The diameter of the isomorphism class of a Banach space

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Dedicated to the memory of V. I. Gurarii

Abstract

We prove that if \( X \) is a separable infinite dimensional Banach space then its isomorphism class has infinite diameter with respect to the Banach-Mazur distance. One step in the proof is to show that if \( X \) is elastic then \( X \) contains an isomorph of \( c_0 \). We call \( X \) elastic if for some \( K < \infty \) for every Banach space \( Y \) which embeds into \( X \), the space \( Y \) is \( K \)-isomorphic to a subspace of \( X \). We also prove that if \( X \) is a separable Banach space such that for some \( K < \infty \) every isomorph of \( X \) is \( K \)-elastic then \( X \) is finite dimensional.

1. Introduction

Given a Banach space \( X \), let \( D(X) \) be the diameter in the Banach-Mazur distance of the class of all Banach spaces which are isomorphic to \( X \); that is,

\[
D(X) = \sup \{ d(X_1, X_2) : X_1, X_2 \text{ are isomorphic to } X \}
\]

where \( d(X_1, X_2) \) is the infimum over all isomorphisms \( T \) from \( X_1 \) onto \( X_2 \) of \( \|T\| \cdot ||T^{-1}|| \). It is well known that if \( X \) is finite (say, \( N \)) dimensional, then \( cN \leq D(X) \leq N \) for some positive constant \( c \) which is independent of \( N \). The upper bound is an immediate consequence of the classical result (see e.g. [T-J, p. 54]) that \( d(Y, \ell^N) \leq \sqrt{N} \) for every \( N \)-dimensional space \( Y \). The lower bound is due to Gluskin [G], [T-J, p. 283].

It is natural to conjecture that \( D(X) \) must be infinite when \( X \) is infinite dimensional, but this problem remains open. As far as we know, this problem was first raised in print in the 1976 book of J. J. Schäffer [S, p. 99]. The problem was recently brought to the attention of the authors by V. I. Gurarii, who

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checked that every infinite dimensional super-reflexive space as well as each of
the common classical Banach spaces has an isomorphism class whose diameter
is infinite. To see these cases, note that if $X$ is infinite dimensional and $E$
is any finite dimensional space, then it is clear that $X$ is isomorphic to $E \oplus 2X_0$
for some space $X_0$. Therefore, if $D(X)$ is finite, then $X$ is finitely complementably
universal; that is, there is a constant $C$ so that every finite dimensional space is
$C$-isomorphic to a $C$-complemented subspace of $X$. This implies that $X$ cannot
have nontrivial type or nontrivial cotype or local unconditional structure or
numerous other structures. In particular, $X$ cannot be any of the classical
spaces or be super-reflexive.

In his unpublished 1968 thesis [Mc], McGuigan conjectured that $D(X)$
must be larger than one when dim $X > 1$. Schäffer [S, p. 99] derived that
$D(X) \geq 6/5$ when dim $X > 1$ as a consequence of other geometrical results
contained in [S], but one can prove directly that $D(X) \geq \sqrt{2}$. Indeed, it is
clearly enough to get an appropriate lower bound on the Banach-Mazur distance
between $X_1 := Y_1 \oplus 1 \mathbb{R}$ and $X_2 := Y_2 \oplus 2 \mathbb{R}$ when $Y$ is a nonzero Banach
space. Now $X_1$ has a one dimensional subspace for which every two dimen-
sional superspace is isometric to $\ell_2^1$. On the other hand, every one dimensional
subspace of $X_2$ is contained in a two dimensional superspace which is isometric
to $\ell_2^2$. It follows that $d(X_1, X_2) \geq d(\ell_2^1, \ell_2^2) = \sqrt{2}$.

The Main Theorem in this paper is a solution to Schäffer’s problem for
separable Banach spaces:

**Main Theorem.** If $X$ is a separable infinite dimensional Banach space,
then $D(X) = \infty$.

Part of the work for proving the Main Theorem involves showing that if $X$
is separable and $D(X) < \infty$, then $X$ contains an isomorph of $c_0$. This proof
is inherently not local in nature, and, strangely enough, local considerations,
such as those mentioned earlier which yield partial results, play no role in our
proof. We do not see how to prove that a nonseparable space $X$ for which
$D(X) < \infty$ must contain an isomorph of $c_0$. Our proof requires Bourgain’s
index theory which in turn requires separability.

Our method of proof involves the concept of an elastic Banach space. Say
that $X$ is $K$-elastic provided that if a Banach space $Y$ embeds into $X$ then $Y$
must $K$-embed into $X$ (that is, there is an isomorphism $T$ from $Y$ into $X$ with

$$\|y\| \leq \|Ty\| \leq K\|y\|$$

for all $y \in Y$). This is the same (by Lemma 2) as saying that every space
isomorphic to $X$ must $K$-embed into $X$. $X$ is said to be elastic if it is
$K$-elastic for some $K < \infty$.

Obviously, if $D(X) < \infty$ then $X$ as well as every isomorph of $X$ is $D(X)$-
elastic. Thus the Main Theorem is an immediate consequence of
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Theorem 1. If $X$ is a separable Banach space and there is a $K$ so that every isomorph of $X$ is $K$-elastic, then $X$ is finite dimensional.

A key step in our argument involves showing that an elastic space $X$ admits a normalized weakly null sequence having a spreading model not equivalent to either the unit vector basis of $c_0$ or $\ell_1$. To achieve this we first prove (Theorem 7) that if $X$ is elastic then $c_0$ embeds into $X$. It is reasonable to conjecture that an elastic infinite dimensional separable Banach space must contain an isomorph of $C[0,1]$. Theorem 1 would be an immediate consequence of this conjecture and the “arbitrary distortability” of $C[0,1]$ proved in [LP]. Our derivation of Theorem 1 from Theorem 7 uses ideas from [LP] as well as [MR].

With the letters $X, Y, Z, \ldots$ we will denote separable infinite dimensional real Banach spaces unless otherwise indicated. $Y \subseteq X$ will mean that $Y$ is a closed (infinite dimensional) subspace of $X$. The closed linear span of the set $A$ is denoted $[A]$. We use standard Banach space theory terminology, as can be found in [LT]. The material we use on spreading models can be found in [BL]. For simplicity we assume real scalars, but all proofs can easily be adapted for complex Banach spaces.

2. The main result

The following lemma [Pe] shows that the two definitions of elastic mentioned in Section 1 are equivalent.

Lemma 2. Let $Y \subseteq (X, \|\cdot\|)$ and let $|\cdot|$ be an equivalent norm on $(Y, \|\cdot\|)$. Then $|\cdot|$ can be extended to an equivalent norm on $X$.

Proof. There exist positive reals $C$ and $d$ with $d\|y\| \leq |y| \leq C\|y\|$ for $y \in Y$. Let $F \subseteq CB_{X^*}$ be a set of Hahn-Banach extensions of all elements of $S_{(Y^*, |\cdot|)}$ to all of $X$. For $x \in X$ define

$$|x| = \sup \{|f(x)| : f \in F\} \vee d\|x\|.$$  

Let $n \in \mathbb{N}$ and $K < \infty$. We shall call a basic sequence $(x_i)$ block $n$-unconditional with constant $K$ if every block basis $(y_i)^{n}_{i=1}$ of $(x_i)$ is $K$-unconditional; that is,

$$\|\sum_{i=1}^{n} \pm a_i y_i\| \leq K \|\sum_{i=1}^{n} a_i y_i\|$$

for all scalars $(a_i)^{n}_{i=1}$ and all choices of $\pm$.

The next lemma is essentially contained in [LP]. In fact, by using the slightly more involved argument in [LP], the conclusion “with constant 2” can
be changed to “with constant $1 + \varepsilon$”, which implies that the constant in the conclusion of Lemma 4 can be changed from $2 + \varepsilon$ to $1 + \varepsilon$.

**Lemma 3.** Let $X$ be a Banach space with a basis $(x_i)$. For every $n$ there is an equivalent norm $|\cdot|_n$ on $X$ so that in $(X, |\cdot|_n)$, $(x_i)$ is block $n$-unconditional with constant 2.

**Proof.** Let $(P_n)$ be the sequence of basis projections associated with $(x_n)$. We may assume, by passing to an equivalent norm on $X$, that $(x_n)$ is bimonomial and hence $\|P_j - P_i\| = 1$ for all $i < j$. Let $\mathcal{S}_n$ be the class of operators $S$ on $X$ of the form $S = \sum_{k=0}^{m}(-1)^k(P_{n_k} - P_{n_{k-1}})$ where $0 \leq n_0 < \cdots < n_m$ and $m \leq n$. Define

$$|x|_n := \sup\{\|Sx\| : S \in \mathcal{S}_n\}.$$  

Thus $\|x\| \leq |x|_n \leq n\|x\|$ for $x \in X$. It suffices to show that for $S \in \mathcal{S}_n$, $|S|_n \leq 2$. Let $x \in \text{span}(x_n)$ and $|Sx|_n = \|TSx\|$ for some $T \in \mathcal{S}_n$. Then since $TS \in \mathcal{S}_{2n} \subseteq \mathcal{S}_n + \mathcal{S}_n$,

$$\|TSx\| \leq 2|x|_n.$$  

**Lemma 4.** For every separable Banach space $X$ and $n \in \mathbb{N}$ there exists an equivalent norm $|\cdot|$ on $X$ so that for every $\varepsilon > 0$, every normalized weakly null sequence in $X$ admits a block $n$-unconditional subsequence with constant $2 + \varepsilon$.

**Proof.** Since $C[0,1]$ has a basis, the lemma follows from Lemma 3 and the classical fact that every separable Banach space 1-embeds into $C[0,1]$.

Lemma 4 is false for some nonseparable spaces. Partington [P] and Talagrand [T] proved that every isomorph of $\ell_\infty$ contains, for every $\varepsilon > 0$, a $1 + \varepsilon$-isometric copy of $\ell_\infty$ and hence of every separable Banach space.

Our next lemma is an extension of the Maurey-Rosenthal construction [MR], or rather the footnote to it given by one of the authors (Example 3 in [MR]). We first recall the construction of spreading models. If $(y_n)$ is a normalized basic sequence then, given $\varepsilon_n \downarrow 0$, one can use Ramsey’s theorem and a diagonal argument to find a subsequence $(x_n)$ of $(y_n)$ with the following property. For all $m$ in $\mathbb{N}$ and $(a_{i_j})_{i=1}^{m} \subseteq [-1,1]$, if $m \leq i_1 < \cdots < i_m$ and $m \leq j_1 < \cdots < j_m$, then

$$\left\|\sum_{k=1}^{m}a_k x_{i_k}\right\| - \left\|\sum_{k=1}^{m}a_k x_{j_k}\right\| < \varepsilon_m.$$  

It follows that for all $m$ and $(a_{i_j})_{i=1}^{m} \subseteq \mathbb{R}$,

$$\lim_{i_1 \to \infty} \cdots \lim_{i_m \to \infty} \left\|\sum_{k=1}^{m}a_k x_{i_k}\right\| = \left\|\sum_{k=1}^{m}a_k \tilde{x}_k\right\|.$$  

exists. The sequence \((\tilde{x}_i)\) is then a basis for the completion of \((\text{span}(\tilde{x}_i), \| \cdot \|)\) and \((\tilde{x}_i)\) is called a spreading model of \((x_i)\). If \((x_i)\) is weakly null, then 

\((\tilde{x}_i)\) is 2-unconditional. One shows this by checking that \((\tilde{x}_i)\) is suppression 1-unconditional, which means that for all scalars \(a_i \in \mathbb{R}\) and \(F \subset \{1, \ldots, m\}\),

\[
\left\| \sum_{i \in F} a_i \tilde{x}_i \right\| \leq \left\| \sum_{i=1}^{m} a_i \tilde{x}_i \right\|.
\]

Also, \((x_i)\) is 1-spreading, which means that for all scalars \(a_i \in \mathbb{R}\) and all \(n(1) < \cdots < n(m)\),

\[
\left\| \sum_{i=1}^{m} a_i \tilde{x}_i \right\| = \left\| \sum_{i=1}^{m} a_i \tilde{x}_{n(i)} \right\|.
\]

It is not difficult to see that, when \((x_i)\) is weakly null, \((\tilde{x}_i)\) is not equivalent to the unit vector basis of \(c_0\) (respectively, \(\ell_1\)) if and only if \(\lim_{m} \left\| \sum_{i=1}^{m} \tilde{x}_i \right\| = \infty\) (respectively, \(\lim_{m} \left\| \sum_{i=1}^{m} \tilde{x}_i \right\| / m = 0\)). All of these facts can be found in [BL].

**Lemma 5.** Let \((x_n)\) be a normalized weakly null basic sequence with spreading model \((\tilde{x}_n)\). Assume that \((\tilde{x}_n)\) is not equivalent to either the unit vector basis of \(\ell_1\) or the unit vector basis of \(c_0\). Then for all \(C < \infty\) there exist \(n \in \mathbb{N}\), a subsequence \((y_i)\) of \((x_i)\), and an equivalent norm \(\| \cdot \|\) on \([y_i]\) so that \((y_i)\) is \(\| \cdot \|\)-normalized and no subsequence of \((y_i)\) is block \(n\)-unconditional with constant \(C\) for the norm \(\| \cdot \|\).

**Proof.** Recall that if \((e_i)\) is normalized and 1-spreading and bimonotone then \(\| \sum_{i=1}^{n} e_i \| \leq 2n\) where \((e_i^*)\) is biorthogonal to \((e_i)\) [LT, p. 118]. Thus \(e = \sum_{i=1}^{n} e_i / \| \sum_{i=1}^{n} e_i \|\) is normed by \(f = \| \sum_{i=1}^{n} e_i \| \sum_{i=1}^{n} e_i^*\), precisely \(f(e) = 1 = \| e \|\), and \(\| f \| \leq 2\). These facts allow us to deduce that there is a subsequence \((y_i)\) of \((x_i)\) so that if \(F \subseteq \mathbb{N}\) is admissible (that is, \(| F | \leq \min F \)) then

\[
f_F = \left( \frac{\sum_{i \in F} y_i}{| F |} \right) \left( \sum_{i \in F} y_i^* \right)
\]

satisfies \(\| f_F \| \leq 5\) and \(f_F(y_F) = 1\), where

\[
y_F = \frac{\sum_{i \in F} y_i}{\| \sum_{i \in F} y_i \|}.
\]

Indeed, \((\tilde{x}_i)\) is 1-spreading and suppression 1-unconditional (since \((x_i)\) is weakly null). Given \(1/2 > \varepsilon > 0\) we can find \((y_i) \subseteq (x_i)\) so that if \(F \subseteq \mathbb{N}\) is admissible then \((y_i)_{i \in F}\) is \(1 + \varepsilon\)-equivalent to \((\tilde{x}_i)^{| F |}\). Furthermore we can choose \((y_i)\) so that if \(F\) is admissible then for \(y = \sum a_i y_i\) and \(\| \sum_{i \in F} a_i y_i \| \leq (2 + \varepsilon) \| y \|\) (for a proof see [O] or [BL]). Hence \(\| f_F \| \leq (2 + \varepsilon) \| f_F \| \| y_F \| \leq 5\) for sufficiently small \(\varepsilon\) by our above remarks.

We are ready to produce a Maurey-Rosenthal type renorming. Choose \(n\) so that \(n > 7C\) and let \(\varepsilon > 0\) satisfy \(n^2 \varepsilon < 1\). We choose a subsequence
$M = (m_j)_{j=1}^{\infty}$ of $\mathbb{N}$ so that $m_1 = 1$ and for $i \neq j$ and for all admissible sets $F$ and $G$ with $|F| = m_i$ and $|G| = m_j$,

a) $\left\| \sum_{k \in F} y_k \right\| < \varepsilon$, if $m_i < m_j$ and

b) $\left\| \frac{\sum_{k \in F} y_k}{\sum_{k \in G} y_k} \right\| \cdot \frac{m_j}{m_i} < \varepsilon$, if $m_i > m_j$.

Indeed, we may assume that $(\tilde{x}_i)$ is not equivalent to the unit vector basis of $c_0$ (and is unconditional) $\lim_{m} \left\| \sum_{i=1}^{m} \tilde{x}_i \right\| = \infty$ so that a) will be satisfied if $(m_k)$ increases sufficiently rapidly. Furthermore, since $(\tilde{x}_i)$ is not equivalent to the unit vector basis of $\ell_1$, $\lim_{m} \left\| \sum_{i=1}^{m} \tilde{x}_i \right\| = 0$ and so b) can also be achieved.

For $i \in \mathbb{N}$ set $A_i = \{ y_F : F$ is admissible and $|F| = m_i \}$ and $A_i^c = \{ f_F : F$ is admissible and $|F| = m_i \}$. Let $\phi$ be an injection into $M$ from the collection of all $(F_1, \ldots, F_i)$ where $i < n$ and $F_1 < F_2 < \cdots < F_i$ are finite subsets of $\mathbb{N}$. Here $F < G$ means $\max F < \min G$. Let

$$\mathcal{F} = \left\{ \sum_{i=1}^{n} f_{F_i} : F_1 < \cdots < F_n, \; \text{each } F_i \text{ is admissible} \right\}$$

and $|F_{i+1}| = \phi(F_1, \ldots, F_i)$ for $1 \leq i < n$.

For $y \in \{(y_i)\}$ let

$$\|y\|_\mathcal{F} = \sup \left\{ |f(y)| : f \in \mathcal{F} \right\}$$

and set

$$|y| = |y|_\mathcal{F} \vee \varepsilon \|y\|.$$ 

This is an equivalent norm since for $f \in \mathcal{F}$, $\|f\| \leq 5n$. Also, $|y_i| = 1$ for all $i$.

Note that if $f_F \in A_i^c$ and $y_F \in A_j$ with $i \neq j$ then

$$|f_F(y_G)| = \left\| \sum_{k \in F} y_k \right\| \sum_{k \in F} y_k \left( \left\| \sum_{k \in G} y_k \right\| \right) \leq \left\| \sum_{k \in F} y_k \right\| \frac{m_i \wedge m_j}{m_i}. $$

If $m_i < m_j$ then $|f_F(y_G)| < \varepsilon$ by a). If $m_i > m_j$ then $|f_F(y_G)| < \varepsilon$ by b).

It follows that if $y = \sum_{i=1}^{n} y_i$ and $f = \sum_{i=1}^{n} f_{F_i} \in \mathcal{F}$ then $|y| \geq f(y) = n$ and if $z = \sum_{i=1}^{n} (-1)^i y_i$, then for all $g = \sum_{j=1}^{n} f_{G_j} \in \mathcal{F}$, $|g(z)| \leq 6 + n^2 \varepsilon < 7$.

Indeed, we may assume that $g \neq f$ and if $F_1 \neq G_1$ then $|G_j| \neq |F_i|$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$ and so by a), b),

$$|g(z)| = \left| \sum_{j=1}^{n} f_{G_j} \left( \sum_{i=1}^{n} (-1)^i y_{F_i} \right) \right| \leq \sum_{j=1}^{n} \sum_{i=1}^{n} |f_{G_j}(y_{F_i})| < n^2 \varepsilon.$$
Otherwise there exists $1 \leq j_0 < n$ so that $F_j = G_j$ for $j \leq j_0$, $|F_{j_0+1}| = |G_{j_0+1}|$ and $|F_i| \neq |G_j|$ for $j_0 + 1 < i, j \leq n$. Using $f_{G_{j_0+1}}(z) \leq 5 + n \varepsilon$ we obtain

$$|g(z)| \leq \left| \sum_{j=1}^{j_0} f_{G_j}(z) \right| + |f_{G_{j_0+1}}(z)| + \left| \sum_{j > j_0+1} f_{G_j}(z) \right|$$

$$< 1 + 5 + n \varepsilon + (n - (j_0 + 1)) n \varepsilon < 6 + n^2 \varepsilon ~.$$ Hence $|z| \leq 7$ follows and the lemma is proved since $n/7 > C$ and such vectors $y$ and $z$ can be produced in any subsequence of $(y_i)$. 

Our next lemma follows from Proposition 3.2 in [AOST].

**Lemma 6.** Let $X$ be a Banach space. Assume that for all $n$, $(x^n_i)_{i=1}^{\infty}$ is a normalized weakly null sequence in $X$ having spreading model $(\tilde{x}_i^n)$ which is not equivalent to the unit vector basis of $\ell_1$. Then there exists a normalized weakly null sequence $(y_i) \subseteq X$ with spreading model $(\tilde{y}_i)$ such that $(\tilde{y}_i)$ is not equivalent to the unit vector basis of $\ell_1$. Moreover, there exists $\lambda > 0$ so that for all $n$,

$$\lambda 2^{-n} \left\| \sum a_i \tilde{x}_i^n \right\| \leq \left\| \sum a_i \tilde{y}_i \right\|$$

for all $(a_i) \subseteq \mathbb{R}$.

**Theorem 7.** Let $X$ be elastic, separable and infinite dimensional. Then $c_0$ embeds into $X$.

We postpone the proof to complete first the

**Proof of Theorem 1.** Assume that $X$ is infinite dimensional and every isomorph of $X$ is $K$-elastic. Then by Theorem 7, $c_0$ embeds into $X$. Choose $k_n \uparrow \infty$ so that $2^{-n}k_n \to \infty$. Using the renormings of $c_0$ by

$$|(a_i)|_n = \sup \left\{ \left\| \sum a_i \right\| : \left\| F \right\| = k_n \right\}$$

and that $X$ is $K$-elastic we can find for all $n$ a normalized weakly null sequence $(x^n_i)_{i=1}^{\infty} \subseteq X$ with spreading model $(\tilde{x}_i^n)_{i=1}^{\infty}$ satisfying

$$\left\| \sum a_i \tilde{x}_i^n \right\| \geq K^{-1} |(a_i)|_n$$

and moreover each $(\tilde{x}_i^n)$ is equivalent to the unit vector basis of $c_0$. Thus by Lemma 6 there exists a normalized weakly null sequence $(y_i)$ in $X$ having spreading model $(\tilde{y}_i)$ which is not equivalent to the unit vector basis of $\ell_1$ and which satisfies for all $n$,

$$\left\| \sum_{i=1}^{k_n} \tilde{y}_i \right\| \geq \lambda K^{-1} 2^{-n} k_n \to \infty .$$

Thus $(\tilde{y}_i)$ is not equivalent to the unit vector basis of $c_0$ as well.
By Lemmas 2 and 5, for all $C < \infty$ we can find $n \in \mathbb{N}$ and a renorming $Y$ of $X$ so that $Y$ contains a normalized weakly null sequence admitting no subsequence which is block $n$-unconditional with constant $C$. By the assumption on $X$, the space $Y$ must $K$-embed into every isomorph of $X$. But if $C$ is large enough this contradicts Lemma 4.

It remains to prove Theorem 7. We shall employ an index argument involving $\ell_\infty$-trees defined on Banach spaces. If $Y$ is a Banach space our trees $T$ on $Y$ will be countable. For some $C$ the nodes of $T$ will be elements $(y_i)^n \subseteq Y$ with $(y_i)^n$ bimonotone basic and satisfying $1 \leq \|y_i\|$ and $\|\sum_1^n \pm y_i\| \leq C$ for all choices of sign. Thus $(y_i)^n$ is $C$-equivalent to the unit vector basis of $\ell^n_\infty$. $T$ is partially ordered by $(x_i)^n \leq (y_i)^m$ if $n \leq m$ and $x_i = y_i$ for $i \leq n$. The order $o(T)$ is given as follows. If $T$ is not well founded (i.e., $T$ has an infinite branch), then $o(T) = \omega_1$. Otherwise we set for such a tree $S$, $S' = \{(x_i)^n \in S : (x_i)^n$ is not a maximal node$\}$. Set $T_0 = T$, $T_1 = T'$ and in general $T_{\alpha+1} = (T_\alpha)'$ and $T_\alpha = \cap_{\beta<\alpha} T_\beta$ if $\alpha$ is a limit ordinal. Then

$$o(T) = \inf \{\alpha : T_\alpha = \phi \}.$$ 

By Bourgain’s index theory [B1], [B2] (see also [AGR]), if $X$ is separable and contains for all $\beta < \omega_1$ such a tree of index at least $\beta$, then $c_0$ embeds into $X$.

We now complete the

**Proof of Theorem 7.** Without loss of generality we may assume that $X \subseteq Z$ where $Z$ has a bimonotone basis $(z_i)$. Let $X$ be $K$-elastic. We will often use semi-normalized sequences in $X$ which are a tiny perturbation of a block basis of $(z_i)$ and to simplify the estimates we will assume below that they are in fact a block basis of $(z_i)$.

For example, if $(y_i)$ is a normalized basic sequence in $X$ then we call $(d_i)$ a *difference sequence of* $(y_i)$ if $d_i = y_{k(2i)} - y_{k(2i+1)}$ for some $k_1 < k_2 < \cdots$. We can always choose such a $(d_i)$ to be a semi-normalized perturbation of a block basis of $(z_i)$ by first passing to a subsequence $(y'_i)$ of $(y_i)$ so that $\lim_{i \to \infty} z^*_j(y'_i)$ exists for all $j$, where $(z^*_j)$ is biorthogonal to $(z_i)$, and taking $(d_i)$ to be a suitable difference sequence of $(y'_i)$. We will assume then that $(d_i)$ is in fact a block basis of $(z_i)$.

We inductively construct for each *limit* ordinal $\beta < \omega_1$, a Banach space $Y_\beta$ that embeds into $X$. $Y_\beta$ will have a normalized bimonotone basis $(y_{\beta}^i)$ that can be enumerated as $(y_{\beta}^i)^\infty_{i=1} = \{y_{\beta}^{\rho,n} : \rho \in C_\beta, n \in \mathbb{N}, i \in \mathbb{N}\}$ where $C_\beta$ is some countable set. The order is such that $(y_{\beta}^{\rho,n})^\infty_{i=1}$ is a subsequence of $(y_{\beta}^i)$ for fixed $\rho$ and $n$.

Before stating the remaining properties of $(y_{\beta}^i)$ we need some terminology. We say that $(w_i)$ is a *compatible difference sequence of* $(y_{\beta}^i)$ of order 1 if $(w_i)$
is a difference sequence of \((y_i^\beta)\) that can be enumerated as follows,

\[
(w_i) = \{w_i^{\beta,\rho,n} : \rho \in C, \, n, i \in \mathbb{N}\}
\]

and such that for fixed \(\rho\) and \(n\),

\[
(w_i^{\beta,\rho,n})_i \text{ is a difference sequence of } (y_i^{\beta,\rho,n+1})_i.
\]

If \((v_i)\) is a compatible difference sequence of \((w_i)\) of order 1, in the above sense, \((v_i)\) will be called a compatible difference sequence of \((y_i^\beta)\) of order 2, and so on. \((y_i^\beta)\) will be said to have order 0.

Let \((v_i)\) be a compatible difference sequence of \((y_i^\beta)\) of some finite order.

We set

\[
T((v_i)) = \left\{(u_i)^s : \text{the } u_i's \text{ are distinct elements of } \{v_i\}^\infty, \text{possibly in different order, and } \left| \sum_{1}^{s} \pm u_i \right| = 1 \text{ for all choices of sign} \right\}.
\]

\(T((v_i))\) is then an \(\ell^\infty\)-tree as described above with \(C = 1\). The inductive condition on \(Y^\beta\), or should we say on \((y_i^{\beta,\rho,n})\), is that for all compatible difference sequences \((v_i)\) of \((y_i^{\beta,\rho,n})\) of finite order,

\[
o(T((v_i))) \geq \beta.
\]

Before proceeding we have an elementary but key

**Sublemma.** Let \(C < \infty\) and let \((w_i)\) be a block basis of a bimonotone basis \((z_i)\) with \(1 \leq \|w_i\| \leq C\) for all \(i\) and let

\[
A = \{F \subseteq \mathbb{N} : F \text{ is finite and } \left| \sum_{i \in F} \pm w_i \right| \leq C \text{ for all choices of sign}\}.
\]

Then there exists an equivalent norm \(\cdot\) on \([([w_i])]\) so that \((w_i)\) is a bimonotone normalized basis such that for all \(F \in A\),

\[
\left| \sum_{F} \pm w_i \right| = 1.
\]

**Proof.** Define \(\sum a_i w_i = \|(a_i)\|_\infty \vee C^{-1} \sum a_i w_i\). \(\Box\)

We begin by constructing \(Y_w\). Let \((x_i) \subseteq X\) be a normalized block basis of \((z_i)\). For \(n \in \mathbb{N}\), let \(\cdot|_n\) be an equivalent norm on \([([x_i])]\) given by the sublemma for \(C = 2^n\). Thus \(\sum_F \pm x_i|_n = 1\) if \(|F| \leq 2^n\).

Since \(X\) is \(K\)-elastic, for all \(n\), \((([x_i]), \cdot|_n)\) \(K\)-embeds into \(X\). We thus obtain for \(n \in \mathbb{N}\), a sequence \((x_n^i)_i \subseteq X\) with \(1 \leq \|x_n^i\| \leq K\) for all \(i\) and such that \(1 \leq \|\sum_{i \in F} \pm x_n^i\| \leq K\) for all \(|F| \leq 2^n\) and all choices of sign. Furthermore
(x^n_i) is K-basic. By standard perturbation and diagonal arguments we may for each n pass to a difference sequence (d^n_i) of (x^{n+1}_i) so that enumerating, (d_i) = \{d^n_i : n \in \mathbb{N}\} is a block basis of (z_i) with 1 \leq \|d_i\| \leq K and with each (d^n_i)_i being a subsequence of (d_i). We have that for |F| \leq 2^n and all signs,

$$1 \leq \left\| \sum_{i \in F} \pm d^n_i \right\| \leq K .$$

We renorm [(d_i)] by the sublemma for C = K and let the ensuing space be Y_\omega. We change the name of (d_i) to (y^\omega_i) in this new norm and let (y^\omega_i)_i = (d^n_i)_i. (y^\omega_i) has the property that if (w_i) is a compatible difference sequence of (y^\omega_i) of finite order, then o(T((w_i))) \geq \omega. Indeed, if (w_i) = \{w^\omega_i : n \in \mathbb{N}, i \in \mathbb{N}\} then for |F| \leq 2^n, \| \sum_{i \in F} w^\omega_i \| = 1.

Assume that Y_\beta has been constructed for the limit ordinal \beta with basis (y^\beta_i) = \{y^\beta_i : p \in C_\beta, n, i \in \mathbb{N}\} with the requisite properties above.

Let U : Y_\omega \to X and V : Y_\beta \to X be K-embeddings. Since in total we are dealing with a countable set of sequences, namely (y^\omega_i)_i for n \in \mathbb{N} and (y^\beta_i)_i for p \in C_\beta, n \in \mathbb{N}, by diagonalization and perturbation we can find a compatible difference sequence (w^\omega_i)_i of (y^\omega_i)_i of order 1 and a compatible difference sequence (w^\beta_i)_i of (y^\beta_i)_i of order 1 so that under a suitable reordering, (d_i) = (Uw^\omega_i)_i \cup (Vw^\beta_i)_i is a block basis of (z_i). Moreover each (Uw^\omega_i)_i and (Vw^\beta_i)_i is a subsequence of (d_i)_i.

Adjoin a new point p_0 to C_\beta and set C_{\beta+\omega} = C_\beta \cup \{p_0\} . Let d^\beta+\omega,\rho,n_i = Vw^\beta_i \rho,n for \rho \in C_\beta and d^\beta+\omega,\rho_0,n_i = Uw^\omega_i \rho_0,n . For |F| \leq 2^n and G \subseteq C_\beta \times \mathbb{N} \times \mathbb{N} for which \| \sum_{(\rho, n, i) \in G} \pm w^\beta_i \| = 1, we have by the triangle inequality that

\[ (*) \quad \left\| \sum_{i \in F} \pm d^\beta+\omega,\rho_0,n_i + \sum_{(\rho, n, i) \in G} \pm d^\beta+\omega,\rho,n_i \right\| \leq 2K . \]

It follows that if we let (y^\beta+\omega_i) be the basis (d^\beta+\omega_i), renormed by the sublemma for C = 2K, that Y_{\beta+\omega} = [(y^\beta+\omega_i)] has the required properties. Indeed (*) yields that o(T((y^\beta+\omega)_i)) \geq \beta + 2^n for n \in \mathbb{N} and so o(T((y^\beta+\omega)_i)) \geq \beta + 2^n. Moreover by our induction hypothesis and choice of (d^\beta+\omega,\rho_0,n_i) the analogue of (*) holds for a compatible difference sequence of order 1 or of any finite order. Thus o(T(z^\beta+\omega_i)) \geq \beta + \omega for any compatible difference sequence of (y^\beta+\omega) of finite order.

If \beta is a limit ordinal not of the form \alpha + \omega we let U_\alpha : Y_\alpha \to X be a K-embedding for each limit ordinal \alpha < \beta. We again diagonalize to form compatible difference sequences (w^\alpha_i)_i of (y^\alpha_i) of order 1 for each such \alpha so that (U_\alpha w^\alpha_i)_{\alpha,i} is a block basis of (z_i) in some order. We let C_\beta be a disjoint
union of the $C_{\alpha}$’s and in the manner above obtain $Y_{\beta}$. Again by our induction hypothesis we will have that $\sigma(T((y_{\beta}^i))) \geq \alpha$ for all limit ordinals $\alpha < \beta$ and the same holds for all compatible difference sequences of $(y_{\beta}^i)$ of finite order.

3. Concluding remarks and problems

$C[0,1]$ is, of course, 1-elastic. By virtue of Lemma 4 (or Theorem 1), for all $K$, $C[0,1]$ can be renormed to be elastic but not $K$-elastic. Are there other examples of separable elastic spaces?

**Problem 8.** Let $X$ be elastic (and separable, say). Does $C[0,1]$ embed into $X$?

Using index arguments, we have the following partial result.

**Proposition 9.** Let $X$ be a separable Banach space, and suppose that $Y = \sum X_n$ is a symmetric decomposition of a space $Y$ into spaces uniformly isomorphic to $X$. If $Y$ is elastic, then $C[0,1]$ embeds into $X$.

In particular, if $1 \leq p < \infty$ and $\ell_p(X)$ is elastic, then $C[0,1]$ embeds into $X$.

**Proof.** Let us first observe that if $C[0,1]$ embeds into $Y$, then $C[0,1]$ embeds into $X$. Since this is surely well known, we just sketch a proof (which, incidentally, uses only that the decomposition $Y = \sum X_n$ of $Y$ is unconditional): Let $P_n$ be the projection from $Y$ onto $X_n$ and let $W$ be a subspace of $Y$ which is isomorphic to $C[0,1]$. By a theorem of Rosenthal’s [R], it is enough to show that for some $n$, the adjoint $P_n^*|W$ of the restriction of $P_n$ to $W$ has nonseparable range. This will be true if there is an $m$ so that $S_m^*|W$ has nonseparable range, where $S_n := \sum_{i=1}^m P_i$. Let $Z$ be a subspace of $W$ which is isomorphic to $\ell_1$. If no such $m$ exists, then for every $m$, the restriction of $S_m$ to $Z$ is strictly singular (that is, not an isomorphism on any infinite dimensional subspace of $Z$), and it then follows that $Z$ contains a sequence $(x_n)$ of unit vectors which is an arbitrarily small perturbation of a sequence $(y_n)$ which is disjointly supported. The sequence $(y_n)$, a fortiori $(x_n)$, is then easily seen to be equivalent to the unit vector basis of $\ell_1$ and its closed span is complemented in $Y$ since the decomposition is unconditional. It follows that $\ell_1$ is isomorphic to a complemented subspace of $C[0,1]$, which of course is false.

To complete the proof of Proposition 9, we assume that $Y$ is $K$-elastic and prove that $C[0,1]$ embeds into $Y$. The proof is similar to, but simpler than, the proof of Theorem 7. First we recall the definition of certain canonical trees $T_\alpha$ of order $\alpha$ for $\alpha < \omega_1$ (see e.g. [JO]). These form the frames upon which we will hang our bases. The tree $T_1$ is a single node. If $T_\alpha$ has been defined, we choose a new node $z \not\in T_\alpha$ and set $T_{\alpha+1} := T_\alpha \cup \{z\}$, ordered by $z < t$ for
all \( t \in T_\alpha \) and with \( T_\alpha \) preserving its order. If \( \beta < \omega_1 \) is a limit ordinal, we let \( T_\beta \) be the disjoint union of \( \{ T_\alpha : \alpha < \beta \} \). Then if \( s,t \) are in \( T_\beta \), we say that \( s \leq t \) if and only if \( s,t \) are both in \( T_\alpha \) for some \( \alpha < \beta \) and \( s \leq t \) in \( T_\alpha \).

We shall prove by transfinite induction that if \( (x_i)_{i=1}^\infty \) is any normalized bimonotone basic sequence and \( \alpha < \omega_1 \), there is a Banach space \( Y_\alpha = Y_\alpha (x_i) \) with a normalized bimonotone basis \( (y^\alpha_i)_{i=1}^\infty \) so that if \( (\gamma_i)_{i=1}^n \) is any branch in \( T_\alpha \) then \( (y^\alpha_{\gamma_i})_{i=1}^n \) is 1-equivalent to \( (x_i)_{i=1}^n \) and for all scalars \( (a_i)_{i=1}^n \), \( \| \sum_{i=1}^n a_i y^\alpha_{\gamma_i} \| \leq \| \sum_{i=1}^n a_i y_i \| \). Furthermore, each \( Y_\alpha \) will \( K \)-embed into \( Y \).

Just as in the proof of Theorem 7, it then follows from index theory that the Banach space spanned by \( (x_i)_{i=1}^\infty \) embeds into \( Y \).

Fix any normalized bimonotone basic sequence \( (x_i)_{i=1}^\infty \). Suppose that \( \beta \) is a limit ordinal and \( Y_\alpha = Y_\alpha (x_i)_{i=1}^\infty \) has been defined for all \( \alpha < \beta \). In view of the hypotheses, the space \( Y \) has a symmetric decomposition into spaces uniformly isomorphic \( Y \), which we can index as \( Y = \sum_{\alpha < \beta} X_\alpha \). For each \( \alpha \), there is an isomorphism \( L_\alpha \) from \( Y_\alpha \) into \( X_\alpha \) so that \( \| L_\alpha \| = 1 \) and \( \| L_\alpha^{-1} \| \) is bounded independently of \( \alpha \). We can put an equivalent norm on \( Y = \sum_{\alpha < \beta} X_\alpha \) to make each \( L_\alpha \) an isometry and make the decomposition 1-unconditional (but not necessarily 1-symmetric). Define \( Y_\beta \) to be the closed linear span of \( \{ L_\alpha Y_\alpha : \alpha < \beta \} \) in \( Y \) with its new norm. The space \( Y_\beta \) has the desired basis indexed by \( T_\beta \) and \( Y_\alpha \) must \( K \)-embed into \( Y \) with its original norm because \( Y \) is \( K \)-elastic.

If \( \beta = \alpha + 1 \), we let \( Y_\beta (x_i)_{i=1}^\infty = \mathbb{R} \oplus Y_\alpha (x_i)_{i=2}^\infty \) with the norm given by

\[
\|(a,y)\| := \|y\| \vee \sup\{\|a x_1 + \sum_{i=2}^n y^\alpha_{\gamma_i}(y) y^\alpha_{\gamma_i}\| : (\gamma_i)_{i=2}^n \text{ is a branch in } T_\alpha\}.
\]

Again it is clear that \( Y_\beta \) must \( K \)-embed into \( Y \) and that \( Y_\beta \) has the desired basis.

Given metric spaces \( X \) and \( Y \), the Lipschitz distance \( d_L(X,Y) \) between them is defined to be the infimum of \( \text{Lip}(f) \cdot \text{Lip}(f^{-1}) \), where the infimum is taken over all biLipschitz mappings \( f \) from \( X \) onto \( Y \) (the infimum of the empty set is \( \infty \)). Call a Banach space \( X \) Lipschitz \( K \)-elastic provided that every isomorph of \( X \) has Lipschitz distance at most \( K \) to some subset of \( X \), and say that \( X \) is Lipschitz elastic if \( X \) is Lipschitz \( K \)-elastic for some \( K < \infty \). It is interesting to note that the analogue of Theorem 1 for Lipschitz elasticity is false. Indeed, Aharoni [A], [BeLi, Theorem 7.11] proved that every separable metric space has, for each \( \epsilon > 0 \), Lipschitz distance less than \( 3 + \epsilon \) to some subset of \( c_0 \), while James [J], [BeLi, Proposition 13.6] proved that for each \( \epsilon > 0 \), the space \( c_0 \) is \( 1 + \epsilon \)-equivalent to a subspace of every isomorph of \( c_0 \). Consequently, any separable Banach space which contains an isomorphic copy of \( c_0 \) is Lipschitz \( 3 + \epsilon \)-elastic.
**Problem 10.** If $X$ is a separable, infinite dimensional Banach space and there is a $K$ so that every isomorph of $X$ is Lipschitz $K$-elastic, must $X$ contain a subspace isomorphic to $c_0$?

We also do not know whether the Main Theorem has a Lipschitz analogue:

**Problem 11.** If $X$ is a separable Banach space and there exists $K < \infty$ so that $d_L(X, Y) < K$ for all spaces $Y$ which are isomorphic (or biLipschitz equivalent) to $X$, then must $X$ be finite dimensional?

The space $c_0$ does not have the property mentioned in Problem 11 (use, for example, the fact [GKL] that for every $K < \infty$ there exists $K' < \infty$ so that if a Banach space $X$ is $K$ biLipschitz equivalent to $c_0$ then $d(X, c_0) \leq K'$).

Since Schäffer’s problem remains open in the nonseparable setting, we mention it explicitly as:

**Problem 12.** Does there exist a nonseparable Banach space $X$ for which $D(X) < \infty$?

Also open is:

**Problem 13.** Does there exist a nonseparable Banach space $X$ and a $K < \infty$ so that all isomorphs of $X$ are $K$-elastic?

Under the generalized continuum hypothesis, it follows from [E] that for every cardinal $\aleph$ there is a Banach space $X$ of density character $\aleph$ so that every Banach space of density character $\aleph$ is isometrically isomorphic to a subspace of $X$. These provide the only known examples of elastic Banach spaces.

**Problem 14.** If $X$ is elastic and infinite dimensional, does $X$ contain an isomorph of every Banach space whose density character is the same as the density character of $X$?

As was mentioned earlier, the space $\ell_\infty$ is, for every $\epsilon > 0$, $1+\epsilon$-isomorphic to a subspace of every isomorph of itself (see [P] and [T]). However, $\ell_\infty$ is not elastic. To see this, take a biorthogonal system $(x_\alpha, x^*_\alpha)_{\alpha < 2^{2^{\aleph}}}$ in $\ell_\infty$ with $\|x_\alpha\| = 1$ and $\|x^*_\alpha\|$ bounded. Given a natural number $m$, renorm $\ell_\infty$ by

$$
\|x\|_m := \max \left\{ \frac{1}{m} \|x\|_\infty, \sup_{|\sigma| = m} \left( \sum_{\alpha \in \sigma} |x^*_\alpha(x)|^2 \right)^{1/2} \right\}.
$$

It is not hard to see that $(\ell_\infty, \|\cdot\|_m)$ is not better than $\sqrt{m}$-normed by any countable set of norm one functionals, which implies that $d((\ell_\infty, \|\cdot\|_m), Y) \geq \sqrt{m}$ for any subspace $Y$ of $\ell_\infty$. A similar argument shows that $\ell_\infty(\aleph)$ is not elastic for any infinite cardinal number $\aleph$. 
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