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The diameter of the isomorphism class of a Banach space

By W. B. Johnson and E. Odell*

Dedicated to the memory of V. I. Gurarii

Abstract

We prove that if X is a separable infinite dimensional Banach space then its isomorphism class has infinite diameter with respect to the Banach-Mazur distance. One step in the proof is to show that if X is elastic then X contains an isomorph of c_0. We call X elastic if for some $K < \infty$ for every Banach space Y which embeds into X, the space Y is K-isomorphic to a subspace of X. We also prove that if X is a separable Banach space such that for some $K < \infty$ every isomorph of X is K-elastic then X is finite dimensional.

1. Introduction

Given a Banach space X, let $D(X)$ be the diameter in the Banach-Mazur distance of the class of all Banach spaces which are isomorphic to X; that is,

$$D(X) = \sup\{d(X_1, X_2) : X_1, X_2 \text{ are isomorphic to } X\}$$

where $d(X_1, X_2)$ is the infimum over all isomorphisms T from X_1 onto X_2 of $\|T\| \cdot \|T^{-1}\|$. It is well known that if X is finite (say, N) dimensional, then $cN \leq D(X) \leq N$ for some positive constant c which is independent of N. The upper bound is an immediate consequence of the classical result (see e.g. [T-J, p. 54]) that $d(Y, \ell_2^N) \leq \sqrt{N}$ for every N dimensional space Y. The lower bound is due to Gluskin [G], [T-J, p. 283].

It is natural to conjecture that $D(X)$ must be infinite when X is infinite dimensional, but this problem remains open. As far as we know, this problem was first raised in print in the 1976 book of J. J. Schäffer [S, p. 99]. The problem was recently brought to the attention of the authors by V. I. Gurarii, who

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checked that every infinite dimensional super-reflexive space as well as each of the common classical Banach spaces has an isomorphism class whose diameter is infinite. To see these cases, note that if X is infinite dimensional and E is any finite dimensional space, then it is clear that X is isomorphic to $E \oplus_2 X_0$ for some space X_0. Therefore, if $D(X)$ is finite, then X is finitely complementably universal; that is, there is a constant C so that every finite dimensional space is C-isomorphic to a C-complemented subspace of X. This implies that X cannot have nontrivial type or nontrivial cotype or local unconditional structure or numerous other structures. In particular, X cannot be any of the classical spaces or be super-reflexive.

In his unpublished 1968 thesis [Mc], McGuigan conjectured that $D(X)$ must be larger than one when $\dim X > 1$. Schäffer [S, p. 99] derived that $D(X) \geq 6/5$ when $\dim X > 1$ as a consequence of other geometrical results contained in [S], but one can prove directly that $D(X) \geq \sqrt{2}$. Indeed, it is clearly enough to get an appropriate lower bound on the Banach-Mazur distance between $X_1 := Y \oplus_1 \mathbb{R}$ and $X_2 := Y \oplus_2 \mathbb{R}$ when Y is a nonzero Banach space. Now X_1 has a one dimensional subspace for which every two dimensional superspace is isometric to ℓ_1^2. On the other hand, every one dimensional subspace of X_2 is contained in a two dimensional superspace which is isometric to ℓ_2^2. It follows that $d(X_1, X_2) \geq d(\ell_1^2, \ell_2^2) = \sqrt{2}$.

The Main Theorem in this paper is a solution to Schäffer's problem for separable Banach spaces:

Main Theorem. *If X is a separable infinite dimensional Banach space, then $D(X) = \infty$.*

Part of the work for proving the Main Theorem involves showing that if X is separable and $D(X) < \infty$, then X contains an isomorph of c_0. This proof is inherently not local in nature, and, strangely enough, local considerations, such as those mentioned earlier which yield partial results, play no role in our proof. We do not see how to prove that a nonseparable space X for which $D(X) < \infty$ must contain an isomorph of c_0. Our proof requires Bourgain's index theory which in turn requires separability.

Our method of proof involves the concept of an *elastic Banach space*. Say that X is K-*elastic* provided that if a Banach space Y embeds into X then Y must K-*embed into* X (that is, there is an isomorphism T from Y into X with

$$\|y\| \leq \|Ty\| \leq K\|y\|$$

for all $y \in Y$). This is the same (by Lemma 2) as saying that every space isomorphic to X must K-embed into X. X is said to be *elastic* if it is K-elastic for some $K < \infty$.

Obviously, if $D(X) < \infty$ then X as well as every isomorph of X is $D(X)$-elastic. Thus the Main Theorem is an immediate consequence of

THEOREM 1. *If X is a separable Banach space and there is a K so that every isomorph of X is K-elastic, then X is finite dimensional.*

A key step in our argument involves showing that an elastic space X admits a normalized weakly null sequence having a spreading model not equivalent to either the unit vector basis of c_0 or ℓ_1. To achieve this we first prove (Theorem 7) that if X is elastic then c_0 embeds into X. It is reasonable to conjecture that an elastic infinite dimensional separable Banach space must contain an isomorph of $C[0,1]$. Theorem 1 would be an immediate consequence of this conjecture and the "arbitrary distortability" of $C[0,1]$ proved in [LP]. Our derivation of Theorem 1 from Theorem 7 uses ideas from [LP] as well as [MR].

With the letters $X, Y, Z, \dots$ we will denote separable infinite dimensional real Banach spaces unless otherwise indicated. $Y \subseteq X$ will mean that Y is a closed (infinite dimensional) subspace of X. The closed linear span of the set A is denoted $[A]$. We use standard Banach space theory terminology, as can be found in [LT]. The material we use on spreading models can be found in [BL]. For simplicity we assume real scalars, but all proofs can easily be adapted for complex Banach spaces.

2. The main result

The following lemma [Pe] shows that the two definitions of elastic mentioned in Section 1 are equivalent.

LEMMA 2. *Let $Y \subseteq (X, \|\cdot\|)$ and let $|\cdot|$ be an equivalent norm on $(Y, \|\cdot\|)$. Then $|\cdot|$ can be extended to an equivalent norm on X.*

Proof. There exist positive reals C and d with $d\|y\| \le |y| \le C\|y\|$ for $y \in Y$. Let $F \subseteq CB_{X^*}$ be a set of Hahn-Banach extensions of all elements of $S_{(Y^*,|\cdot|)}$ to all of X. For $x \in X$ define

$$|x| = \sup \big\{ |f(x)| : f \in F \big\} \vee d\|x\| \ . \qquad \square$$

Let $n \in \mathbb{N}$ and $K < \infty$. We shall call a basic sequence (x_i) *block n-unconditional with constant K* if every block basis $(y_i)_{i=1}^n$ of (x_i) is K-unconditional; that is,

$$\|\sum_{i=1}^n \pm a_i y_i\| \le K \|\sum_{i=1}^n a_i y_i\|$$

for all scalars $(a_i)_{i=1}^n$ and all choices of $\pm$.

The next lemma is essentially contained in [LP]. In fact, by using the slightly more involved argument in [LP], the conclusion "with constant 2" can be changed to "with constant $1+\varepsilon$", which implies that the constant in the conclusion of Lemma 4 can be changed from $2+\varepsilon$ to $1+\varepsilon$.

Lemma 3. *Let X be a Banach space with a basis (x_i). For every n there is an equivalent norm $|\cdot|_n$ on X so that in $(X,|\cdot|_n)$, (x_i) is block n-unconditional with constant 2.*

Proof. Let (P_n) be the sequence of basis projections associated with (x_n). We may assume, by passing to an equivalent norm on X, that (x_n) is bimonotone and hence $\|P_j - P_i\| = 1$ for all $i<j$. Let $\mathcal{S}_n$ be the class of operators S on X of the form $S=\sum_{k=1}^m (-1)^k (P_{n_k} - P_{n_{k-1}})$ where $0 \le n_0 < \cdots < n_m$ and $m \le n$. Define

$$|x|_n := \sup\{\|Sx\| : S \in \mathcal{S}_n\}\ .$$

Thus $\|x\| \le |x|_n \le n\|x\|$ for $x \in X$. It suffices to show that for $S \in \mathcal{S}_n$, $|S|_n \le 2$. Let $x \in \operatorname{span}(x_n)$ and $|Sx|_n = \|TSx\|$ for some $T \in S_n$. Then since $TS \in \mathcal{S}_{2n} \subseteq \mathcal{S}_n + \mathcal{S}_n$,

$$\|TSx\| \le 2|x|_n\,. \qquad \square$$

Lemma 4. *For every separable Banach space X and $n \in \mathbb{N}$ there exists an equivalent norm $|\cdot|$ on X so that for every $\varepsilon > 0$, every normalized weakly null sequence in X admits a block n-unconditional subsequence with constant $2+\varepsilon$.*

Proof. Since $C[0,1]$ has a basis, the lemma follows from Lemma 3 and the the classical fact that every separable Banach space 1-embeds into $C[0,1]$. $\square$

Lemma 4 is false for some nonseparable spaces. Partington [P] and Talagrand [T] proved that every isomorph of ℓ_∞ contains, for every $\varepsilon > 0$, a $1+\varepsilon$-isometric copy of ℓ_∞ and hence of every separable Banach space.

Our next lemma is an extension of the Maurey-Rosenthal construction [MR], or rather the footnote to it given by one of the authors (Example 3 in [MR]). We first recall the construction of *spreading models*. If (y_n) is a normalized basic sequence then, given $\varepsilon_n \downarrow 0$, one can use Ramsey's theorem and a diagonal argument to find a subsequence (x_n) of (y_n) with the following property. For all m in $\mathbb{N}$ and $(a_i)_{i=1}^m \subset [-1,1]$, if $m \le i_1 < \cdots < i_m$ and $m \le j_1 < \cdots < j_m$, then

$$\left| \Big\| \sum_{k=1}^m a_k x_{i_k} \Big\| - \Big\| \sum_{k=1}^m a_k x_{j_k} \Big\| \right| < \varepsilon_m.$$

It follows that for all m and $(a_i)_{i=1}^m \subset \mathbb{R}$,

$$\lim_{i_1\to\infty} \cdots \lim_{i_m\to\infty} \Big\| \sum_{k=1}^m a_k x_{i_k} \Big\| \equiv \Big\| \sum_{k=1}^m a_k \tilde{x}_k \Big\|$$

exists. The sequence $(\tilde{x}_i)$ is then a basis for the completion of $(\operatorname{span}(\tilde{x}_i), \|\cdot\|)$ and $(\tilde{x}_i)$ is called a *spreading model* of (x_i). If (x_i) is weakly null, then $(\tilde{x}_i)$ is 2-unconditional. One shows this by checking that $(\tilde{x}_i)$ is suppression 1-unconditional, which means that for all scalars $(a_i)_{i=1}^m$ and $F \subset \{1, \ldots, m\}$,

$$\left\|\sum_{i \in F} a_i \tilde{x}_i\right\| \leq \left\|\sum_{i=1}^m a_i \tilde{x}_i\right\|.$$

Also, (x_i) is 1-spreading, which means that for all scalars $(a_i)_{i=1}^m$ and all $n(1) < \cdots < n(m)$,

$$\left\|\sum_{i=1}^m a_i \tilde{x}_i\right\| = \left\|\sum_{i=1}^m a_i \tilde{x}_{n(i)}\right\|.$$

It is not difficult to see that, when (x_i) is weakly null, $(\tilde{x}_i)$ is not equivalent to the unit vector basis of c_0 (respectively, ℓ_1) if and only if $\lim_m \|\sum_{i=1}^m \tilde{x}_i\| = \infty$ (respectively, $\lim_m \|\sum_{i=1}^m \tilde{x}_i\|/m = 0$). All of these facts can be found in [BL].

LEMMA 5. *Let (x_n) be a normalized weakly null basic sequence with spreading model $(\tilde{x}_n)$. Assume that $(\tilde{x}_n)$ is not equivalent to either the unit vector basis of ℓ_1 or the unit vector basis of c_0. Then for all $C < \infty$ there exist $n \in \mathbb{N}$, a subsequence (y_i) of (x_i), and an equivalent norm $|\cdot|$ on $[(y_i)]$ so that (y_i) is $|\cdot|$-normalized and no subsequence of (y_i) is block n-unconditional with constant C for the norm $|\cdot|$.*

Proof. Recall that if $(e_i)_1^n$ is normalized and 1-spreading and bimonotone then $\|\sum_1^n e_i\| \, \|\sum_1^n e_i^*\| \leq 2n$ where $(e_i^*)_1^n$ is biorthogonal to $(e_i)_1^n$ [LT, p. 118]. Thus $e = \frac{\sum_1^n e_i}{\|\sum_1^n e_i\|}$ is normed by $f = \frac{\|\sum_1^n e_i\|}{n} \sum_{i=1}^n e_i^*$, precisely $f(e) = 1 = \|e\|$, and $\|f\| \leq 2$. These facts allow us to deduce that there is a subsequence (y_i) of (x_i) so that if $F \subseteq \mathbb{N}$ is *admissible* (that is, $|F| \leq \min F$) then

$$f_F \equiv \frac{\|\sum_{i \in F} y_i\|}{|F|} \Big(\sum_{i \in F} y_i^*\Big)$$

satisfies $\|f_F\| \leq 5$ and $f_F(y_F) = 1$, where

$$y_F \equiv \frac{\sum_{i \in F} y_i}{\|\sum_{i \in F} y_i\|}\ .$$

Indeed, $(\tilde{x}_i)$ is 1-spreading and suppression 1-unconditional (since (x_i) is weakly null). Given $1/2 > \varepsilon > 0$ we can find $(y_i) \subseteq (x_i)$ so that if $F \subseteq \mathbb{N}$ is admissible then $(y_i)_{i \in F}$ is $1+\varepsilon$-equivalent to $(\tilde{x}_i)_{i=1}^{|F|}$. Furthermore we can choose (y_i) so that if F is admissible then for $y = \sum a_i y_i$, $\|\sum_{i \in F} a_i y_i\| \leq (2+\varepsilon)\|y\|$ (for a proof see [O] or [BL]). Hence $\|f_F\| \leq (2+\varepsilon)\|f_F\,|_{[y_i]_{i \in F}}\| < 5$ for sufficiently small ε by our above remarks.

We are ready to produce a Maurey-Rosenthal type renorming. Choose n so that $n > 7C$ and let $\varepsilon > 0$ satisfy $n^2\varepsilon < 1$. We choose a subsequence

$M = (m_j)_{j=1}^\infty$ of $\mathbb{N}$ so that $m_1 = 1$ and for $i \neq j$ and for all admissible sets F and G with $|F| = m_i$ and $|G| = m_j$,

a) $\dfrac{\|\sum_{k\in F} y_k\|}{\|\sum_{k\in G} y_k\|} < \varepsilon$, if $m_i < m_j$ and

b) $\dfrac{\|\sum_{k\in F} y_k\|}{\|\sum_{k\in G} y_k\|}\ \dfrac{m_j}{m_i} < \varepsilon$, if $m_i > m_j$.

Indeed, we have chosen (y_i) so that

$$\frac{1}{2}\Big\|\sum_{i=1}^{|F|} \tilde{x}_i\Big\| \leq \Big\|\sum_{i\in F} y_i\Big\| \leq 2\Big\|\sum_{i=1}^{|F|} \tilde{x}_i\Big\|$$

and similarly for G. Since $(\tilde{x}_i)$ is not equivalent to the unit vector basis of c_0 (and is unconditional) $\lim_m \|\sum_1^m \tilde{x}_i\| = \infty$ so that a) will be satisfied if (m_k) increases sufficiently rapidly. Furthermore, since $(\tilde{x}_i)$ is not equivalent to the unit vector basis of ℓ_1, $\lim_m \frac{\|\sum_1^m \tilde{x}_i\|}{m} = 0$ and so b) can also be achieved.

For $i \in \mathbb{N}$ set $\mathcal{A}_i = \{y_F : F$ is admissible and $|F| = m_i\}$ and $\mathcal{A}_i^* = \{f_F : F$ is admissible and $|F| = m_i\}$. Let ϕ be an injection into M from the collection of all $(F_1, \dots, F_i)$ where $i < n$ and $F_1 < F_2 < \cdots < F_i$ are finite subsets of $\mathbb{N}$. Here $F < G$ means $\max F < \min G$. Let

$$\mathcal{F} = \Big\{ \sum_{i=1}^n f_{F_i} : F_1 < \cdots < F_n,\ |F_1| = m_1 = 1,\ \text{each } F_i \text{ is admissible}$$
$$\text{and } |F_{i+1}| = \phi(F_1, \dots, F_i) \text{ for } 1 \leq i < n \Big\}.$$

For $y \in [(y_i)]$ let

$$\|y\|_{\mathcal{F}} = \sup\Big\{|f(y)| : f \in \mathcal{F}\Big\}$$

and set

$$|y| = \|y\|_{\mathcal{F}} \vee \varepsilon\|y\| .$$

This is an equivalent norm since for $f \in \mathcal{F}$, $\|f\| \leq 5n$. Also, $|y_i| = 1$ for all i.

Note that if $f_F \in \mathcal{A}_i^*$ and $y_G \in \mathcal{A}_j$ with $i \neq j$ then

$$|f_F(y_G)| = \frac{\|\sum_{k\in F} y_k\|}{m_i} \sum_{k\in F} y_k^* \left(\frac{\sum_{k\in G} y_k}{\|\sum_{k\in G} y_k\|}\right) \leq \frac{\|\sum_{k\in F} y_k\|}{\|\sum_{k\in G} y_k\|} \frac{m_i \wedge m_j}{m_i} .$$

If $m_i < m_j$ then $|f_F(y_G)| < \varepsilon$ by a). If $m_i > m_j$ then $|f_F(y_G)| < \varepsilon$ by b).

It follows that if $y = \sum_{i=1}^n y_{F_i}$ and $f = \sum_{i=1}^n f_{F_i} \in \mathcal{F}$ then $|y| \geq f(y) = n$ and if $z = \sum_{i=1}^n (-1)^i y_{F_i}$, then for all $g = \sum_{j=1}^n f_{G_j} \in \mathcal{F}$, $|g(z)| \leq 6 + n^2\varepsilon < 7$. Indeed, we may assume that $g \neq f$ and if $F_1 \neq G_1$ then $|G_j| \neq |F_i|$ for all $1 < i \leq n$ and $1 \leq j \leq n$ and so by a), b),

$$|g(z)| = \left|\sum_{j=1}^n f_{G_j}\left(\sum_{i=1}^n (-1)^i y_{F_i}\right)\right| \leq \sum_{j=1}^n \sum_{i=1}^n |f_{G_j}(y_{F_i})| < n^2\varepsilon .$$

Otherwise there exists $1 \le j_0 < n$ so that $F_j = G_j$ for $j \le j_0$, $|F_{j_0+1}| = |G_{j_0+1}|$ and $|F_i| \ne |G_j|$ for $j_0 + 1 < i, j \le n$. Using $f_{G_{j_0+1}}(z) \le 5 + n\varepsilon$ we obtain

$$\begin{aligned} |g(z)| &\le \Big|\sum_{j=1}^{j_0} f_{G_j}(z)\Big| + |f_{G_{j_0+1}}(z)| + \Big|\sum_{j>j_0+1}^{n} f_{G_j}(z)\Big| \\ &< 1 + 5 + n\varepsilon + (n-(j_0+1))n\varepsilon < 6 + n^2\varepsilon \ . \end{aligned}$$

Hence $|z| \le 7$ follows and the lemma is proved since $n/7 > C$ and such vectors y and z can be produced in any subsequence of (y_i). □

Our next lemma follows from Proposition 3.2 in [AOST].

Lemma 6. *Let X be a Banach space. Assume that for all n, $(x_i^n)_{i=1}^\infty$ is a normalized weakly null sequence in X having spreading model $(\tilde{x}_i^n)$ which is not equivalent to the unit vector basis of ℓ_1. Then there exists a normalized weakly null sequence $(y_i) \subseteq X$ with spreading model $(\tilde{y}_i)$ such that $(\tilde{y}_i)$ is not equivalent to the unit vector basis of ℓ_1. Moreover, there exists $\lambda > 0$ so that for all n,*

$$\lambda 2^{-n} \|\sum a_i \tilde{x}_i^n\| \le \|\sum a_i \tilde{y}_i\|$$

for all $(a_i) \subseteq \mathbb{R}$.

Theorem 7. *Let X be elastic, separable and infinite dimensional. Then c_0 embeds into X.*

We postpone the proof to complete first the

Proof of Theorem 1. Assume that X is infinite dimensional and every isomorph of X is K-elastic. Then by Theorem 7, c_0 embeds into X. Choose $k_n \uparrow \infty$ so that $2^{-n} k_n \to \infty$. Using the renormings of c_0 by

$$|(a_i)|_n = \sup\Big\{\Big|\sum_F a_i\Big| : |F| = k_n\Big\}$$

and that X is K-elastic we can find for all n a normalized weakly null sequence $(x_i^n)_{i=1}^\infty \subseteq X$ with spreading model $(\tilde{x}_i^n)_{i=1}^\infty$ satisfying

$$\|\sum a_i \tilde{x}_i^n\| \ge K^{-1} |(a_i)|_n$$

and moreover each $(\tilde{x}_i^n)$ is equivalent to the unit vector basis of c_0. Thus by Lemma 6 there exists a normalized weakly null sequence (y_i) in X having spreading model $(\tilde{y}_i)$ which is not equivalent to the unit vector basis of ℓ_1 and which satisfies for all n,

$$\Big\|\sum_1^{k_n} \tilde{y}_i\Big\| \ge \lambda K^{-1} 2^{-n} k_n \to \infty \ .$$

Thus $(\tilde{y}_i)$ is not equivalent to the unit vector basis of c_0 as well.

By Lemmas 2 and 5, for all $C < \infty$ we can find $n \in \mathbb{N}$ and a renorming Y of X so that Y contains a normalized weakly null sequence admitting no subsequence which is block n-unconditional with constant C. By the assumption on X, the space Y must K-embed into every isomorph of X. But if C is large enough this contradicts Lemma 4. $\square$

It remains to prove Theorem 7. We shall employ an index argument involving ℓ_∞-trees defined on Banach spaces. If Y is a Banach space our trees T on Y will be countable. For some C the nodes of T will be elements $(y_i)_1^n \subseteq Y$ with $(y_i)_1^n$ bimonotone basic and satisfying $1 \le \|y_i\|$ and $\|\sum_1^n \pm y_i\| \le C$ for all choices of sign. Thus $(y_i)_1^n$ is C-equivalent to the unit vector basis of ℓ_∞^n. T is partially ordered by $(x_i)_1^n \le (y_i)_1^m$ if $n \le m$ and $x_i = y_i$ for $i \le n$. The order $o(T)$ is given as follows. If T is not well founded (i.e., T has an infinite branch), then $o(T) = \omega_1$. Otherwise we set for such a tree S, $S' = \{(x_i)_1^n \in S : (x_i)_1^n$ is not a maximal node$\}$. Set $T_0 = T$, $T_1 = T'$ and in general $T_{\alpha+1} = (T_\alpha)'$ and $T_\alpha = \cap_{\beta<\alpha} T_\beta$ if α is a limit ordinal. Then

$$o(T) = \inf\{\alpha : T_\alpha = \phi\} \ .$$

By Bourgain's index theory [B1], [B2] (see also [AGR]), if X is separable and contains for all $\beta < \omega_1$ such a tree of index at least β, then c_0 embeds into X.

We now complete the

Proof of Theorem 7. Without loss of generality we may assume that $X \subseteq Z$ where Z has a bimonotone basis (z_i). Let X be K-elastic. We will often use semi-normalized sequences in X which are a tiny perturbation of a block basis of (z_i) and to simplify the estimates we will assume below that they are in fact a block basis of (z_i).

For example, if (y_i) is a normalized basic sequence in X then we call (d_i) a *difference sequence* of (y_i) if $d_i = y_{k(2i)} - y_{k(2i+1)}$ for some $k_1 < k_2 < \cdots$. We can always choose such a (d_i) to be a semi-normalized perturbation of a block basis of (z_i) by first passing to a subsequence (y_i') of (y_i) so that $\lim_{i\to\infty} z_j^*(y_i')$ exists for all j, where (z_i^*) is biorthogonal to (z_i), and taking (d_i) to be a suitable difference sequence of (y_i'). We will assume then that (d_i) is in fact a block basis of (z_i).

We inductively construct for each *limit* ordinal $\beta < \omega_1$, a Banach space Y_β that embeds into X. Y_β will have a normalized bimonotone basis (y_i^β) that can be enumerated as $(y_i^\beta)_{i=1}^\infty = \{y_i^{\beta,\rho,n} : \rho \in C_\beta,\, n \in \mathbb{N},\, i \in \mathbb{N}\}$ where C_β is some countable set. The order is such that $(y_i^{\beta,\rho,n})_{i=1}^\infty$ is a subsequence of (y_i^β) for fixed ρ and n.

Before stating the remaining properties of (y_i^β) we need some terminology. We say that (w_i) is a *compatible difference sequence of* (y_i^β) *of order* 1 if (w_i)

is a difference sequence of (y_i^β) that can be enumerated as follows,

$$(w_i) = \{w_i^{\beta,\rho,n} : \rho \in C_\beta,\ n, i \in \mathbb{N}\}$$

and such that for fixed ρ and n,

$$(w_i^{\beta,\rho,n})_i \text{ is a difference sequence of } (y_i^{\beta,\rho,n+1})_i\ .$$

If (v_i) is a compatible difference sequence of (w_i) of order 1, in the above sense, (v_i) will be called a *compatible difference sequence of* (y_i^β) *of order* 2, and so on. (y_i^β) will be said to have order 0.

Let (v_i) be a compatible difference sequence of (y_i^β) of some finite order. We set

$$T\big((v_i)\big) = \Big\{(u_i)_1^s : \text{ the } u_i\text{'s are distinct elements of } \{v_i\}_1^\infty, \text{ possibly in different order, and } \Big\|\sum_1^s \pm u_i\Big\| = 1 \text{ for all choices of sign}\Big\}\ .$$

$T\big((v_i)\big)$ is then an ℓ_∞-tree as described above with $C = 1$. The inductive condition on Y_β, or should we say on $(y_i^{\beta,\rho,n})$, is that for all compatible difference sequences (v_i) of $(y_i^{\beta,\rho,n})$ of finite order,

$$o(T((v_i))) \geq \beta\ .$$

Before proceeding we have an elementary but key

Sublemma. *Let $C < \infty$ and let (w_i) be a block basis of a bimonotone basis (z_i) with $1 \leq \|w_i\| \leq C$ for all i and let*

$$\mathcal{A} = \{F \subseteq \mathbb{N} : F \textit{ is finite and } \|\sum_{i\in F} \pm w_i\| \leq C \textit{ for all choices of sign}\}.$$

Then there exists an equivalent norm $|\cdot|$ on $[(w_i)]$ so that (w_i) is a bimonotone normalized basis such that for all $F \in \mathcal{A}$,

$$\Big|\sum_F \pm w_i\Big| = 1\ .$$

Proof. Define $|\sum a_i w_i| = \|(a_i)\|_\infty \vee C^{-1}\|\sum a_i w_i\|$. $\square$

We begin by constructing Y_ω. Let $(x_i) \subseteq X$ be a normalized block basis of (z_i). For $n \in \mathbb{N}$, let $|\cdot|_n$ be an equivalent norm on $[(x_i)]$ given by the sublemma for $C = 2^n$. Thus $|\sum_F \pm x_i|_n = 1$ if $|F| \leq 2^n$.

Since X is K-elastic, for all n, $([(x_i)], |\cdot|_n)$ K-embeds into X. We thus obtain for $n \in \mathbb{N}$, a sequence $(x_i^n)_i \subseteq X$ with $1 \leq \|x_i^n\| \leq K$ for all i and such that $1 \leq \|\sum_{i\in F} \pm x_i^n\| \leq K$ for all $|F| \leq 2^n$ and all choices of sign. Furthermore

$(x_i^n)_i$ is K-basic. By standard perturbation and diagonal arguments we may for each n pass to a difference sequence $(d_i^n)_i$ of $(x_i^{n+1})_i$ so that enumerating, $(d_i) = \{d_i^n : n, i \in \mathbb{N}\}$ is a block basis of (z_i) with $1 \le \|d_i\| \le K$ and with each $(d_i^n)_i$ being a subsequence of (d_i). We have that for $|F| \le 2^n$ and all signs,

$$1 \le \left\| \sum_{i\in F} \pm d_i^n \right\| \le K \ .$$

We renorm $[(d_i)]$ by the sublemma for $C = K$ and let the ensuing space be Y_ω. We change the name of (d_i) to (y_i^ω) in this new norm and let $(y_i^{\omega,1,n})_i = (d_i^n)_i$. (y_i^ω) has the property that if (w_i) is a compatible difference sequence of (y_i^ω) of finite order, then $o(T((w_i))) \ge \omega$. Indeed, if $(w_i) = \{w_i^{\omega,1,n} : n \in \mathbb{N}, i \in \mathbb{N}\}$ then for $|F| \le 2^n$, $\| \sum_{i\in F} \pm w_i^{\omega,1,n} \| = 1$.

Assume that Y_β has been constructed for the limit ordinal β with basis $(y_i^\beta) = \{y_i^{\beta,\rho,n} : p \in C_\beta,\ n, i \in \mathbb{N}\}$ with the requisite properties above.

Let $U : Y_\omega \to X$ and $V : Y_\beta \to X$ be K-embeddings. Since in total we are dealing with a countable set of sequences, namely $(y_i^{\omega,1,n})_i$ for $n \in \mathbb{N}$ and $(y_i^{\beta,\rho,n})_i$ for $p \in C_\beta$, $n \in \mathbb{N}$, by diagonalization and perturbation we can find a compatible difference sequence $(w_i^\omega)_i$ of $(y_i^\omega)_i$ of order 1 and a compatible difference sequence $(w_i^\beta)_i$ of $(y_i^\beta)_i$ of order 1 so that under a suitable reordering, $(d_i) = (Uw_i^\omega)_i \cup (Vw_i^\beta)_i$ is a block basis of (z_i). Moreover each $(Uw_i^{\omega,1,n})_i$ and $(Vw_i^{\beta,\rho,n})_i$ is a subsequence of $(d_i)_i$.

Adjoin a new point p_0 to C_β and set $C_{\beta+\omega} = C_\beta \cup \{\rho_0\}$. Let $d_i^{\beta+\omega,\rho,n} = Vw_i^{\beta,\rho,n}$ for $\rho \in C_\beta$ and $d_i^{\beta+\omega,\rho_0,n} = Uw_i^{\omega,1,n}$. For $|F| \le 2^n$ and $G \subseteq C_\beta \times \mathbb{N} \times \mathbb{N}$ for which $\| \sum_{(\rho,n,i)\in G} \pm w_i^{\beta,\rho,n} \| = 1$, we have by the triangle inequality that

$$(*) \qquad \left\| \sum_{i\in F} \pm d_i^{\beta+\omega,\rho_0,n} + \sum_{(\rho,n,i)\in G} \pm d_i^{\beta+\omega,\rho,n} \right\| \le 2K \ .$$

It follows that if we let $(y_i^{\beta+\omega})$ be the basis $(d_i^{\beta+\omega})$, renormed by the sublemma for $C = 2K$, that $Y_{\beta+\omega} = [(y_i^{\beta+\omega})]$ has the required properties. Indeed $(*)$ yields that $o(T((y_i^{\beta+\omega}))) \ge \beta + 2^n$ for $n \in \mathbb{N}$ and so $o(T((y_i^{\beta+\omega}))) \ge \beta + \omega$. Moreover by our induction hypothesis and choice of $(d_i^{\beta+\omega,p_0,n})$ the analogue of $(*)$ holds for a compatible difference sequence of order 1 or of any finite order. Thus $o(T(z_i^{\beta+\omega})) \ge \beta + \omega$ for any compatible difference sequence of $(y_i^{\beta+\omega})$ of finite order.

If β is a limit ordinal not of the form $\alpha + \omega$ we let $U_\alpha : Y_\alpha \to X$ be a K-embedding for each limit ordinal $\alpha < \beta$. We again diagonalize to form compatible difference sequences $(w_i^\alpha)_i$ of (y_i^α) of order 1 for each such α so that $(U_\alpha w_i^\alpha)_{\alpha,i}$ is a block basis of (z_i) in some order. We let C_β be a disjoint union of the C_α's and in the manner above obtain Y_β. Again by our induction hypothesis we will have that $o(T((y_i^\beta))) \geq \alpha$ for all limit ordinals $\alpha < \beta$ and the same holds for all compatible difference sequences of (y_i^β) of finite order.□

3. Concluding remarks and problems

$C[0,1]$ is, of course, 1-elastic. By virtue of Lemma 4 (or Theorem 1), for all K, $C[0,1]$ can be renormed to be elastic but not K-elastic. Are there other examples of separable elastic spaces?

Problem 8. Let X be elastic (and separable, say). Does $C[0,1]$ embed into X?

Using index arguments, we have the following partial result.

PROPOSITION 9. *Let X be a separable Banach space, and suppose that $Y = \sum X_n$ is a symmetric decomposition of a space Y into spaces uniformly isomorphic to X. If Y is elastic, then $C[0,1]$ embeds into X.*

In particular, if $1 \leq p < \infty$ and $\ell_p(X)$ is elastic, then $C[0,1]$ embeds into X.

Proof. Let us first observe that if $C[0,1]$ embeds into Y, then $C[0,1]$ embeds into X. Since this is surely well known, we just sketch a proof (which, incidentally, uses only that the decomposition $Y = \sum X_n$ of Y is unconditional): Let P_n be the projection from Y onto X_n and let W be a subspace of Y which is isomorphic to $C[0,1]$. By a theorem of Rosenthal's [R], it is enough to show that for some n, the adjoint $P_{n|W}^*$ of the restriction of P_n to W has nonseparable range. This will be true if there is an m so that $S_{m|W}^*$ has nonseparable range, where $S_m := \sum_{i=1}^m P_i$. Let Z be a subspace of W which is isomorphic to ℓ_1. If no such m exists, then for every m, the restriction of S_m to Z is strictly singular (that is, not an isomorphism on any infinite dimensional subspace of Z), and it then follows that Z contains a sequence (x_n) of unit vectors which is an arbitrarily small perturbation of a sequence (y_n) which is disjointly supported. The sequence (y_n), *a fortiori* (x_n), is then easily seen to be equivalent to the unit vector basis of ℓ_1 and its closed span is complemented in Y since the decomposition is unconditional. It follows that ℓ_1 is isomorphic to a complemented subspace of $C[0,1]$, which of course is false.

To complete the proof of Proposition 9, we assume that Y is K-elastic and prove that $C[0,1]$ embeds into Y. The proof is similar to, but simpler than, the proof of Theorem 7. First we recall the definition of certain canonical trees T_α of order α for $\alpha < \omega_1$ (see e.g. [JO]). These form the frames upon which we will hang our bases. The tree T_1 is a single node. If T_α has been defined, we choose a new node $z \notin T_\alpha$ and set $T_{\alpha+1} := T_\alpha \cup \{z\}$, ordered by $z < t$ for all $t \in T_\alpha$ and with T_α preserving its order. If $\beta < \omega_1$ is a limit ordinal, we let T_β be the disjoint union of $\{T_\alpha : \alpha < \beta\}$. Then if s, t are in T_β, we say that $s \leq t$ if and only if s, t are both in T_α for some $\alpha < \beta$ and $s \leq t$ in T_α.

We shall prove by transfinite induction that if $(x_i)_{i=1}^\infty$ is any normalized bimonotone basic sequence and $\alpha < \omega_1$, there is a Banach space $Y_\alpha = Y_\alpha(x_i)$ with a normalized bimonotone basis $(y_t^\alpha)_{t\in T_\alpha}$ so that if $(\gamma_i)_{i=1}^n$ is any branch in T_α then $(y_{\gamma_i}^\alpha)_{i=1}^n$ is 1-equivalent to $(x_i)_{i=1}^n$ and for all scalars $(a_r)_{t\in T_\alpha}$, $\|\sum_{i=1}^n a_{\gamma_i} y_{\gamma_i}^\alpha\| \leq \|\sum_{t\in T_\alpha} a_t y_t^\alpha\|$. Furthermore, each Y_α will K-embed into Y. Just as in the proof of Theorem 7, it then follows from index theory that the Banach space spanned by $(x_i)_{i=1}^\infty$ embeds into Y.

Fix any normalized bimonotone basic sequence $(x_i)_{i=1}^\infty$. Suppose that β is a limit ordinal and $Y_\alpha = Y_\alpha(x_i)_{i=1}^\infty$ has been defined for all $\alpha < \beta$. In view of the hypotheses, the space Y has a symmetric decomposition into spaces uniformly isomorphic Y, which we can index as $Y = \sum_{\alpha<\beta} X_\alpha$. For each α, there is an isomorphism L_α from Y_α into X_α so that $\|L_\alpha\| = 1$ and $\|L_\alpha^{-1}\|$ is bounded independently of α. We can put an equivalent norm on $Y = \sum_{\alpha<\beta} X_\alpha$ to make each L_α an isometry and make the decomposition 1-unconditional (but not necessarily 1-symmetric). Define Y_β to be the closed linear span of $\{L_\alpha Y_\alpha : \alpha < \beta\}$ in Y with its new norm. The space Y_β has the desired basis indexed by T_β and Y_α must K-embed into Y with its original norm because Y is K-elastic.

If $\beta = \alpha + 1$, we let $Y_\beta(x_i)_{i=1}^\infty = \mathbb{R} \oplus Y_\alpha(x_i)_{i=2}^\infty$ with the norm given by

$$\|(a, y)\| := \|y\| \vee \sup\{\|ax_1 + \sum_{i=2}^n y_{\gamma_i}^{\alpha *}(y) y_{\gamma_i}^\alpha\| : \ (\gamma_i)_{i=2}^n \text{ is a branch in } T_\alpha\}.$$

Again it is clear that Y_β must K-embed into Y and that Y_β has the desired basis. $\square$

Given metric spaces X and Y, the Lipschitz distance $d_L(X, Y)$ between them is defined to be the infimum of $\mathrm{Lip}\,(f) \cdot \mathrm{Lip}\,(f^{-1})$, where the infimum is taken over all biLipschitz mappings f from X onto Y (the infimum of the empty set is ∞). Call a Banach space X *Lipschitz K-elastic* provided that every isomorph of X has Lipschitz distance at most K to some subset of X, and say that X is *Lipschitz elastic* if X is Lipschitz K-elastic for some $K < \infty$. It is interesting to note that the analogue of Theorem 1 for Lipschitz elasticity is false. Indeed, Aharoni [A], [BeLi, Theorem 7.11] proved that every separable metric space has, for each $\epsilon > 0$, Lipschitz distance less than $3 + \epsilon$ to some subset of c_0, while James [J], [BeLi, Proposition 13.6] proved that for each $\epsilon > 0$, the space c_0 is $1 + \epsilon$-equivalent to a subspace of every isomorph of c_0. Consequently, any separable Banach space which contains an isomorphic copy of c_0 is Lipschitz $3 + \epsilon$-elastic.

Problem 10. If X is a separable, infinite dimensional Banach space and there is a K so that every isomorph of X is Lipschitz K-elastic, must X contain a subspace isomorphic to c_0?

We also do not know whether the Main Theorem has a Lipschitz analogue:

Problem 11. If X is a separable Banach space and there exists $K < \infty$ so that $d_L(X, Y) < K$ for all spaces Y which are isomorphic (or biLipschitz equivalent) to X, then must X be finite dimensional?

The space c_0 does not have the property mentioned in Problem 11 (use, for example, the fact [GKL] that for every $K < \infty$ there exists $K' < \infty$ so that if a Banach space X is K biLipschitz equivalent to c_0 then $d(X, c_0) \leq K'$).

Since Schäffer's problem remains open in the nonseparable setting, we mention it explicitly as:

Problem 12. Does there exist a nonseparable Banach space X for which $D(X) < \infty$?

Also open is:

Problem 13. Does there exist a nonseparable Banach space X and a $K < \infty$ so that all isomorphs of X are K-elastic?

Under the generalized continuum hypothesis, it follows from [E] that for every cardinal $\aleph$ there is a Banach space X of density character $\aleph$ so that every Banach space of density character $\aleph$ is isometrically isomorphic to a subspace of X. These provide the only known examples of elastic Banach spaces.

Problem 14. If X is elastic and infinite dimensional, does X contain an isomorph of every Banach space whose density character is the same as the density character of X?

As was mentioned earlier, the space ℓ_∞ is, for every $\epsilon > 0$, $1+\epsilon$-isomorphic to a subspace of every isomorph of itself (see [P] and [T]). However, ℓ_∞ is not elastic. To see this, take a biorthogonal system $(x_\alpha, x_\alpha^*)_{\alpha < 2^{\aleph_0}}$ in ℓ_∞ with $\|x_\alpha\| = 1$ and $\|x_\alpha^*\|$ bounded. Given a natural number m, renorm ℓ_∞ by

$$\|x\|_m := \max\left\{\frac{1}{m}\|x\|_\infty, \sup_{|\sigma|=m}\left(\sum_{\alpha\in\sigma}|x_\alpha^*(x)|^2\right)^{1/2}\right\}.$$

It is not hard to see that $(\ell_\infty, \|\cdot\|_m)$ is not better than $\sqrt{m}$-normed by any countable set of norm one functionals, which implies that $d((\ell_\infty, \|\cdot\|_m), Y) \geq \sqrt{m}$ for any subspace Y of ℓ_∞. A similar argument shows that $\ell_\infty(\aleph)$ is not elastic for any infinite cardinal number $\aleph$.

Texas A&M University, College Station, TX
E-mail address: johnson@math.tamu.edu

The University of Texas at Austin, Austin, TX
E-mail address: odell@math.utexas.edu

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