^* u\|^2 &= (u, X_i X_i^* u) = \|X_i u\|^2 + (u, [X_i, X_i^*]u) \\ &= \|X_i u\|^2 + (\Lambda^{\frac{1}{2}} u, \Lambda^{-\frac{1}{2}}[X_i, X_i^*]u) \\ &\leq \|X_i u\|^2 + C\|u\|^2_{\frac{1}{2}}.\end{aligned}$$

To estimate $\|u\|^2_{\frac{1}{2}}$ by $C(\sum \|X_k u\|^2 + \|u\|^2)$ we will estimate $\|Du\|^2_{-\frac{1}{2}}$ by $C(\sum \|X_k u\|^2 + \|u\|^2)$ for all first order operators D. Thus it suffices to estimate Du when $D = X_i$ and when $D = [X_i, X_j]$. The estimate is clearly satisfied if $D = X_i$, if $D = [X_i, X_j]$ we have

$$\begin{aligned}\|[X_i, X_j]u\|^2_{-\frac{1}{2}} &= (X_i X_j u, \Lambda^{-1}[X_i, X_j]u) - (X_j X_i u, \Lambda^{-1}[X_i, X_j]u) \\ &= (X_i X_j u, P^0 u) - (X_j X_i u, P^0 u);\end{aligned}$$

the first term on the right is estimated by

$$\begin{aligned}(X_i X_j u, P^0 u) &= (X_j u, X_i^* P^0 u) = -(X_j u, P^0 X_i^* u) + O(\|u\|^2 + \|X_j u\|^2) \\ &\leq C(\|X_j u\|\|X_i^* u\| + \|u\|^2 + \|X_j u\|^2) \\ &\leq \text{l.c.} \sum (\|X_k u\|^2 + \text{s.c.}\|X_i^* u\|^2 + C\|u\|^2\end{aligned}$$

and the second term on the right is estimated similarly. Combining these we have

$$\|u\|^2_{\frac{1}{2}} \leq C(\sum \|\frac{\partial u}{\partial x_i}\|^2_{-\frac{1}{2}} + \|u\|^2) \leq C(\sum \|X_k u\|^2 + \|u\|^2) + \text{s.c.}\|u\|^2_{\frac{1}{2}};$$

hence

$$\|u\|^2_{\frac{1}{2}} \leq C\big(\sum \|X_k u\|^2 + \|u\|^2\big)$$

which concludes the proof of theorem A.

3. The operators E_k

In this section we define the operators: $L, \bar{L}, X_{1k}, X_2$, and E_k.

Let $\mathfrak{H}$ be the hypersurface in $\mathbb{C}^2$ given by:

$$\Re(z_2) = -|z_1|^2.$$

We identify $\mathbb{R}^3$ with the Heisenberg group represented by $\mathfrak{H}$ using the mapping $\mathfrak{H} \to \mathbb{R}^3$ given by $x = \Re z_1$, $y = \Im z_1$, $t = \Im z_2$. Let $z = x + \sqrt{-1}\, y$. Let

$$L = \frac{\partial}{\partial z_1} - 2\bar{z}_1 \frac{\partial}{\partial z_2} = \frac{\partial}{\partial z} + \sqrt{-1}\bar{z}\frac{\partial}{\partial t}$$

and

$$\bar{L} = \frac{\partial}{\partial \bar{z}_1} - 2z_1 \frac{\partial}{\partial \bar{z}_2} = \frac{\partial}{\partial \bar{z}} - \sqrt{-1}\, z\frac{\partial}{\partial t}.$$

Let X_{1k} and X_2 be the restrictions to $\mathfrak{H}$ of the operators

$$X_{1k} = \bar{z}_1^k L = \bar{z}^k \frac{\partial}{\partial z} + \sqrt{-1}\, \bar{z}^{k+1}\frac{\partial}{\partial t}.$$

We set

$$X_2 = \bar{L} = \frac{\partial}{\partial \bar{z}} - \sqrt{-1}\, z\frac{\partial}{\partial t}$$

and

$$E_k = X_{1k}^* X_{1k} + X_2^* X_2 = -\bar{L}|z|^{2k}L - L\bar{L}.$$

By induction on j we define the commutators A_k^j setting $A_k^1 = [X_{1k}, X_2]$ and $A_k^j = [A_k^{j-1}, X_2]$. Note that X_2, A_k^k and A_k^{k+1} span the tangent space of $\mathbb{R}^3$.

4. Loss of derivatives (part I)

In this section we prove that the subelliptic estimate (1) does not hold when $k \geq 1$. We also prove a proposition which gives the loss of derivatives in the sup norms which is part of Theorem B. To complete the proof of Theorem B, by establishing loss in the Sobolev norms, we will use additional microlocal analysis of E_k, the proof of Theorem B is completed in Section 6.

Definition 4. If U is a neighborhood of the origin then $\varrho \in C_0^\infty(U)$ is real-valued and is defined as follows $\varrho(z,t) = \eta(z)\tau(t)$, where $\eta \in C_0^\infty(\{z \in \mathbb{C} \mid |z| < 2\})$ with $\eta(z) = 1$ when $|z| \leq 1$ and $\tau \in C_0^\infty(\{t \in \mathbb{R} \mid |t| < 2a\})$ with $\tau(t) = 1$ when $|t| \leq a$.

The following proposition shows that the subelliptic estimate (1) does not hold when $k > 0$.

Proposition 1. *If $k \geq 1$ and if there exist a neighborhood U of the origin and constants s and C such that*

$$\|u\|_s^2 \leq C(\|\bar{z}^k Lu\|^2 + \|\bar{L}u\|^2),$$

for all $u \in C_0^\infty(U)$, then $s \leq 0$.

Proof. Let λ_0 and a be sufficiently large so that the support of $\varrho(\lambda z, t)$ lies in U when $\lambda \geq \lambda_0$. We define g_λ by

$$g_\lambda(z,t) = \varrho(\lambda z, t)\exp(-\lambda^{\frac{5}{2}}(|z|^2 - it)).$$

Note that $L\eta(z) = \bar{L}\eta(z) = 0$ when $|z| \leq 1$, that $L(\tau) = i\bar{z}\tau'$, and that $\bar{L}(\tau) = -iz\tau'$. Setting $R^\lambda v(z,t) = v(\lambda z, t)$, we have:

$$\bar{z}^k L(g_\lambda) = (\lambda \bar{z}^k (R^\lambda L\eta)\tau + i\bar{z}(R^\lambda \eta)\tau' + \lambda^{\frac{5}{2}}\bar{z}R^\lambda \varrho)\exp(-\lambda^{\frac{5}{2}}(|z|^2 + it)) \tag{2}$$

and

$$\bar{L}(g_\lambda) = (\lambda(R^\lambda \bar{L}\eta)\tau - iz(R^\lambda \eta)\tau')\exp(-\lambda^{\frac{5}{2}}(|z|^2 + it)). \tag{3}$$

Note that the restriction of $|g_\lambda|$ to $\mathfrak{H}$ is

$$|g_\lambda(z,t)| = \varrho(\lambda z, t)\exp(-\lambda^{\frac{5}{2}}|z|^2).$$

Now we have, using the changes of variables: first $(z,t) \mapsto (\lambda^{-1}z, t)$ and then $z \mapsto \lambda^{-\frac{1}{4}}z$

$$\begin{aligned}
\|g_\lambda\|^2 &= \frac{C}{\lambda^2}\int_{\mathbb{R}^2} \eta(z)^2 \exp(-2\lambda^{\frac{1}{2}}|z|^2)dxdy \\
&\geq \frac{C}{\lambda^2}\int_{\mathbb{R}^2} \exp(-2\lambda^{\frac{1}{2}}|z|^2)dxdy - \frac{C}{\lambda^2}\int_{|z|\geq 1} \exp(-2\lambda^{\frac{1}{2}}|z|^2)dxdy \\
&\geq \frac{C}{\lambda^{\frac{5}{2}}} - \frac{C}{\lambda^2}\exp(-\lambda^{\frac{1}{2}})\int_{\mathbb{R}^2} \exp(-\lambda^{\frac{1}{2}}|z|^2)dxdy \\
&\geq \frac{C}{\lambda^{\frac{5}{2}}} - \frac{C}{\lambda^{\frac{5}{2}}}\exp(-\lambda^{\frac{1}{2}}).
\end{aligned}$$

Then we have

$$\|g_\lambda\|^2 \geq \frac{\text{const.}}{\lambda^{\frac{5}{2}}}$$

for sufficiently large λ. Further, using the above coordinate changes to estimate the individual terms in (2) and in (3), we have

$$\begin{aligned}
&\|\bar{z}^k \lambda(R^\lambda L\eta)\tau \exp(-\lambda^{\frac{5}{2}}(|z|^2 - it)\|^2 + \|\lambda(R^\lambda \bar{L}\eta)\tau\exp(-\lambda^{\frac{5}{2}}(|z|^2 - it)\|^2 \\
&\qquad \leq C\exp(-\lambda^{\frac{1}{2}})\int_{|z|\geq 1} \exp(-\lambda^{\frac{1}{2}}|z|^2)dxdy \leq \frac{C}{\lambda^{\frac{1}{2}}}\exp(-\lambda^{\frac{1}{2}}),
\end{aligned}$$

$$\||z|(R^\lambda\eta)\tau')\exp(-\lambda^{\frac{5}{2}}(|z|^2+it))\|^2 \\ \leq \frac{C}{\lambda^2}\int |z|^2\exp(-2\lambda^{\frac{1}{2}}|z|^2)dxdy \leq \frac{C}{\lambda^4},$$

and

$$\|\lambda^{\frac{5}{2}}\bar{z}^{k+1}R^\lambda\varrho)\exp(-\lambda^{\frac{5}{2}}(|z|^2+it))\|^2 \\ \leq C\lambda^{1-2k}\int |z|^{2k+2}\exp(-2\lambda^{\frac{1}{2}}|z|^2)dxdy \leq \frac{C}{\lambda^{\frac{5k}{2}}}.$$

Hence, if $k \geq 1$, we have

$$\|\bar{z}^k Lg_\lambda\|^2 + \|\bar{L}g_\lambda\|^2 \leq \frac{C}{\lambda^{\frac{5}{2}}}.$$

Since $|x| \leq \frac{2}{\lambda}$ on the support of g_λ then we conclude, from the lemma proved below, that given ε there there exists C such that

$$\lambda^\varepsilon\|g_\lambda\| \leq C\|g_\lambda\|_\varepsilon,$$

for sufficiently large λ. It the follows that, if $k \geq 1$ then the subelliptic estimate

$$\|g_\lambda\|_\varepsilon^2 \leq C(\|\bar{z}^k Lg_\lambda\|^2 + \|\bar{L}g_\lambda\|^2)$$

implies that $\lambda^{2\varepsilon-\frac{5}{2}} \leq C\lambda^{-\frac{5}{2}}$ which is a contradiction and thus the proposition follows. The following lemma then completes the proof. For completeness we include a proof which is along the lines given in [ChK].

Lemma 1. *Let Q_δ denote a bounded open set contained in the "slab" $\{x \in \mathbb{R}^n \mid |x_1| \leq \delta\}$. Then, for each $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that*

$$\|u\| \leq C\delta^\varepsilon\|u\|_\varepsilon, \tag{4}$$

for all $u \in C_0^\infty(Q_\delta)$ and $\delta > 0$.

Proof. Note that the general case follows from the case of $n = 1$. Since, writing $x = (x_1, x')$, if for each fixed x' we have $\|u(\cdot, x')\| \leq C\delta^\varepsilon\|u(\cdot, x')\|_\varepsilon$ then, after integrating with respect to x' and noting that $(1+\xi_1^2)^\varepsilon \leq (1+|\xi|^2)^\varepsilon$, we obtain the desired estimate. So we will assume that $n = 1$ and set $x = x_1$ and $\xi = \xi_1$. We define $|||u|||_s$ by

$$|||u|||_s^2 = \int |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi.$$

We will show that, if $s \geq 0$, there exists a constant C such that

$$\|u\|_s \leq C|||u|||_s,$$

for all $u \in C_0^\infty((-1,1))$. First we have

$$|\hat{u}(\xi)| = |\int e^{-ix\xi}u(x)dx| \leq \sqrt{2}\|u\|.$$

Next, if $|\xi| \leq a \leq 1$,

$$(1+\xi^2)^s|\hat{u}(\xi)|^2 \leq 2^{s+1}\|u\|^2$$

and

$$\int_{-\infty}^{\infty}(1+\xi^2)^s|\hat{u}(\xi)|^2 d\xi$$
$$= \int_{|\xi|\leq a}\cdots + \int_{|\xi|>a}\cdots \leq 2^{s+2}a\|u\|^2 + \left(\frac{1}{a^2}+1\right)^s |||u|||_s^2.$$

Hence if a is small we obtain $\|u\|_s \leq C|||u|||_s$, as required. If $\mathrm{supp}(u) \subset (-\delta,\delta)$ then set $u_\delta(x) = u(\delta x)$ so that $\mathrm{supp}(u_\delta) \subset (-1,1)$. Now

$$\hat{u_\delta}(\xi) = \frac{1}{\delta}\hat{u}\left(\frac{\xi}{\delta}\right)$$

so that $\|u_\delta\|^2 = \frac{1}{\delta}\|u\|^2$ and $|||u_\delta|||_s^2 = \delta^{2s-1}|||u|||_s^2$ which concludes the proof.

Next we prove that E_k loses at least k derivatives in the sup norms.

Proposition 2. *If for some open sets U and U', with $\bar{U} \subset U'$, and for each s_0 there exists a constant $C = C(s_0)$ such that*

$$\sum_{|\alpha|\leq p}\sup_{x\in U}|D^\alpha u(x)| \leq C\Big(\sum_{|\beta|\leq q}\sup_{x\in U'}|D^\beta E_k u(x)| + \|u\|_{-s_0}\Big), \tag{5}$$

for all $u \in C_0^\infty(\mathbb{R}^3)$, then $q \geq p+k$.

Proof. If $\delta > 0$ define u_δ by

$$u_\delta = (|z|^2 - \sqrt{-1}t)^p \log(|z|^2 + \delta - \sqrt{-1}t),$$

where log denotes the branch of the logarithm that takes reals into reals. Since u_δ is the restriction of $(-z_2)^p\log(-z_2+\delta)$ to $\mathfrak{H}$ we have $\bar{L}u_\delta = 0$. Then we have

$$\lim_{\delta\to 0}|D_t^p u_\delta(0)| = \infty.$$

Further

$$\begin{aligned} E_k u_\delta &= -\bar{L}|z|^{2k}Lu_\delta \\ &= 2k|z_1|^{2k}\Big(-p(-z_2)^{p-1}\log(-z_2+\delta) + (-z_2)^p\log(-z_2+\delta) + \frac{(-z_2)^p}{(-z_2+\delta)}\Big) \\ &= 2k|z|^{2k}\Big(p(|z|^2-\sqrt{-1}t)^{p-1}\log(|z|^2+\delta-\sqrt{-1}t) + \frac{(|z|^2-\sqrt{-1}t)^p}{|z|^2+\delta-\sqrt{-1}t}\Big). \end{aligned}$$

Note that $\|u_\delta\|_{-s_0}$ is bounded independently of δ when $s_0 \geq 3$. Thus, when $q \leq p+k-1$, we have

$$\sum_{|\beta|\leq q} \sup_{x\in U'} |D^\beta E_k u_\delta(x)| \leq \text{const.}$$

with the constant independent of δ. Hence, applying (5) to u_δ we obtain $q \geq p+k$. This concludes the proof of the proposition.

5. Notation

In this section we set down some notation which will be used throughout the rest of the paper.

1. Associated to the cutoff function ϱ defined in Definition 1, is a C^∞ function μ such that $L\varrho = \bar{z}\mu$ and $\bar{L}\varrho = z\bar{\mu}$ (Such a μ exists since

   $$L\varrho(z,t) = D_z\eta(z)\tau(t) + i\bar{z}\eta(z)D_t\tau(t).$$

   Since $D_z\eta(z) = 0$ in a neighborhood of $z = 0$ we can set $\mu(z,t) = \frac{D_z\eta(z)}{\bar{z}}\tau(t) + i\eta(z)D_t\tau(t)$.)

2. Given cutoff functions ϱ, ϱ', as in Definition 1, with $\varrho' = 1$ in a neighborhood of the support of ϱ, then we denote by $\{\varrho_i\}$ a special sequence of cutoff functions, each of which satisfies Definition 1 and such that: $\varrho_1 = \varrho$, $\varrho' = 1$ in a neighborhood of $\bigcup \varrho_i$, and $\varrho_{i+1} = 1$ in a neighborhood of the support of ϱ_i.

3. The abbreviations "s.c." and "l.c." will be used for "small constant" and "large constant", respectively in the following sense. $\mathcal{A}u \leq \text{s.c.}\mathcal{B}u + \text{l.c.}\mathcal{C}u$ means that given any constant s.c. there exists a constant l.c. such that the inequality holds for all u in some specified class.

4. We will use $\|u\|_{-\infty}$ to denote the following. Given $\mathcal{A}u$, the expression $\mathcal{A}u \leq \|u\|_{-\infty}$ means that: if for any s_o there exist a constant $C = C(s_o)$ such that $\mathcal{A}u \leq C\|u\|_{-s_o}$ holds for all u in some specified class.

6. Microlocalization on the Heisenberg group

Denote by T the vector field defined by

$$T = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}.$$

Then

$$[L,\bar{L}] = [\frac{\partial}{\partial z} + \sqrt{-1}\,\bar{z}\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{z}} - \sqrt{-1}\,z\frac{\partial}{\partial t}] = 2T.$$

The following simple formula, which is obtained by integration by parts, is the starting point of all the estimates connected with the operators E_k.

LEMMA 2. *For* $u \in C_0^\infty(\mathbb{R}^3)$ *we have*

$$\|Lu\|^2 = 2(Tu,u) + \|\bar{L}u\|^2. \tag{6}$$

Proof. Since $L^* = -\bar{L}$ and $\bar{L}^* = -L$, we have

$$\|Lu\|^2 = (Lu,Lu) = -(\bar{L}Lu,u) = -([\bar{L},L]u,u) - (L\bar{L}u,u) = 2(Tu,u) + \|\bar{L}u\|^2.$$

We set $x_1 = \Re z, x_2 = \Im z$, and $x_3 = t$ and denote the dual coordinates by ξ_1, ξ_2, and ξ_3. For $(\alpha, t_0) \in \mathbb{C} \times \mathbb{R}$ we define

$$z^\alpha = z - \alpha \quad \text{and} \quad x_3^\alpha = -2\alpha_2 x_1 + 2\alpha_1 x_2 + x_3 - t_0,$$

where $\alpha_1 = \Re\alpha$ and $\alpha_2 = \Im\alpha$. Then

$$L = \frac{\partial}{\partial z^\alpha} + i\bar{z}^\alpha \frac{\partial}{\partial x_3^\alpha}$$

and

$$\bar{L} = \frac{\partial}{\partial \bar{z}^\alpha} - i z^\alpha \frac{\partial}{\partial x_3^\alpha}.$$

We set $x_1^\alpha = x_1 - \alpha_1$, $x_2^\alpha = x_2 - \alpha_2$, and $x^\alpha = (x_1^\alpha, x_2^\alpha, x_3^\alpha)$. Let $\mathcal{F}_\alpha$ denote the the Fourier transform in the x_j^α coordinates; that is

$$\mathcal{F}_\alpha u(\xi) = \int e^{-ix^\alpha\cdot\xi} u(x^\alpha) dx_1^\alpha x_2^\alpha x_3^\alpha.$$

Definition 5. Let $S^2 = \{\xi \in \mathbb{R}^3 \mid |\xi| = 1\}$ be the unit sphere. Suppose that $\mathcal{U}, \mathcal{U}_1$ are open subsets of S^2 with $\bar{\mathcal{U}}_1 \subset \mathcal{U}$. For each such pair of open sets we define a set of $\gamma \in C^\infty(\mathbb{R}^3)$, with $\gamma \geq 0$, such that

1. $\gamma(\frac{\xi}{|\xi|}) = \gamma(\xi)$ when $|\xi| \geq 1$.
2. $\gamma(\xi) = 1$ when $\xi \in \mathcal{U}_1$.
3. $\gamma(\xi) = 0$ when $\xi \in S^2 - \mathcal{U}$.

To such a γ and $\alpha \in \mathbb{C}$ we associate the operator Γ_α defined by

$$\mathcal{F}_\alpha \Gamma_\alpha u(\xi) = \gamma(\xi)\mathcal{F}_\alpha u(\xi).$$

Let $\mathcal{U}^+, \mathcal{U}_1^+, \mathcal{U}^0, \mathcal{U}_1^0, \mathcal{U}^-$, and $\mathcal{U}_1^-$ be open subsets of S^2 defined as follows.

$$\mathcal{U}^+ = \Big\{\xi \in S^2 \mid \xi_3 > \frac{5}{9}\Big\}, \quad \mathcal{U}_1^+ = \Big\{\xi \in S^2 \mid \xi_3 > \frac{4}{9}\Big\},$$
$$\mathcal{U}^0 = \Big\{\xi \in S^2 \mid |\xi_3| < \frac{5}{6}\Big\}, \quad \mathcal{U}_1^0 = \Big\{\xi \in S^2 \mid |\xi_3| < \frac{2}{3}\Big\},$$
$$\mathcal{U}^- = \{\xi \in S^2 \mid -\xi \in \mathcal{U}^+\}, \text{ and } \mathcal{U}_1^- = \{\xi \in S^2 \mid -\xi \in \mathcal{U}_1^+\}.$$

We denote by γ^+, γ^0, and γ^- the corresponding functions and require further that $\gamma^+(\xi) = \gamma^-(\xi) = 0$ when $|\xi| \le \frac{1}{2}$ and $\gamma^0(\xi) = 1$ when $\frac{\xi}{|\xi|} \in \mathcal{U}_1^0$. The sets of these functions will be denoted by $\mathcal{G}^+$, $\mathcal{G}^0$, and $\mathcal{G}^-$, respectively. The corresponding operators are denoted by Γ_α^+, Γ_α^0, and Γ_α^-. The sets of these operators will be denoted by $\mathfrak{G}_\alpha^+$, $\mathfrak{G}_\alpha^0$, and $\mathfrak{G}_\alpha^-$, respectively. Given $(\alpha, t_0) \in \mathbb{C}\times\mathbb{R}$ the functions $\Gamma_\alpha^+ u$, $\Gamma_\alpha^0 u$, and $\Gamma_\alpha^- u$ will be referred to as microlocalizations of u at (α, t_0) in the regions $+$, 0, and $-$, respectively.

The following lemma shows that the 0 microlocalization is elliptic for the operators L and $\bar{L}$. In our estimates we will often encounter error terms which can be bounded by $C_{s_0}\|u\|_{-s_0}$ for every s_0; abusing notation we will bound such terms by "$\|u\|_{-\infty}$".

Lemma 3. *If U is a neighborhood of (α, t_0) and if $\gamma^0, \tilde{\gamma}^0 \in \mathcal{G}^0$ with $\tilde{\gamma}^0 = 1$ in a neighborhood of the support of γ^0 then there exist constants $a > 0$ and $C > 0$ such that, if $|z-\alpha| < a$ on U, then*

$$\|\Gamma_\alpha^0 u\|_1 \le C(\|\Gamma_\alpha^0 L u\| + \|\tilde{\Gamma}_\alpha^0 u\| + \|u\|_{-\infty})$$

and

$$\|\Gamma_\alpha^0 u\|_1 \le C(\|\Gamma_\alpha^0 \bar{L} u\| + \|\tilde{\Gamma}_\alpha^0 u\| + \|u\|_{-\infty}),$$

for all $u \in C_0^\infty(U)$.

Proof. If $\xi \in \mathcal{U}^0$ and if $|\xi| \ge 1$ then $|\xi_3| \le \frac{5}{6}|\xi|$. Thus, if $\xi \in \mathcal{U}^0$, then $|\xi| \le 6(|\xi_1| + |\xi_2|) + 1$. Now,

$$\|\Gamma_\alpha^0 u\|_1^2 \le C\Big(\sum_1^2 \|\frac{\partial}{\partial x_j^\alpha}\Gamma_\alpha^0 u\|^2 + \|\tilde{\Gamma}_\alpha^0 u\|^2 + \|u\|_{-\infty}^2\Big).$$

Let $U' \supset \bar{U}$ be an open set such that $|z-\alpha| > 2a$ on U' and let $\varphi \in C_0^\infty(U')$ satisfying $\varphi = 1$ in a neighborhood of $\bar{U}$. Then

$$\begin{aligned}
\|\Gamma_\alpha^0 u\|_1^2 &\le C\Big(\sum_1^2 \|\frac{\partial}{\partial x_j^\alpha}\Gamma_\alpha^0 \varphi u\|^2 + \|\tilde{\Gamma}_\alpha^0 u\|^2 + \|u\|_{-\infty}^2\Big) \\
&\le C'\Big(\sum_1^2 \|\frac{\partial}{\partial x_j^\alpha}\varphi\Gamma_\alpha^0 u\|^2 + \|\tilde{\Gamma}_\alpha^0 u\|^2 + \|u\|_{-\infty}^2\Big) \\
&\le C''(\|L\varphi\Gamma_\alpha^0 u\|^2 + \|\bar{L}\varphi\Gamma_\alpha^0 u\|^2 \\
&\qquad + \max_{U'}|z-\alpha|^2\|\frac{\partial}{\partial x_3^\alpha}\Gamma_\alpha^0 u\|^2 + \|\tilde{\Gamma}_\alpha^0 u\|^2 + \|u\|_{-\infty}^2) \\
&\le C''(\|\Gamma_\alpha^0 L u\|^2 + \|\Gamma_\alpha^0 \bar{L} u\|^2 + 4a^2\|\Gamma_\alpha^0 u\|_1^2 + \|\tilde{\Gamma}_\alpha^0 u\|^2 + \|u\|_{-\infty}^2).
\end{aligned}$$

Hence, taking a suitably small we obtain

$$\|\Gamma_\alpha^0 u\|_1^2 \le C(\|\Gamma_\alpha^0 L u\|^2 + \|\Gamma_\alpha^0 \bar{L} u\|^2 + \|\tilde{\Gamma}_\alpha^0 u\|^2 + \|u\|_{-\infty}^2).$$

Furthermore, substituting $\varphi\Gamma^0_\alpha u$ for u in (6), we have

$$\begin{aligned}\|L\varphi\Gamma^0_\alpha u\|^2 &= 2(T\varphi\Gamma^0_\alpha u, \varphi\Gamma^0_\alpha u) + \|\bar{L}\varphi\Gamma^0_\alpha u\|^2\\ &\le \text{s.c.}\|\frac{\partial}{\partial x_3^\alpha}\Gamma^0_\alpha u\|^2 + \text{l.c.}(\|\tilde{\Gamma}^0_\alpha u\|^2 + \|u\|^2_{-\infty}) + C\|\Gamma^0_\alpha \bar{L}u\|^2\\ &\le \text{s.c.}\|\Gamma^0_\alpha u\|^2_1 + \text{l.c.}(\|\tilde{\Gamma}^0_\alpha u\|^2 + \|u\|^2_{-\infty}) + C\|\Gamma^0_\alpha \bar{L}u\|^2,\end{aligned}$$

and since

$$\|L\varphi\Gamma^0_\alpha u\|^2 \le C(\|\Gamma^0_\alpha Lu\|^2 + \|\tilde{\Gamma}^0_\alpha u\|^2 + \|u\|^2_{-\infty})$$

we get

$$\|\Gamma^0_\alpha u\|_1 \le C(\|\Gamma^0_\alpha \bar{L}u\| + \|\tilde{\Gamma}^0_\alpha u\|^2 + \|u\|^2_{-\infty}).$$

Similarly we obtain

$$\|\Gamma^0_\alpha u\|_1 \le C(\|\Gamma^0_\alpha Lu\| + \|\tilde{\Gamma}^0_\alpha u\|^2 + \|u\|^2_{-\infty}).$$

This completes the proof of the lemma.

LEMMA 4. *If R^s is a pseudodifferential operator of order s then there exists C such that*

$$\|[R^s, \Gamma^+_\alpha]u\| \le C(\|\Gamma^0_\alpha u\|_{s-1} + \|u\|_{-\infty})$$

and

$$|[R^s, \Gamma^-_\alpha]u\| \le C(\|\Gamma^0_\alpha u\|_{s-1} + \|u\|_{-\infty}).$$

Proof. Since $\gamma^0 = 1$ on a neighborhood of the support of the derivatives of γ^+ it also equals one on a neighborhood of the support of the symbol of $[R^s, \Gamma^+_\alpha]$. Hence $[R^s, \Gamma^+_\alpha] = [R^s, \Gamma^+_\alpha]\Gamma^0_\alpha + R^{-\infty}$, where $R^{-\infty}$ is a pseudodifferential operator whose symbol is identically zero. The same argument works for the term $[R^s, \Gamma^-_\alpha]$ and the lemma follows.

Definition 6. For each $s \in \mathbb{R}$ we define the operator Ψ^s_α as follows. Let $\mathcal{U}^*$ and $\mathcal{U}^*_1$ be open sets in S^2 such that $\mathcal{U}^* = \{\xi \in S^2 \mid |\xi_3| > \frac{1}{6}$ and $\mathcal{U}^*_1 = \{\xi \in S^2 \mid |\xi_3| > \frac{1}{3}$. Let γ^* be the function on R^3 associated with $\mathcal{U}^*$, $\mathcal{U}^*$ such that $\gamma^*(\xi) = 0$ when $|\xi| \le \frac{1}{3}$ and $\gamma^*(\xi) = 1$ in the region $\{\xi \in \mathbb{R}^3 \mid \frac{\xi}{|\xi|} \in \mathcal{U}^*_1$ and $|\xi| \ge \frac{1}{2}\}$. Then we set $\psi^s(\xi) = (1 + |\xi_3|^2)^{\frac{s}{2}}\gamma^*(\xi)$ and define Ψ^s_α by

$$\mathcal{F}_\alpha \Psi^s_\alpha u(\xi) = \psi^s(\xi)\mathcal{F}_\alpha u(\xi).$$

Note that there exist positive constants c and C such that

$$c(1 + |\xi|^2)^{\frac{s}{2}}\gamma^*(\xi) \le \psi^s(\xi) \le C(1 + |\xi|^2)^{\frac{s}{2}}\gamma^*(\xi).$$

Hence $\|\Psi^s_\alpha \Gamma^+_\alpha u\| \sim \|\Gamma^+_\alpha u\|_s$ and $\|\Psi^s_\alpha \Gamma^-_\alpha u\| \sim \|\Gamma^-_\alpha u\|_s$; by $\sim$ we mean that they differ by an operator of order $-\infty$. Also, since $\gamma^* = 1$ on the supports of γ^+ and γ^-, we have

$$\Psi^s \Psi^{s'} \Gamma^+_\alpha \sim \Psi^{s+s'}\Gamma^+_\alpha \text{ and } \Psi^s \Psi^{s'} \Gamma^-_\alpha \sim \Psi^{s+s'}\Gamma^-_\alpha.$$

Lemma 5. *There exists C such that*

$$\|\Gamma^+_\alpha \bar{L}u\|^2 + \|\Gamma^+_\alpha u\|^2_{\frac{1}{2}} \leq C(\|\Gamma^+_\alpha Lu\|^2 + \tilde{\Gamma}^+_\alpha u\|^2 + \|u\|^2_{-\infty}),$$

and

$$\|\Gamma^-_\alpha Lu\|^2 + \|\Gamma^-_\alpha u\|^2_{\frac{1}{2}} \leq C(\|\Gamma^-_\alpha \bar{L}u\|^2 + \tilde{\Gamma}^-_\alpha u\|^2 + \|u\|^2_{-\infty}),$$

for all $u \in C^\infty_0(U)$.

Proof. Taking $\varphi \in C^\infty_0$ with $\varphi = 1$ in a neighborhood of $\bar{U}$ we substitute $\varphi\Gamma^+_\alpha u$ for u in (6) and obtain

$$\|L\varphi\Gamma^+_\alpha u\|^2 = 2(T\varphi\Gamma^+_\alpha u, \varphi\Gamma^+_\alpha u) + \|\bar{L}\varphi\Gamma^+_\alpha u\|^2.$$

Now, we have

$$(T\varphi\Gamma^+_\alpha u, \varphi\Gamma^+_\alpha u) = (T\Gamma^+_\alpha \varphi u, \varphi\Gamma^+_\alpha u) + O(\tilde{\Gamma}^+_\alpha u\|^2 + \|u\|^2_{-\infty}).$$

Since $\mathcal{F}_\alpha(Tu) = \xi_3\mathcal{F}_\alpha(u)$ we have $T\Gamma^+_\alpha \sim \Psi^1\Gamma^+_\alpha \sim \Psi^{\frac{1}{2}}\Psi^{\frac{1}{2}}\Gamma^+_\alpha$ and

$$(T\varphi\Gamma^+_\alpha u, \varphi\Gamma^+_\alpha u) = \|\Psi^{\frac{1}{2}}_\alpha\Gamma^+_\alpha u\|^2 + O(\tilde{\Gamma}^+_\alpha u\|^2 + \|u\|^2_{-\infty}).$$

This proves the first part of the lemma, the second follows from the fact that $|\xi_3|\gamma^-(\xi) = -\xi_3\gamma^+(-\xi)$. Then $\Psi^1\Gamma^-_\alpha \sim \Psi^{\frac{1}{2}}\Psi^{\frac{1}{2}}\Gamma^-_\alpha$, thus concluding the proof.

7. Loss of derivatives (part II) Conclusion of the proof of Theorem B

In this section we conclude the proof of Theorem B by showing that if $k \geq 2$ then E_k loses at least $k-1$ derivatives in the Sobolev norms.

Proposition 3. *Suppose that there exist two neighborhoods of the origin U and U', with $\bar{U} \subset U'$, and real numbers s_1 and s_2 such that if $\varrho, \varrho' \in C^\infty_0(U')$ with $\varrho = 1$ on U and $\varrho' = 1$ in a neighborhood of the support of ϱ, and if for any real number s_0 there exists a constant $C = C(\varrho, \varrho', s_0)$ such that*

$$\|\varrho u\|_{s_1} \leq C(\|\varrho' E_k u\|_{s_2} + \|u\|_{-s_0}), \tag{7}$$

for all $u \in \mathcal{S}$, then $s_2 \geq s_1 + k - 1$. Here $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions.

Proof. Let $\{\varrho_i\}$ and $\{\varrho'_i\}$ be sequences of cutoff functions in $C^\infty_0(U)$ and $C^\infty_0(U')$, respectively. We assume that $\varrho_(z,t) = \eta_i(|z|)\tau_i(t)$ and $\varrho'_i(z,t) = \eta'_i(|z|)\tau'_i(t)$ as in Definition 1. We further assume that $\varrho_0 = \varrho$, $\varrho'_0 = \varrho'$, $\varrho_{i+1} = 1$ in a neighborhood of the support of ϱ_i, and $\varrho'_{i+1} = 1$ in a neighborhood of the support of ϱ'_i and that the $\eta_i(|z|)$ are monotone decreasing in $|z|$. We also choose $\{\gamma^+_i\}$ and $\{\gamma^0_i\}$ such that $\gamma^+_i \in \mathcal{G}^+$, $\gamma^+_{i+1} = 1$, and $\gamma^0_i \in \mathcal{G}^0$ and $\gamma^0_{i+1} = 1$ in neighborhoods of the supports of γ^+_i and γ^0_i, respectively. Further we require that $\gamma_i^0 = 1$ in a neighborhood of the support of derivatives of γ_i^+. Substituting $\Psi^{-s_1}\Gamma_0^+ u$ for u in (7), replacing $s_0 + s_1$ by s_0, we have

$$\|\varrho\Psi^{-s_1}\Gamma_0^+ u\|_{s_1} \le C(\|\varrho' E_k \Psi^{-s_1}\Gamma_0^+ u\|_{s_2} + \|u\|_{-s_0}).$$

Since $\gamma_1^+ \varrho \gamma_0^+ = \varrho\gamma_0^+$,

$$\begin{aligned}\|\varrho\Psi^{-s_1}\Gamma_0^+ u\|_{s_1} &= \|\Psi^{s_1}\Gamma_1^+ \varrho \Psi^{-s_1}\Gamma_0^+ u\| + O(\|u\|_{-1}) \\ &= \|\varrho\Gamma_0^+ u\| + \|\Psi^{s_1}\Gamma_1^+[\varrho, \Psi^{s_1}]\Gamma_0^+ u\| + O(\|u\|_{-s_0}).\end{aligned}$$

Furthermore, $\Psi^{s_1}\Gamma_1^+[\varrho, \Psi^{s_1}]\Gamma_0^+$ is an operator of order -1; hence we get

$$\|\Psi^{s_1}\Gamma_1^+[\varrho, \Gamma_0^+]\Psi^{-s_1} u\| \le C(\|u\|_{-s_0})$$

and

$$\|\varrho\Gamma_0^+ u\| \le C(\|\varrho' E_k \Psi^{-s_1}\Gamma_0^+ u\|_{s_2} + \|u\|_{-s_0}).$$

Next we have

$$\|\varrho' E_k \Psi^{-s_1}\Gamma_0^+ u\|_{s_2} \le \|\Psi^{s_2 - s_1}\Gamma_0^+ \varrho' E_k u\| + \|[\varrho' E_k, \Psi^{-s_1}\Gamma_0^+]u\|_{s_2}.$$

Since the symbol of $\gamma_1^0\gamma_1^+\varrho_1' = 1$ in a neighborhood of the symbol of $[\varrho' E_k, \Psi^{-s_1}\Gamma_0^+]$ and since the order of $[\varrho' E_k, \Psi^{-s_1}\Gamma_0^+]$ is $-s_1 + 1$, we have

$$\|[\varrho' E_k, \Psi^{-s_1}\Gamma_0^+]u\|_{s_2} \le C(\|\varrho_1'\Gamma_1^0\Gamma_1^+ u\|_{s_2 - s_1 + 1} + \|u\|_{-s_0}).$$

Applying Proposition 3, we have

$$\|\varrho_1'\Gamma_1^0\Gamma_1^+ u\|_{s_2 - s_1 + 1} \le C(\|\varrho_1' E_k \Gamma_1^+ u\|_{s_2 - s_1 - 1} + \|\varrho_2'\Gamma_2^0\Gamma_1^+ u\|_{s_2 - s_1} + \|u\|_{-\infty})$$

so that

$$\begin{aligned}&\|\varrho' E_k \Psi^{-s_1}\Gamma_0^+ u\|_{s_2} \\ &\quad \le C(\|\Psi^{s_2 - s_1}\Gamma_0^+ \varrho' E_k u\| + \|\varrho_1'\Gamma_2^0\Gamma_1^+ u\|_{s_2 - s_1} + \|\varrho_1' E_k \Gamma_1^+ u\|_{s_2 - s_1 - 1} + \|u\|_{-s_0}).\end{aligned}$$

Therefore

$$\begin{aligned}\|\varrho\Gamma_0^+ u\| \le C(&\|\Psi^{s_2 - s_1}\Gamma_0^+ \varrho' E_k u\| \\ &+ \|\varrho_1' E_k \Gamma_1^+ u\|_{s_2 - s_1 - 1} + \|\varrho_1' E_k \Gamma_1^+ u\|_{s_2 - s_1 - 1} + \|u\|_{-s_0}).\end{aligned}$$

Now we have

$$\|\varrho_1' E_k \Gamma_1^+ u\|_{s_2 - s_1 - 1} \le \|\Psi^{s_2 - s_1 - 1}\Gamma_1^+ \varrho_1' E_k u\| + \|[\varrho_1' E_k, \Gamma_1^+]u\|_{s_2 - s_1 - 1},$$

again since $[\varrho_1' E_k, \Gamma_1^+]$ is an operator of order one and since $\varrho_2' 2\gamma_2^0\gamma_2^+ = 1$ in a neighborhood of its symbol, we get

$$\|[\varrho_2' E_k, \Gamma_1^+]u\|_{s_2 - s_1 - 1} \le C(\|\varrho_2'\Gamma_2^0\Gamma_2^+ u\|_{s_2 - s_1} + \|u\|_{-s_0}).$$

Then, again applying Proposition 3, we have

$$\|\varrho_2'\Gamma_2^0\Gamma_2^+ u\|_{s_2 - s_1} \le C(\|\varrho_3'\Gamma_3^0 E_k \Gamma_2^+ u\|_{s_2 - s_1 - 2} + \|u\|_{-\infty})$$

so that

$$\|\varrho_2' E_k \Gamma_1^+ u\|_{s_2-s_1-1} \leq C(\|\Psi^{s_2-s_1-1}\Gamma_1^+ \varrho_2' E_k u\| + \|\varrho_3' \Gamma_3^0 E_k \Gamma_2^+ u\|_{s_2-s_1-2} + \|u\|_{-s_0}).$$

Hence

$$\|\varrho \Gamma^+ u\| \leq C(\|\Psi^{s_2-s_1}\Gamma_0^+ \varrho' E_k u\| + \|\Psi^{s_2-s_1-1}\Gamma_1^+ \varrho_2' E_k u\| + \|\varrho_4' \Gamma_3^0 E_k \Gamma_2^+ u\|_{s_2-s_1-2} + \|u\|_{-s_0}).$$

Proceeding inductively we obtain

$$\|\varrho \Gamma^+ u\| \leq C\Big(\sum_{i=0}^{N} \|\Psi^{s_2-s_1-i}\Gamma_i^+ \varrho_i' E_k u\| + \|\varrho_{N+3}' \Gamma_{N+2}^0 E_k \Gamma_{N+1}^+ u\|_{s_2-s_1-N-1} + \|u\|_{-s_0}\Big).$$

Since $\|[\Psi^{s_2-s_1-i}\Gamma_i^+, \eta_i']E_k u\|$ can be incorporated in the successive terms, we get, by choosing $N \geq s_2 - s_1 + 1 - s_0$

$$\|\varrho \Gamma^+ u\| \leq C\Big(\sum_{i=0}^{N} \|\Psi^{s_2-s_1-i}\Gamma_i^+ \tau_i' E_k u\| + \|u\|_{-s_0}\Big).$$

Let $\tilde{\tau} \in C_0^\infty$ with $\tilde{\tau} = 1$ on the support of τ_N'; then $\tau_i' E_k u = \tau_i' E_k \tilde{\tau} u$ when $i \leq N$ so that replacing u by $\tilde{\tau} u$ we obtain

$$\|\varrho \Gamma^+ \tilde{\tau} u\| \leq C\Big(\sum_{i=0}^{N} \|\Psi^{s_2-s_1-i}\Gamma_i^+ \tau_i' E_k u\| + \|\tilde{\tau} u\|_{-s_o}\Big).$$

Hence, since $\gamma^0 = 1$ in a neighborhood of the support of the symbol of $[\Gamma^+, \tilde{\tau}]$ and thus can be incorporated in the estimate as above, we have

$$\|\varrho \Gamma^+ u\|^2 \leq C\Big(\sum_{i=0}^{N} \|\Psi^{s_2-s_1-i}\Gamma_i^+ \tau_i' E_k u\|^2 + \|\tilde{\tau} u\|_{-s_o}^2\Big).$$

Choosing $\tilde{\gamma}^+$ so that $\tilde{\gamma}^+ = 1$ in a neighborhood of the supports of the γ_i^+, we have $\tilde{\tau}\tilde{\gamma}^+ = 1$ in a neighborhood of the support of the symbol of $\Psi^{s_2-s_1-i}\Gamma_i^+ \tau_i' E_k$. Then we obtain

$$\begin{aligned}\|\varrho \Gamma^+ u\|^2 &\leq C\Big(\sum_{i=0}^{N} \|\Psi^{s_2-s_1-i}\tilde{\Gamma}^+ \tilde{\tau} E_k u\|^2 + \|\tilde{\tau} u\|_{-s_o}^2\Big)\\ &\leq C(\|\Psi^{s_2-s_1}\tilde{\Gamma}^+ \tilde{\tau} E_k u\|^2 + \|\tilde{\tau} u\|_{-s_o}^2).\end{aligned}$$

We define h_λ by

$$h_\lambda(z,t) = \exp\left(-\lambda^2(|z|^2 - it)\right),$$

since $\tilde{\tau} h_\lambda \in \mathcal{S}$ and obtain

$$\|\varrho \Gamma^+ h_\lambda\| \leq C\left(\|\Psi^{s_2-s_1}\tilde{\Gamma}_i^+ \tilde{\tau} E_k h_\lambda\| + \|\tilde{\tau} h_\lambda\|_{-s_o})\right).$$

Assuming that $\eta(|z|)$ is monotone decreasing we have $\eta(|z|) \geq \eta(\lambda|z|)$; hence, setting $\eta_\lambda(z) = \eta(\lambda|z|)$, we obtain

$$\|\varrho\Gamma^+ h_\lambda\| \geq \|\eta_\lambda \tau \Gamma^+ h_\lambda\|.$$

Then, setting $x' = (x_1, x_2)$, $y' = (y_1, y_2)$, and $\xi' = (\xi_1, \xi_2)$ and changing variables $\lambda y' \to y'$, $\xi' \to \lambda\xi'$, and $\xi_3' \to \xi_3' + \lambda^2$, we get

$$\begin{aligned}
\eta_\lambda \tau \Gamma^+ h_\lambda(x) &= \int \exp(i(x-y)\cdot\xi)\tau(y_3)\gamma^+(\xi)\exp(-\lambda^2(|y'|^2 - iy_3))dyd\xi \\
&= \int \exp(i(x'-y')\cdot\xi' + x_3\xi_3 - y_3(\xi_3 - \lambda^2))\tau(y_3)\gamma^+(\xi)\exp(-\lambda^2|y'|^2)dyd\xi \\
&= \lambda^{-2}\int \exp(i(\lambda x' - y')\cdot\xi') \\
&\quad + (x_3 - y_3)\xi_3)\tau(y_3)\gamma^+(\lambda\xi', \lambda^2 + \xi_3)\exp(-|y'|^2)dyd\xi.
\end{aligned}$$

Making the change of variables $\lambda x' \to x'$ we have

$$\begin{aligned}
&\|\eta_\lambda \tau \Gamma^+ h_{\lambda,\delta}\|^2 \\
&\quad = \frac{1}{\lambda^6}\int |\int \exp(i(x-y)\cdot\xi)\tau(y_3)\gamma^+(\lambda\xi', \lambda^2 + \xi_3)\exp(-|y'|^2)dyd\xi|^2 dx.
\end{aligned}$$

Given $(\xi', \xi_3) \in \operatorname{supp}(\gamma^+)$ we have

$$\lim_{\lambda\to\infty} |\frac{\lambda\xi'}{\xi_3 + \lambda^2}| = 0,$$

and there exists $\tilde\lambda$ such that $\gamma^+(\lambda\xi', \lambda^2 + \xi_3) = 1$ when $\lambda \geq \tilde\lambda$. Hence we have $\lim_{\lambda\to\infty} \gamma^+(\lambda\xi', \lambda^2 + \xi_3) = 1$; thus there exist λ_0 such that

$$\|\eta_\lambda \tau \Gamma^+ h_\lambda\|^2 \geq \frac{1}{2\lambda^6}\int |\int \exp(i(x-y)\cdot\xi)\tau(y_3)\exp(-|y'|^2)dyd\xi|^2 dx,$$

when $\lambda \geq \lambda_0$, therefore there exists C independent of λ such that

$$\|\varrho\Gamma^+ h_\lambda\| \geq \frac{C}{\lambda^3},$$

when $\lambda \geq \lambda_0$.

Next, we will estimate the term $\|\tilde\tau h_\lambda\|_{-s_o}$. We will use the facts that $\frac{1}{m!}\bar{L}^m(\bar{z}^m)$
$= 1$ and that $\bar{L}(h_\lambda) = 0$. Taking $m \leq s_o$, we have

$$\begin{aligned}
\mathcal{F}(\Lambda^{-s_o}\tilde\tau h_\lambda)(\xi) &= \int (1 + |\xi|^2)^{\frac{-s_o}{2}}\tilde\tau(x_3)\exp(-i(x\cdot\xi - \lambda^2 x_3) - \lambda^2|z|^2)dx \\
&= \frac{1}{m!}\int \bar{L}^m(\bar{z}^m)(1 + |\xi|^2)^{\frac{-s_o}{2}}\tilde\tau)\exp(-i(x\cdot\xi - \lambda^2 x_3) - \lambda^2|z|^2)dx \\
&= -\frac{1}{m!}\int \bar{z}^m(1 + |\xi|^2)^{\frac{-s_o}{2}}\bar{L}^m(\tilde\tau(x_3)\exp(-i(x\cdot\xi - \lambda^2 x_3) - \lambda^2|z|^2)dx
\end{aligned}$$

$$= -\frac{1}{m!}\int \bar{z}^m(1+|\xi|^2)^{\frac{-s_o}{2}}\exp(-\lambda^2(|z|^2 - ix_3))\bar{L}^m(\tilde{\tau}(x_3)\exp(-ix\cdot\xi))dx$$
$$= -\frac{1}{m!}\int \bar{z}^m(1+|\xi|^2)^{\frac{-s_o}{2}}\exp(-\lambda^2(|z|^2))\bar{L}^m(\tilde{\tau}(x_3)$$
$$\cdot \exp(-ix'\cdot\xi' - ix_3(\xi_3-\lambda^2))dx$$

and

$$\bar{L}^m\left(\tilde{\tau}(x_3)\exp(-ix'\cdot\xi' + i - ix_3(\xi_3-\lambda^2))\right)$$
$$= \sum_{j=0}^{m} a_j(x_3)z^j(i\xi_1+\xi_2-2z\xi_3)^{m-j}\exp(-ix'\cdot\xi' - ix_3(\xi_3-\lambda^2)).$$

Thus, setting $w^{(m)}(x,\xi) = \sum_{j=0}^m a_j(x_3)z^j(i\xi_1+\xi_2-2z\xi_3)^{m-j}$ and denoting the corresponding pseudodifferential operator by $W^{(m)}$, we have

$$\|\tilde{\tau}h_\lambda\|_{-s_o} = C\|W^{(m)}\bar{z}^m h_\lambda\|_{-s_o} \le C\|z^m\tilde{\tau}'h_\lambda\|_{m-s_o} \le C\|z^m\tilde{\tau}'h_\lambda\|,$$

where $\tilde{\tau}' \in C_0^\infty(\mathbb{R})$ and $\tilde{\tau}' = 1$ in a neighborhood of the support of $\tilde{\tau}$. Now, changing coordinates $\lambda z \to z$, we get

$$\|z^m\tilde{\tau}'h_\lambda\|^2 = \int |z|^{2m}\tilde{\tau}'(x_3)^2\exp(-2\lambda^2|z|^2)dx \le \frac{C}{\lambda^{2m+2}}.$$

To estimate the remaining terms we have

$$E_k h_\lambda(z,t) = -2(k+1)\lambda^2|z|^{2k}h_\lambda(z,t).$$

Therefore, with the coordinate change $\lambda x' \to x'$, we get

$$\mathcal{F}\left(\Psi^s\Gamma^+\tau E_k h_\lambda\right)(\xi)$$
$$= C\mathcal{F}\left(\Psi^s\Gamma^+\tau\lambda^2|z|^{2k}h_\lambda\right)(\xi)$$
$$= C\lambda^{-2}(1+\xi_3^2)^{\frac{s}{2}}\gamma^+(\xi)\mathcal{F}\left(\tau(x_3)|z|^{2k}\exp(-\lambda^2|z|^2)\right)(\xi',\xi_3-\lambda^2)$$
$$= C\lambda^{-2k-2}(1+\xi_3^2)^{\frac{s}{2}}\gamma^+(\xi)\hat{\tau}(\xi_3-\lambda^2)\mathcal{F}\left(|z|^{2k}\exp(-2|z|^2)\right)(\lambda^{-1}\xi').$$

Then, integrating and making the changes of coordinates $\xi' \to \lambda\xi'$, $\xi_3 \to \xi_3+\lambda^2$, we get

$$\|\Psi^s\Gamma^+\tau E_k h_\lambda\|^2$$
$$\le C\lambda^{-4k-4}\int(1+\xi_3^2)^s\gamma^+(\xi)\hat{\tau}(\xi_3-\lambda^2)|^2|\mathcal{F}\left(|z|^{2k}\exp(-|z|^2)\right)(\lambda^{-1}\xi')|^2d\xi$$
$$\le C\lambda^{-4k-2}$$
$$\cdot\int(1+(\xi_3+\lambda^2)^2)^s\gamma^+(\lambda\xi',\xi_3+\lambda^2)\hat{\tau}(\xi_3)|^2|\mathcal{F}\left(|z|^{2k}\exp(-|z|^2)\right)(\xi')|^2d\xi.$$

Then if $s \ge 0$ and if λ is sufficiently large we have

$$\|\Psi^s\Gamma^+\tau E_k h_\lambda\|^2 \le C\lambda^{4s-4k-2}.$$

We assume $k \geq 1$; if $s_2 - s_1 < 0$ then

$$\|\Psi^{s_1-s_2}\varrho\Gamma^+ h_\lambda\|^2 \leq C\left(\|\tilde{\Gamma}_i^+\tilde{\tau}E_k h_\lambda\|^2 + \|\tilde{\tau}h_\lambda\|^2_{-s_o}\right) \leq C\lambda^{-4k-2}$$

and, by Lemma 1,

$$\begin{aligned}\|\Psi^{s_1-s_2}\varrho\Gamma^+ h_\lambda\|^2 &= \|\Psi^{s_1-s_2}\eta_\lambda\tau\Gamma^+ h_\lambda\|^2 + O(\|\tau\Gamma^+ h_\lambda\ |^2_{-s_o})\\ &\geq C\lambda^{2s_2-2s_1}\|\eta_\lambda\tau\Gamma^+ h_\lambda\|^2 - C'(\|\tau\Gamma^+ h_\lambda\|^2_{-s_o})\\ &\geq C(\lambda^{2s_2-2s_1-2} + \lambda^{-2m-2}).\end{aligned}$$

This implies that for large λ we have $\lambda^{2s_2-2s_1-2} \leq C(\lambda^{-4k-2} + \lambda^{-2m-2})$, which is a contradiction, so that $s_2 - s_1 \geq 0$ and

$$\begin{aligned}C_1\lambda^{-6} \leq C_2\|\varrho\Gamma^+ h_\lambda\|^2 &\leq C\left(\|\Psi^{s_2-s_1}\tilde{\Gamma}_i^+\tilde{\tau}E_k h_\lambda\|^2 + \|\tilde{\tau}h_\lambda\|^2_{-s_o}\right)\\ &\leq C_3(\lambda^{4s_2-4s_1-4k-2} + \lambda^{-2m-2}).\end{aligned}$$

Therefore, if m large we get $C_1 \leq 2C_3\lambda^{4(s_2-s_1-k+1)}$ for large λ. Hence $s_2 - s_1 - k + 1 \geq 0$, which concludes the proof of the proposition and also of Theorem B.

8. Elliptic and subelliptic microlocalizations

In this section we will show that the *a priori* estimates for the operator E_k gain two derivatives in the 0 microlocalization and gains one derivative in the $-$ microlocalization, these gains are in the Sobolev norms. Without loss of generality we will deal only with microlocalizations near the origin, taking $\alpha = 0$ and setting $\mathfrak{G}^0 = \mathfrak{G}^0_0$ and $\mathfrak{G}^- = \mathfrak{G}^-_0$. The subscript α will be dropped from the corresponding operators.

PROPOSITION 4. *Let U and U' be neighborhoods of the origin with $\bar{U} \subset U'$ and $|z| \leq a$ on U', where a is sufficiently small as in Lemma 3. Suppose that $\varrho \in C_0^\infty(U)$ and $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ on a neighborhood of $\bar{U}$. Further suppose that $\gamma^0, \tilde{\gamma}^0 \in \mathcal{G}^0$ with $\tilde{\gamma}^0 = 1$ on a neighborhood of the support of γ^0. Then, given $s, s_0 \in \mathbb{R}$, there exists $C = C(\varrho, \varrho', \gamma^0, \tilde{\gamma}^0, s, s_0)$ such that*

$$\|\varrho\Gamma^0 u\|^2_{s+2} + \|\varrho\Gamma^0\bar{z}^k L u\|^2_{s+1} + \|\varrho\Gamma^0\bar{L}u\|^2_{s+1} \leq C(\|\varrho'\tilde{\Gamma}^0 E_k u\|^2_s + \|u\|^2_{-s_0}),$$

for all $u \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz class of rapidly decreasing functions.

Proof. Let $\{\varrho_i\}$ be a sequence of functions such that $\varrho_i \in C_0^\infty(U)$, $\varrho_0 = \varrho$, $\varrho_{i+1} = 1$ in a neighborhood of the support of ϱ_i, and such that $\varrho' = 1$ in a neighborhood of the supports of all the ϱ_i. Let $\{\gamma_i^0\}$ be a sequence in $\mathcal{G}^0$ such that $\gamma_0^0 = \gamma^0$, $\gamma_{i+1}^0 = 1$ in a neighborhood of the support of γ_i, and $\tilde{\gamma}^0 = 1$ in a neighborhood of the supports of all the γ_i^0. Then substituting $\varrho\Lambda^{s+1}\Gamma_1^0 u$ for u in Lemma 3 we have

$$\|\Gamma^0\varrho\Lambda^{s+1}\Gamma_1^0 u\|_1^2 \leq C(\|\Gamma^0\bar{L}\varrho\Lambda^{s+1}\Gamma_1^0 u\|^2 + \|\varrho\Lambda^{s+1}\Gamma_1^0 u\|^2).$$

Hence

$$\|\Gamma^0\varrho\Lambda^{s+1}\Gamma^0_1 u\|_1^2 \le C(\|\Gamma^0\bar z^k L\varrho\Lambda^{s+1}\Gamma^0_1 u\|^2 + \|\Gamma^0\bar L\varrho\Lambda^{s+1}\Gamma^0_1 u\|^2 + \|\varrho\Lambda^{s+1}\Gamma^0_1 u\|^2).$$

Then

$$\|\Gamma^0\varrho\Lambda^{s+1}\Gamma^0_1 u\|_1^2 = \|\varrho\Gamma^0 u\|_{s+2}^2 + O(\|\Lambda^1[\Gamma^0\varrho,\Lambda^{s+1}]\Gamma^0_1 u\|^2 + \|\Lambda^{s+2}[\Gamma^0,\varrho]\Gamma^0_1 u\|^2 + \|\Lambda^{s+2}\varrho(\Gamma^0\Gamma^0_1 - \Gamma^0)u\|^2).$$

Since $[\Gamma^0\varrho,\Lambda^{s+1}]$ is a pseudodifferential operator of order $s+1$ and since $\varrho_1 = 1$ on the support of its symbol, we have

$$\|\Lambda^1[\Gamma^0\varrho,\Lambda^{s+1}]\Gamma^0_1 u\|^2 \le C(\|\varrho_1\Gamma^0_1 u\|_{s+1}^2 + \|\Gamma^0_1 u\|_{-\infty}^2).$$

The operator $[\Gamma^0,\varrho]$ is of order -1 and $\varrho_1 = 1$ on the support of its symbol, so that

$$\|\Lambda^{s+2}[\Gamma^0,\varrho]\Gamma^0_1 u\|^2 \le C(\|\varrho_1\Gamma^0_1 u\|_{s+1}^2 + \|\Gamma^0_1 u\|_{-\infty}^2).$$

The symbol of the operator $\Lambda^{s+2}\varrho(\Gamma^0\Gamma^0_1 - \Gamma^0)$ is zero so that

$$\|\Lambda^{s+2}\varrho(\Gamma^0\Gamma^0_1 - \Gamma^0)u\|^2 \le C\|u\|_{-\infty}^2.$$

Then we obtain

$$\|\varrho\Gamma^0 u\|_{s+2}^2 \le C(\|\Gamma^0\varrho\Lambda^{s+1}\Gamma^0_1 u\|_1^2 + \|\varrho_1\Gamma^0_1 u\|_{s+1}^2 + \|u\|_{-\infty}^2),$$

so that

$$\|\varrho\Gamma^0 u\|_{s+2}^2 \le C(\|\Gamma^0\bar z^k L\varrho\Lambda^{s+1}\Gamma^0_1 u\|^2 + \|\Gamma^0\bar L\varrho\Lambda^{s+1}\Gamma^0_1 u\|^2 + \|\varrho_1\Gamma^0_1 u\|_{s+1}^2 + \|u\|_{-\infty}^2).$$

The following lemma which involves a vector field X will be applied with $X = \bar z^k L$ and $X = \bar L$.

LEMMA 6. *If X is a complex vector field on $\mathbb{R}^3$ then*

$$\|\Gamma^0 X\varrho\Lambda^{s+1}\Gamma^0_1 u\|^2 \le C(\|\varrho\Gamma^0 X u\|_{s+1}^2 + \|\varrho_1\Gamma^0_1 u\|_{s+1}^2 + \|u\|_{-\infty}^2)$$

and

$$\begin{aligned}\|\varrho\Gamma^0 X u\|_{s+1}^2 = {} & (\Lambda^s\varrho\Gamma^0 X^* X u, \Lambda^{s+2}\varrho\Gamma^0 u) \\ & + O(\|\varrho_1\Gamma^0_1 u\|_{s+1}^2 + \|\varrho\Gamma^0 u\|_{s+2}^2\|\varrho_1\Gamma^0_1 u\|_{s+1}^2 \\ & + \|\varrho_1\Gamma^0_1 X u\|_s^2 + \|\varrho_1\Gamma^0_1 u\|_{s+1}\|\varrho\Gamma^0 X u\|_{s+1} + \|u\|_{-\infty}^2),\end{aligned}$$

for all $u \in \mathcal{S}$.

Proof. We have

$$\|\Gamma^0 X\varrho\Lambda^{s+1}\Gamma^0_1 u\| \le \|\Gamma^0\varrho\Lambda^{s+1}\Gamma^0_1 X u\| + \|\Gamma^0[X,\varrho\Lambda^{s+1}\Gamma^0_1]u\|.$$

The operator $P = \Gamma^0\varrho\Lambda^{s+1}\Gamma^0_1 X - \Lambda^{s+1}\Gamma^0_1\varrho\Gamma^0 X$ is of order $s+1$ and $\varrho_1\gamma^0_1 = 1$ in a neighborhood of the symbol of P; hence

$$\|Pu\| \le C(\|P\varrho_1\Gamma^0_1 u\| + \|u\|_{-\infty}) \le C(\|\varrho_1\Gamma^0_1 u\|_{s+1} + \|u\|_{-\infty}).$$

Since $\gamma_1^0 = 1$ in a neighborhood of the support of the symbol of $\varrho\Gamma^0$, we get

$$\|\Gamma^0\varrho\Lambda^{s+1}\Gamma_1^0 Xu\| \le C(\|\varrho\Gamma^0 Xu\|_{s+1} + \|u\|_{-\infty}).$$

Furthermore, $\Gamma^0[X, \varrho\Lambda^{s+1}\Gamma_1^0]$ is of order $s+1$ and $\varrho_1\gamma_1^0 = 1$ in a neighborhood of the support of its symbol so that

$$\|\Gamma^0[X, \varrho\Lambda^{s+1}\Gamma_1^0]u\| \le C(\|\varrho_1\Gamma_1^0 u\|_{s+1} + \|u\|_{-\infty}),$$

which proves the first part of the lemma.

For the second part of the lemma we write

$$\begin{aligned}\|\varrho\Gamma^0 Xu\|_{s+1}^2 &= (\Lambda^{s+1}\varrho\Gamma^0 Xu, \Lambda^{s+1}\varrho\Gamma^0 Xu)\\ &= (\Lambda^{s+1}\varrho\Gamma^0 Xu, [\Lambda^{s+1}\varrho\Gamma^0, X]u) + ([X^*, \Lambda^{s+1}\varrho\Gamma^0]Xu, \Lambda^{s+1}\varrho\Gamma^0 u)\\ &\quad + (\Lambda^s\varrho\Gamma^0 X^* Xu, \Lambda^{s+2}\varrho\Gamma^0 u).\end{aligned}$$

Then, since $[\Lambda^{s+1}\varrho\Gamma^0, X]$ is of order $s+1$ and $\varrho_1\gamma_1^0 = 1$ in a neighborhood of its symbol,

$$\|[\Lambda^{s+1}\varrho\Gamma^0, X]u\|^2 \le C(\|\varrho_1\Gamma_1^0 u\|_{s+1}^2 + \|u\|_{-\infty}).$$

Then

$$\begin{aligned}&([X^*, \Lambda^{s+1}\varrho\Gamma^0]Xu, \Lambda^{s+1}\varrho\Gamma^0 u)\\ &= ([(\Lambda^{s+1}\varrho\Gamma^0)^*, [X^*, \Lambda^{s+1}\varrho\Gamma^0]]Xu, u) + ((\Lambda^{s+1}\varrho\Gamma^0)^* Xu, [X^*, \Lambda^{s+1}\varrho\Gamma^0]^* u).\end{aligned}$$

Let $Q = [(\Lambda^{s+1}\varrho\Gamma^0)^*, [X^*, \Lambda^{s+1}\varrho\Gamma^0]]$; then Q has order $2s+1$ and $\varrho_1\gamma_1^0 = 1$ in a neighborhood of its symbol. Thus

$$\begin{aligned}|(QXu, u)| &\le C(|(Q\varrho_1\Gamma_1^0 Xu, \varrho_1\Gamma_1^0 u)| + \|u\|_{-\infty}^2)\\ &\le C(\|\varrho_1\Gamma_1^0 Xu\|_s^2 + \|\varrho_1\Gamma_1^0 u\|_{s+1}^2 + \|u\|_{-\infty}^2).\end{aligned}$$

The symbol of the operator $(\Lambda^{s+1}\varrho\Gamma^0)^* - \Lambda^{s+1}\varrho\Gamma^0$ is zero, the order of $[X^*, \Lambda^{s+1}\varrho\Gamma^0]^*$ is $s+1$ and $\varrho_1\gamma_1^0 = 1$ on a neighborhood of its support. Hence

$$|((\Lambda^{s+1}\varrho\Gamma^0)^* Xu, [X^*, \Lambda^{s+1}\varrho\Gamma^0]^* u)| \le C(\|\varrho\Gamma^0 Xu\|_{s+2}\|\varrho_1\Gamma_1^0 u\|_{s+1} + \|u\|_{-\infty}^2).$$

Combining these we conclude the proof of the lemma.

Returning to the proof of the proposition, by using the above lemma, when $X = \bar{z}^k L$ and when $X = \bar{L}$, we obtain

$$\begin{aligned}&\|\varrho\Gamma^0 u\|_{s+2}^2 + \|\varrho\Gamma^0\bar{z}^k Lu\|_{s+1}^2 + \|\varrho\Gamma^0\bar{L}u\|_{s+1}^2\\ &\qquad\le C(\|\varrho\Gamma^0 E_k u\|_s^2 + \|\varrho_1\Gamma_1^0\bar{z}^k Lu\|_s^2 + \|\varrho_1\Gamma_1^0\bar{L}u\|_s^2 + \|u\|_{-\infty}^2).\end{aligned}$$

Replacing ϱ by ϱ_i, ϱ_1 by ϱ_{i+1}, Γ^0 by Γ_i^0, Γ_1^0 by Γ_{i+1}^0, and s by $s-i$ we obtain

$$\begin{aligned}&\|\varrho_i\Gamma_i^0 u\|_{s+2-i}^2 + \|\varrho_i\Gamma_i^0\bar{z}^k Lu\|_{s+1-i}^2 + \|\varrho_i\Gamma_i^0\bar{L}u\|_{s+1-i}^2\\ &\quad\le C(\|\varrho_i\Gamma_i E_k u\|_{s-i}^2 + \|\varrho_{i+1}\Gamma_{i+1}^0\bar{z}^k Lu\|_{s-i}^2 + \|\varrho_{i+1}^0\Gamma_{i+1}^0\bar{L}u\|_{s-i}^2 + \|u\|_{-\infty}^2).\end{aligned}$$

Proceeding inductively, we obtain

$$\|\varrho\Gamma^0 u\|_{s+2}^2 + \|\varrho\Gamma^0 \bar{z}^k Lu\|_{s+1}^2 + \|\varrho\Gamma^0 \bar{L}u\|_{s+1}^2$$
$$\leq C\Big(\sum_{i=0}^{N} \|\varrho_i\Gamma_i^0 E_k u\|_{s-i}^2 + \|\varrho_{N+1}\Gamma_{N+1}^0 \bar{z}^k Lu\|_{s-N}^2 + \|\varrho_{i+N}^0\Gamma_{i+N}^0 \bar{L}u\|_{s-N}^2 + \|u\|_{-\infty}^2\Big).$$

Setting $N \geq s_0 + s + 1$ we conclude the proof of the proposition since

$$\|\varrho_i\Gamma_i^0 E_k u\|_{s-i}^2 \leq C(\|\varrho'\tilde{\Gamma}^0 E_k u\|_s^2 + \|u\|_{-\infty}^2).$$

Proposition 5. *Given neighborhoods of the origin U and U' with $\bar{U} \subset U'$; suppose that $\varrho \in C_0^\infty(U)$ and $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ on a neighborhood of $\bar{U}$. Further suppose that $\gamma^-, \tilde{\gamma}^- \in \mathcal{G}^-$ with $\tilde{\gamma}^- = 1$ on a neighborhood of the support of γ^-. Then, given $s, s_0 \in \mathbb{R}$, there exists $C = C(\varrho, \varrho', \gamma^-, \tilde{\gamma}^-, s, s_0)$ such that*

$$\|\varrho\Gamma^- u\|_{s+1}^2 + \|\varrho\Gamma^- \bar{z}^k Lu\|_{s+\frac{1}{2}}^2 + \|\varrho\Gamma^- \bar{L}u\|_{s+\frac{1}{2}}^2 \leq C(\|\varrho'\tilde{\Gamma}^- E_k u\|_s^2 + \|u\|_{-s_0}^2),$$

for all $u \in \mathcal{S}$.

Proof. The proof is entirely analogous to that of the above proposition. We use Lemma 5 in place of Lemma 3 and substitute $\varrho\Lambda^{s+\frac{1}{2}}\Gamma_1^- u$ for u we obtain

$$\|\Gamma^- \varrho\Lambda^{s+\frac{1}{2}}\Gamma_1^- u\|_{\frac{1}{2}}^2 \leq C(\|\Gamma^0 \bar{L}\varrho\Lambda^{s+\frac{1}{2}}\Gamma_1^- u\|^2 + \|\varrho\Lambda^{s+\frac{1}{2}}\Gamma_1^0 u\|^2).$$

Then one proceeds exactly as above to obtain the proof.

In the case $k = 0$ the vectorfields L and $\bar{L}$ play exactly the same role and so we obtain the following.

Proposition 6. *Given neighborhoods of the origin U and U' with $\bar{U} \subset U'$. Suppose that $\varrho \in C_0^\infty(U)$ and $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ on a neighborhood of $\bar{U}$. Further suppose that $\gamma^+, \tilde{\gamma}^+ \in \mathcal{G}^+$ with $\tilde{\gamma}^+ = 1$ on a neighborhood of the support of γ^+. Then, given $s, s_0 \in \mathbb{R}$, there exists $C = C(\varrho, \varrho', \gamma^+, \tilde{\gamma}^+, s, s_0)$ such that*

$$\|\varrho\Gamma^+ u\|_{s+1}^2 + \|\varrho\Gamma^+ Lu\|_{s+\frac{1}{2}}^2 + \|\varrho\Gamma^+ \bar{L}u\|_{s+\frac{1}{2}}^2 \leq C(\|\varrho'\tilde{\Gamma}^+ E_0 u\|_s^2 + \|u\|_{-s_0}^2),$$

for all $u \in \mathcal{S}$.

9. The operator E_0 and gain of derivatives

Since E_0 is a real operator, it can be written as $E_0 = -X^2 - Y^2$, where $X = \frac{1}{\sqrt{2}}\Re L$ and $Y = \frac{1}{\sqrt{2}}\Im L$. Thus it is one of the simplest operators that satisfy Hörmander's condition and it is well understood. Nevertheless, it is instructive to write it in terms of L and $\bar{L}$ and analyze it microlocally in the framework of the previous section. The operator E_0 gains one derivative. As we have seen the operators E_k do not gain derivatives when $k > 0$ and $z = 0$; in a neighborhood on which $z \neq 0$ they do gain derivatives and they also gain in the 0 and $-$ microlocalizations.

In the analysis of E_0 we can assume, without loss of generality, that $\alpha = 0$ and we set $\gamma = \gamma_0$, and $\Gamma = \Gamma_0$. The basic observation is that the gain of derivatives in the + and − microlocalizations is controlled by the operators $\bar{L}L$ and $L\bar{L}$, respectively. In the 0 microlocalization the gain of derivatives is controlled by both $\bar{L}L$ and $L\bar{L}$ independently. Propositions 4 and 5 give *a priori* estimates for E_k in the 0 and − microlocalizations, respectively. Proposition 6 gives these estimates for the + microlocalization. Here we show how to go from the *a priori* estimates to hypoellipticity. In particular we prove that E_0 is hypoelliptic and that E_k is hypoelliptic on open sets on which $z \neq 0$ and that the 0 and − microlocalizations of the operators E_k are hypoelliptic.

PROPOSITION 7. *If u is a distribution such that for some open set $V \subset \mathbb{R}^3$ the restriction of $E_0 u$ to V is in $C^\infty(V)$ then the restriction of u to V is also in $C^\infty(U)$. More precisely, if $E_0 u \in H^s_{\mathrm{loc}}(V)$ then $u \in H^{s+1}_{\mathrm{loc}}(V)$.*

Proof. Assuming that $E_0 u \in H^s_{\mathrm{loc}}(V)$, it suffices to show that any $P \in V$ has a neighborhood $U \subset V$ such that for any $\varrho \in C_0^\infty(U)$ we have $\varrho u \in H^{s+1}(\mathbb{R}^3)$. Without loss of generality we may assume that $P = 0$. Now choose neighborhoods U and U' of P such that $\bar{U} \subset U'$ and $|z| \leq a$ on U', as in Proposition 4. Let $\varrho \in C_0^\infty(U)$, let $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ in a neighborhood of the support of ϱ, and let $\theta \in C_0^\infty(\mathbb{R}^3)$ such that $\theta = 1$ on a neighborhood of $\bar{U}'$. Since u is a distribution there exists an $s_0 \in \mathbb{R}$ such that $\theta u \in H^{-s_0}(\mathbb{R}^3)$. Then, choosing γ^+, γ^0, and γ^- such that $\gamma^+ + \gamma^0 + \gamma^- \geq$ const. > 0 and combining Propositions 4, 5, and 6 we obtain the *a priori* estimate

$$\|\varrho u\|^2_{s+1} + \|\varrho L u\|^2_{s+\frac{1}{2}} + \|\varrho \bar{L} u\|^2_{s+\frac{1}{2}} \leq C(\|\varrho' E_0 u\|^2_s + \|u\|^2_{-s_0}),$$

for all $u \in C^\infty(\mathbb{R}^3)$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\chi(0) = 1$. For $\delta > 0$ we define the smoothing operator S_δ by $\mathcal{F}(S_\delta u)(\xi) = \chi(\delta\xi)\hat{u}(\xi)$. The important facts are that:

1. If $\delta > 0$ then for any distribution v the function $S_\delta v \in C^\infty(\mathbb{R})$.
2. If v is a distribution and if $\|S_\delta v\|_s$ is bounded independently of δ then $v \in H^s(\mathbb{R}^3)$.
3. If $v \in H^s(\mathbb{R}^3)$ then $\lim_{\delta\to 0} \|S_\delta v - v\|_s = 0$.
4. For $\delta \geq 0$ the operator S_δ is a pseudodifferential operator which is uniformly of order zero.

Replacing u by $S_\delta \theta u$ in Lemma 6 and in the proofs of Propositions 4, 5, and 6 and using item 4 above we obtain

$$\|S_\delta \varrho u\|^2_{s+1} \leq C(\|S_\delta \varrho' E_0 u\|^2_s + \|\tilde{S}_\delta \varrho' u\|^2_{s+\frac{1}{2}} + \|\tilde{S}_\delta \tilde{\theta} u\|^2_{-s_0}),$$

where $\tilde{S}_\delta$ has the symbol $\tilde{\chi}(\delta\xi)$ with $\tilde{\chi} = 1$ in a neighborhood of the support of χ. Choose m so that $-s_0 \geq s+1-m$, then substituting $s+1-m+j$ for s above we obtain, by induction on j, that $\|S_\delta \varrho u\|_{s+1}^2$ is bounded independently of δ. Hence $\varrho u \in H^{s+1}(\mathbb{R}^3)$ thus concluding the proof.

Next we will show that in any region in which $z \neq 0$ the operator E_k is hypoelliptic with a gain of one derivative.

Proposition 8. *If $V \subset \mathbb{R}^3$ is an open set, with the property that $z \neq 0$ on V, and if u is a distribution such that the restriction of $E_k u$ to V is in $C^\infty(V)$, then the restriction of u to V is also in $C^\infty(U)$. More precisely, if $E_k u \in H^s_{\mathrm{loc}}(V)$ then $u \in H^{s+1}_{\mathrm{loc}}(V)$.*

Proof. Let $P \in V$ then $P = (\alpha, t_0)$ with $\alpha \neq 0$. Let U be a neighborhood of P such that on U we have $|z-\alpha| < a$, where a is chosen as in Lemma 3, and also such that on U we have $|z| \geq b > 0$. Then

$$\|Lu\|^2 \leq b^{-2k}\|\bar{z}^k Lu\|^2,$$

for all $u \in C_0^\infty(U)$. Hence Propositions 4, 5, and 6 hold with γ replaced by γ_α. The proof is then concluded using the same argument as above, replacing S_δ with $S_{\alpha,\delta}$, which is defined by $\mathcal{F}_\alpha(S_{\alpha,\delta}u)(\xi) = \chi(\delta\xi)\mathcal{F}_\alpha u(\xi)$.

Now we prove microlocal hypoellipticity in the 0 and $-$ microlocalizations.

Proposition 9. *Given neighborhoods of the origin U and U' with $\bar{U} \subset U'$ and $|z| \leq a$ on U', where a is sufficiently small as in Lemma 3, suppose that $\varrho \in C_0^\infty(U)$ and $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ on a neighborhood of $\bar{U}$. Further suppose that $\gamma^0 \in \mathcal{G}^0$. Then, given $s \in \mathbb{R}$, if u is a distribution such that $\varrho' E_k u \in H^s(\mathbb{R}^3)$ then $\varrho\Gamma^0 u \in H^{s+2}(\mathbb{R}^3)$.*

Proof. The proof consists of proving the following estimate

$$\|S_\delta \varrho \Gamma^0 u\|_{s+2}^2 \leq C(\|\varrho' E_k u\|_s^2 + \|u\|_{-s_0}^2).$$

Its proof is exactly analogous to the proof of Proposition 4. Replacing u by $S_\delta u$ the same proof as of Lemma 6 using XS_δ instead of X gives

$$\begin{aligned}\|S_\delta\varrho\Gamma^0 Xu\|_{s+1}^2 = {}& (\Lambda^s S_\delta \varrho \Gamma^0 X^* Xu, \Lambda^{s+2} S_\delta \varrho \Gamma^0 u)\\ &+ O(\|\tilde{S}_\delta \varrho_1 \Gamma_1^0 u\|_{s+1}^2 + \|S_\delta \varrho \Gamma^0 u\|_{s+2}^2 \|\tilde{S}_\delta \varrho_1 \Gamma_1^0 u\|_{s+1}^2\\ &+ \|\tilde{S}_\delta \varrho_1 \Gamma_1^0 Xu\|_s^2 + \|\tilde{S}_\delta \varrho_1 \Gamma_1^0 u\|_{s+1}\|\varrho\Gamma^0 Xu\|_{s+1} + \|u\|_{-\infty}^2).\end{aligned}$$

The argument then proceeds exactly as in Proposition 4 and shows that $\|S_\delta \varrho \Gamma^0 u\|_{s+2}^2$ is bounded independently of δ completing the proof.

For the $-$ microlocalization we the following result follows from an argument entirely analogous to the above proposition.

PROPOSITION 10. *Given neighborhoods of the origin U and U' with $\bar{U} \subset U'$ and $|z| \leq a$ on U', where a is sufficiently small as in Lemma 3. Suppose that $\varrho \in C_0^\infty(U)$ and $\varrho' \in C_0^\infty(U')$ with $\varrho' = 1$ on a neighborhood of $\bar{U}$. Further suppose that $\gamma^- \in \mathcal{G}^0$. Then, given $s \in \mathbb{R}$, if u is a distribution such that $\varrho' E_k u \in H^s(\mathbb{R}^3)$ then $\varrho \Gamma^- u \in H^{s+1}(\mathbb{R}^3)$.*

10. The operator E_1: no loss, no gain

As was shown in Section 5 the operator E_1 does not gain any derivatives. Here we will give a proof of an *a priori* estimate which shows that it does not lose any derivatives. More precisely, the estimate will show that E_1 does not lose any derivatives after it is proved that E_1 is hypoelliptic. This will be done using the same estimate with an appropriate smoothing operator in Section 14. As we have seen all the operators E_k gain a derivative in regions where $z \neq 0$ and in the 0 and $-$ microlocalizations. Thus the remaining case is the $+$ microlocalization when $z = 0$. Since the operators E_k are invariant under translation in the t direction it will suffice to consider neighborhoods of the origin. In this section we will present a direct proof of the *a priori* estimates for E_1 which will rely on the following lemma. This proof however cannot be adopted to prove the corresponding *a priori* estimate for the operator $F_1 = E_1 + c$ unless $c \geq 0$. In fact the same estimates will be proved when we treat the general case of E_k with $k \geq 1$. However that treatment is much more complicated so it might be worthwhile to note this simpler proof.

In the previous section we showed that the elliptic microlocalization $\Gamma^0 u$ is smooth whenever $E_k u$ is smooth. Thus we do not have to keep track of just which microlocalizing operator in $\mathfrak{G}^0$ is used; in order to simplify the calculations we will write u^0 instead of $\Gamma^0 u$. Similarly, since all the commutators with Γ^+ that arise are dominated as follows $\|[\Gamma^+, R^s]u\| \leq C(\|\Gamma^0 u\|_{s-1} + \|u\|_{-\infty}$, we will write u^+ instead of Γ^+.

LEMMA 7. *Given a bounded open set $U \subset \mathbb{R}^3$ there exists $C > 0$ such that*

$$\|u\|^2 \leq C(\|\bar{z}Lu\|^2 + \|\bar{L}u\|^2),$$

for all $u \in C_0^\infty(U)$.

Proof. If $u \in C_0^\infty(U)$ we have

$$\begin{aligned}\|u\|^2 &= (L(z)u, u) = -(zLu, u) - (zu, \bar{L}u) \leq \|\bar{z}Lu\|\|u\| + \|zu\|\|\bar{L}u\| \\ &\leq \text{s.c.}\|u\|^2 + \text{l.c.}(\|\bar{z}Lu\|^2 + \|\bar{L}u\|^2).\end{aligned}$$

Absorbing the first term on the right into the left-hand side completes the proof.

The other estimate we will use here is given in Lemma 5 with $\alpha = 0$, namely

$$\|\bar{L}u^+\|^2 + \|u^+\|^2_{\frac{1}{2}} \le C(\|Lu^+\|^2 + \|u^+\|^2 + \|u\|^2_{-\infty}), \tag{8}$$

for all $u \in C_0^\infty(U)$.

Proposition 11. *Let U be a bounded neighborhood of the origin such that $|z| \le a$ on U, let $\varrho, \varrho' \in C_0^\infty(U)$ with $\varrho' = 1$ in a neighborhood of the support of ϱ. Then, given $s, s_0 \in \mathbb{R}$ there exists $C = C(\varrho, \varrho', s, s_0)$ such that*

$$\|\Psi^{s+\frac{1}{2}}\varrho u^+\| \le C(\|\Psi^{s+\frac{1}{2}}\varrho' E_1 u\| + \|\Psi^s \varrho' u\| + \|u\|_{-s_o}),$$

for all $u \in C_0^\infty(\mathbb{R}^3)$.

Proof. We assume that $u \in C_0^\infty(\mathbb{R}^3)$ and replace u in (8) by $\varrho'\bar{z}\Psi^s u$. Then, following the method of Proposition 4, we get

$$\begin{aligned}\|\varrho'\bar{z}\Psi^s u^+\|^2_{\frac{1}{2}} &\le C(\|\bar{z}L\varrho'\Psi^s u^+\|^2 + \|\bar{L}\varrho'\Psi^s u^+\|^2 + \|\varrho''\Psi^s u^+\|^2 + \|u\|^2_{-\infty})\\ &\le C(|(\varrho'\Psi^s(E_k u)^+, \varrho\Psi^s u^+)| + \|\varrho'' u^0\|^2_s + \|\varrho''\Psi^s u^+\|^2 + \|u\|^2_{-\infty})\\ &\le C(\|\varrho' E_k u\|^2_s + \|\varrho'' u\|^2_s + \|u\|^2_{-\infty}).\end{aligned}$$

Next, we replace u by $\varrho\Psi^{s+\frac{1}{2}}u^+$ in Lemma 8 and, with the use of Lemma 1 and the fact that

$$\|\varrho'\bar{z}\Psi^s u^+\|^2_{\frac{1}{2}} = \|z\varrho'\Psi^{s+\frac{1}{2}}u^+\|^2 + O(\|u^0\|^2_{s-\frac{1}{2}} + \|u\|^2_{-\infty}),$$

we obtain

$$\begin{aligned}\|\varrho\Psi^{s+\frac{1}{2}}u^+\|^2 &\le C(\|\bar{z}L\varrho\Psi^{s+\frac{1}{2}}u^+\|^2 + \|\bar{L}\varrho\Psi^{s+\frac{1}{2}}u^+\|^2 + \|u^0\|^2_s + \|u\|^2_{-\infty})\\ &\le C(\|\varrho E_k u\|^2_{s+\frac{1}{2}}\|^2 + \|L(\varrho)\Psi^{s+\frac{1}{2}}u^+\|^2\\ &\qquad + \|\bar{L}(\varrho)\Psi^{s+\frac{1}{2}}u^+\|^2 + \|\varrho' u^0\|^2_{s+\frac{1}{2}} + \|u\|^2_{-\infty})\\ &\le C(\|\varrho E_k u\|^2_{s+\frac{1}{2}}\|^2 + \|z\varrho'\Psi^{s+\frac{1}{2}}u^+\|^2 + \|\varrho' u^0\|^2_{s-\frac{1}{2}} + \|u\|^2_{-\infty})\\ &\le C(\|\varrho' E_k u\|^2_{s+\frac{1}{2}}\|^2 + \|\varrho'' u\|^2_s + \|u\|^2_{-\infty}).\end{aligned}$$

Then, redefining ϱ' and ϱ'', we conclude the proof.

11. Estimates of $\varrho\bar{L}u^+$ and of $\varrho L\bar{L}u^+$

In this section we begin to prove the *a priori* estimates for the operators E_k with $k \ge 1$. These will be derived from the estimate (8) and the estimates in the 0 microlocalization. The main difficulty is the localization in space; one cannot have a term with the cutoff function ϱ between u and L, or $\bar{L}$, unless the term also contains suitable powers of z and $\bar{z}$. Substituting $\varrho\Psi^s\bar{L}u$ for u in (8) we have

$$\|\bar{L}\varrho\Psi^s\bar{L}u^+\|^2 + \|\varrho\Psi^s\bar{L}u^+\|^2_{\frac{1}{2}} \le C(\|L\varrho\Psi^s\bar{L}u^+\|^2 + \|\varrho\Psi^s\bar{L}u^+\|^2 + \|u\|^2_{-\infty}),$$

so that,

$$\begin{aligned}
&\|\varrho\Psi^s L\bar{L}u^+\|^2 + \|\varrho\Psi^s\bar{L}^2u^+\|^2 + \|\varrho\Psi^{s+\frac{1}{2}}\bar{L}u^+\| \\
&\quad \le C(\|\varrho\Psi^s L\bar{L}u^+\|^2 + \|\varrho'\Psi^s\bar{L}u^+\|^2 + \|\varrho' u^0\|^2_{s+1} + \|u\|^2_{-\infty}) \\
&\quad \le C(|(\varrho\Psi^s\bar{L}L\bar{L}u^+, \varrho\Psi^s\bar{L}u^+)| + \|\varrho'\Psi^s\bar{L}u^+\|^2 + \|\varrho'' E_k u\|^2_{s-1} + \|u\|^2_{-\infty}).
\end{aligned}$$

Since $\bar{L}L\bar{L} = -\bar{L}E_k - \bar{L}^2|z|^{2k}L$, we have

$$\begin{aligned}
&|(\varrho\Psi^s\bar{L}L\bar{L}u^+, \varrho\Psi^s\bar{L}u^+)| \\
&\quad \le C(|(\varrho\Psi^s\bar{L}E_k u^+, \varrho\Psi^s\bar{L}u^+)| + |(\varrho\Psi^s\bar{L}^2|z|^{2k}Lu^+, \varrho\Psi^s\bar{L}u^+)| \\
&\quad \le \text{l.c.}\|\varrho' E_k u\|^2_s + \text{s.c.}\|\varrho\Psi^s L\bar{L}u^+\|^2 + C|(\varrho\Psi^s\bar{L}^2|z|^{2k}Lu^+, \varrho\Psi^s\bar{L}u^+)|.
\end{aligned}$$

Then, to estimate $|(\varrho\Psi^s\bar{L}^2|z|^{2k}Lu^+, \varrho\Psi^s\bar{L}u^+)|$, we have

$$\begin{aligned}
\bar{L}^2|z|^{2k}L &= -k\bar{L}z^k\bar{z}^{k-1}L + \bar{L}|z|^{2k}\bar{L}L \\
&= -k^2\bar{L}|z|^{2(k-1)} + \bar{L}Lz^k\bar{z}^{k-1} - 2kz^k\bar{z}^{k-1}T + \bar{L}|z|^{2k}L\bar{L} - 2|z|^{2k}T\bar{L} \\
&= -k^2\bar{L}|z|^{2(k-1)} - 4kz^k\bar{z}^{k-1}T + k(k-1)Lz^k\bar{z}^{k-2} + kLz^k\bar{z}^{k-1}\bar{L} \\
&\quad + \bar{L}|z|^{2k}L\bar{L} - 2|z|^{2k}T\bar{L},
\end{aligned}$$

and, using integration by parts, we get

$$\begin{aligned}
|(\varrho\Psi^s\bar{L}|z|^{2(k-1)}u^+, \varrho\Psi^s\bar{L}u^+)| &\le \text{l.c.}\|z^{2(k-1)}\varrho\Psi^s u^+\|^2 + \mathcal{E}_1, \\
|(\varrho\Psi^s z^k\bar{z}^{k-1}Tu^+, \varrho\Psi^s\bar{L}u^+)| &\le \text{l.c.}\|z^{2k-1}\varrho\Psi^{s+\frac{1}{2}}u^+\|^2 + \mathcal{E}_2, \\
(k-1)|(\varrho\Psi^s Lz^k\bar{z}^{k-2}u^+, \varrho\Psi^s\bar{L}u^+)| &\le (k-1)(\text{l.c.}\|z^{2(k-1)}\varrho\Psi^s u^+\|^2 + \mathcal{E}_3), \\
|(\varrho\Psi^s Lz^k\bar{z}^{k-1}\bar{L}u^+, \varrho\Psi^s\bar{L}u^+)| &\le \text{l.c.}\|z^{2k-1}\varrho\Psi^s u^+\|^2 + \mathcal{E}_4, \\
|(\varrho\Psi^s\bar{L}|z|^{2k}L\bar{L}u^+, \varrho\Psi^s\bar{L}u^+)| &\le \mathcal{E}_4,
\end{aligned}$$

and

$$|(\varrho\Psi^s|z|^{2k}T\bar{L}u^+, \varrho\Psi^s\bar{L}u^+)| \le \mathcal{E}_2,$$

where

$$\begin{aligned}
\mathcal{E}_1 &\sim \|\varrho' u^0\|^2_s + \|\varrho'\Psi^s\bar{L}u^+\|^2 + \|u\|^2_{-\infty}, \\
\mathcal{E}_2 &\sim \text{s.c.}\|\varrho\Psi^{s+\frac{1}{2}}\bar{L}u^+\|^2 + \mathcal{E}_1, \\
\mathcal{E}_3 &\sim \text{s.c.}\|\varrho\Psi^s\bar{L}^2u^+\|^2 + \mathcal{E}_1,
\end{aligned}$$

and

$$\mathcal{E}_4 \sim \text{s.c.}\|\varrho\Psi^s L\bar{L}u^+\|^2 + \mathcal{E}_1.$$

$$|(\Psi^s\varrho\bar{L}|z|^{2(k-1)}u^+,\Psi^s\varrho\bar{L}u^+)|\le C(\|z^{2k-2}\Psi^s\varrho u^+\|^2+\mathcal{E}_2),$$
$$|(\Psi^s\varrho\bar{L}z^{k-1}\bar{z}^k\bar{L}u^+,\Psi^s\varrho\bar{L}u^+)|\le C\mathcal{E}_1$$
$$|(\Psi^s z^k\bar{z}^{k-1}\varrho Tu^+,\Psi^s\varrho\bar{L}u^+)|\le C(\|z^{2k-1}\Psi^{s+\frac{1}{2}}\varrho u^+\|^2+\mathcal{E}_1+\mathcal{E}_3)$$
$$|(\Psi^s|z|^{2k}T\varrho\bar{L}u^+,\Psi^s\varrho\bar{L}u^+)|\le C(\mathcal{E}_1+\mathcal{E}_2)$$
$$|(\Psi^s Lz^k\bar{z}^{k-2}\varrho u^+,\Psi^s\varrho\bar{L}u^+)|\le C(\|z^{2k-2}\Psi^s\varrho u^+\|^2+\mathcal{E}_1+\mathcal{E}_4)$$
$$|(\Psi^s Lz^k\bar{z}^{k-1}\varrho\bar{L}u^+,\Psi^s\varrho\bar{L}u^+)|\le C(\mathcal{E}_1+\mathcal{E}_4),$$

and

$$|(\Psi^s L|z|^{2k}\varrho\bar{L}^2u^+,\Psi^s\varrho\bar{L}u^+)|\le C(\mathcal{E}_1+\mathcal{E}_4).$$

Again, let $\{\varrho_i\}$ be a sequence of cutoff functions as defined in Section 2. Then substituting ϱ_i for ϱ, $s-\frac{i-1}{2}$ for s, and ϱ_{i+1} for ϱ', we get

$$\begin{aligned}&\|\varrho_i\Psi^{s-\frac{i-1}{2}}L\bar{L}u^+\|^2+\|\varrho_i\Psi^{s+1-\frac{i}{2}}\bar{L}u^+\|^2\\&\quad\le C(\|\varrho' E_k u\|^2_{s-\frac{i-1}{2}}+\|z^{2k-2}\Psi^{s-\frac{i-1}{2}}\varrho_i u^+\|^2\\&\qquad+\|z^{2k-1}\Psi^{s+1-\frac{i}{2}}\varrho_i u^+\|^2+\|\varrho_{i+1}\Psi^{s-\frac{i-1}{2}}\bar{L}u^+\|^2+\|u\|^2_{-\infty}).\end{aligned}$$

Then we obtain the following, by substituting these inequalities into each other for successive i

$$\begin{aligned}&\|\varrho\Psi^s L\bar{L}u^+\|^2+\|\varrho\Psi^{s+\frac{1}{2}}\bar{L}u^+\|^2\\&\le C\Big(\sum_{i=1}^{N}\big(\|\varrho_i\Psi^{s-\frac{i-1}{2}}\bar{L}E_k u^+\|^2+\|z^{2k-2}\Psi^{s-\frac{i-1}{2}}\varrho_i u^+\|^2+\|z^{2k-1}\Psi^{s+1-\frac{i}{2}}\varrho_i u^+\|^2\big)\\&\quad+\|\varrho_{N+1}\Psi^{s-\frac{N-1}{2}}\bar{L}u^+\|^2+\|\varrho_N E_k u\|^2_{s-1}+\|u\|^2_{-\infty}\Big).\end{aligned}$$

Given s_o we choose $N>2(s-s_o)+1$ then we obtain the following estimate which will be repeatedly used in establishing the *a priori* estimates for E_k

$$\begin{aligned}\text{(9)}\qquad&\|\varrho\Psi^s L\bar{L}u^+\|^2+\|\varrho\Psi^{s+\frac{1}{2}}\bar{L}u^+\|^2\le C(\|\varrho' E_k u^+\|^2_s+\|z^{2k-2}\Psi^s\varrho' u^+\|^2\\&\qquad+\|z^{2k-1}\Psi^{s+\frac{1}{2}}\varrho' u^+\|^2+\|u\|^2_{-s_o}).\end{aligned}$$

12. Estimates of $\|z^j\Psi^{s+ja}\varrho u^+\|$

LEMMA 8. *If $a>0$ then for $m\in\mathbb{Z}^+$ and a small constant* s.c. *there exists a constant* l.c. *such that*

$$\sum_{j=1}^{m-1}\|z^j\Psi^{s+ja}\varrho u^+\|^2\le \text{l.c.}\|z^m\Psi^{s+ma}\varrho u^+\|^2+\text{s.c.}\|\Psi^s\varrho u\|^2 \\ +C(\|\varrho' u^0\|^2_{s+(m-1)a-1}+\|u\|^2_{-\infty}),$$

for all $u\in C^\infty(U)$.

Proof. For $m=2$ we have

$$\|z\Psi^{s+a}\varrho u^+\|^2=(|z|^2\Psi^{s+2a}\varrho u^+,\Psi^s\varrho u^+)+O(\|\varrho' u^0\|^2_{s+a-1}+\|u\|^2_{-\infty}) \\ \le \text{l.c.}\|z^2\Psi^{s+2a}\varrho u^+\|^2+\text{s.c.}\|\Psi^s\varrho u^+\|^2+C(\|\varrho' u^0\|^2_{s+a-1}+\|u\|^2_{-\infty}).$$

For $m>2$ we assume

$$\sum_{j=1}^{m-2}\|z^j\Psi^{s+ja}\varrho u^+\|^2\le \text{l.c.}\|z^{m-1}\Psi^{s+(m-1)a}\varrho u^+\|^2+\text{s.c.}\|\Psi^s\varrho u\|^2 \\ +C(\|\varrho u^0\|^2_{s+(m-2)a-1}+\|u\|^2_{-\infty}),$$

and we have

$$\|z^{m-1}\Psi^{s+(m-1)a}\varrho u^+\|^2=(z^m\bar z\Psi^{s+ma}\varrho u^+,z^{m-2}\Psi^{s+(m-2)a}\varrho u^+) \\ +O(\|\varrho u^0\|^2_{s+(m-1)a-1}+\|u\|^2_{-\infty}) \\ \le \text{l.c.}\|z^m\Psi^{s+ma}\varrho u^+\|^2+\text{s.c.}\|z^{m-2}\Psi^{s+(m-2)a}\varrho u^+\|^2 \\ +C(\|\varrho u^0\|^2_{s+(m-1)a-1}+\|u\|^2_{-\infty}).$$

Adding this to the above and absorbing the term multiplied by s.c. in the right-hand side we conclude the proof.

LEMMA 9. *If $0<j<m$ and if $\frac{mA}{j}<B$ then for any* s.c. *and any N there exists C_N such that*

$$\|z^j\Psi^{s+A}\varrho u^+\|^2\le \text{s.c.}(\|z^m\Psi^{s+B}\varrho u^+\|^2+\|\Psi^s\varrho u^+\|^2) \\ +C(\|\varrho u^0\|^2_{s+B-1}+C_N\|u^+\|^2_{-N}+C(\|\varrho u^0\|^2_{s+B-1}+\|u\|^2_{-\infty}),$$

for all $u\in C_0^\infty(U)$.

Proof. With $a=\frac{A}{j}$ we have

$$\|z^j\Psi^{s+A}\varrho u^+\|^2\le \text{l.c.}\|z^m\Psi^{s+ma}\varrho u^+\|^2+s.c\|\Psi^s\varrho u^+\|^2 \\ +C(\|\varrho u^0\|^2_{s+ma-1}+\|u\|^2_{-\infty}).$$

Since $ma=\frac{mA}{j}<B$,

$$\psi^{s+ma}(\xi)\le \text{s.c.}\psi^{s+B}(\xi)+\text{l.c.}(1+|\xi|^2)^{-\frac{N}{2}}.$$

Then

$$\begin{aligned}\|z^m\Psi^{s+ma}\varrho u^+\|^2 &= \|\Psi^{s+ma}z^m\varrho u^+\|^2 + O(\|u^0\|^2_{s+ma-1} + \|u\|^2_{-\infty})\\ &\le \text{s.c.}\|\Psi^{s+B}z^m\varrho u^+\|^2 + C_N\|u^+\|^2_{-N}\\ &\quad + O(\|\varrho u^0\|^2_{s+ma-1} + \|u\|^2_{-\infty})\\ &\le \text{s.c.}\|z^m\Psi^{s+B}\varrho u^+\|^2 + C_N\|u^+\|^2_{-N}\\ &\quad + O(\|\varrho u^0\|^2_{s+B-1} + \|u\|^2_{-\infty}).\end{aligned}$$

Combining with the above we conclude the proof of the lemma.

LEMMA 10. *If* $\sigma = \frac{1}{2k}$ *and if* $1 \le j \le k$ *then*

$$\|z^j\Psi^{s+j\sigma}\varrho u^+\|^2 \le C(\|\varrho' E_k u\|^2_s + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}),$$

for all $u \in C_0^\infty(U)$.

Proof. First note that

$$\|\varrho z^k\Psi^s u^+\|^2 \le C(\|\varrho' E_k u\|^2_{s-\frac{1}{2}} + \|\varrho'\Psi^{s-\frac{1}{2}}u^+\|^2 + \|u\|^2_{-\infty}).$$

Then, replacing s by $s + k\sigma$, since $k\sigma - \frac{1}{2} = 0$, we have

$$\begin{aligned}\|z^j\Psi^{s+j\sigma}\varrho u^+\|^2 &\le C(\|z^k\Psi^{s+k\sigma}\varrho u^+\|^2 + \|\Psi^s\varrho u\|^2\\ &\quad + C\|\varrho' u^0\|^2_{s+(k-1)a-1} + \|u\|^2_{-\infty})\\ &\le C(\|\bar z^k L\Psi^s\varrho u^+\|^2 + \|\bar L\Psi^s\varrho u^+\|^2 + \|\varrho' u^+\|^2_s + \|u\|^2_{-\infty})\\ &\le C(\|\varrho' E_k u\|^2_s + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}).\end{aligned}$$

13. Estimate of $\|\varrho\Psi^{s+\sigma}u^+\|$

LEMMA 11. *There exists a* $C > 0$ *such that*

$$\|\varrho\Psi^{s+\sigma}u^+\| \le C(\|\varrho' E_k u\|^2_{s+\sigma+k-1} + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}),$$

for all $u \in C_0^\infty(U)$.

Proof.

$$\begin{aligned}\|\Psi^{s+\sigma}\varrho u^+\|^2 &= (L(z)\Psi^{s+\sigma}\varrho u^+, \Psi^{s+\sigma}\varrho u^+)\\ &= -(zL\Psi^{s+\sigma}\varrho u^+, \Psi^{s+\sigma}\varrho u^+) - (z\Psi^{s+\sigma}\varrho u^+, \bar L\Psi^{s+\sigma}\varrho u^+)\\ &\le \text{l.c.}\|z\Psi^{s+\sigma}\varrho L u^+\|^2 + C\|\Psi^{s+\sigma}\varrho\bar L u^+\|^2 + \text{“}error\text{”},\end{aligned}$$

where,

$$\text{“error”} \le \text{s.c.}\|\Psi^{s+\sigma}\varrho u^+\|^2 + C(\|z\Psi^{s+\sigma}\varrho' u^+\|^2 + \|\varrho u^0\|^2_{s+\sigma} + \|u\|^2_{-\infty}).$$

In the estimate of the “error” the first term on the right gets absorbed and the other terms are estimated as follows.

$$\|z\Psi^{s+\sigma}\varrho' u^+\|^2 \le C(\|\varrho' E u\|^2_s + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}).$$

The third term, which is microlocalized in the elliptic region, is estimated by

$$\|\varrho u^0\|_{s+\sigma}^2 \le C(\|\varrho Eu\|_{s+\sigma-2}^2 + \|\varrho' u\|_s^2).$$

Hence we get

$$\|\Psi^{s+\sigma}\varrho u^+\|^2 \le C\Big(\|z\Psi^{s+\sigma}\varrho L u^+\|^2 + \|\Psi^{s+\sigma}\varrho \bar{L} u^+\|^2 + \|\varrho Eu\|_s^2 + \|\varrho' u\|_s^2 + \|u\|_{-\infty}^2\Big).$$

From (9) we have

$$\|\Psi^{s+\sigma}\varrho \bar{L} u^+\|^2 \le C(\|\varrho Eu\|_{s+\sigma-\frac{1}{2}}^2 + \|\varrho' u\|_s^2 + \|u\|_{-\infty}^2).$$

So the term that remains to be estimated is $\|z\Psi^{s+\sigma}\varrho L u^+\|^2$, and we have

$$\begin{aligned}\|z\Psi^{s+\sigma}\varrho L u^+\|^2 &= (|z|^2\Psi^{s+\sigma+\frac{1}{2}}\varrho L u^+, \Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+)\\ &\quad + O(\|u^0\|_{s+\sigma-2}^2 + \|u\|_{-\infty}^2)\\ &\le \text{l.c.}\|z^2\Psi^{s+\sigma+\frac{1}{2}}\varrho L u^+\|^2 + \text{s.c.}\|\Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+\|^2\\ &\quad + O(\|u^0\|_{s+\sigma-2}^2 + \|u\|_{-\infty}^2)\end{aligned}$$

and

$$\begin{aligned}\|\Psi^{s+\sigma-\frac{1}{2}}\varrho L u^+\|^2 &= (\Psi^{s+\sigma}\varrho L u^+, \Psi^{s+\sigma-1}\varrho L u^+)\\ &= -(\Psi^{s+\sigma}\varrho u^+, \bar{L}\Psi^{s+\sigma-1}\varrho L u^+) + \mathcal{E}_1\\ &= -(\Psi^{s+\sigma}\varrho u^+, [\bar{L}, \Psi^{s+\sigma-1}\varrho L] u^+)\\ &\quad - (\Psi^{s+\sigma}\varrho u^+, \Psi^{s+\sigma-1}\varrho L\bar{L} u^+) + \mathcal{E}_1\\ &\le C(\|\Psi^{s+\sigma}\varrho u^+\|^2 + \|[\bar{L}, \Psi^{s+\sigma-1}\varrho L] u^+\|^2\\ &\quad + \|\Psi^{s+\sigma-1}\varrho L\bar{L} u^+\|^2 + \mathcal{E}_1.\end{aligned}$$

The second term is estimated as follows

$$\begin{aligned}[\bar{L}, \Psi^{s+\sigma-1}\varrho L] u^+ &= [\bar{L}, \Psi^{s+\sigma-1}]\varrho L u^+ + \Psi^{s+\sigma-1}\bar{L}(\varrho) L u^+\\ &\quad - 2\Psi^{s+\sigma-1}\varrho T u^+ + \Psi^{s+\sigma-1}\varrho L\bar{L} u^+\end{aligned}$$

so that

$$\begin{aligned}\|[\bar{L}, \Psi^{s+\sigma-1}]\varrho L u^+\|^2 &\le C(\|\Psi^{s+\sigma-1}\Gamma^0\varrho u^+\|^2 + \|u\|_{-\infty}^2)\\ &\le C(\|\varrho' u^0\|_{s+\sigma}^2 + \|u\|_{-\infty}^2),\end{aligned}$$

and

$$\|\Psi^{s+\sigma-1}\bar{L}(\varrho) L u^+\|^2 + \|\Psi^{s+\sigma-1}\varrho T u^+\|^2 \le C(\|z\Psi^{s+\sigma}\varrho' u^+\|^2 + \|\Psi^{s+\sigma}\varrho u\|^2) + \mathcal{E}_2.$$

Furthermore we have

$$\|\Psi^{s+\sigma-1}\varrho L\bar{L} u^+\|^2 \le C\|\varrho' E_k u\|_{s+\sigma-\frac{1}{2}}^2 + \mathcal{E}_3.$$

The terms $\mathcal{E}$ are bounded as follows

$$\mathcal{E}_1 \leq C(\|u^0\|^2_{s+\sigma} + \|z\Psi^{s+\sigma}\varrho' u^+\|^2 + \|u\|^2_{-\infty}).$$

By Lemma 10 we get

$$\begin{aligned}\mathcal{E}_1 &\leq C(\|\varrho' E_k u\|^2_s + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}),\\ \mathcal{E}_2 &\leq C(\varrho' u\|^2_{s+\sigma-1} + \mathcal{E}_1) \leq C'\mathcal{E}_1,\end{aligned}$$

and

$$\mathcal{E}_3 \leq C(\|z^{2k-1}\varrho u^+\|^2_s + \|z^{2k-2}\varrho u^+\|^2_{s-\frac{1}{2}} + \mathcal{E}_2) \leq C'\mathcal{E}_2.$$

Hence we have

$$\begin{aligned}&\|\Psi^{s+\sigma}\varrho u^+\|^2 + \|z\Psi^{s+\sigma}\varrho L u^+\|^2\\ &\qquad \leq C(\|z^2\Psi^{s+\sigma+\frac{1}{2}}\varrho L u^+\|^2 + \|\varrho' E_k u\|^2_s + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}).\end{aligned}$$

To estimate the first term on the right we will use Lemma 8 as follows.

$$\|z^2\Psi^{s+\sigma+\frac{1}{2}}\varrho L u^+\|^2 \leq C(\|z\Psi^{s+\sigma+\frac{1}{2}}\varrho z L u^+\|^2 + \|\varrho u^0\|^2_{s+\sigma} + \|u\|^2_{-\infty}).$$

We apply Lemma 8 with $a = \frac{1}{2}$, $m = k-1$, s replaced by $s+\sigma$, and u replaced by zLu to obtain

$$\begin{aligned}\|z^2\Psi^{s+\sigma+\frac{1}{2}}\varrho L u^+\|^2 &\leq \text{l.c.}\|z^{k-1}\Psi^{s+\sigma+\frac{k-1}{2}}\varrho z L u^+\|^2 + \text{s.c.}\|z\Psi^{s+\sigma}\varrho L u^+\|^2\\ &\qquad + \|\varrho u^0\|^2_{s+\sigma+\frac{k-1}{2}} + \|u\|^2_{-\infty})\\ &\leq \text{l.c.}\|z^k\Psi^{s+\sigma+\frac{k-1}{2}}\varrho L u^+\|^2 + \text{s.c.}\|z\Psi^{s+\sigma}\varrho L u^+\|^2\\ &\qquad + C(\|\varrho u^0\|^2_{s+\sigma+\frac{k-1}{2}} + \|u\|^2_{-\infty}).\end{aligned}$$

Therefore we have

$$\begin{aligned}&\|\Psi^{s+\sigma}\varrho u^+\|^2\\ &\quad \leq C(\|z^k\Psi^{s+\sigma+\frac{k-1}{2}}\varrho L u^+\|^2 + \|\varrho' E_k u\|^2_{s+\sigma+\frac{k-1}{2}-2} + \|\varrho' u\|^2_s + \|u\|^2_{-\infty})\\ &\quad \leq C(\|\Psi^{s+\sigma+\frac{k-1}{2}}\varrho \bar{z}^k L u^+\|^2 + \|\varrho' E_k u\|^2_{s+\sigma+\frac{k-1}{2}-2} + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}).\end{aligned}$$

Next, from Lemma 8 with $m = k$, $a = \frac{1}{2}$ and s replaced by $s+\sigma$, we have

$$\begin{aligned}\|z\Psi^{s+\sigma+\frac{1}{2}}u\|^2 &\leq \text{l.c.}\|z^k\Psi^{s+\sigma+\frac{k}{2}}\varrho u^+\|^2\\ &\qquad + \text{s.c.}\|\Psi^{s+\sigma}\varrho u^+\|^2 + C(\|\varrho' u^0\|^2_{s+\sigma+\frac{k-1}{2}-1} + \|u\|^2_{-\infty})\\ &\leq C\|\Psi^{s+\sigma+\frac{k-1}{2}}\varrho \bar{z}^k L u^+\|^2 + \mathcal{E}_1\end{aligned}$$

and

$$\begin{aligned}\|z^k\Psi^{s+\sigma+\frac{k}{2}}\varrho u^+\|^2 &\le C\|\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{z}^k L u^+\|^2+\mathcal{E}_1\\ &= -C(\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{L}|z|^{2k}Lu^+,\Psi^{s+\sigma+\frac{k-1}{2}}\varrho u^+)\\ &\quad -2C(\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{z}^kLu^+,\Psi^{s+\sigma+\frac{k-1}{2}}\bar{z}^{k+1}\mu u^+)+\mathcal{E}_2\end{aligned}$$

since

$$\begin{aligned}|(\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{z}^kLu^+,\Psi^{s+\sigma+\frac{k-1}{2}}\bar{z}^{k+1}\mu u^+)| &\le \text{s.c.}\|z^k\Psi^{s+\sigma+\frac{k}{2}}\varrho Lu^+\|^2\\ &\quad +\text{l.c.}\|z^{k+1}\Psi^{s+\sigma+\frac{k-2}{2}}\varrho' u^+\|^2+\mathcal{E}_2.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\|z^k\Psi^{s+\sigma+\frac{k}{2}}\varrho u^+\|^2 &\le C\|\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{z}^kLu^+\|^2+\mathcal{E}_2\\ &\le C|(\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{L}|z|^{2k}Lu^+,\Psi^{s+\sigma+\frac{k-1}{2}}\varrho u^+)|+\mathcal{E}_3\\ &\le C|(\Psi^{s+\sigma+\frac{k-1}{2}}\varrho E_k u^+,\Psi^{s+\sigma+\frac{k-1}{2}}\varrho u^+)|\\ &\quad +|(\Psi^{s+\sigma+\frac{k-1}{2}}\varrho L\bar{L}u^+,\Psi^{s+\sigma+\frac{k-1}{2}}\varrho u^+)|+\mathcal{E}_3\\ &\le C(\|\Psi^{s+\sigma+k-1}\varrho E_k u\|^2\\ &\quad +|(\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{L}u^+,\Psi^{s+\sigma+\frac{k-1}{2}}\bar{z}\mu u^+)|)+\mathcal{E}_4\\ &\le C(\|\Psi^{s+\sigma+k-1}\varrho E_k u\|^2+\|\Psi^{s+\sigma+k-1}\varrho\bar{L}u^+\|^2)+\mathcal{E}_5\\ &\le C(\|\varrho E_k u\|^2_{s+\sigma+k-1}+\|z^{2k-1}\Psi^{s+\sigma+k-1}\varrho' u^+\|^2\\ &\quad +\|z^{2k-2}\Psi^{s+\sigma+k-\frac{3}{2}}\varrho' u^+\|^2)+\mathcal{E}_6.\end{aligned}$$

Thus, applying Lemma 9 with $m=2k-1$, $j=2k-2$, $A=k-\frac{3}{2}$, $B=k-1$, and s replaced by $s+\sigma$, we have

$$\frac{mA}{j}=\frac{2k-1}{2k-2}(k-\frac{3}{2})<k-1=B.$$

Now,

$$\|z^{2k-2}\Psi^{s+\sigma+k-\frac{3}{2}}\varrho' u^+\|^2\le \text{s.c.}\|z^{2k-1}\Psi^{s+\sigma+k-1}\varrho' u^+\|^2+\mathcal{E}_7.$$

Replacing ϱu^+ by $\bar{z}^{k-1}\varrho' u^+$ and s by $s+\frac{k-2}{2}$ we obtain

$$\begin{aligned}&\|z^{2k-1}\Psi^{s+\sigma+k-1}\varrho' u^+\|^2\\ &\quad\le C(\|\Psi^{s+\sigma+k-\frac{3}{2}}\varrho'\bar{z}^{2k-1}Lu^+\|^2+\mathcal{E}_8\\ &\quad\le C(|(\Psi^{s+\sigma+k-\frac{3}{2}}\bar{L}\varrho'|z|^{2k}Lu^+,|z|^{2k-2}\Psi^{s+\sigma+k-\frac{3}{2}}\varrho' u)|\\ &\qquad +\text{s.c.}\|\Psi^{s+\sigma+k-1}\varrho'\bar{z}^{2k-1}Lu^+\|^2+\text{l.c.}\|z^{2k-1}\Psi^{s+\sigma+k-2}\varrho'' u\|^2)+\mathcal{E}_8\\ &\quad\le C(\|\varrho' E_k u\|^2_{s+\sigma+2k-3}+|(\Psi^{s+\sigma+k-\frac{3}{2}}\varrho' L\bar{L}u^+,|z|^{2k-2}\Psi^{s+\sigma+k-\frac{3}{2}}\varrho' u)|)+\mathcal{E}_9\\ &\quad\le C(\|\varrho' E_k u\|^2_{s+\sigma+2k-3}+|(\Psi^{s+\sigma+k-2}\varrho''\bar{L}u^+,z|z|^{2k-2}\Psi^{s+\sigma+k-1}\varrho' u)|\\ &\qquad +|(\Psi^{s+\sigma+k-1}\varrho'\bar{L}u^+,z^{k-1}\bar{z}^{k-2}\Psi^{s+\sigma+k-2}\varrho' u)|)+\mathcal{E}_9\end{aligned}$$

$$\leq C(\|\varrho' E_k u\|^2_{s+\sigma+2k-3} + \text{l.c.}\|\Psi^{s+\sigma+k-2}\varrho''\bar{L}u^+\|^2 + \text{s.c.}\|z^{2k-1}\Psi^{s+\sigma+k-1}\varrho' u\|^2 + \text{s.c.}\|\Psi^{s+\sigma+k-1}\varrho'\bar{L}u^+\|^2 + \text{l.c.}\|z^{2k-3}\Psi^{s+\sigma+k-2}\varrho' u^+\|^2) + \mathcal{E}_9.$$

Now applying Lemma 9 as above but with $j = 2k-3$ and $A = k-2$, we get

$$\frac{mA}{j} = \frac{2k-1}{2k-3}(k-2) < k-1 = B.$$

Hence

$$\|z^{2k-3}\Psi^{s+\sigma+k-2}\varrho' u^+\|^2 \leq \text{s.c.}\|z^{2k-1}\Psi^{s+\sigma+k-1}\varrho' u^+\|^2 + \mathcal{E}_9.$$

Combining the above we obtain

$$\|\Psi^{s+\sigma}\varrho u^+\|^2 \leq C\|\varrho' E_k u\|^2_{s+\sigma+k-1} + \mathcal{E}_{10}.$$

To complete the proof of the *a priori* estimate we will analyze the error terms:

$$\mathcal{E}_1 \sim \|z^k\Psi^{s+\sigma+\frac{k-1}{2}}\varrho u^+\|^2 + \|\varrho' u^0\|^2_{s+\sigma+\frac{k-1}{2}} + \text{s.c.}\|\Psi^{s+\sigma}\varrho u^+\|^2 + \|u\|^2_{-\infty},$$

$$\mathcal{E}_2 \sim \mathcal{E}_1 + \|z^{k+1}\Psi^{s+\sigma+\frac{k-1}{2}}\varrho' u^+\|^2,$$

$$\mathcal{E}_3 \sim \mathcal{E}_2 + \|\varrho' u^0\|^2_{s+\sigma+\frac{k}{2}} + \text{s.c.}\|\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{z}^k L u^+\|^2,$$

$$\mathcal{E}_4 \sim \mathcal{E}_3 + \|\Psi^{s+\sigma+\frac{k-1}{2}}\varrho\bar{L}u^+\|^2,$$

$$\mathcal{E}_5 \sim \mathcal{E}_4 + \|z\Psi^{s+\sigma}\varrho' u^+\|^2$$

$$\mathcal{E}_6 \sim \mathcal{E}_5 + \|z^{2k-1}\Psi^{s+\sigma+k-\frac{3}{2}}\varrho' u^+\|^2 + \|z^{2k-2}\Psi^{s+\sigma+k-2}\varrho' u^+\|^2$$

$$\mathcal{E}_7 \sim \mathcal{E}_6 + \|\varrho u^0\|^2_{s+k-2} + \|u^+\|^2_{-N}$$

$$\mathcal{E}_8 \sim \mathcal{E}_7 + \|\Psi^{s+\sigma+k-\frac{3}{2}}\varrho'\bar{z}^{2k}Lu^+\|^2$$

$$\mathcal{E}_9 \sim \mathcal{E}_8 + \text{s.c.}\|\Psi^{s+\sigma+k-1}\varrho\bar{z}^{2k-1}Lu^+\|^2 + \|z^{2k-1}\Psi^{s+\sigma+k-2}\varrho' u\|^2,$$

and

$$\mathcal{E}_{10} \sim \mathcal{E}_9 + \text{s.c.}\|\Psi^{s+\sigma+k-1}\varrho\bar{L}u^+\|^2 + \|\Psi^{s+\sigma+k-2}\varrho'\bar{L}u^+\|^2.$$

The "admissible" errors are $\|\varrho' u\|^2_s + \|u\|^2_{-\infty}$. The terms involving u^0 are all bounded by const.$\|\varrho' E_k u\|^2_{s+\sigma+k-2}$ modulo admissible errors. The terms involving a small constant s.c. are absorbed in the left. The term $\|z\Psi^{s+\sigma}\varrho' u^+\|$ is bounded by const.$\|\varrho' E_k u\|^2_s$, and the remaining terms can be bounded by a constant times $\mathcal{A}(s, \varrho')$, where $\mathcal{A}(s, \varrho')$ is defined by

$$\mathcal{A}(s,\varrho') = \|z^k\Psi^{s+\sigma+\frac{k-1}{2}}\varrho' u^+\|^2 + \|z^{2k-1}\Psi^{s+\sigma+k-\frac{3}{2}}\varrho' u^+\|^2 + \|\Psi^{s+\sigma+k-2}\varrho'\bar{L}u^+\|^2.$$

Repeating the same estimates with s replaced by $s - \frac{1}{2}$ we replace the error $\mathcal{A}(s, \varrho')$ by $\mathcal{A}(s - \frac{1}{2}, \varrho'')$. Repeating this process $2k-2$ times (and redefining ϱ') we obtain the desired *a priori* estimate, namely:

$$\|\Psi^{s+\sigma}\varrho u^+\|^2 \leq C(\|\varrho' E_k u\|^2_{s+\sigma+k-1} + \|\varrho' u\|^2_s + \|u\|^2_{-\infty}). \tag{10}$$

14. Smoothing

To conclude the proof of Theorem C we will apply the above estimate to the smoothing of a solution. Given a distribution solution u of $E_k u = f$ with f whose restriction to U is in $C^\infty(U)$, we wish to show that the restriction of u to U is in C^∞. Without loss of generality we assume that the distribution u has compact support and lies in $H^{-s_0}(\mathbb{R}^3)$. For $\delta > 0$ we will define a smoothing operator K_δ such that $K_\delta u \in C^\infty$ and $\lim_{\delta\to 0} K_\delta(\varrho u^+) \sim \varrho u^+$.

Definition 7. Let $\omega \in C_0^\infty(\mathbb{R})$, with $\omega(0) = 1$ and let $\kappa_\delta(\xi) = \omega(\delta\xi_3)\gamma^+(\xi)$ and

$$\widehat{K_\delta u}(\xi) = \kappa_\delta(\xi)\hat{u}(\xi),$$

where $\gamma^+(\xi) = 1$ in a neighborhood of the support of $\hat{u}^+$.

LEMMA 12. *If $\|K_\delta(\varrho u^+)\|_s \le C$ and if $\varrho' u^0 \in H^s$ then $\varrho u^+ \in H^s$.*

Proof: We have

$$\|K_\delta(\varrho u^+) - \varrho u^+\|_s \le \|K_\delta((\varrho u)^+) - (\varrho u)^+)\|_s + C\|\varrho' u^0\|_s$$

and

$$\lim_{\delta\to 0}(1+|\xi|^2)^{\frac{s}{2}}\omega(\delta\xi_3)\widehat{(\varrho u)^+}(\xi)) = (1+|\xi|^2)^{\frac{s}{2}}\widehat{(\varrho u)^+}(\xi)).$$

Then $(\varrho u)^+ \in H^s$ and since $\widehat{(\varrho u)^+} - \widehat{\varrho u^+}$ is supported in the elliptic region $\mathfrak{U}^0$ we have

$$\|\varrho u^+\|_s \le \|(\varrho u)^+\|_s + C\|\varrho' u^0\|_s,$$

thus concluding the proof.

LEMMA 13. *For $\delta > 0$, K_δ is a pseudodifferential operator of order $-\infty$ which is of order zero uniformly in δ. K_δ has the following commutation properties.*

1. $[E, K_\delta](I - \Gamma^0)$ *is a pseudodifferential operator of order $-\infty$ uniformly in δ.*
2. *If R^s is a pseudodifferential operator of order s then*

$$[R^s, K_\delta] = \Gamma^0 R_\delta^{s-1} + \Psi^{s-1} R_\delta^0 + R_\delta^{-\infty},$$

where R_δ^{s-1}, R_δ^0, and $R_\delta^{-\infty}$ are pseudodifferential operators of orders $-\infty$ for $\delta > 0$ and of orders $s-1$ and 0 uniformly in δ.

Proof. Number 1 follows from the fact that when $|\xi| \ge 1$ then $\gamma^0(\xi) = 1$ on the support of these symbols. To deal with number 2 we write the principal symbol of $[R^s, K_\delta]$. Setting $x_1 = x,\ x_2 = y$ and $x_3 = t$, we have

$$\sum_j \frac{\partial\kappa_\delta}{\partial\xi_j}\frac{\partial r^s}{\partial x_j} = \delta\omega'(\delta\xi_3)\tilde{\gamma}^+\frac{\partial r^s}{\partial x_3} + \sum_j \omega(\delta\xi_3)\frac{\partial\tilde{\gamma}^+}{\partial\xi_j}\frac{\partial r^s}{\partial x_j}.$$

The lemma then follows, since

$$\delta\omega'(\delta\xi_3)\tilde{\gamma}^+\frac{\partial r^s}{\partial x_3} = \xi_3^{s-1}\gamma^+\left\{\tilde{\gamma}^+\xi_3^{-s}\delta\xi_3\omega'(\delta\xi_3)\frac{\partial r^s}{\partial x_3}\right\},$$

where $\tilde{\gamma}^+ = 1$ in a neighborhood of the support of γ^+ and equals zero in a neighborhood of the origin. The expression in braces is the symbol of an operator of order zero uniformly in δ.

Conclusion of proof of Theorem C. Substituting $K_\delta u$ for u in (10) we obtain

$$\|\Psi^{s+\sigma}\varrho K_\delta u^+\|^2 \le C(\|\varrho' E_k K_\delta u\|^2_{s+\sigma+k-1} + \|\varrho' K_\delta u\|^2_s + \|K_\delta u\|^2_{-\infty}).$$

Then we have

$$\begin{aligned}\|K_\delta(\varrho u^+)\|^2_{s+\sigma} &\le C(\|\Psi^{s+\sigma}\varrho K_\delta u^+\|^2 + \|\varrho' u\|^2_{s+\sigma-1}),\\ \|[\varrho' E_k, K_\delta]u\|^2_{s+\sigma+k-1} &\le C(\|\varrho'' u^0\|^2_{s+\sigma+k-1} + \|u\|^2_{-\infty}),\\ \|\varrho' K_\delta u\|^2_s &\le C\|\varrho' u\|^2_s,\end{aligned}$$

and

$$\|K_\delta u\|^2_{-\infty} \le C\|u\|^2_{-\infty}.$$

Further

$$\|\varrho'' u^0\|^2_{s+\sigma+k-1} \le C(\|\varrho''' E_k u\|^2_{s+\sigma+k-3} + \|u\|^2_{-\infty}).$$

Therefore, changing notation for the cutoff functions, we get

$$\|K_\delta(\varrho u^+)\|^2_{s+\sigma} \le C(\|\varrho' E_k u\|^2_{s+\sigma+k-1} + \|\varrho' u\|^2_s + \|u\|^2_{-s_0}).$$

Therefore, if $u \in H^{-s_0}$, if $u^+ \in H^s_{\mathrm{loc}}(U)$, and if $E_k u \in H^{s+\sigma+k-1}_{\mathrm{loc}}(U)$ then $u^+ \in H^{s+\sigma}_{\mathrm{loc}}(U)$. It then follows that if $u \in H^{-s_0}$ and if $E_k u \in H^{s_1}_{\mathrm{loc}}(U)$ then $u^+ \in H^{s_1-k+1}_{\mathrm{loc}}(U)$. Since, under the same assumptions, we have $u^0 \in H^{s_1+2}_{\mathrm{loc}}(U)$ and $u^- \in H^{s_1+1}_{\mathrm{loc}}(U)$ we conclude that $u \in H^{s_1-k+1}_{\mathrm{loc}}(U)$, thus proving Theorem C.

15. Local existence in L^2

The *a priori* estimates for E_k imply the following local existence result.

THEOREM. *If $P \in U \subset \mathbb{R}^3$ with U an open set, then there exists a neighborhood $U_1 \subset \bar{U}_1 \subset U$, with $P \in U_1$, such that if $f \in H^{k-1}_{\mathrm{loc}}(U)$ then there exists $u \in L^2(U_1)$ and $E_k u = f$ in U_1.*

Proof. Let U_1 be a small neighborhood of P. In Lemma 11 set $\varrho = 1$ in a neighborhood of $\bar{U}_1$ and set $u = v \in C_0^\infty(U_1)$ so that $\varrho u = v$ and $[\Psi^{s+\sigma}, \Gamma^+]$ is an operator of order $-\infty$ on $C_0^\infty(U_1)$. Hence we obtain

$$\|\Psi^{s+\sigma} v^+\|^2 \le C(\|E_k v\|^2_{s+\sigma+k} + \|v\|^2_s),$$

for all $v \in C_0^\infty(U_1)$. Setting $s + \sigma + k = 0$ and combinig with the estimates for v^0 and v^-, we obtain

$$\|v\|^2_{-k+1} \leq C(\|E_k v\|^2 + \|v\|^2_{-k+1-\sigma}).$$

Then, if the diameter of U_1 is sufficiently small, we have

$$\|v\|^2_{-k+1-\sigma} \leq \text{small const. } \|v\|^2_{-k+1}.$$

Hence

$$\|v\|_{-k+1} \leq \text{const. } \|E_k v\|,$$

for all $v \in C_0^\infty(U_1)$.

Let $\mathcal{W} = C_0^\infty(U_1)$ and let $K : \mathcal{W} \to \mathbb{C}$ be the linear functional defined by $Kw = (v, f)$ with $w = E_k v$. Then

$$|Kw| = |(v, f)| \leq \|v\|_{-k+1}\|f\|_{k-1} \leq C\|w\|.$$

So K is bounded on $\mathcal{W}$; hence it can be extended to a bounded linear functional on $L^2(U_1)$. Therefore there exists $u \in L^2(U_1)$ such that $Kw = (w, u)$, that is $(v, f) = (E_k v, u) = (v, E_k u)$. Thus $E_k u = f$ in $L^2(U_1)$, which completes the proof.

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Appendix: Analyticity and loss of derivatives

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Abstract

In [2], J. J. Kohn proves C^∞ hypoellipticity for a sum of squares of complex vector fields which exhibit a large loss of derivatives. Here, we prove analytic hypoellipticity for this operator.

1. Introduction and outline

In [2], J. J. Kohn proves hypoellipticity for the operator

$$P = LL^* + (\overline{z}^k L)^*(\overline{z}^k L), \qquad L = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t},$$

for which there is a large loss of derivatives — indeed in the *a priori* estimate one bounds only the Sobolev norm of order $-(k-1)/2$, and thus there is a loss of $k-1$ derivatives: $Pu \in H^s_{\mathrm{loc}} \implies u \in H^{s-(k-1)}_{\mathrm{loc}}$.

We show in this note that solutions of $Pu = f$ with f real analytic are themselves real analytic in any open set where f is. In so doing we use an *a priori* estimate which follows easily from that established by Kohn for this operator, namely for test functions v of small support near the origin:

$$\|\overline{L}v\|_0^2 + \|\overline{z}^k Lv\|_0^2 + \|v\|^2_{-\frac{k-1}{2}} \lesssim |(Pv, v)_{L^2}|. \tag{1.1}$$

In fact, in [5] (see also [1]), we give a rapid and direct derivation of (1.1) for this operator and similar estimates for more degenerate operators.

The first two terms on the left of this estimate exhibit maximal control in $\overline{L}$ and $\overline{z}^k L$, but only these complex directions. Hence in obtaining recursive bounds for derivatives it is essential to keep one of these vector fields available for as long as possible. For this, we will construct a carefully balanced localization of high powers of $T = -2i\partial/\partial t$ and use the estimate repeatedly, reducing the order of powers of T but accumulating derivatives on the localizing functions. These Ehrenpreis type localizing functions work 'as if analytic' up to a prescribed order, with all constants independent of that order, as in [3], [4], but eventually the good derivatives ($\overline{L}$ or $\overline{z}^k L$) are lost and we must use the third term on the left of the estimate, absorb the loss of $\frac{k-1}{2}$ derivatives, introduce a new localizing function of larger support and start the whole process again, but with only a (fixed) fraction of the original power of T.

2. Observations and simplifications

Our first observation is that we know the analyticity of the solution for z different from 0 from the earlier work of the second author [3], [4] and Trèves [6]. Thus, modulo brackets with localizing functions whose derivatives are supported in the known analytic hypoelliptic region, we take all localizing functions independent of z.

Our second observation is that it suffices to bound derivatives measured in terms of high powers of the vector fields L and $\overline{L}$ in L^2 norm, by standard arguments, and indeed estimating high powers of L can be reduced to bounding high powers of $\overline{L}$ and powers of T of half the order, by repeated integration by parts. Thus our overall scheme will be to start with high powers (order $2p$) of

L or $\overline{L}$, use integration by parts and the *a priori* estimate repeatedly to reduce to treating $T^p u$ in a slightly larger set.

And to do this, we introduce a new special localization of T^p adapted to this problem.

3. The localization of high powers of T

The new localization of T^p may be written in the form:

$$(T^{p_1,p_2})_\varphi = \sum_{\substack{a\le p_1\\ b\le p_2}} \frac{L^a\circ z^a\circ T^{p_1-a}\circ\varphi^{(a+b)}\circ T^{p_2-b}\circ\overline{z}^b\circ\overline{L}^b}{a!b!}.$$

Here by $\varphi^{(r)}$ we mean $(-i\partial/\partial t)^r\varphi(t)$ since near $z=0$ we have seen that we may take the localizing function independent of z. Note that the leading term (with $a+b=0$) is merely $T^{p_1}\varphi T^{p_2}$ which equals $T^{p_1+p_2}$ on the initial open set Ω_0 where $\varphi\equiv 1$.

We have the commutation relations:

$$[L,(T^{p_1,p_2})_\varphi]\equiv L\circ(T^{p_1-1,p_2})_{\varphi'},$$
$$[\overline{L},(T^{p_1,p_2})_\varphi]\equiv(T^{p_1,p_2-1})_{\varphi'}\circ\overline{L},$$
$$[(T^{p_1,p_2})_\varphi,z]=(T^{p_1-1,p_2})_{\varphi'}\circ z,$$

and

$$[(T^{p_1,p_2})_\varphi,\overline{z}]=\overline{z}\circ(T^{p_1,p_2-1})_{\varphi'},$$

where the $\equiv$ denotes modulo $C^{p_1-p_1'+p_2-p_2'}$ terms of the form

$$\frac{L^{p_1-p_1'}\circ z^{p_1-p_1'}\circ T^{p_1'}\circ\varphi^{(p_1-p_1'+p_2-p_2'+1)}\circ T^{p_2'}\circ\overline{z}^{p_2-p_2'}\circ\overline{L}^{p_2-p_2'}}{(p_1-p_1')!(p_2-p_2')!} \tag{3.2}$$

with either $p_1'=0$ or $p_2'=0$, i.e., terms where all free T derivatives have been eliminated on one side of φ or the other. Thus if we start with $p_1=p_2=p/2$, and iteratively apply these commutation relations, the number of T derivatives not necessarily applied to φ is eventually at most $p/2$.

4. The recursion

We insert first $v=(T^{\frac{p}{2},\frac{p}{2}})_\varphi u$ in the *a priori* inequality, then bring $(T^{\frac{p}{2},\frac{p}{2}})_\varphi$ to the left of $P=-L\overline{L}-\overline{L}z^k\overline{z}^kL$ since Pu is known and analytic. We have, omitting for now the 'subelliptic' term,

$$\|\overline{L}(T^{\frac{p}{2},\frac{p}{2}})_\varphi u\|_0^2+\|\overline{z}^kL(T^{\frac{p}{2},\frac{p}{2}})_\varphi u\|_0^2\lesssim|(P(T^{\frac{p}{2},\frac{p}{2}})_\varphi u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)_{L^2}|$$
$$\lesssim|((T^{\frac{p}{2},\frac{p}{2}})_\varphi Pu,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)_{L^2}|+|(([P,(T^{\frac{p}{2},\frac{p}{2}})_\varphi]u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)_{L^2}|$$

and, by the above bracket relations,

$$\begin{aligned}
&([P,(T^{\frac{p}{2},\frac{p}{2}})_\varphi]u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)\\
&\quad= -([L\overline{L},(T^{\frac{p}{2},\frac{p}{2}})_\varphi]u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u) - ([\overline{L}z^k\overline{z}^kL,(T^{\frac{p}{2},\frac{p}{2}})_\varphi]u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)\\
&\quad\equiv -(L(T^{\frac{p}{2},\frac{p}{2}-1})_{\varphi'}\overline{L}u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u) - (L(T^{\frac{p}{2}-1,\frac{p}{2}})_{\varphi'}\overline{L}u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)\\
&\qquad -((T^{\frac{p}{2}-1,\frac{p}{2}})_{\varphi'}\overline{L}z^k\overline{z}^kLu,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)\\
&\qquad -\sum_{k'=1}^{k}(\overline{L}z^{k'}(T^{\frac{p}{2},\frac{p}{2}-1})_{\varphi'}z^{k-k'}\overline{z}^kLu,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)\\
&\qquad -\sum_{k'=0}^{k-1}(\overline{L}z^k\overline{z}^{k'}(T^{\frac{p}{2}-1,\frac{p}{2}})_{\varphi'}\overline{z}^{k-k'}Lu,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u)\\
&\qquad -(\overline{L}z^k\overline{z}^kL(T^{\frac{p}{2},\frac{p}{2}-1})_{\varphi'}u,(T^{\frac{p}{2},\frac{p}{2}})_\varphi u),
\end{aligned}$$

with the same meaning for $\equiv$ as above. In every term, no powers of z or $\overline{z}$ have been lost, though some may need to be brought to the left of the $(T^{q_1,q_2})_{\tilde{\varphi}}$ with again no loss of powers of z or $\overline{z}$ and a further reduction in order, every bracket reduces the order of the sum of the two indices p_1 and p_2 by one (here we started with $p_1 = p_2 = p/2$), picks up one derivative on φ, and leave the vector fields over which we have maximal control in the estimate intact and in the correct order. Thus we may bring either $\overline{L}z^k$ or L to the right as $\overline{z}^kL$ or $\overline{L}$, and use a weighted Schwarz inequality on the result to take maximal advantage of the *a priori* inequality. Iterations of all of this continue until there remain at most $p/2$ free T derivatives (i.e., the T derivatives on at least one side of φ are all 'corrected' by good vector fields) and perhaps as many as $p/2$ L or $\overline{L}$ derivatives, and we may continue further until, at worst, these remaining L or $\overline{L}$ derivatives bracket two at a time to produce more T's, with corresponding combinatorial factors. After all of this, there will be at most $T^{\frac{3p}{4}}$ remaining, and a factor of $\frac{p}{2}!! \sim \frac{p}{4}!$

It is here that the final term on the left of the *a priori* inequality is used, in order to bring the localizing function out of the norm after creating another balanced localization of $T^{3p/4}$ with a new localizing function of Ehrenpreis type with slightly larger support, geared, roughly, to $3p/4$ instead of to p.

Recall that such such localizing functions ψ may be constructed for any N and satisfy

$$\left|\psi^{(r)}\right| \leq \left(\frac{C}{e}\right)^{r+1} N^r, \quad r \leq 2N$$

where C is independent of N and $e = \mathrm{dist}(\{\psi \equiv 1\}, (\mathrm{supp}\,\psi)^c)$.

5. Conclusion of the proof

Finally, this entire process, which reduced the order from p to at most $3p/4$, (or more precisely to at most $3p/4 + (k-1)/2$), is repeated, over and over, each time essentially reducing the order by a factor of 3/4. After at most $\log_{4/3} p$ such iterations we are reduced to a bounded number of derivatives, and, as in [3] and [4], all of these nested open sets may be chosen to fit in the one open set Ω' where Pu is known to be analytic, and all constants chosen independent of p (but depending on Pu). The fact that in those works one full iteration reduced the order by half played no essential role — a factor of 3/4 works just as well.

To be precise, the sequence of open sets, $\{\Omega_j\}$, each compactly contained in the next, with $\Omega_{\log_{4/3} p} = \Omega'$, have separations $d_j = \text{dist}(\Omega_j, \Omega_{j+1}^c)$, with $\sum d_j = \text{dist}(\Omega_0, \Omega'^c) = d$, which need to be picked carefully. The localizing functions $\{\varphi_j\}$ with $\varphi_j \in C_0^\infty(\Omega_{j+1}) \equiv 1$ on Ω_j satisfy

$$\left|\varphi_j^{(r)}\right| \leq (C/d_j)^{r+1}((3/4)^j p)^r, \qquad r \leq 2(3/4)^j p. \tag{5.3}$$

We shall take the $d_j = \frac{1}{(j+1)^2}/d \sum \frac{1}{(j+1)^2}$.

Now at most $(3/4)^j p$ derivatives will fall on φ_j, and most of the effect of the derivatives will be balanced by corresponding factorials in the denominator, as in (3.2), roughly the powers of $(3/4)^j p$ in (5.3) in view of Stirling's formula. In addition, as noted immediately before the last paragraph in Section 4, there will be factorials corresponding to the diminution of powers of T. What will *not* be balanced are the powers of d_j^{-1}, but the product of these factors will contribute

$$\Pi_{j=1}^{\log_{4/3} p} \left(j^2\right)^{(3/4)^j p} = \left(\Pi_{j=1}^{\log_{4/3} p} j^{(3/4)^j}\right)^{2p} = C^p,$$

which, together with the factorials just mentioned, proves the analyticity of the solution in Ω_0.

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