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Well-posedness for the motion of an incompressible liquid with free surface boundary

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Abstract

We study the motion of an incompressible perfect liquid body in vacuum. This can be thought of as a model for the motion of the ocean or a star. The free surface moves with the velocity of the liquid and the pressure vanishes on the free surface. This leads to a free boundary problem for Euler's equations, where the regularity of the boundary enters to highest order. We prove local existence in Sobolev spaces assuming a "physical condition", related to the fact that the pressure of a fluid has to be positive.

1. Introduction

We consider Euler's equations describing the motion of a perfect incompressible fluid in vacuum:

$$\big(\partial_t + V^k\partial_k\big)v_j + \partial_j p = 0, \quad j=1,\dots,n \quad \text{in} \quad \mathcal{D}, \tag{1.1}$$

$$\operatorname{div} V = \partial_k V^k = 0 \quad \text{in} \quad \mathcal{D} \tag{1.2}$$

where $\partial_i = \partial/\partial x^i$ and $\mathcal{D} = \cup_{0\le t\le T}\,\{t\}\times\mathcal{D}_t$, $\mathcal{D}_t \subset \mathbb{R}^n$. Here $V^k = \delta^{ki}v_i = v_k$, and we use the convention that repeated upper and lower indices are summed over. V is the velocity vector field of the fluid, p is the pressure and $\mathcal{D}_t$ is the domain the fluid occupies at time t. We also require boundary conditions on the free boundary $\partial\mathcal{D} = \cup_{0\le t\le T}\,\{t\}\times\partial\mathcal{D}_t$;

$$p = 0, \quad \text{on} \quad \partial\mathcal{D}, \tag{1.3}$$

$$(\partial_t + V^k\partial_k)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}). \tag{1.4}$$

Condition (1.3) says that the pressure p vanishes outside the domain and condition (1.4) says that the boundary moves with the velocity V of the fluid particles at the boundary.

Given a domain $\mathcal{D}_0 \subset \mathbb{R}^n$, that is homeomorphic to the unit ball, and initial data v_0, satisfying the constraint (1.2), we want to find a set $\mathcal{D} =$

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$\cup_{0\le t\le T}\{t\}\times\mathcal{D}_t$, $\mathcal{D}_t\subset\mathbb{R}^n$ and a vector field v solving (1.1)–(1.4) with initial conditions

$$\{x;\,(0,x)\in\mathcal{D}\}=\mathcal{D}_0,\qquad\text{and}\qquad v=v_0,\quad\text{on}\quad\{0\}\times\mathcal{D}_0. \tag{1.5}$$

Let $\mathcal{N}$ be the exterior unit normal to the free surface $\partial\mathcal{D}_t$. Christodoulou[C2] conjectured that the initial value problem (1.1)–(1.5), is well-posed in Sobolev spaces if

$$\nabla_{\mathcal{N}}\,p\le -c_0<0,\quad\text{on}\quad\partial\mathcal{D},\qquad\text{where}\quad\nabla_{\mathcal{N}}=\mathcal{N}^i\partial_{x^i}. \tag{1.6}$$

Condition (1.6) is a natural *physical condition* since the pressure p has to be positive in the interior of the fluid. It is essential for the well-posedness in Sobolev spaces. A condition related to Rayleigh-Taylor instability in [BHL], [W1] turns out to be equivalent to (1.6); see [W2]. With the divergence of (1.1)

$$-\triangle p=(\partial_j V^k)\partial_k V^j,\qquad\text{in}\quad\mathcal{D}_t,\qquad p=0,\quad\text{on}\quad\partial\mathcal{D}_t. \tag{1.7}$$

In the irrotational case, when $\operatorname{curl} v_{ij}=\partial_i v_j-\partial_j v_i=0$, then $\triangle p\le 0$ so that $p\ge 0$ and (1.6) holds by the strong maximum principle. Furthermore Ebin [E1] showed that the equations are ill-posed when (1.6) is not satisfied and the pressure is negative and Ebin [E2] announced an existence result when one adds surface tension to the boundary condition which has a regularizing effect so that (1.6) is not needed.

The incompressible perfect fluid is to be thought of as an idealization of a liquid. For small bodies like water drops surface tension should help holding it together and for larger denser bodies like stars its own gravity should play a role. Here we neglect the influence of such forces. Instead it is the incompressibility condition that prevents the body from expanding and it is the fact that the pressure is positive that prevents the body from breaking up in the interior. Let us also point out that, from a physical point of view one can alternatively think of the pressure as being a small positive constant on the boundary instead of vanishing. What makes this problem difficult is that the regularity of the boundary enters to highest order. Roughly speaking, the velocity tells the boundary where to move and the boundary is the zero set of the pressure that determines the acceleration.

In general it is possible to prove local existence for analytic data for the free interface between two fluids. However, this type of problem might be subject to instability in Sobolev norms, in particular Rayleigh-Taylor instability, which occurs when a heavier fluid is on top of a lighter fluid. Condition (1.6) prevents Rayleigh-Taylor instability from occurring. Indeed, if condition (1.6) is violated Rayleigh-Taylor instability occurs in a linearized analysis.

Some existence results in Sobolev spaces were known in the irrotational case, for the closely related water wave problem which describes the motion of the surface of the ocean under the influence of earth's gravity. The gravitational field can be considered as uniform and it reduces to our problem by going to an accelerated frame. The domain $\mathcal{D}_t$ is unbounded for the water wave problem coinciding with a half-space in the case of still water. Nalimov [Na] and Yosihara [Y] proved local existence in Sobolev spaces in two space dimensions for initial conditions sufficiently close to still water. Beale, Hou and Lowengrab [BHL] have given an argument to show that this problem is linearly well-posed in a weak sense in Sobolev spaces, assuming a condition, which can be shown to be equivalent to (1.6). The condition (1.6) prevents the Rayleigh-Taylor instability from occurring when the water wave turns over. Finally Wu [W1], [W2] proved local existence in the general irrotational case in two and three dimensions for the water wave problem. The methods of proofs in these papers use the facts that the vector field is irrotational to reduce to equations on the boundary and they do not generalize to deal with the case of nonvanishing curl.

We consider the general case of nonvanishing curl. With Christodoulou [CL] we proved local *a priori* bounds in Sobolev spaces in the general case of nonvanishing curl, assuming (1.6) holds initially. Usually if one has *a priori* estimates, existence follows from similar estimates for some regularization or iteration scheme for the equation, but the sharp estimates in [CL] use all the symmetries of the equations and so only hold for perturbations of the equations that preserve the symmetries. In [L1] we proved existence for the linearized equations, but the estimates for the solution of the linearized equations lose regularity compared to the solution we linearize around, and so existence for the nonlinear problem does not follow directly. Here we use improvements of the estimates in [L1] together with the Nash-Moser technique to show local existence for the nonlinear problem in the smooth class:

THEOREM 1.1. *Suppose that* v_0 *and* $\partial\mathcal{D}_0$ *in* (1.5) *are smooth*, $\mathcal{D}_0$ *is diffeomorphic to the unit ball*, *and that* (1.6) *holds initially when* $t = 0$. *Then there is a* $T > 0$ *such that* (1.1)–(1.5) *has a smooth solution for* $0 \le t \le T$, *and* (1.6) *holds with* c_0 *replaced by* $c_0/2$ *for* $0 \le t \le T$.

In [CL] we proved local energy bounds in Sobolev spaces. It now follows from the bounds there that the solution remains smooth as long as it is C^2 and the physical condition (1.6) holds. The existence for smooth data now implies existence in the Sobolev spaces considered in [CL]. Moreover, the method here also works for the compressible case [L2], [L3].

Let us now describe the main ideas and difficulties in the proof. In order to construct an iteration scheme we must first introduce some parametrization in which the moving domain becomes fixed. We express Euler's equations in this fixed domain. This is achieved by the Lagrangian coordinates given by following the flow lines of the velocity vector field of the fluid particles.

In [L1] we studied the linearized equations of Euler's equations expressed in Lagrangian coordinates. We proved that the linearized operator is invertible at a solution of Euler's equations. The linearized equations become an evolution equation for what we call the normal operator, (2.17). The normal operator is unbounded and not elliptic but it is symmetric and positive on divergence-free vector fields if (1.6) holds. This leads to energy bounds; existence for the linearized equations follows from a delicate regularization argument. The solution of the linearized equations however loses regularity compared to the solution we linearize around so that existence for the nonlinear problem does not follow directly from an inverse function theorem in a Banach space, but we must use the Nash-Moser technique.

We first define a nonlinear functional whose zero will be a solution of Euler's equations expressed in the Lagrangian coordinates. Instead of defining our map by the left-hand sides of (1.1) and (1.2) expressed in the Lagrangian coordinates, we let our map be given by the left-hand side of (1.1) and we let pressure be implicitly defined by (1.7) satisfying the boundary condition (1.3). This is because one has to make sure that the pressure vanishes on the boundary at each step of an iteration or else the linearized operator is ill-posed. One can see this by looking at the irrotational case where one gets an evolution equation on the boundary. If the pressure vanishes on the boundary then one has an evolution equation for a positive elliptic operator but if it does not vanish on the boundary there will also be some tangential derivative, no matter how small the coefficients they come with, the equation will have exponentially growing Fourier modes.

In order to use the Nash-Moser technique one has to be able to invert the linearized operator in a neighborhood of a solution of Euler's equations or at least do so up to a quadratic error [Ha]. In this paper we generalize the existence in [L1] so that the linearized operator is invertible in a neighborhood of a solution of Euler's equations and outside the class of divergence-free vector fields. This does present a difficulty because the normal operator, introduced in [L1], is only symmetric on divergence-free vector fields and in general it loses regularity. Overcoming this difficulty requires two new observations. The first is that, also for the linearized equations, there is an identity for the curl that gives a bound that is better than expected. The second is that one can bound any first order derivative of a vector field by the curl, the divergence and the normal operator times one over the constant c_0 in (1.6). Although the normal operator is not elliptic on general vector fields it is elliptic on irrotational divergence-free vector fields and in general one can invert it if one also has bounds for the curl and the divergence.

The methods here and in [CL] are on a technical level very different but there are philosophical similarities. First we fix the boundary by introducing Lagrangian coordinates. Secondly, we take the geometry of the boundary into account: here, in terms of the normal operator and Lie derivatives with respect to tangential vector fields and in [CL], in terms of the second fundamental form of the boundary and tangential components of the tensor of higher order derivatives. Thirdly, we use interior estimates to pick up the curl and the divergence. Lastly, we get rid of a difficult term, the highest order derivative of the pressure, by projecting. Here we use the orthogonal projection onto divergence-free vector fields whereas in [CL] we used the local projection of a tensor onto the tangent space of the boundary.

The paper is organized as follows. In Section 2 we reformulate the problem in the Lagrangian coordinates and give the nonlinear functional of which a solution of Euler's equation is a zero, and we derive the linearized equations in this formulation. In Section 2 we also give an outline of the proof and state the main steps to be proved. The main part of the paper, Sections 3 to 13 are devoted to proving existence and tame energy estimates for the inverse of the linearized operator. Once this is proven, the remaining Sections 14 to 18 are devoted to setting up the Nash-Moser theorem we are using.

2. Lagrangian coordinates and the linearized operator

Let us first introduce the Lagrangian coordinates in which the boundary becomes fixed. By a scaling we may assume that $\mathcal{D}_0$ has the volume of the unit ball Ω and since we assumed that $\mathcal{D}_0$ is diffeomorphic to the unit ball we can, by a theorem in [DM], find a volume-preserving diffeomorphism $f_0 : \Omega \to \mathcal{D}_0$, i.e. det $(\partial f_0/\partial y) = 1$. Assume that $v(t,x)$, $p(t,x)$, $(t,x) \in \mathcal{D}$ are given satisfying the boundary conditions (1.3)–(1.4). The Lagrangian coordinates $x = x(t,y) = f_t(y)$ are given by solving

$$\frac{dx(t,y)}{dt} = V(t, x(t,y)), \qquad x(0,y) = f_0(y), \quad y \in \Omega. \tag{2.1}$$

Then $f_t : \Omega \to \mathcal{D}_t$ is a volume-preserving diffeomorphism, if $\operatorname{div} V = 0$, and the boundary becomes fixed in the new y coordinates. Let us introduce the material derivative:

$$D_t = \frac{\partial}{\partial t}\Big|_{y=\text{constant}} = \frac{\partial}{\partial t}\Big|_{x=\text{constant}} + V^k \frac{\partial}{\partial x^k}. \tag{2.2}$$

The partial derivatives $\partial_i = \partial/\partial x^i$ can then be expressed in terms of partial derivatives $\partial_a = \partial/\partial y^a$ in the Lagrangian coordinates. We will use letters $a, b, c, \dots, f$ to denote partial differentiation in the Lagrangian coordinates and $i, j, k, \dots$ to denote partial differentiation in the Eulerian frame.

In these coordinates Euler's equation (1.1) become

$$D_t^2 x_i + \partial_i p = 0, \qquad (t,y) \in [0,T] \times \Omega, \tag{2.3}$$

where now $x_i = x_i(t,y)$ and $p = p(t,y)$ are functions on $[0,T] \times \Omega$, D_t is just the partial derivative with respect to t and $\partial_i = (\partial y^a/\partial x^i)\partial_a$, where ∂_a is differentiation with respect to y^a. Now, (1.7) becomes

$$\triangle p + (\partial_i V^k)\partial_k V^i = 0, \qquad p\Big|_{\partial\Omega} = 0, \qquad \text{where} \quad V^i = D_t x^i. \tag{2.4}$$

Here

$$\triangle p = \sum_{i=1}^n \partial_i^2 p = \kappa^{-1}\partial_a\big(\kappa g^{ab}\partial_b p\big) \qquad \text{where} \quad g_{ab} = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}, \tag{2.5}$$

and g^{ab} is the inverse of the metric g_{ab} and $\kappa = \det(\partial x/\partial y) = \sqrt{\det g}$. The initial conditions (1.5) becomes

$$x\Big|_{t=0} = f_0, \qquad D_t\, x\Big|_{t=0} = v_0. \tag{2.6}$$

Christodoulou's physical condition (1.6) becomes

$$\nabla_{\mathcal{N}}\, p \le -c_0 < 0, \quad \text{on} \quad \partial\Omega, \qquad \text{where} \quad \nabla_{\mathcal{N}} = \mathcal{N}^i\partial_{x^i}. \tag{2.7}$$

This is needed in the proof for the normal operator (2.17) to be positive which leads to energy bounds. In addition to (2.7) we also need to assume a coordinate condition having to do with the facts that we are looking for a solution in the Lagrangian coordinates and we are starting by composing with a particular diffeomorphism. The coordinate conditions are

$$|\partial x/\partial y|^2 + |\partial y/\partial x|^2 \le c_1^2, \qquad \sum_{a,b=1}^n \big(|g^{ab}| + |g_{ab}|\big) \le n c_1^2, \tag{2.8}$$

where $|\partial x/\partial y|^2 = \sum_{i,a=1}^n (\partial x^i/\partial y^a)^2$. This is needed for (2.5) to be invertible. We note that the second condition in (2.8) follows from the first and the first follows from the second with a larger constant. We remark that this condition is fulfilled initially since we are composing with a diffeomorphism. Furthermore, for a solution of Euler's equations, $\operatorname{div} V = 0$, so the volume form κ is preserved and hence an upper bound for the metric also implies a lower bound for the eigenvalues; an upper bound for the inverse of the metric follows. However, in the iteration, we will go outside the divergence-free class and hence we must make sure that both (2.7) and (2.8) hold at each step of the iteration. We will prove the following theorem:

THEOREM 2.1. *Suppose that initial data* (2.6) *are smooth*, v_0 *satisfy the constraint* (1.2), *and that* (2.7) *and* (2.8) *hold when* $t = 0$. *Then there is* $T > 0$ *such that* (2.3), (2.4) *have a solution* $x, p \in C^\infty([0,T]\times\overline{\Omega})$. *Furthermore*, (2.7), (2.8) *hold, for* $0 \le t \le T$, *with* c_0 *replaced by* $c_0/2$ *and* c_1 *replaced by* $2c_1$.

Theorem 1.1 follows from Theorem 2.1. In fact, the assumption that $\mathcal{D}_0$ is diffeomorphic to the unit ball, together with the fact that one then can find a volume-preserving diffeomorphism guarantees that (2.8) holds initially. Once we obtain a solution to (2.3)–(2.4), we can hence follow the flow lines of V

in (2.1). This defines a diffeomorphism of $[0, T] \times \Omega$ to $\mathcal{D}$, and so we obtain smoothness of V as a function of (t, x) from the smoothness as a function of (t, y).

In this section we first define a nonlinear functional whose zero is a solution of Euler's equations, (2.9)–(2.13). Then we derive the linearized operator in Lemma 2.2. The existence will follow from the Nash-Moser inverse function theorem, once we prove that the linearized operator is invertible and so-called tame estimates exist for the inverse stated in Theorem 2.3. Proving that the linearized operator is invertible away from a solution of Euler's equations and outside the divergence-free class is the main difficulty of the paper. This is because the normal operator (2.17) is only symmetric and positive within the divergence-free class and in general it looses regularity. In order to prove that the linearized operator is invertible and estimates exist for its inverse we introduce a modification (2.31) of the linearized operator that preserves the divergence-free condition, and first prove that the modification is invertible and estimates for its inverse, stated in Theorem 2.4. The difference between the linearized operator and the modification is lower order and the estimates for the inverse of the modified linearized operator lead to existence and estimates also for the inverse of the linearized operator.

Proving the estimates for the inverse of the modified linearized operator, stated in Theorem 2.4, takes up most of the paper, Sections 3 to 13. In this section we also derive certain identities for the curl and the divergence; see (2.29), (2.30), needed for the proof of Theorem 2.4. Here we also transform the vector field to the Lagrangian frame and express the operators and identities there; see Lemma 2.5. The estimates in Theorem 2.4 will be derived in the Lagrangian frame since commutators of the normal operator with certain differential operators are better behaved in this frame.

In Section 3, we introduce the orthogonal projection onto divergence-free vector fields and decompose the modified linearized equation into a divergence-free part and an equation for the divergence. This is needed to prove Theorem 2.4 because the normal operator is only symmetric on divergence-free vector fields and in general loses regularity. However, we have a better equation for the divergence which will allow us to obtain the same space regularity for the divergence as for the vector field itself.

In Section 4 we introduce the tangential vector fields and Lie derivatives and calculate commutators between these and the operators that occur in the modified linearized equation, in particular the normal operator. In Section 5 we show that any derivative of a vector field can be estimated by derivatives of the curl and of the divergence, and tangential derivatives or tangential derivatives of the normal operator. Section 6 introduces the L^∞ norms that we will use and states the interpolation inequalities that we will use. In Sections 7 and 8 we give the tame $L^2\infty$ and L^∞ estimates for the Dirichlet problem.

In Section 9 we give the equations and estimates for the curl to be used. In Section 10 we show existence for the modified linearized equations in the divergence class. In Section 11 we give the improved estimates for the inverse of the modified linearized operator within the divergence-free class. These are needed in Section 12 to prove existence and estimates for the inverse of the modified linearized operator. Finally in Section 13 we use this to prove existence and estimates for the inverse of the linearized operator.

In Section 14 we explain what is needed to ensure that the physical and coordinate conditions (2.7) and (2.8) continue to hold. In Section 15 we summarize the tame estimates for the inverse of the linearized operator in the formulation used with the Nash-Moser theorem. In Section 16 we derive the tame estimates for the second variational derivative. In Section 17 we give the smoothing operators needed for the proof of the Nash-Moser theorem on a bounded domain. Finally, in Section 18 we state and prove the Nash-Moser theorem in the form that we will use.

Let us now define the nonlinear map, needed to find a solution of Euler's equations. Let

$$\Phi_i = \Phi_i(x) = D_t^2 x_i + \partial_i p, \qquad \text{where} \quad \partial_i = (\partial y^a/\partial x^i)\partial_a; \tag{2.9}$$

$p = \Psi(x)$ is given by solving

$$\triangle p = -(\partial_i V^k)\partial_k V^i, \qquad p\big|_{\partial\Omega} = 0, \qquad \text{where} \quad V = D_t x. \tag{2.10}$$

A solution to Euler's equations is given by

$$\Phi(x) = 0, \quad \text{for} \quad 0 \le t \le T, \qquad x\big|_{t=0} = f_0, \quad D_t x\big|_{t=0} = v_0. \tag{2.11}$$

We will find $T > 0$ and a smooth function x satisfying (2.11) using the Nash-Moser iteration scheme.

First we turn (2.11) into a problem with vanishing initial data and a small inhomogeneous term using a trick from [Ha] as follows. It is easy to construct a formal power series solution x_0 as $t \to 0$:

$$D_t^k \Phi(x_0)\big|_{t=0} = 0, \qquad k \ge 0, \qquad x_0\big|_{t=0} = f_0, \quad D_t x_0\big|_{t=0} = v_0. \tag{2.12}$$

In fact, the equation (2.10) for the pressure p only depends on one time derivative of the coordinate x so that commuting through time derivatives in (2.10) gives a Dirichlet problem for $D_t^k p$ depending only on $D_t^m x$, for $m \le k+1$ and $D_t^\ell p$, for $\ell \le k-1$. Similarly commuting through time derivatives in Euler's equation, (2.11), gives $D_t^{2+k} x$ in terms of $D_t^m x$, for $m \le k$, and $D_t^\ell p$, for $\ell \le k$. We can hence construct a formal power series solution in t at $t = 0$ and by a standard trick we can find a smooth function x_0 having this as its power series; see Section 10. We will now solve for u in

$$\tilde{\Phi}(u) = \Phi(u + x_0) - \Phi(x_0) = F_\delta - F_0 = f_\delta, \qquad u\big|_{t=0} = D_t u\big|_{t=0} = 0 \tag{2.13}$$

where F_δ is constructed as follows. Let $F_0 = \Phi(x_0)$ and let $F_\delta(t,y) = F_0(t-\delta,y)$, when $t \geq \delta$ and $F_\delta(t,y)=0$, when $t \leq \delta$. Then F_δ is smooth and $f_\delta = F_\delta - F_0$ tends to 0 in C^∞ when $\delta \to 0$. Furthermore, f_δ vanishes to infinite order as $t \to 0$. Now, $\tilde{\Phi}(0)=0$ and so it will follow from the Nash-Moser inverse function theorem that $\tilde{\Phi}(u) = f_\delta$ has a smooth solution u if δ is sufficiently small. Then $x = u + x_0$ satisfies (2.11) for $0 \leq t \leq \delta$.

In order to solve (2.11) or (2.13) we must show that the linearized operator is invertible. Let us therefore first calculate the linearized equations. Let δ be the Lagrangian variation, i.e. derivative with respect to some parameter r when (t,y) are fixed. We have:

Lemma 2.2. *Let $\overline{x} = \overline{x}(r,t,y)$ be a smooth function of $(r,t,y) \in K = [-\varepsilon,\varepsilon]\times[0,T]\times\overline{\Omega}$, $\varepsilon > 0$, such that $\overline{x}\big|_{r=0} = x$. Then $\Phi(\overline{x})$ is a smooth function of $(r,t,y)\in K$, such that $\partial\Phi(\overline{x})/\partial r\big|_{r=0} = \Phi'(x)\delta x$, where $\delta x = \partial\overline{x}/\partial r\big|_{r=0}$ and the linear map $L_0 = \Phi'(x)$ is given by*

$$\Phi'(x)\delta x_i = D_t^2\delta x_i + (\partial_k\partial_i p)\delta x^k + \partial_i\delta p_0 + \partial_i\big(\delta p_1 - \delta x^k\partial_k p\big), \tag{2.14}$$

where p satisfies (2.10) *and δp_i, $i=0,1$, are given by solving*

$$\triangle\big(\delta p_1 - \delta x^k\partial_k p\big) = 0, \qquad \delta p_1\big|_{\partial\Omega} = 0, \tag{2.15}$$

$$\triangle\delta p_0 = -2(\partial_k V^i)\partial_i\big(\delta V^k - \delta x^l\partial_l V^k\big), \qquad \delta p_0\big|_{\partial\Omega} = 0, \tag{2.16}$$

where $\delta v = D_t\delta x$. Here, the normal operator

$$A\delta x_i = -\partial_i\big(\partial_k p\ \delta x^k - \delta p_1\big) \tag{2.17}$$

restricted to divergence-free vector fields is symmetric and positive, in the inner product $\langle u,w\rangle = \int_{\mathcal{D}_t}\delta^{ij}u_i w_j\,dx$, if the physical condition (2.7) *holds.*

Proof. That $\Phi(\overline{x})$ is a smooth function follows from the fact that the solution of (2.10) is a smooth function if $\overline{x}$ is; see Section 16. Let us now calculate $\Phi'(x)$. Since $[\delta, \partial/\partial y^a] = 0$ it follows that

$$[\delta,\partial_i] = \Big(\delta\frac{\partial y^a}{\partial x^i}\Big)\frac{\partial}{\partial y^a} - (\partial_i\delta x^l)\partial_l, \tag{2.18}$$

where we used the formula for the derivative of the inverse of a matrix $\delta A^{-1} = -A^{-1}(\delta A)A^{-1}$. It follows that $[\delta - \delta x^l\partial_l, \partial_i] = 0$ ($\delta - \delta x^l\partial_l$ is the Eulerian variation). Hence

$$\delta\Phi_i - \delta x^k\partial_k\Phi_i = D_t^2\delta x_i - (\partial_k D_t^2 x_i)\delta x^k + \partial_i\big(\delta p - \delta x^k\partial_k p\big), \tag{2.19}$$

where

$$\begin{aligned}\triangle\big(\delta p - \delta x^k\partial_k p\big) &= (\delta - \delta x^k\partial_k)\triangle p\\ &= -2(\partial_k V^i)\partial_i\big(\delta V^k - \delta x^l\partial_l V^k\big), \qquad \delta p\big|_{\partial\Omega} = 0.\end{aligned} \tag{2.20}$$

The symmetry and positivity of A were proven in [L1]; see also Section 3 here. $\square$

In order to use the Nash-Moser iteration scheme to obtain a solution of (2.13) we must show that the linearized operator is invertible and that the inverse satisfies tame estimates:

THEOREM 2.3. *Let*

$$|||u|||_{a,k} = \sup_{0\le t\le T} \|u(t,\cdot)\|_{a,\infty} + \cdots + \|D_t^k u(t,\cdot)\|_{a,\infty} \tag{2.21}$$

where $\|u(t,\cdot)\|_{a,\infty}$ *are the Hölder norms in* $\overline{\Omega}$; *see* (17.1).

Suppose that (2.7) *and* (2.8) *hold initially, where* p *is given by* (2.10), *and let* $x_0 \in C^\infty\big([0,T]\times\overline{\Omega}\big)$ *satisfy* (2.12). *Then there is a* $T_0 = T(x_0) > 0$, *depending only on upper bounds for* $|||x_0|||_{4,2}$, c_0^{-1} *and* c_1, *such that the following hold. If* $x \in C^\infty\big([0,T]\times\overline{\Omega}\big)$, p *is defined by* (2.10),

$$T \le T_0, \quad |||x - x_0|||_{4,2} \le 1, \quad \text{and} \quad (x-x_0)\big|_{t=0} = D_t\,(x-x_0)\big|_{t=0} = 0, \tag{2.22}$$

then (2.7) *and* (2.8) *hold for* $0 \le t \le T$ *with* c_0 *replaced by* $c_0/2$ *and* c_1 *replaced by* $2c_1$. *Furthermore, linearized equations*

$$\Phi'(x)\delta x = \delta\Phi, \qquad \text{in} \quad [0,T]\times\overline{\Omega}, \qquad \delta x\big|_{t=0} = D_t\,\delta x\big|_{t=0} = 0, \tag{2.23}$$

where $\delta\Phi \in C^\infty\big([0,T]\times\overline{\Omega}\big)$ *have a solution* $\delta x \in C^\infty\big([0,T]\times\overline{\Omega}\big)$. *The solution satisfies the estimate*

$$|||\delta x|||_{a,2} \le C_a\big(|||\delta\Phi|||_{a+r_0+2,0} + |||\delta\Phi|||_{1,0}\,|||x-x_0|||_{a+r_0+6,2}\big), \qquad a \ge 0 \tag{2.24}$$

where $C_a = C_a(x_0)$ *is bounded when* a *is bounded, and in fact depends only on upper bounds for* $|||x_0|||_{a+r_0+6,2}$, c_0^{-1} *and* c_1. *Here* $r_0 = [n/2]+1$, *where* n *is the number of space dimensions.*

Furthermore Φ *is twice differentiable and the second derivative satisfies the estimate*

$$\begin{aligned} |||\Phi''(x)(\delta x,\epsilon x)|||_{a,0} \le C_a\Big(&|||\delta x|||_{a+4,1}|||\epsilon x|||_{2,1} + |||\delta x|||_{2,1}|||\epsilon x|||_{a+4,1} \\ &+ |||x-x_0|||_{a+4,1}|||\delta x|||_{2,1}|||\epsilon x|||_{2,1}\Big). \end{aligned} \tag{2.25}$$

The proof of Theorem 2.1 follows from Theorem 2.3 and Proposition 18.1. In Theorem 2.3 we use norms that only have two time derivatives and our Nash-Moser theorem, Proposition 18.1, gives a solution of (2.13) $u \in C^2\big([0,T], C^\infty(\overline{\Omega})\big)$. However, additional regularity in time follows from differentiating the equations with respect to time. In fact, if $x \in C^k\big([0,T], C^\infty(\overline{\Omega})\big)$ then $D_t^2 x = -\partial_i p \in C^{k-1}\big([0,T], C^\infty(\overline{\Omega})\big)$, since (2.10) only depends on one time derivative of x; see the proof of Lemma 6.7. It follows that $x \in C^{k+1}\big([0,T], C^\infty(\overline{\Omega})\big)$.

Theorem 2.3 follows from Lemma 14.1, Proposition 15.1 and Proposition 16.1. The main point is existence for (2.23) and the tame estimate (2.24) given in Proposition 15.1. We will now discuss how to prove existence and estimates for the linearized equations. The terms $(\partial_k\partial_i p)\delta x^k$ and $\partial_i\delta p_0$ in (2.14) are order zero in δx and $D_t\delta x$. The last term is a positive symmetric operator but only on divergence-free vector fields and in general it is an unbounded operator that loses regularity. In general δx is not going to be divergence-free but we will derive evolution equations for the divergence and the curl of δx, that gain regularity. These evolution equations come from the fact that the divergence and the curl of the velocity v are conserved, expressed in the Lagrangian coordinates for a solution of Euler's equation, $\Phi(x)=0$. In fact, since $[D_t,\partial_i]=-(\partial_i V^k)\partial_k$ it follows from (2.9) that

$$D_t \operatorname{div} V = \operatorname{div}\Phi, \qquad\qquad \mathcal{L}_{D_t}\operatorname{curl} v = \operatorname{curl}\Phi \tag{2.26}$$

where curl $v_{ij}=\partial_i v_j-\partial_j v_i$ and $\mathcal{L}_{D_t}$ is the space-time Lie derivative with respect to $D_t=(1,V)$:

$$\mathcal{L}_{D_t}\sigma_{ij} = D_t\,\sigma_{ij} + (\partial_i V^l)\sigma_{lj} + (\partial_j V^l)\sigma_{il} \tag{2.27}$$

restricted to the space components. Expressing the two form σ in the Lagrangian frame we see that this is just the time derivative:

$$D_t\big(a_a^i a_b^j\sigma_{ij}\big) = a_a^i a_b^j\mathcal{L}_{D_t}\sigma_{ij}, \qquad \text{where} \quad a_a^i = \partial x^i/\partial y^a. \tag{2.28}$$

We have the following evolution equations for the divergence and the curl of the linearized operator

$$\operatorname{div}\big(\Phi'(x)\delta x\big) = D_t^2\operatorname{div}\delta x + (\partial_i\delta x^k)\partial_k\Phi^i, \tag{2.29}$$

$$\begin{aligned}\operatorname{curl}(\Phi'(x)\delta x) = {}& \mathcal{L}_{D_t}\operatorname{curl}\big(D_t\,\delta x - \delta x^k\partial v_k\big)\\ &+(\partial_i\delta x^k)\partial_j\Phi_k - (\partial_j\delta x^k)\partial_i\Phi_k.\end{aligned} \tag{2.30}$$

In fact, since $[\delta,\partial_i] = -(\partial_i\delta x^k)\partial_k$ and $[D_t,\partial_i] = -(\partial_i V^k)\partial_k$ it follows that $\delta\operatorname{div} D_t x = D_t\operatorname{div}\delta x$ so that by (2.26) $D_t^2\operatorname{div}\delta x = \delta\operatorname{div}\Phi$ and (2.29) follows. To prove (2.30) we note that $[\delta, a_a^i a_b^j\partial_i] = [\delta, a_b^j\partial_a] = (\delta a_b^j)\partial_a = (\partial_b\delta x^j)\partial_a = a_a^i a_b^k(\partial_k\delta x^j)\partial_i$ so that

$$\begin{aligned}\delta\big(a_a^i a_b^j\operatorname{curl} v_{ij}\big) &= a_a^i a_b^j\big(\operatorname{curl}\delta v_{ij} + (\partial_j\delta x^k)\partial_i v_k - (\partial_i\delta x^k)\partial_k v_k\big)\\ &= a_a^i a_b^j\operatorname{curl}(\delta v - \delta x_k\partial V^k)_{ij}\end{aligned} \tag{2.31}$$

where $\operatorname{curl}(\delta v - \delta x_k\partial V^k)_{ij} = \partial_i(\delta v_j - \delta x^k\partial_j v_k) - \partial_j(\delta v_i - \delta x^k\partial_i v_k)$ and (2.30) follows since by (2.26)–(2.28)

$$\mathcal{L}_{D_t}\operatorname{curl}(\delta v - \delta x_k\partial V^k) = \operatorname{curl}(\delta\Phi - \delta x_k\partial\Phi^k). \tag{2.32}$$

In [L1] we proved existence and estimates for the inverse of the linearized operator at a solution of Euler's equation and within the divergence-free class.

We only inverted $\Phi'(x)\delta x = \delta\Phi$ when $\delta\Phi$ was divergence-free and $\Phi(x) = 0$, in which case by (2.29) δx is also divergence-free. In order to use the Nash-Moser iteration scheme we will show that the linearized operator is invertible away from a solution of Euler's equation and outside the divergence-free class. This does present a problem since the normal operator is only symmetric on divergence-free vector fields. So for general vector fields we lose a derivative. In order to recover this loss we will use the fact that one has better evolution equations for the divergence and for the curl that do not lose regularity. Now, (2.29), (2.30), say that we can get bounds for the divergence and the curl of $D_t\,\delta x$ if we have bounds for all first order derivatives of δx. In fact (2.29), (2.30) can be integrated even without knowing a bound for first order derivatives of $D_t\,\delta x$.

We will now first modify the linearized operator so as to remove the term $(\partial_i \delta x^k)\partial_k \Phi^i$ in (2.29) without making (2.30) worse. Without this term, (2.29) will give us an evolution equation that allows us to control the divergence. This together with the fact that the normal operator (2.17) is symmetric and positive on divergence-free vector fields will give us existence for the inverse of the modified linearized operator. The modified linearized operator is given by

$$\begin{aligned} L_1 \delta x^i &= \Phi'(x)\delta x^i - \delta x^k \partial_k \Phi^i + \delta x^i \operatorname{div}\Phi \\ &= D_t^2 \delta x^i - (\partial_k D_t^2 x^i)\delta x^k + \partial_i\big(\delta p_1 - \delta x^k \partial_k p\big) + \delta x^i \operatorname{div}\Phi + \partial_i \delta p_0. \end{aligned} \tag{2.33}$$

It follows from (2.29) that

$$\operatorname{div}(L_1\delta x) = D_t^2 \operatorname{div}\delta x + \operatorname{div}\Phi\, \operatorname{div}\delta x. \tag{2.34}$$

The operator L_1 reduces to the linearized operator $L_0 = \Phi'(x)$ when $\Phi(x) = 0$ and the difference $L_1 - L_0$ is lower order. Furthermore, L_1 preserves the divergence-free condition. We will first prove existence for the inverse of the modified linearized operator and the existence of the inverse of the linearized operator follows since the difference is lower order. The main part of the typescript is devoted to proving the following existence and energy estimates:

Theorem 2.4. *Suppose that x is smooth and that the physical condition* (2.7) *and the coordinate condition* (2.8) *hold for* $0 \le t \le T$. *Then*

$$L_1 \delta x = \delta\Phi, \qquad 0 \le t \le T, \qquad \delta x\big|_{t=0} = D_t\,\delta x\big|_{t=0} = 0 \tag{2.35}$$

has a smooth solution δx if $\delta\Phi$ is smooth.

Furthermore, there are constants K_4 depending only on upper bounds for T, c_0^{-1}, c_1, r and $|||x|||_{4,2}$ such that the following estimates hold. If $\operatorname{div}\delta\Phi = 0$ *then* $\operatorname{div}\delta x = 0$ *and*

$$\|D_t \delta x\|_r + \|\delta x\|_r \le K_4 \int_0^t \big(\|\delta\Phi\|_r + |||x|||_{r+3,1}\|\delta\Phi\|_0\big)\, d\tau, \qquad r \ge 0. \tag{2.36}$$

If $\operatorname{div}\delta\Phi = 0$, $\operatorname{curl}\delta\Phi = 0$ *and* $\delta\Phi\big|_{t=0} = 0$ *then*

$$\|D_t^2\delta x\|_r + \|D_t\delta x\|_r + \|\delta x\|_r + c_0\|\delta x\|_{r+1} \le K_4 \int_0^t \big(\|D_t\delta\Phi\|_r + \|\delta\Phi\|_r + |||x|||_{r+3,2}(\|D_t\delta\Phi\|_0 + \|\delta\Phi\|_0)\big)\, d\tau, \qquad r \ge 0. \tag{2.37}$$

In general

$$\|D_t\delta x\|_{r-1} + \|\delta x\|_r \le K_4 \int_0^t \big(\|\delta\Phi\|_r + |||x|||_{r+3,2}\|\delta\Phi\|_1\big)\, d\tau, \qquad r \ge 1. \tag{2.38}$$

Here $|||x|||_{r,k}$ *is as in Theorem* 2.3 *and*

$$\|\delta x\|_r = \|\delta x(t,\cdot)\|_r = \sum_{|\alpha|\le r} \Big(\int_\Omega |\partial_y^\alpha \delta x(t,y)|^2\, dy\Big)^{1/2}. \tag{2.39}$$

The proof of the existence for (2.23) and the tame estimate (2.24) for the inverse of the linearized operator in Theorem 2.3 follows from Theorem 2.4. In fact, since the difference $(L_1 - \Phi'(x))\delta x = O(\delta x)$ is lower order, the estimate (2.38) will then allow us to get existence and the same estimate also for the inverse of the linearized operator (2.23), by iteration. In (2.38) we only have estimates for a one time derivative, but we also get estimates for an additional time derivative from using the equation. The L^2 estimates for (2.23) so obtained then give the L^∞ estimates (2.24) by also using Sobolev's lemma.

The proof of Theorem 2.4 takes up most of the manuscript. The proof of (2.36) uses the symmetry and positivity of the normal operator (2.17) within the divergence-free class. This leads to energy estimates within the divergence-free class. The proof of (2.37) is obtained by first differentiating the equation with respect to time and then by using the fact that a bound for two time derivatives also gives a bound for the normal operator (2.17) using the equation. The normal operator is not elliptic acting on general vector fields. However, it is elliptic acting on divergence and curl free vector fields and in general one can invert it and gain a space derivative if one also has bounds for the curl and the divergence; see Lemma 5.4. Here we also need to use the improved estimate for the curl coming from (2.30). To prove (2.38) we first subtract from a vector field picking up the divergence. The equation for the divergence from (2.34):

$$D_t^2 \operatorname{div}\delta x + \operatorname{div}\Phi\, \operatorname{div}\delta x = \operatorname{div}\delta\Phi \tag{2.40}$$

is just an ordinary differential equation that does not lose regularity and in fact the estimates for (2.40) gain an extra time derivative compared to the estimate (2.36). Once we control the divergence we use the orthogonal projection onto divergence-free vector fields to obtain an equation for the divergence-free part by projecting the equation (2.35); see Section 3. The equation so obtained is of the form (2.35) with $\operatorname{div}\delta\Phi = 0$ and $\delta\Phi$ depending also on the divergence $\operatorname{div}\delta x$ just calculated. The interaction term coming from the divergence part loses a space derivative but it is in the form of a gradient so that we can recover this loss by using the gain of a space derivative in (2.37).

In order to prove the energy estimates needed to prove Theorem 2.4 one has to express the vector fields in the Lagrangian frame; see (2.43). Theorem 2.4, expressed in the Lagrangian frame, follows from Theorem 10.1, Theorem 11.1 and Theorem 12.1. Below, we will express equation (2.35) in the Lagrangian frame and in Section 3 we outline the main ideas of how to decompose the equation into a divergence-free part and an equation for the divergence using the orthogonal projection onto divergence-free vector fields. We also show the basic energy estimate within the divergence-free class.

As described above we now want to invert the modified linearized operator (2.35) by decomposing it into an operator on the divergence-free part and the ordinary differential equation (2.40) for the divergence. Hence we first want to be able to invert L_1 in the divergence-free class. The normal operator A, the third term on the second row in (2.33), maps divergence-free vector fields onto divergence-free vector fields. We also want to modify the time derivative by adding a lower order term so it preserves the divergence-free condition. Let the Lie derivative and modified Lie derivative with respect to the time derivative acting on vector fields be defined by

(2.41)
$$\mathcal{L}_{D_t}\delta x^i = D_t\delta x^i - (\partial_k V^i)\delta x^k \qquad \text{and} \qquad \hat{\mathcal{L}}_{D_t}\delta x^i = \mathcal{L}_{D_t}\delta x^i + \operatorname{div} V\, \delta x^i.$$

As before, $\mathcal{L}_{D_t}$ is the space time Lie derivative restricted to the space components. Then

$$\operatorname{div} \hat{\mathcal{L}}_{D_t}\delta x = \hat{D}_t \operatorname{div}\delta x, \qquad \text{where} \qquad \hat{D}_t = D_t + \operatorname{div} V; \tag{2.42}$$

i.e., $\hat{D}_t f = D_t f + (\operatorname{div} V)\, f$.

This is easier to see if we express the vector field in the Lagrangian frame. Let

$$W^a = \frac{\partial y^a}{\partial x^i}\delta x^i. \tag{2.43}$$

Then,

$$\begin{aligned} D_t\, \delta x^i &= D_t\big(W^b \partial x^i/\partial y^b\big) = (D_t W^b)\partial x^i/\partial y^b + W^b \partial V^i/\partial y^b \\ &= (D_t W^b)\partial x^i/\partial y^b + \delta x^k \partial_k V^i \end{aligned} \tag{2.44}$$

and multiplying by the inverse $\partial y^a/\partial x^i$ gives

$$D_t W^a = \frac{\partial y^a}{\partial x^i}\mathcal{L}_{D_t}\delta x^i \qquad \text{and} \qquad \hat{D}_t W^a = \frac{\partial y^a}{\partial x^i}\hat{\mathcal{L}}_{D_t}\delta x^i. \tag{2.45}$$

With $\kappa = \det(\partial x/\partial y)$,

$$\dot{W}^a = \hat{D}_t W^a = D_t W^a + (\operatorname{div} V) W^a = \kappa^{-1} D_t(\kappa W^a) \tag{2.46}$$

since $D_t\,\kappa = \kappa \operatorname{div} V$; see [L1]. Since the divergence is invariant,

$$\operatorname{div} \delta x = \operatorname{div} W = \kappa^{-1}\partial_a\big(\kappa W^a\big), \tag{2.47}$$

it therefore follows that

$$\operatorname{div} \hat{D}_t W = \hat{D}_t \operatorname{div} W. \tag{2.48}$$

The idea is now to replace the time derivatives D_t in (2.33) by $\hat{\mathcal{L}}_{D_t}$ or equivalently express L_1 in the Lagrangian frame and use the modified time derivatives $\hat{D}_t$. Expressing the operator L_1 in the Lagrangian frame we get:

Lemma 2.5. *Let $\dot{W} = \hat{D}_t W$ and $\ddot{W} = \hat{D}_t^2 W$. Then* (2.35) *can be written as $L_1 W = F$, where W is given by* (2.43), *$F^a = \delta\Phi^i \partial y^a/\partial x^i$ and*

$$L_1 W^a = \ddot{W}^a + AW^a - B(W,\dot{W})^a, \qquad B(W,\dot{W})^a = B_0 W^a + B_1 \dot{W}^a. \tag{2.49}$$

Here

$$g_{ab} A W^b = -\partial_a\big((\partial_c p)W^c - q_1\big), \qquad \operatorname{div} AW = 0 \tag{2.50}$$

$$g_{ab} B_0 W^b = \dot{\sigma}\big(D_t g_{ac} - \omega_{ac} - \dot{\sigma} g_{ac}\big)W^c - \partial_a q_3, \qquad \operatorname{div} B_0 W = -\,\dot{\sigma}^2 \operatorname{div} W \tag{2.51}$$

$$g_{ab} B_1 \dot{W}^b = -\big(D_t g_{ac} - \omega_{ac} - 2\dot{\sigma} g_{ac}\big)\dot{W}^c - \partial_a q_2, \qquad \operatorname{div} B_1 \dot{W} = 2\dot{\sigma} \operatorname{div} \dot{W}, \tag{2.52}$$

where q_i, for $i = 1,2,3$, are given by solving the Dirichlet problem $q_i\big|_{\partial\Omega} = 0$ where $\triangle q_i$ are given by the equations for the divergences above, $\sigma = \ln \kappa$, $\dot{\sigma} = D_t\,\sigma = \operatorname{div} V$, $\ddot{\sigma} = D_t^2 \sigma$ and

$$D_t\, g_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\big(\partial_i v_j + \partial_j v_i\big), \qquad \omega_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\big(\partial_i v_j - \partial_j v_i\big). \tag{2.53}$$

Now,

$$\operatorname{div}(L_1 W) = D_t^2 \operatorname{div} W + \ddot{\sigma} \operatorname{div} W. \tag{2.54}$$

Let $\underline{L}_1 W_a = g_{ab} L_1 W^b$, $\dot{w}_a = g_{ab}\dot{W}^b$ and $\tilde{w}_a = \dot{w}_a - (\omega_{ab} + \dot{\sigma} g_{ab})W^b$. Then

$$\operatorname{curl}(\underline{L_1}W) = D_t \operatorname{curl} \tilde{w} + \operatorname{curl} \underline{B}_4 W \tag{2.55}$$

$$\operatorname{curl}(\underline{L_1}W) = D_t \operatorname{curl} \dot{w} + \operatorname{curl} \underline{B}_5 \dot{W} + \operatorname{curl} \underline{B}_6 W \tag{2.56}$$

where $\underline{B}_4 W_a = (D_t\,\omega_{ab} + \ddot{\sigma} g_{ab})W^b$, $\underline{B}_5 \dot{W}_a = -(\omega_{ab} + \dot{\sigma} g_{ab})\dot{W}^b$ and $\underline{B}_6 W_a = -\dot{\sigma}(D_t\, g_{ab} - \omega_{ab} - \dot{\sigma} g_{ab})W^b$.

Furthermore $L_0 = \Phi'(x)$ expressed in the Lagrangian frame is given by

$$L_0 W^a = L_1 W^a - B_3 W^a, \qquad \text{where} \qquad B_3 W^a = -W^c \nabla_c \Phi^a + W^a \operatorname{div} \Phi, \tag{2.57}$$

where ∇_c is covariant differentiation with respect to the metric g_{ab} and $\Phi^a = \Phi^i \partial y^a/\partial x^i$; i.e., $\nabla_c \Phi^a = (\partial x^i/\partial y^c)(\partial y^a/\partial x^j)\partial_i \Phi^j$.

Proof. Differentiating (2.44) once more gives

$$D_t^2\delta x^i - (\partial_k D_t V^i)\delta x^k = (D_t^2 W^b)\partial x^i/\partial y^b + 2(D_t W^b)\partial V^i/\partial y^b. \tag{2.58}$$

It follows that

(2.59)
$$\begin{aligned}\frac{\partial x^i}{\partial y^a}\big(D_t^2\delta x^i - (\partial_k D_t V^i)\delta x^k\big) &= \frac{\partial x^i}{\partial y^a}\frac{\partial x^i}{\partial y^b}D_t^2 W^b + 2(D_t W^b)\frac{\partial x^i}{\partial y^b}\frac{\partial x^j}{\partial y^a}\partial_i v_j\\ &= g_{ab}D_t^2 W^b + (D_t\, g_{ab} - \omega_{ab})D_t W^b.\end{aligned}$$

Then, from (2.33),

(2.60)
$$\begin{aligned}g_{ab}L_1 W^b &= g_{ab}D_t^2 W^b - \partial_a\big((\partial_c p)W^c - q\big) + (D_t g_{ac} - \omega_{ac})D_t W^c + \ddot\sigma g_{ab}W^b\\ &= D_t\big(g_{ab}D_t W^b - \omega_{ab}W^b\big) - \partial_a\big((\partial_c p)W^c - q\big) + D_t\omega_{ab}W^b + \ddot\sigma g_{ab}W^b\end{aligned}$$

where $q = \delta p$ is chosen so that the divergence is equal to $\operatorname{div} L_1 W = D_t^2 \operatorname{div} W + \operatorname{div} W\, D_t \operatorname{div} V$ in order for it to be consistent with (2.34). We have $\hat D_t^2 = (D_t + \operatorname{div} V)(D_t + \operatorname{div} V) = D_t^2 + 2\dot\sigma D_t + \dot\sigma^2 + \ddot\sigma = D_t^2 + 2\dot\sigma \hat D_t + \ddot\sigma - \dot\sigma^2$ so that

$$D_t^2 = \hat D_t^2 - 2\dot\sigma\hat D_t + \dot\sigma^2 - \ddot\sigma, \qquad D_t = \hat D_t - \dot\sigma. \tag{2.61}$$

Hence, with $\dot W = \hat D_t W$ and $\ddot W = \hat D_t^2 W$, we can write the equation (2.60) as

$$L_1 W^a = \ddot W^a - g^{ab}\partial_b\big((\partial_c p)W^c - q_1\big) - B^a(W,\dot W) \tag{2.62}$$

where q_1 is chosen so that the divergence of the second term on the right vanishes and

$$\begin{aligned}g_{ab}B^b(W,\dot W) = &-\big(D_t g_{ac} - \omega_{ac} - 2\dot\sigma g_{ac}\big)\dot W^c\\ &+\big(\dot\sigma\big(D_t g_{ac} - \omega_{ac} - \dot\sigma g_{ac}\big)W^c - \partial_a q_0.\end{aligned} \tag{2.63}$$

Here q_0 is chosen as follows so that $\operatorname{div} L_1 W = \hat D_t^2 \operatorname{div} W - \operatorname{div} B = D_t^2 \operatorname{div} W + \operatorname{div} W\,\ddot\sigma$. But $\hat D_t^2 \operatorname{div} W = D_t^2 \operatorname{div} W + 2\dot\sigma \hat D_t \operatorname{div} W + (\ddot\sigma - \dot\sigma^2)\operatorname{div} W$ so we must have $\operatorname{div} B = 2\dot\sigma\hat D_t \operatorname{div} W - \dot\sigma^2 \operatorname{div} W$. Hence q_0 is chosen so that this is fulfilled and (2.49) follows by writing $q_0 = q_2 + q_3$. Now, (2.54) follows from (2.34) or (2.49) and then from (2.49) we write L_1 in the two alternative forms:

$$\begin{aligned}g_{ab}L_1 W^b = D_t\big(g_{ab}\dot W^b - (\omega_{ab} + \dot\sigma g_{ab})W^b\big) - \partial_a\big((\partial_c p)W^c - q_1\big)\\ +(D_t\omega_{ab} + \ddot\sigma g_{ab})W^b + \partial_a q_0;\end{aligned} \tag{2.64}$$

$$\begin{aligned}g_{ab}L_1 W^b = D_t\big(g_{ab}\dot W^b\big) - \partial_a\big((\partial_c p)W^c - q_1\big)\\ -(\omega_{ab} + \dot\sigma g_{ab})(\dot W^b - \dot\sigma W^b) - \dot\sigma D_t g_{ab}W^b + \partial_a q_0.\end{aligned} \tag{2.65}$$

(2.55) and (2.56) follow from these. Finally, we also want to express $L_0 = \Phi'(x)$ is these coordinates. In order to do this we must transform the term $\delta x^k\partial_k\Phi^i$ in (2.33) to the Lagrangian frame. If $\Phi^a = \Phi^i\partial y^a/\partial x^i$, then $(\delta x^k\partial_k\Phi^i)\partial y^a/\partial x^i = W^c\nabla_c\Phi^a$, where ∇_c is covariant differentiation; see e.g. [CL], and then (2.57) follows. □

3. The projection onto divergence-free vector fields and the normal operator

Let us now also define the projection P onto divergence-free vector fields by

$$PU^a = U^a - g^{ab}\partial_b p_U, \qquad \triangle p_U = \operatorname{div} U, \qquad p_U\big|_{\partial\Omega} = 0. \tag{3.1}$$

(Here $\triangle q = \kappa^{-1}\partial_a\big(\kappa g^{ab}\partial_b q\big)$.) P is the orthogonal projection in the inner product

$$\langle U, W\rangle = \int_\Omega g_{ab} U^a W^b \kappa dy \tag{3.2}$$

and its operator norm is one:

$$\|PW\| \leq \|W\|, \qquad \text{where} \quad \|W\| = \langle W, W\rangle^{1/2}. \tag{3.3}$$

For a function f that vanishes on the boundary define $A_f W^a = g^{ab}\underline{A}_f W_b$, where

$$\underline{A}_f W_a = -\partial_a\big((\partial_c f)W^c - q\big), \;\; \triangle\big((\partial_c f)W^c - q\big) = 0, \;\; q\big|_{\partial\Omega} = 0, \tag{3.4}$$

i.e. $A_f W$ is the projection of $-g^{ab}\partial_b\big((\partial_c f)W^c\big)$. This is defined for general vector fields but it is only symmetric in the divergence-free class. Next,

$$\langle U, A_f W\rangle = \int_{\partial\Omega} n_a\, U^a (-\partial_c f) W^c\, dS, \quad \text{if} \quad \operatorname{div} U = \operatorname{div} W = 0, \tag{3.5}$$

where n is the unit conormal. If $f\big|_{\partial\Omega} = 0$ then $-\partial_c f\big|_{\partial\Omega} = (-\nabla_N f) n_c$. It follows that A_f is a symmetric operator on divergence-free vector fields, and in particular, the normal operator in (2.50)

$$A = A_p \tag{3.6}$$

is positive since we assumed that $-\nabla_N p \geq c > 0$ on the boundary. Now,

$$|\langle U, A_f W\rangle| \leq \|\nabla_N f/\nabla_N p\|_{L^\infty(\partial\Omega)} \langle U, AU\rangle^{1/2}\langle W, AW\rangle^{1/2}, \qquad \text{if} \quad \operatorname{div} U = \operatorname{div} W = 0. \tag{3.7}$$

Since the projection has norm one it follows from (3.4) that

$$\|A_f W\| \leq \|\partial^2 f\|_{L^\infty(\Omega)}\|W\| + \|\partial f\|_{L^\infty(\Omega)}\|\partial W\|. \tag{3.8}$$

Note also that A_f acting on divergence-free vector fields by (3.5) depends only on $\nabla_N f\big|_{\partial\Omega}$; i.e., $A_{\tilde f} = A_f$ if $\nabla_N \tilde f\big|_{\partial\Omega} = \nabla_N f\big|_{\partial\Omega}$. We can therefore replace f by the Taylor expansion of order one in the distance to the boundary in polar coordinates multiplied by a smooth function that is one close to the boundary and vanishes close to the origin. It follows that

$$\|A_f W\| \le C \sum_{S\in\mathcal{S}} \|\nabla_N S f\|_{L^\infty(\partial\Omega)}\|W\| + C\|\nabla_N f\|_{L^\infty(\partial\Omega)}(\|\partial W\| + \|W\|), \qquad \text{if} \quad \operatorname{div} W = 0, \tag{3.9}$$

where $\mathcal{S}$ is a set of vector fields that span the tangent space of $\partial\Omega$; see Section 4.

In order to prove existence for the linearized equations we (in [L1]) replaced the normal operator A by a smoothed out bounded operator that still has the same positive properties as A and commutators with Lie derivatives, and which also has vanishing divergence and curl away from the boundary. This makes it possible to pass to the limit and obtain existence for the linearized equations. The smoothed out normal operator is defined as follows. Let $\rho = \rho(d)$ be a smoothed out version of the distance function to the boundary $d(y) = \operatorname{dist}(y, \partial\Omega) = 1 - |y|$ in the standard Euclidean metric $\delta_{ij} dy^i dy^j$ in the y coordinates, $\rho' \ge 0$, $\rho(d) = d$, when $d \le 1/4$ and $\rho(d) = 1/2$ when $d \ge 3/4$. Then we can alternatively express A_f as

$$\underline{A}_f W_a = -\partial_a\big((f/\rho)(\partial_c\rho)W^c - q\big), \ \ \triangle\big((f/\rho)(\partial_c\rho)W^c - q\big) = 0, \ \ q\big|_{\partial\Omega} = 0. \tag{3.10}$$

Let $\chi(\rho)$ be a smooth function such that $\chi' \ge 0$, $\chi(\rho) = 0$ when $\rho \le 1/4$, $\chi(\rho) = 1$ when $\rho \ge 3/4$. Since A_f is unbounded we now define an approximation that is a bounded operator: $A_f^\varepsilon W^a = g^{ab} \underline{A}_f^\varepsilon W_b$, where

$$\begin{aligned} \underline{A}_f^\varepsilon W_a &= -\chi_\varepsilon \partial_a\big((f/\rho)(\partial_c\rho)W^c\big) + \partial_a q, \\ \triangle q &= \kappa^{-1}\partial_a\big(g^{ab}\kappa\chi_\varepsilon\partial_b\big((f/\rho)(\partial_c\rho)W^c\big)\big), \quad q\big|_{\partial\Omega} = 0, \end{aligned} \tag{3.11}$$

where $\chi_\varepsilon(\rho) = \chi(\rho/\varepsilon)$. We have

$$\langle U, A_f^\varepsilon W\rangle = \int_\Omega (f/\rho)\chi_\varepsilon'(\partial_a\rho)\, U^a(\partial_c\rho)W^c\,\kappa dy, \quad \text{if } \operatorname{div} U = \operatorname{div} W = 0, \tag{3.12}$$

from which it follows that A_f^ε is also symmetric. And in particular $A^\varepsilon = A_p^\varepsilon$ is positive since we assumed that $p \ge 0$, at least close to the boundary. Now,

$$|\langle U, A_f^\varepsilon W\rangle| \le \|f/p\|_{L^\infty(\Omega\setminus\Omega_{\varepsilon/4})}\langle U, A^\varepsilon U\rangle^{1/2}\langle W, A^\varepsilon W\rangle^{1/2}, \qquad \text{if } \operatorname{div} U = \operatorname{div} W = 0, \tag{3.13}$$

where $\Omega_\varepsilon = \{y \in \Omega; d(y, \partial\Omega) < \varepsilon\}$. It also follows from (3.12) that another expression for $\underline{A}_f^\varepsilon$ is

$$\begin{aligned} \underline{A}_f^\varepsilon W_a &= (f/\rho)\chi_\varepsilon'(\partial_a\rho)\,(\partial_c\rho)W^c - \partial_a q, \\ \triangle q &= \kappa^{-1}\partial_a\big(\kappa g^{ab}(f/\rho)\chi_\varepsilon'(\partial_b\rho)\,(\partial_c\rho)W^c\big), \quad q\big|_{\partial\Omega} = 0, \end{aligned} \tag{3.14}$$

acting on divergence-free vector fields. Furthermore, by (3.12),

(3.15)
$$\|D_t^k A^\varepsilon W\|_r \le C_\varepsilon \sum_{j=0}^{k} \|D_t^j W\|_r, \quad \text{where} \quad \|W\|_r = \sum_{|\alpha|\le r} \|\partial_y^\alpha W(t,\cdot)\|_{L^2(\Omega)}.$$

Let us also define the projected multiplication operators M_β with a two form β by

$$\underline{M}_\beta W_a = \underline{P}(\beta_{ab} W^b). \tag{3.16}$$

Since the projection has norm one it follows that

$$\|M_\beta W\| \le \|\beta\|_\infty \|W\|. \tag{3.17}$$

Furthermore we define the operator taking vector fields to one forms by

$$\underline{G} W_a = \underline{M}_g W_a = P(g_{ab} W^b). \tag{3.18}$$

Then G acting on divergence-free vector fields is just the identity I.

Let L_1 be the modified linearized operator in (2.49) and let $\dot W = \hat D_t W = D_t W + (\operatorname{div} V) W = \kappa^{-1} D_t(\kappa W)$, $\ddot W = \hat D_t^2 W$. We want to prove existence of a solution W to

$$L_1 W = \ddot W + AW - B_0 W - B_1 \dot W = F, \qquad W\big|_{t=0} = \dot W\big|_{t=0} = 0 \tag{3.19}$$

for general vector fields F that are not necessarily divergence-free. To do this we first subtract off a vector field W_1 that picks up the divergence and then solve (3.19) in the divergence-free class. Let us decompose a vector field into a divergence-free part and a gradient using the orthogonal projection:

$$W = W_0 + W_1, \qquad W_0 = PW, \qquad W_1^a = g^{ab}\partial_b q_1, \qquad q_1\big|_{\partial\Omega} = 0. \tag{3.20}$$

Then if $\dot g_{ab} = \check D_t g_{ab}$, where $\check D_t = D_t - \dot\sigma$, we have $\partial_a D_t\, q_1 = D_t(g_{ab} W_1^b) = \dot g_{ab} W_1^b + g_{ab} \dot W_1^b$ and $\partial_a D_t^2 q_1 = \ddot g_{ab} W_1^b + 2\dot g_{ab} \dot W_1^b + g_{ab} \ddot W_1^b$, where $\ddot g_{ab} = \check D_t^2 g_{ab}$. Hence

$$\ddot W_1^a = g^{ab}\partial_b D_t^2 q_1 - 2 g^{ab} \dot g_{bc} \dot W_1^c - g^{ab} \ddot g_{bc} W_1^c. \tag{3.21}$$

Since $D_t^2 q_1\big|_{\partial\Omega} = 0$ and the projection of a gradient of a function that vanishes on the boundary vanishes,

(3.22)
$$P\ddot W_1^a = B_2(W_1, \dot W_1)^a, \quad \text{where} \quad B_2(W_1,\dot W_1)^a = -P\big(2g^{ab}\dot g_{bc}\dot W_1^c + g^{ab}\ddot g_{bc} W_1^c\big).$$

Since $\operatorname{div} W_0 = 0$ it follows that $\operatorname{div} \dot W_0 = \operatorname{div} \ddot W_0 = 0$ and hence by Lemma 2.5

$$PL_1 W_0 = L_1 W_0 = \ddot W_0 + AW_0 - B_1 \dot W_0 - B_0 W_0 \tag{3.23}$$

$$PL_1 W_1 = AW_1 - B_{11} \dot W_1 - B_{01} W_1 \tag{3.24}$$

where

$$B_{11}\dot{W}^a = PB_1\dot{W}^a + 2P\big(g^{ab}\dot{g}_{bc}\dot{W}^c\big),\qquad (3.25)$$
$$B_{01}W^a = PB_0W^a + P\big(g^{ab}\ddot{g}_{bc}W^c\big).$$

Hence projection of (3.19) gives

$$L_1W_0 = -PL_1W_1 + PF = -AW_1 + B_{11}\dot{W}_1 + B_{01}W_1 + PF. \qquad (3.26)$$

Here, by (2.54)

$$W_1^a = g^{ab}\partial_b q_1, \qquad \triangle\, q_1 = \varphi, \qquad q_1\big|_{\partial\Omega} = 0, \qquad (3.27)$$

where

$$D_t^2\varphi + \ddot{\sigma}\,\varphi = \operatorname{div} F. \qquad (3.28)$$

By (3.23), (3.24) we also have

$$(I-P)L_1W_0 = 0 \qquad (3.29)$$
$$(I-P)L_1W_1 = \ddot{W}_1 - B_2(W_1,\dot{W}_1) - (I-P)B_0W + (I-P)B_1\dot{W}_1. \qquad (3.30)$$

Summing up, we have proved:

Lemma 3.1. *Suppose that W satisfies $L_1W = F$. Let $W_0 = PW$, $W_1 = (I-P)W$, $F_0 = PF$ and $F_1 = (I-P)F$. Then*

$$L_1W_0 = F_0 - AW_1 + B_{11}\dot{W}_1 + B_{01}W_1 \qquad (3.31)$$
$$\ddot{W}_{11} = F_1 + B_2(W_1,\dot{W}_1) + (I-P)B_0W_1 + (I-P)B_1\dot{W}_1 \qquad (3.32)$$

where B_{01} and B_{11} are given by (3.25), *B_2 is given by* (3.22) *and B_0, B_1 are as in* (2.51), (2.52). *Furthermore*

$$D_t^2 \operatorname{div} W_1 + \ddot{\sigma}\operatorname{div} W_1 = \operatorname{div} F. \qquad (3.33)$$

We now find a solution of (3.19) by first solving the ordinary differential equation (3.28) and then solving the Dirichlet problem for q_1 and defining W_1 by (3.27). Finally we solve (3.26) for W_0 within the divergence-free class. This gives existence of solutions for (3.19) for general vector fields F once we can solve them for divergence-free vector fields. However, we also need estimates for (3.19) that do not lose regularity going from F to W in order to show existence also for the linearized equations (2.57):

$$L_0W = L_1W - B_3W = F, \qquad W\big|_{t=0} = \dot{W}\big|_{t=0} = 0, \qquad (3.34)$$

by iteration. It seems as if there is a loss of regularity in the term $-AW_1$ in (3.26). However, $\operatorname{curl} AW_1 = 0$ and there is an improved estimate, for (3.19) when $\operatorname{div} F = 0$ and $\operatorname{curl} F = 0$, obtained by differentiating with respect to time and using the fact that an estimate for two time derivatives also gives an estimate for the operator A through the equation (3.19). We can estimate any first order derivative of a vector field in terms of the curl, the divergence and the normal operator A and there is an identity for the curl.

Let us now also derive the basic energy estimate which will be used to prove existence and estimates for (3.19) within the divergence-free class:

$$\ddot{W} + AW = H, \qquad W\big|_{t=0} = \dot{W}\big|_{t=0} = 0, \qquad \operatorname{div} H = 0, \tag{3.35}$$

where A is the normal operator or the smoothed version. For any symmetric operator B we have

$$\frac{d}{dt}\langle W, BW\rangle = \frac{d}{dt}\int_\Omega \kappa W^a \underline{B} W_a \, dy = 2\langle \dot{W}, BW\rangle + \langle W, \dot{B}W\rangle \tag{3.36}$$

where $\dot{W} = \kappa^{-1} D_t(\kappa W)$ and $\dot{B}$ is the time derivative of the operator B considered as an operator from the divergence-free vector fields to the one forms corresponding to divergence-free vector fields:

$$\dot{B}W^a = P\big(g^{ab}(D_t \underline{B} W_b - \underline{B}\dot{W}_b)\big), \qquad \underline{B}W_b = g_{bc} B W^c; \tag{3.37}$$

see Section 4. The projection comes up here since we take the inner product with a divergence-free vector field in (3.37). Let the lowest order energy $E_0 = E(W)$ be defined by

$$E(W) = \langle \dot{W}, \dot{W}\rangle + \langle W, (A+I)W\rangle. \tag{3.38}$$

Since $\langle W, W\rangle = \langle W, GW\rangle$, where G is the projection onto divergence-free vector fields given by (3.18), it follows that

$$\dot{E}_0 = 2\langle \dot{W}, \ddot{W} + (A+I)W\rangle + \langle \dot{W}, \dot{G}\dot{W}\rangle + \langle W, (\dot{A} + \dot{G})W\rangle. \tag{3.39}$$

In particular it follows from (3.4) or (3.10), respectively (3.16) and (3.18), that

$$\dot{A}_f = A_{\dot{f}}, \ \ \dot{G} = M_{\dot{g}}, \ \text{ where } \dot{f} = \kappa D_t(\kappa^{-1} f) \ \text{ and } \ \dot{g} = \kappa D_t(\kappa^{-1} g). \tag{3.40}$$

In fact the time derivate of an operator, as defined by (3.37), commutes with the projection since $D_t\, \partial_a q = \partial_a D_t\, q$, where $D_t\, q\big|_{\partial\Omega} = 0$ if $q\big|_{\partial\Omega} = 0$, and the projection of the gradient of functions that vanishes on the boundary vanishes. It therefore follows from (3.7) or (3.12) and (3.17) that

$$|\langle W, \dot{A}W\rangle| \le \|\dot{p}/p\|_\infty \langle W, AW\rangle, \qquad |\langle W, \dot{G}W\rangle| \le \|\dot{g}\|_\infty \langle W, W\rangle. \tag{3.41}$$

The last two terms in (3.38) are hence bounded by a constant times the energy so it follows that

$$|\dot{E}_0| \le \sqrt{E_0}\big(2\|H\| + c\sqrt{E_0}\big), \qquad c = \|\dot{p}/p\|_\infty + \|\dot{g}\|_\infty + 2 \tag{3.42}$$

from which a bound for the lowest order energy follows.

Similarly, we get higher order energy estimates for vector fields that are tangential at the boundary; see Section 10. Once we have these estimates we use the fact that any derivative of a vector field can be bounded by tangential derivatives and derivatives of the divergence and the curl; see Section 5. The divergence vanishes and we can get estimates for the curl as follows. Let $w_a = g_{ab}W^b$, $\dot{w}_a = g_{ab}\dot{W}^b$ and $\ddot{w}_a = g_{ab}\ddot{W}^b$. Then $D_t w_a = \dot{g}_{ab}W^b + \dot{w}_a$ and $D_t\dot{w}_a = \dot{g}_{ab}\dot{W}^b + \ddot{w}_a$ where $\dot{g}_{ab} = \check{D}_t g_{ab} = \kappa D_t(\kappa g_{ab})$. Since

$$\ddot{w} + \underline{A}W = \underline{H}, \qquad H = B_0 W + B_1\dot{W} + F \tag{3.43}$$

where $\operatorname{curl}\underline{A}W = 0$,

$$|D_t \operatorname{curl} w| + |D_t \operatorname{curl}\dot{w}| \le C\big(|\partial W| + |W| + |\partial\dot{W}| + |\dot{W}| + |\operatorname{curl}\underline{F}|\big). \tag{3.44}$$

Note that the estimate for the curl is actually very strong. The higher order operator A vanishes so that there is no loss of regularity anymore and furthermore the estimate is point wise. This crude estimate suffices for the most part. However, there is an additional cancellation, whereas one would not need to assume estimate for $|\partial\dot{W}|$ in the right-hand side of (3.41). The improved estimate is for $\dot{w}_a$ replaced by $\tilde{w}_a = \dot{w}_a - \omega_{ab}W^b$, where $\omega_{ab} = \partial_a v_b - \partial_b v_a$. It follows from Lemma 2.5 that

$$\begin{aligned} |D_t \operatorname{curl} w| + |D_t \operatorname{curl}\tilde{w}| &\le C\big(|\operatorname{curl}\tilde{w}| + |\partial W| + |W| + |\operatorname{curl}\underline{F}|\big),\\ |\operatorname{curl}(\tilde{w} - \dot{w})| &\le C\big(|W| + |\partial W|\big). \end{aligned} \tag{3.45}$$

4. The tangential vector fields, Lie derivatives and commutators

Following [L1], we now construct the tangential vector fields that are time independent expressed in the Lagrangian coordinates, i.e. that commute with D_t. This means that in the Lagrangian coordinates they are of the form $S^a(y)\partial/\partial y^a$. Furthermore, they will satisfy,

$$\partial_a S^a = 0. \tag{4.1}$$

Since Ω is the unit ball in $\mathbf{R}^n$ the vector fields can be explicitly given. The vector fields

$$y^a\partial/\partial y^b - y^b\partial/\partial y^a \tag{4.2}$$

corresponding to rotations, span the tangent space of the boundary and are divergence-free in the interior. Furthermore they span the tangent space of the level sets of the distance function from the boundary in the Lagrangian coordinates:

$$d(y) = \operatorname{dist}(y, \partial\Omega) = 1 - |y| \tag{4.3}$$

away from the origin $y \ne 0$. We will denote this set of vector fields by $\mathcal{S}_0$ We also construct a set of divergence-free vector fields that span the full tangent space at distance $d(y) \ge d_0$ and that are compactly supported in the interior at a fixed distance $d_0/2$ from the boundary. The basic one is

$$h(y^3, \dots, y^n)\Big(f(y^1)g'(y^2)\partial/\partial y^1 - f'(y^1)g(y^2)\partial/\partial y^2\Big), \tag{4.4}$$

which satisfies (4.1). Furthermore we can choose f, g, h such that it is equal to $\partial/\partial y^1$ when $|y^i| \leq 1/4$, for $i = 1, \ldots, n$ and so that it is 0 when $|y^i| \geq 1/2$ for some i. In fact let f and g be smooth functions such that $f(s) = 1$ when $|s| \leq 1/4$ and $f(s) = 0$ when $|s| \geq 1/2$ and $g'(s) = 1$ when $|s| \leq 1/4$ and $g(s) = 0$ when $|s| \geq 1/2$. Finally let $h(y^3, \ldots, y^n) = f(y^3) \ldots f(y^n)$. By scaling, translation and rotation of these vector fields we can obviously construct a finite set of vector fields that span the tangent space when $d \geq d_0$ and are compactly supported in the set where $d \geq d_0/2$. We will denote this set of vector fields by $\mathcal{S}_1$. Let $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ denote the family of tangential space vector fields and let $\mathcal{T} = \mathcal{S} \cup \{D_t\}$ denote the family of space time tangential vector fields.

Let the radial vector field be

$$R = y^a \partial/\partial y^a. \tag{4.5}$$

Now,

$$\partial_a R^a = n \tag{4.6}$$

is not 0 but for our purposes it suffices that it is constant. Let $\mathcal{R} = \mathcal{S} \cup \{R\}$. Note that $\mathcal{R}$ spans the full tangent space of the space everywhere. Let $\mathcal{U} = \mathcal{S} \cup \{R\} \cup \{D_t\}$ denote the family of all vector fields. Note also that the radial vector field commutes with the rotations;

$$[R, S] = 0, \qquad S \in \mathcal{S}_0. \tag{4.7}$$

Furthermore, the commutators of two vector fields in $\mathcal{S}_0$ is just $\pm$ another vector field in $\mathcal{S}_0$. Therefore, for $i = 0, 1$, let $\mathcal{R}_i = \mathcal{S}_i \cup \{R\}$, $\mathcal{T}_i = \mathcal{S}_i \cup \{D_t\}$ and $\mathcal{U}_i = \mathcal{S}_i \cup \{R\} \cup \{D_t\}$.

Let us now introduce the Lie derivative of the vector field W with respect to the vector field T;

$$\mathcal{L}_T W^a = T W^a - (\partial_c T^a) W^c. \tag{4.8}$$

We will only deal with Lie derivatives with respect to the vector fields T constructed above. For those vector fields T we have

$$[D_t, T], \qquad \text{and} \qquad [D_t, \mathcal{L}_T] = 0. \tag{4.9}$$

The Lie derivative of a one form is defined by

$$\mathcal{L}_T \alpha_a = T \alpha_a + (\partial_a T^c) \alpha_c. \tag{4.10}$$

The Lie derivative also commutes with exterior differentiation, $[\mathcal{L}_T, d] = 0$ so that

$$\mathcal{L}_T \partial_a q = \partial_a T q \tag{4.11}$$

if q is a function. The Lie derivative of a two form is given by

$$\mathcal{L}_T \beta_{ab} = T \beta_{ab} + (\partial_a T^c) \beta_{cb} + (\partial_b T^c) \beta_{ac}. \tag{4.12}$$

Furthermore if w is a one form and $\operatorname{curl} w_{ab} = dw_{ab} = \partial_a w_b - \partial_b w_a$ then since the Lie derivative commutes with exterior differentiation:

$$\mathcal{L}_T \operatorname{curl} w_{ab} = \operatorname{curl} \mathcal{L}_T w_{ab}. \tag{4.13}$$

We will also use the fact that the Lie derivative satisfies Leibniz's rule, e.g.

$$\mathcal{L}_T(\alpha_c W^c) = (\mathcal{L}_T \alpha_c) W^c + \alpha_c \mathcal{L}_T W^c, \quad \mathcal{L}_T(\beta_{ac} W^c) = (\mathcal{L}_T \beta_{ac}) W^c + \beta_{ac} \mathcal{L}_T W^c. \tag{4.14}$$

Furthermore, we will also treat D_t as if it were a Lie derivative and set

$$\mathcal{L}_{D_t} = D_t. \tag{4.15}$$

Now of course this is not a space Lie derivative. It can however be interpreted as a space time Lie derivative restricted to the space components. It satisfies the same properties (4.9)–(4.14) as the other Lie derivatives we are considering. The reason we want to call it $\mathcal{L}_{D_t}$ is simply that we will apply products of Lie derivatives and D_t and since they behave in exactly the same way it is more efficient to have one notation for them.

The modification of the Lie derivative

$$\tilde{\mathcal{L}}_U W = \mathcal{L}_U W + (\operatorname{div} U) W, \tag{4.16}$$

preserves the divergence-free condition:

$$\operatorname{div} \tilde{\mathcal{L}}_U W = \tilde{U} \operatorname{div} W, \qquad \text{where} \qquad \tilde{U} f = U f + (\operatorname{div} U) f, \tag{4.17}$$

if f is a function. Note that (4.16) is invariant and (4.17) holds for any vector field U. However, since we are considering Lie derivatives only with respect to the vector fields constructed above and only expressed in the Lagrangian coordinates it is simpler to use the modification

$$\hat{\mathcal{L}}_U W = \kappa^{-1} \mathcal{L}_U(\kappa W) = \mathcal{L}_U W + (U\sigma) W, \qquad \text{where} \qquad \sigma = \ln \kappa. \tag{4.18}$$

Due to (4.1), $\operatorname{div} S = \kappa^{-1} \partial_a(\kappa S^a) = S\sigma$, if S is any of the tangential vector fields and $\operatorname{div} R = R\sigma + n$, if R is the radial vector field. For any of our tangential vector fields it follows that

$$\operatorname{div} \hat{\mathcal{L}}_U W = \hat{U} \operatorname{div} W, \qquad \text{where} \qquad \hat{U} f = U f + (U\sigma) f = \kappa^{-1} U(\kappa f). \tag{4.19}$$

This has several advantages. The commutators satisfy $[\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_T] = \hat{\mathcal{L}}_{[U,T]}$, since this is true for the usual Lie derivative. Furthermore, this definition is consistent with our previous definition of $\hat{D}_t$.

However, when we apply this to one-forms we want to use the regular definition of the Lie derivative. Also, when applying this to two-forms, most of the time we use the regular definition: However, when applied to two forms it turns out to be sometimes convenient to use the opposite modification:

$$\check{\mathcal{L}}_T \beta_{ab} = \mathcal{L}_T \beta_{ab} - (U\sigma) \beta_{ab}. \tag{4.20}$$

We will most of the time apply the Lie derivative to products of the form $\alpha_a = \beta_{ab} W^b$:

$$\mathcal{L}_T\big(\beta_{ab} W^b\big) = (\check{\mathcal{L}}_T \beta_{ab}) W^b + \beta_{ab} \hat{\mathcal{L}}_T W \tag{4.21}$$

since the usual Lie derivative satisfies Leibniz's rule. Using the modified Lie derivative we indicated in [L2] how to extend the existence theorem in [L1] to the case when κ is no longer constant, i.e. $D_t \sigma = \operatorname{div} V \neq 0$. This will be carried out in more detail here.

Let $\mathcal{U} = \{U_i\}_{i=1}^M$ be some labeling of our family of vector fields. We will also use multi-indices $I = (i_1, \dots, i_r)$ of length $|I| = r$. Let $U^I = U_{i_1} \dots U_{i_r}$ and $\mathcal{L}_U^I = \mathcal{L}_{U_{i_1}} \dots \mathcal{L}_{U_{i_r}}$, where $\mathcal{L}_U$ is the Lie derivative. Similarly let $\hat{U}^I f = \hat{U}_{i_1} \dots \hat{U}_{i_r} f = \kappa^{-1} U^I (\kappa f)$ and $\hat{\mathcal{L}}_U^I W = \hat{\mathcal{L}}_{U_{i_1}} \dots \hat{\mathcal{L}}_{U_{i_r}} W = \kappa^{-1} \mathcal{L}_U^I (\kappa W)$, where $\hat{\mathcal{L}}_U$ is the modified Lie derivative. Sometimes we will also write $\mathcal{L}_U^I$, where $U \in \mathcal{S}_0$ or $I \in \mathcal{S}_0$, meaning that $U_{i_k} \in \mathcal{S}_0$ for all of the indices in I.

We will now calculate the commutator between Lie derivatives and the operator defined in Section 3, i.e. the normal operator and the projected multiplication operators. It is easier to calculate the commutator with Lie derivatives of these operators considered as operators with values in the one-forms. The one-form w corresponding to the vector fields W is given by lowering the indices

$$w_a = \underline{W}_a = g_{ab} W^b. \tag{4.22}$$

For an operator B on vector fields we denote the corresponding operator with values in the one-forms by $\underline{B}$. These are related by

$$\underline{B} W_a = g_{ab} B W^b, \qquad B W^a = g^{ab} \underline{B}_a. \tag{4.23}$$

Most operators that we consider will map onto the divergence-free vector fields and so we will project the result afterwards to stay in this class. Furthermore, in order to preserve the divergence-free condition we will use the modified Lie derivative. If the modified Lie derivative is applied to a divergence-free vector field then the result is divergence-free and so projecting after commuting does not change the result. As pointed out above, for our operators it is easier to commute Lie derivatives with the corresponding operators from the divergence-free vector fields to the one forms. Let B_T be defined by

$$B_T W^a = P\big(g^{ab}\big(\mathcal{L}_T \underline{B} W_b - \underline{B}_b \hat{\mathcal{L}}_T W\big)\big). \tag{4.24}$$

In particular if B is a multiplication operator $\underline{B}_a W = P(\beta_{ab} W^b) = \beta_{ab} W^b - \partial_a q$, where q vanishes on the boundary is chosen so that $\operatorname{div} BW = 0$ then

$$\mathcal{L}_T \underline{B}_a W = \beta_{ab} \hat{\mathcal{L}}_T W^b + (\check{\mathcal{L}}_T \beta_{ab}) W^b + \partial_a T q \tag{4.25}$$

and if we project to the divergence-free vector fields then the term $\partial_a T q$ vanishes since if T is a tangential vector field then $Tq = 0$ as well. It therefore follows that B_T is another multiplication operator:

$$\underline{B}_T W_a = P\big((\check{\mathcal{L}}_T \beta_{ab}) W^b\big). \tag{4.26}$$

In particular, we will denote the time derivative of an operator by $\dot{B} = B_{D_t}$ and for a multiplication operator this is

$$\dot{B} W = B_{D_t} W = P\big(\big(\check{D}_t \beta_{ab}) W^b\big). \tag{4.27}$$

If B maps onto the divergence-free vector fields

$$\hat{\mathcal{L}}_T B W^a = \hat{\mathcal{L}}_T (g^{ab} \underline{B}_a W) = (\hat{\mathcal{L}}_T g^{ab}) \underline{B}_a W + g^{ab} \mathcal{L}_T \underline{B}_a W. \tag{4.28}$$

Here $\hat{\mathcal{L}}_T g^{ab} = -g^{ac} g^{bd} \check{\mathcal{L}}_T g_{cd}$. If B maps onto the divergence-free vector fields then $\hat{\mathcal{L}}_T B$ is also divergence-free so the left-hand side is unchanged if we project:

$$\hat{\mathcal{L}}_T B W^a = -P\big(g^{ab} (\check{\mathcal{L}}_T g_{bc}) \underline{B} W^c\big) + P\big(g^{ab} \big(\mathcal{L}_T \underline{B}_a W - \underline{B}_a \hat{\mathcal{L}}_T W\big)\big) + B \hat{\mathcal{L}}_T W^a. \tag{4.29}$$

By (4.26) applied the $\underline{G}_{ab} = P(g_{ab} W^b)$ we see that $G_T W = P\big((g^{ab} \check{\mathcal{L}}_T g_{bc}) W^c)\big)$ so the first term in the right of (4.29) is $G_T B W^a$. The second term is, by definition (4.24), $B_T W$ so we get

$$\hat{\mathcal{L}}_T B W = B \hat{\mathcal{L}}_T W + B_T W - G_T B W. \tag{4.30}$$

The most important property of the projection is that it almost commutes with Lie derivatives with respect to tangential vector fields. If $\underline{P} u_a = u_a - \partial_a p_U$ then

$$\underline{P} \mathcal{L}_T \underline{P} u_a = \underline{P} \mathcal{L}_T u_a \tag{4.31}$$

since $\mathcal{L}_T \partial_a p_U = \partial_a T p_U$ vanishes when we project again since $T p_U$ vanishes on the boundary, a fact used just above. We have already calculated commutators between Lie derivatives and the multiplication operators; so let us now also calculate the commutator between the Lie derivative with respect to tangential vector fields and the normal operator, defined by $A_f W^a = g^{ab} \underline{A}_f W_b$, where

$$\underline{A}_f W_a = -\partial_a \big((\partial_c f) W^c - q\big), \qquad \triangle\big((\partial_c f) W^c - q\big) = 0, \qquad q\big|_{\partial\Omega} = 0 \tag{4.32}$$

and f was the function that vanished on the boundary. Since the Lie derivative commutes with exterior differentiation it follows that

$$\mathcal{L}_T \underline{A}_f W_a = -\partial_a \big((\partial_c f) \hat{\mathcal{L}}_T W^c + (\partial_c \check{T} f) W^c + (\partial_c T \sigma) f W^c - T q\big). \tag{4.33}$$

Since q vanishes on the boundary it follows that Tq also vanishes on the boundary and so does $(\partial_c T \sigma) f W^c$. Therefore the last two terms vanish when we project again and so we get

$$P\big(g^{ab} \mathcal{L}_T \underline{A}_f W_b\big) = P\big(g^{ab} \underline{A}_f \hat{\mathcal{L}}_T W_b\big) + P\big(g^{ab} \underline{A}_{\check{T} f} W_b\big). \tag{4.34}$$

Let us now change notation so that $A = A_p$, where p is the pressure. Then having just calculated A_T defined by (4.24) to be $A_T = A_{\check{T}p}$, we have

$$A_T = A_{\check{T}p}, \qquad \text{if} \qquad A = A_p. \tag{4.35}$$

In particular, if $T = D_t$ is the time derivative we will use the notation $\dot{A} = A_{D_t}$ which then is

$$\dot{A}W = A_{D_t}W = A_{\check{D}_t p}W. \tag{4.36}$$

Exactly the same formulas hold for A^ε_f. By (3.14)

$$\begin{aligned} \underline{A}^\varepsilon_f W_a &= (f/\rho)\chi'_\varepsilon(\partial_a\rho)\,(\partial_c\rho)W^c - \partial_a q, \\ \triangle q &= \kappa^{-1}\partial_a\big(\kappa g^{ab}(f/\rho)\chi'_\varepsilon(\partial_b\rho)\,(\partial_c\rho)W^c\big), \quad q\big|_{\partial\Omega} = 0 \end{aligned} \tag{4.37}$$

where $\rho = \rho(d)$, $d(y) = \operatorname{dist}(y, \partial\Omega)$. It follows that $T\rho =$, if $T \in \mathcal{T}$. Furthermore $S \in \mathcal{S}_1 = \mathcal{S} \setminus \mathcal{S}_0$ vanishes close to the boundary when $d(y) \leq d_0/2$ and $\chi'_\varepsilon = 0$ when $d(y) \geq \varepsilon$ so it follows that

$$\mathcal{L}_T\underline{A}^\varepsilon_f W_a = ((\check{T}f)/\rho)\chi'_\varepsilon(\partial_a\rho)\,(\partial_c\rho)W^c - (f/\rho)\chi'_\varepsilon(\partial_a\rho)\,(\partial_c\rho)\hat{\mathcal{L}}_T W^c - \partial_a Tq. \tag{4.38}$$

Hence

$$P\big(g^{ab}\mathcal{L}_T\underline{A}^\varepsilon_f W_b\big) = P\big(g^{ab}\underline{A}^\varepsilon_f\hat{\mathcal{L}}_T W_b\big) + P\big(g^{ab}\underline{A}^\varepsilon_{\check{T}f}W_b\big). \tag{4.39}$$

We can now also calculate higher order commutators:

Definition 4.1. If T is a vector fields let B_T be defined by (4.24). If T and S are two tangential vector fields we define $B_{TS} = (B_S)_T$ to be the operator obtained by first using (4.24) to define B_S and then define $(B_S)_T$ to be the operator obtained from (4.24) with B_S in place of B. Similarly if $S^I = S^{i_2}\dots S^{i_r}$ is a product of $r = |I|$ vector fields then we define

$$B_I = \big(\dots(B_{S^{i_1}})\dots\big)_{S^{i_r}}. \tag{4.40}$$

If B is a projected multiplication operator $BW^a = P\big(g^{ab}\beta_{bc}W^c\big)$ then

$$B_I W = P\big(g^{ab}(\check{\mathcal{L}}^I_T\beta_{bc})W^c\big). \tag{4.41}$$

In particular if $GW^a = P\big(g^{ab}g_{bc}W^c\big)$ then

$$G_I W = P\big(g^{ab}(\check{\mathcal{L}}^I_T g_{bc})W^c\big). \tag{4.42}$$

If A is the normal operator then

$$A_I W^a = P\big(g^{ab}\partial_b\big((\partial_c\check{T}^I p)W^c\big)\big). \tag{4.43}$$

With B_T as in (4.4) we have proved that if B maps onto the divergence-free vector fields then

$$\hat{\mathcal{L}}_T BW = BW_T + B_T W - G_T BW, \qquad W_T = \hat{\mathcal{L}}_T W. \tag{4.44}$$

Repeating this gives, for a product of modified Lie derivatives:

$$\hat{\mathcal{L}}_T^I BW = c_I^{I_1\dots I_k} G_{I_3}\dots G_{I_k} B_{I_1} W_{I_2}, \quad W_J = \hat{\mathcal{L}}_T^J W, \tag{4.45}$$

where the sum is over all combinations of $I = I_1 + \cdots + I_k$, and $c_I^{I_1\dots I_k}$ are some constants such that $c_I^{I_1\dots I_k} = 1$ if $I_1 + I_2 = I$. Let us then also introduce the notation

$$G_I^{I_1 I_2} = c_I^{I_1\dots I_k} G_{I_3}\dots G_{I_k}, \tag{4.46}$$

where the sum is over all combinations such that $I_3 + \dots I_k = I - I_1 - I_2$. With this notation we can write (4.41)

$$\hat{\mathcal{L}}_T^I BW = G_I^{I_1 I_2} B_{I_1} W_{I_2} \tag{4.47}$$

where again $G_I^{I_1 I_2} = 1$ if $I_1 + I_2 = I$. Also let

$$\tilde{G}_I^{I_1\dots I_k} = 0, \quad \text{if} \quad I_2 = I, \qquad \text{and} \qquad \tilde{G}_I^{I_1\dots I_k} = G_I^{I_1\dots I_k}, \quad \text{otherwise}. \tag{4.48}$$

Then we also have

$$\hat{\mathcal{L}}_T^I BW = BW_I + \tilde{G}_I^{I_1 I_2} B_{I_1} W_{I_2}. \tag{4.49}$$

5. Estimating derivatives of a vector field in terms of the curl, the divergence and tangential derivatives or the normal operator

The first part of the lemma below says that one can get a pointwise estimate of any first order derivative of a vector field by the curl, the divergence and derivatives that are tangential at the boundary. The second part say that one can get L^2 estimates with a normal derivative instead of tangential derivatives. The last part says that we can get the estimate for the normal derivative from the normal operator. The lemma is formulated in the Eulerian frame, i.e. in terms of the Euclidean coordinates. Later we will reformulate it in the Lagrangian frame and get similar estimates for higher derivatives.

Lemma 5.1. *Let $\tilde{\mathcal{N}}$ be a vector field that is equal to the normal $\mathcal{N}$ at the boundary $\partial\mathcal{D}_t$ and satisfies $|\tilde{\mathcal{N}}| \le 1$ and $|\partial\tilde{\mathcal{N}}| \le K$. Let $q^{ij} = \delta^{ij} - \tilde{\mathcal{N}}^i\tilde{\mathcal{N}}^j$. Then*

$$|\partial\beta|^2 \le C\big(q^{kl}\delta^{ij}\partial_k\beta_i\,\partial_l\beta_j + |\operatorname{curl}\beta|^2 + |\operatorname{div}\beta|^2\big) \tag{5.1}$$

$$\int_{\mathcal{D}_t} |\partial\beta|^2 dx \le C\int_{\mathcal{D}_t}\big(\delta^{ij}\tilde{\mathcal{N}}^k\tilde{\mathcal{N}}^l\partial_i\beta_k\,\partial_j\beta_l + |\operatorname{curl}\beta|^2 + |\operatorname{div}\beta|^2 + K^2|\beta|^2\big)\,dx. \tag{5.2}$$

Suppose that $\delta^{ij}\alpha_j$ is another vector field that is normal at the boundary and let $A\beta_i = \partial_i(\alpha_k\beta^k - q)$ and q is chosen so that $\operatorname{div} A\beta = 0$ and $q|_{\partial\Omega} = 0$. Then

$$\begin{aligned} \int_{\mathcal{D}_t} &\delta^{ij}\alpha_k\alpha_l\,\partial_i\beta^k\,\partial_j\beta^l\,dx \\ &\le C\int_{\mathcal{D}_t}\big(\delta^{ij}A\beta_i\,A\beta_j + |\alpha|^2\big(|\operatorname{curl}\beta|^2 + |\operatorname{div}\beta|^2\big) + |\partial\alpha|^2|\beta|^2\big)\,dx. \end{aligned} \tag{5.3}$$

Proof. (5.1) follows from the pointwise estimate

$$\delta^{ij}\delta^{kl}w_{ki}w_{lj} \le C\big(\delta^{ij}q^{kl}w_{ki}w_{lj} + |\hat{w}|^2 + (\operatorname{tr} w)^2\big) \tag{5.4}$$

$$\delta^{ij}\delta^{kl}w_{ki}w_{lj} \le C\big(\tilde{\mathcal{N}}^i\tilde{\mathcal{N}}^j\delta^{kl}w_{ki}w_{lj} + (q^{ij}q^{kl} - q^{ik}q^{jl})w_{ki}w_{lj} + |\hat{w}|^2 + (\operatorname{tr} w)^2\big) \tag{5.5}$$

where $\hat{w}_{ij} = w_{ij} - w_{ji}$ is the antisymmetric part and $\operatorname{tr} w = \delta^{ij}w_{ij}$ is the trace. To prove (5.4), (5.5) we may assume that w is symmetric and traceless. Writing $\delta^{ij} = q^{ij} + \tilde{\mathcal{N}}^i\tilde{\mathcal{N}}^j$ we see that (5.4) for such tensors follows from the estimate $\tilde{\mathcal{N}}^i\tilde{\mathcal{N}}^j\tilde{\mathcal{N}}^k\tilde{\mathcal{N}}^l w_{ki}w_{lj} = (\tilde{\mathcal{N}}^i\tilde{\mathcal{N}}^k w_{ki})^2 = (q^{ik}w_{ki})^2 \le nq^{ij}q^{kl}w_{ki}w_{lj}$. (This says that $(\operatorname{tr}(QW))^2 \le n\operatorname{tr}(QWQW)$ which is obvious if one writes it out and uses the symmetry.) Now, (5.5) follows since $(\delta^{ij}q^{kl} - \tilde{\mathcal{N}}^i\tilde{\mathcal{N}}^j\delta^{kl})w_{ki}w_{lj} = (q^{ij}q^{kl} - \tilde{\mathcal{N}}^i\tilde{\mathcal{N}}^j\tilde{\mathcal{N}}^k\tilde{\mathcal{N}}^l)w_{ki}w_{lj} = (q^{ij}q^{kl} - q^{ik}q^{jl})w_{ki}w_{lj}$. Also, (5.2) follows from (5.5) and integration by parts using the fact that the boundary terms vanish, since we assumed that $\tilde{\mathcal{N}} = \mathcal{N}$ there, and that $(q^{ij}q^{kl} - q^{ik}q^{jl})\beta_i\,\partial_k\partial_j\beta_l = 0$:

$$\int_{\mathcal{D}_t}(q^{ij}q^{kl} - q^{ik}q^{jl})\partial_k\beta_i\,\partial_j\beta_l\,dx = -\int_{\mathcal{D}_t}\partial_k(q^{ij}q^{kl} - q^{ik}q^{jl})\,\beta_i\,\partial_j\beta_l\,dx. \tag{5.6}$$

We have $A\beta_i = (\partial_i\alpha_k)\beta^k + \alpha_k\partial_i\beta^k - \partial_i q$ and so to prove (5.3) we must estimate $\|\partial q\|_{L^2}$. Since $0 = \partial_i A\beta^i = \triangle(\alpha_k\beta^k) - \triangle q$ it follows that $\triangle q = \triangle\big(\alpha_k\beta^k\big) = 2\partial_i\big((\partial^i\alpha_k)\beta^k\big) + \alpha_k\triangle\beta^k - (\triangle\alpha_k)\beta^k$ and $\alpha_k\triangle\beta^k = \partial_i\big(\alpha^i\operatorname{div}\beta + \alpha_k\operatorname{curl}\beta^{ik}\big) - \operatorname{div}\alpha\,\operatorname{div}\beta - (\partial_k\alpha_i)\partial^k\beta^i + (\partial_k\alpha_i)\partial^i\beta^k$, and hence $\triangle(\alpha_k\beta^k) = \partial_i\Big(2(\partial^i\alpha_k)\beta^k + \alpha^i\operatorname{div}\beta + \alpha_k\operatorname{curl}\beta^{ik} - \operatorname{div}\alpha\,\beta^i - \operatorname{curl}\alpha^i{}_k\,\beta^k\Big)$. It follows that

$$\begin{aligned}\int_\Omega |\partial q|^2\,dx &= -\int_\Omega q\triangle q\,dx \\ &= -\int_\Omega q\partial_i\big(\alpha^i\operatorname{div}\beta + \alpha_k\operatorname{curl}\beta^{ik} + (\partial^i\alpha_k + \partial_k\alpha^i)\beta^k - \operatorname{div}\alpha\,\beta^i\big)\,dx\end{aligned} \tag{5.7}$$

and integration by parts again gives $\|\partial q\|_{L^2} \le C\big(\|\,|\alpha|\operatorname{div}\beta\|_{L^2} + \|\,|\alpha|\operatorname{curl}\beta\|_{L^2} + \|\,|\partial\alpha|\beta\|_{L^2}\big)$. □

Definition 5.1. For $\mathcal{V}$, any of the family of vector fields introduced in [L1], and for β a two form, a one form, a function or a vector field we define

$$|\beta|_r^{\mathcal{V}} = \sum_{|I|\le r,\, I\in\mathcal{V}} |\mathcal{L}_U^I\beta\,|, \qquad [\beta]_r^{\mathcal{V}} = \sum_{r_1+\ldots r_k\le r,\, r_i\ge 1} |\beta|_{r_1}^{\mathcal{V}}\cdots|\beta|_{r_k}^{\mathcal{V}} \tag{5.8}$$

and $[\beta]_0^{\mathcal{V}} = 1$. Furthermore,

$$|\beta|_r = \sum_{|\alpha|\le r} |\partial_y^\alpha\beta|. \tag{5.9}$$

If β is a function then $\mathcal{L}_U\beta = U\beta$ and in general it is equal to this plus terms proportional to β. Hence (5.8) is equivalent to just the sum $\sum_{|I|\le r,\, I\in\mathcal{V}} |U^I\beta|$. In particular if $\mathcal{R}$ denotes the family of space vector fields then $|\beta|_r^{\mathcal{R}}$ is equivalent to $|\beta|_r$ with a constant of equivalence independent of the metric. Note also that if β is the one form $\beta_a = \partial_a q$ then $\mathcal{L}_U^I\beta = \partial U^I q$ so that $|\partial q|_r^{\mathcal{V}} = \sum_{|I|\le r,\, I\in\mathcal{V}} |\partial U^I q|$.

Definition 5.2. Let c_1 be a constant such that

$$|\partial x/\partial y|^2 + |\partial y/\partial x|^2 \le c_1^2, \qquad \sum_{a,b=1}^{n} \big(|g_{ab}| + |g^{ab}|\big) \le nc_1^2, \tag{5.10}$$

and let K_1 denote a continuous function of c_1.

We note that the second condition in (5.10) follows from the first and the first follows from the second with a larger constant. We remark that this condition is fulfilled initially since we are composing with a diffeomorphism. Furthermore, for solution of Euler's equations, $\operatorname{div} V = 0$, so the volume form κ is preserved and hence an upper bound for the metric also implies a lower bound for the eigenvalues and an upper bound for the inverse of the metric follows.

In what follows it will be convenient to consider the norms of $\hat{\mathcal{L}}_U^I W = \kappa^{-1}\mathcal{L}_U^I(\kappa W)$ if W is a vector field and of $\check{\mathcal{L}}_U^I g = \kappa\mathcal{L}_U^I(\kappa^{-1}g)$, if g is the metric. The reason for this is simply that $\operatorname{div}(\hat{\mathcal{L}}_U^I W) = \hat{U}^I \operatorname{div} W$ and $\mathcal{L}_U^I \operatorname{curl} w = \operatorname{curl}(\mathcal{L}_U^I w)$ and when we lower indices $w_a = g_{ab}W^b = (\kappa^{-1}g_{ab})(\kappa W^b)$ and apply the Lie derivative to the product we get $\mathcal{L}_U w_a = (\check{\mathcal{L}}_U g_{ab})W^b + g_{ab}\hat{\mathcal{L}}_U W^b$.

Lemma 5.2. *Let W be a vector field and let $w_a = g_{ab}W^b$ be the corresponding one form. Let $\kappa = \det(\partial x/\partial y) = \sqrt{\det g}$. Then*

$$|\kappa| + |\kappa^{-1}| \le K_1, \qquad |U^I\kappa| + |U^I\kappa^{-1}| \le K_1 c^{I_1\dots I_k}|U^{I_1}g|\dots|U^{I_k}g| \tag{5.11}$$

where the sum is over all $I_1 + \dots I_k = I$.

With notation as in Definition 5.1 *and Section* 4,

$$|\kappa W|_r^{\mathcal{R}} \le K_1\big(|\operatorname{curl} w|_{r-1}^{\mathcal{R}} + |\kappa \operatorname{div} W|_{r-1}^{\mathcal{R}} + |\kappa W|_r^{\mathcal{S}} + \sum_{s=0}^{r-1} |g/\kappa|_{r-s}^{\mathcal{R}}|\kappa W|_s^{\mathcal{R}}\big). \tag{5.12}$$

Also,

$$|\kappa W|_r^{\mathcal{R}} \le K_1 \sum_{s=0}^{r} [g/\kappa]_s^{\mathcal{R}}\big(|\operatorname{curl} w|_{r-1-s}^{\mathcal{R}} + |\kappa \operatorname{div} W|_{r-1-s}^{\mathcal{R}} + |\kappa W|_{r-s}^{\mathcal{S}}\big), \tag{5.13}$$

where for $s = r$ *there is the convention that* $|\operatorname{curl} w|^{\mathcal{V}}_{-1} = |\kappa \operatorname{div} W|^{\mathcal{V}}_{-1} = 0$. *Furthermore* (5.12), (5.13) *hold without the factors* κ *and* $1/\kappa$; *i.e.*,

$$(5.14)\qquad |W|^{\mathcal{R}}_r \leq K_1 \sum_{s=0}^{r} [g]^{\mathcal{R}}_s \big(|\operatorname{curl} w|^{\mathcal{R}}_{r-1-s} + |\operatorname{div} W|^{\mathcal{R}}_{r-1-s} + |W|^{\mathcal{S}}_{r-s} \big).$$

(5.12), (5.13) *also hold for the vector field* W *replaced by a one form* w; *i.e.*,

$$(5.15)\qquad |w|^{\mathcal{R}}_r \leq K_1 \sum_{s=0}^{r} [g]^{\mathcal{R}}_s \big(|\operatorname{curl} w|^{\mathcal{R}}_{r-1-s} + |\operatorname{div} W|^{\mathcal{R}}_{r-1-s} + |w|^{\mathcal{S}}_{r-s} \big).$$

Moreover, the inequalities (5.12)–(5.15) *also hold with* $(\mathcal{R}, \mathcal{S})$ *replaced by* $(\mathcal{U}, \mathcal{T})$.

Proof. If $\sigma = \ln \kappa = (\ln \det g)/2$ then $U\sigma = \operatorname{tr} \mathcal{L}_U g/2 = g^{ab} \mathcal{L}_U g_{ab}/2$ and $\mathcal{L}_U g^{ab} = -g^{ac} g^{bd} \mathcal{L}_U g_{cd}$. An easy consequence of Lemma 5.1, see [L1], is: In the Lagrangian frame we have, with $w_a = \underline{W}_a = g_{ab} W^b$,

$$(5.16)\qquad |\hat{\mathcal{L}}_U W| \leq K_1 \Big(|\operatorname{curl} \underline{W}| + |\operatorname{div} W| + \textstyle\sum_{S \in \mathcal{S}} |\hat{\mathcal{L}}_S W| + [g]_1 |W| \Big), \qquad U \in \mathcal{R},$$

$$(5.17)\qquad |\hat{\mathcal{L}}_U W| \leq K_1 \Big(|\operatorname{curl} \underline{W}| + |\operatorname{div} W| + \textstyle\sum_{T \in \mathcal{T}} |\hat{\mathcal{L}}_T W| + [g]_1 |W| \Big), \qquad U \in \mathcal{U}$$

where $[g]_1 = 1 + |\partial g|$. Furthermore

$$(5.18)\qquad |\partial W| \leq K_1 \Big(|\hat{\mathcal{L}}_R W| + \textstyle\sum_{S \in \mathcal{S}} |\hat{\mathcal{L}}_S W| + [g]_1 |W| \Big).$$

When $d(y) \leq d_0$ we may replace the sums over $\mathcal{S}$ by the sums over $\mathcal{S}_0$ and the sum over $\mathcal{T}$ by the sum over $\mathcal{T}_0$. In [L1] this was proved for $\hat{\mathcal{L}}_U$ replaced by $\mathcal{L}_U$, but the difference is just a lower order term.

We claim that

$$(5.19)\qquad \sum_{|I|=r, U \in \mathcal{R}} |\hat{\mathcal{L}}^I_U W| \leq K_1 \sum_{|J|=r-1, U \in \mathcal{R}} \big(|\operatorname{curl} \underline{\hat{\mathcal{L}}^J_U W}| + |\operatorname{div} \hat{\mathcal{L}}^J_U W| + [g]_1 |\hat{\mathcal{L}}^J_U W| \big) + K_1 \sum_{|I|=r, S \in \mathcal{S}} |\hat{\mathcal{L}}^I_S W|.$$

First we note that there is nothing to prove if $d(y) \geq d_0$ since then the $\mathcal{S}$ span the full tangent space. Therefore, it suffices to prove (5.19) when $d(y) \leq d_0$ and with $\mathcal{S}$ replaced by $\mathcal{S}_0$ and $\mathcal{R}$ replaced by $\mathcal{R}_0$. Then (5.19) follows from (5.16) if $r = 1$. Assuming that it is true for r replaced by $r-1$ we will prove that it holds for r. If we apply (5.16) to $\hat{\mathcal{L}}^J_U W$, where $|J| = r-1$, we get

$$(5.20)\qquad |\hat{\mathcal{L}}_U \hat{\mathcal{L}}^J_U W| \leq K_1 \big(|\operatorname{curl} \underline{\hat{\mathcal{L}}^J_U W}| + |\operatorname{div} \hat{\mathcal{L}}^J_U W| + \sum_{S \in \mathcal{S}} |\hat{\mathcal{L}}_S \hat{\mathcal{L}}^J_U W| + [g]_1 |\hat{\mathcal{L}}^J_U W| \big).$$

If $\hat{\mathcal{L}}_U^J$ consist of all tangential derivatives then it follows that $|\hat{\mathcal{L}}_U\hat{\mathcal{L}}_U^J W|$ is bounded by the right-hand side of (5.19). If $\hat{\mathcal{L}}_U^J$ does not consist of only tangential derivatives then, since $[\hat{\mathcal{L}}_R,\hat{\mathcal{L}}_S]=\hat{\mathcal{L}}_{[R,S]}=0$, if $S\in\mathcal{S}_0$, we can write $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_U^J W=\hat{\mathcal{L}}_U^K\hat{\mathcal{L}}_{S'}W$, for some $S'\in\mathcal{S}_0$. If we now apply (5.19) with r replaced by $r-1$ to $\hat{\mathcal{L}}_{S'}W$, (5.19) follows also for r.

In Lemma 5.2 we have $\mathcal{L}_U^I\operatorname{curl} w=\operatorname{curl}\mathcal{L}_U^I w$ which however is different from curl $\underline{\hat{\mathcal{L}}_U^I W}$. Now,

(5.21)
$$\mathcal{L}_U^J w_a=\mathcal{L}_U^J(g_{ab}W^b)=-g_{ab}\hat{\mathcal{L}}_U^J W^b+\tilde{c}^{\,J}_{J_1J_2}g^{J_1}_{ab}\hat{\mathcal{L}}_U^{J_2}W^b,\ \text{ where }\ g^J_{ab}=\check{\mathcal{L}}_U^J g_{ab}$$

and the sum is over all $J_1+J_2=J$ and $\tilde{c}^{\,J}_{J_1J_2}=1$ for $|J_2|<|J|$ $c^{\,J}_{J_1J_2}=0$ if $J_2=J$. It follows that

(5.22)
$$|\operatorname{curl}\underline{\hat{\mathcal{L}}_U^J W}-\operatorname{curl}\mathcal{L}_U^J w|\le 2\tilde{c}^{J}_{J_1J_2}\big(|\partial g^{J_1}||\hat{\mathcal{L}}_U^{J_2}W|+|g^{J_1}||\partial\hat{\mathcal{L}}_U^{J_2}W|\big),\ \ |J_2|<|J|,$$

where the partial derivative can be estimated by Lie derivatives. Furthermore, in Lemma 5.2, we have $|U^I(\kappa\operatorname{div}W)|=\kappa^{-1}|\hat{U}^I\operatorname{div}W|=\kappa^{-1}|\operatorname{div}\hat{\mathcal{L}}_U^I W|$. (5.13) follows by induction from (5.12). □

Definition 5.3. For $\mathcal{V}$ any of the family of vector fields introduced in [L1] let

$$(5.23)\qquad \|W\|_r^{\mathcal{V}}=\sum_{|I|\le r,\,I\in\mathcal{V}}\|\mathcal{L}_U^I W\|,\qquad\qquad \|W\|_{r,\infty}^{\mathcal{V}}=\sum_{|I|\le r,\,I\in\mathcal{V}}\|\mathcal{L}_U^I W\|_\infty$$

and let

$$(5.24)\qquad \|W\|_r=\sum_{|\alpha|\le r}\|\partial_y^\alpha W\|,\qquad\qquad \|W\|_{r,\infty}=\sum_{|\alpha|\le r}\|\partial_y^\alpha W\|_\infty$$

where $\|W\|=\|W\|_{L^2(\Omega)}$, $\|W\|_\infty=\|W\|_{L^\infty(\Omega)}$.

It follows from the discussion after Definition 5.1 and (5.11) that $\|W\|_r$ is equivalent to $\|W\|_r^{\mathcal{R}}$ with a constant of equivalence independent of the metric. As with the pointwise estimates it will sometimes be convenient to use $\|\hat{\mathcal{L}}_U^I W\|=\|\kappa^{-1}\mathcal{L}_U^I(\kappa W)\|$ instead. This is in particular true for the family of space tangential vector fields $\mathcal{S}$. However instead of introducing special notation we write $\|\kappa W\|_r^{\mathcal{S}}$. Since κ is bounded from above and below by a constant K_1 this is equivalent to a constant of equivalence K_1. Furthermore, by interpolation $\|\kappa W\|_r^{\mathcal{S}}\le K_1(\|g\|_r\|W\|+\|W\|_r^{\mathcal{S}})$ and $\|W\|_r^{\mathcal{S}}\le K_1(\|g\|_r\|W\|+\|\kappa W\|_r^{\mathcal{S}})$, and our inequalities anyway contain lower order terms of this form, and so the inequalities below are true either with or without κ.

Lemma 5.3. *With a constant* K_1 *as in Definition* 5.1:

(5.25)
$$\|W\|_r \leq K_1\big(\|\operatorname{curl} w\|_{r-1} + \|\kappa \operatorname{div} W\|_{r-1} + \|\kappa W\|_r^{\mathcal{S}} + K_1 \sum_{s=0}^{r-1} \|g\|_{r-s,\infty}\|W\|_s\big)$$

and, with the convention that $\|\operatorname{curl} w\|_{-1} + \|\operatorname{div} W\|_{-1} = 0$,

$$\|W\|_r \leq K_1 \sum_{s=0}^{r} \|g\|_{r-s,\infty}\big(\|\operatorname{curl} w\|_{s-1} + \|\kappa \operatorname{div} W\|_{s-1} + \|\kappa W\|_s^{\mathcal{S}}\big). \tag{5.26}$$

Proof. This follows from Lemma 5.2 and the interpolation inequalities below in Lemma 6.2. □

We can also bound derivatives of a vector field by the curl, the divergence and the normal operator:

Lemma 5.4. *Let* $c_0 > 0$ *be a constant such that* $|\nabla_N p| \geq c_0 > 0$, *let* K_2 *and* K_3 *be constants such that* $\|\nabla_N p\|_{L^\infty(\partial\Omega)} \leq K_2$ *and* $\sum_{S\in\mathcal{S}} \|\nabla_N S p\|_{L^\infty(\partial\Omega)} \leq K_3$. *Then*

$$c_0\|\partial W\| \leq C\big(\|AW\| + K_2(\|\operatorname{curl} w\| + \|\operatorname{div} W\|) + (K_3 + [g]_1)\|W\|\big). \tag{5.27}$$

Proof. We want to express (5.2) and (5.3) in the Lagrangian frame. We also want to pick an extension of the normal to the interior. If $d(y)$ is the distance to the boundary in the Lagrangian frame, since Ω is the unit ball this is just $1 - |y|$. Let $\chi_1(d)$ be a smooth function that is 1 close to 0, and 0 when $d > 1/2$. If $u_c = \partial_c d$ then $n_c = u_c/\sqrt{g^{ab}u_a u_b}$ is the unit conormal at the boundary and $\tilde{n}_c = \chi_1(d) n_c$ defines an extension to the interior and $\tilde{N}^a = g^{ab}\tilde{n}_b$ is an extension of the unit normal to the interior. Similarly, by the remarks in Section 3, the normal operator only depends on $\nabla_N p$ restricted to the boundary. Let us define $\alpha_b = \chi_2(d) f \partial_b d$, where f is a function that is equal to $N^c\partial_c p = \nabla_N p$ at the boundary and extended to be constant along rays through the origin, and χ_2 is a function that is 1 on the support of χ_1 and 0 when $d > 3/4$. Then $\underline{A}W^a = P(g^{ab}\partial_b((\partial_c p)W^c)) = P(g^{ab}\partial_b(\alpha_c W^c))$ by the remarks in Section 3. Now, we express (5.2) and (5.3) in the Lagrangian coordinates and partial differentiation becomes covariant differentiation. So we will pick up a constant coming from the Christoffel symbols, i.e. one derivative of the metric $[g]_1 = 1 + |\partial g|$. Similarly, one derivative of the normal N^a also gives rise to one derivative of the metric. Hence (5.2) and (5.3) become

$$\|\partial W\| \leq C\big(\|\chi_1(n_c \partial W^c)\| + \|\operatorname{curl} w\| + \|\operatorname{div} W\| + [g]_1\|W\|\big) \tag{5.28}$$

and

(5.29)

$$\|f\chi_2(n_c\partial W^c)\| \le \big(\|AW\| + \|f\,\mathrm{curl}\,w\| + \|f\,\mathrm{div}\,W\| + [g]_1\|fW\| + \|\,|\partial f|W\|\big).$$

Since $|f| \ge c_0$ and $\chi_2 = 1$ in a neighborhood of the support of χ_1, the lemma follows. □

By Lemma 5.4 we have

$$c_0\|\partial\hat{\mathcal{L}}_S^J W\| \le K_3\big(\|\,\mathrm{curl}\,\underline{\hat{\mathcal{L}}_S^J W}\| + \|\,\mathrm{div}\,\hat{\mathcal{L}}_S^J W\| + \|A\hat{\mathcal{L}}_S^J W\| + \|\hat{\mathcal{L}}_S^J W\|\big) \tag{5.30}$$

where K_3 is as in Definition 6.1 and c_0 as in the physical condition (1.6). Here, the curl of $(\underline{\hat{\mathcal{L}}_S^J W})_a = g_{ab}\hat{\mathcal{L}}_S^J W^b$ is by (5.22) equal to the curl of $\mathcal{L}_S^J w$ plus lower order terms. In particular we see that we can get any space tangential derivative in this way. Thus we also get:

Lemma 5.5. *With K_3 as in Definition* 6.1,

(5.31)

$$c_0\|W\|_r \le K_3\big(\|\,\mathrm{curl}\,w\|_{r-1} + \|\,\mathrm{div}\,W\|_{r-1} + \|W\|_{r-1,A}^{\mathcal{S}} + \sum_{s=0}^{r-1}\|g\|_{r-s,\infty}\|W\|_s\big)$$

where

$$\|W\|_{s,A}^{\mathcal{S}} = \sum_{|I|=s,I\in\mathcal{S}}\|A\hat{\mathcal{L}}_S^I W\|. \tag{5.32}$$

6. Interpolation, the L^∞ estimates for the pressure in terms of the coordinate and the L^∞ norms

Let us now first state the interpolation inequalities to be used:

Lemma 6.1. *Let β be a two form, a function or a vector field. Let $\|\beta\|_r$ be L^2-Sobolev norms and $\|\beta\|_{r,\infty}$ be the C^k norms on the unit ball Ω in $\mathbf{R}^n$. Then if $0 \le s \le r$ and $j \ge 0$*

$$\|\beta\|_{j+s,\infty} \le C\|\beta\|_{j,\infty}^{1-s/r}\|\beta\|_{j+r,\infty}^{s/r}, \tag{6.1}$$

$$\|\beta\|_s \le C\|\beta\|_0^{1-s/r}\|\beta\|_r^{s/r}. \tag{6.2}$$

For a proof see e.g. [H1], [H2], for the L^∞ norm and [CL], for the L^2 norms. ((6.1) for $j > 0$ follows from (6.1) for $j = 0$ applied to ∂_y^α for $|\alpha| \le j$.) A consequence is:

Lemma 6.2. *With the same assumptions as in Lemma* 6.1,

$$\|\alpha\|_{j+r-s,\infty}\|\beta\|_{j+s,\infty} \le \big(\|\alpha\|_{j,\infty}\|\beta\|_{j+r,\infty} + \|\beta\|_{j,\infty}\|\alpha\|_{j+r,\infty}\big), \tag{6.3}$$

$$\|\beta\|_{r-s,\infty}\|W\|_s \le C\big(\|\beta\|_{0,\infty}\|W\|_r + \|\beta\|_{r,\infty}\|W\|_0\big), \tag{6.4}$$

$$\|f_1\|_{j+s_1,\infty}\cdots\|f_k\|_{j+s_k,\infty} \leq C\sum_{i=1}^{k}\|f_1\|_{j,\infty}\cdots\|f_{i-1}\|_{j,\infty}\|f_i\|_{j+s_1+\cdots+s_k,\infty}\|f_{i+1}\|_{j,\infty}\cdots\|f_k\|_{j,\infty}. \tag{6.5}$$

Proof. This follows from use of Lemma 6.1 on each factor and the inequality $A^{s/r}B^{1-s/r}\leq A+B$, e.g.

$$\begin{aligned}\|\beta\|_{r-s,\infty}\|W\|_s &\leq C\|\beta\|_{0,\infty}^{s/r}\|\beta\|_{r,\infty}^{1-s/r}\|W\|_0^{1-s/r}\|W\|_r^{s/r}\\ &= C\big(\|\beta\|_{0,\infty}\|W\|_r\big)^{s/r}\big(\|\beta\|_{r,\infty}\|W\|_0\big)^{1-s/r}\\ &\leq C\big(\|\beta\|_{0,\infty}\|W\|_r+\|\beta\|_{r,\infty}\|W\|_0\big).\end{aligned} \tag{6.6}$$

This proves (6.4). The proof of (6.3) is exactly the same, (6.5) follows from (6.3) by induction. □

Let us now introduce some notation to be used in subsequent sections. We will derive tame estimates involving the higher norms of the coordinate x with constants that are bounded if some lower norms of the coordinate x are bounded: Recall Definition 5.2 of c_1:

$$|\partial x/\partial y|^2+|\partial y/\partial x|^2\leq c_1^2, \qquad \sum_{a,b=1}^{n}\big(|g_{ab}|+|g^{ab}|\big)\leq nc_1^2, \tag{6.7}$$

and K_1 denotes a continuous function of c_1.

Definition 6.1. Let c_1 be as in Definition 5.2 and for $i=2,3,4$ let $c_i\geq c_1$ be a constant such that

$$\|x\|_{2,\infty}+\|\dot x\|_{1,\infty}\leq c_2, \tag{6.8}$$

$$\|x\|_{3,\infty}+\|\dot x\|_{2,\infty}+\|\ddot x\|_{1,\infty}\leq c_3, \tag{6.9}$$

$$\|x\|_{4,\infty}+\|\dot x\|_{3,\infty}+\|\ddot x\|_{2,\infty}\leq c_4. \tag{6.10}$$

Let $K_i\geq 1$ denote a constant that depends continuously on c_i.

Now from Lemma 6.2, we have:

Lemma 6.3. *With K_1 as in Definition* 5.1,

$$\|\partial y/\partial x\|_{r,\infty}\leq K_1\|x\|_{r+1,\infty}. \tag{6.11}$$

If $\partial_i=\partial/\partial x^i=(\partial y^a/\partial x^i)\partial/\partial y^a$ and $\alpha=(\alpha_1,\dots,\alpha_n)$ let $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$. For any function f,

$$\|\partial^\alpha f\|_{r,\infty}\leq K_1\big(\|f\|_{r+k,\infty}+\|x\|_{r+k,\infty}\|f\|_{1,\infty}\big), \qquad k=|\alpha|. \tag{6.12}$$

Moreover,

$$\begin{aligned}(6.13)\quad &\|(\partial_{i_1} f_1)\cdots(\partial_{i_n} f_n)\|_{r,\infty}\\ &\quad\le K_1\sum_{i=1}^{n}\|f_1\|_{1,\infty}\cdots\|f_{i-1}\|_{1,\infty}\|f_i\|_{r+1,\infty}\|f_{i+1}\|_{1,\infty}\cdots\|f_n\|_{1,\infty}\\ &\qquad+K_1\|x\|_{r+1,\infty}\|f_1\|_{1,\infty}\cdots\|f_n\|_{1,\infty},\end{aligned}$$

$$\begin{aligned}(6.14)\quad &\|(\partial_{i_0}\partial_{i_1} f_1)(\partial_{i_2} f_2)\cdots(\partial_{i_n} f_n)\|_{r,\infty}\\ &\quad\le K_1\sum_{i=1}^{n}\|f_1\|_{1,\infty}\cdots\|f_{i-1}\|_{1,\infty}\|f_i\|_{r+2,\infty}\|f_{i+1}\|_{1,\infty}\cdots\|f_n\|_{1,\infty}\\ &\qquad+K_1\|x\|_{r+2,\infty}\|f_1\|_{1,\infty}\cdots\|f_n\|_{1,\infty}.\end{aligned}$$

Proof. Let A be the matrix $\partial x^i/\partial y^a$. Using the formula for the derivative of a matrix $\partial_a A^{-1} = -A^{-1}(\partial_a A)A^{-1}$ we see that $\partial_y^\alpha A^{-1}$ is a sum of terms of the form

$$(6.15)\qquad A^{-1}(\partial_y^{\alpha_1}A)A^{-1}\cdots(\partial_y^{\alpha_k}A)A^{-1},\qquad |\alpha_1|+\cdots+|\alpha_k| = |\alpha| = r.$$

Since $|A^{-1}| \le Cc_1$ we see from (6.5) with $j=0$ that this is bounded by $K_1|A|_{r,\infty}$ which proves (6.11). Now $\partial_y^\gamma\partial_x^\alpha f$ is sum of terms of the form

$$(6.16)\qquad \begin{gathered}A^{-1}(\partial_y^{\beta_1}A^{-1})\cdots(\partial_y^{\beta_{k-1}}A^{-1})\partial_y^{\beta_k}(\partial_y f),\\ |\beta_1|+\cdots+|\beta_k| = |\gamma|+|\alpha| = r-1+k.\end{gathered}$$

By (6.5), this is bounded by $K_1\|\partial_y f\|_{r-1+k,\infty}+K_1\|A\|_{r-1+k,\infty}\|\partial_y f\|_{0,\infty}$ which proves (6.12). Also, (6.13) follows from (6.5) with $j=0$, and (6.12) with $k=1$. (Note by (6.12) $\|\partial f\|_{0,\infty}\le K_1\|f\|_{1,\infty}$.) Similarly, by (6.5) with $j=0$ and (6.12) we can bound the left side of (6.14) by

(6.17)

$$\begin{aligned}&\big(\|f_1\|_{r+2,\infty}+\|x\|_{r+2,\infty}\big)\|f_2\|_{1,\infty}\cdots\|f_n\|_{1,\infty}+\sum_{i=2}^{n}(\|f_1\|_{2,\infty}+\|x\|_{2,\infty})\|f_1\|_{1,\infty}\\ &\qquad\cdots\|f_{i-1}\|_{1,\infty}\big(\|f_i\|_{r+1,\infty}+\|x\|_{r+1,\infty}\big)\|f_{i+1}\|_{1,\infty}\cdots\|f_n\|_{1,\infty}.\end{aligned}$$

The first term is of the form on the right side of (6.14). The terms in the sum become a sum of four terms of the form $K_1\|h_1\|_{2,\infty}\|h_2\|_{r+1,\infty}$ multiplied by factors of the form $\|f_k\|_{1,\infty}$. Using (6.5) with $j=1$ we can bound $\|h_1\|_{2,\infty}\|h_2\|_{r+1,\infty}\le C\|h_1\|_{1,\infty}\|h_2\|_{r+2,\infty}+\|h_1\|_{r+1,\infty}\|h_2\|_{1,\infty}$. This also proves (6.14). $\square$

Lemma 6.4. *Let p be the solution of $\triangle p = -(\partial_i V^j)(\partial_j V^i)$, where $v^i = D_t x^i$ and let $\dot p = D_t p$. Then for $r\ge 1$,*

$$(6.18)\qquad \|p\|_{r,\infty}\le K_3(\|\dot x\|_{r,\infty}+\|x\|_{r+1,\infty})$$

$$(6.19)\qquad \|\dot p\|_{r,\infty}\le K_3(\|\ddot x\|_{r,\infty}+\|\dot x\|_{r+1,\infty}+\|x\|_{r+2,\infty}).$$

Proof. We just apply Proposition 7.1 to

$$\triangle p = -(\partial_i V^j)\partial_j V^i, \qquad v^i = D_t x^i, \qquad p\big|_{\partial\Omega} = 0 \tag{6.20}$$

using Lemma 6.3 to estimate the product. (Recall that $\|V\|_{2,\infty} \le K_3$.) Since $D_t = \partial_t + V^k\partial_k$, where $\partial_t = \partial_t\big|_{x=const}$, we have

$$\triangle \dot p = \triangle\big((\partial_t + V^k\partial_k)p\big) = D_t\triangle p + (\triangle V^k)\partial_k p + 2\delta^{ij}(\partial_i V^k)\partial_j\partial_k p \tag{6.21}$$

and

$$D_t \triangle p = -(\partial_t + V^k\partial_k)\big((\partial_i V^j)(\partial_j V^i)\big) = -2(\partial_i V^j)(\partial_j \dot V^i) + 2(\partial_i V^j)(\partial_j V^k)\partial_k V^i \tag{6.22}$$

so that

$$\triangle \dot p = -2(\partial_i V^j)(\partial_j \dot V^i) + 2(\partial_i V^j)(\partial_j V^k)\partial_k V^i + (\triangle V^k)\partial_k p + 2\delta^{ij}(\partial_i V^k)\partial_j\partial_k p. \tag{6.23}$$

The second part of the lemma now follows from Proposition 7.1 using Lemma 6.3 and the first part of Lemma 6.4. □

Let us now introduce the L^∞ norms to be used:

Definition 6.2.

$$m_s(t) = \|x(t,\cdot)\|_{1+s,\infty}, \tag{6.24}$$

$$\dot m_s(t) = \|x(t,\cdot)\|_{2+s,\infty} + \|\dot x(t,\cdot)\|_{1+s,\infty}, \tag{6.25}$$

$$\ddot m_s(t) = \|x(t,\cdot)\|_{3+s,\infty} + \|\dot x(t,\cdot)\|_{2+s,\infty} + \|\ddot x(t,\cdot)\|_{1+s,\infty}, \tag{6.26}$$

$$n_s(t) = \|x(t,\cdot)\|_{4+s,\infty} + \|\dot x(t,\cdot)\|_{3+s,\infty} + \|\ddot x(t,\cdot)\|_{2+s,\infty}. \tag{6.27}$$

We remark that in Definition 5.2 we made an assumption that the inverses of g and $\partial y/\partial x$ are bounded. This means that m_0 etc. are all bounded from below as well. We note that the corresponding bounds for the metrics $g_{ab} = \delta_{ij}(\partial x^i/\partial y^a)(\partial x^j/\partial y^b)$ and $\omega_{ab} = (\operatorname{curl} v)_{ab} = (\partial x^i/\partial y^a)(\partial x^j/\partial y^b)(\partial_i v_j - \partial_j v_i)$ follow from the bounds for x, $\dot x$, and $\ddot x$:

$$\|g\|_{r,\infty} \le K_1 m_r, \;\; \|\dot g\|_{r,\infty} + \|\omega\|_{r,\infty} \le K_2 \dot m_r, \;\; \|\ddot g\|_{r,\infty} + \|\dot\omega\|_{r,\infty} \le K_3 \ddot m_r. \tag{6.28}$$

The proof of this uses the interpolation inequality (6.5) in Lemma 6.2 applied to each term we get when we differentiate. In view of the coordinate condition, see Definition 5.1, the same bounds also hold for g replaced by the inverse of g.

Furthermore, we will now prove that the corresponding bounds for the pressure follow from this. We will actually lose a derivative when passing to the bounds for the pressure because we will go over Hölder spaces, but this does not matter. In Lemma 6.4, we proved that

$$\|p(t,\cdot)\|_{r+1,\infty} \le K_3 \dot m_r(t), \tag{6.29}$$

$$\|p(t,\cdot)\|_{r+2,\infty} + \|\dot p(t,\cdot)\|_{r+1,\infty} \le K_3 \ddot m_r(t), \tag{6.30}$$

$$\|p(t,\cdot)\|_{r+3,\infty} + \|\dot p(t,\cdot)\|_{r+2,\infty} \le K_3 n_r(t). \tag{6.31}$$

In particular

$$\|p\|_{2,\infty} + \|\dot{p}\|_{1,\infty} \le K_3, \qquad \|\dot{p}\|_{2,\infty} \le K_4. \tag{6.32}$$

We will frequently use interpolation, e.g.

$$m_r \dot{m}_s \le C(m_{r+s}\dot{m}_0 + m_0 \dot{m}_{r+s}) \le K_2 \dot{m}_{r+s}, \tag{6.33}$$

which follows from Lemma 6.1 and the proof of Lemma 6.2 applied to each term we get when multiplying any of the expressions (6.24)–(6.27) together.

We must also ensure that if the physical condition (2.7) and coordinate condition (2.8) hold initially they will hold for some small time $0 \le t \le T$, with c_0 replaced by $c_0/2$ and c_1 replaced by $2c_1$. This will be proved in Section 11, and until then we will just assume that T is so small that these conditions hold for $0 \le t \le T$. Furthermore, we will also assume that $T \le c_0 \le 1$ since the estimates derived then will be independent of T and c_0.

7. The L^∞ estimates for the Dirichlet problem

In this section, we give tame Hölder estimates for the solution of the Dirichlet problem:

$$\triangle q = F, \qquad q\big|_{\partial\Omega} = 0. \tag{7.1}$$

Our Hölder estimates lose a derivative since we want to use them for integer values. This is not important and with an additional loss of regularity, we could have avoided using Hölder estimates altogether and just gotten the C^k estimates from the Sobolev estimates, proved in the next section, using Sobolev's lemma. Apart from getting estimates for the solution of (7.1) we also need estimates for time derivatives and variational derivatives. For this we need to know that the solution of (7.1) depends smoothly on parameters if the metric and the inhomogeneous term do. We remark that the coordinate condition is critical since it is needed in order to invert the Laplacian.

One can also use the results in Section 5 to get tame estimates for the solution of the Dirichlet problem. In fact if we take $W^a = g^{ab}\partial_b q$, and $w_a = \partial_q q$, then $\operatorname{div} W = \triangle q$ and $\operatorname{curl} w = 0$. Applying Lemma 5.2 to W therefore gives:

$$|W|_r^{\mathcal{R}} \le K_1 \sum_{s=0}^{r} [g]_s^{\mathcal{R}} \big(|\triangle q|_{r-1-s}^{\mathcal{R}} + |W|_{r-s}^{\mathcal{S}} \big), \tag{7.2}$$

where for $s = r$ we should interpret $|\triangle q|_{-1} = 0$, and

$$|\partial q|_r^{\mathcal{R}} \le K_1 \sum_{s=0}^{r} [g]_s^{\mathcal{R}} \big(|\triangle q|_{r-1-s}^{\mathcal{R}} + |\partial q|_{r-s}^{\mathcal{S}} \big). \tag{7.3}$$

Therefore it suffices to obtain estimates for tangential derivatives only. This is easier because the Dirichlet boundary condition is preserved by tangential derivatives. If $q\big|_{\partial\Omega} = 0$ then $S^I q\big|_{\partial\Omega} = 0$. The L^∞ estimates use the standard Schauder estimates for the Dirichlet problem. Because we want to have the final result in terms of C^k norms instead of Hölder norms these results lose a derivative.

PROPOSITION 7.1. *If $q\big|_{\partial\Omega} = 0$ then for $r \geq 1$*

$$\|q\|_{r,\infty} \leq K_3\big(\|\triangle q\|_{r-1,\infty} + \|g\|_{r,\infty}\|\triangle q\|_{0,\infty}\big). \tag{7.4}$$

Proof. If we apply Lie derivatives $\hat{\mathcal{L}}_S^I$ to $W^a = g^{ab}\partial_b q$ we get

(7.5)
$$W_I^a = g^{ab}\partial_b S^I q + \tilde{c}^I_{I_1 I_2}\hat{g}^{I_1 ab}\,\partial_b S^{I_2} q, \qquad \hat{g}^{I\,ab} = \hat{\mathcal{L}}_S^I g^{ab}, \qquad W_I = \hat{\mathcal{L}}_S^I W$$

and the sum is over all combinations $I = I_1 + I_2$, and the $\tilde{c}^I_{I_1 I_2}$ are constants such that $\tilde{c}^I_{I_1 I_2} = 0$ if $I_2 = I$. Since $\operatorname{div} W_I = \operatorname{div}\hat{\mathcal{L}}_S^I W = \hat{S}^I \operatorname{div} W = \hat{S}^I \triangle q = \kappa^{-1} S^I(\kappa\triangle q)$ it follows from the divergence of (7.5) that

$$\triangle(S^I q) = \hat{S}^I\triangle q - \kappa^{-1}\partial_a\big(\tilde{c}^I_{I_1 I_2}\hat{g}^{I_1 ab}\,\partial_b S^{I_2} q\big), \qquad \hat{g}^{I\,ab} = \hat{\mathcal{L}}_U^I g^{ab}. \tag{7.6}$$

Let $\|u\|_{2+\alpha,\infty}$ denote Hölder norms, see Section 17 and Proposition 7.2. By Proposition 7.2 we have

(7.7)
$$\|S^I q\|_{2+\alpha,\infty} \leq K_1\big(\|\hat{S}^I\triangle q\|_{1,\infty} + \tilde{c}^I_{I_1 I_2}(\|\hat{g}^{I_1}\|_{1,\infty}\|S^{I_2} q\|_{2+\alpha,\infty} + \|\hat{g}^{I_1}\|_{2,\infty}\|S^{I_2} q\|_{1+\alpha,\infty})\big).$$

If we let $M_r = \sum_{|I|\leq r-2}\|S^I q\|_{2+\alpha,\infty}$, $r \geq 2$, $M_r = \|q\|_{r+\alpha,\infty}$ for $r = 0, 1$ it follows from Proposition 7.2 that $M_0 + M_1 \leq K_3\|\triangle q\|$, $M_2 \leq K_3\|\triangle q\|_{1,\infty}$ and for $r \geq 3$ we have:

$$M_r \leq K_3\big(\|\triangle q\|_{r-1,\infty} + \sum_{s=1}^{r-1}\|g\|_{r+1-s,\infty} M_s\big). \tag{7.8}$$

Inductively it follows that

$$\begin{aligned} M_r &\leq K_3\big(\|\triangle q\|_{r-1,\infty} + \sum_{s=0}^{r-2}\|g\|_{r-s,\infty}\|\triangle q\|_{s,\infty}\big) \\ &\leq K_3\big(\|\triangle q\|_{r-1,\infty} + \|g\|_r\|\triangle q\|_{0,\infty}\big) \end{aligned} \tag{7.9}$$

where we used interpolation. With $I \in \mathcal{S}$, $|I| = r - 2$ we hence get from differentiating (7.5) and using what we used proved

$$\|\partial W_I\|_{0,\infty} \leq K_3\big(\|\triangle q\|_{r-1,\infty} + \|g\|_r\|\triangle q\|_{0,\infty}\big). \tag{7.10}$$

However, once we have bounds for the tangential components, the bounds for all components in terms of these and $\hat{R}^I\triangle p$ follow from Lemma 5.2.

Theorem 6.6 in [GT], together with Theorem 8.16, and Theorem 8.33 in [GT], in our setting reads:

PROPOSITION 7.2. *Suppose that* $\|\phi\|_{k+\alpha,\infty}$ *denotes the Hölder norms and* $0<\alpha<1$, *and* k *is an integer (see Section* 17*). Then*

$$\triangle p = g^{ab}\partial_a\partial_b p + \kappa^{-1}(\partial_a(\kappa g^{ab}))\partial_b p = \kappa^{-1}\partial_a\big(\kappa g^{ab}\partial_b p\big) \tag{7.11}$$

where

$$\|g^{ab}\|_{0+\alpha,\infty} + \|\partial g^{ab}\|_{0+\alpha,\infty} \le \Lambda, \qquad \sum_{a,b}|g^{ab}| + |g_{ab}| \le \lambda. \tag{7.12}$$

Suppose that $p\big|_{\partial\Omega}=0$. *Then*

$$\|p\|_{2+\alpha,\infty} \le C\big(\|p\|_\infty + \|\triangle p\|_{0+\alpha,\infty}\big) \tag{7.13}$$

where $C=C(n,\alpha,\lambda,\Lambda)$ *and*

$$\|p\|_\infty \le C\|\triangle p\|_\infty. \tag{7.14}$$

And if $\triangle p = F + \kappa^{-1}\partial_a(\kappa G^a)$, *then*

$$\|p\|_{1+\alpha,\infty} \le C\big(\|p\|_\infty + \|F\|_\infty + \|G\|_{0+\alpha,\infty}\big). \tag{7.15}$$

Furthermore

$$\begin{aligned} \|uv\|_{\alpha,\infty} &\le C\|u\|_{\gamma,\infty}\|v\|_{\alpha,\infty}, \qquad \gamma\ge\alpha,\\ \|uv\|_{\alpha,\infty} &\le C\big(\|u\|_{0,\infty}\|v\|_{\alpha,\infty} + \|v\|_{0,\infty}\|u\|_{\alpha,\infty}\big). \end{aligned} \tag{7.16}$$

Note that if we multiply by κ then the operator is also in the divergence form that [GT] has in Theorem 10.33. Anyway, in our case it is equivalent to a domain in $\mathcal{D}_t$ with the usual metric.

Let us now prove that the solution of (7.1) depends smoothly on parameters if the metric g and the inhomogeneous term F do. Let us assume that the parameter is time t. We have:

LEMMA 7.3. *Let* ϕ *be the solution of*

$$\triangle\phi = \kappa^{-1}\partial_a\big(\kappa g^{ab}\partial_b\phi\big) = F, \qquad \phi\big|_{\partial\Omega}=0, \tag{7.17}$$

where $\kappa=\sqrt{\det g}$, *and* g *satisfies the coordinate condition* (2.8) *on* $[0,T]$. *Suppose that* $g_{ab}, F\in C^k([0,T],C^\infty(\overline{\Omega}))$. *Then* $\phi\in C^k([0,T],C^\infty(\overline{\Omega}))$.

Proof. Let us write ϕ_t, g_t, F_t, and $\triangle_t=\triangle_{g_t}$, to indicate the dependence of t. Our initial assumption is that $g_t, F_t\in C^k([0,T],C^\infty(\overline{\Omega}))$. That

$$\triangle_t\phi_t = F_t, \qquad \phi_t\big|_{\partial\Omega}=0, \tag{7.18}$$

has a solution $\phi_t \in C^\infty(\overline{\Omega})$ if $F_t, g_t \in C^\infty(\overline{\Omega})$ and the coordinate condition is fulfilled is well known. We will prove that $F_t, g_t \in C^1([0,T], C^\infty(\overline{\Omega}))$ implies that $\phi_t \in C^1([0,T], C^\infty(\overline{\Omega}))$. If this is the case, then $\dot{\phi}_t = D_t \phi_t$ satisfies

$$\triangle_t \dot{\phi}_t = \dot{F}_t - \dot{\triangle}_t \phi_t, \qquad \dot{\phi}_t\big|_{\partial\Omega} = 0, \tag{7.19}$$

where $\dot{\triangle}_t = [D_t, \triangle]$ and $\dot{F}_t = D_t F_t$. Since the right-hand side of (7.19) is also in $C^1([0,T], C^\infty(\overline{\Omega}))$ we can repeat the argument to conclude that $\dot{\phi}_t \in C^1([0,T], C^\infty(\overline{\Omega}))$; i.e., $\phi_t \in C^2([0,T], C^\infty(\overline{\Omega}))$. In general we can then use induction to conclude that $\phi_t \in C^k([0,T], C^\infty(\overline{\Omega}))$ since

$$\triangle_t D_t^k \phi_t = D_t^k F - \sum_{j=0}^{k-1} c_l^k \triangle_t^{(k-j)} D_t^j \phi_t, \tag{7.20}$$

where the $\triangle_t^{(k)}$ are the repeated commutators defined inductively by $\triangle_t^{(k)} = [D_t, \triangle_t^{(k-1)}]$, $\triangle_t^{(0)} = \triangle_t$.

First we will show that $F_t, g_t \in C([0,T], C^\infty(\overline{\Omega}))$ implies that $\phi_t \in C([0,T], C^\infty(\overline{\Omega}))$. We will only prove this for $t = 0$ since the proof in general is just a notational difference from the proof for $t = 0$:

$$\triangle_t(\phi_t - \phi_0) = F_t - F_0 - (\triangle_t - \triangle_0)\phi_0. \tag{7.21}$$

Since the $C^m(\overline{\Omega})$ or $H^m(\overline{\Omega})$ norm of the right-hand side tends to 0 as $t \to 0$ for any m and since we have uniform bounds for $\triangle_t^{-1}$, in Lemma 7.3, it follows that the $C^m(\overline{\Omega})$ or $H^m(\overline{\Omega})$ norm of $\phi_t - \phi_0$ tends to 0 as $t \to 0$ for any m. Hence $\phi_t \in C([0,T], C^\infty(\overline{\Omega}))$. Now, let $\dot{\phi}_t$ be defined by (7.19). By the same argument it follows that $\dot{\phi}_t \in C([0,T], C^\infty(\overline{\Omega}))$. It remains to prove that ϕ_t is differentiable. We have

$$\triangle_t\big(\phi_t - \phi_0 - t\dot{\phi}_0\big) = F_t - F_0 - t\dot{F}_0 + \big(\triangle_t - \triangle_0 - t\dot{\triangle}_0\big)\phi_0 + t(\triangle_t - \triangle_0)\dot{\phi}_0. \tag{7.22}$$

Since F_t and g_t are differentiable as functions of t it follows that the $C^m(\overline{\Omega})$ or $H^m(\overline{\Omega})$ norm of the right-hand side divided by t tends to 0 as $t \to 0$ for any m. Since we also have bounds for the inverse of $\triangle_t$ that are uniform in t we conclude that any C^m or H^m norm of $\phi_t - \phi_0 - t\dot{\phi}_0$ divided by t also tends to 0 as $t \to 0$ for any m. It follows that $\phi_t \in C^1([0,T], C^\infty(\overline{\Omega}))$. ☐

8. The L^2 estimates for the Dirichlet problem

In this section, we give tame L^2-Sobolev estimates for the solution of the Dirichlet problem:

$$\triangle q = F, \qquad q\big|_{\partial\Omega} = 0. \tag{8.1}$$

We also remark that the coordinate condition is critical in all the estimates in this section since it is needed in order to invert the Laplacian $\triangle$. As pointed out in the beginning of Section 7, if we use the results from Section 5 it suffices to obtain estimates for tangential derivatives only, which is easier because the Dirichlet boundary condition is preserved by tangential derivatives. If $q\big|_{\partial\Omega} = 0$ then $S^I q\big|_{\partial\Omega} = 0$.

PROPOSITION 8.1. *Suppose that q is a solution of the Dirichlet problem, $q\big|_{\partial\Omega} = 0$, and $W^a = g^{ab}\partial_b q$. Then if $r \geq 0$,*

$$\|W\|_r \leq K_1 \sum_{s=0}^{r-1} \|g\|_{r-1-s,\infty} \|\triangle q\|_s + K_1 \|g\|_{r,\infty} \|W\|, \tag{8.2}$$

and if $r \geq 1$,

$$\|W\|_r + \|q\|_{r+1} \leq K_1 \sum_{s=0}^{r-1} \|g\|_{r-s,\infty} \|\triangle q\|_s. \tag{8.3}$$

Furthermore, for $i \leq 2$ and $r \geq 0$,

$$\|\hat{D}_t^i W\|_r \leq K_3 \sum_{s=0}^{r-1} \sum_{j+k\leq i} \|\check{D}_t^k g\|_{r-s,\infty} \|\hat{D}_t^j \triangle q\|_s + K_3 \sum_{j+k\leq i} \|\check{D}_t^k g\|_{r,\infty} \|\hat{D}_t^j W\| \tag{8.4}$$

and if $r \geq 1$,

$$\|\hat{D}_t^i W\|_r + \|D_t^i q\|_{r+1} \leq K_3 \sum_{s=0}^{r-1} \sum_{j+k\leq i} \|\check{D}_t^k g\|_{r-s,\infty} \|\hat{D}_t^j \triangle q\|_s. \tag{8.5}$$

Moreover if P is the orthogonal projection onto divergence-free vector fields and W is any vector field then, for $r \geq 0$,

$$\|PW\|_r \leq K_1 \sum_{s=0}^{r} \|g\|_{r-s,\infty} \|W\|_s \tag{8.6}$$

and, for $i \leq 2$,

$$\|\hat{D}_t^i PW\|_r \leq K_3 \sum_{s=0}^{r} \sum_{j+k\leq i} \|\check{D}_t^k g\|_{r-s,\infty} \|\hat{D}_t^j W\|_s. \tag{8.7}$$

Before the proof we have a useful lemma:

LEMMA 8.2. *Suppose that $S \in \mathcal{S}$ and $q\big|_{\partial\Omega} = 0$, and*

$$\hat{\mathcal{L}}_S W^a = g^{ab}\partial_b q + F^a. \tag{8.8}$$

Then

$$\|\hat{\mathcal{L}}_S W\| \leq K_1 (\|\operatorname{div} W\| + \|F\|). \tag{8.9}$$

Proof. Let $W_S = \hat{\mathcal{L}}_S W$. Then

$$\int_\Omega g_{ab} W_S^a W_S^b \,\kappa\, dy = \int_\Omega W_S^a \partial_a q \;\kappa dy + \int_\Omega W_S^a g_{ab} F^b \kappa dy. \tag{8.10}$$

If we integrate by parts in the first integral on the right, using the fact that q vanishes on the boundary, we get

$$\int_\Omega W_S^a\, \partial_a q \,\kappa\, dy = -\int_\Omega \operatorname{div}(W_S)\, q\, \kappa\, dy. \tag{8.11}$$

However $\operatorname{div} W_S = \hat{S} \operatorname{div} W$. Then we can integrate by parts in the angular direction: $S = S^a \partial_a$, $\hat{S} = S + \operatorname{div} S$ so we get $\int_\Omega (\hat{S} f)\, \kappa dy = \int_\Omega \partial_a \big(S^a f \kappa\big)\, dy = 0$, where $\partial_a S^a = 0$. Hence,

$$\int_\Omega W_S^a\, \partial_a q\, \kappa\, dy = \int_\Omega \operatorname{div}(W)\,(Sq) \kappa\, dy. \tag{8.12}$$

Here $|Sq| \leq K_1 |\partial q|$ so it follows that

$$\|W_S\|^2 \leq K_1 \|\operatorname{div} W\| \big(\|W_S\| + \|F\|\big) + K_1 \|W_S\| \|F\| \tag{8.13}$$

and so

$$\|W_S\| \leq K_1 (\|\operatorname{div} W\| + \|F\|). \tag{8.14}$$

□

Proof of Proposition 8.1. If we apply $\mathcal{L}_S^I$ to $w_a = g_{ab} W^b$ we get

$$\partial_a S^I q = g_{ab} W_I^b + \tilde{c}^{I_1 I_2} \breve{g}_{I_2\, ab} W_{I_2}^b, \qquad W_I = \hat{\mathcal{L}}_S^I W, \qquad \breve{g}_{I\, ab} = \check{\mathcal{L}}_S^I g_{ab}. \tag{8.15}$$

The sum is over all combinations $I = I_1 + I_2$, $\tilde{c}^{I_1 I_2}$ are constants such that $\tilde{c}_I^{I_1 I_2} = 0$ if $I_2 = I$. If we write $S^I = S S^J$, $W_I = \hat{\mathcal{L}}_S W_J$, and use Lemma 8.2 we get (since $\operatorname{div} W_J = \hat{S}^J \operatorname{div} W = \hat{S}^J \triangle q = \kappa^{-1} S^J (\kappa \triangle q)$)

$$\|W_I\| \leq K_1 \|\hat{S}^J \triangle q\| + K_1 \tilde{c}_I^{I_1 I_2} \|\breve{g}_{I_1}\|_\infty \|W_{I_2}\|; \tag{8.16}$$

or if we sum over all of them and use interpolation,

$$\|\kappa W\|_r^{\mathcal{S}} \leq K_1 \|\kappa \triangle q\|_{r-1} + K_1 \sum_{s=0}^{r-1} \|g\|_{r-s,\infty} \|\kappa W\|_s^{\mathcal{S}}, \qquad r \geq 1. \tag{8.17}$$

We now want to apply Lemma 5.3 to $W^a = g^{ab} \partial_b q$. Then $\operatorname{curl} w = 0$ and $\operatorname{div} W = \triangle q$ so that

(8.18)

$$\|W\|_r \leq K_1 \|\kappa \triangle q\|_{r-1} + K_1 \|\kappa W\|_r^{\mathcal{S}} + K_1 \sum_{s=0}^{r-1} \|g\|_{r-s,\infty} \|W\|_s, \qquad r \geq 1.$$

We now use (8.17) to replace the term $K_1 \|\kappa W\|_r^{\mathcal{S}}$ by the terms of the form already on the right-hand side of (8.18) and we also replace κ by 1 which just produces more terms of the same form. We use induction on r and interpolation (8.2) follows.

We also need an estimate for the lowest order term:

$$\|W\|^2 = \int_\Omega g^{ab}(\partial_a q)(\partial_b q)\,\kappa\,dy = -\int_\Omega (\triangle q)q\,\kappa\,dy.$$

However, there is a constant depending just on the volume of Ω, i.e. $\int_\Omega \kappa\,dy$, such that $\|q\| \le K_1\|\triangle q\|$; see [SY]. Therefore in addition we have

$$\|W\| + \|q\| + \|\partial q\| \le K_1\|\triangle q\|, \qquad \text{if} \quad W^a = g^{ab}\partial_b q. \tag{8.19}$$

Now, we inductively get from using interpolation again:

$$\|\kappa W\|_r \le K_1 \sum_{s=0}^{r-1} \|g\|_{r-s,\infty}\|\triangle q\|_s, \qquad r \ge 1. \tag{8.20}$$

Note that we can remove κ on the left since doing so just produces lower order terms of the same form. This proves the estimate for W in (8.3) and the estimate for q follows from this since $W^a = g^{ab}\partial_b q$. In fact by (8.15) with S^I replaced by any space vector fields R^I, $I \in \mathcal{R}$, $\|\partial q\|_r \le K_1 \sum_{s=0}^r \|g\|_{r-s,\infty}\|W\|_s$ and by (8.19), we also have an estimate for $\|q\|$.

It remains to prove the estimate with time derivatives. We can now repeat the argument with i of the tangential derivatives being the time derivative, $\hat{\mathcal{L}}_{D_t}^i = \hat{D}_t^i$ and $\check{\mathcal{L}}_{D_t}^i = \check{D}_t^i$. This gives

$$\partial_a S^I D_t^i\, q = g_{ab}\hat{D}_t^i W_I^b + \tilde{c}_{Ii}^{I_1 i_1 I_2 i_2}(\check{D}_t^{i_1}\check{g}_{I_2\,ab})\hat{D}_t^{i_2}W_{I_2}^b, \tag{8.21}$$

where $\tilde{c}_I^{I_1 i_1 I_2 i_2} = 1$ for all $(I_1 + I_2, i_1 + i_2) = (I, i)$ such that $|I_2| + i_2 < |I| + i$ and 0 otherwise. By Lemma 8.2 again

$$\|\hat{D}_t^i W_I\| \le C\|\hat{S}^J\hat{D}_t^i\triangle q\| + C\tilde{c}_{Ii}^{I_1 i_1 I_2 i_2}\|\check{D}_t^{i_1}\check{g}_{I_1}\|_\infty\|\hat{D}_t^{i_2}W_{I_2}\| \tag{8.22}$$

where $|J| = |I| - 1$. Hence

$$\begin{aligned}\|D_t^i(\kappa W)\|_r^{\mathcal{S}} \le {} & K_1\|D_t^i(\kappa\triangle q)\|_{r-1} \\ & + K_1 \sum_{s\le r,\, j\le i,\, s+j<r+i} \|D_t^{i-j}(\kappa^{-1}g)\|_{r-s,\infty}\|D_t^j(\kappa W)\|_s^{\mathcal{S}}.\end{aligned} \tag{8.23}$$

By Lemma 8.3 below and (8.23) it follows that for $i \le 2$

$$\begin{aligned}\|D_t^i(\kappa W)\|_r \le K_3\Big(& \|D_t^i(\kappa\triangle q)\|_{r-1} \\ & + \sum_{s\le r,\, j\le i,\, s+j<r+i} \|D_t^{i-j}(\kappa^{-1}g)\|_{r-s,\infty}\|D_t^j(\kappa W)\|_s\Big).\end{aligned} \tag{8.24}$$

(8.4) now follows from this and interpolation.

Since $\hat{D}_t\triangle q = \triangle D_t\, q + \kappa^{-1}\partial_a\big(\kappa(\hat{D}_t g^{ab})\partial_b q\big)$,

$$\begin{aligned}\|\triangle D_t q\| & \le K_2\big(\|\hat{D}_t\triangle q\| + \|\check{D}_t g\|_{0,\infty}\|\partial^2 q\| + \|\check{D}_t g\|_{1,\infty}\|\partial q\|\big) \\ & \le K_3\big(\|\hat{D}_t\triangle q\| + \|\triangle q\|\big),\end{aligned} \tag{8.25}$$

where we used (8.3) with $r = 2$. Therefore using (8.19) applied to $D_t q$ in place of q we also get an estimate for the lowest order norm:

$$\|\hat{D}_t W\| + \|\partial D_t q\| + \|D_t q\| \le K_3\big(\|\hat{D}_t \triangle q\| + \|\triangle q\|\big). \tag{8.26}$$

Using this, (8.4) and interpolation gives (8.5) for a one-time derivative, apart from the estimate for $\|D_t q\|_{r+1}$. By (8.21) with S^I replaced by any space vector fields R^I, $I \in \mathcal{R}$, $\|\partial D_t\, q\|_r \le K_1\big(\sum_{s=0}^r \|g\|_{r-s,\infty}\|\dot{W}\|_s + \sum_{s=0}^r \|\check{D}_t g\|_{r-s,\infty}\|W\|_s\big)$ and by (8.26) we also have an estimate for $\|D_t q\|$.

Since $\hat{D}_t^2 \triangle q = \triangle D_t^2\, q + 2\kappa^{-1}\partial_a\big(\kappa(\hat{D}_t g^{ab})\partial_b D_t q\big) + \kappa^{-1}\partial_a\big(\kappa(\hat{D}_t^2 g^{ab})\partial_b q\big)$,

$$\begin{aligned}\|\triangle D_t^2 q\| &\le K_2\big(\|\hat{D}_t^2\triangle q\| + \|\check{D}_t^2 g\|_{0,\infty}\|\partial^2 q\| + \|\check{D}_t^2 g\|_{1,\infty}\|\partial q\| \\ &\qquad + \|\check{D}_t g\|_{0,\infty}\|\partial^2 D_t\, q\| + \|\check{D}_t g\|_{1,\infty}\|\partial D_t\, q\|\big) \\ &\le K_3\big(\|\hat{D}_t^2\triangle q\| + \|\hat{D}_t\triangle q\| + \|\triangle q\|\big)\end{aligned} \tag{8.27}$$

where we used (8.5) for $i \le 1$. Therefore we also get an estimate for the lowest order norm:

$$\|\hat{D}_t^2 W\| + \|\partial D_t^2 q\| + \|D_t^2 q\| \le K_3\big(\|\hat{D}_t^2\triangle q\| + \|\hat{D}_t\triangle q\| + \|\triangle q\|\big). \tag{8.28}$$

Using this, (8.4) and interpolation gives (8.5) also for a two-time derivative, apart from the estimate for $\|D_t^2 q\|_{r+1}$. By (8.21) with S^I replaced by any space vector fields R^I, $I \in \mathcal{R}$,

$$\|\partial D_t^2\, q\|_r \le K_3 \sum_{s=0}^r \big(\|g\|_{r-s,\infty}\|\ddot{W}\|_s + \|\check{D}_t g\|_{r-s,\infty}\|\dot{W}\|_s + \|\check{D}_t^2 g\|_{r-s,\infty}\|W\|_s\big)$$

and by (8.28) we also have an estimate for $\|D_t^2 q\|$.

It remains to prove the estimates for the projections (8.6), (8.7). We have $W = W_0 + W_1$, where $W_0 = PW$, and $W_1 = g^{ab}\partial_b q$ where $\triangle q = \operatorname{div} W$ and $q\big|_{\partial\Omega} = 0$. Proving (8.6), (8.7) for $r \ge 1$ reduces to proving it for $r = 0$ by using (8.3), (8.5), since $\hat{R}^I \triangle q = \operatorname{div}(\hat{\mathcal{L}}_R^I W)$ and replacing κ by 1 just produces more terms of the same form . (8.6) for $r = 0$ follows since the projection has norm 1, $\|PW\| \le \|W\|$. Since the projection of $g^{ab} D_t^k w_{1b} = g^{ab}\partial_b D_t^k q$ vanishes we obtain from Lemma 8.3 below:

$$\|P\hat{D}_t^i W_1\| \le K_1 \sum_{j=0}^{i-1} \|\check{D}_t^{i-j} g\|\, \|\hat{D}_t^j W_1\|. \tag{8.29}$$

Since also $P\hat{D}_t^i W_0 = \hat{D}_t^i W_0$,

$$(I - P)\hat{D}_t^i W_1 = (I - P) D_t^i W, \tag{8.30}$$

and since the projection has norm one,

$$\|\hat{D}_t^i W_1\| + \|\hat{D}_t^i W_0\| \le K_1\|\hat{D}_t^i W\| + K_1 \sum_{j=0}^{i-1} \|\check{D}_t^{i-j} g\|\, \|\hat{D}_t^j W_1\|. \tag{8.31}$$

Hence for $i = 0, 1, 2$ it inductively follows that

$$\|\hat{D}_t^i W_0\| + \|\hat{D}_t^i W_1\| \le K_3 \sum_{j=0}^{i} \|\hat{D}_t^j W\|, \qquad i \le 2. \tag{8.32}$$

Since as before replacing κ by 1 just produces more terms of the same form this proves (8.7) also for $r = 0$. (6.4), (6.5) follows from interpolation. □

LEMMA 8.3. *Let $W^a = g^{ab} w_b$. Then*

$$\hat{D}_t^i W^a = g^{ab} D_t^i w_b - \sum_{j=0}^{i-1} \binom{i}{j} g^{ab} (\check{D}_t^{i-j} g_{bc}) \hat{D}_t^j W^c. \tag{8.33}$$

Furthermore if $W^a = g^{ab}\partial_b q$ then

$$|D_t^i(\kappa W)|_r^{\mathcal{R}} \le K_1 \Big(|D_t^i(\kappa \operatorname{div} W)|_{r-1}^{\mathcal{R}} + |\kappa W|_r^{\mathcal{S}} + \sum_{s\le r,\, j\le i, s+j<r+i} |D_t^{i-j}(\kappa g)|_{r-s}^{\mathcal{R}} |D_t^j(\kappa W)|_s^{\mathcal{R}} \Big). \tag{8.34}$$

Proof. We have

$$D_t^i w_b = D_t^i\big(\kappa^{-1} g_{bc}\, \kappa W^c\big) = \sum_{j=0}^{i} \binom{i}{j} \big(D_t^{i-j}(\kappa^{-1} g_{bc})\big) \hat{D}_t^j(\kappa W^c),$$

which proves (8.33).

Now, (8.34) follows from (5.12) by (8.33) and the fact that the curl of $w_a = \partial_a$ vanishes which estimates the curl of $g_{ab}\hat{D}_t^i W^b$. □

9. The estimates for the curl

We are studying an equation of the general form

$$\ddot{W} + AW = H, \qquad H = B(W, \dot{W}) + F. \tag{9.1}$$

Here B is a linear combination of multiplication operators. Here A is the normal operator and it projects to the divergence-free vector fields even if W is not divergence-free. We have $\operatorname{curl} AW = 0$ and $\operatorname{div} AW = 0$ so that

$$\operatorname{div} \ddot{W} = \operatorname{div} H, \qquad \operatorname{curl} \ddot{w} = \operatorname{curl} \underline{H} \tag{9.2}$$

where $\ddot{w}_a = g_{ab}\ddot{W}^b$ as defined. Now recall that $\dot{w}_a = g_{ab}\dot{W}^b$; it follows that

$$|D_t \operatorname{curl} \dot{w}| + |D_t \operatorname{curl} w| \le C\big(|\partial D_t g||W| + |\partial g|\partial W| + |\partial \dot{W}| + |\operatorname{curl} \ddot{w}|\big). \tag{9.3}$$

Similarly

$$|\hat{D}_t \operatorname{div} \dot{W}| + |\hat{D}_t \operatorname{div} W| \le C\big(|\operatorname{div} \dot{W}| + |\operatorname{div} \ddot{W}|\big). \tag{9.4}$$

Hence

$$|D_t \operatorname{curl} \dot w| + |D_t \operatorname{curl} w| + |\hat D_t \operatorname{div} \dot W| + |\hat D_t \operatorname{div} W| + |\operatorname{curl} \ddot w| + |\operatorname{div} \ddot W| \le C\big(|\partial \dot W| + |\partial W| + |\partial g||\dot W| + |\partial g||W| + |\operatorname{div} H| + |\operatorname{curl} \underline{H}|\big). \tag{9.5}$$

Since B is of order one and in fact is a multiplication operator it follows that the terms in $\operatorname{curl} B(W, \dot W)$ and $\operatorname{div} B(W, \dot W)$ are also going to be of the form on the right-hand side of (9.5). However, we need to take a closer look at what the operator B really is because on the one hand it will give an improved estimate and on the other hand we want to find out exactly what the constants above depend on:

LEMMA 9.1. *Suppose that $L_1 W = F$, where $L_1 W = \ddot W + AW - B(W, \dot W)$ is as given by Lemma 2.4, $\dot W^a = \hat D_t W^a$ and $\ddot W^a = \hat D_t^2 W^a$. Let $w_a = g_{ab} W^b$, $\dot w_a = g_{ab} \dot W^b$, $\ddot w_a = g_{ab} \ddot W^b$. Then*

$$D_t \operatorname{curl} w_{ab} = \operatorname{curl} \dot w_{ab} + \partial_a\big(\dot g_{bc} W^c\big) - \partial_b\big(\dot g_{ac} W^c\big) \tag{9.6}$$

$$D_t \operatorname{curl} \dot w_{ab} = \operatorname{curl} \ddot w_{ab} + \partial_a\big(\dot g_{bc} \dot W^c\big) - \partial_b\big(\dot g_{ac} \dot W^c\big) \tag{9.7}$$

$$\operatorname{curl} \ddot w_{ab} = \operatorname{curl} \underline{F}_{ab} + \operatorname{curl} \underline{B}(\dot W, W)_{ab} \tag{9.8}$$

where $\dot g_{ab} = \check D_t g_{ab} = D_t\, g_{ab} - \dot\sigma g_{ab}$ and

$$\underline{B}_a(W, \dot W) = -\big(\dot g_{ac} - \omega_{ac} - \dot\sigma g_{ac}\big)\dot W^c + \dot\sigma\big(\dot g_{ac} - \omega_{ac}\big) W^c - \partial_a q_0 \tag{9.9}$$

and $L_1 W = F$. On the other hand, if $\tilde w_a = \dot w_a - (\dot\sigma g_{ab} + \omega_{ab}) W^b$ and L_1 is given by (2.54) *then*

$$D_t \operatorname{curl} w_{ab} = \operatorname{curl} \tilde w_{ab} + \partial_a\big((\dot g_{bc} + \omega_{bc} + \dot\sigma g_{bc}) W^c\big) - \partial_b\big((\dot g_{ac} + \omega_{ac} + \dot\sigma g_{ac}) W^c\big), \tag{9.10}$$

$$D_t \operatorname{curl} \tilde w_{ab} = -\partial_a\big((D_t\, \omega_{bc} + \ddot\sigma g_{bc}) W^c\big) + \partial_b\big((D_t\, \omega_{ac} + \ddot\sigma g_{ac}) W^c\big) + \operatorname{curl} \underline{F}_{ab}, \tag{9.11}$$

$$\operatorname{curl} \dot w_{ab} = \operatorname{curl} \tilde w_{ab} + \partial_a\big((\dot\sigma g_{bc} + \omega_{bc}) W^c\big) - \partial_b\big((\dot\sigma g_{ac} + \omega_{ac}) W^c\big). \tag{9.12}$$

Proof. The proof uses Lemma 2.5 and the identity $D_t\, w_a = D_t(g_{ab} W^b) = \dot g_{ab} W^b + g_{ab} \dot W^b$ and (2.54). □

Now we want to commute with Lie derivatives $\mathcal{L}_R^I$, since the Lie derivative commutes with the curl: $\mathcal{L}_R \operatorname{curl} w = \operatorname{curl} \mathcal{L}_R w$.

By Lemma 5.2 the next follows from Lemma 9.1 and Lemma 6.2:

Lemma 9.2. *With notation as in Lemma* 9.1 *and Definition* 6.1 *we have*

$$\|D_t \operatorname{curl} w\|_{r-1} + \|D_t \operatorname{curl} \dot{w}\|_{r-1} + \|\operatorname{curl} \ddot{w}\|_{r-1} \leq 2\|\operatorname{curl} \underline{F}\|_{r-1} + K_2 \sum_{s=0}^{r} \big(\|x\|_{r+1-s,\infty} + \|\dot{x}\|_{r+1-s,\infty}\big)\big(\|W\|_s + \|\dot{W}\|_s\big). \tag{9.13}$$

Also,

$$\begin{aligned}\|D_t \operatorname{curl} w\|_{r-1} &+ \|D_t \operatorname{curl} \tilde{w}\|_{r-1} \\ &\leq \|\operatorname{curl} \tilde{w}\|_{r-1} + \|\operatorname{curl} \underline{F}\|_{r-1} \\ &\quad + K_3 \sum_{s=0}^{r} \big(\|x\|_{r+1-s,\infty} + \|\dot{x}\|_{r+1-s,\infty} + \|\ddot{x}\|_{r+1-s,\infty}\big)\|W\|_s\end{aligned} \tag{9.14}$$

and

$$\big|\, \|\operatorname{curl} \dot{w}\|_{r-1} - \|\operatorname{curl} \tilde{w}\|_{r-1} \,\big| \leq K_2 \sum_{s=0}^{r} \big(\|x\|_{r+1-s,\infty} + \|\dot{x}\|_{r+1-s,\infty}\big)\|W\|_s. \tag{9.15}$$

Remark. The difference between (9.13) on the one hand and (9.14), (9.15) on the other hand is that the latter does not require estimates for $\|\dot{W}\|_s$ but instead requires an extra time derivative of the coordinate. However, we do control two time derivatives of the coordinates.

10. Existence for the inverse of the modified linearized operator in the divergence-free class

We first want to show that

$$L_1 W = \ddot{W} + AW - B_0 W - B_1 \dot{W} = F, \qquad W\big|_{t=0} = \tilde{W}_0, \quad \dot{W}\big|_{t=0} = \tilde{W}_1, \tag{10.1}$$

has a smooth solution W:

Theorem 10.1. *Suppose that x and p are smooth, $p\big|_{\partial\Omega} = 0$ and that the coordinate and physical condition* (2.8) *and* (2.7) *hold for $0 \leq t \leq T$. Let L_1 be defined by* (2.49) *and suppose that $\tilde{W}_0$, $\tilde{W}_1$ and F are smooth and divergence-free. Then* (10.1) *has a smooth solution for $0 \leq t \leq T$.*

In case, $\operatorname{div} V = 0$ and $\operatorname{div} F = 0$, existence for (10.1) was proved in [L1]. We now want to generalize this result to prove existence when $\operatorname{div} V \neq 0$ and $\operatorname{div} F = 0$. This is just a minor modification of the proof in [L1], with mostly notational differences, multiplying with $\kappa = \det(\partial x/\partial y)$ and κ^{-1} since $\operatorname{div} W = \kappa \partial_a(\kappa W^a)$. We will just give an outline of the proof.

First we note that we can reduce to the case with vanishing initial data and F vanishing to all orders as $t \to 0$. In fact, we can get all higher time derivatives by differentiating the equation with an inhomogeneous term

$$\hat{D}_t^{k+2} W = B_k\big(W, ., \hat{D}_t^{k+1} W, \partial W, \dots, \partial \hat{D}_t^k W\big) + \hat{D}_t^k F, \tag{10.2}$$

where B_k are some linear functions followed by projections; see (10.8) with I consisting of just time derivatives. Let us therefore define functions of space only by

(10.3)
$$\tilde{W}^{k+2} = B_k\big(\tilde{W}^0, \dots, \tilde{W}^{k+1}, \partial \tilde{W}^0, \dots, \partial \tilde{W}^k\big)\big|_{t=0} + \hat{D}_t^k F\big|_{t=0}, \qquad k \geq 0.$$

Then

$$\tilde{W}(t, y) = \frac{\kappa(0, y)}{\kappa(t, y)} \sum_{k=0}^{m-1} \tilde{W}^k(y) t^k / k! \tag{10.4}$$

defines a formal power series solution at $t = 0$. What we are doing is just expanding κW in a formal power series in t, since $D_t(\kappa W) = \kappa \hat{D}_t W$. Since $\operatorname{div} \tilde{W}^k = 0$ it follows that $\operatorname{div} \tilde{W} = 0$. We also note that if the initial data are smooth then we can construct a smooth approximate solution $\tilde{W}$ that satisfies the equation to all orders as $t \to 0$. This is obtained by multiplying the k^{th} term in (10.4) by a smooth cutoff $\chi(t/\varepsilon_k)$, to be chosen below, and by summing up the infinite series. Here χ is smooth, $\chi(s) = 1$, for $|s| \leq 1/2$ and $\chi(s) = 0$ for $|s| \geq 1$. The sequence $\varepsilon_k > 0$ can then be chosen small enough so that the series converges in $C^n([0, T], H^m)$ for any n and m if we take $(\|\tilde{W}^k\|_k + 1)\varepsilon_k \leq 1/2$. By replacing W by $W - \tilde{W}$ and F by $F - L_1 \tilde{W}$ in (10.1) we reduce to the situation with vanishing initial data and an inhomogeneous term F vanishing to all orders as $t \to 0$.

We will therefore assume that initial data in (10.1) vanish and that F is smooth, divergence-free and vanishes to all order as $t \to 0$. Then existence of a solution W_ε for the equation where we replace the normal operator A by the smoothed out normal operator A^ε, $\varepsilon > 0$, in (10.1)

$$L_1^\varepsilon W_\varepsilon = \ddot{W}_\varepsilon + A^\varepsilon W_\varepsilon - B_0 W_\varepsilon - B_1 \dot{W}_\varepsilon = F \tag{10.5}$$

follows since all the operators are bounded on $H^r(\Omega)$; see (3.15). Thus it is just an ordinary differential equation in $H^r(\Omega)$, for any $r > 0$. Additional regularity in time follows by application of time derivatives. This was proved in [L1]. Lowering the indices in (10.5), we obtain

$$\underline{G} \ddot{W}_\varepsilon + \underline{A}^\varepsilon W_\varepsilon - \underline{B}_0 W_\varepsilon - \underline{B}_1 \dot{W}_\varepsilon = \underline{G} F. \tag{10.6}$$

Let $\hat{\mathcal{L}}_T^I$, $I \in \mathcal{T}$, stand for a product of modified Lie derivatives, see Section 4, of $|I|$ vector fields in $\mathcal{T}$ and let $W_{\varepsilon I} = \hat{\mathcal{L}}_T^I W_\varepsilon$. If we repeatedly apply Lie derivatives $\mathcal{L}_T$ and the projection, see Section 4,

$$c_I^{I_1 I_2}\big(\underline{G}_{I_1} \ddot{W}_{\varepsilon I_2} + \underline{A}_{I_1}^\varepsilon W_{\varepsilon I_2} - \underline{B}_{0 I_1} W_{\varepsilon I_2} - \underline{B}_{1 I_1} \dot{W}_{\varepsilon I_2} - \underline{G}_{I_1} F_{I_2}\big) = 0 \tag{10.7}$$

where the sum is over all combinations of $I_1+I_2=I$ and $c_I^{I_1I_2}=1$. If we raise the indices again we get

$$\ddot{W}_{\varepsilon I}+A^\varepsilon W_{\varepsilon I}=-\tilde{c}_I^{I_1I_2}\big(G_{I_1}\ddot{W}_{\varepsilon I_2}+A^\varepsilon_{I_1}W_{I_2}\big) +c_I^{I_1I_2}\big(B_{0I_1}W_{\varepsilon I_2}+B_{1I_1}\dot{W}_{\varepsilon I_2}+G_{I_1}F_{I_2}\big) \tag{10.8}$$

where $\tilde{c}_I^{I_1I_2}=1$, if $|I_2|<|I|$, and $\tilde{c}_I^{I_1I_2}=0$ if $|I_2|=|I|$.

Let us define energies

$$E_I=E(W_{\varepsilon I})=\langle\dot{W}_{\varepsilon I},\dot{W}_{\varepsilon I}\rangle+\langle W_{\varepsilon I},(A^\varepsilon+I)W_{\varepsilon I}\rangle,\quad E_s^{\mathcal{T}}=\sum_{|I|\le s, I\in\mathcal{T}}\sqrt{E_I}. \tag{10.9}$$

Note that in the sum we also included all time derivatives $\hat{\mathcal{L}}_{D_t}$. The reason for this is that when calculating commutators second order time derivatives show up in the first term on the right in (10.7). As for (3.38), by differentiating (10.9) we get

$$\dot{E}_I=2\langle\dot{W}_{\varepsilon I},\ddot{W}_{\varepsilon I}+(A^\varepsilon+I)W_{\varepsilon I}\rangle+\langle\dot{W}_{\varepsilon I},\dot{G}\dot{W}_{\varepsilon I}\rangle+\langle W_{\varepsilon I},(\dot{A}^\varepsilon+\dot{G})W_{\varepsilon I}\rangle. \tag{10.10}$$

Now, $\dot{G}$ is a bounded operator by (3.17). The last term can be bounded by $\langle W_I,(A^\varepsilon+I)W_{\varepsilon I}\rangle$ using (4.43) which also holds for A^ε by (4.37), and (3.13). Therefore, the last two terms are bounded by $E_r^{\mathcal{T}}$, where $r=|I|$. Using (10.8) to estimate the first term we see that the L^2 norm of the last term on the right of (10.8) is bounded by a constant times $E_r^{\mathcal{T}}$ plus $\|F\|_r^{\mathcal{T}}$ which $=\sum_{|I|\le r,\, I\in\mathcal{T}}\|\hat{\mathcal{L}}_T^I F\|$, and $\|F\|=\langle F,F\rangle^{1/2}$. The same is true with the first part of the first term on the right in (10.8) since $|I_2|<|I|$ there and since we have included all time derivatives in the definition of $E_s^{\mathcal{T}}$. It only remains to deal with the second part of the first term on the right of (10.8). This term comes from the commutators of $\hat{\mathcal{L}}_T^I$ and A^ε and we add an additional term to the energy in order to pick it up. Let

$$D_I=2\tilde{c}_I^{I_1I_2}\langle W_{\varepsilon I},A^\varepsilon_{I_1}W_{\varepsilon I_2}\rangle \tag{10.11}$$

where the sum is over all $I_1+I_2=I$, $|I_2|<|I|$ and $\tilde{c}_I^{I_1I_2}=1$. This term is lower order and is again bounded by using (3.13) on the energies $CE_r^{\mathcal{T}}E_{r-1}^{\mathcal{T}}$. Furthermore,

$$\dot{D}_I=2\tilde{c}_I^{I_1I_2}\langle\dot{W}_{\varepsilon I},A^\varepsilon_{I_1}W_{\varepsilon I_2}\rangle+\langle W_{\varepsilon I},\dot{A}^\varepsilon_{I_1}W_{\varepsilon I_2}\rangle+\langle W_{\varepsilon I},A^\varepsilon_{I_1}\dot{W}_{\varepsilon I_2}\rangle \tag{10.12}$$

where, by (3.13) the second to last term is bounded by $CE_r^{\mathcal{T}}E_{r-1}^{\mathcal{T}}$ and the last term is bounded by $CE_r^{\mathcal{T}}E_r^{\mathcal{T}}$, since we have included all time derivatives in the definition (10.9). Hence, we have proved that

$$|\dot{E}_I+\dot{D}_I|\le CE_r^{\mathcal{T}}\big(E_r^{\mathcal{T}}+\|F\|_r^{\mathcal{T}}\big),\ \ |D_I|\le CE_r^{\mathcal{T}}E_{r-1}^{\mathcal{T}},\ \ r\ge 0,\quad E_{-1}^{\mathcal{T}}=0. \tag{10.13}$$

Using induction and a Grönwall type of argument, see [L1], we have

LEMMA 10.2. *There is a constant C depending only on t and on $x(t,y)$ but not on ε such that*

$$E_r^{\mathcal{T}} \leq C \int_0^t \|F\|_r^{\mathcal{T}}\, d\tau. \tag{10.14}$$

In fact, integrating the first inequality in (10.13) from 0 to t, using the fact that $E_I(0) = D_I(0) = 0$, summing over $|I| \leq r$, and using the second inequality we have $(E_r^{\mathcal{T}})^2 \leq C E_r^{\mathcal{T}} E_{r-1}^{\mathcal{T}} + C \int_0^t E_r^{\mathcal{T}} (E_r^{\mathcal{T}} + \|F\|_r^{\mathcal{T}})\, d\tau$. Hence

$$\overline{E}_r \leq C\overline{E}_{r-1} + C \int_0^t (\overline{E}_r + \|F\|_r^{\mathcal{T}})\, d\tau, \quad \text{where} \quad \overline{E}_r(t) = \sup_{0 \leq \tau \leq t} E_r(\tau). \tag{10.15}$$

Introduction of $M_r = \int_0^t \overline{E}_r\, d\tau$ gives $\dot{M}_r - CM_r \leq C\overline{E}_{r-1} + C\int_0^t \|F\|_r^{\mathcal{T}}\, d\tau$. Multiplying by the integrating factor e^{-Ct}, we see that M_r is bounded by some constant, depending on t times $C\overline{E}_{r-1} + C\int_0^t \|F\|_r^{\mathcal{T}}\, d\tau$. Hence for some other constant, $\overline{E}_r \leq C\overline{E}_{r-1} + C\int_0^t \|F\|_r^{\mathcal{T}}\, d\tau$ and (10.14) follows by induction.

From the uniform energy bounds in Lemma 10.2 it follows that $\|W_\varepsilon\| \leq C$, where C is independent of ε so that we can choose a weakly convergent subsequence W_{ε_n} that converges weakly in the inner product to W which is also in that space. Let U be a smooth divergence-free vector field where $0 < t < T$ in the support. Then

$$\int_0^T \int_\Omega g_{ab}(L_1^\varepsilon W_\varepsilon^a) U^b\, \kappa dy\, d\tau = \int_0^T \int_\Omega g_{ab} W_\varepsilon^a (L_1^{\varepsilon *} U^b)\, \kappa dy\, d\tau \tag{10.16}$$

where $L_1^{\varepsilon *}$ is the space time adjoint. The only term that depends on ε in $L_1^{\varepsilon *}$ is A^ε, since A^ε is self-adjoint. Since the projection is continuous it also follows that $A^\varepsilon U \to AU$, as $\varepsilon \to 0$, strongly in L^2 if $U \in H^1$. Then the right-hand side of (10.16) converges so we get

$$\int_0^T \int_\Omega g_{ab} W^a (L_1^* U^b)\, \kappa dy\, d\tau = \int_0^T \int_\Omega g_{ab} F^a U^b\, \kappa dy\, d\tau \tag{10.17}$$

where now W is the weak limit. Hence W is a weak solution of the equation. Furthermore W_ε is divergence-free so it follows that W is weakly divergence-free; i.e.,

$$\int_0^T \int_\Omega W^a (\partial_a q)\, \kappa\, dy\, d\tau = 0 \tag{10.18}$$

for all smooth functions q that vanish on the boundary. We now conclude that W has additional regularity so that we can integrate by parts and conclude that W is a regular solution.

Note that since the curl of a gradient vanishes

$$\operatorname{curl} \underline{A}^\varepsilon W_\varepsilon = 0, \qquad \text{when} \quad d(y) \geq \varepsilon. \tag{10.19}$$

It follows that the formulas in Lemma 9.1 hold true for $d(y) \geq \varepsilon$ and we have

Lemma 10.3. *When* $d(y) \geq \varepsilon$,

$$|D_t \operatorname{curl} \dot{w}_\varepsilon|_{r-1}^{\mathcal{U}} \leq C\big(|W_\varepsilon|_r^{\mathcal{U}} + |\dot{W}_\varepsilon|_r^{\mathcal{U}}\big) + |\operatorname{curl} \underline{F}|_{r-1}^{\mathcal{U}}, \tag{10.20}$$

$$|D_t \operatorname{curl} w_\varepsilon|_{r-1}^{\mathcal{U}} \leq C\big(|W_\varepsilon|_r^{\mathcal{U}} + |\dot{W}_\varepsilon|_r^{\mathcal{U}}\big). \tag{10.21}$$

By Lemma 5.2 (see the last statement):

Lemma 10.4.

$$|W|_r^{\mathcal{U}} \leq C\big(|\operatorname{curl} w|_{r-1}^{\mathcal{U}} + |\operatorname{div} W|_{r-1}^{\mathcal{U}} + |W|_r^{\mathcal{T}}\big), \tag{10.22}$$

$$|\dot{W}|_r^{\mathcal{U}} \leq C\big(|\operatorname{curl} \dot{w}|_{r-1}^{\mathcal{U}} + |\operatorname{div} \dot{W}|_{r-1}^{\mathcal{U}} + |\dot{W}|_r^{\mathcal{T}}\big). \tag{10.23}$$

Let $C_0^{\mathcal{U},\varepsilon} = 0$ and for $r \geq 1$ let

$$C_r^{\mathcal{U},\varepsilon} = \|\operatorname{curl} \dot{w}_\varepsilon\|_{\mathcal{U}^{r-1}(\Omega_\varepsilon)} + \|\operatorname{curl} w_\varepsilon\|_{\mathcal{U}^{r-1}(\Omega_\varepsilon)}, \tag{10.24}$$

where

$$\|\beta\|_{\mathcal{U}^r(\Omega_\varepsilon)} = \sqrt{\int_{\Omega_\varepsilon} (|\beta|_r^{\mathcal{U}})^2 \, \kappa dy}$$

and $\Omega_\varepsilon = \{y \in \Omega; d(y) > \varepsilon\}$. Since $\operatorname{div} W = \operatorname{div} \dot{W} = 0$ and since $d(y) \geq \varepsilon$ in the domain of integration in (10.24) it therefore follows from Lemma 10.3 and Lemma 10.4 that

$$|\dot{C}_r^{\mathcal{U},\varepsilon}| \leq C(C_r^{\mathcal{U},\varepsilon} + E_r^{\mathcal{T}}) + C\|\underline{F}\|_r^{\mathcal{U}} \tag{10.25}$$

where C depends on t and $x(t,y)$ but is independent of ε. This together with Lemma 10.2 and Lemma 10.4 now gives us uniform bounds:

Lemma 10.5.

$$\|\dot{W}_\varepsilon\|_{\mathcal{U}^r(\Omega_\varepsilon)} + \|W_\varepsilon\|_{\mathcal{U}^r(\Omega_\varepsilon)} + E_r^{\mathcal{T}} \leq C \int_0^t \|F\|_r^{\mathcal{U}} \, d\tau. \tag{10.26}$$

We can hence pass to the limit and conclude that the limit W also satisfies the same estimate and therefore if we integrate by parts in (10.17) and (10.18) conclude that W in fact is a smooth solution:

Proposition 10.6. *Suppose* $x(t,y)$ *is smooth and* (2.7) *and* (2.8) *hold for* $0 \leq t \leq T$. *Suppose also that* F *is smooth for* $0 \leq t \leq T$, $\operatorname{div} F = 0$ *and* F *vanishes to all orders as* $t \to 0$. *Then the modified linearized equation* (10.1) *with vanishing initial data*, $\tilde{W}_0 = \tilde{W}_1 = 0$, *has a smooth solution* W *for* $0 \leq t \leq T$, *satisfying* $\operatorname{div} W = 0$. *Furthermore*, *the solution satisfies the estimate*:

$$\|\dot{W}\|_r^{\mathcal{U}} + \|W\|_r^{\mathcal{U}} + E_r^{\mathcal{T}} \leq C_r \int_0^t \|F\|_r^{\mathcal{U}} \, d\tau, \qquad r \geq 0. \tag{10.27}$$

11. Estimates for the inverse of the modified linearized operator in the divergence-free class

We will now give improved estimates for the modified linearized equation

$$L_1 W = \ddot{W} + AW - B_0 W - B_1 \dot{W} = F, \tag{11.1}$$

within the divergence-free class.

THEOREM 11.1. *Suppose that x and p are smooth, $p\big|_{\partial\Omega} = 0$ and that the coordinate and physical conditions* (2.8) *and* (2.7) *hold for $0 \le t \le T$. Let L_1 be defined by* (2.49) *and suppose that W and F are smooth and divergence-free satisfying* (11.1) *for $0 \le t \le T$. Then if $W\big|_{t=0} = \dot{W}\big|_{t=0} = 0$,*

$$\|\dot{W}\|_r + \|W\|_r \le K_3 e^{K_3(1+c_0^{-1})T} \sum_{s=0}^{r} \underline{n}_{r-1-s} \int_0^t \|F\|_s \, d\tau, \tag{11.2}$$

$$\|\ddot{W}\|_{r-1} \le K_3 e^{K_3(1+c_0^{-1})T} \Big(\sum_{s=0}^{r} \underline{n}_{r-1-s} \int_0^t \|F\|_s \, d\tau + \sum_{s=0}^{r} \underline{n}_{r-1-s} \|F\|_s \Big) \tag{11.3}$$

for $0 \le t \le T$. If in addition $F\big|_{t=0} = 0$ then for $r \ge 1$ and $0 \le t \le T$,

$$\begin{aligned} &\|\ddot{W}\|_{r-1} + \|\dot{W}\|_{r-1} + \|W\|_{r-1} + c_0 \|W\|_r \\ &\qquad \le K_4 e^{K_4(1+c_0^{-1})T} \sum_{s=0}^{r-1} \underline{n}_{r-1-s} \int_0^t \big(\|\dot{F}\|_s + \|F\|_s + \|\operatorname{curl} F\|_s \big) \, d\tau. \end{aligned} \tag{11.4}$$

Here $c_0 > 0$ is the constant in (2.7), *where*

$$\underline{n}_s = \sup_{0 \le t \le T} n_s(t), \tag{11.5}$$

$$n_s(t) = \|x(t,\cdot)\|_{4+s,\infty} + \|\dot{x}(t,\cdot)\|_{3+s,\infty} + \|\ddot{x}(t,\cdot)\|_{2+s,\infty} \tag{11.6}$$

and K_3 is a constant, which depends on $\underline{n}_{-1} + c_1$, where c_1 is the constant in (2.8).

For $r = 0$, (11.2) is the basic energy estimate from Section 3. For $r \ge 1$, (11.2) follows from first applying Lie derivatives with respect to space tangential vector fields to the equation and estimating the energy for these as well as using the evolution equation for the curl and the fact that we can estimate any derivative by the curl, the divergence and tangential derivatives. The difference between (11.2) and (10.27) is, apart from the fact that we keep track of how the constants depend on the solution we linearize around, that we only have space derivatives in the norms in (11.2). The commutators in the energy estimate are now estimated using the curl as well as the energies of tangential derivatives. (11.3) follows from (11.2) by use of (11.1) to estimate $\ddot{W}$. (11.2) and (11.3) follow from Proposition 11.4 below.

The importance of (11.4) is that one gets control of an additional space derivative $c_0\|W\|_r$ by only controlling an additional time derivative and the curl of the right-hand side. (11.4) without the term $c_0\|W\|_r$ on the left and $\|\operatorname{curl} F\|_s$ on the right, in principle follows from (11.2) applied to the equation one gets for $\dot W$ by commuting $\hat D_t$ through L_1 in (11.1). The commutator term $\dot A W$ can in principle be controlled by the energy of the same order but in order not to get constants depending on $\ddot A$ we will bound it using an additional space derivative. $c_0\|W\|_r$ can be controlled as follows. Using the estimate without this term in (11.1) gives control of $\|AW\|_{r-1}$. By Lemma 5.4 this gives us control of $c_0\|W\|_r$ if we also control $\|\operatorname{curl} w\|_{r-1}$. We then use the fact that there is an improved evolution equation for the curl which only requires control of $\|W\|_r$ and not $\|\dot W\|_r$, by Lemma 9.2. (11.4) follows from Proposition 11.9.

Let us rewrite (11.1) slightly:

$$\ddot W + AW = H, \qquad \text{where} \qquad H = B_0 W + B_1 \dot W + F \tag{11.7}$$

and by (2.51), (2.52) the operators By_1 and B_0 are on divergence-free vector fields

$$\begin{aligned} B_1 \dot W^a &= -P\big(g^{ab}\big(D_t\, g_{bc} - \omega_{bc} - 2\dot\sigma g_{bc}\big)\dot W^c\big),\\ B_0 W^a &= P\big(g^{ab}\dot\sigma\big(D_t\, g_{bc} - \omega_{bc} - \dot\sigma g_{bc}\big)W^c\big). \end{aligned} \tag{11.8}$$

It follows from (4.47) and (4.49) that

$$\begin{gathered} \ddot W_I + AW_I = H_I + K_I, \qquad K_I = \tilde G_I^{I_1 I_2} A_{I_1} W_{I_2},\\ H_I = G_I^{I_1 I_2}\big(B_{0I_1} W_{I_2} + B_{1I_1}\dot W_{I_2}\big) + F_I \end{gathered} \tag{11.9}$$

where $W_I = \mathcal{L}_S^I W$, $F_I = \mathcal{L}_S^I F$, and A_I and B_{iI} are given by (4.41) and (4.43). Let

$$E_I = E(W_I) = \langle \dot W_I, \dot W_I\rangle + \langle W_I, (A+I)W_I\rangle. \tag{11.10}$$

Then

$$\begin{aligned} \dot E_I &= 2\langle \dot W_I, \ddot W_I + (A+I)W_I\rangle + \langle \dot W_I, \dot G \dot W_I\rangle + \langle W_I, (\dot A + \dot G)W_I\rangle\\ &= 2\langle \dot W_I, K_I + H_I\rangle + \langle \dot W_I, W_I\rangle + \langle \dot W_I, \dot G\dot W_I\rangle + \langle W_I, (\dot A + \dot G) W_I\rangle. \end{aligned} \tag{11.11}$$

The last three terms can be estimated by E_I, by (3.42), and so we get

$$|\dot E_I| \le 2\sqrt{E_I}\,\|K_I + H_I\| + K_3(1 + c_0^{-1})E_I. \tag{11.12}$$

However, in order to estimate the first term we must estimate $\|K_I\| + \|H_I\|$:

Lemma 11.2. *Let c_i, K_i, for $i = 1, 2, 3$, m_s and $\dot m_s$ be as in Definitions 5.2 and 7.1. Now,*

$$\|G_I^{I_1 I_1} W\| \le K_1 m_s \|W\|, \qquad s = |I| - |I_1| - |I|_2, \tag{11.13}$$

$$\|B_{iI_1} W\| \le K_2 \dot m_s \|W\|, \qquad s = |I_1|, \quad i = 0, 1, \tag{11.14}$$

$$\|A_{I_1} W\| \le K_3\big(\dot m_{s+1}\|W\| + \dot m_s \|W\|_1\big), \qquad s = |I_1|, \tag{11.15}$$

and if $r = |I|$ *then*

$$\|K_I\| \le K_3 \sum_{s=0}^{r} \dot{m}_{r+1-s} \|W\|_s, \tag{11.16}$$

$$\|H_I\| \le K_2 \sum_{s=0}^{r} \dot{m}_{r-s} \big(\|W\|_s + \|\dot{W}\|_s\big) + \|F\|_r. \tag{11.17}$$

Proof. The proof of (11.14) and (11.15) uses (4.41) and (4.43) for A_I and B_I, the bounds (3.9) and (3.3) and (7.10), (7.11) to estimate the pressure in terms of the coordinate. The proof of (11.13) also uses the interpolation inequalities in Lemma 6.2. (11.16) and (11.17) are just a combination of (11.14) and (11.15) with (11.13) and the interpolation inequalities in Lemma 6.2. Note the remark after Definition 5.2 that $\|W_I\| \le K_1(\|W\|_s + \|g\|_s \|W\|)$ if $s = |I|$. A remark about the estimate (11.15) for A_{I_1} is required. By (4.43) $A_I = A_{\check{S}^I p}$ which can be estimated by (3.9) if we control $\|\nabla_N S \check{S}^I p\|_{L^\infty(\partial\Omega)}$. In (11.17) we claim that this will involve at most $s+2$ space derivatives of the metric. In fact, $\check{S}^I p = S^I p + C^{I_1 \ldots I_k} (S^{I_1}\sigma) \ldots (S^{I_{k-1}}\sigma) S^{I_k} p)$ and $S^{I_k} p = 0$ on the boundary and so it follows that the normal derivative must fall on $S^{I_k} p$ so the factors $S^{I_j}\sigma$ never get differentiated by ∇_N. $\square$

Definition 11.1. Let

$$\begin{aligned} E_r^{\mathcal{S}} &= \Big(\sum_{|I| \le r, S \in \mathcal{S}} E_I \Big)^{1/2}, \\ C_r^{\mathcal{R}} &= \| \operatorname{curl} w \|_{r-1}^{\mathcal{R}} + \| \operatorname{curl} \dot{w} \|_{r-1}^{\mathcal{R}}, \\ \langle W \rangle_{A,r} &= \sum_{|I| \le r, I \in \mathcal{S}} \langle W_I, A W_I \rangle \end{aligned} \tag{11.18}$$

where $C_0^{\mathcal{R}}$ should be interpreted as 0.

Summing up the results in Lemma 11.2, Lemma 9.2 and Lemma 5.3 we have:

LEMMA 11.3.

$$|\dot{E}_r^{\mathcal{S}}| \le K_3 (1 + c_0^{-1}) E_r^{\mathcal{S}} + \sum_{s=0}^{r} \big(K_2 \dot{m}_{r-s} \|\dot{W}\|_s + K_3 \dot{m}_{r+1-s} \|W\|_s \big) + \|F\|_r, \tag{11.19}$$

and

$$|\dot{C}_r^{\mathcal{R}}| + \| \operatorname{curl} \ddot{w} \|_{r-1} \le K_2 \sum_{s=0}^{r} \dot{m}_{r-s} \big(\|\dot{W}\|_s + \|W\|_s \big) + K_1 \sum_{s=0}^{r} m_{r-s} \|F\|_s, \tag{11.20}$$

$$\|\dot{W}\|_r + \|W\|_r + \langle W\rangle_{A,r} \le K_1 \sum_{s=0}^{r} m_{r-s}\big(C_s^{\mathcal{R}} + E_s^{\mathcal{S}}\big), \tag{11.21}$$

and

$$C_r^{\mathcal{R}} + E_r^{\mathcal{S}} \le K_1 \sum_{s=0}^{r} m_{r-s}\big(\|\dot{W}\|_s + \|W\|_s\big) + \langle W\rangle_{A,s}. \tag{11.22}$$

Proof. The first inequality follows from (11.12) and Lemma 11.2, the second from Lemma 9.2 and the third from Lemma 5.3. The last inequality is just due to that $E_r^{\mathcal{S}}$ contains $\|\kappa W\|_s^{\mathcal{S}}$ and differentiating κ produces lower order terms. □

Proposition 11.4. *Let $c_0 > 0$ and $c_1 < \infty$ be such that* (2.7) *and* (2.8) *hold and x is smooth for $0 \le t \le T$. Let $\underline{\dot{m}}_s = \sup_{0\le t\le T} \dot{m}_s(t)$, where $\dot{m}_s$ is as in Definition* 6.2, *and set*

$$E_r = \|\dot{W}\|_r + \|W\|_r + \langle W\rangle_{A,r}. \tag{11.23}$$

Then for $r \ge 0$, there is K_3, as in Definition 6.1, *such that, for $0 \le t \le T$,*

$$E_r(t) \le K_3 e^{K_3(1+c_0^{-1})T} \sum_{s=0}^{r} \underline{\dot{m}}_{r+1-s}\Big(E_s(0) + \int_0^t \|F\|_s \, d\tau\Big), \tag{11.24}$$

and for $r \ge 1$,

$$\|\ddot{W}\|_{r-1} \le K_3 e^{K_3(1+c_0^{-1})T}\Big(\sum_{s=0}^{r} \underline{\dot{m}}_{r+1-s}\Big(E_s(0) + \int_0^t \|F\|_s \, d\tau\Big) + \sum_{s=0}^{r-1} m_{r-1-s}\|F\|_s\Big). \tag{11.25}$$

Proof. We will prove the estimate for $\tilde{E}_r = E_r^{\mathcal{S}} + C_r^{\mathcal{R}}$, in place of E_r. In view of Lemma 11.3 and interpolation, $\dot{m}_r m_s \le K_1 \dot{m}_{r+s}$, the estimate for E_r follows from this. By Lemma 11.3 and interpolation, $\dot{m}_r m_s \le K_1 \dot{m}_{r+s}$, we also have

$$\frac{d\tilde{E}_r}{dt} \le K_3(1 + c_0^{-1})\tilde{E}_r + K_3 \sum_{s=0}^{r-1} \dot{m}_{r+1-s}\tilde{E}_s + K_1 \sum_{s=0}^{r} m_{r-s}\|F\|_s \tag{11.26}$$

where we also used the fact that $\dot{m}_1 \le c_3$. Let $a = K_3(1 + c_0^{-1})$. Multiplying by the integrating factor we get

$$(\tilde{E}_r e^{-at})' \le e^{-at} K_3\big(\sum_{s=0}^{r-1} \dot{m}_{r+1-s}(\tilde{E}_s + \|F\|_s) + m_0\|F\|_r\big), \tag{11.27}$$

where this is to be interpreted as the absence of the sum if $r = 0$. Integration of this gives

(11.28)
$$\tilde{E}_r(t) \leq K_3 e^{aT}\Big(\tilde{E}_r(0) + \int_0^t \big(\sum_{s=0}^{r-1} \dot{m}_{r+1-s}(\tilde{E}_s + \|F\|_s) + n_0\|F\|_r\big)\, d\tau\Big), \quad t \leq T,$$

where the sum is to be interpreted as absent if $r = 0$. The proof of the estimate with $\tilde{E}_r$ in place of E_r is by induction. Since the sum is absent if $r = 0$ it follows for $r = 0$. In general we use the interpolation: $\dot{m}_{r+1-s}\dot{m}_{s+1-t} \leq C\dot{m}_1\dot{m}_{r+1-t} \leq K_3\dot{m}_{r+1-t}$.

To prove the estimate for $\|\ddot{W}\|_{r-1}$ we note that by Lemma 5.3 it is bounded by the curl and the tangential components:

$$\|\ddot{W}\|_r \leq K_1 \sum_{s=0}^{r} m_{r-s}\big(\|\operatorname{curl}\ddot{w}\|_{s-1} + \sum_{|I|=s, I\in\mathcal{S}} \|\ddot{W}_I\|\big) \tag{11.29}$$

where the curl is as estimated in Lemma 11.3 and the tangential components can be estimated using the equation $\ddot{W}_I = AW_I + K_I + H_I$ and Lemma 11.2:

(11.30)
$$\sum_{|I|\leq r, I\in\mathcal{S}} \|\ddot{W}_I\| \leq \sum_{s=0}^{r} \big(K_2\dot{m}_{r-s}\|\dot{W}\|_s + K_3\dot{m}_{r+1-s}\|W\|_s\big) + \dot{m}_0\|W\|_{r+1} + \|F\|_r.$$

Hence by (11.29), (11.20) and (11.30)

(11.31)
$$\|\ddot{W}\|_r \leq K_2 \sum_{s=0}^{r} \dot{m}_{r-s}\|\dot{W}\|_s + K_3 \sum_{s=0}^{r+1} \dot{m}_{r+1-s}\|W\|_s + K_1 \sum_{s=0}^{r} m_{r-s}\|F\|_s.$$

(11.25) follows from this with r replaced by $r - 1$. $\square$

We now want to get estimates for an additional time derivative by differentiating the equation. This gives an estimate for the normal operator through the equation and together with estimates for the curl gives the estimate for the additional space derivative sought. Recall that

$$\ddot{W} + AW = H, \qquad \text{where} \qquad H = B(W, \dot{W}) + F, \tag{11.32}$$

and where B, given by (2.49) or (2.63), is

$$\underline{B}_a(W, \dot{W}) = -\big(\dot{g}_{ac} - \omega_{ac} - \dot{\sigma} g_{ac}\big)\dot{W}^c + \dot{\sigma}\big(\dot{g}_{ac} - \omega_{ac}\big)W^c - \partial_a q_0 \tag{11.33}$$

where $\dot{g}_{ab} = \check{D}_t g_{ab}$. In order to differentiate with respect to time let us now write this in the form $\underline{G}\ddot{W} + \underline{A}W = \underline{H}$:

$$g_{ac}\ddot{W}^c + \underline{A}_a W = \underline{B}_a(W, \dot{W}) + g_{ac}F^c. \tag{11.34}$$

Differentiating the equation gives

$$g_{ac}\dddot{W}^c + \underline{A}_a\dot{W} + \underline{\dot{A}}_a W = D_t\,\underline{B}_a(\dot{W}, W) - \dot{g}_{ac}\ddot{W}^c + \dot{g}_{ac}F^c + g_{ac}\dot{F}^c. \tag{11.35}$$

Now,

$$\begin{aligned} D_t\,\underline{B}_a(W,\dot{W}) = &-\big(\dot{g}_{ac} - \omega_{ac} - \dot{\sigma} g_{ac}\big)\ddot{W}^c \\ &-\big(\ddot{g}_{ac} - \dot{\omega}_{ac} + \dot{\sigma}\omega_{ab} - \dot{\sigma}\dot{g}_{ac} - \ddot{\sigma} g_{ac}\big)\dot{W}^c + \dot{\sigma}\big(\dot{g}_{ac} - \omega_{ac}\big)\dot{W}^c \\ &+\big(\dot{\sigma}(\ddot{g}_{ac} - \dot{\omega}_{ac} + \dot{\sigma}\omega_{ac}) - \ddot{\sigma}(\dot{g}_{ac} - \omega_{ac})\big)W^c - \partial_q D_t q_0. \end{aligned} \tag{11.36}$$

In conclusion we get

$$\dddot{W} + A\dot{W} + \dot{A}W = H_1 \ \text{ where } \ H_1 = B_9\ddot{W} + B_8\dot{W} + B_7 W + \dot{G}F + \dot{F} \tag{11.37}$$

where

$$B_9\ddot{W}^b = -P\big(g^{ba}\big(2\dot{g}_{ac} - \omega_{ac} - \dot{\sigma} g_{ac}\big)\ddot{W}^c\big), \tag{11.38}$$

$$B_8\dot{W}^b = P\big(g^{ba}\big(2\dot{\sigma}(\dot{g}_{ac} - \omega_{ac}) - \ddot{g}_{ac} + \dot{\omega}_{ac} + \ddot{\sigma} g_{ac}\big)\dot{W}^c\big), \tag{11.39}$$

$$B_7 W^b = P\big(g^{ba}\big(\ddot{\sigma}(\dot{g}_{ac} - \omega_{ac}) + \dot{\sigma}(\ddot{g}_{ac} - \dot{\omega}_{ac} + \dot{\sigma}\omega_{ac})\big)W^c\big). \tag{11.40}$$

Applying vector fields to (11.37) gives

$$\dddot{W}_I + A\dot{W}_I + \dot{A}W_I = H_{1I} + K_{1I}, \ \text{ where } \ K_{1I} = -\tilde{G}_I^{I_1 I_2}\big(A_{I_1}\dot{W}_{I_2} + \dot{A}_{I_1}W_{I_2}\big) \tag{11.41}$$

and

$$H_{1I} = G_I^{I_1 I_2}\big(B_{6I_1}W_{I_2} + B_{7I_1}\dot{W}_{I_2} + B_{8I_1}\ddot{W}_{I_2}\big) + \dot{F}_I + G_I^{I_1 I_2}\dot{G}_{I_1}F_{I_2}. \tag{11.42}$$

Let

$$E_{1I} = E(\dot{W}_I) = \langle \ddot{W}_I, \ddot{W}_I\rangle + \langle \dot{W}_I, (A+I)\dot{W}_I\rangle. \tag{11.43}$$

Then

$$\begin{aligned} \dot{E}_{1I} &= 2\langle \ddot{W}_I, \dddot{W}_I + (A+I)\dot{W}_I\rangle + \langle \ddot{W}_I, \dot{G}\ddot{W}_I\rangle + \langle \dot{W}_I, (\dot{A}+\dot{G})\dot{W}_I\rangle \\ &= -2\langle \ddot{W}_I, \dot{A}W_I\rangle + 2\langle \ddot{W}_I, K_{1I} + H_{1I}\rangle + \langle \ddot{W}_I, \dot{W}_I\rangle \\ &\quad + \langle \ddot{W}_I, \dot{G}\ddot{W}_I\rangle + \langle \dot{W}_I, (\dot{A}+\dot{G})\dot{W}_I\rangle. \end{aligned} \tag{11.44}$$

The last three terms are estimated by E_{1I} and the second term is estimated as before by lower energies:

$$|\dot{E}_{1I}| \le 2\sqrt{E}_{1I}\|\dot{A}W_I\| + 2\sqrt{E_{1I}}\|K_{1I} + H_{1I}\| + K_3(1 + c_0^{-1})E_I. \tag{11.45}$$

However, the estimate of the first term $-2\langle \ddot{W}_I, \dot{A}W_I\rangle$ requires some new observations. This term could be absorbed by adding $2\langle \dot{W}_I, \dot{A}W_I\rangle$ to the energy, which instead would produce $2\langle \dot{W}_I, \dot{A}W_I\rangle$ and $2\langle \dot{W}_I, \ddot{A}W_I\rangle$. However, we want to have an estimate that only requires two time derivatives of the coordinate and this would require an estimate for $\ddot{A}$, which requires three time derivatives of the coordinates. Instead we will use the fact that, by Lemmas 5.4 and 5.5,

$$\|\dot{A}W_I\| \le K_3(\|\partial W_I\| + \|W_I\|) \le K_3(1 + c_0^{-1})\big(\|AW_I\| + \|\operatorname{curl}\underline{W_I}\| + \|W_I\|\big). \tag{11.46}$$

Then there appears to be a loss of regularity, but remember that we now have an estimate also for $\|\ddot{W}_I\|$ in the energy; by the equation (11.2) we can estimate $\|AW_I\| \le \|\ddot{W}_I\| + \|K_I\| + \|H_I\|$, where the last two terms were estimated in Lemma 11.2. $\operatorname{curl}\underline{W_I}$ is by (5.22) up to terms of lower order equal to $\mathcal{L}_S^I \operatorname{curl} w$. At this point we have to use the fact that we have an improved evolution equation for the curl.

LEMMA 11.5.

$$\|B_{iI_1}W\| \le K_3 \dddot{m}_s \|W\|, \qquad s = |I_1|, \quad i = 7, 8, 9, \tag{11.47}$$

$$\|\dot{A}_{I_1}W\| \le K_3\big(\dddot{m}_{s+1}\|W\| + \dddot{m}_s\|W\|_1\big), \qquad s = |I_1|, \tag{11.48}$$

and if $r = |I|$ *then*

$$\|K_{1I}\| \le K_3 \sum_{s=0}^{r} \dddot{m}_{r+1-s}\big(\|\dot{W}\|_s + \|W\|_s\big) \tag{11.49}$$

$$\|H_{1I}\| \le K_3 \sum_{s=0}^{r} \dddot{m}_{r-s}\big(\|\ddot{W}\|_s + \|\dot{W}\|_s + \|W\|_s\big) + \|\dot{F}\|_r + K_2 \sum_{s=0}^{r} \dot{m}_{r-s}\|F\|_s. \tag{11.50}$$

Definition 11.2. Let

$$E_{r,1}^{\mathcal{S}} = \Big(\sum_{|I|\le r, S\in\mathcal{S}} E_{I,1}\Big)^{1/2}, \qquad C_{r,1}^{\mathcal{R}} = \|\operatorname{curl} w\|_r^{\mathcal{R}} + \|\operatorname{curl}\tilde{w}\|_r^{\mathcal{R}}, \tag{11.51}$$

where $\tilde{w}$ is as in Lemma 9.1.

Summing up the results in Lemmas 11.5, 9.2 and 5.5 yields

LEMMA 11.6.

$$\begin{aligned}\dot{E}_{r,1}^{\mathcal{S}} \le{}& K_3(1 + c_0^{-1})E_{r,1}^{\mathcal{S}} + K_3 \sum_{s=0}^{r} \dddot{m}_{r-s}\big(\|\ddot{W}\|_s + \|\dot{W}\|_s\big)\\ &+K_3 \sum_{s=0}^{r+1} \dddot{m}_{r+1-s}\|W\|_s + \|\dot{F}\|_r + K_2 \sum_{s=0}^{r} \dot{m}_{r-s}\|F\|_s\end{aligned} \tag{11.52}$$

and

$$\dot{C}_{r,1}^{\mathcal{R}} \le C_{r,1}^{\mathcal{R}} + K_2 \sum_{s=0}^{r+1} \dddot{m}_{r+1-s}\|W\|_s + K_1\|\operatorname{curl}\underline{F}\|_r. \tag{11.53}$$

Now,

(11.54)

$$\|\ddot W\|_r \le K_1 \sum_{s=0}^r m_{r-s} E^{\mathcal{S}}_{s,1} + K_3 \sum_{s=0}^r \dot m_{r-s}\big(\|\dot W\|_s + \|W\|_s\big) + K_1 \sum_{s=1}^r m_{r-s}\|F\|_s$$

and

$$c_0\|W\|_{r+1} \le K_3\big(C^{\mathcal{R}}_{r,1} + E^{\mathcal{S}}_{r,1} + K_2 \sum_{s=0}^r \dot m_{r+1-s}(\|\dot W\|_s + \|W\|_s)\big). \tag{11.55}$$

Proof. The energy estimate (11.52) follows from the energy estimate (11.45) and the estimates in Lemma 11.5. The estimate for the curl (11.53) follows from Lemma 9.2. The estimate for $\ddot W$ (11.54) uses Lemma 5.3:

$$\|\ddot W\|_r \le K_1 \sum_{s=1}^r m_{r-s}\big(\|\operatorname{curl} \ddot w\|_{s-1} + E^{\mathcal{S}}_{s,1}\big) + m_r E^{\mathcal{S}}_{0,1} \tag{11.56}$$

where the estimate for the curl follows from Lemma 11.3, and we use interpolation, $m_s \dot m_r \le K_3 \dot m_{s+r}$. Let us now prove the additional space regularity (11.56). We have from the equation, $AW_I = -\ddot W_I + H_I + K_I$, and Lemma 11.2,

$$\begin{aligned} \|W\|^{\mathcal{S}}_{r,A} &\le E^{\mathcal{S}}_{r,1} + K_2 \sum_{s=0}^r \dot m_{r+1-s}(\|W\|_s + \|\dot W\|_s) + \|F\|_r, \\ \|W\|^{\mathcal{S}}_{s,A} &= \sum_{|I|=s, I\in\mathcal{S}} \|A\hat{\mathcal{L}}^I_S W\| \end{aligned} \tag{11.57}$$

and (11.56) follows since by Lemma 5.5:

$$c_0\|W\|_{r+1} \le K_3\big(C^{\mathcal{R}}_{r,1} + \|W\|^{\mathcal{S}}_{r,A} + \sum_{s=0}^r m_{r+1-s}\|W\|_s\big). \tag{11.58}$$

□

PROPOSITION 11.7. *Let* $c_0 > 0$ *and* $c_1 < \infty$ *be such that* (2.7) *and* (2.8) *hold and* x *is smooth for* $0 \le t \le T$. *Let* $\underline{\ddot m}_s = \sup_{0\le t\le T} \ddot m_s(t)$, *where* $\ddot m_s$ *is as in Definition* 6.2, *and set*

$$\begin{aligned} E_{r,1} = {}& \|\ddot W\|_r + \|\dot W\|_r + \langle \dot W\rangle_{A,r} + \|W\|_r + \langle W\rangle_{A,r} \\ &+ \|\operatorname{curl} \tilde w\|_r + \|\operatorname{curl} w\|_r + c_0\|W\|_{r+1}, \end{aligned} \tag{11.59}$$

where $\tilde w$ *is as in Lemma* 9.1. *Then for* $r \ge 0$ *there is* K_4, *as in Definition* 6.1, *such that, for* $0 \le t \le T$,

$$\begin{aligned} E_{r,1}(t) \le K_4 e^{K_4(1+c_0^{-1})T} \sum_{s=0}^r \underline{\ddot m}_{r+1-s}\Big(E_{s,1}(0) + \int_0^t \big(\|\dot F\|_s + \|F\|_s \\ + \|\operatorname{curl} F\|_s\big)\, d\tau\Big) \end{aligned} \tag{11.60}$$

and for $r \geq 1$

$$\|\ddot{W}\|_{r-1} \leq K_4 e^{K_4(1+c_0^{-1})T} \sum_{s=0}^{r} \underline{\ddot{m}}_{r+1-s} \cdot \Big(E_{s,1}(0) + \int_0^t \big(\|\dot{F}\|_s + \|F\|_s + \|\operatorname{curl} F\|_s\big)\, d\tau\Big) + K_4 e^{K_4(1+1/c_0)T} \sum_{s=0}^{r-1} \underline{\dot{m}}_{r-1-s}(\|F\|_s + \|\dot{F}\|_s). \tag{11.61}$$

Proof. The proof would be the same as the proof of Proposition 11.4 apart from the fact that we must worry more about the possibility of the constant c_0 being small. As in the proof of Proposition 11.4 the estimate for $E_{r,1}$ would follow from the same estimate for $\tilde{E}_{r,1} = E^{\mathcal{S}}_{r,1} + C^{\mathcal{R}}_{r,1} + \tilde{E}_r$, where $\tilde{E}_r = E^{\mathcal{S}}_r + C^{\mathcal{R}}_r$ is as in the proof of Proposition 11.4. The critical term with c_0 is by Lemma 11.6 and Lemma 11.3 bounded by the other terms plus lower order terms. Note that by Lemma 11.3 and interpolation $\sum_{s=0}^{r} \ddot{m}_{r+1-s}\tilde{E}_r$ bounds the lower order terms with $\|\dot{W}\|_s$ and $\|W\|_s$, for $s \leq r$ in Lemma 11.6. By Lemma 11.6 and the proof of Proposition 11.4 we have

$$\frac{d\tilde{E}_{r,1}}{dt} \leq K_4(1 + c_0^{-1})\tilde{E}_{r,1} + K_3(1 + c_0^{-1}) \sum_{s=0}^{r-1} \ddot{m}_{r+1-s}\tilde{E}_{s,1} + K_1 \sum_{s=0}^{r} m_{r-s}\|F\|_s + C\|\dot{F}\|_r \tag{11.62}$$

where we also used the fact that $\ddot{m}_1 \leq c_4$. Let $a = K_4(1 + c_0^{-1})$. Multiplying by the integrating factor we get

$$(\tilde{E}_{r,1}e^{-at})' \leq e^{-at} K_4 \Big((1 + c_0^{-1}) \sum_{s=0}^{r-1} \ddot{m}_{r+1-s}\tilde{E}_{s,1} + \sum_{s=0}^{r} m_{r-s}\|F\|_s + \|\dot{F}\|_r\Big), \tag{11.63}$$

where the sum is absent if $r = 0$. Integration of this gives

$$\tilde{E}_{r,1}(t) \leq K_4 e^{aT}\Big(\tilde{E}_{r,1}(0) + \int_0^t \big((1 + c_0^{-1}) \sum_{s=0}^{r-1} \ddot{m}_{r+1-s}\tilde{E}_{s,1} + \sum_{s=0}^{r} m_{r-s}\|F\|_s + \|\dot{F}\|_r\big)\, d\tau\Big), \tag{11.64}$$

for $t \leq T$, where the sum is absent if $r = 0$. The proof of the estimate (11.60) with $\tilde{E}_{r,1}$ in place of $E_{r,1}$ follows by induction from (11.64). Since the sum in (11.64) is absent if $r = 0$ it follows that it is true for $r = 0$. In general we use interpolation, $\ddot{m}_{r+1-s}\ddot{m}_{s+1-t} \leq C\ddot{m}_1\dot{m}_{r+1-t} \leq K_4\ddot{m}_{r+1-t}$. (11.61) follows as in the proof of (11.25). $\square$

12. Existence and estimates for the inverse of the modified linearized operator for general vector fields

In this section we will show that the modified linearized operator can be solved for general vector fields outside the divergence-free class; i.e., we solve

$$L_1 W = F, \qquad W\big|_{t=0} = \dot{W}\big|_{t=0} = 0 \tag{12.1}$$

when F is not necessarily divergence-free. Below we give estimates for the solution of (12.1) that are good enough that the linearized operator can be considered as a lower order modification of (12.1). In the next section we will use these to prove existence and estimates also for the inverse of the linearized operator by iteration. One gets a new iterate by substituting the previous iterate into the right-hand side of (12.3) and solving for the new iterate on the left-hand side. We want estimates that are good enough that we get the same regularity for the new iterate and so we need estimates for (12.1) that do not loose regularity going from F to W.

THEOREM 12.1. *Let $0 < T \le c_0 \le 1$ and $0 < c_1 < \infty$ be such that* (2.7), (2.8) *hold and x is smooth for $0 \le t \le T$. Let $\underline{n}_s = \sup_{0\le t\le T} n_s(t)$, where n_s is as in Definition* 6.2. *Then the equation* (12.1), *with F smooth, has a smooth solution W, for $0 \le t \le T$. Furthermore, there is K_4 as in Definition* 6.1, *such that, for $0 \le t \le T$,*

$$\|\dot{W}\|_{r-1} + \|W\|_r \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau, \qquad r \ge 1, \tag{12.2}$$

and

$$\|\ddot{W}\|_{r-1} \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau + K_4 \sum_{s=0}^{r-1} \underline{n}_{r-1-s} \|F\|_s, \qquad r \ge 1. \tag{12.3}$$

As in Section 3 we can decompose $W = W_0 + W_1$ where W_0 is divergence-free and W_1 is the gradient of a function vanishing at the boundary. By (3.26) W_0 satisfies

$$L_1 W_0 = -AW_1 + B_{11}\dot{W}_1 + B_{01}W_1 + PF, \qquad W_0\big|_{t=0} = \dot{W}_0\big|_{t=0} = 0, \tag{12.4}$$

where all the terms on the right-hand side are divergence-free and B_{01} and B_{11} are bounded operators given by (3.25). By (3.27) and (3.28) W_1 satisfies

$$W_1^a = g^{ab}\partial_b q_1, \qquad \triangle\, q_1 = \varphi, \qquad q_1\big|_{\partial\Omega} = 0, \tag{12.5}$$

where

$$D_t^2 \varphi + \ddot{\sigma}\varphi = \operatorname{div} F, \qquad \varphi\big|_{t=0} = D_t\varphi\big|_{t=0} = 0. \tag{12.6}$$

The solution of (12.6) is a smooth function if F is smooth so it follows that that W_1 is smooth and hence (12.4) has a smooth solution W_0 by Theorem 10.1. Therefore, we have proved that the modified linearized operator (12.1) has a smooth solution W if F and x are smooth and the coordinate and physical conditions are satisfied for $0 \leq t \leq T$. However, on the right-hand side of (12.4) the term AW_1 loses space regularity since A is order one. If we just use Proposition 11.4 and Proposition 12.3 below we are going to get an estimate that loses space regularity going from F to W in (12.1). However, because the curl of AW_1 vanishes we can use the improved estimate in Proposition 11.7 that gains an extra space derivative to handle the term $-AW_1$. Let us first prove the estimate for (12.5), (12.6):

LEMMA 12.2. *Suppose that*

$$D_t^2\varphi + \ddot{\sigma}\varphi = \hat{D}_t^2\varphi - 2\dot{\sigma}\hat{D}_t\varphi + \dot{\sigma}^2\varphi = f. \tag{12.7}$$

Let $T < 1$ *and set* $\underline{\ddot{m}}_s = \sup_{0\leq t\leq T}\ddot{m}_s(t)$, *where* $\ddot{m}_s$ *is as in Definition* 6.2. *Then, there is* K_3, *as in Definition* 6.1, *such that for* $0 \leq t \leq T$ *and* $r \geq 1$,

$$\|\dot{\varphi}\|_{r-1} + \|\varphi\|_{r-1} \leq K_3 \sum_{s=0}^{r-1} \underline{\ddot{m}}_{r-1-s} \int_0^t \|f\|_s \, d\tau, \tag{12.8}$$

$$\|\ddot{\varphi}\|_{r-1} \leq K_3 \sum_{s=0}^{r-1} \underline{\ddot{m}}_{r-1-s} \Big(\|f\|_s + \int_0^t \|f\|_s \, d\tau \Big). \tag{12.9}$$

Proof. (12.4) is just an ordinary differential equation for each space coordinate however one just has to make sure to integrate it in such a way that we do not get more than two time derivatives on the metric.

$$\begin{aligned} D_t\big((\hat{D}_t\varphi)^2 + \dot{\sigma}^2\varphi^2\big) &= 2\dot{\sigma}(\hat{D}_t\varphi)^2 + 2\dot{\sigma}(\ddot{\sigma} - \dot{\sigma}^2)\varphi^2 + 2(\hat{D}_t\varphi)f, \\ \hat{D}_t^2\varphi - 2\dot{\sigma}\hat{D}_t\varphi + \dot{\sigma}^2\varphi &= f. \end{aligned} \tag{12.10}$$

Integrating this in time and space gives the lowest order energy estimate in (12.8). The lowest order estimate in (12.9) follows from this since once we have estimates for the φ and $\hat{D}_t\varphi$ we get an estimate for $\hat{D}_t^2\varphi$ from the equation.

In order to get (12.8) and (12.9) for higher derivatives we commute through $\hat{R}^I$, defined in Section 4 by $\hat{R}^I f = \kappa^{-1}R^I(\kappa f)$, where $I = (i_1, \ldots, i_r)$ is a multi-index and $R^I = R_{i_1} \ldots R_{i_r}$ is a product of the vector fields in $\mathcal{R}$ defined in Section 4. Then $[\hat{D}_t, \hat{R}^I] = 0$ and with $\varphi_I = \hat{R}^I\varphi$ and $\dot{\sigma}_I = \hat{R}^I\dot{\sigma}$, we obtain

$$\hat{D}_t^2\varphi_I - 2\dot{\sigma}\hat{D}_t\varphi_I + \dot{\sigma}^2\varphi_I = f_I, \quad f_I = 2\tilde{c}^{I_1 I_2}\dot{\sigma}_{I_1}\hat{D}_t\varphi_{I_2} - \tilde{d}^{I_0 I_1 I_2}\dot{\sigma}_{I_0}\dot{\sigma}_{I_1}\varphi_{I_2} + \hat{R}^I f \tag{12.11}$$

where the sums are over all combinations of $I_1 + I_2 = I$, respectively $I_0 + I_1 + I_2 = I$ and $\tilde{c}^{I_1 I_2} = 1$ and $\tilde{d}^{I_0 I_1 I_2} = 1$ unless $I_2 = I$ in which case they are 0.

We can now use (12.10) applied to f_I in place of f and φ_I in place of φ. Here the terms in f_I are lower order. $\square$

Once we get the corresponding bounds for φ in terms of $\operatorname{div} F$, the bounds for W_1 follow from Proposition 6.1.

PROPOSITION 12.3. *Suppose that* $W_1^a = g^{ab}\partial_b q$, *where* $q\big|_{\partial\Omega} = 0$ *and* $\triangle q = \varphi$ *where* φ *satisfies*

$$D_t^2\varphi + \ddot{\sigma}\varphi = \operatorname{div} F. \tag{12.12}$$

Let $T < 1$ *and set* $\underline{\ddot{m}}_s = \sup_{0\leq t\leq T} \ddot{m}_s(t)$, *where* $\ddot{m}_s$ *is as in Definition* 6.2. *Then, there is* K_3, *as in Definition* 6.1, *such that, for* $0 \leq t \leq T$,

$$\begin{aligned} &\|\dot{W}_1(t)\|_r + \|W_1(t)\|_r \\ &\qquad \leq K_3 \sum_{s=1}^{r} \underline{\ddot{m}}_{r-s}\big(\|\dot{W}_1(0)\|_s + \|W_1(0)\|_s + \int_0^t \|F\|_s\, d\tau\big), \qquad r\geq 1, \end{aligned} \tag{12.13}$$

and

$$\begin{aligned} \|\ddot{W}_1(t)\|_r \leq K_3 \sum_{s=1}^{r} \underline{\ddot{m}}_{r-s}\Big(&\|\dot{W}_1(0)\|_s + \|W_1(0)\|_s \\ &+ \int_0^t \|F\|_s\, d\tau + \|F(t)\|_s\Big), \qquad r\geq 1. \end{aligned} \tag{12.14}$$

We will now further decompose the solution of (12.4) into two parts $W_0 = W_{00} + W_{01}$, where

$$L_1 W_{01} = -AW_1 = F_{01}, \qquad W_{01}\big|_{t=0} = \dot{W}_{01}\big|_{t=0} = 0, \tag{12.15}$$

and

$$L_1 W_{00} = PF + B_{11}\dot{W}_1 + B_{01}W_1 = F_{00}, \qquad W_{00}\big|_{t=0} = \dot{W}_{00}\big|_{t=0} = 0. \tag{12.16}$$

For (12.15) we use the estimate in Proposition 11.7 and for (12.16) we use Proposition 11.4. This together with Proposition 12.3 gives Corollary 12.4 below. Our solution to (12.1) is now obtained as a sum of $W = W_1 + W_{01} + W_{00}$ and so it will satisfy the worst of the estimates in Corollary 12.4. This proves Theorem 12.1.

COROLLARY 12.4. *Let* $0 < T \leq c_0 \leq 1$ *and* $c_1 < \infty$ *be such that* (2.7) *and* (2.8) *hold and* x *is smooth for* $0 \leq t \leq T$. *Let* $\underline{n}_s = \sup_{0\leq t\leq T} n_s(t)$, *where* n_s *is as in Definition* 6.2. *Let* W_1 *be the solution of* (12.5), (12.6), *with* W_{01} *the solution of* (12.15) *and* W_{00} *the solution of* (12.16). *Then there is* K_4, *as in*

Definition 6.1, *such that, for* $0 \le t \le T$ *and* $r \ge 1$,

$$
(12.17) \qquad \|\dot W_1\|_r + \|W_1\|_r \le K_4 \sum_{s=1}^{r} \underline{n}_{r-1-s} \int_0^t \|F\|_s \, d\tau,
$$

$$
\|\ddot W_1\|_r \le K_4 \sum_{s=1}^{r} \underline{n}_{r-1-s} \Big(\int_0^t \|F\|_s \, d\tau + \|F\|_s \Big),
$$

$$
(12.18) \qquad \|\ddot W_{01}\|_{r-1} + \|\dot W_{01}\|_{r-1} + \|W_{01}\|_r \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau,
$$

(12.19)

$$
\|\dot W_{00}\|_r + \|W_{00}\|_r \le K_4 \sum_{s=0}^{r} \underline{n}_{r-1-s} \int_0^t \|F\|_s \, d\tau,
$$

$$
\|\ddot W_{00}\|_{r-1} \le K_4 \sum_{s=0}^{r} \underline{n}_{r-1-s} \int_0^t \|F\|_s \, d\tau + K_3 \sum_{s=0}^{r-1} \underline{n}_{r-1-s} \|F\|_s .
$$

Proof. (12.17) follows from Proposition 12.3. By Proposition 11.7 we have for $r \ge 1$

$$
(12.20) \qquad \|\ddot W_{01}\|_{r-1} + \|\dot W_{01}\|_{r-1} + c_0 \|W_{01}\|_r + \|W_{01}\|_{r-1}
\le K_4 \sum_{s=0}^{r-1} \underline{n}_{r-1-s} \int_0^t \big(\|\dot F_{01}\|_s + \|F_{01}\|_s + \|\operatorname{curl} F_{01}\|_s \big) \, d\tau.
$$

It follows that also $\ddot W_{01}\big|_{t=0} = 0$ since $AW_1\big|_{t=0} = 0$. Here the curl of $F_{01} = AW_1$ vanishes and $\hat D_t A W_1 = A\dot W_1 + \dot A W_1 - \dot G A W_1$ so that

$$
(12.21) \qquad \|\dot F_{01}\|_{r-1} + \|F_{01}\|_{r-1} \le K_4 \sum_{s=0}^{r} \underline{n}_{r-1-s} \big(\|\dot W_1\|_s + \|W_1\|_s \big).
$$

Using (12.17), (12.20) and (12.21) we obtain (12.18). Note that the constant c_0 in (12.20) can be replaced by 1 since we have two consecutive integrals and we assumed that $0 \le t \le T \le c_0$. Finally from Proposition 11.4 we get

$$
(12.22) \qquad \|\dot W_{00}\|_r + \|W_{00}\|_r \le K_3 \sum_{s=0}^{r} \underline{n}_{r-1-s} \int_0^t \|F_{00}\|_s \, d\tau,
$$

$$
\|\ddot W_{00}\|_{r-1} \le K_3 \sum_{s=0}^{r} \underline{n}_{r-1-s} \int_0^t \|F_{00}\|_s \, d\tau + K_3 \|F_{00}\|_{r-1}.
$$

Now, the operators B_{01} and B_{11} in (12.16) are bounded, which is given by (3.25), of the same form already studied in Section 9, and PF, the projection, is bounded by the estimates in Proposition 6.38; so it follows that

$$\|F_{00}\|_r \le K_4 \sum_{s=0}^{r} n_{r-s}(\|\dot W_1\|_s + \|W_1\|_s) + K_1 \sum_{s=0}^{r} n_{r-s-2}\|F\|_s. \tag{12.23}$$

Combining these inequalities, using interpolation as usual, also gives (12.19). □

13. Existence and L^2 estimates for the inverse of the linearized operator

In this section we finally prove existence and estimates for the the inverse of the linearized operator

$$L_0 W = F, \qquad W\big|_{t=0} = \dot W\big|_{t=0} = 0, \tag{13.1}$$

where $L_0 = \Phi'(x)$ is given by (2.14). Now, (13.1) can be written

$$L_1 W = B_3 W + F, \qquad W\big|_{t=0} = \dot W\big|_{t=0} = 0, \tag{13.2}$$

where the modified linearized operator L_1 is given by (2.49) and B_3 is given by (2.57). In the previous section we proved existence and estimates for the modified linearized operator L_1:

$$L_1 W = F, \qquad W\big|_{t=0} = \dot W\big|_{t=0} = 0. \tag{13.3}$$

The existence and estimates for (13.3) can now be used to prove existence and estimates for (13.2), and hence for (13.1), by iteration. We simply define a sequence by $W_0 = 0$ and for $k \ge 1$:

$$L_1 W_k = B_3 W_{k-1} + F, \qquad W_k\big|_{t=0} = \dot W_k\big|_{t=0} = 0. \tag{13.4}$$

We will use the estimates for (13.3) to show that W_k converges to a solution of (13.2) and that the solution of (13.2) satisfies the same estimates as the solution of (13.3).

THEOREM 13.1. *Let $0 < T \le c_0 \le 1$ and $0 < c_1 < \infty$ be such that (2.7) and (2.8) hold and x is smooth for $0 \le t \le T$. Let $\underline{n}_s = \sup_{0\le t\le T} n_s(t)$, where n_s is as in Definition 6.2. Then the equation (13.1), with F smooth, has a smooth solution W, for $0 \le t \le T$. Furthermore, there is K_4 as in Definition 6.1, such that, for $0 \le t \le T$,*

$$\|\dot W\|_{r-1} + \|W\|_r \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau, \qquad r \ge 1, \tag{13.5}$$

and

$$\|\ddot W\|_{r-1} \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau + K_4 \sum_{s=0}^{r-1} \underline{n}_{r-1-s}\|F\|_s, \qquad r \ge 1. \tag{13.6}$$

Proof. The existence and the estimates in the theorem for (13.3) were given in Theorem 12.1. The estimate for (13.1) follows from the estimate for (13.3) by writing (13.1) in the form (13.2). If W satisfies

$$L_1 W = B_3 \tilde{W} + F, \qquad W\big|_{t=0} = \dot{W}\big|_{t=0} = 0, \tag{13.7}$$

where B_3 is given by (2.57), then by (13.5) for (13.3),

$$\|\dot{W}\|_{r-1} + \|W\|_r \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \big(\|F\|_s + \|\tilde{W}\|_s\big)\, d\tau, \qquad r \ge 1. \tag{13.8}$$

We claim that (13.5) for (13.3) follows from this with $\tilde{W} = W$, by induction, for some other K_4. In fact, assume that (13.5) is true for $r \le k-1$; then it follows from (13.8) and interpolation that

$$\begin{aligned} \|\dot{W}\|_{r-1} + \|W\|_r \le{}& K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau + K_4 \int_0^t \|W\|_r \, d\tau \\ &+ K_4 \sum_{s=1}^{r-1} \sum_{k=1}^{s} \underline{n}_{r-s} \underline{n}_{s-k} \int_0^t \int_0^\tau \|F\|_k \, dz \; d\tau \\ \le{}& K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau + K_4 \int_0^t \|W\|_r \, d\tau \end{aligned} \tag{13.9}$$

for some other K_4. By a standard Gronwall argument we can get rid of the $\|W\|_r$, replacing K_4 by some other K_4. Let $g(t) = \int_0^t \|W\|_r \, d\tau$ and $f(t) = \sum_{s=0}^r \underline{n}_{r-s} \int_0^t \|F\|_s \, d\tau$. Then $g'(t) \le K_4 g + K_4 f$ so that $\big(g e^{-K_4 t}\big)' \le K_4 f$ and integrating this up gives $g \le K_4 \int_0^t f \, d\tau$ for some other K_4 and for $t \le T$.

Similarly it follows from (13.5) that the solution of (13.7) satisfies

$$\begin{aligned} \|\ddot{W}\|_{r-1} \le{}& K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \big(\|F\|_s + \|\tilde{W}\|_s\big)\, d\tau \\ &+ K_4 \sum_{s=0}^{r-1} \underline{n}_{r-1-s} \big(\|F\|_s + \|\tilde{W}\|_s\big), \qquad r \ge 1 \end{aligned} \tag{13.10}$$

(13.6) for (13.1) follows from this with $\tilde{W} = W$ by the estimate (13.5) just obtained.

It remains to prove existence for (12.2). We put up an iteration $W_0 = 0$ and $L_1 W_k = F + (L_1 - L_0) W_{k-1}$, for $k \ge 1$. Then $L_1 W_1 = F$ so that W_1 satisfies the desired estimate and is smooth. Let $\overline{W}_k = W_k - W_{k-1}$, for $k \ge 1$. Then $\overline{W}_1 = W_1$ and $L_1 \overline{W}_k = (L_1 - L_0)\overline{W}_{k-1}$, for $k \ge 2$. In conclusion

$$L_1 \overline{W}_1 = F, \qquad L_1 \overline{W}_k = B_3 \overline{W}_{k-1}, \qquad k \ge 2, \qquad \overline{W}_k\big|_{t=0} = \dot{\overline{W}}_k\big|_{t=0} = 0 \tag{13.11}$$

where B_3 is a bounded operator given by (2.57). By estimate (13.8) for each k,

$$\sum_{k=1}^{N} \sup_{0\le\tau\le t} \big(\|\dot{\overline{W}}_k(\tau,\cdot)\|_{r-1} + \|\overline{W}_k(\tau,\cdot)\|_r\big) \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \big(\|F\|_s + \sum_{k=1}^{N-1} \|\overline{W}_k\|_s\big)\, d\tau, \qquad r \ge 1. \tag{13.12}$$

Note that the supremum is inside the sum since we use (13.8) for each $\overline{W}_k$ and since on the left of (13.8) we may take the supremum of $\tau \le t$. The same argument that leads to the proof of the estimate (13.5) for (13.1) from (13.8) now gives the uniform estimate

$$\sum_{k=1}^{N} \sup_{0\le\tau\le t} \big(\|\dot{\overline{W}}_k(\tau,\cdot)\|_{r-1} + \|\overline{W}_k(\tau,\cdot)\|_r\big) \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s\, d\tau, \quad r \ge 1, \tag{13.13}$$

where K_4 is independent of N. (One replaces the sum on the right of (13.12) by the larger sum on the left of (13.12).) Similarly the uniform estimates corresponding to (13.5) also hold, as is seen by using (13.10) for each k and replacing the sum on the right by the larger sum on the left and using (13.13):

$$\sum_{k=1}^{N} \sup_{0\le\tau\le t} \|\ddot{\overline{W}}_k(\tau,\cdot)\|_{r-1} \le K_4 \sum_{s=1}^{r} \underline{n}_{r-s} \int_0^t \|F\|_s\, d\tau + K_4 \sum_{s=0}^{r-1} \underline{n}_{r-1-s} \sup_{0\le\tau\le t} \|F(\tau,\cdot)\|_s, \quad r \ge 1. \tag{13.14}$$

It follows that $W_N = \sum_{k=1}^N \overline{W}_k$ is a Cauchy sequence in $C^2([0,T], H^{r-1}(\overline{\Omega}))$, for any T, and hence there is a limit $W \in C^2([0,T], H^{r-1}(\overline{\Omega}))$, for any T. Additional regularity in time follows from differentiating this equation. We have already proved that $\hat{D}_t^2 W = AW + B_0 W + B_1 \dot{W} + B_3 W \in C^1([0,T], H^{r-2}(\overline{\Omega}))$; i.e., $\hat{D}_t^2 W$ is continuously differentiable with respect to time so that $W \in C^3\big([0,T], H^{r-2}(\overline{\Omega})\big)$ and so on. Since this argument is true for any r it follows that W is smooth. □

14. Estimates for the physical and coordinate conditions

We assume that the physical condition and the coordinate condition hold initially at time 0 for some constants $c_0 > 0$ and $c_1 < \infty$ and we need to show that this implies that they will hold with c_0 replaced by $c_0/2$ and c_1 replaced by $2c_1$, for $0 \le t \le T$, if T is sufficiently small.

Let us introduce the space time norms:

$$|||u|||_r = \sup_{0\le t\le T} \|u(t,\cdot)\|_{r,\infty}, \qquad |||u|||_{r,k} = |||u|||_r + \cdots + |||D_t^k u|||_r. \tag{14.1}$$

LEMMA 14.1. *Let* $M(t) = \sup_{y\in\Omega} \sqrt{|\partial x/\partial y|^2 + |\partial y/\partial x|^2}$. *Then*

$$M(t) \le 2M(0), \quad \text{for} \quad t \le T, \quad \text{if} \quad T|||\dot{x}|||_1 M(0) \le 1/8. \tag{14.2}$$

Let $N(t) = \sup_{y\in\partial\Omega} |\nabla_N p|^{-1}$. *Then when* T *is so small that* (14.2) *holds,*

$$N(t) \le 2N(0) \quad \text{for} \quad t \le T, \quad \text{if} \quad T|||\dot{p}|||_1 M(0)N(0) \le 1/8. \tag{14.3}$$

Proof. We have $|D_t\, \partial x/\partial y| \le |||\dot{x}|||_1$ and $|D_t\, \partial y/\partial x| \le |\partial y/\partial x|^2 |D_t \partial x/\partial y|$ so that $M'(t) \le (1 + M^2)|||\dot{x}|||_1 \le 2M^2|||\dot{x}|||_1$, since also $M(t) \ge 1$. Hence

$$M(t) \le M(0)\big(1 - 2|||\dot{x}|||_1 M(0)t\big)^{-1}, \quad \text{when} \quad 2|||\dot{x}|||_1 M(0)t < 1. \tag{14.4}$$

Now, $\nabla_N p = N^a \partial_a p$, where N is the unit normal, so that $D_t\nabla_N p = \nabla_N D_t p + (D_t N^a)\partial_a p = \nabla_N D_t p + (D_t N^a) g_{ab} N^b \nabla_N p$, since $p\big|_{\partial\Omega} = 0$. Furthermore $0 = D_t(g_{ab}N^a N^b) = 2g_{ab}(D_t N^a)N^b + (D_t g_{ab})N^a N^b$ and $N^a = (\partial y^a/\partial x^i)N^i$, where $\delta_{ij}N^i N^j = 1$. Hence $|D_t\nabla_N p| \le M\big(|\partial D_t p| + |\partial D_t x||\nabla_N p|\big)$. Therefore if $N(t) = \sup_{y\in\partial\Omega} |\nabla_N p|^{-1}$, we have $N' \le M|||\dot{p}|||_1 N^2 + M|||\dot{x}|||_1 N/2$ and if we use (14.2) and multiply with the integrating factor, $\tilde{N}(t) = N(t)e^{-tM(0)|||\dot{x}|||_1}$, we get $\tilde{N}' \le 2e^{1/8}M(0)|||\dot{p}|||_1 \tilde{N}^2$. Hence

$$N(t) \le N(0)e^{1/8}\big(1 - N(0)2e^{1/8}M(0)|||\dot{p}|||_1 t\big)^{-1}, \qquad \text{when} \quad N(0)2e^{1/8}M(0)|||\dot{p}|||_1 t < 1. \tag{14.5}$$

This proves the lemma. □

It now follows from Lemma 14.2:

LEMMA 14.2. *Let* x_0 *be the approximate solution satisfying* (2.12) *and suppose that* (2.7) *and* (2.8) *holds when* $t = 0$. *Then there is a* $T_0 > 0$, *depending only on an upper bound for* $|||x_0|||_{4,2}$, c_1 *and* c_0^{-1} *such that* (2.7) *and* (2.8) *hold for* $0 \le t \le T$ *with* c_0 *replaced by* $c_0/2$ *and* c_1 *replaced by* $2c_1$ *provided that*

$$0 < T \le T_0, \ \text{and} \ |||x - x_0|||_{4,2} \le 1, \ \text{and} \ (x - x_0)\big|_{t=0} = D_t(x - x_0)\big|_{t=0} = 0. \tag{14.6}$$

Proof. We need to satisfy the conditions (14.2) and (14.3) in Lemma 14.1. Since $|||\dot{x}|||_1 \le |||x_0|||_{4,2}+1$, (14.2) holds if $T \le (8c_1(|||x_0|||_{4,2}+1))^{-1}$. To satisfy (14.3) we use the estimate in Lemma 6.4, where K_3 is as in Definition 6.1, to obtain $\|\dot{p}\|_{1,\infty} \le F\big(\|x\|_{3,\infty} + \|\dot{x}\|_{2,\infty} + \|\ddot{x}\|_{1,\infty}\big)$ for some increasing function F. Hence (14.3) holds if $T \le c_0(8c_1 + F(|||x_0|||_{4,2} + 1))^{-1}$. □

15. Tame L^∞ estimates for the inverse of the linearized operator

We are now going to modify the estimate for the inverse of the linearized operator in Theorem 13.1 so that it can be used with the Nash-Moser inverse function theorem in Section 18. We want tame estimates for the inverse of the linearized operator

$$\Phi'(x)\delta x = \delta\Phi, \qquad 0 \le t \le T, \qquad \delta x\big|_{t=0} = D_t\,\delta x\big|_{t=0} = 0, \tag{15.1}$$

but the norms in Theorem 13.1 are in terms of $W^a = \delta x^i \partial y^a/\partial x^i$ and $F^a = \delta\Phi^i \partial y^a/\partial x^i$ and we like to see our operator as an operator on δx. Using interpolation and Theorem 13.1 we get

$$\begin{aligned}
&\|\delta\ddot{x}\|_r + \|\delta\dot{x}\|_r + \|\delta x\|_r \\
&\quad\le K_2(\|\ddot{W}\|_r + \|\dot{W}\|_r + \|W\|_r) \\
&\qquad + K_2(\|\ddot{x}\|_{r+1} + \|\dot{x}\|_{r+1} + \|x\|_{r+1})(\|\ddot{W}\| + \|\dot{W}\| + \|W\|) \\
&\quad\le K_4 \sup_{0\le\tau\le t} \|F(\tau,\cdot)\|_{r+1} + K_4 \sup_{0\le\tau\le t} (\|\ddot{x}(\tau,\cdot)\|_{r+4,\infty} \\
&\qquad + \|\dot{x}(\tau,\cdot)\|_{r+4,\infty} + \|x(\tau,\cdot)\|_{r+4,\infty}) \sup_{0\le\tau\le t} \|F(\tau,\cdot)\|_1 \\
&\quad\le K_4 \sup_{0\le\tau\le t} \|\delta\Phi(\tau,\cdot)\|_{r+1} + K_4 \sup_{0\le\tau\le t} (\|\ddot{x}(\tau,\cdot)\|_{r+4,\infty} \\
&\qquad + \|\dot{x}(\tau,\cdot)\|_{r+4,\infty} + \|x(\tau,\cdot)\|_{r+4,\infty}) \sup_{0\le\tau\le t} \|\delta\Phi(\tau,\cdot)\|_1.
\end{aligned} \tag{15.2}$$

Another issue is that we have L^2 estimates of δx but we need L^∞ estimates for x. The L^2 norm is bounded by the L^∞ norm and the L^∞ norm is, by Sobolev's lemma, bounded by the L^2 norm of an additional $n/2$ derivatives so one can obviously turn one into the other with an additional loss:

$$\|u(t,\cdot)\|_r \le c_r \|u(t,\cdot)\|_{r,\infty} \le C_r \|u(t,\cdot)\|_{r+r_0}, \qquad r_0 = [n/2]+1. \tag{15.3}$$

Furthermore, the Nash-Moser theorems to follow (§18) are in terms of Hölder spaces, but one can obviously also turn Hölder norms into L^∞ norms with a loss of an additional derivative:

$$C_k^{-1}\|u(t,\cdot)\|_{k,\infty} \le \|u(t,\cdot)\|_{a,\infty} \le C_k \|u(t,\cdot)\|_{k+1,\infty}, \qquad k \le a \le k+1 \tag{15.4}$$

where $\|u(t,\cdot)\|_{a,\infty}$ denotes the Hölder norms in Section 17. Let us now introduce the norms

$$|||u|||_{a,k} = |||u|||_a + \cdots + |||D_t^k u|||_a,, \qquad \text{where} \quad |||u|||_a = \sup_{0\le t\le T} \|u(t,\cdot)\|_{a,\infty}. \tag{15.5}$$

It follows that if (2.7) and (2.8) hold then (15.1) has a solution that satisfies

$$|||\delta x|||_{a,2} \le K_4\big(|||\delta\Phi|||_{a+r_0+2} + |||\delta\Phi|||_1\, |||x|||_{a+r_0+6,2}\big), \qquad a \ge 0. \tag{15.6}$$

We in fact want to solve for u in (2.13):

$$\tilde{\Phi}(u) = \Phi(u + x_0) - \Phi(x_0) = f_\delta. \tag{15.7}$$

Then $\tilde{\Phi}'(u) = \Phi'(u + x_0)$ and the norm of x in (15.6) may be replaced by the norm of $u = x - x_0$ since

$$|||x|||_{a,2} \le |||x - x_0|||_{a,2} + |||x_0|||_{a,2} \le |||x - x_0|||_{a,2} + C_a \tag{15.8}$$

for some constant C_a depending on x_0. Hence we have proved:

Proposition 15.1. *Suppose that x is smooth for $0 \le t \le T$ and that the conditions in Lemma 14.2 hold. Then if $\delta\Phi$ is smooth for $0 \le t \le T$, (15.1) has a smooth solution δx. Furthermore there are constants C_a, depending on the approximate solution x_0, on (c_0, c_1) in (2.7), (2.8) and on a, such that*

$$|||\delta x|||_{a,2} \le C_a\big(|||\delta\Phi|||_{a+r_0+2} + |||x - x_0|||_{a+r_0+6,2}\|\delta\Phi|||_1\big), \qquad a \ge 0, \tag{15.9}$$

provided that

$$|||x - x_0|||_{4,2} \le 1. \tag{15.10}$$

16. Regularity properties of the Euler map and tame estimates for the second variational derivative

Recall that the Euler map is given by

$$\Phi(x)_i = D_t^2 x_i + \partial_i p, \quad \text{in} \quad [0,T]\times\Omega, \quad \text{where} \quad \partial_i = \frac{\partial y^a}{\partial x^i}\partial_a \tag{16.1}$$

and where $p = \Psi(x)$ is solved by setting

$$\triangle p = -(\partial_i V^k)\partial_k V^i, \qquad V^i = D_t\, x^i, \qquad p\big|_{\partial\Omega} = 0. \tag{16.2}$$

We will now discuss the regularity properties of Φ needed and the definition of derivatives of Φ: Let $\mathcal{F} = C^\infty\big([0,T]\times\overline{\Omega}\big)$, $\mathcal{F}_M = \{x \in \mathcal{F};\, |\partial x/\partial y| + |\partial y/\partial x| < M\}$ and let $I_k = I\times\cdots\times I$ be k copies of $I = [-\varepsilon,\varepsilon]$, $\varepsilon > 0$. Suppose that $\overline{x} \in C^m(I_k, \mathcal{F}_M)$, $m \ge k$; then we claim that $\Phi(\overline{x}) \in C^m(I_k, \mathcal{F})$. In fact, by the proof of Lemma 7.3, $\overline{p} = \Psi(\overline{x}) \in C^m(I_k, \mathcal{F})$, since there, $t \in R$ was just any parameter and we can replace it by $t \in \mathbf{R}^k$ and replace the derivatives with respect to t by partial derivatives.

Definition 16.1. Suppose that $x \in \mathcal{F} = C^\infty\big([0,T]\times\overline{\Omega}\big)$ and $w_j \in \mathcal{F}$, for $j \le k$. Set $\overline{x} = x + r_1 w_1 + \cdots + r_k w_k$ and suppose that $\Phi(\overline{x})$ is a C^k function of $(r_1,\dots,r_k)$ close to $(0,\dots,0)$ with values in $\mathcal{F}$. We define the k^{th} (directional) derivative of Φ at the point x in the directions w_i, $i = 1,..,k$ by

(16.3)

$$\Phi^{(k)}(x)(w_1,\dots,w_k) = \frac{\partial}{\partial r_1}\cdots\frac{\partial}{\partial r_k}\Phi(\overline{x})\big|_{r_1=\cdots=r_k=0}, \quad \overline{x} = x + r_1 w_1 + \cdots + r_k w_k.$$

We say that $\Phi(x)$ is k times differentiable at x if $\Phi(\overline{x})$ is a C^k function of $(r_1,\dots,r_k)$ close to $(0,\dots,0)$ with values in $\mathcal{F}$, and if $\Phi^{(j)}(x)(w_1,\dots,w_j)$ is linear in each of the arguments $w_1,\dots,w_j$, for $j\le k$.

It is clear that (16.3) is independent of the order of differentiation, but to conclude that it is multi linear in $w_1,\dots,w_k$ one also needs to assume that it is continuous as a functional of $x,w_1,\dots w_k$; see [Ha]. We instead take (16.3) as the definition of the derivative and once we calculate it the linearity follows by inspection, in our case. We will assume that Φ is twice differentiable in which case it follows from the above definition that Taylor's formula with integral remainder of order two holds:

$$(16.4)\qquad \big(\Phi'(v)-\Phi'(u)\big)w=\int_0^1 \Phi''\big(u+s(v-u)\big)(v-u,w)\,ds,$$

$$(16.5)\qquad \Phi(v)-\Phi(u)-\Phi'(u)(v-u) \\ =\int_0^1(1-s)\Phi''\big(u+s(v-u)\big)(v-u,v-u)\,ds.$$

The Nash-Moser technique uses these remainder formulas together with tame estimates for the second variational derivative that we will now derive:

PROPOSITION 16.1. *Suppose that x is smooth for $0\le t\le T$ and that the conditions in Lemma 14.3 hold. Then Φ is twice differentiable and the second derivative satisfies the estimates*

$$(16.6)\qquad |||\Phi''(\delta x,\epsilon x)|||_a\le C_a\Big(|||\delta x|||_{a+4,1}|||\epsilon x|||_{1,1}+|||\delta x|||_{1,1}|||\epsilon x|||_{a+4,1} \\ +C_a|||x-x_0|||_{a+5,1}|||\delta x|||_{1,1}|||\epsilon x|||_{1,1}\Big)$$

provided that

$$(16.7)\qquad |||x-x_0|||_{4,2}\le 1.$$

Here the norms are as in (15.5).

Let us now calculate the second derivative of Φ and afterwards prove the tame estimates for it. Let us first recall the commutator identities:

LEMMA 16.2.

$$(16.8)\qquad [\delta,\partial_i]=-(\partial_i\delta x^k)\partial_k,$$

$$(16.9)\qquad [\delta,\partial_i\partial_j]=-(\partial_i\delta x^k)\partial_j\partial_k-(\partial_j\delta x^k)\partial_i\partial_k-(\partial_i\partial_j\delta x^k)\partial_k.$$

Furthermore

$$(16.10)\qquad [\delta,\triangle]=-(\triangle\delta x^k)\partial_k-2(\partial^i\delta x^j)\partial_i\partial_j$$

and if ε is another variation then

$$[\delta,[\epsilon,\triangle]] = \big((\triangle\delta x^l)\partial_l\epsilon x^k + (\partial_l\partial_m\delta x^k)\partial_l\epsilon x^m + (\triangle\epsilon x^l)\partial_l\delta x^k + (\partial_l\partial_m\epsilon x^k)\partial_l\delta x^m\big)\partial_k + 2\big((\partial^k\delta x^m)\partial_m\epsilon x^l + (\partial^k\epsilon x^m)\partial_m\delta x^l + (\partial^m\delta x^k)\partial_m\epsilon x^l\big)\partial_k\partial_l. \tag{16.11}$$

Proof. (16.8) was proved in Lemma 2.2 and (16.9) follows from this since $[\delta,\partial_i\partial_j] = [\delta,\partial_i]\partial_j + \partial_i[\delta,\partial_j]$. (16.10) follows from contracting (16.9). (16.11) follows from using (16.9) and (16.10) applied to δ as well as ϵ in place of δ. □

Let $\overline{x}(t,y,r) = x(t,y) + r\,\delta x(t,y)$. The first variational derivative $\Phi'(x)$ of the Euler map

$$\Phi'(x)\delta x_i = \delta\Phi(x)_i = \frac{\partial\Phi(\overline{x})_i}{\partial r}\Big|_{r=0} \tag{16.12}$$

is given by

Lemma 16.3.

$$\Phi'(x)\delta x_i = D_t^2\delta x_i - \partial_k p\;\partial_i\delta x^k + \partial_i\,p'(\delta x). \tag{16.13}$$

Here $\delta p = p'(\delta x) = \Psi'(x)\delta x$ satisfies

$$\triangle\delta p = \delta\triangle p + \partial_k p\,\triangle\delta x^k + 2(\partial_i\partial_k p)\partial^i\delta x^k, \qquad \textit{where} \tag{16.14}$$

$$\delta\triangle p = 2\partial_k V^i\,\partial_i\delta x^l\,\partial_l V^k - 2\partial_k V^i\,\partial_i\delta v^k \tag{16.15}$$

where $\delta v = D_t\delta x$ and $\delta p\big|_{\partial\Omega} = 0$.

Proof. This follows from a calculation using the fact that $\delta - \delta x^k\partial_k$ commutes with ∂_i and hence with $\triangle$ or by (16.9). □

Let $\overline{x}(t,y,r,s) = x(t,y) + r\delta x(t,y) + s\,\epsilon x(t,y)$. Then the second variational derivative is given by

$$\Phi''(x)(\delta x,\epsilon x)_i = \epsilon\delta\Phi(x)_i = \frac{\partial^2\Phi_i(\overline{x})}{\partial r\partial s}\Big|_{r=s=0}, \tag{16.16}$$

which in turn is given by:

Lemma 16.4. *Let $\delta v = D_t\delta x$ and $\epsilon v = D_t\epsilon x$. Then*

$$\Phi''(\delta x,\epsilon x)_i = \partial_k p\,\big(\partial_i\epsilon x^l\,\partial_l\delta x^k + \partial_i\delta x^l\,\partial_l\epsilon x^k\big) - \partial_k p'(\epsilon x)\;\partial_i\delta x^k - \partial_k p'(\delta x)\;\partial_i\epsilon x^k + \partial_i\,p''(\delta x,\epsilon x) \tag{16.17}$$

where $\delta p = p'(\delta x) = \Psi'(x)\delta x$ and $\delta\epsilon p = p''(\delta x,\epsilon x) = \Psi''(x)(\delta x,\epsilon x)$ satisfies

$$\triangle(\delta\epsilon p) = [\triangle,\delta\epsilon]p + \delta\epsilon\triangle p, \qquad [\triangle,\delta\epsilon]p = f_1 + 2f_2 - f_3 - 2f_4, \qquad \delta\epsilon\triangle p = -2f_5 + 2f_6 - 2f_7 \tag{16.18}$$

where:

$$
\begin{aligned}
f_1 &= (\triangle \delta x^i)(\partial_i \epsilon p) + (\triangle \epsilon x^i)(\partial_i \delta p),\\
f_2 &= (\partial_i \partial_j \delta p)(\partial_j \epsilon x^i) + (\partial_i \partial_j \epsilon p)(\partial_j \delta x^i),\\
f_3 &= \partial_j p\{(\partial_i \delta x^j)(\triangle \epsilon x^i) + (\partial_i \epsilon x^j)(\triangle \delta x^i)\\
&\quad +2(\partial_k \delta x^i)(\partial_k \partial_i \epsilon x^j) + 2(\partial_k \epsilon x^i)(\partial_k \partial_i \delta x^j)\},\\
f_4 &= \partial_i \partial_j p\{(\partial_k \delta x^j)(\partial_k \epsilon x^i) + (\partial_k \delta x^i)(\partial_j \epsilon x^k) + (\partial_k \epsilon x^i)(\partial_j \delta x^k)\},\\
f_5 &= (\partial_k v^l)(\partial_l v^j)\{(\partial_i \delta x^k)(\partial_j \epsilon x^i) + (\partial_i \epsilon x^k)(\partial_j \delta x^i)\}\\
&\quad +(\partial_i v^k)(\partial_j v^l)(\partial_k \delta x^j)(\partial_l \epsilon x^i),\\
f_6 &= (\partial_k v^j)\{(\partial_i \delta v^k)(\partial_j \epsilon x^i) + (\partial_i \epsilon v^k)(\partial_j \delta x^i)\\
&\quad +(\partial_j \delta v^i)(\partial_i \epsilon x^k) + (\partial_j \epsilon v^i)(\partial_i \delta x^k)\},\\
f_7 &= (\partial_i \delta v^j)(\partial_j \epsilon v^i),
\end{aligned}
$$

and $\delta\epsilon p\big|_{\partial\Omega} = 0.$

Proof. A calculation using the fact that $[\delta, \partial_i] = -(\partial_i \delta x^k)\partial_k$ and $\epsilon\delta x = 0$ gives (16.17). (16.18) follows from Lemma 16.2 and

$$\triangle \delta\epsilon p = [\delta, \triangle]\epsilon p + [\epsilon, \triangle]\delta p + \big[\delta, [\epsilon, \triangle]\big] + \delta\epsilon\triangle p. \tag{16.19}$$ □

The estimates for the first and second derivatives of $p = \Psi(x)$ are given in the following lemma:

LEMMA 16.5. *Let* $p = \Psi(x)$ *be the solution of* $\triangle p = -(\partial_i V^j)\partial_j V^i$, $p\big|_{\partial\Omega} = 0$, *where* $V = D_t x$. *Let* $\delta p = p'(\delta x) = \Psi'(x)\delta x$ *be the variational derivative. When* $D_t \delta x = \delta v$ *and* $D_t \epsilon x = \epsilon v$,

$$\|\delta p\|_{r,\infty} \le K_3\Big(\|\delta v\|_{r,\infty} + \|\delta x\|_{r+1,\infty} + \big(\|x\|_{r+2,\infty} + \|v\|_{r+1,\infty}\big)\big(\|\delta x\|_{1,\infty} + \|\delta v\|_{1,\infty}\big)\Big) \tag{16.20}$$

and when $p''(\delta x, \epsilon x) = \Psi''(x)(\delta x, \varepsilon x)$ *is the second variational derivative,*

$$
\begin{aligned}
&\|p''(\delta x, \epsilon x)\|_{r,\infty}\\
&\quad \le K_3(\|\delta v\|_{r+1,\infty} + \|\delta x\|_{r+2,\infty})(\|\epsilon x\|_{1,\infty} + \|\epsilon v\|_{1,\infty})\\
&\qquad +K_3(\|\epsilon v\|_{r+1,\infty} + \|\epsilon x\|_{r+2,\infty})(\|\delta x\|_{1,\infty} + \|\delta v\|_{1,\infty})\\
&\qquad +K_3(\|v\|_{r+2,\infty} + \|x\|_{r+3,\infty})(\|\epsilon x\|_{1,\infty} + \|\epsilon v\|_{1,\infty})(\|\delta x\|_{1,\infty} + \|\delta v\|_{1,\infty}).
\end{aligned}
\tag{16.21}
$$

Proof. The proof of (16.20) is similar to the estimate of a time derivative in the proof of Lemma 6.4. By Lemma 16.3, Lemma 6.3 and Lemma 6.4,

(16.22)
$$\begin{aligned}\|\triangle\delta p-\delta\triangle p\|_{r-1,\infty}&\le K_1\|\delta x\|_{r+1,\infty}\|p\|_{1,\infty}+K_1\|p\|_{r+1,\infty}\|\delta x\|_{1,\infty}\\&\quad+K_1\|x\|_{r+1,\infty}\|p\|_{1,\infty}\|\delta x\|_{1,\infty}\\&\le K_3\|\delta x\|_{r+1,\infty}+\big(\|v\|_{r+1,\infty}+\|x\|_{r+2,\infty}\big)\|\delta x\|_{1,\infty}\end{aligned}$$

and

$$\begin{aligned}\|\delta\triangle p\|_{r-1,\infty}&\le K_3\big(\|\delta x\|_{r,\infty}+\|\delta v\|_{r,\infty}\big)\\&\quad+K_3\big(\|v\|_{r,\infty}+\|x\|_{r,\infty}\big)\big(\|\delta x\|_{1,\infty}+\|\delta v\|_{1,\infty}\big),\end{aligned}\tag{16.23}$$

which proves (16.20). Similarly by Lemma 16.4, Lemma 6.3, Lemma 6.4 and (16.20),

$$\begin{aligned}\|[\triangle,\delta\epsilon]p\|_{r-1,\infty}&\le K_1\|\delta x\|_{r+1,\infty}\|\epsilon p\|_{1,\infty}+K_1\|\epsilon p\|_{r+1,\infty}\|\delta x\|_{1,\infty}\\&\quad+K_1\|x\|_{r+1,\infty}\|\epsilon p\|_{1,\infty}\|\delta x\|_{1,\infty}\\&\quad+K_1\|\delta p\|_{r+1,\infty}\|\epsilon x\|_{1,\infty}+K_1\|\epsilon x\|_{r+1,\infty}\|\delta p\|_{1,\infty}\\&\quad+K_1\|x\|_{r+1,\infty}\|\epsilon x\|_{1,\infty}\|\delta p\|_{1,\infty}\\&\quad+K_1\|\delta x\|_{r+1,\infty}\|\epsilon x\|_{1,\infty}+K_1\|\epsilon x\|_{r+1,\infty}\|\delta x\|_{1,\infty}\\&\quad+K_1(\|x\|_{r+1,\infty}+\|p\|_{r+1,\infty})\|\epsilon x\|_{1,\infty}\|\delta x\|_{1,\infty}\\&\le K_1\big(\|\epsilon x\|_{r+2,\infty}+\|\epsilon v\|_{r+1,\infty}\big)\|\delta x\|_{1,\infty}\\&\quad+K_1\big(\|\delta x\|_{r+2,\infty}+\|\delta v\|_{r+1,\infty}\big)\|\epsilon x\|_{1,\infty}\\&\quad+K_1(\|x\|_{r+3,\infty}+\|v\|_{r+2,\infty})\|\epsilon x\|_{1,\infty}\|\delta x\|_{1,\infty}\end{aligned}\tag{16.24}$$

and

(16.25)
$$\begin{aligned}&\|\delta\epsilon\triangle p\|_{r-1,\infty}\\&\quad\le K_3\big(\|\delta x\|_{r,\infty}+\|\delta v\|_{r,\infty}\big)\big(\|\epsilon x\|_{1,\infty}+\|\epsilon v\|_{1,\infty}\big)\\&\qquad+K_3\big(\|\epsilon x\|_{r,\infty}+\|\epsilon v\|_{r,\infty}\big)\big(\|\delta x\|_{1,\infty}+\|\delta v\|_{1,\infty}\big)\\&\qquad+K_3\big(\|v\|_{r,\infty}+\|x\|_{r,\infty}\big)\big(\|\delta x\|_{1,\infty}+\|\delta v\|_{1,\infty}\big)\big(\|\epsilon x\|_{1,\infty}+\|\epsilon x\|_{1,\infty}\big)\end{aligned}$$

which proves (16.21). □

From Lemma 16.1, Lemma 16.6, the fact that $\partial_i=(\partial y^a/\partial x^i)\partial/\partial y^a$ and interpolation we have, LEMMA 16.6.

(16.26)
$$\begin{aligned}\|\Phi''(\epsilon x,\delta x)_i\|_{r,\infty}\\ \le K_3(\|\delta v\|_{r+2,\infty}+\|\delta x\|_{r+3,\infty})(\|\epsilon x\|_{1,\infty}+\|\epsilon v\|_{1,\infty})\\ +K_3(\|\epsilon v\|_{r+2,\infty}+\|\epsilon x\|_{r+3,\infty})(\|\delta x\|_{1,\infty}+\|\delta v\|_{1,\infty})\\ +K_3(\|v\|_{r+3,\infty}+\|x\|_{r+4,\infty})(\|\epsilon x\|_{1,\infty}+\|\epsilon v\|_{1,\infty})(\|\delta x\|_{1,\infty}+\|\delta v\|_{1,\infty}).\end{aligned}$$

Finally, also using (15.8) we get Proposition 16.1.

17. The smoothing operators

We will work in Hölder spaces since the standard proof of the Nash-Moser theorem uses them. The Hölder norms for functions defined on a compact convex set B are given by, if $k < a \le k+1$, where $k \ge 0$ is an integer,

$$\|u\|_{a,\infty} = \|u\|_{H^a} = \sup_{x,y\in B} \sum_{|\alpha|=k} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{a-k}} + \sup_{x\in B} |u(x)| \tag{17.1}$$

and $\|u\|_{H^0} = \sup_{x\in B} |u(x)|$. Since we use the same notation for the C^k norms, $\|u\|_{k,\infty} = \|u\|_{C^k}$, we will distinguish these by simply using letters a, b, c, d, e, f etc. for the Hölder norms and $i, j, k, l, ..$ for the C^k norms. However, since a Lipschitz continuous function is differentiable almost everywhere and the norm of the derivative at these points is bounded by the Lipschitz constant, we conclude that for integer values this is the same as the L^∞ norm of $\partial^\alpha u$ for $|\alpha| \le k$, and furthermore, since all our functions are smooth it is the same as the supremum norm. Our tame estimates for the inverse of the linearized operator and the second variational derivative are only for C^k norms with integer exponents, with $B = \overline{\Omega}$. However, since $\|u\|_{k,\infty} \le C\|u\|_{a,\infty} \le C\|u\|_{k+1,\infty}$, if $k \le a \le k+1$, see (17.2), they also hold for noninteger values with a loss of one more derivative.

The Hölder norms satisfy

$$\|u\|_{a,\infty} \le C\|u\|_{b,\infty}, \qquad a \le b, \tag{17.2}$$

and they also satisfy the interpolation inequality

$$\|u\|_{c,\infty} \le C\|u\|_{a,\infty}^{\lambda} \|u\|_{b,\infty}^{1-\lambda} \tag{17.3}$$

where $a \le c \le b$, $0 \le \lambda \le 1$ and $\lambda a + (1-\lambda)b = c$.

We will use norms which consist of Hölder norms in space and supremum C^k norms only in time

$$|||u|||_{a,k} = \sup_{0\le t\le T} \big(\|u(t,\cdot)\|_{a,\infty} + \|D_t u(t,\cdot)\|_{a,\infty} + \cdots + \|D_t^k u(t,\cdot)\|_{a,\infty}\big). \tag{17.4}$$

For the Nash-Moser technique, apart from tame estimates one also needs smoothing operators S_θ that satisfy the properties below with respect to the Hölder norms, and in fact also with respect to the norms above since the smoothing operators will be invariant under time translation. We have:

PROPOSITION 17.1.

$$\|S_\theta u\|_{a,\infty} \le C\|u\|_{b,\infty}, \qquad a \le b \tag{17.5}$$

$$\|S_\theta u\|_{a,\infty} \le C\theta^{a-b}\|u\|_{b,\infty}, \qquad a \ge b \tag{17.6}$$

$$\|(I - S_\theta)u\|_{a,\infty} \le C\theta^{a-b}\|u\|_{b,\infty}, \qquad a \le b \tag{17.7}$$

$$\|(S_{2\theta} - S_\theta)u\|_{a,\infty} \le C\theta^{a-b}\|u\|_{b,\infty} \qquad a, b \ge 0 \tag{17.8}$$

where the constants C only depend on the dimension and an upper bound for a and b.

Moreover, these estimates hold with the norms replaced by the norms (17.4) *for fixed k.*

First we note that (17.8) follows from (17.6), when $a \ge b$ and from (17.7), when $a \le b$. (This alternatively follows from an additional property $\|d/d\theta S_\theta u\|_{a,\infty} \le C\theta^{a-b-1}\|u\|_{b,\infty}$, $a \ge 0$, that also holds.)

For compactly supported functions on $\mathbf{R}^n$ there are standard smoothing operators, see [H1], that satisfy the above properties (17.5)–(17.8), with respect to the norms defined in (17.1). However we have functions defined on the compact set $\overline{\Omega}$ that do not have compact support in Ω. Therefore we need to extend these functions to have compact support in some larger set, without increasing the Hölder norms more than with a multiplicative constant. There is a standard extension operator in [S] that turns out to have these properties; see Lemma 17.2 below. If $\tilde{S}_\theta$ is the standard smoothing operator mentioned above, that satisfies (17.5)–(17.8), we define our smoothing operator by

$$S_\theta u = \tilde{S}_\theta \tilde{u}\big|_{\Omega}, \qquad \text{where} \qquad \tilde{u} = \mathcal{E}xt(u). \tag{17.9}$$

Since $\tilde{S}_\theta$ satisfies (17.5)–(17.8) and since $\|\tilde{u}\|_{b,\infty} \le C\|u\|_{b,\infty}$, by Lemma 17.2, it follows that S_θ satisfies (17.5)–(17.8).

LEMMA 17.2. *There is a linear extension operator $\mathcal{E}xt$ such that $\mathcal{E}xt(f) = f$ in $\{y;\, |y| \le 1\}$,* supp $\mathcal{E}xt(f) \subset \{y;\, |y| \le 2\}$ *and*

$$\|\mathcal{E}xt(f)\|_{a,\infty} \le C\|f\|_{a,\infty} \tag{17.10}$$

where the norms on the left are Hölder norms in $\{y;\, |y| \le 2\}$ and the norms on the right are Hölder norms in $\{y;\, |y| \le 1\}$, and C is bounded when a is bounded.

Proof. We will introduce polar coordinates and for fixed angular variables ω extend a function defined for the radial variable $r \leq 1$ to $r \geq 1$. Away from the origin, the change of variables given by polar coordinates is a diffeomorphism and Hölder continuity is preserved under composition with a diffeomorphism κ:

$$\|f \circ \kappa\|_{a,\infty} \leq C_a \|f\|_{a,\infty}. \tag{17.11}$$

Therefore, let us first remove the origin by a partition of unity. Let $\chi_0 \in C_0^\infty(\mathbf{R})$ satisfy $\chi_0(|y|) = 1$, when $|y| \leq 1/2$ and $\chi_0(|y|) = 0$, when $|y| \geq 3/4$, and let $\chi_1 = 1 - \chi_0$. Furthermore, we multiply with another cutoff function so that the extension has compact support in $|y| \leq 2$. Let $\chi_2 \in C_0^\infty(\mathbf{R})$ satisfy $\chi_2(|y|) = 1$, when $|y| \leq 5/4$ and $\chi_2(y) = 0$, when $|y| \geq 3/2$. If $\mathcal{E}xt_1(f)$ is the extension operator in the radial variable, defined in (17.14) below, we now define the extension $\mathcal{E}xt(f)$ of f to be

$$\mathcal{E}xt(f) = \chi_2 \mathcal{E}xt_1(\chi_1 f) + \chi_0 f. \tag{17.12}$$

Hölder continuity in (r, ω) follows from Hölder continuity of $\mathcal{E}xt_1(f)$ in the radial variable and the linearity and invariance under rotations of $\mathcal{E}xt_1(f)$, using the triangle inequality. In fact if $f_\omega(r) = f(r, \omega)$ then $\partial_\omega^\alpha \mathcal{E}xt_1(f_\omega) = \mathcal{E}xt_1(f_\omega^\alpha)$, where $f_\omega^\alpha = \partial_\omega^\alpha f_\omega$ and if $j + |\alpha| = k < a \leq k+1$ then by (17.18) and (17.17)

$$\begin{aligned} &|\partial_r^j \mathcal{E}xt_1(f_\omega^\alpha)(r) - \partial_r^j \mathcal{E}xt_1(f_\sigma^\alpha)(\rho)| \\ &\quad\leq |\partial_r^j \mathcal{E}xt_1(f_\omega^\alpha)(r) - \partial_r^j \mathcal{E}xt_1(f_\omega^\alpha)(\rho)| \\ &\qquad + |\partial_r^j \mathcal{E}xt_1(f_\omega^\alpha - f_\sigma^\alpha)(\rho)| \\ &\quad\leq \sup_{r',\rho'} \frac{|\partial_r^j \partial_\omega^\alpha f(r', \omega) - \partial_r^j \partial_\omega^\alpha f(\rho', \omega)|}{|r' - \rho'|^{a-k}} \; |r - \rho|^{a-k} \\ &\qquad + \sup_{\rho} \sup_{\omega',\sigma'} \frac{|\partial_r^j \partial_\omega^\alpha f(\rho, \omega') - \partial_r^j \partial_\omega^\alpha f(\rho, \sigma')|}{|\omega' - \sigma'|^{a-k}} \; |\omega - \sigma|^{a-k}. \end{aligned} \tag{17.13}$$

It therefore remains to prove the estimates (17.17) and (17.18) for the extension in the radial variable only given by (17.14).

Suppose that $f(r)$ is a function defined for $r \leq 1$. We define the extension f by $\mathcal{E}xt_1(f)(r) = f(r)$, when $r \leq 1$, and

$$\mathcal{E}xt_1(f)(r) = \int_1^\infty f(r - 2\lambda(r-1))\, \psi_1(\lambda)\, d\lambda, \qquad r \geq 1, \tag{17.14}$$

where ψ_1 is a continuous function on $[1, \infty)$, such that

$$\begin{gathered} \int_1^\infty \psi_1(\lambda)\, d\lambda = 1, \qquad \int_1^\infty \lambda^k \psi_1(\lambda)\, d\lambda = 0, \quad k > 0, \\ |\psi_1(\lambda)| \leq C_N (1+\lambda)^{-N}, \quad N \geq 0. \end{gathered} \tag{17.15}$$

The existence of such a function was proved in [S] where the extension operator was also introduced. In [S] it was proved that this operator is continuous on the Sobolev spaces but it was not proved there that it is continuous on the Hölder spaces; so we must prove this. As pointed out above, we only need to prove that it is Hölder continuous with respect to the radial variable.

First we note that if $f \in C^k$ then the extension is in C^k. In fact

$$\partial_r^j \,\mathcal{E}xt_1(f)(r) = \int_1^\infty f^{(j)}(r - 2\lambda(r-1))(1-2\lambda)^j \,\psi_1(\lambda)\,d\lambda, \qquad r \geq 1. \tag{17.16}$$

From the continuity of $\partial_r^j f$ and (17.14), (17.15) it follows that

$$\lim_{r\to 1} \partial_r^j \,\mathcal{E}xt_1(f)(r) = \partial_r^j \,f(1),$$

that $\mathcal{E}xt_1(f)$ is in C^k, and that for k, an integer

$$\sup_r |\partial_r^k \mathcal{E}xt_1(f)(r)| \leq C_k \sup_r |f^{(k)}(r)|. \tag{17.17}$$

Suppose now that $k < a \leq k+1$ where k is an integer. We will prove that

$$\sup_{r,\rho} \frac{|\partial_r^k \,\mathcal{E}xt_1(f)(r) - \partial_r^k \,\mathcal{E}xt_1(f)(\rho)|}{|r-\rho|^{a-k}} \leq C_a \sup_{r,\rho} \frac{|f^{(k)}(r) - f^{(k)}(\rho)|}{|r-\rho|^{a-k}}. \tag{17.18}$$

If $r \leq 1$ and $\rho \leq 1$ there is nothing to prove. Also if $r < 1 < \rho$ or $\rho < 1 < r$, then $|r-\rho| \geq |1-\rho|$ and $|r-\rho| \geq |1-r|$ so in this case, we can reduce (17.18) to two estimates with either $r = 1$ or $\rho = 1$. Also it is symmetric in r and ρ so it only remains to prove the assertion when $r > \rho \geq 1$. Then we have

$$\begin{aligned} &\Big| \int_1^\infty \big(f^{(k)}(r - 2\lambda(r-1)) - f^{(k)}(\rho - 2\lambda(\rho-1))\big)(1-2\lambda)^k \,\psi_1(\lambda)\,d\lambda \Big| \\ &\qquad \leq \sup_{r',\rho'} \frac{|f^{(k)}(r') - f^{(k)}(\rho')|}{|r'-\rho'|^{a-k}} \; |r-\rho|^{a-k} \int_1^\infty |(1-2\lambda)^a \,\psi_1(\lambda)|\,d\lambda \end{aligned} \tag{17.19}$$

and using the last estimate in (17.15), (17.18) follows. □

18. The Nash-Moser iteration

At this point, given the results stated in Sections 11–14, the problem is now reduced to a completely standard application of the Nash-Moser technique. One can just follow the steps of the proofs of [AG], [H1], [H2], [K1] replacing their norms with our norms. The main difference is that we have a boundary, but we have constructed smoothing operators that satisfy the required properties for the case with a boundary. Furthermore, we avoid doing smoothing in the time direction; a similar approach was followed in [K2]. Alternatively, one could follow the approach of [Ha], where it is proved that C^∞ of a compact manifold with a boundary is also a tame space. Just one small detail is missing which is that the the set $[0,T]\times\overline{\Omega}$ is not smooth at $\{0\}\times\partial\Omega$, and again we get back to the situation were it is preferable just to do smoothing in the space directions only.

We will follow the formulation from [AG] which is similar to [H1], [H2]. The theorem there is stated in terms of Hölder norms, with a slightly different definition of the Hölder norms for integer values. However, the only properties that are used from the norms are the smoothing properties, (17.5)–(17.8) and the interpolation property (17.3) which we proved with the usual definition, i.e. the one used in [H1].

Let us also change notation and call $\tilde{\Phi}(u)$ in (2.13) $\Phi(u)$. Let

$$\||u\||_{a,k}=\sup_{0\le t\le T}\|u(t,\cdot)\|_{a,\infty}+\cdots+\|D_t^k u(t,\cdot)\|_{a,\infty},\qquad \||u\||_a=\||u\||_{a,0}, \tag{18.1}$$

where $\|u(t,\cdot)\|_a$ are the Hölder norms; see (17.1). Proposition 15.1 and Proposition 16.1 now say that the conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ below hold:

$(\mathcal{H}_1)$: Φ, is twice differentiable and satisfies

$$\begin{aligned}\||\Phi''(u)(v_1,v_2)\||_a\le C_a\Big(\||v_1\||_{a+\mu,2}\||v_2\||_{\mu,2}+\||v_1\||_{\mu,2}\||v_2\||_{a+\mu,2}\Big)\\ +C_a\||u\||_{a+\mu,2}\||v_1\||_{\mu,2}\||v_2\||_{\mu,2},\end{aligned} \tag{18.2}$$

where $\mu=5$, for $u,v_1,v_2\in C^\infty([0,T],C^\infty(\overline{\Omega}))$, if

$$\||u\||_{\mu,2}\le 1,\qquad \mu=5. \tag{18.3}$$

$(\mathcal{H}_2)$: If $u\in C^\infty([0,T],C^\infty(\overline{\Omega}))$ satisfies (18.3) then there is a linear map $\psi(u)$ from $C^\infty([0,T],C^\infty(\overline{\Omega}))$ to $C^\infty([0,T],C^\infty(\overline{\Omega}))$ such that $\Phi'(u)\psi(u)=\mathrm{Id}$ and

$$\||\psi(u)g\||_{a,2}\le C_a\big(\||g\||_{a+\lambda}+\||g\||_\lambda\,\||u\||_{a+d,2}\big), \tag{18.4}$$

where $\lambda=[n/2]+3$ and $d=[n/2]+7$.

Proposition 18.1. *Suppose that Φ satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $\Phi(0)=0$. Let $\alpha>\mu$, $\alpha>d$, $\alpha>\lambda+2\mu$, $\alpha\notin\mathbb{N}$. Then:*

There is neighborhood $W_\delta=\{f\in C^\infty([0,T],C^\infty(\overline{\Omega}));\ \||f\||_{\alpha+\lambda}\le\delta^2\}$, $\delta>0$, such that, for $f\in W_\delta$, the equation

$$\Phi(u)=f \tag{18.5}$$

has a solution $u=u(f)\in C^2([0,T],C^\infty(\overline{\Omega}))$. Furthermore,

$$\||u(f)\||_{a,2}\le C\||f\||_{\alpha+\lambda},\qquad a<\alpha. \tag{18.6}$$

In the proof, we construct a sequence $u_j \in C^\infty([0,T], C^\infty(\overline{\Omega}))$ converging to u, that satisfies $|||u_j|||_{\mu,2} \le 1$ and $|||S_i u_i|||_{\mu,2} \le 1$, for all j, where S_i is the smoothing operator in (18.7). The estimates (18.2) and (18.4) will only be used for convex combinations of these and hence within the domain (18.3) for which these estimates hold.

Following [H1], [H2], [AG], [K1], [K2] we set

(18.7)
$$u_{i+1} = u_i + \delta u_i, \;\; \delta u_i = \psi(S_i u_i) g_i, \;\; u_0 = 0, \;\; S_i = S_{\theta_i}, \;\; \theta_i = \theta_0 2^i, \;\; \theta_0 \ge 1.$$

The g_i are to be defined so that u_i formally converges to a solution and then

$$\begin{aligned}(18.8)\;\; \Phi(u_{i+1}) - \Phi(u_i) &= \Phi'(u_i)(u_{i+1} - u_i) + e_i'' = \Phi'(u_i)\psi(S_i u_i) g_i + e_i'' \\ &= (\Phi'(u_i) - \Phi'(S_i u_i))\psi(S_i u_i) g_i + g_i + e_i'' = e_i' + e_i'' + g_i,\end{aligned}$$

where

$$e_i' = (\Phi'(u_i) - \Phi'(S_i u_i))\delta u_i, \tag{18.9}$$
$$e_i'' = \Phi(u_{i+1}) - \Phi(u_i) - \Phi'(u_i)\delta u_i, \tag{18.10}$$
$$e_i = e_i' + e_i''. \tag{18.11}$$

Therefore

$$\Phi(u_{i+1}) - \Phi(u_i) = e_i + g_i \tag{18.12}$$

and adding, we get

$$\Phi(u_{i+1}) = \sum_{j=0}^{i} g_j + S_i E_i + e_i + (I - S_i)E_i, \qquad E_i = \sum_{j=0}^{i-1} e_j. \tag{18.13}$$

To ensure that $\Phi(u_i) \to f$ we must have

$$\sum_{j=0}^{i} g_j + S_i E_i = S_i f. \tag{18.14}$$

Thus

$$g_0 = S_0 f, \qquad g_i = (S_i - S_{i-1})(f - E_{i-1}) - S_i e_{i-1} \tag{18.15}$$

and

$$\Phi(u_i) = S_i f + e_i + (I - S_i)E_i. \tag{18.16}$$

Given $u_0, u_1, \dots, u_i$ these determine $\delta u_0, \delta u_1, \dots, \delta u_i$ which by (18.9), (18.10) determine $e_1, \dots, e_{i-1}$, which by (18.15) determine g_i. The new term u_{i+1} is determined by (18.7).

Lemma 18.2. *Assume that* $|||u_i|||_{\mu,2} \le 1$, $|||u_{i+1}|||_{\mu,2} \le 1$ *and* $|||S_i u_i|||_{\mu,2} \le 1$. *Then*

(18.17)
$$\begin{aligned}|||e_i'|||_a \le C_a\Big(&|||(I-S_i)u_i|||_{a+\mu,2}|||\delta u_i|||_{\mu,2} + |||(I-S_i)u_i|||_{\mu,2}|||\delta u_i|||_{a+\mu,2}\Big)\\ &+C_r|||S_i u_i|||_{a+\mu,2}|||(I-S_i)u_i|||_{\mu,2}|||\delta u_i|||_{\mu,2}\end{aligned}$$

and

$$|||e_i''|||_a \le C_r\Big(|||\delta u_i|||_{a+\mu,2}|||\delta u_i|||_{\mu,2} + |||u_i|||_{a+\mu,2}|||\delta u_i|||_{\mu,2}^2\Big). \tag{18.18}$$

Proof. The proof of (18.17) makes use of

(18.19)
$$(\Phi'(u_i) - \Phi'(S_i u_i))\delta u_i = \int_0^1 \Phi''(S_i u_i + s(I-S_i)u_i)(u_i - S_i u_i, \delta u_i)\, ds$$

together with (18.2). Note that from the third term in (18.2) we get a term that is not present in (18.17) since it can be bounded by the others using the assumptions. In fact, since $|||u_i|||_{\mu,2} + |||S_i u_i|||_{\mu,2} \le 2$,

$$|||(I-S_i)u_i|||_{a+\mu,2}|||(I-S_i)u_i|||_{\mu,2}|||\delta u_i|||_{\mu,2} \le 2|||(I-S_i)u_i|||_{a+\mu,2}|||\delta u_i|||_{\mu,2}.$$

(18.18) makes use of

$$\Phi(u_{i+1}) - \Phi(u_i) - \Phi'(u_i)\delta u_i = \int_0^1 (1-s)\Phi''(u_i + s\delta u_i)(\delta u_i, \delta u_i)\, ds \tag{18.20}$$

together with (18.2). Here we used the fact that $|||\delta u_i|||_{a+\mu,2}|||\delta u_i|||_{\mu,2}^2 \le 2|||\delta u_i|||_{a+\mu,2}|||\delta u_i|||_{\mu,2}$. □

Let $\tilde\alpha > \alpha$ and $\tilde\alpha - \mu > 2(\alpha-\mu)$. Throughout the proof C_a will stand for constants that depend on a but independent of n in (18.21).

Our inductive assumption (H_n) is,

$$|||\delta u_i|||_{a,2} \le \delta\theta_i^{a-\alpha}, \qquad 0 \le a \le \tilde\alpha, \qquad i \le n. \tag{18.21}$$

If $n = 0$ then if $a \le \tilde\alpha$, we have $|||\delta u_0|||_{a,2} \le C_{\tilde\alpha}|||f|||_{\alpha+\lambda} \le C_{\tilde\alpha}\delta^2$, so it follows that (18.21) holds for $n=0$ if we choose δ so small that $C_{\tilde\alpha}\delta \le \theta_0^{\tilde\alpha-\alpha}$. We must now prove that (H_n) implies (H_{n+1}) if $C_{\tilde\alpha}'\delta \le 1$, where $C_{\tilde\alpha}'$ is some constant that only depends on $\tilde\alpha$ but is independent of n.

Lemma 18.3. *If* (18.21) *holds then for* $i \le n$

(18.22)
$$\sum_{j=0}^{i} |||\delta u_j|||_{a,2} \le C_a\delta\big(\min(i, 1/|\alpha-a|)+1)(\theta_i^{a-\alpha}+1\big), \qquad 0 \le a \le \tilde\alpha.$$

Proof. Using (18.21) we get $\sum_{j=0}^{i} |||\delta u_j|||_{a,2} \le C_a\delta \sum_{j=0}^{i} 2^{j(a-\alpha)}$ and when $\sum_{j=0}^{i} 2^{-sj} \le C(\min(1+1/s, i)+1)$, if $s>0$, (18.22) follows. □

LEMMA 18.4. *If* (H_n), *i.e.* (18.21), *hold and* $\tilde\alpha > \alpha$, *then for* $i \le n+1$,

$$|||u_i|||_{a,2} \le C_a\delta(\min(i, 1/|\alpha-a|)+1)(\theta_i^{a-\alpha}+1),\ 0 \le a \le \tilde\alpha, \tag{18.23}$$

$$|||S_i u_i|||_{a,2} \le C_a\delta(\min(i, 1/|\alpha-a|)+1)(\theta_i^{a-\alpha}+1),\ a \ge 0, \tag{18.24}$$

$$|||(I-S_i)u_i|||_{a,2} \le C_a\delta\theta_i^{a-\alpha},\ 0 \le a \le \tilde\alpha. \tag{18.25}$$

Proof. The proof of (18.23) is just summing up the series $u_{i+1} = \sum_{j=0}^{i} \delta u_j$, using Lemma 18.3. (18.24) follows from (18.22) using (17.5) for $a \le \tilde\alpha$ and (17.6) with $b = \tilde\alpha$ for $a \ge \tilde\alpha$. (18.25) follows from (17.7) with $b = \tilde\alpha$ and (18.23) with $a = \tilde\alpha$. □

Having assumed that $\alpha > \mu$, we note that in particular,

$$|||u_i|||_{\mu,2} \le 1 \quad \text{and} \quad |||S_i u_i|||_{\mu,2} \le 1, \qquad \text{for} \quad i \le n+1 \quad \text{if} \quad C_\mu\delta \le 1. \tag{18.26}$$

As a consequence of Lemma 18.4 and Lemma 18.2 we get

LEMMA 18.5. *If* (H_n) *is satisfied and* $\alpha > \mu$, *then for* $i \le n$,

$$|||e_i'|||_a \le C_a\delta^2\theta_i^{a-2(\alpha-\mu)}, \qquad 0 \le a \le \tilde\alpha - \mu, \tag{18.27}$$

$$|||e_i''|||_a \le C_a\delta^2\theta_i^{a-2(\alpha-\mu)}, \qquad 0 \le a \le \tilde\alpha - \mu. \tag{18.28}$$

As a consequence of Lemma 18.5 and (17.8) we get

LEMMA 18.6. *If* (H_n) *is satisfied*, *then for* $i \le n+1$,

$$|||\, S_i e_{i-1}|||_a \le C_a\delta^2\theta_i^{a-2(\alpha-\mu)}, \qquad a \ge 0, \tag{18.29}$$

$$|||(S_i - S_{i-1})f|||_a \le C_a\theta_i^{a-\beta}|||f|||_\beta, \qquad a \ge 0, \tag{18.30}$$

$$|||(I-S_i)f|||_a \le C_a\theta_i^{a-\beta}|||f|||_\beta, \qquad 0 \le a \le \beta. \tag{18.31}$$

Furthermore, *if* $\tilde\alpha - \mu > 2(\alpha-\mu)$:

$$|||(S_i - S_{i-1})E_{i-1}|||_a \le C_a\delta^2\theta_i^{a-2(\alpha-\mu)}, \qquad a \ge 0 \tag{18.32}$$

$$|||(I-S_i)E_i|||_a \le C_a\delta^2\theta_i^{a-2(\alpha-\mu)}, \qquad 0 \le a \le \tilde\alpha - \mu. \tag{18.33}$$

Proof. (18.29) follows from (18.27). For $a \le \tilde\alpha - \mu$ we use (17.5) with $b = a$ and for $a \ge \tilde\alpha - \mu$, we use (17.6) with $b = \tilde\alpha - \mu$. (18.30) follows from (17.8) and (18.31) follows from (17.7). Now, $E_i = \sum_{j=0}^{i-1} e_j$ and so by Lemma 18.5 $|||E_i|||_{\tilde\alpha-\mu} \le C_a\delta^2 \sum_{j=0}^{i-1} \theta_j^{\tilde\alpha-\mu-2(\alpha-\mu)} \le C_a'\delta^2\theta_i^{\tilde\alpha-\mu-2(\alpha-\mu)}$, since we assumed that the exponent is positive. (18.32) follows from this and (17.8) with $b = \tilde\alpha - \mu$ and similarly (18.33) follows from (17.7) with $b = \tilde\alpha - \mu$. □

Now, we have

Lemma 18.7. *If* (H_n) *is satisfied,* $\tilde{\alpha}-\mu > 2(\alpha-\mu)$*, and* $\alpha>\mu$ *then for* $i \le n+1$,

$$|||g_i|||_a \le C_a \delta^2 \theta_i^{a-2(\alpha-\mu)} + C_a \theta_i^{a-\beta} |||f|||_\beta, \qquad a \ge 0. \tag{18.34}$$

Using this lemma and (18.4) we get

Lemma 18.8. *If* (H_n) *holds,* $\tilde{\alpha}-\mu > 2(\alpha-\mu)$, $\alpha>\mu$, $\alpha>d$ *then, for* $i \le n+1$,

$$|||\delta u_i|||_{a,2} \le C_a \delta^2 \theta_i^{a+\lambda-2(\alpha-\mu)} + C_a |||f|||_\beta \theta_i^{a+\lambda-\beta}, \qquad a \ge 0. \tag{18.35}$$

Proof. Using (18.7), (18.4), (18.34) and (18.24) we get

$$\begin{aligned} |||\delta u_i|||_{a,2} \le\, & C_a\big(\delta^2\theta_i^{a+\lambda-2(\alpha-\mu)} + |||f|||_\beta \theta_i^{a+\lambda-\beta}\big) \\ & + C_a\big(\delta^2\theta_i^{\lambda-2(\alpha-\mu)} + |||f|||_\beta \theta_i^{\lambda-\beta}\big) \\ & \cdot \delta(\min(i, 1/|\alpha-a-d|)+1)(\theta_i^{a+d-\alpha}+1). \end{aligned} \tag{18.36}$$

The lemma follows from the fact that

$$\min(i, 1/|\alpha-a-d|) + 1 \le C\theta_i^a/(\theta_i^{a+d-\alpha}+1),$$

where C is a constant depending on $\alpha-d>0$ but independent of i. □

If, we now pick $\beta=\alpha+\lambda$, and use the assumptions that $\lambda+\alpha < 2(\alpha-\mu)$, and $|||f|||_{\alpha+\lambda} \le \delta^2$, we get that for $i \le n+1$,

$$|||\delta u_i|||_{a,2} \le C_a \delta^2 \theta_i^{a-\alpha}, \qquad a \ge 0. \tag{18.37}$$

If we pick $\delta > 0$ so small that

$$C_{\tilde{\alpha}}\delta \le 1, \tag{18.38}$$

the assumption (H_{n+1}) is proven.

The convergence of the u_i is an immediate consequence of Lemma 18.2:

$$\sum_{i=0}^{\infty} |||u_{i+1}-u_i|||_{a,2} \le C_a\delta, \qquad a<\alpha. \tag{18.39}$$

It follows from Lemma 18.6 that

$$|||\Phi(u_i) - f|||_a \le C_a\delta^2\theta_i^{a-\alpha-\lambda} \tag{18.40}$$

which tends to 0, as $i\to\infty$, if $a < \alpha+\lambda$.

It remains to prove $u \in C^2([0,T], C^\infty(\overline{\Omega}))$. Note that in Lemma 18.8 we proved a better estimate than (H_n). In fact if we let $\gamma = 2(\alpha-\mu)-(\alpha+\lambda) > 0$ and $\alpha' = \alpha+\gamma$, then $|||f|||_{\alpha'+\lambda} \le C$ implies that

$$|||\delta u_i|||_{a,2} \le C_a \theta_i^{a-\alpha'}, \qquad a \ge 0. \tag{18.41}$$

Using this new estimate, in place of (H_n), we can take Lemma 18.3 to Lemma 18.8 and replace α by α' and δ by 1. Then it follows from Lemma 18.8 that

$$\||\delta u_i\||_{a,2} \le C_a \theta_i^{a+\lambda-2(\alpha'-\mu)} + C_a \theta_i^{a+\lambda-\beta} \||f\||_\beta \tag{18.42}$$

and if we now pick $\gamma' = 2(\alpha'-\mu) - (\lambda-\alpha') = 2\gamma$ and $\alpha'' = \alpha' + \gamma' = \alpha + 2\gamma$, and use the fact that $\||f\||_{\alpha'+\gamma'} \le C$, we see that

$$\||\delta u_i\||_{a,2} \le C_a \theta_i^{a-\alpha''}, \qquad a \ge 0. \tag{18.43}$$

Since the gain $\gamma > 0$ is constant, repeating this process yields that (18.41) holds for any α' and hence that (18.39), (18.40) hold for any $\alpha \ge 0$ (with δ replaced by 1). It follows that u_j is a Cauchy sequence in $C^2\big([0,T], C^k(\overline{\Omega})\big)$, for any k, and hence that $u_j \to u \in C^2\big([0,T], C^\infty(\overline{\Omega})\big)$ and $\Phi(u_j) \to f \in C\big([0,T], C^\infty(\overline{\Omega})\big)$. (18.6) follows from (18.37) with $\delta^2 = \||f\||_{\alpha+\lambda}$. This concludes the proof of Proposition 18.1.

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