

Stable ergodicity of certain linear automorphisms of the torus

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Abstract

We find a class of ergodic linear automorphisms of \mathbb{T}^N that are stably ergodic. This class includes all non-Anosov ergodic automorphisms when $N = 4$. As a corollary, we obtain the fact that all ergodic linear automorphism of \mathbb{T}^N are stably ergodic when $N \leq 5$.

1. Introduction

The purpose of this paper is to give sufficient conditions for a linear automorphism on the torus to be stably ergodic. By stable ergodicity we mean that any small perturbation remains ergodic. So, let a linear automorphism on the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ be generated by a matrix $A \in \mathrm{SL}(N, \mathbb{Z})$ in the canonical way. We shall denote also by A the induced linear automorphism. It is known after Halmos [Ha] that A is ergodic if and only if no root of unity is an eigenvalue of A . However, it was Anosov [An] who provided the first examples of stably ergodic linear automorphisms. Indeed, the so-called Anosov diffeomorphisms (of which hyperbolic linear automorphisms are a particular case) are both ergodic and \mathcal{C}^1 -open which gives rise to their stable ergodicity.

Circa 1969, Pugh and Shub began studying stable ergodicity of diffeomorphisms. They wondered, for instance, whether

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

was stably ergodic in \mathbb{T}^4 . More generally, Hirsh, Pugh and Shub posed in [HPS] the following question:

Question 1. Is every ergodic linear automorphism of \mathbb{T}^N stably ergodic?

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This paper gives a positive answer to this question under some restrictions. Let us introduce some notation to be more precise. We shall call A *pseudo-Anosov* if it verifies the following conditions: A is ergodic, its characteristic polynomial p_A is irreducible over the integers, and p_A cannot be written as a polynomial in t^n for any $n \geq 2$. There is a reason for calling such an A a pseudo-Anosov linear automorphism. Indeed, if h is a homeomorphism of a surface S , then it induces an action h_* over the first homology group of S , $H_1(S, \mathbb{Z})$. Since $H_1(S, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$, where g is the genus of S , we can consider h_* as inducing a linear automorphism A_h on \mathbb{T}^{2g} in the canonical way. But if A_h is a pseudo-Anosov linear automorphism on \mathbb{T}^{2g} , then h is isotopic to a pseudo-Anosov homeomorphism of S (see for instance [CB]).

We shall denote by E^c the eigenspace corresponding to the eigenvalues of modulus one, and call it the *center space*. We obtain the following results:

THEOREM 1.1. *All pseudo-Anosov linear automorphisms $A : \mathbb{T}^N \rightarrow \mathbb{T}^N$ such that $\dim E^c = 2$ are \mathcal{C}^5 -stably ergodic if $N \geq 6$.*

THEOREM 1.2. *All pseudo-Anosov linear automorphisms $A : \mathbb{T}^4 \rightarrow \mathbb{T}^4$ are \mathcal{C}^{22} -stably ergodic.*

Moreover, as we shall have after Corollaries A.7 and A.5 of Appendix A, all ergodic A acting on \mathbb{T}^4 are either Anosov or pseudo-Anosov and all ergodic A acting on \mathbb{T}^5 are Anosov. Hence, we get as a corollary:

THEOREM 1.3. *All ergodic linear automorphism of \mathbb{T}^N are stably ergodic for $N \leq 5$.*

In this way, we solve Question 1 about stable ergodicity on \mathbb{T}^N for $N \leq 5$. We wonder if, in fact, the assumption about differentiability could be reduced. There are clues indicating this is a reasonable result to expect. One of them is, for instance, that what we obtain with our hypothesis is far stronger than ergodicity. On the other hand, we may find analogies with the case of diffeomorphisms with irrational rotation number of the circle, where a \mathcal{C}^2 hypothesis implies ergodicity with respect to Lebesgue measure [He].

The same remark about the assumption of differentiability holds for $N \geq 6$. We believe that techniques in this paper could be used to show Theorem 1.1 holds even when dropping the assumption that A is pseudo-Anosov. Moreover, though maybe requiring tools in the spirit of [RH] and [Vi], Theorem 1.1 might follow equally well from the less restrictive assumption of $A|_{E^c}$ being an isometry. We point out that Shub and Wilkinson have proved, under this assumption, that any ergodic linear automorphism is approximated by a stably ergodic diffeomorphism [SW].

As a final remark, observe that Theorem 1.1 makes sense only for N even, since N odd implies all pseudo-Anosov linear automorphisms are Anosov (see Corollary A.4 of Appendix A). Observe also that there exist matrices in the hypothesis of Theorem 1.1 for any even $N \geq 6$ (see Proposition A.8 of Appendix A).

The following theorem will be our starting point:

THEOREM A ([PS]). *If $f \in \text{Diff}_m^2(M)$ is a center bunched, partially hyperbolic, dynamically coherent diffeomorphism with the essential accessibility property then f is ergodic.*

What we shall see is that for A in the hypotheses of Theorems 1.1 and 1.2 there exists a \mathcal{C}^r neighborhood of diffeomorphisms verifying conditions of Theorem A. But before getting deeper into the sketch of the proof we shall briefly explain the meaning of these conditions.

A *partially hyperbolic* diffeomorphism f is one that admits a Df -invariant decomposition of the tangent bundle $TM = E_f^s \oplus E_f^c \oplus E_f^u$, such that $Df|_{E_f^s}$ and $Df^{-1}|_{E_f^u}$ are contractions and moreover they contract more sharply than Df on the center bundle E_f^c . This is a \mathcal{C}^1 open condition. Now as any ergodic linear automorphism is partially hyperbolic (see [Pa]), there will be a \mathcal{C}^1 neighborhood of A consisting of partially hyperbolic diffeomorphisms.

A partially hyperbolic diffeomorphism f is said to be *center bunched* if it satisfies a rather technical condition, which states basically that the behavior of Df along the center bundle is almost an isometry compared with the rate of expansion and contraction of the other spaces. Again this is a \mathcal{C}^1 open condition and as the center bundle of an ergodic linear automorphism is the center space, it follows that any ergodic linear automorphism is center bunched and so are its perturbations.

The *dynamic coherence* condition deals with the integrability of the center bundle. It is not *a priori* an open condition. However, it becomes an open condition if, for instance, the center bundle is tangent to a \mathcal{C}^1 foliation ([HPS, Ths. 7.1, 7.2]). This is the case of the ergodic linear automorphisms, where the center, if not trivial, is tangent to the foliation by planes parallel to the center space.

So we are left to check the essential accessibility property, which is, in fact, the main task in this paper. Let us introduce its definition. Consider a partially hyperbolic diffeomorphism f and let $\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u$ be the invariant foliations tangent to E_f^s, E_f^u respectively. We shall say that a point $\tilde{y} \in \mathbb{T}^N$ is su-accessible from $\tilde{x} \in \mathbb{T}^N$ if there exists a path $\gamma : I \rightarrow \mathbb{T}^N$, from now on an *su-path*, piecewise contained in s- or u-leaves. This defines an equivalence relation on \mathbb{T}^N . We shall say f verifies the *accessibility property* if the torus itself is an su-class. More generally, we say that f has the *essential accessibility property* if each su-saturated set in \mathbb{T}^N has either null or full Lebesgue measure.

We observe that an ergodic linear automorphism A has the accessibility property if and only if A is Anosov. So A does not have the accessibility property, but it has the essential accessibility property. We mention that the linear automorphisms we deal with, are the first examples of partially hyperbolic, stably ergodic systems not having the accessibility property. However, there are stably ergodic systems that are not partially hyperbolic. We can mention the example in [BV] where there is a dominated splitting with an expanding invariant bundle; or even the example in [Ta] where there is no hyperbolic invariant subbundle at all. We must point out that these examples are nonuniformly hyperbolic and moreover, they display some kind of accessibility.

To prove the essential accessibility property, we first prove that the partition by accessibility classes is essentially minimal; that is, an open saturated set (satisfying some extra condition) is either the empty set or the whole space. Then we show that each accessibility class is essentially a manifold and that the dimension of the accessibility classes depends semicontinuously. Using this we show that either there is only one accessibility class, and hence f has the accessibility property, or else the partition into accessibility classes is in fact a foliation. In this case we use KAM theory to prove that this foliation is *smoothly* conjugated to the corresponding foliation for A , the linear automorphism. As the foliation for A is ergodic (see [Pa]), we get the essential accessibility property. We get also, as a corollary, that in case there is not accessibility, the perturbation must be topologically conjugated to A .

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2. Preliminaries

We say that a diffeomorphism $f : M \rightarrow M$ is partially hyperbolic if there is a continuous Df -invariant splitting

$$TM = E_f^u \oplus E_f^c \oplus E_f^s$$

in which E_f^s and E_f^u are nontrivial bundles and

$$\begin{aligned} m(D^u f) &> \|D^c f\| \geq m(D^c f) > \|D^s f\|, \\ m(D^u f) &> 1 > \|D^s f\|, \end{aligned}$$

where $D^\sigma f$ is the restriction of Df to E_f^σ for $\sigma = s, c$ or u ,

$$\|D^\sigma f\| = \sup_{x,v \neq 0} \frac{|D_x^\sigma f(v)|}{|v|}$$

is the norm of this linear operator and $m(D^\sigma f)$ is the conorm of the linear operator; i.e.,

$$m(D^\sigma f) = \inf_{x,v \neq 0} \frac{|D_x^\sigma f(v)|}{|v|}.$$

For a partially hyperbolic diffeomorphism define $E_f^{cs} = E_f^c \oplus E_f^s$ and $E_f^{cu} = E_f^c \oplus E_f^u$. A partially hyperbolic diffeomorphism is dynamically coherent if the distributions E_f^c, E_f^{cs} and E_f^{cu} are all integrable, with the integral manifolds of E_f^{cs} and E_f^{cu} foliated, respectively, by the integral manifolds of E_f^c and E_f^s and by the integral manifolds of E_f^c and E_f^u . As observed in the introduction, any C^1 perturbation $f : \mathbb{T}^N \rightarrow \mathbb{T}^N$ of an ergodic linear automorphism A is partially hyperbolic, center bunched and dynamically coherent.

Let us recall some definitions and results: first of all the existence of the invariant foliations $\tilde{\mathcal{F}}^\sigma$ in \mathbb{T}^N tangent to the E_f^σ invariant bundles respectively for $\sigma = s, u, c, cs, cu$. The foliations are *a priori* only continuous but each leaf is as differentiable as f , and depends continuously with f . Also, as we shall work mostly in the universal covering of the torus, i.e. \mathbb{R}^N , let us denote by $p : \mathbb{R}^N \rightarrow \mathbb{T}^N$ the covering projection. We call \mathcal{F}^σ , $\sigma = s, u, c, cs, cu$, the lift of the corresponding invariant foliations of the torus to \mathbb{R}^N . Notice that each leaf of \mathcal{F}^σ is not the preimage by p of the corresponding leaf of $\tilde{\mathcal{F}}^\sigma$ in \mathbb{T}^N but only a connected component of this preimage. Call the leaf of \mathcal{F}^σ through the point x , $W^\sigma(x)$ for $\sigma = s, u, c, cs, cu$ and the leaf of $\tilde{\mathcal{F}}^\sigma$ through the point \tilde{x} , $\tilde{W}^\sigma(x)$.

We have defined the su-accessibility relation in the introduction, let us define the same relation in \mathbb{R}^N , that is, $y \in \mathbb{R}^N$ is su-accessible from $x \in \mathbb{R}^N$ if there exists a path $\gamma : I \rightarrow \mathbb{R}^N$, (an su-path), piecewise contained in s- or u-leaves. Let us call the accessibility class of a point x in \mathbb{R}^N by $C(x)$. Notice again that for a point $x \in \mathbb{R}^N$, $C(x)$ is the lift of the accessibility class of the corresponding point $\tilde{x} = p(x) \in \mathbb{T}^N$ and not the preimage of this accessibility class by the covering projection.

Also call $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ a lift of f assuming without loss of generality that $F(0) = 0$.

Call $E^\sigma = E_A^\sigma$, $\sigma = s, u, c, cs, cu$, and $E^{su} = E^s \oplus E^u$, the invariant spaces of A . The same methods of construction of the invariant foliations of [HPS] allow us to write (see Appendix B, Proposition B.1)

$$\begin{aligned} \gamma^s : \mathbb{R}^N \times E^s &\rightarrow E^{cu}, & \gamma^{cs} : \mathbb{R}^N \times E^{cs} &\rightarrow E^u, \\ \gamma^u : \mathbb{R}^N \times E^u &\rightarrow E^{cs}, & \gamma^{cu} : \mathbb{R}^N \times E^{cu} &\rightarrow E^s, \\ \gamma^c : \mathbb{R}^N \times E^c &\rightarrow E^{su}, \end{aligned}$$

such that if $\gamma^\sigma(x, \cdot) = \gamma_x^\sigma$, $\sigma = s, u, c, cs, cu$ then

$$W^\sigma(x) = x + \text{graph}(\gamma_x^\sigma) = \{x + v + \gamma_x^\sigma(v), v \in E^\sigma\},$$

$\gamma^\sigma(x + n, v) = \gamma^\sigma(x, v)$ and $\gamma^\sigma(x, 0) = 0$. Put in E^s, E^u, E^c some norm making $A|_{E^s}$ and $A^{-1}|_{E^u}$ contractions and $A|_{E^c}$ an isometry. Let us define for $v \in \mathbb{R}^N$, $|v| = |v^s| + |v^u| + |v^c|$ where $v = v^s + v^u + v^c$ with respect to $\mathbb{R}^N = E^s \oplus E^u \oplus E^c$. In the same way define for $v \in E^{cs}$, $|v| = |v^s| + |v^c|$ and the same for E^{cu} . It is not hard to verify (see Appendix B) the following:

LEMMA 2.1. *There exist $\kappa = \kappa(f)$ such that $\kappa(f) \rightarrow 0$ as $f \xrightarrow{C^1} A$ and $C > 0$ that only depends on the C^1 size of the neighborhood of A such that for $v \in E^\sigma$,*

- (1) $|\gamma_x^\sigma(v)| \leq C \log |v|$ for $\sigma = s, u, |v| \geq 2$,
- (2) $|\gamma_x^\sigma(v)| \leq \kappa$ for $\sigma = c, cs, cu$ for any v ,
- (3) $|(\gamma_x^u(v))^s| \leq \kappa$ for any v ,
- (4) $|(\gamma_x^s(v))^u| \leq \kappa$ for any v ,
- (5) $|\gamma_x^\sigma(v)| \leq \kappa|v|$ for $\sigma = s, u, c, cs, cu$ for any v .

We have another lemma which will be proved in Appendix B

LEMMA 2.2. *For any $x, y \in \mathbb{R}^N$,*

- (1) $\#W^s(x) \cap W^{cu}(y) = 1$,
- (2) $\#W^u(x) \cap W^{cs}(y) = 1$.

Define

$$\pi^s : \mathbb{R}^N \rightarrow W^{cu}(0), \quad \pi^s(x) = W^s(x) \cap W^{cu}(0),$$

$\pi^u : \mathbb{R}^N \rightarrow W^{cs}(0)$ in the same way and

$$\pi^{su} : \mathbb{R}^N \rightarrow W^c(0), \quad \pi^{su} = \pi^s \circ \pi^u.$$

Define also $j_x^\sigma : E^\sigma \rightarrow \mathbb{R}^N$, $j_x^\sigma(v) = x + v + \gamma_x^\sigma(v)$, $\sigma = s, u, c, cs, cu$ the parametrizations of the invariant manifolds.

On the other hand we have that if f is C^r and sufficiently C^1 near A then \mathcal{F}^s restricted to $W^{cs}(x)$ is a C^r foliation and the same holds for the \mathcal{F}^u foliation (see [PSW] and Appendix B). Moreover, given $C > 0$, if f is C^r close to A then the s and u holonomy maps between the center manifolds of points whose center manifolds are at distance less than C , whenever defined, are uniformly C^r close to the ones of A . More precisely:

LEMMA 2.3. *Given $C > 0$ and $\varepsilon > 0$ there is a neighborhood of A in the \mathcal{C}^r topology such that for any f in this neighborhood, x and y with $|x - y| \leq C$, $x \in W^{cu}(y)$,*

$$\begin{aligned} \pi_{xy}^u : W^c(x) &\rightarrow W^c(y), & \pi_{xy}^u(z) &= W^u(z) \cap W^c(y) \\ P_{xy}^u : E^c &\rightarrow E^c, & P_{xy}^u &= (j_y^c)^{-1} \circ \pi_{xy}^u \circ j_x^c \end{aligned}$$

and if

$$P_{xy}^u(z) = z + (x - y)^c + \varphi_{xy}^u(z)$$

then $\|\varphi_{xy}^u\|_{\mathcal{C}^r} < \varepsilon$ where the sup-norm in all derivatives of order less than or equal to r is used. The same holds for the s-holonomy; that is, given $x \in W^{cs}(y)$,

$$\begin{aligned} \pi_{xy}^s : W^c(x) &\rightarrow W^c(y), & \pi_{xy}^s(z) &= W^s(z) \cap W^c(y), \\ P_{xy}^s : E^c &\rightarrow E^c, & P_{xy}^s &= (j_y^c)^{-1} \circ \pi_{xy}^s \circ j_x^c \end{aligned}$$

and if

$$P_{xy}^s(z) = z + (x - y)^c + \varphi_{xy}^s(z)$$

then $\|\varphi_{xy}^s\|_{\mathcal{C}^r} < \varepsilon$.

Proof. See Appendix B. □

For $n \in \mathbb{Z}^N$ define

$$\begin{aligned} x_n &= W^u(n) \cap W^{cs}(0), & \pi_n^u : W^c(n) &\rightarrow W^c(x_n) \\ & & \pi_n^s : W^c(x_n) &\rightarrow W^c(0) \end{aligned}$$

as above and

$$T_n : E^c \rightarrow E^c, \quad T_n = (j_0^c)^{-1} \circ \pi_n^s \circ \pi_n^u \circ L_n \circ j_0^c$$

where $L_n : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $L_n(x) = x + n$. The T_n 's work as holonomies of the partition by accessibility classes; that is, if you take an su-path from $p(0)$ to $\tilde{W}^c(p(0))$ formed by two legs, the first unstable and the second stable such that closing the su-path with an arc inside $\tilde{W}^c(p(0))$ joining the final point of the su-path to $p(0)$ it is homotopic to the curve generated by $-n$, then T_n is just holonomy along this su-path. We make this choice of path, there is not a canonical choice of path, and any reasonable choice should work.

As a consequence of the preceding lemma we have

COROLLARY 2.4. *T_n is \mathcal{C}^r for all $n \in \mathbb{Z}^N$; moreover,*

$$T_n(z) = z + n^c + \varphi_n(z)$$

and for any $\varepsilon > 0$ and $R > 0$ there is a neighborhood of A in the \mathcal{C}^r topology such that if f is in this neighborhood, then $\|\varphi_n\|_{\mathcal{C}^r} < \varepsilon$ whenever $|n| \leq R$.

In the case the partition by accessibility classes is in fact a foliation, which means that the T_n 's commute (i.e. $T_n \circ T_m = T_m \circ T_n = T_{n+m}$), we shall use the linearization theorem of Arnold and Moser (see [He]) to get a smooth conjugacy of the T_n 's to the corresponding T_n^A 's of the linear automorphism. Then we shall build the smooth conjugacy of the foliation by accessibility classes of f to the one of A using this conjugacy.

Define for $x \in \mathbb{R}$, $\|x\| = \inf_{k \in \mathbb{Z}} |x + k|$. As usual, we say that $\alpha \in \mathbb{R}^c$ satisfies a diophantine condition with exponent β if $\|n \cdot \alpha\| \geq \frac{c}{|n|^{c+\beta}}$ for some $c > 0$ and for any $n \in \mathbb{Z}^c$, $n \neq 0$, where $x \cdot y$ denotes the standard inner product on \mathbb{R}^c and $|n| = \sum_{i=1}^c |n_i|$.

For $\alpha \in \mathbb{R}^c$, define $R_\alpha : \mathbb{R}^c \rightarrow \mathbb{R}^c$, $R_\alpha(x) = x + \alpha$. Also, denote $\mathcal{C}_b^r(\mathbb{R}^c, \mathbb{R}^c)$ by the set of \mathcal{C}^r bounded functions.

THEOREM 2.5 (KAM ([He, p. 203])). *Given $\beta > 0$, $\beta \notin \mathbb{Z}$, $\alpha \in \mathbb{R}^c$ satisfying a diophantine condition with exponent β and $\theta = c + \beta$, there is $V \subset \mathcal{C}_b^{2\theta}(\mathbb{R}^c, \mathbb{R}^c)$ a neighborhood of the 0 function such that given $\varphi \in V$ satisfying $\varphi(x + n) = \varphi(x)$ for $n \in \mathbb{Z}^c$, there exist $\lambda \in \mathbb{R}^c$ and $\eta \in \mathcal{C}_b^\theta(\mathbb{R}^c, \mathbb{R}^c)$ satisfying $\eta(x + n) = \eta(x)$ for any $n \in \mathbb{Z}^c$, $\eta(0) = 0$ and such that $h = \text{id} + \eta$, h is a diffeomorphism and $Q = R_\alpha + \varphi$ then $Q = R_\lambda \circ h^{-1} \circ R_\alpha \circ h$. Moreover given $\varepsilon > 0$ there is $\delta > 0$ such that if the $\mathcal{C}^{2\theta}$ size of φ is less than δ then the \mathcal{C}^θ size of η and the modulus of λ is less than ε .*

Let us list some properties of A .

LEMMA 2.6. *For any $n \in \mathbb{Z}^N$, $n \neq 0$, and $l \in \mathbb{Z}$, $l \neq 0$, $S = \{\sum_{i=0}^{N-1} k_i A^{il} n : k_i \in \mathbb{Z} \text{ for } i = 0, \dots, N-1\}$ is a subgroup of maximal rank.*

Proof. The proof follows easily from the fact that the characteristic polynomial of A^l is irreducible for any nonzero l . See Appendix A, Lemma A.9 for more details. \square

Moreover, we may suppose without loss of generality that A satisfies the following:

- (1) $Ae_i = e_{i+1}$ for $i = 1, \dots, N-1$,
- (2) $Ae_N = -\sum_{i=0}^{N-1} p_i e_{i+1}$, $P_A(z) = \sum_{i=0}^N p_i z^i$ the characteristic polynomial of A .

Indeed, taking $n \in \mathbb{Z}^N$, $n \neq 0$, defining $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ by $L(e_i) = A^{i-1}n$ for $i = 1, \dots, N$ and taking $B = L^{-1}AL$ we easily see that B induces a linear automorphism and satisfies the properties listed above. Besides, given f isotopic to A , we have its lift $F = A + \varphi$, where φ is \mathbb{Z}^N -periodic and we may work with $G = B + \hat{\varphi}$ where $\hat{\varphi} = L^{-1}\varphi \circ L$ is \mathbb{Z}^N -periodic, and ergodicity of G would imply ergodicity of f as is easily seen.

In this paper, C stands for a generic constant that only depends on the size of the neighborhood of A .

3. Holonomies

In this section we shall prove some properties about the holonomies needed in the following sections. We recommend that the reader omit this section in a first reading.

PROPOSITION 3.1. *There exists $C > 0$ only depending on the C^1 size of the neighborhood of A and $\beta = \beta(f)$ such that $\beta(f) \rightarrow 0$ as $f \xrightarrow{C^1} A$ and such that, given $x, y \in \mathbb{R}^N$, $x \in W^s(y)$, the following properties are satisfied for $\pi^s : W^c(x) \rightarrow W^c(y)$,*

- (1) *If $d^s(x, y) \leq 2$ then $\text{Lip}(\pi^s) \leq C$.*
- (2) *If $d^s(x, y) \geq 1$ then $\text{Lip}(\pi^s) \leq C(d^s(x, y))^\beta$.*

And the same properties hold if $x \in W^u(y)$ when u and s are interchanged.

Proof. The proof of (1) is a consequence of Lemma 2.3. Let us prove (2). Take $0 < \lambda < 1$ such that $|DF|_{E^s} < \lambda$ and $0 < \gamma = \gamma(f)$ such that $\exp(-\gamma) < |DF|_{E^c} < \exp(\gamma)$ and we may suppose that $\gamma(f) \rightarrow 0$ as $f \xrightarrow{C^1} A$. Let us take $n \geq 0$ the first integer that satisfies $d^s(F^n(y), F^n(x)) < 1$. Then we have that given $w, z \in W^c(x)$, $d^c(F^n(w), F^n(z)) \leq \exp(n\gamma)d^c(w, z)$. Now, using (1), we have that

$$d^c(\pi^s(F^n(w)), \pi^s(F^n(z))) \leq C d^c(F^n(w), F^n(z)),$$

and so

$$\begin{aligned} d^c(\pi^s(w), \pi^s(z)) &= d^c(F^{-n}(\pi^s(F^n(w))), F^{-n}(\pi^s(F^n(z)))) \\ &\leq \exp(n\gamma) d^c(\pi^s(F^n(w)), \pi^s(F^n(z))) \\ &\leq C \exp(n\gamma) d^c(F^n(w), F^n(z)) \\ &\leq C \exp(2n\gamma) d^c(w, z). \end{aligned}$$

Let us estimate n . By the definition of n we get that $n \leq \frac{\log d^s(y, x)}{-\log \lambda} + 1$ and so, calling $\beta = -\frac{2\gamma}{\log \lambda}$ we get

$$d^c(\pi^s(w), \pi^s(z)) \leq C \exp(2n\gamma) d^c(w, z) \leq C \exp(\gamma) (d^s(x, y))^\beta d^c(z, w)$$

which is the desired claim. \square

COROLLARY 3.2. *There exists $C > 0$ that only depends on the neighborhood of A such that for any $n \in \mathbb{Z}^N$*

- (1) *If $|n^s|, |n^u| \geq 2$ then $\text{Lip}(T_n) \leq C(|n^s||n^u|)^\beta$,*

- (2) If $|n^s| \leq 2$ and $|n^u| \geq 2$ then $\text{Lip}(T_n) \leq C|n^u|^\beta$,
- (3) If $|n^u| \leq 2$ and $|n^s| \geq 2$ then $\text{Lip}(T_n) \leq C|n^s|^\beta$,
- (4) If $|n^s|, |n^u| \leq 2$ then $\text{Lip}(T_n) \leq C$,

where β is as in Proposition 3.1.

Proof. We prove the first affirmation, the others follow by the same method. We have $x_n = W^u(n) \cap W^{cs}(0)$ and $y_n = W^s(x_n) \cap W^c(0)$. So, using Proposition 3.1, we only have to estimate $d^u(n, x_n)$ and $d^s(x_n, y_n)$. Now we have that $x_n = n + v^u + \gamma_0^u(v_u) = v^{cs} + \gamma_0^{cs}(v^{cs})$ and $y_n = x_n + v^s + \gamma_{x_n}^s(v_s) = v^c + \gamma_0^c(v^c)$. So, by Lemma 2.1, $|(x_n - n)^u| \leq |n^u| + \kappa$ and $|(x_n - y_n)^s| \leq |n^s| + 2\kappa$. The corollary follows from the fact that $\frac{1}{C}|(x - y)^\sigma| \leq d^\sigma(x, y) \leq C|(x - y)^\sigma|$ for $\sigma = s, u, c, cs, cu$ and some constant $C > 0$ that only depends on the \mathcal{C}^1 size of the neighborhood of A . \square

For $L > 0$ and $x \in \mathbb{R}^N$ define $W_L^\sigma(x) = j_x^\sigma(B_L^\sigma(0))$ for $\sigma = s, u, c, cs, cu$ where $j_x^\sigma : E^\sigma \rightarrow \mathbb{R}^N$, $j_x^\sigma(v) = x + v + \gamma_x^\sigma(v)$, $\sigma = s, u, c, cs, cu$ are the parametrizations of the invariant manifolds. Moreover, $W_L^\sigma(A) = \bigcup_{x \in A} W_L^\sigma(x)$. Given $S \subset \mathbb{Z}^N$, a subgroup of maximal rank, let us define $\mathbb{T}_S^N = \mathbb{R}^N/S$ the torus generated by the lattice S . Set $\nu(S) = \text{vol}(\mathbb{T}_S^N)$.

LEMMA 3.3. *There is $b > 0$ depending only on the size of the neighborhood of A such that if $L(\varepsilon) = \varepsilon^{-b}$ then, given $x \in \mathbb{R}^N$ and $S \subset \mathbb{Z}^N$ a subgroup of maximal rank, for $\varepsilon > 0$ small enough,*

$$W_\varepsilon^s(W_{L(\varepsilon)}^u(W_\varepsilon^c(x))) \cap (W_\varepsilon^s(W_{L(\varepsilon)}^u(W_\varepsilon^c(x))) + n) \neq \emptyset$$

for some $n \in S$, $n \neq 0$.

Proof. We only have to prove that there is some set $V \subset W_\varepsilon^s(W_{L(\varepsilon)}^u(W_\varepsilon^c(x)))$ such that $\text{vol}(V) > \nu(S)$. Call $W = W_\varepsilon^s(W_{L(\varepsilon)}^u(W_\varepsilon^c(x)))$. We have the following:

CLAIM 1. *There is a constant $C > 0$ depending only on the \mathcal{C}^1 distance of f to A such that for any $z \in W_{\frac{L(\varepsilon)}{2}}^u(x)$, with $\delta = CL(\varepsilon)^{-\beta}\varepsilon$, where β is as in Proposition 3.1, $B_\delta(z) \subset W$.*

Let us leave the proof of the claim until the end, and show how the lemma follows from this claim. Using the fact that $W_{\frac{L(\varepsilon)}{2}}^u(x) = j_x^u(B_{\frac{L(\varepsilon)}{2}}^u(x))$ we see easily that there are points $z_1, \dots, z_n \in W_{\frac{L(\varepsilon)}{2}}^u(x)$, $n \geq C(L(\varepsilon)\delta^{-1})^u$, where C is some constant that only depends on the \mathcal{C}^1 size of the neighborhood of A and $\dim E^u = u$ such that $W_\delta^u(z_i) \cap W_\delta^u(z_j) = \emptyset$ if $i \neq j$. Now we claim that

$B_{\frac{\delta}{3}}(z_i) \cap B_{\frac{\delta}{3}}(z_j) = \emptyset$ if $i \neq j$. To prove this, we have $z_i = z_j + a^u + \gamma_{z_j}^u(a^u)$ and hence

$$|j_{z_j}^{-1}(z_i) - j_{z_j}^{-1}(z_j)| = |a_u| = |(z_j - z_i)^u| \leq |z_i - z_j|;$$

so if the balls in \mathbb{R}^N intersect, then $W_\delta^u(z_i)$ and $W_\delta^u(z_j)$ must intersect contradicting the choice of the z_i . Call $V = \bigcup_{i=1}^n B_{\frac{\delta}{3}}(z_i) \subset W$. Let us estimate the volume of V . Call $\gamma = b(u - \beta(N - u)) - (N - u)$. If β is small enough which means if f is close enough to A and if b is big enough then we have that $\gamma > 0$, and for instance $b = \frac{N}{u}$, $\beta \leq \frac{u^2}{2N(N-u)}$ so that $\gamma \geq \frac{u}{2}$. Thus,

$$\text{vol}(V) = \sum_{i=1}^n \text{vol}(B_{\frac{\delta}{3}}(z_i)) \geq Cn\delta^N \geq C(L(\varepsilon)\delta^{-1})^u \delta^N = C\varepsilon^{-\gamma}.$$

If ε is small enough, as $\gamma > 0$, we get that $\text{vol}(V) > \nu(S)$. So we are left with the proof of the claim. Let us prove that for any $z \in W_{\frac{L(\varepsilon)}{2}}^u(x)$, and for any $y \in W_\varepsilon^c(x)$, $W_{\frac{2L(\varepsilon)}{3}}^u(y) \cap W^c(z) \neq \emptyset$. Call $w = W^u(y) \cap W^c(z) \neq \emptyset$ and let us show that $w \in W_{\frac{2L(\varepsilon)}{3}}^u(y)$. We have that

$$\begin{aligned} w &= y + b^u + \gamma_y^u(b^u) = z + h^c + \gamma_z^c(h^c), \\ z &= x + a^u + \gamma_x^u(a^u), \quad y = x + r^c + \gamma_x^c(r^c), \end{aligned}$$

and we have to estimate $|b^u|$. Now,

$$b^u = z^u - y^u + (\gamma_z^c(h^c))^u = a^u - (\gamma_x^c(r^c))^u + (\gamma_z^c(h^c))^u$$

and so $|b^u| \leq |a^u| + 2\kappa \leq \frac{2L(\varepsilon)}{3}$ if ε is small enough which gives us the intersection. Call $\pi_z^u : W^c(z) \rightarrow W^c(x)$ the unstable holonomy map. By Proposition 3.1 we have that $\text{Lip}(\pi_z^u) \leq CL(\varepsilon)^\beta$ and so calling $\delta_1 = \frac{1}{C}L(\varepsilon)^{-\beta}\varepsilon$, we get $\pi_z^u(W_{\delta_1}^c(z)) \subset W_\varepsilon^c(x)$ and hence that $W_{\delta_1}^c(z) \subset W_{\frac{2L(\varepsilon)}{3}}^u(W_\varepsilon^c(x))$. Take now y such that $|y - z| \leq c\delta_1$ for some positive c to be fixed, and define

$$w = W^s(y) \cap W^{cu}(z), \quad r' = W^u(w) \cap W^c(x), \quad r = W^u(w) \cap W^c(z).$$

So we want to prove that $y \in W_\varepsilon^s(w)$, $w \in W_{L(\varepsilon)}^u(r')$ and $r' \in W_\varepsilon^c(x)$. To this end, we use r and so, we prove that $r \in W_{\delta_1}^c(z)$ and that $d^u(w, r)$ is small enough so that $w \in W_{L(\varepsilon)}^u(r')$. Now,

$$y = w + a^s + \gamma_w^s(a^s), \quad w = z + b^{cu} + \gamma_z^{cu}(b^{cu})$$

and so

$$\begin{aligned} a^s &= y^s - z^s - \gamma_z^{cu}(b^{cu}), \\ b^{cu} &= (w - z)^{cu} = y^{cu} - z^{cu} - \gamma_w^s(a^s) \end{aligned}$$

and by Lemma 2.1

$$\begin{aligned} |a^s| &\leq |y^s - z^s| + \kappa|b^{cu}|, \\ |b^{cu}| &\leq |y^{cu} - z^{cu}| + \kappa|a^s|, \end{aligned}$$

which gives us

$$|a^s| + |b^{cu}| \leq \frac{1}{1-\kappa} |y-z|$$

and hence

$$\begin{aligned} |a^s| &\leq c_1 \delta_1, \\ |b^{cu}| &\leq c_1 \delta_1 \end{aligned}$$

where $c_1 = \frac{c}{1-\kappa}$. Thus, we get $y \in W_{c_1 \delta_1}^s(w) \subset W_\varepsilon^s(w)$ if ε is small enough. On the other hand,

$$r = z + g^c + \gamma_z^c(g^c) = w + h^u + \gamma_w^u(h^u)$$

so that

$$\begin{aligned} g^c &= w^c - z^c + (\gamma_w^u(h^u))^c = (b^{cu})^c + (\gamma_w^u(h^u))^c, \\ h^u &= z^u - w^u + (\gamma_z^c(g^c))^u = (b^{cu})^u + (\gamma_z^c(g^c))^u. \end{aligned}$$

Hence

$$\begin{aligned} |g^c| &\leq c_1 \delta_1 + \kappa |h^u|, \\ |h^u| &\leq c_1 \delta_1 + \kappa |g^c|, \end{aligned}$$

which gives us

$$|g^c| + |h^u| \leq \frac{c_1 \delta_1}{1-\kappa}$$

or

$$\begin{aligned} |g^c| &\leq \frac{c}{(1-\kappa)^2} \delta_1, \\ |h^u| &\leq \frac{c}{(1-\kappa)^2} \delta_1. \end{aligned}$$

So, taking c sufficiently small, we get $r \in W_{\delta_1}^c(z)$ and $r \in W_{\delta_1}^u(w) \subset W_\varepsilon^u(w)$. Finally, as $r \in W_{\delta_1}^c(z)$, we have $r' \in W_\varepsilon^c(x)$ and

$$r = r' + g^u + \gamma_{r'}^u(g^u).$$

Now, $|g^u| \leq \frac{2L(\varepsilon)}{3}$ and hence, as

$$w = r + t^u + \gamma_r^u(t^u) = r' + l^u + \gamma_{r'}^u(l^u),$$

we have $l^u = t^u + g^u$ and $t^u = -h^u$, and so

$$|l^u| \leq \varepsilon + \frac{2L(\varepsilon)}{3} < L(\varepsilon)$$

if ε is small enough. Thus, $w \in W_{L(\varepsilon)}^u(r')$. \square

COROLLARY 3.4. *Fix $\varepsilon > 0$ and n as in Lemma 3.3; then $|n^u| \leq 3L(\varepsilon)$, $|n^s| \leq 4\kappa$ and $|n^c| \leq C|n^{su}|$.*

Proof. It follows in the same spirit as the proof of Corollary 3.2. \square

We have another lemma.

LEMMA 3.5. *There is $C > 0$ that only depends on the C^1 size of the neighborhood of A such that given $x \in E^c$, $n \in \mathbb{Z}^N$,*

- (1) $|T_n(x) - (x + n^c)| \leq C \log(|n^s| |n^u|) + C$, if $|n^s|, |n^u| \geq 3$,
- (2) $|T_n(x) - (x + n^c)| \leq C \log |n^u| + C$, if $|n^u| \geq 3$ and $|n^s| \leq 3$,
- (3) $|T_n(x) - (x + n^c)| \leq C \log |n^s| + C$, if $|n^s| \geq 3$ and $|n^u| \leq 3$,
- (4) $|T_n(x) - (x + n^c)| \leq C$, if $|n^u|, |n^s| \leq 3$.

Proof. We prove the first one; the other follow in the same way. Fix $n \in \mathbb{Z}^n$, suppose $|n^s|, |n^u| \geq 3$; take $x^c \in E^c$ and $x = j_0^c(x^c)$, $y = W^u(x+n) \cap W^{cs}(0)$ and $z = W^s(y) \cap W^{cu}(0)$. Then we have that $T_n(x) = z^c$. Now

$$\begin{aligned} z - (x + n) &= y - (x + n) + v^s + \gamma_y^s(v^s) = -(x + n) + w^{cu} + \gamma_0^{cu}(w^{cu}), \\ y - (x + n) &= a^u + \gamma_x^u(a^u) = -(x + n) + b^{cs} + \gamma_0^{cs}(b^{cs}). \end{aligned}$$

Hence

$$\begin{aligned} (z - (x + n))^c &= (y - (x + n))^c + (\gamma_y^s(v^s))^c, \\ (y - (x + n))^c &= (\gamma_x^u(a^u))^c, \\ a^u &= -x^u - n^u + \gamma_0^{cs}(b^{cs}), \\ v^s &= -y^s + \gamma_0^{cu}(w^{cu}), \\ y^s &= x^s + n^s + (\gamma_x^u(a^u))^s. \end{aligned}$$

As $x \in W^c(0)$ we have that $|x^s|, |x^u| \leq \kappa$. So by Lemma 2.1 of Section 2,

$$\begin{aligned} |(z - (x + n))^c| &\leq |(y - (x + n))^c| + |\gamma_y^s(v^s)| \\ &\leq C \log |a^u| + C \log |v^s| \\ &\leq C \log(|n^u| + 2\kappa) + C \log(\kappa + |y^s|) \\ &\leq C \log(|n^u| + 2\kappa) + C \log(|n^s| + 3\kappa) \end{aligned}$$

from which the result follows. \square

4. A minimal property of the system

THEOREM 4.1. *Let U be a nonempty open connected su-saturated subset of \mathbb{R}^N and suppose there is $S \subset \mathbb{Z}^N$ a subgroup of \mathbb{Z}^N of maximal rank such that $U + S = U$. Then $U = \mathbb{R}^N$.*

For the proof of the theorem we need the following proposition. In this proposition, $\pi_q(U)$ are the q^{th} homotopy groups of U .

PROPOSITION 4.2. *Let U be a nonempty, open, connected subset of \mathbb{R}^N and suppose U satisfies the following properties:*

- a) $\pi_q(U) = \{0\}$ for any $q \geq 1$,
- b) $U + S = U$ for some subgroup $S \subset \mathbb{Z}^N$ of maximal rank,

then $U = \mathbb{R}^N$.

Proof. Without loss of generality we may suppose $S = \mathbb{Z}^N$. Call $\tilde{U} = p(U)$ where $p : \mathbb{R}^N \rightarrow \mathbb{T}^N$ is the covering projection. Now, we have that $p : U \rightarrow \tilde{U}$, the restriction of p to U , is a covering projection too. So, as $\pi_q(U) = \{0\}$ for any $q \geq 1$, we get, by Corollary 11 in Chapter 7, Section 2 of [Sp], that $\pi_q(\tilde{U}) = \{0\}$ for $q \geq 2$. Moreover, it is not hard to see that $i_{\#} : \pi_1(\tilde{U}) \rightarrow \pi_1(\mathbb{T}^N) = \mathbb{Z}^N$ is an isomorphism where $i_{\#}$ is the action of the inclusion map $i : \tilde{U} \rightarrow \mathbb{T}^N$ in the homotopy groups. Because \tilde{U} is open and connected and $\pi_q(\mathbb{T}^N) = \{0\}$ for $q \geq 2$ we get that $i : \tilde{U} \rightarrow \mathbb{T}^N$ is a weak homotopy equivalence as defined after Corollary 18 in Chapter 7, Section 6 of [Sp]. As \mathbb{T}^N is a CW complex, using Corollary 23 in Chapter 7, Section 6 of [Sp] we get that $i_{\#} : [\mathbb{T}^N; \tilde{U}] \rightarrow [\mathbb{T}^N; \mathbb{T}^N]$ is an isomorphism, where $[P; X]$ is the set of homotopy classes of maps from P to X . Hence, there is $g : \mathbb{T}^N \rightarrow \tilde{U}$ such that $i \circ g$ is homotopic to $id : \mathbb{T}^N \rightarrow \mathbb{T}^N$. Now, by degree theory, this implies that $i \circ g$ must be surjective and hence $\tilde{U} = \mathbb{T}^N$ which is equivalent to $U = \mathbb{R}^N$. \square

So, we only have to prove property a) of the proposition. To this end, we first prove that $\pi_q(U) = \{0\}$ for $q \geq 2$ and then that $\pi_1(U) = \{0\}$. This last property is the hard one.

LEMMA 4.3. $\pi^s : \mathbb{R}^N \rightarrow W^{cu}(0)$, $\pi^u : \mathbb{R}^N \rightarrow W^{cs}(0)$ and $\pi^{su} : \mathbb{R}^N \rightarrow W^c(0)$ are fibrations (or Hurewicz fiber spaces) as defined at the beginning of Section 2 in Chapter 2 of [Sp], and so they are weak fibrations (or Serre fiber spaces) as defined after Corollary 4 in Chapter 7, Section 2 of [Sp].

Proof. Once we prove the lemma for π^s and π^u , the case of π^{su} follows from Theorem 6 in Chapter 2, Section 2 of [Sp]. Let us prove then that π^s is a fibration. Take X a topological space, $g' : X \rightarrow \mathbb{R}^N$ and $G : X \times I \rightarrow W^{cu}(0)$ such that $G(x, 0) = \pi^s \circ g'(x)$ for $x \in X$. We have to prove that there exists $G' : X \times I \rightarrow \mathbb{R}^N$ such that $G'(x, 0) = g'(x)$ for $x \in X$ and $\pi^s \circ G' = G$. Define $G'(x, t) = W^s(G(x, t)) \cap W^{cu}(g'(x))$. It is not hard to see that this G' makes the desired properties. The case of π^u is completely analogous. \square

LEMMA 4.4. *Given any open and connected s-saturated set E , $\pi_q(E) = \pi_q(E \cap W^{cu}(0))$ for any $q \geq 1$. The same property holds if E is u-saturated when W^{cu} is replaced by W^{cs} . If E is su-saturated, then $\pi_q(E) = \pi_q(E \cap W^c(0))$ for any $q \geq 1$.*

Proof. Since E is s-saturated, it is not hard to see that $\pi^s|_E$ is a weak fibration and $\pi^s(E) = E \cap W^{cu}(0)$. So, take $x \in E \cap W^{cu}(0)$. As $(\pi^s)^{-1}(x) = W^s(x)$ is contractible since it is homeomorphic to \mathbb{R}^s we have by Theorem 10 of Chapter 7, Section 2 of [Sp], the following sequence

$$0 = \pi_q((\pi^s)^{-1}(x)) \xrightarrow{i\#} \pi_q(E) \xrightarrow{\pi^s\#} \pi_q(E \cap W^{cu}(0)) \xrightarrow{\bar{\partial}} \pi_{q-1}((\pi^s)^{-1}(x)) = 0$$

which is exact and hence we have the desired result. The proof when E is u-saturated is analogous and the case E is su-saturated follows by application of the same method to $\pi^u|_{E \cap W^{cu}(0)}$. \square

COROLLARY 4.5. *Any U as in Theorem 4.1 satisfies $\pi_q(U) = \{0\}$ for $q \geq 2$.*

Proof. By the preceding lemma $\pi_q(U) = \pi_q(U \cap W^c(0))$ for any $q \geq 1$. Because $W^c(0)$ is homeomorphic to \mathbb{R}^2 we have $\pi_q(U) = \pi_q(U \cap W^c(0)) = \{0\}$ for any $q \geq 2$. \square

Now, we want to prove that $D = U \cap W^c(0)$ is simply connected which is equivalent to proving that the complement of D in the Riemann sphere is connected (regarding $W^c(0)$ as \mathbb{R}^2), or what is equivalent, that any connected component of the complement of D is not bounded.

Recall the definition of $T_n : E^c \rightarrow E^c$,

$$T_n = (j_0^c)^{-1} \circ \pi_n^s \circ \pi_n^u \circ L_n \circ j_0^c$$

where $L_n : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $L_n(x) = x + n$, for $n \in \mathbb{Z}^N$, $x_n = W^u(n) \cap W^{cs}(0)$, $\pi_n^u = \pi^u|_{W^c(x_n)}$, $\pi_n^s = \pi^s|_{W^c(x_n)}$ and $j_0^c : E^c \rightarrow \mathbb{R}^N$, $j_0^c(v) = v + \gamma_0^c(v)$ is the parametrization of the center manifold of 0, $W^c(0)$.

Let us call $D^c = (j_0^c)^{-1}(D)$ and recall that $C(y)$ is the accessibility class of y . Let us state the following proposition which solves our problem.

PROPOSITION 4.6. *For any $x \in E^c$ and $\delta > 0$ there are $n \in S$, $n \neq 0$, $k \in \mathbb{Z}$, $k > 0$ and $\eta_i : [0, 1] \rightarrow E^c$, $i = 0, \dots, k-1$ such that $\eta_i([0, 1]) \subset j_0^{-1}((C(j_0(x)) + S) \cap W^c(0))$, $\eta_i(0) \in B_\delta^c(T_{in}(x))$ and $\eta_i(1) = T_{(i+1)n}(x)$. Moreover $|T_{kn}(x) - x| \rightarrow \infty$ as $\delta \rightarrow 0$.*

Before the proof of this proposition, let us show how it solves our problem.

COROLLARY 4.7. *Any connected component of the complement of D is not bounded.*

Proof. Take $B \subset W^c(0)$ a connected component of the complement of D and call $B^c = j_0^{-1}(B)$. Take $x \in B^c$ and suppose by contradiction that B is bounded. Let $R > 0$ be such that $B^c \subset B_R^c(x)$, the ball of center x and radius R . Using the preceding proposition we have that for any $\delta > 0$ there are

$n \in S$, $n \neq 0$ and $k \in \mathbb{Z}$, $k > 0$ such that $C_\delta = \bigcup_{i=0}^k \overline{B_\delta^c}(T_{in}(x)) \cup \bigcup_{i=0}^{k-1} \eta_i[0, 1]$ is connected and $|T_{kn}(x) - x| \rightarrow \infty$ as $\delta \rightarrow 0$. So, for δ small enough we get that \hat{C}_δ , the connected component of $C_\delta \cap \overline{B_{2R}^c}(x)$ that contains x , satisfies $\hat{C}_\delta \cap S_{2R}^c(x) \neq \emptyset$, where $S_{2R}^c(x)$ is the boundary of $B_{2R}^c(x)$. Then, looking at the Hausdorff space of the compact subsets of $\overline{B_{2R}^c}(x)$ we see that there is a subsequence $\delta_i \rightarrow 0$ such that $\hat{C}_{\delta_i} \rightarrow \hat{C}$ in the Hausdorff topology. Because of the properties of the Hausdorff topology, we get that \hat{C} is connected, $x \in \hat{C}$, $\hat{C} \subset (E^c \setminus D^c)$, and so $\hat{C} \subset B^c$, and $\hat{C} \cap S_{2R}^c(x) \neq \emptyset$, thus contradicting the boundedness of B . \square

Let us begin the proof of Proposition 4.6.

LEMMA 4.8. *There is a constant $c > 0$ that only depends on A such that $r = \frac{N-1}{2}$, $|n^c| \geq \frac{c}{|n|^r}$ for any $n \in \mathbb{Z}^N$, $n \neq 0$.*

Proof. See Lemma 3 of [Ka] or Lemma A.10 of Appendix A. \square

Proof of Proposition 4.6. Take $\delta > 0$ and define $\varepsilon > 0$ by $\delta = \varepsilon^\gamma$, $\gamma = 1 - \beta(s + 4b)$, where β is as in Proposition 3.1, b is as in Lemma 3.3, $s = rb + 1$ and r is as in Lemma 4.8. Moreover, we may suppose, if f is sufficiently close to A that $\gamma > 0$. Take $n \in S$ as in Lemma 3.3 for this ε . Also, take $\frac{\varepsilon^{-s}}{2} \leq k \leq \varepsilon^{-s}$. Thus, by (2) of Lemma 3.5 we have that

$$\begin{aligned} |T_{kn}(x) - x| &\geq |kn^c| - C \log |kn^u| - C \\ &\geq k \frac{C}{|n|^r} - bC \log 3C\varepsilon^{-1} - C \log k - C \\ &\geq \varepsilon^{-s} \frac{C}{|n^{su}|^r} - bC \log \varepsilon^{-1} - sC \log \varepsilon^{-1} - C - bC \log 3C \\ &\geq \varepsilon^{-s} C \varepsilon^{rb} - (b + s)C \log \varepsilon^{-1} - C - bC \log 3C \\ &= C\varepsilon^{-1} - (b + s)C \log \varepsilon^{-1} - C. \end{aligned}$$

Since $\varepsilon^\gamma = \delta$, we have $|T_{kn}(x) - x| \rightarrow \infty$ as $\delta \rightarrow 0$. Let us prove now the other part of the lemma. By Lemma 3.3, we have

$$W_\varepsilon^s(W_{L(\varepsilon)}^u(W_\varepsilon^c(j_0(x)))) \cap W_\varepsilon^s(W_{L(\varepsilon)}^u(W_\varepsilon^c(j_0(x) + n))) \neq \emptyset.$$

Take z in this intersection. Then there are points y, w, y', w' such that

$$z \in W_\varepsilon^s(y), z \in W_\varepsilon^s(y'), y \in W_{L(\varepsilon)}^u(w), y' \in W_{L(\varepsilon)}^u(w' + n)$$

and $j_0^{-1}(w) \in B_\varepsilon^c(x)$, $j_0^{-1}(w') \in B_\varepsilon^c(x)$. Now, let us define

$$S : W^c(n) \rightarrow W^c(0), \quad S = \pi_2^u \circ \pi^s \circ \pi_1^u,$$

where

$$\pi_1^u : W^c(n) \rightarrow W^c(y'), \quad \pi^s : W^c(y') \rightarrow W^c(y), \quad \pi_2^u : W^c(y) \rightarrow W^c(0)$$

are the respective holonomies. By hypothesis we have that $S(w' + n) = w$. Moreover, using Proposition 3.1, we have $\text{Lip}(S) \leq CL(\varepsilon)^{2\beta}$. Furthermore,

$$S(j_0(x) + n) \in C(j_0(x) + n) \cap W^c(0)$$

and hence

$$j_0 \circ T_{in} \circ j_0^{-1} \circ S(j_0(x) + n) \in C(j_0(x) + (i+1)n) \cap W^c(0).$$

Now, take $0 \leq i \leq k-1$ and call $\hat{x}_{i+1} = T_{in} \circ j_0^{-1}(S(j_0(x) + n))$. Then

$$\begin{aligned} d^c(\hat{x}_{i+1}, T_{in}(x)) &\leq d^c(\hat{x}_{i+1}, T_{in} \circ j_0^{-1} S(w' + n)) \\ &\quad + d^c(T_{in}(j_0^{-1}(w)), T_{in}(x)) \\ &\leq \text{Lip}(T_{in}) \text{Lip}(j_0^{-1}) \text{Lip}(S) \text{Lip}(j_0) d^c(j_0^{-1}(w'), x) \\ &\quad + \text{Lip}(T_{in}) d^c(j_0^{-1}(w), x) \\ &\leq \text{Lip}(T_{in}) (CLip(S) + 1) \varepsilon. \end{aligned}$$

Now using Corollary 3.2 we get

$$d^c(\hat{x}_{i+1}, T_{in}(x)) \leq (CKL(\varepsilon))^\beta (CL(\varepsilon)^{2\beta} + 1) \varepsilon \leq C\varepsilon^{1-\beta(s+3b)}.$$

If ε is small enough, we obtain

$$d^c(\hat{x}_{i+1}, T_{in}(x)) < \varepsilon^{1-\beta(s+4b)} = \varepsilon^\gamma = \delta.$$

Finally, as we shall see in the next section, Lemma 5.5, there is a path $\hat{\eta}_i : [0, 1] \rightarrow W^c(0)$, $\hat{\eta}_i([0, 1]) \subset C(j_0(x) + (i+1)n)$, such that $\hat{\eta}_i(0) = j_0(\hat{x}_{i+1})$ and $\hat{\eta}_i(1) = j_0(T_{(i+1)n}(x))$. So, taking $\eta_i = j_0^{-1} \circ \hat{\eta}_i$ we get the desired result. \square

As a corollary of the proof of Lemma 3.3 we have the following:

COROLLARY 4.9. *Any su-saturated open subset of \mathbb{R}^N has infinite volume.*

COROLLARY 4.10. *For any open su-saturated $U \subset \mathbb{R}^N$ and $S \subset \mathbb{Z}^N$ subgroup of maximal rank, there is $0 \neq n \in S$ such that $U \cap U + n \neq \emptyset$.*

Proof. $p_S : \mathbb{R}^N \rightarrow \mathbb{T}_S^N$, the covering projection to the torus generated by the lattice S , cannot be injective when restricted to U because if it were injective we would get $\text{vol}_{\mathbb{T}_S^N}(p_S(U)) = \text{vol}_{\mathbb{R}^N}(U) = \infty$. \square

COROLLARY 4.11. *Any open or closed F -invariant su-saturated $U \subset \mathbb{R}^N$ satisfying $U + \mathbb{Z}^N = U$ is either empty or the whole \mathbb{R}^N .*

Proof. We prove the case U is open; the case U is closed follows from work with the complement. Take $V \subset U$ a connected component of U . As V is open and su-saturated we have by Corollary 4.10 that there is $n \in \mathbb{Z}^N$, $n \neq 0$, such that $V + n \cap V \neq \emptyset$ and so $V = V + n$ since $V + n \subset U$. Moreover,

as the nonwandering set of f is \mathbb{T}^N we have that there are $k \in \mathbb{Z}$, $k \neq 0$ and $l \in \mathbb{Z}^N$ such that $[F^k(V) + l] \cap V \neq \emptyset$ and hence $F^k(V) + l = V$ because $F^k(V) + l \subset U$. From this, and the properties of A , it is not hard to see that there is a subgroup $S \subset \mathbb{Z}^N$ of maximal rank satisfying $V + S = V$. In fact, $S = \{\sum_{i=0}^{N-1} k_i A^{ik} n : k_i \in \mathbb{Z} \text{ for } i = 0, \dots, N-1\}$. So, using Theorem 4.1 we get the corollary since V is open, connected, su-saturated and $V + S = V$. \square

COROLLARY 4.12. *If $C(0)$ is open then $C(0) = \mathbb{R}^N$. And hence f has the accessibility property.*

Proof. By Corollary 4.10 there is $n \in \mathbb{Z}^N$ such that $C(0) + n \cap C(0) \neq \emptyset$ and so $C(0) + n = C(0)$. Because $F(C(0)) = C(0)$ there is a subgroup $S \subset \mathbb{Z}^N$ of maximal rank satisfying $C(0) + S = C(0)$. Hence, as $C(0)$ is connected, using Theorem 4.1 we get that $C(0) = \mathbb{R}^N$. \square

5. Structure of the accessibility classes

In this section we shall prove that either $C(0)$ is open, and hence the whole \mathbb{R}^N by Corollary 4.12, or $\#(C(x) \cap W^c(0)) = 1$ for any $x \in \mathbb{R}^N$.

THEOREM 5.1. *Either $C(0) = \mathbb{R}^N$ and hence f has the accessibility property, or $\#(C(x) \cap W^c(0)) = 1$ for any $x \in \mathbb{R}^N$.*

The proof of the theorem essentially splits into two propositions:

PROPOSITION 5.2. *For any $x \in \mathbb{R}^N$ one of the followings holds:*

- (1) $C(x)$ is open,
- (2) $C(x) \cap W^c(0)$ is the injective image of either S^1 , $(-1, 1)$, $[0, 1]$ or $[0, 1)$,
- (3) $\#(C(x) \cap W^c(0)) = 1$.

Moreover, denoting M the set of points satisfying property 3, M is closed, su-saturated, F -invariant and $M + \mathbb{Z}^N = M$. Hence by Corollary 4.11 M is either empty or \mathbb{R}^N .

Let us mention that in case (2) more is true; that is, $C(x) \cap W^c(0)$ is a topological one-dimensional manifold without boundary; i.e., it is homeomorphic to either S^1 or $(-1, 1)$. Moreover, by the differentiability of the holonomies between center manifolds, it can be proved that they are in fact differentiable manifolds. But we only need the way we state it. Indeed, we have the following proposition:

PROPOSITION 5.3. *In the above proposition, case (2) cannot hold for 0; i.e., either*

- (1) $C(0)$ is open;
- (2) $\#(C(0) \cap W^c(0)) = 1$.

Before the proof of the propositions, let us prove the theorem:

Proof of Theorem 5.1. We must prove that either $C(0) = \mathbb{R}^N$ or $M = \mathbb{R}^N$. We know that M is either empty or the whole \mathbb{R}^N . Let us suppose that $M \neq \mathbb{R}^N$, hence $M = \emptyset$ and so, 0 must satisfy either (1) or (2) of Proposition 5.2. But by Proposition 5.3 we have that 0 must satisfy (1) and hence $C(0)$ is open and by Corollary 4.12 $C(0) = \mathbb{R}^N$. \square

LEMMA 5.4. *For any $x \in \mathbb{R}^N$, $C(x) \cap W^c(0)$ is open if and only if $C(x)$ is open.*

Proof. If $C(x)$ is open then $C(x) \cap W^c(0)$ is open by definition of relative topology. If $C(0) \cap W^c(x)$ is open, then $(\pi^{su})^{-1}(C(x) \cap W^c(0)) = C(x)$ and hence $C(x)$ is open. \square

LEMMA 5.5. *Given $x \in W^c(0)$ and $y \in C(x) \cap W^c(0)$ there is $\varepsilon_0 > 0$ and $\gamma : W_{\varepsilon_0}^c(x) \times I \rightarrow W^c(0)$ continuous such that $\gamma(x, 0) = x$, $\gamma(x, 1) = y$ and $\gamma(z, I) \subset C(z)$ for any $z \in W_{\varepsilon_0}^c(x)$ where $I = [0, 1]$.*

Proof. We first build a path in $W^c(0)$ from x to y . Since $y \in C(x)$, there is an su path $\eta : I \rightarrow \mathbb{R}^N$ such that $\eta(0) = x$ and $\eta(1) = y$. Take $\pi^{su} \circ \eta$ which gives the desired path. For the construction of γ as in the lemma, just remember that the stable and unstable foliations are continuous, so that if we take a point close enough to x , we can build a path close to η and then project it to $W^c(0)$ as we did with η . \square

LEMMA 5.6. *If $\text{int}(C(x) \cap W^c(0)) \neq \emptyset$ then $C(x) \cap W^c(0)$ is open.*

Proof. Let z and $\varepsilon > 0$ be such that $W_{\varepsilon}^c(z) \subset C(x) \cap W^c(0)$ and take $y \in C(x)$. Then there is $\varepsilon_0 > 0$ and $\gamma : W_{\varepsilon_0}^c(z) \times I \rightarrow W^c(0)$ continuous such that $\gamma(y, 0) = y$, $\gamma(y, 1) = z$ and $\gamma(w, I) \subset C(w)$ for any $w \in W_{\varepsilon_0}^c(y)$. As $\gamma(\cdot, 1) = \tilde{\gamma}$ is continuous, $\tilde{\gamma}^{-1}(W_{\varepsilon}^c(z))$ is open and $y \in \tilde{\gamma}^{-1}(W_{\varepsilon}^c(z)) \subset C(x) \cap W^c(0)$. So $C(x) \cap W^c(0)$ is open. \square

By an arc we mean a homeomorphic image of $[0, 1]$. In what follows let us identify $W^c(0)$ with E^c for the sake of simplicity.

LEMMA 5.7. *Suppose $C(x) \cap W^c(0)$ is not open and let $\eta_i : I \rightarrow C(x) \cap W^c(0)$ be injective $i = 1, 2$. Then $\eta_1(I) \cap \eta_2(I) = \emptyset$ or $\eta_1(I) \cup \eta_2(I)$ is either an arc or a circle.*

Proof. Let x and η_i , $i = 1, 2$, be as in the lemma. Suppose that $\eta_1(I) \cap \eta_2(I) \neq \emptyset$ but that the conclusion of the lemma does not hold. We claim the following:

CLAIM 2. *There are closed subintervals $I_1, I_2 \subset I$, and points $a \in I_1$, $b \in I_2$ such that $\eta_1(I_1) \cap \eta_2(I_2) = \{\eta_1(a)\} = \{\eta_2(b)\}$ and either $a \in \partial I_1$ and $b \in \text{int}(I_2)$ or $a \in \text{int}(I_1)$ and $b \in \partial I_2$.*

We leave the proof of the claim until the end. Without loss of generality we may suppose that $a \in \partial I_1$ and $b \in \text{int}(I_2)$. Moreover, let us make a reparametrization that sends I_1 to $[0, a]$ and I_2 to $[0, 1]$. We use the same notation η_1 and η_2 for these reparametrizations. Take ε_0 and $\gamma : B_{\varepsilon_0}^c(\eta_1(a)) \times I \rightarrow W^c(0)$ as in Lemma 5.5, such that $\gamma(\eta_1(a), 1) = \eta_2(0)$. Given $\varepsilon_1 > 0$ small enough we can define

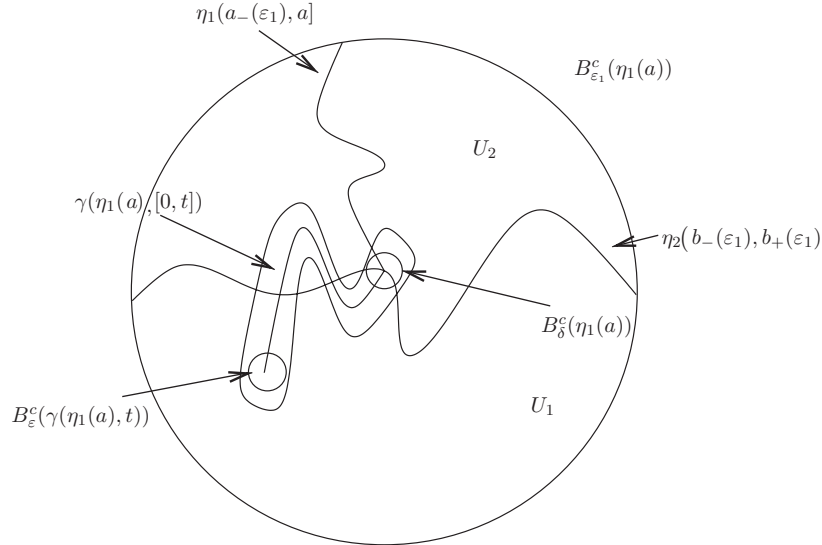
$$b_-(\varepsilon_1) = \sup\{s < b \text{ such that } \eta_2(s) \notin B_{\varepsilon_1}^c(\eta_1(a))\}$$

and

$$b_+(\varepsilon_1) = \inf\{s > b \text{ such that } \eta_2(s) \notin B_{\varepsilon_1}^c(\eta_1(a))\}.$$

Furthermore, define

$$a_-(\varepsilon_1) = \sup\{s < a \text{ such that } \eta_1(s) \notin B_{\varepsilon_1}^c(\eta_1(a))\}.$$



Notice that

$$U = B_{\varepsilon_1}^c(\eta_1(a)) \setminus \left(\eta_1(a_-(\varepsilon_1), a] \cup \eta_2(b_-(\varepsilon_1), b_+(\varepsilon_1)) \right)$$

has exactly three connected components. Suppose now that there is $t > 0$ such that $\gamma(\eta_1(a), [0, t]) \subset B_{\varepsilon_1}^c(\eta_1(a))$ and $\gamma(\eta_1(a), t) \in U_1$ for U_1 a connected

component of U . Take $\varepsilon > 0$ such that

$$B_\varepsilon^c(\gamma(\eta_1(a), [0, t])) \subset B_{\varepsilon_1}^c(\eta_1(a)), \quad B_\varepsilon^c(\gamma(\eta_1(a), t)) \subset U_1$$

and take $\delta > 0$ such that if $z \in B_\delta^c(\eta_1(a))$ then $\gamma(z, s) \in B_\varepsilon^c(\gamma(\eta_1(a), s))$ for $s \in [0, t]$. Take now U_2 another connected component of U ; then for any $z \in U_2 \cap B_\delta^c(\eta_1(a))$ we have that $\gamma(z, [0, t]) \subset B_{\varepsilon_1}^c(\eta_1(a))$ and $\gamma(z, t) \in U_1$. Hence, as $\gamma(z, [0, t])$ is connected we have that

$$\gamma(z, [0, t]) \cap \left(\eta_1(a_-(\varepsilon_1), a] \cup \eta_2(b_-(\varepsilon_1), b_+(\varepsilon_1)) \right) \neq \emptyset$$

and so $U_2 \cap B_\delta^c(\eta_1(a)) \subset C(x) \cap W^c(0)$ contradicting that it has empty interior. So there is not such a t . This implies that when

$$t_+ = \inf\{t > 0 \text{ such that } \gamma(\eta_1(a), t) \notin B_{\varepsilon_1}^c(\eta_1(a))\},$$

then

$$\gamma(\eta_1(a), [0, t_+)) \subset \eta_1(a_-(\varepsilon_1), a] \cup \eta_2(b_-(\varepsilon_1), b_+(\varepsilon_1)).$$

Suppose first that $\gamma(\eta_1(a), [0, t_+)) \cap \eta_2(b_-(\varepsilon_1), b] \neq \emptyset$ (the other cases will follow in a similar way). As before, we have that

$$V = B_{\varepsilon_1}^c(\eta_1(a)) \setminus (\eta_1(a_-(\varepsilon_1), a] \cup \eta_2[b, b_+(\varepsilon_1)))$$

has two connected components and that there is $0 < t' < t_+$ such that $\gamma(\eta_1(a), t') \in \eta_2(b_-(\varepsilon_1), b) \subset V_1$ which is one of the connected components of V . Because $t' < t_+$, there is $\varepsilon > 0$ such that

$$B_\varepsilon(\gamma(\eta_1(a), [0, t'])) \subset B_{\varepsilon_1}^c(\eta_1(a)).$$

Now, this case follows by the same arguments as in the preceding case.

Proof of Claim 2. Call $K_i = \eta_i^{-1}(\eta_1(I) \cap \eta_2(I))$. Taking U as a connected component of the complement of K_1 , we have that U is an interval, with $c < d$ its endpoints. Let us assume that $U \subset (0, 1)$. If $\eta_2^{-1}(\eta_1(c))$ is not an endpoint of I , then take $I_1 = [c, \frac{c+d}{2}]$, $I_2 = I$, $a = c$ and $b = \eta_2^{-1}(\eta_1(c))$. Otherwise, we have that $\eta_2^{-1}(\eta_1(c))$ is an endpoint of I and we may suppose it is 0. If $\eta_2^{-1}(\eta_1(d))$ is not an endpoint of I , take $I_1 = [\frac{c+d}{2}, d]$, $I_2 = I$, $a = d$ and $b = \eta_2^{-1}(\eta_1(d))$. Otherwise, we have that $\eta_2^{-1}(\eta_1(d))$ is an endpoint of I and it must be 1. So we have that $\eta_2(0) = \eta_1(c)$ and $\eta_2(1) = \eta_1(d)$.

Take another connected component of the complement of K_1 , if any, and call it V . Call the endpoints of V $r < s$ and assume $r \in K_1$. Again, if $\eta_2^{-1}(\eta_1(r))$ is not an endpoint of I , we are done; if not, we have that $\eta_1(r) = \eta_2(0)$ or $\eta_1(r) = \eta_2(1)$, and so, $r = c$ or $r = d$, but $r \neq c$ because $V \cap U = \emptyset$. Then, take $I_1 = [\frac{c+d}{2}, \frac{r+s}{2}]$, $I_2 = I$, $a = d$ and $b = 1$. If $r \notin K_1$, then $r = 0$ and $s \in K_1$. In this case, either $\eta_2^{-1}(\eta_1(s))$ is not an endpoint of I and we are done, or $s = c$ and we take $I_1 = [\frac{r+s}{2}, \frac{c+d}{2}]$, $I_2 = I$, $a = c$ and $b = 0$. So if $U \subset (0, 1)$ we may assume that there is not another connected component

and hence $U = K_1^c$ and $\eta_2(0) = \eta_1(c)$ and $\eta_2(1) = \eta_1(d)$. From here it is not hard to see that we can concatenate η_1 and η_2 to build a circle. So we may assume that there is no connected component of the complement of K_1 inside $(0, 1)$. Hence there are at most two connected components. And we may assume that the same holds for K_2 . If $K_1 = I$ then $\eta_1(I) \subset \eta_2(I)$ and we are done. So we may assume that $U = [0, d)$ is a connected component of K_1^c and that $\eta_1(d) = \eta_2(1)$, if not, we can argue as before. But then $1 \in K_2$ and hence, either $K_2 = I$ and we are done again, or $[0, r)$ is a connected component of the complement of K_2 . Now, $\eta_1^{-1}(\eta_2(r))$ is an endpoint of I and also it is in K_1 so that $\eta_2(r) = \eta_1(1)$ because 0 is not in K_1 . Now, $K_1 = [d, 1]$ and $K_2 = [r, 1]$ and from here it is not hard to see that $\eta_1(I) \cup \eta_2(I)$ must be an arc. \square

Proof of Proposition 5.2. Suppose that (1) and (3) do not hold; then by the preceding lemma the proof of the proposition follows in the spirit of the proof that the only one dimensional manifolds are the ones in (2). That M is su-saturated, F -invariant and $M + \mathbb{Z}^N = M$ is almost obvious. To prove that it is closed, we prove that the complement is open. But by Lemma 5.5 it is not hard to see that the complement is in fact open. \square

COROLLARY 5.8. *If f is sufficiently close to A then $C(0) \cap W^c(0)$ is not homeomorphic to $[0, 1]$.*

Proof. As $F(C(0) \cap W^c(0)) = C(0) \cap W^c(0)$, if $C(0) \cap W^c(0)$ were homeomorphic to $[0, 1]$ then we would have to have that either the endpoints are fixed or permuted by F . As the only point fixed by F is 0 and F has no period-two orbits we get that this is impossible. \square

Now, we are going to prove Proposition 5.3. Arguing by contradiction suppose in the sequel that $C(0) \cap W^c(0) = \eta(J)$ where $\eta : J \rightarrow W^c(0)$ is injective, $\eta(0) = 0$ and $J = (-1, 1)$ if $C(0) \cap W^c(0)$ is homeomorphic to $(-1, 1)$, $J = [0, 1)$ otherwise. Notice that in the case $C(0) \cap W^c(0)$ is homeomorphic to a circle, we do not have that η is a homeomorphism. Moreover, define $H = \eta([0, 1))$ and suppose, working with f^2 if necessary, that $F(H) = H$.

We may suppose that $W^c(0)$ has the euclidean structure inherited from E^c . Now, we may chose f close enough to A in order to get the following claim.

CLAIM 3. *For $\theta \in [0, 2\pi)$ define the line with slope θ ,*

$$l(\theta) = \{(r \cos(\theta), r \sin(\theta)) : r \geq 0\},$$

with $S(\theta)$ the sector bounded between $l(\theta)$ and $F(l(\theta))$, and $I(\theta) = \text{int}S(\theta)$. A priori there are two sectors, one satisfying the following: there is $n \geq 2$ such that $F(I(\theta)) \cap I(\theta) = \emptyset$ and $\bigcup_{i=0}^n F^i(S(\theta)) = W^c(0)$. Clearly n does not depend on θ , not on F .

Define $H(t) = \eta((0, t])$. As $F(H) = H$ and the only fixed point of F is 0 we may suppose, working with f^{-1} if necessary, that $F(H(t)) \supset H(t)$.

LEMMA 5.9. *For any θ and $t > 0$, $H(t) \cap l(\theta) \neq \emptyset$.*

Proof. Let θ and $t > 0$ be given and suppose that $H(t) \cap l(\theta) = \emptyset$. Then

$$H(t) \cap F(l(\theta)) \subset F(H(t) \cap l(\theta)) = \emptyset.$$

Hence $H(t) \subset I(\theta)$ or $H(t) \cap S(\theta) = \emptyset$. The first possibility cannot happen because $F(I(\theta)) \cap I(\theta) = \emptyset$ and then $H(t) = H(t) \cap F(H(t)) = \emptyset$. Neither can the second one because in this case

$$F^{-k}(H(t)) \cap S(\theta) \subset H(t) \cap S(\theta) = \emptyset.$$

Hence $H(t) \cap F^k(S(\theta)) = \emptyset$ and $H(t) = H(t) \cap \bigcup_{i=0}^n F^i(S(\theta)) = \emptyset$. \square

COROLLARY 5.10. *For any $\chi : [0, \varepsilon) \rightarrow W^c(0) \mathcal{C}^1$, with $\chi(0) = 0$, $\dot{\chi}(0) \neq 0$, $s > 0$ and $\delta > 0$, $\chi([0, \delta]) \cap H(s) \neq \emptyset$.*

Proof. Take χ, δ and s as in the corollary. Call $\chi' = F \circ \chi$. As $\dot{\chi}(0) \neq \dot{\chi}'(0)$ there is $\rho > 0$ such that $\chi[0, \rho) \cap \chi'[0, \rho) = \{0\}$. Moreover, calling $CC(D, x)$ the connected component of D that contains x , we can take τ small enough so that

$$R = B_\tau^c(0) \setminus [CC(\chi[0, \rho) \cap B_\tau^c(0), 0) \cup CC(\chi'[0, \rho) \cap B_\tau^c(0), 0)]$$

has exactly two connected components. Moreover, if τ is small enough, there are θ_0 and θ_1 such that $l(\theta_0) \cap B_\tau^c(0) \setminus \{0\}$ and $l(\theta_1) \cap B_\tau^c(0) \setminus \{0\}$ do not lie in the same connected component of R . We may suppose that $\rho < \delta$. Take $s_0 < s$ such that $H(s_0) \subset B_\tau^c(0)$. Suppose by contradiction that $H(s_0) \cap \chi[0, \rho) = \emptyset$, then, as $F(H(s_0)) \supset H(s_0)$ we have that $H(s_0) \cap \chi'[0, \rho) = \emptyset$. By the preceding lemma, $H(s_0) \cap l(\theta_0) \neq \emptyset$ and as $H(s_0)$ is connected it must lie in the same connected component in which lies $l(\theta_0) \cap B_\tau^c(0) \setminus \{0\}$, thus contradicting that $H(s_0) \cap l(\theta_1) \neq \emptyset$. \square

Given $x \in C(0) \cap W^c(0)$ there is a \mathcal{C}^1 diffeomorphism $P_x : W^c(0) \rightarrow W^c(0)$ such that $P_x(0) = x$ and $P_x(z) \in C(z)$ for any $z \in W^c(0)$. To build such a diffeomorphism, take an su-path from 0 to x and mark the corners; then define the diffeomorphism sliding along the s or u-foliation from the center manifold of a corner to the center manifold of the following corner. In other words, take $\gamma : [0, 1] \rightarrow \mathbb{R}^N$, call $0 = x_0, x_1, \dots, x_n = x$ the corners of γ enumerated by the order of $[0, 1]$, and define $\pi_0 : W^c(0) \rightarrow W^c(x_1)$ sliding along the s-foliation if the first leg of γ is an s-path or the u-foliation if it is a u-path. Then repeat the procedure from x_1 to x_2 thus defining $\pi_1 : W^c(x_1) \rightarrow W^c(x_2)$ and so on until you reach $x_n = x$. As the holonomies used in the construction are at least \mathcal{C}^1 and the definition of accessibility class, the composition of the π_i 's

give the desired P_x . Notice that P_x is a diffeomorphism because we can make the inverse process in order to get the inverse of P_x .

With this P_x and the corollary above we get the following:

COROLLARY 5.11. *For any $t \in (0, 1)$ and any $\chi : [0, \varepsilon) \rightarrow W^c(0) \mathcal{C}^1$, with $\chi(0) = \eta(t)$, $\dot{\chi}(0) \neq 0$, $s > 0$ and $\delta > 0$, $\chi((0, \delta)) \cap \eta(t - s, t + s) \neq \emptyset$.*

So, let us get the contradiction to the hypothesis that $C(0) \cap W^c(0)$ is neither open nor $\{0\}$.

Proof of Proposition 5.3. Take a point z in $W^c(0)$ that is not in $C(0)$. Now take the line segment from z to 0 and $t_0 \in (0, 1)$. Run along the line segment from z to 0 and stop the first time you touch $\eta[0, t_0]$. Suppose this point is $\eta(t_1)$. Now take the line segment from $\eta(t_1)$ to z and call it l . Then $l \cap \eta[0, t_0] = \{\eta(t_1)\}$ but by the above corollary, this implies that $t_1 = t_0$ and so $\eta(t_0)$ is in the line segment from z to 0. As t_0 was an arbitrary point in $(0, 1)$ we have that $\eta[0, 1)$ is contained in the line segment from z to 0 thus contradicting Lemma 5.9. \square

6. Case $C(0)$ is trivial

We suppose in this section that $\#(C(x) \cap W^c(0)) = 1$ for any $x \in \mathbb{R}^N$.

LEMMA 6.1. $T_n \circ T_m = T_{n+m}$ for all $n \in \mathbb{Z}^N$.

Proof. Notice that $T_n(x) = C(x + n) \cap W^c(0)$. Hence

$$\begin{aligned} T_n \circ T_m(x) &= C(T_m(x) + n) \cap W^c(0) \\ &= C(C(x + m) \cap W^c(0) + n) \cap W^c(0) \\ &= (C(x + m) + n) \cap W^c(0) = T_{n+m}(x) \end{aligned}$$

thus proving the claim. \square

Define the linear transformation $L : E^c \rightarrow \mathbb{R}^2$ by $L(e_1^c) = (1, 0)$ and $L(e_2^c) = (0, 1)$. Let $L(n^c) = \alpha_n$ and $P_i = L \circ T_{e_i} \circ L^{-1}$, $\tilde{Q}_n = L \circ T_n \circ L^{-1}$. Also, take $C > 0$. We choose the \mathcal{C}^r neighborhood of A small enough to obtain the following: There is

$h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $h = x + \eta$ such that:

- (1) $h^{-1} \circ P_1 \circ h = R_{(1,0)}$,
- (2) $h^{-1} \circ P_2 \circ h = R_{(0,1)}$,
- (3) $h^{-1} \circ \tilde{Q}_n \circ h = Q_n$ is in some given \mathcal{C}^r neighborhood of R_{α_n} if $|n| \leq C$, and the \mathcal{C}^r neighborhood of A is small enough.

$$(4) \quad |\eta(z)| \leq C \log^+ |z| + C,$$

$$(5) \quad \eta(0) = 0.$$

Let us show how to build such an h . In the sequel, when we say that a diffeomorphism is \mathcal{C}^r close to the identity, we mean that when the \mathcal{C}^r neighborhood of A is sufficiently small we can take the diffeomorphism as close as wanted to the identity.

Now, $P_1(x, y) = (x, y) + (1, 0) + \varphi_1(x, y)$ and we can take $\|\varphi_1\|_{\mathcal{C}^r}$ as small as wanted. Take $\varepsilon < \frac{1}{2}$ small and define $\psi : \mathbb{R} \rightarrow \mathbb{R}$, \mathcal{C}^∞ such that $\psi(x) = 0$ if $x < \varepsilon$ and $\psi(x) = 1$ if $x > 1 - \varepsilon$. We require (taking ε small enough) that $\|\psi\|_{\mathcal{C}^r} \leq C$ for some fixed constant that only depends on r . Take $\hat{h}_1(x, y) = (x, y) + \psi(x)\varphi_1(x, y)$. If the \mathcal{C}^r norm of φ_1 is sufficiently small, then \hat{h}_1 is a \mathcal{C}^r diffeomorphism, \mathcal{C}^r close to the identity. By definition we have that if $|x| \leq \varepsilon$ then

$$(1) \quad P_1 \circ \hat{h}_1 = \hat{h}_1 \circ R_{(1,0)}.$$

Define now $\hat{h}(x, y) = P_1^{[x]}(\hat{h}_1(x - [x], y))$ where $[x]$ stands for the integral part of x . We claim that $\hat{h}|_{[-\varepsilon < x < 1 + \varepsilon]} = \hat{h}_1$, \hat{h} is a \mathcal{C}^r diffeomorphism and $P_1 \circ \hat{h} = \hat{h} \circ R_{(1,0)}$. The first claim is obvious if $0 \leq x < 1$. If $-\varepsilon < x < 0$ then we have that

$$\hat{h}(x, y) = P_1^{-1}(\hat{h}_1(x + 1, y)) = \hat{h}_1(x, y)$$

by (1). In the same way we get the first claim if $1 \leq x < 1 + \varepsilon$. That \hat{h} is \mathcal{C}^r is essentially by definition, because if (x, y) satisfies that $x \notin \mathbb{Z}$, then there is a neighborhood of (x, y) such that $[x'] = [x]$ for any (x', y') in this neighborhood. Also, if $x \in \mathbb{Z}$ and the neighborhood of (x, y) is such that $|x - x'| < \varepsilon/2$, for (x', y') in this neighborhood, and by the first part of the claim, the property follows. That \hat{h} is in fact a diffeomorphism also follows in the same way; just notice that defining n such that $P_1^{-n}(z, w) \in \hat{h}_1[0 \leq x < 1]$ we have that

$$\hat{h}^{-1}(z, w) = \hat{h}_1^{-1} \circ P_1^{-n}(z, w) + n.$$

Now, $\hat{h}^{-1} \circ P_1 \circ \hat{h} = R_{(1,0)}$. Define $P'_2 = \hat{h}^{-1} \circ P_2 \circ \hat{h}$. By the commutativity, $P'_2(x + (1, 0)) = P'_2(x) + (1, 0)$ so that P'_2 induces a diffeomorphism of the cylinder. Next, taking the circle $[y = 0]$ and working as above, we can build a \mathcal{C}^r diffeomorphism $h' : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $h'(0, 0) = (0, 0)$ such that

$$h'(x + (1, 0)) = h'(x) + (1, 0), \quad P'_2 \circ h' = h' \circ R_{(0,1)}$$

and $h'|_{[-\varepsilon < x, y < 1 + \varepsilon]}$ is \mathcal{C}^r close to the identity. So taking $h = \hat{h} \circ h'$ we have that h is a \mathcal{C}^r diffeomorphism which is restricted to some small neighborhood of the standard square, is \mathcal{C}^r close to the identity and

$$h^{-1} \circ P_1 \circ h = R_{(1,0)}, \quad h^{-1} \circ P_2 \circ h = R_{(0,1)}.$$

Notice that the third condition must be verified only in a neighborhood of the standard square and that it is verified because of Corollary 2.4 and the fact that h may be chosen as close to the identity as desired. That $h(0) = 0$ follows again by construction. Let us prove that h satisfies condition 4. Set $x = (x_1, x_2)$ and $n = [x_1]e_1 + [x_2]e_2$. Then, using the first property and Lemma 3.5, we have that

$$\begin{aligned} |\eta(x)| &= |h(x) - x| = |P_1^{[x_1]} \circ P_2^{[x_2]} \circ h(x - ([x_1], [x_2])) - x| \\ &\leq |LT_n \circ L^{-1}h(x - ([x_1], [x_2])) - x| \\ &\leq \|L\| |T_n \circ L^{-1}h(x - ([x_1], [x_2])) - (L^{-1}h(x - ([x_1], [x_2])) + n^c)| \\ &\quad + |h(x - ([x_1], [x_2])) - (x - ([x_1], [x_2]))| \\ &\leq C\|L\| \log |n| + |\eta(x - ([x_1], [x_2]))| \leq C\|L\| \log |x| + C. \end{aligned}$$

6.1. Case $N \geq 6$.

LEMMA 6.2. *If $N \geq 6$ there is $n \in \mathbb{Z}^N$ such that if the linear transformation $L : E^c \rightarrow \mathbb{R}^2$ is defined by $L(e_1^c) = (1, 0)$, and $L(e_2^c) = (0, 1)$ and $L(n^c) = \alpha$, then α satisfies a diophantine condition with exponent δ for any $\delta > 0$. Clearly, n only depends on A .*

Proof. The proof of the lemma will be carried out in Appendix A. \square

Now, using the KAM theorem, we have that $Q = Q_n = R_\lambda \circ h_1^{-1} \circ R_\alpha \circ h_1$ with $\|h_1 - \text{id}\|_{C^1} < \frac{1}{2}$.

LEMMA 6.3. *Let $Q : \mathbb{R}^c \rightarrow \mathbb{R}^c$, $Q = R_\lambda \circ h_1^{-1} \circ R_\alpha \circ h_1$ and suppose $\|h_1 - \text{id}\|_{C^1} < \delta$. Then*

$$|Q^k(x) - (x + k\alpha)| \geq k|\lambda|(1 - \delta) + \delta(|\lambda| - 2)$$

and

$$|Q^k(0)| \leq k(|\lambda| + |\alpha| + 2\delta)$$

for all $k \geq 0$.

Proof. Denote $h_1 = x + \varphi$ and notice that $Q(x) = x + \alpha + \lambda + \varphi(x) - \varphi(Q(x) - \lambda)$ and so

$$\begin{aligned} Q^k(x) &= x + k\alpha + k\lambda + \sum_{j=0}^{k-1} \varphi(Q^j(x)) - \sum_{j=0}^{k-1} \varphi(Q^{j+1}(x) - \lambda) \\ &= x + k\alpha + k\lambda + \sum_{j=1}^{k-1} [\varphi(Q^j(x)) - \varphi(Q^j(x) - \lambda)] \\ &\quad + \varphi(x) - \varphi(Q^k(x) - \lambda) \end{aligned}$$

so that

$$\left| \sum_{j=1}^{k-1} [\varphi(Q^j(x)) - \varphi(Q^j(x) - \lambda)] + \varphi(x) - \varphi(Q^k(x) - \lambda) \right| \leq (k-1)|\lambda|\delta + 2\delta$$

and thus we get the first estimate. The second one follows easily by the same method. \square

The following lemma gives another bound.

LEMMA 6.4. *There is $C > 0$ such that for any $k \geq 1$, $|\tilde{Q}_n^k(0) - k\alpha| \leq C \log k + C$.*

Proof. It is essentially a special case of Lemma 3.5. \square

Finally,

$$\begin{aligned} |\tilde{Q}_n^k(0) - k\alpha| &= |h(Q^k(0)) - k\alpha| = |Q^k(0) + \eta(Q^k(0)) - k\alpha| \\ &\geq k|\lambda|/2 + \frac{1}{2}(|\lambda| - 2) - |\eta(Q^k(0))| \\ &\geq k|\lambda|/2 + \frac{1}{2}(|\lambda| - 2) - C \log^+ |Q^k(0)| - C \\ &\geq k|\lambda|/2 + \frac{1}{2}(|\lambda| - 2) - C \log k \\ &\quad - C \log(|\lambda| + |\alpha| + 1) - C \\ &= k|\lambda|/2 - C \log k - C \end{aligned}$$

and thus this implies that $\lambda = 0$.

6.2. *Case $N = 4$.* Because Lemma 6.2 is false for $N = 4$, we need to use another argument here. What we do in this case is to show how the proof in [Mo] of the linearization of commuting circle diffeomorphisms applies in our case. We follow the notation of [Mo] in this subsection.

We say that $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\phi(x + n) = \phi(x) + n$ for any $n \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$ has rotation vector $\alpha \in \mathbb{R}^2$ if

$$\lim_{k \rightarrow +\infty} \frac{\phi^k(x) - x}{k} \rightarrow \alpha$$

uniformly in x .

THEOREM 6.5. *Let $\phi_\nu : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, be two C^{22} diffeomorphisms such that when $\phi_\nu(x) = x + \alpha_\nu + \hat{\phi}_\nu(x)$,*

- (1) $\phi_\nu(x + n) = \phi_\nu(x) + n$ for $\nu = 1, 2$, any $n \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$,
- (2) $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1$,
- (3) $\hat{\phi}_\nu$ is close to 0 in the C^{22} -supnorm for $\nu = 1, 2$.

Assume also that ϕ_ν has rotation vector α_ν and that the following diophantine condition holds:

$$\max_{\nu=1,2} \|k \cdot \alpha_\nu\| \geq \frac{c}{|k|^2}$$

for any $k \in \mathbb{Z}^2$, $k \neq 0$. Then there is a C^1 diffeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

- (1) $h(x + n) = h(x) + n$ for any $n \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$,
- (2) $\phi_\nu \circ h = h + \alpha_\nu$ for $i = 1, 2$.

So the theorem says that two commuting diffeomorphisms of the 2-torus with diophantine rotation vectors and close to these diophantine rotations are smoothly and simultaneously conjugated to the rotations.

As we said, for the proof we show how to adapt the proof of Moser to our setting. So, we have $\tau = 2$ in formula (1.3) of page 106 of [Mo]. On page 115 of [Mo] formula (3.5) becomes

$$V_0 = C_0^\infty(\mathbb{T}^2, \mathbb{R}); \quad V_1 = C_0^\infty(\mathbb{T}^2, \mathbb{R}^2); \quad V_2 = C_0^\infty(\mathbb{T}^2, so(2)).$$

The operators L, A, B, L^*, A^*, B^* and M are defined in the same way. In Lemma 3.1 we take $\sigma = 4 + \frac{1}{30}$. Let us show how the proof of Lemma 3.1 applies in our case.

$$\begin{aligned} v &= \sum_{j \in \mathbb{Z}^2, j \neq 0} v_j e^{2\pi i j \cdot x}, \\ Mv &= \sum_{j \in \mathbb{Z}^2, j \neq 0} \mu_j v_j e^{2\pi i j \cdot x}, \\ \mu_j &= 4[\sin^2(\pi \alpha_1 \cdot j) + \sin^2(\pi \alpha_2 \cdot j)] \geq \frac{c}{|j|^2}, \\ |M^{-1}v|_r &\leq C \sum_{j \in \mathbb{Z}^2, j \neq 0} \mu_j^{-1} |v_j| |j|^r, \\ &\leq C \sum_{j \in \mathbb{Z}^2, j \neq 0} |j|^{-(2+\frac{1}{30})} |v|_{\sigma+r} \leq C |v|_{\sigma+r}. \end{aligned}$$

Now, on page 117, the smoothing operators are defined in the same way, with S^1 changed by \mathbb{T}^2 and \mathbb{R} by \mathbb{R}^2 . The construction of the smooth solution \tilde{u} is as well, everything defined componentwise. On page 118, everything works as well. The only difference is the fact that ψ_ν has rotation number α_ν implies $\hat{\psi}_\nu$ has a zero. In our case, we apply Lemma 3.5 which, modulo changing the constants, is invariant under conjugacy. Thus we obtain the fact that each component of $\hat{\psi}_\nu$ has a zero. Hence we have

$$|\hat{\psi}|_0 \leq 2|\hat{\psi} - c|_0.$$

Finally on page 119, we define $\varepsilon_s = \varepsilon^{\kappa^s}$ but now, $\kappa = 1 + \frac{109}{242}$,

$$N_s = \varepsilon_s^{-\xi}, \quad l = 22, \quad \sigma = 4 + \frac{1}{30},$$

where $\xi = \frac{2(l+5)}{l^2} = \frac{27}{242}$ and in formula (3.20) we change $|\hat{u}^{(s)}|_1 < \varepsilon_s^{1/2}$ to $|\hat{u}^{(s)}|_1 < \varepsilon_s^{4/11}$. Hence everything works and we get the conjugacy $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whenever $|\hat{\phi}_\nu|_0$ and $|\hat{\phi}_\nu|_l$ are sufficiently small.

So in our case, $\phi_\nu = Q_{n_\nu}$ for $\nu = 1, 2$. In order to get the smooth conjugacy, we need the following lemma:

LEMMA 6.6. *There exist n_1, n_2 such that if we take the linear transformation $L : E^c \rightarrow \mathbb{R}^2$ defined by $L(e_1^c) = (1, 0)$, and $L(e_2^c) = (0, 1)$ and call $L(n_1^c) = \alpha_1$, $L(n_2^c) = \alpha_2$, then there is a constant $c > 0$ such that*

$$\max_{\nu=1,2} \|k \cdot \alpha_\nu\| \geq \frac{c}{|k|^2}$$

for any $k \in \mathbb{Z}^2$, $k \neq 0$.

The proof of the lemma will be carried out in the appendix.

Thus we fit in Theorem 6.5 and hence get a smooth diffeomorphism $h_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying

- (1) $h_1(x + n) = h_1(x) + n$ for any $n \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$,
- (2) $\phi_\nu \circ h_1 = h_1 + \alpha_\nu$ for $i = 1, 2$.

6.3. *End of the proof.* Call $h_2 = L^{-1} \circ h_1 \circ h^{-1} \circ L$ in either case. As the T_n 's form a commutative group of diffeomorphisms and the R_{n^c} 's acts transitively on E^c we get that $h_2 \circ T_n = R_{n^c} \circ h_2$ for all $n \in \mathbb{Z}^N$. Now define $h_3 : W^c(0) \rightarrow E^c$ by $h_3 = h_2 \circ j_0^{-1}$ and $h^c : \mathbb{R}^N \rightarrow E^c$ by $h^c = h_3 \circ \pi^{su}$. We have that

$$h^c \circ L_n = h_2 \circ j_0^{-1} \circ \pi^{su} \circ L_n = h_2 \circ T_n \circ j_0^{-1} \circ \pi^{su} = R_{n^c} \circ h_2 \circ j_0^{-1} \circ \pi^{su} = R_{n^c} \circ h^c.$$

Moreover, $h^c|_{W^{c\sigma}(x)}$ is C^1 for any $x \in \mathbb{R}^N$, $\sigma = u, s$ and $y \in C(x)$ if and only if $h^c(x) = h^c(y)$. Indeed, it is not hard to see that $\text{Lip}(h^c|_{W^{c\sigma}(x)}) \leq C(L)$ and $\text{Lip}((h^c|_{W^{c\sigma}(x)})^{-1}) \leq C(L)$ for some constant $C(L)$ that only depends on the size of the neighborhood of A and L . We claim that $h^c \circ F = A^c \circ h^c$. By definition, we only have to prove it in $W^c(0)$. Now,

$$h^c(F(\pi^{su}(n))) = h^c(\pi^{su}(An)) = h^c(An) = A^c n^c = A^c h^c(n) = A^c h^c(\pi^{su}(n)).$$

As $\pi^{su}(\mathbb{Z}^N)$ is dense in $W^c(0)$ (this is because $h^c(\pi^{su}(\mathbb{Z}^N)) = \{n^c : n \in \mathbb{Z}^N\}$ and $h^c|_{W^c(0)}$ is a diffeomorphism), we get the desired claim. Now, denoting $F = A + \psi$ and solving the cohomological equations

$$A^s \varphi^s - \varphi^s \circ F = \psi^s \quad \text{and} \quad A^u \varphi^u - \varphi^u \circ F = \psi^u$$

which can be solved as in the Anosov case or as in Hartman-Grobman's theorem, we get $h^s(x) = x^s + \varphi^s(x)$ and $h^u(x) = x^u + \varphi^u(x)$. Defining $H_1 : \mathbb{R}^N \rightarrow \mathbb{R}^N$ by $H_1 = h^s + h^u + h^c$, H_1 is a homeomorphism, $H_1(x+n) = H_1(x)+n$ for all $n \in \mathbb{Z}^N$ which means that H_1 induces a homeomorphism of the torus and that $H_1 \circ F = A \circ H_1$ and hence f is conjugated to A . The problem now is that, as in the Anosov case, *a priori*, H_1 has no regularity property other than just being continuous; hence we now define $H_2 : \mathbb{R}^N \rightarrow \mathbb{R}^N$ by $H_2(x) = x^s + x^u + h^c(x)$. H_2 is again a homeomorphism, and $H_2(x+n) = H_2(x)+n$ for all $n \in \mathbb{Z}^N$ and so it induces a homeomorphism of the torus. Because of the properties listed above we have $H_2(C(x)) = H_2(x) + E^{su}$. So if we prove some regularity property for H_2 , using the fact that $x + E^{su}$, $x \in \mathbb{R}^N$, induces an ergodic foliation of the torus, we get the essential accessibility property. We claim that H_2 is bi-Lipschitz. To prove this claim, notice that, as H_2 induces a homeomorphism of the torus, it only has to be proved in a neighborhood of a fundamental domain of the torus. Moreover, we only have to prove that it is locally bi-Lipschitz by compactness. So, take x, y and $\hat{x} = W^s(x) \cap W^{cu}(y)$; then $h^c(x) = h^c(\hat{x})$. Moreover, $\hat{x} = x + v^s + \gamma_x^s(v^s) = y + w^{cu} + \gamma_y^{cu}(w^{cu})$. Thus,

$$\begin{aligned} |\hat{x} - y| &= |w^{cu} + \gamma_y^{cu}(w^{cu})| \leq (1 + \kappa)|w^{cu}|, \\ |w^{cu}| &= |(x - y)^{cu} + \gamma_x^s(v^s)| \leq |(x - y)^{cu}| + \kappa|v^s|, \\ |v^s| &= |(y - x)^s + \gamma_y^{cu}(w^{cu})| \leq |(y - x)^s| + \kappa|w^{cu}|, \end{aligned}$$

and

$$|w^{cu}| + |v^s| \leq \frac{1}{1 - \kappa}|x - y|,$$

and $|w^{cu}| \leq \frac{1}{1 - \kappa}|x - y|$. Hence

$$|\hat{x} - y| \leq \frac{1 + \kappa}{1 - \kappa}|x - y|$$

and so we may suppose that x and y are close enough to get $\hat{x} \in W_1^{cu}(y)$ and thus, $|h^c(\hat{x}) - h^c(y)| \leq C(1)|\hat{x} - y|$. As $H_2(x) = x^s + x^u + h^c(x)$ we get that H_2 is Lipschitz.

Let us prove now that $|H_2(x) - H_2(y)| \geq c_0|x - y|$ for some constant c_0 . As $|H_2(x) - H_2(y)| \geq |(x - y)^{su}|$, we may suppose that $|(x - y)^{su}| \leq |(x - y)^c|$. Define $x' = W^u(\hat{x}) \cap W^c(y)$; then $x' = \hat{x} + v^u + \gamma_{\hat{x}}^u(v^u) = y + w^c + \gamma_y^c(w^c)$. So, again we may suppose x and y so close that

$$\begin{aligned} |h^c(x) - h^c(y)| &= |h^c(x') - h^c(y)| \geq \frac{1}{C(1)}|x' - y|, \\ |x - y| &\leq 2|(x - y)^c|, \\ |(x - y)^c| &= |(w^{cu} - \gamma_x^s(v^s))^c| \leq |(\hat{x} - y)^c| + \kappa|(\hat{x} - y)^s|, \\ |(\hat{x} - y)^c| &= |w^c - (\gamma_{\hat{x}}^u(v^u))^c| \leq |(x' - y)^c| + \kappa|(x' - y)^u|, \\ |(\hat{x} - y)^s| &= |(\gamma_y^c(w^c) - \gamma_{\hat{x}}^u(v^u))^s| \leq \kappa(|(x' - y)^c| + |(x' - y)^u|). \end{aligned}$$

Hence

$$|(x - y)^c| \leq |(x' - y)^c| + \kappa|(x' - y)^u| + \kappa^2(|(x' - y)^c| + |(x' - y)^u|) \leq 2|x' - y|.$$

And so, taking $c_0 = \frac{1}{4C(1)}$ we get that H_2^{-1} is Lipschitz and hence that H_2 is bi-Lipschitz. In fact, it can be proved that H_2 is a \mathcal{C}^1 diffeomorphism. To do this, just notice that $h^c|_{W^c(x)}$ is a \mathcal{C}^1 diffeomorphism and $h^c|_{W^\sigma(x)}$, $\sigma = s, u$ is constant for any $x \in \mathbb{R}^N$. Thus, the partial derivatives are continuous and hence h^c is \mathcal{C}^1 and so H_2 is \mathcal{C}^1 . Working in the same way with H_2^{-1} we get the desired claim.

Appendix A. Diophantine approximations

In this appendix we will prove some results about diophantine approximations.

THEOREM A.1. *Let α_i , $i = 1, \dots, n$, be real algebraic numbers and suppose that $1, \alpha_1, \dots, \alpha_n$ are linearly independent over the rationals. Then, given $\delta > 0$ there is a constant $c = c(\delta, \alpha_1, \dots, \alpha_n)$ such that for any $n + 1$ integers q_1, \dots, q_n, p with $q = \max(|q_1|, \dots, |q_n|) > 0$*

$$|q_1\alpha_1 + \dots + q_n\alpha_n + p| \geq \frac{c}{q^{n+\delta}}.$$

Proof. See Chapter VI, Corollary 1E of [Sc]. □

PROPOSITION A.2. *Let P be a polynomial of degree N , with integer coefficients, irreducible over the integers. Suppose that one root of P is a complex number of modulus one, say λ . Set $c_1 = 2\Re(\lambda)$ where \Re stands for the real part of the number. Then, for any Q with integer coefficients such that $Q(c_1) = 0$, $\deg Q \geq \frac{N}{2}$.*

Proof. Take Q such that $Q(c_1) = 0$ and $\deg Q = d$. Then, as $c_1 = \lambda + \lambda^{-1}$, we get that $T(x) = x^d Q(x + x^{-1})$, $\deg T = 2d$ and $T(\lambda) = 0$. As P is irreducible, $2d \geq N$. □

Let us define some tools that will be useful in what follows. For any given $\theta \in \mathbb{C}$, $|\theta| = 1$ let us denote $c_k(\theta) = 2\Re(\theta^k)$ and $a_k(\theta) = \frac{\Im(\theta^k)}{\Im(\theta)}$ where \Im stands for the imaginary part of the number. We have that a_k and c_k satisfy the following recurrence relation:

- (1) $a_0 = 0, a_1 = 1, c_0 = 2,$
- (2) $a_{k+1} = c_k + a_{k-1}$ for $k \geq 2,$
- (3) $c_k = c_1 a_k - 2a_{k-1}$ for $k \geq 1.$

From this recurrence relation we obtain polynomials with integer coefficients R_k and I_k that do not depend on θ such that $a_k = I_k(c_1)$ and $c_k = R_k(c_1)$. Moreover, $\deg(R_k) = k$, $\deg(I_k) = k-1$ and when α_i^k and β_i^k are the coefficients of R_k and I_k respectively, we have the following:

- (1) $\alpha_k^k = 1$ and $\beta_{k-1}^k = 1$ for $k \geq 1$,
- (2) $\alpha_{k-2i-1}^k = 0$ and $\beta_{k-2i-2}^k = 0$ for $k \geq 1$, $0 \leq 2i \leq k-1$.

Given a polynomial P with a root λ with modulus 1 set $c_k = c_k(\lambda)$ and $a_k = a_k(\lambda)$.

COROLLARY A.3. *If P is a polynomial with integer coefficients, irreducible over the integers and $\deg P$ is odd then it has no root of modulus one.*

Proof. Take P a polynomial with integer coefficients, suppose $\deg P = 2r+1$ and that λ is a root of P with modulus 1. Write $P(z) = \sum_{k=0}^{2r+1} p_k z^k$. Then

$$\begin{aligned} 0 = \lambda^{-r} P(\lambda) &= \sum_{k=0}^r p_k \bar{\lambda}^{r-k} + \sum_{k=r+1}^{2r+1} p_k \lambda^{k-r} \\ &= \sum_{k=0}^r p_{r-k} \bar{\lambda}^k + \sum_{k=1}^{r+1} p_{k+r} \lambda^k \end{aligned}$$

where $\bar{\lambda}$ is the conjugate of λ . As $\bar{\lambda}$ is also a root of P , we obtain, in the same way

$$0 = \bar{\lambda}^{-r} P(\bar{\lambda}) = \sum_{k=0}^r p_{r-k} \lambda^k + \sum_{k=1}^{r+1} p_{k+r} \bar{\lambda}^k.$$

So, from both we obtain that

$$\begin{aligned} 0 &= \sum_{k=0}^r p_{r-k} (\bar{\lambda}^k - \lambda^k) + \sum_{k=1}^{r+1} p_{k+r} (\lambda^k - \bar{\lambda}^k) \\ &= p_{2r+1} (\lambda^{r+1} - \bar{\lambda}^{r+1}) + \sum_{k=1}^r (p_{r+k} - p_{r-k}) (\lambda^k - \bar{\lambda}^k) \\ &= 2i \left[p_{2r+1} \Im(\lambda^{r+1}) + \sum_{k=1}^r (p_{r+k} - p_{r-k}) \Im(\lambda^k) \right]. \end{aligned}$$

Now,

$$0 = p_{2r+1} I_{r+1}(c_1) + \sum_{k=1}^r (p_{r+k} - p_{r-k}) I_k(c_1) = Q(c_1).$$

Hence, as $\deg I_k = k - 1$ and $p_{2r+1} \neq 0$, since $\deg P = 2r + 1$, $\deg Q = r$. Thus, using Proposition A.2 we get a contradiction, thus proving the corollary. \square

COROLLARY A.4. *If N is odd and $A \in \mathrm{SL}(N, \mathbb{Z})$ has irreducible characteristic polynomial then, A is Anosov.*

Proof. This is clear from the preceding corollary. \square

COROLLARY A.5. *Any ergodic linear automorphism of \mathbb{T}^5 is Anosov.*

Proof. Taking a power, we may suppose that $\det A = 1$. If the characteristic polynomial of A is irreducible, then the result follows from the preceding corollary; so let us assume that it is reducible. Then $P = LQ$ where either $\deg Q = 1, \deg L = 4$ or $\deg Q = 2, \deg L = 3$. In the first case 1 or -1 must be a root of Q and hence of A contradicting ergodicity; so we cannot have this decomposition. In the second case the leading coefficient of Q is 1 and the independent term is ± 1 . Hence, if Q has a root with modulus 1, it is a root of unity, contradicting ergodicity; so the roots of Q do not have modulus 1. As the independent term of L is also ± 1 , if it has a root with modulus 1, it cannot be real. Hence the conjugate is also a root and then, ± 1 must be a root of L , again contradicting ergodicity. So in this case A is Anosov too. \square

COROLLARY A.6. *If P is a polynomial of even degree, $\deg P = 2r$, with integral coefficients, with a root λ of modulus one, then there is a polynomial Q with integral coefficients such that $Q(c_1) = 0$ and $\deg Q = r$. Moreover, if P is irreducible, with $P(z) = \sum_{k=0}^{2r} p_k z^k$, $p_{r+k} = p_{r-k}$ for $k = 1, \dots, r$, Q is irreducible, and then Q is such that its leading coefficient equals p_{2r} .*

Proof. Here we work as in the proof of Corollary A.3. Write $P(z) = \sum_{k=0}^{2r} p_k z^k$. Then

$$\begin{aligned} 0 &= \lambda^{-r} P(\lambda) = p_r + \sum_{k=0}^{r-1} p_k \bar{\lambda}^{r-k} + \sum_{k=r+1}^{2r} p_k \lambda^{k-r} \\ &= p_r + \sum_{k=1}^r p_{r-k} \bar{\lambda}^k + \sum_{k=1}^r p_{k+r} \lambda^k \end{aligned}$$

and as $\bar{\lambda}$ is also a root of P , we obtain, in the same way

$$0 = \bar{\lambda}^{-r} P(\bar{\lambda}) = p_r + \sum_{k=1}^r p_{r-k} \lambda^k + \sum_{k=1}^r p_{k+r} \bar{\lambda}^k.$$

Thus, from both statements we obtain that

$$\begin{aligned}
0 &= 2p_r + \sum_{k=1}^r (p_{r+k} + p_{r-k})(\lambda^k + \bar{\lambda}^k) \\
&= 2p_r + \sum_{k=1}^r (p_{r+k} + p_{r-k})2\Re\epsilon(\lambda^k) \\
&= 2p_r + \sum_{k=1}^r (p_{r+k} + p_{r-k})R_k(c_1) = 2Q(c_1).
\end{aligned}$$

As $\deg R_k = k$, $\deg Q \leq r$ and thus we get the first part of the corollary.

For the second part we have

$$\begin{aligned}
0 &= \sum_{k=1}^r (p_{r+k} - p_{r-k})(\lambda^k - \bar{\lambda}^k) \\
&= 2i \left[\sum_{k=1}^r (p_{r+k} - p_{r-k})\Im(\lambda^k) \right] \\
&= 2i\Im(\lambda) \left[\sum_{k=1}^r (p_{r+k} - p_{r-k})I_k(c_1) \right] = 2i\Im(\lambda)L(c_1).
\end{aligned}$$

As $\deg I_k = k - 1$, $\deg L \leq r - 1$. So, by Proposition A.2 we get that $L \equiv 0$ and so $p_{r+k} = p_{r-k}$ for $k = 1, \dots, r$. Now,

$$Q(z) = p_r + \sum_{k=1}^r p_{r+k}R_k(z)$$

has degree r and hence has minimal degree among the allowed, and so it must be irreducible. As $\alpha_r^r = 1$, we get that the leading coefficient of Q is p_{2r} . \square

COROLLARY A.7. *Any ergodic linear automorphism of \mathbb{T}^4 is Anosov or pseudo-Anosov.*

Proof. We work as in the case of \mathbb{T}^5 . Now, with A^2 we may suppose its determinant is 1. Suppose it is neither Anosov nor pseudo-Anosov. If its characteristic polynomial is irreducible then we have that $P(z) = z^4 + az^2 + 1$ but then, if λ is a root of P , it must be a root of unity, or its modulus must be different from 1. So the characteristic polynomial must be reducible, $P = LQ$, but then, $\deg L = 2$, $\deg Q = 2$ and we are done. \square

PROPOSITION A.8. *For any $d \geq 3$ there is a linear automorphism of \mathbb{T}^{2d} as in the hypothesis of Theorem 1.1.*

Proof. Here we will work as in Lemma A.6. For d odd, define $Q(z) = z^d - 2$ and for d even, define $Q(z) = z^d - d2^{d-1}z + 2$. We shall prove that for any d ,

there are polynomials satisfying the required properties such that when λ is the root of modulus one and $c_1 = 2\Re(\lambda)$, then $Q(c_1) = 0$. We claim that for any d , Q has one and only one real root c_1 with modulus less than or equal to 2 and it satisfies $|c_1| < 2$. For d odd this is obvious as the only real root of Q is $2^{1/d}$. For d even, notice that Q has only one minimum and it is 2 and $Q(2) = 2 - (d-1)2^d < 0$ so Q has exactly two real roots, one less than 2 and the other bigger than 2. Moreover, Q has only positive real roots and so we get the desired claim.

We claim now that for any d , Q is irreducible. Suppose by contradiction that Q is reducible, $Q = LR$. We may suppose that the absolute value of the leading coefficients of L and R are both 1 and that $L(0) = 2$ and $R(0) = 1$. But this will imply (as is not hard to see) that all the coefficients of L must be even, thus contradicting that its leading coefficient is ± 1 . Let us define $P(x) = \sum_{k=0}^{2d} p_k x^k$ where $p_{d+k} = p_{d-k}$ for $k = 1, \dots, d$, and $Q(z) = p_d + \sum_{k=1}^d p_{d+k} R_k(z)$ where the R_k are defined as after the proof of Proposition A.2.

Now in order to find the p_k 's, we have to solve a $(d+1) \times (d+1)$ -equation, with integral coefficients (the coefficients of the R_k) and this is written in a triangular form and has only ones in the diagonal. So, it has a solution in the integers, and we may choose a solution having $p_{2d} = p_0 = 1$. We claim that P has only two roots with modulus one (λ and $\bar{\lambda}$) that are not roots of unity, that P is irreducible and is not a polynomial of a power. To prove this claim, first notice that Q is just the polynomial found in Corollary A.6. So, if P has another root with modulus 1 other than λ and $\bar{\lambda}$ then Q must have another real root with modulus less than or equal to 2. Moreover, it must have in fact two roots, because if not, $\lambda = \bar{\lambda}$ and hence $\lambda = \pm 1$ and so $|c_1| = 2$. If P were reducible, then there would be a polynomial P' with $\deg P' < \deg P$ and $P'(\lambda) = 0$ and then we would get a polynomial Q' with $\deg Q' < \deg Q$ with $Q'(c_1) = 0$, thus contradicting the irreducibility of Q . If it were a polynomial of a power, then it would have to have more than two roots with modulus 1. In fact, if $P(x) = T(x^n)$, $n \geq 2$, and μ is such that $\mu^n = \lambda^n$, if the only such μ are λ and $\bar{\lambda}$ then we must have that $n = 2$ and that $\lambda^2 \pm 1 = 0$ and thus, as $2d \geq 4$, this contradicts the irreducibility of P . If λ were a root of unity, then by the irreducibility of P , all the roots of P must be roots of unity and as P has exactly two roots with modulus 1, the multiplicity of λ must be bigger than 1 thus contradicting the irreducibility. Now, defining A by

- (1) $Ae_i = e_{i+1}$ for $i = 1, \dots, N-1$,
- (2) $Ae_N = -\sum_{i=0}^{N-1} p_i e_{i+1}$,

we see that the characteristic polynomial of A is just P . □

LEMMA A.9. *A is pseudo-Anosov if and only if the characteristic polynomial of A^l is irreducible for any $l \in \mathbb{Z}$, $l > 0$.*

Proof. If the characteristic polynomial of A^l is irreducible for any $l > 0$ then the characteristic polynomial of A , P_A , is irreducible. Suppose that $P_A(x) = Q(x^n)$ for some $n \geq 2$. We have that $P_{A^n}(x^n) = P_A(x)H(x) = Q(x^n)H(x)$ where $H(x) = \det\left(\sum_{k=0}^{n-1} x^k A^{-k}\right)$. But then, it is not hard to see that $H(x) = T(x^n)$ for some polynomial T and hence that $P_{A^n} = QT$, thus contradicting that P_{A^n} is irreducible.

Suppose now that A is pseudo-Anosov but P_{A^l} is reducible for some $l > 0$. Then, it is not hard to see that there is a nontrivial subgroup $S \subset \mathbb{Z}^N$ such that $A^l S = S$. Moreover there is a subgroup R such that:

$$\mathbb{R}^N = [S] \oplus A[S] \oplus \cdots \oplus A^{l-1}[S] \oplus [R]$$

where $[S]$ is the subspace generated by S , and hence $P_A(x) = Q(x^l)T(x)$, with Q and T the characteristics polynomials of $A^l|_S$ and $A|_R$ respectively. As P_A is irreducible and is not a polynomial of a power we get a contradiction. \square

LEMMA A.10 (Lemma 4.8). *For any $\delta > 0$ there is a constant $c = c(A, \delta)$ such that $r = \frac{N}{2} - 1$, $|n^c| \geq \frac{c}{|n|^{r+\delta}}$ for any $n \in \mathbb{Z}^N$, $n \neq 0$.*

Proof. Call λ the eigenvalue with modulus 1 and e_k the standard basis of \mathbb{R}^N . Then, because of the form of A we have $A^k e_1 = e_{k+1}$ for $k = 0, \dots, N-1$. So, given $n \in \mathbb{Z}^N$, $n = \sum_{k=0}^{N-1} n_{k+1} e_{k+1}$, we have $n^c = (\sum_{k=0}^{N-1} n_{k+1} \lambda^k) e_1^c$. Then $|n^c| \geq C |\sum_{k=0}^{N-1} n_{k+1} \lambda^k|$. Now, by Corollary A.6, $c_1^k = P_k(c_1)$ for any $k \geq 0$ where P_k is a polynomial with integral coefficients of degree less than or equal to $\frac{N}{2} - 1$. So, we can write

$$2\Re\left(\sum_{k=0}^{N-1} n_{k+1} \lambda^k\right) = \sum_{k=0}^{N-1} n_{k+1} R_k(c_1) = \sum_{k=0}^{N/2-1} L_k^1(n) c_1^k$$

and

$$\Im\left(\sum_{k=0}^{N-1} n_{k+1} \lambda^k\right) = \Im(\lambda) \sum_{k=0}^{N-1} n_{k+1} I_k(c_1) = \Im(\lambda) \sum_{k=0}^{N/2-1} L_k^2(n) c_1^k$$

where L_k^i is a homogeneous form for $k = 0, \dots, \frac{N}{2} - 1$, $i = 1, 2$. Finally, as c_1 is the root of a polynomial with integral coefficients, irreducible over the integers and with degree $\frac{N}{2}$, we can use Theorem A.1 and thus get that whenever $M_1 = \max(|L_1^1(n)|, \dots, |L_{N/2-1}^1(n)|) > 0$,

$$\left|\sum_{k=0}^{N-1} n_{k+1} \lambda^k\right| \geq \left|\Re\left(\sum_{k=0}^{N-1} n_{k+1} \lambda^k\right)\right| = \frac{1}{2} \left|\sum_{k=0}^{N/2-1} L_k^1(n) c_1^k\right| \geq \frac{c}{M_1^{r+\delta}}$$

and whenever $M_2 = \max(|L_1^2(n)|, \dots, |L_{N/2-1}^2(n)|) > 0$

$$\left|\sum_{k=0}^{N-1} n_{k+1} \lambda^k\right| \geq \left|\Im\left(\sum_{k=0}^{N-1} n_{k+1} \lambda^k\right)\right| = \frac{1}{\Im(\lambda)} \left|\sum_{k=0}^{N/2-1} L_k^2(n) c_1^k\right| \geq \frac{c}{M_2^{r+\delta}}$$

as $|L_k^i(n)| \leq C|n|$ for any $k, i = 1, 2$. Now, C does not depend on n , and the result follows whenever there is some $k \geq 1$ and i is such that $L_k^i(n) \neq 0$. If $L_k^i(n) = 0$ for any $k \geq 1$ and $i = 1, 2$ but $L_0^i(n) \neq 0$ the result also follows. So we must deal with the case that $L_k^i(n) = 0$ for any $k \geq 0$ and $i = 1, 2$, but this implies that $n^c = 0$ and this cannot happen since in this case we have that $n \in E^{su}$, and the irreducibility of A implies that $E^{su} = \mathbb{R}^N$ which contradicts the assumption. \square

LEMMA A.11 (Lemma 6.2). *If $N \geq 6$ there is $n \in \mathbb{Z}^N$ such that if the linear transformation $L : E^c \rightarrow \mathbb{R}^2$ is defined by $L(e_1^c) = (1, 0)$, and $L(e_2^c) = (0, 1)$ and $L(n^c) = \alpha$, then α satisfies a diophantine condition with exponent δ for any $\delta > 0$.*

Proof. Take $n = e_2 + e_4$. Once we define the linear transformation that sends e_1^c to $(1, 0)$, e_2^c to $(0, 1)$, it is not hard to see that it sends n^c to $(-c_1, c_1^2)$. Now, by Proposition A.2 we have that $1, -c_1, c_1^2$ are linearly independent over the rationals. So, using Theorem A.1 we get the lemma. \square

LEMMA A.12 (Lemma 6.6). *If $N = 4$ there exist n_1, n_2 such that if the linear transformation $L : E^c \rightarrow \mathbb{R}^2$ defined by $L(e_1^c) = (1, 0)$, and $L(e_2^c) = (0, 1)$ and $L(n_1^c) = \alpha_1$, $L(n_2^c) = \alpha_2$, then there is a constant $c > 0$ such that*

$$\max_{\nu=1,2} \| \|k \cdot \alpha_\nu\| \| \geq \frac{c}{|k|^2}$$

for any $k \in \mathbb{Z}^2$, $k \neq 0$.

Proof. Take $n_1 = (-p_1, 1 - p_2, -p_1, -1)$ and $n_2 = (1, 0, 1, 0)$ where the characteristic polynomial of A is $P(z) = z^4 + p_1z^3 + p_2z^2 + p_1z + 1$. Then we have that $L(n_1^c) = (c_1, 0)$ and $L(n_2^c) = (0, c_1)$ where c_1 is as before for λ the root of P of modulus 1. Now we have $Q(c_1) = c_1^2 + p_1c_1 + p_2 - 1 = 0$ and as Q is irreducible, this implies that there exists $c > 0$ such that $\| \|qc_1\| \| \geq \frac{c}{|q|^2}$ for any $q \in \mathbb{Z}$, $q \neq 0$. Given $k \in \mathbb{Z}^2$, $k \neq 0$ we get that $\| \|k \cdot \alpha_1\| \| \geq \frac{c}{|k_1|^2} \geq \frac{c}{|k|^2}$ if $k_1 \neq 0$ and the same holds for α_2 if $k_2 \neq 0$. As $k \neq 0$ we have the desired result. \square

Appendix B. Invariant manifolds

In this section we will show how to get the invariant foliations and how to prove regularity properties in their holonomies. We will follow [HPS] and [PSW].

PROPOSITION B.1. *If f is sufficiently C^r close to A then there exists*

$$\begin{aligned}\gamma^s &: \mathbb{R}^N \times E^s \rightarrow E^{cu}, & \gamma^{cs} &: \mathbb{R}^N \times E^{cs} \rightarrow E^u, \\ \gamma^u &: \mathbb{R}^N \times E^u \rightarrow E^{cs}, & \gamma^{cu} &: \mathbb{R}^N \times E^{cu} \rightarrow E^s, \\ \gamma^c &: \mathbb{R}^N \times E^c \rightarrow E^{su},\end{aligned}$$

such that if $\gamma^\sigma(x, \cdot) = \gamma_x^\sigma$, $\sigma = s, u, c, cs, cu$ then

$$W^\sigma(x) = x + \text{graph}(\gamma_x^\sigma) = \{x + v + \gamma_x^\sigma(v), v \in E^\sigma\},$$

$\gamma^\sigma(x + n, v) = \gamma^\sigma(x, v)$ and $\gamma^\sigma(x, 0) = 0$. Each γ^σ is continuous in the first variable and C^r in the second one. Moreover, $\text{Lip}(\gamma_x^\sigma) \leq \kappa$ where $\kappa = \kappa(f)$ and $\kappa(f) \rightarrow 0$ as $f \xrightarrow{C^1} A$.

Proof. By the invariant manifold theory, it is known that there exist $\varepsilon > 0$ and $\gamma^\sigma : \mathbb{T}^N \times E^\sigma(\varepsilon) \rightarrow E^\nu$, where σ and ν are related in the obvious way, with all the desired regularities. So we only have to prove the existence of the global transformations, i.e. that the invariant manifolds are locally a graph is a known fact. What is new here is that they are global graphs. We shall prove the existence of γ^u . The existence of the others follows in an analogous way changing the spaces accordingly. Let us define the space

$$G = \left\{ \begin{array}{l} \gamma : E^u \rightarrow E^{cs} \text{ continuous, such that} \\ |\gamma|_* < \infty, \text{Lip}(\gamma) < \infty \text{ and } |\gamma|_1 < \infty \end{array} \right\}$$

where $|\gamma|_* = \sup_{v \neq 0} \frac{|\gamma(v)|}{|v|}$, $\text{Lip}(\gamma)$ is the Lipschitz constant of γ and $|\gamma|_1 = \sup_{|v| \geq 2} \frac{|\gamma(v)|}{\log |v|}$. It is not hard to see that G with the norm $|\cdot|_*$ is a Banach space.

Define (on the Banach bundle $\mathbb{T}^N \times G$) the graph transform Γ :

$$\begin{array}{ccc} \mathbb{T}^N \times G & \xrightarrow{\Gamma} & \mathbb{T}^N \times G \\ \downarrow & & \downarrow \\ \mathbb{T}^N & \xrightarrow{f} & \mathbb{T}^N \end{array}$$

defined in the following way: for x and $\gamma \in G$,

$$g_{x,\gamma}^u : E^u \rightarrow E^u \quad g_{x,\gamma}^u(w) = A^u w + \varphi^u(x + w + \gamma(w)) - \varphi^u(x)$$

and

$$\Gamma(\gamma)(x, v) = A^{cs} \gamma((g_{x,\gamma}^u)^{-1}(v)) + \varphi^{cs}\left(x + (g_{x,\gamma}^u)^{-1}(v) + \gamma((g_{x,\gamma}^u)^{-1}(v))\right) - \varphi^{cs}(x).$$

There are constants $\kappa = \kappa(f)$ satisfying $\kappa(f) \rightarrow 0$ as $f \xrightarrow{C^1} A$ and $C > 0$ that only depend on the C^1 size of the neighborhood of A such that

$$G(\kappa, C) = \{\gamma \text{ such that } |\gamma|_* \leq \kappa, \text{Lip}(\gamma) \leq \kappa \text{ and } |\gamma|_1 \leq C\};$$

$G(\kappa, C)$ is closed in G and invariant under the action of the graph transform. Moreover, Γ acts here as a contraction. So there exists a section $\eta : \mathbb{T}^N \rightarrow G(\kappa, C)$ invariant under the graph transform. Define $\gamma^u(x, v) = \eta(p(x))(v)$ where $p : \mathbb{R}^N \rightarrow \mathbb{T}^N$ is the covering projection. The continuous dependence on f follows from the continuity of the invariant section in section theorems. \square

LEMMA B.2 (Lemma 2.2). *For any $x, y \in \mathbb{R}^N$,*

$$(1) \#W^s(x) \cap W^{cu}(y) = 1,$$

$$(2) \#W^u(x) \cap W^{cs}(y) = 1.$$

Proof. As always we are going to prove only the first one. To prove that $x, y \in \mathbb{R}^N$ intersect we must solve the following equation:

$$x + v^s + \gamma_x^s(v^s) = y + w^{cu} + \gamma_y^{cu}(w^{cu}).$$

Let $v^s = y^s - x^s + \gamma_y^{cu}(w^{cu})$ and define $l : E^{cu} \rightarrow E^{cu}$ by

$$l(w^{cu}) = w^{cu} + x^{cu} - y^{cu} + \gamma_x^s(y^s - x^s + \gamma_y^{cu}(w^{cu})) = w^{cu} + r(w^{cu}).$$

As we see easily, using the preceding proposition, $\text{Lip}(r) \leq \kappa^2$; so if $\kappa < 1$ we have that l is a homeomorphism. Hence there exists w^{cu} such that $l(w^{cu}) = 0$. It is not hard to see now that this w^{cu} and $v^s = y^s - x^s + \gamma_y^{cu}(w^{cu})$ are the only ones solving the above equation. \square

LEMMA B.3 (Lemma 2.1). *There exists $\kappa = \kappa(f)$ such that $\kappa(f) \rightarrow 0$ as $f \xrightarrow{\mathcal{C}^1} A$ and $C > 0$ that only depends on the \mathcal{C}^1 size of the neighborhood of A such that for $v \in E^\sigma$,*

$$(1) |\gamma_x^\sigma(v)| \leq C \log |v| \text{ for } \sigma = s, u, |v| \geq 2,$$

$$(2) |\gamma_x^\sigma(v)| \leq \kappa \text{ for } \sigma = c, cs, cu \text{ for any } v,$$

$$(3) |(\gamma_x^u(v))^s| \leq \kappa \text{ for any } v,$$

$$(4) |(\gamma_x^s(v))^u| \leq \kappa \text{ for any } v,$$

$$(5) |\gamma_x^\sigma(v)| \leq \kappa |v| \text{ for } \sigma = s, u, c, cs, cu \text{ for any } v.$$

Proof. The proof of (1) and (5) follows from Proposition B.1. The proof of (3) and (4) follows from (2) as the stable and unstable manifolds subfoliate the center-stable and center-unstable manifolds. And the proof of (2) follows essentially by the proof of the stability of the plaque expansive foliations in [HPS]. Nevertheless, let us give a proof of (2) that is a little bit simpler in this case. It follows essentially the idea of the proof of the Hartman-Grobman theorem.

Let us prove it for the case of $\sigma = cu$, the case of cs works as well. Denote $F = A + \psi$ and let us solve the cohomological equation

$$A^s \varphi^s - \varphi^s \circ F = \psi^s.$$

Now,

$$\varphi^s = - \sum_{k=0}^{+\infty} (A^s)^k \psi^s \circ F^{-(k+1)}$$

and $\varphi^s(x+n) = \varphi^s(x)$ for any $n \in \mathbb{Z}^N$, so that

$$\|\varphi^s\|_0 \leq \frac{1}{1 - \|A^s\|} \|\psi^s\|_0$$

where $\|\cdot\|_0$ is the sup-norm. Thus, if f is sufficiently \mathcal{C}^0 close to A then we may suppose $\|\varphi^s\| \leq \kappa/2$. Define $h^s : \mathbb{R}^N \rightarrow E^s$ by $h^s(x) = x^s + \varphi^s(x)$. Then $h^s \circ F = A^s h^s$. We claim that if $x \in W^{cu}(y)$ then $h^s(x) = h^s(y)$. Indeed

$$\begin{aligned} |h^s(x) - h^s(y)| &= |(A^s)^n h^s(F^{-n}(x)) - (A^s)^n h^s(F^{-n}(y))| \\ &\leq \|A^s\|^n (|F^{-n}(x) - F^{-n}(y)| + \kappa) \\ &\leq \|A^s\|^n (\mu^n |x - y| + \kappa) \end{aligned}$$

where $\mu = \sup \|DF^{-1}|_{E_f^{cu}}\|$ is as close to 1 as we want if f is \mathcal{C}^1 close to A . Hence as $\|A^s\|\mu < 1$ we have that $h^s(x) = h^s(y)$. Now, we claim that if $h^s(x) = h^s(y)$ then $W^{cu}(x) = W^{cu}(y)$. Take x and y such that $h^s(x) = h^s(y)$. When $z = W^s(x) \cap W^{cu}(y)$, we claim that $z = x$. As $z \in W^{cu}(y)$,

$$\begin{aligned} 0 &= |h^s(F^n(x)) - h^s(F^n(z))| \\ &= |(F^n(x) - F^n(z))^s + (\varphi^s(F^n(x)) - \varphi^s(F^n(z)))| \\ &\geq |(F^n(x) - F^n(z))^s| - \kappa \geq C|F^n(x) - F^n(z)| - \kappa. \end{aligned}$$

This last inequality follows because $z \in W^s(x)$. But then, letting $n \rightarrow -\infty$ we get a contradiction if $x \neq z$. So we get that $h^s(x) = h^s(y)$ if and only if $W^{cu}(x) = W^{cu}(y)$. Call now $H^s : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $H^s(x) = x^{cu} + h^s(x)$. Using those last properties, it is not hard to see that $H^s(x+n) = H^s(x) + n$ and that H^s is a homeomorphism. Moreover, if $y \in W^{cu}(x)$, then $H^s(y) = (y-x)^{cu} + H^s(x)$; thus we get that $H^s(W^{cu}(x)) \subset H^s(x) + E^{cu}$. On the other hand, if $z \in H^s(x) + E^{cu}$ then $z^s = (H^s(x))^s = h^s(x)$, and so, if $H^s(y) = z$ then $h^s(y) = (H^s(y))^s = z^s = h^s(x)$ and hence, $y \in W^{cu}(x)$. So we have that $H^s(W^{cu}(x)) = H^s(x) + E^{cu}$. Also, $(H^s)^{-1}(y) = \hat{h}^s(y) + y^{cu}$ for some $\hat{h}^s(y) = y^s + \hat{\varphi}^s(y)$, and so,

$$\begin{aligned} W^{cu}(x) &= (H^s)^{-1}(H^s(x) + E^{cu}) \\ &= \{x + v + \varphi^s(x) + \hat{\varphi}^s(H^s(x) + v) \text{ st } v \in E^{cu}\}. \end{aligned}$$

Hence we get that $\gamma_x^{cu}(v) = \varphi^s(x) + \hat{\varphi}^s(H^s(x) + v)$, and the proof then follows from the fact that $\hat{\varphi}^s(y) = -\varphi^s((H^s)^{-1}(y))$.

Now for the case $\sigma = c$, working in the same way, call φ^u the solution of the cohomological equation

$$A^u \varphi^u - \varphi^u \circ F = \psi^u.$$

Define $h^{su} : \mathbb{R}^N \rightarrow E^{su}$ by $h^{su}(x) = x^s + x^u + \varphi^s(x) + \varphi^u(x) = h^s(x) + h^u(x)$. Now, $h^{su}(x) = h^{su}(y)$ if and only if $W^c(x) = W^c(y)$, just because $W^{cs}(x) \cap W^{cu}(x) = W^c(x)$. Next, let $H^{su} : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $H^{su}(x) = x^c + h^{su}(x)$. It follows from the properties above that $H^{su}(x+n) = H^{su}(x) + n$ and that H^{su} is a homeomorphism. Moreover we have $H^{su}(W^c(x)) = H^{su}(x) + E^c$ and $(H^{su})^{-1}(y) = \hat{h}^{su}(y) + y^c$ for some $\hat{h}^{su}(y) = y^{su} + \hat{\varphi}^{su}(y)$, and so,

$$W^c(x) = (H^{su})^{-1}(H^{su}(x) + E^c) = \{x + v + \varphi^{su}(x) + \hat{\varphi}^{su}(H^{su}(x) + v) \text{ st } v \in E^c\}.$$

Hence, $\gamma_x^c(v) = \varphi^{su}(x) + \hat{\varphi}^{su}(H^{su}(x) + v)$, and the proof then follows from the fact that $\hat{\varphi}^{su}(y) = -\varphi^{su}((H^{su})^{-1}(y))$. \square

LEMMA B.4 (Lemma 2.3). *Given $C > 0$ and $\varepsilon > 0$ there is a neighborhood of A in the C^r topology such that for any f in this neighborhood, x and y with $|x - y| \leq C$, $x \in W^{cu}(y)$,*

$$\begin{aligned} \pi_{xy}^u : W^c(x) &\rightarrow W^c(y), & \pi_{xy}^u(z) &= W^u(z) \cap W^c(y), \\ P_{xy}^u : E^c &\rightarrow E^c, & P_{xy}^u &= (j_y^c)^{-1} \circ \pi_{xy}^u \circ j_x^c, \end{aligned}$$

and if

$$P_{xy}^u(z) = z + (x - y)^c + \varphi_{xy}(z),$$

then $\|\varphi_{xy}\|_{C^r} < \varepsilon$ where the sup-norm in all derivatives of order less than or equal to r is used. The same holds for the s-holonomy.

Proof. To prove this lemma, we will use the H^s built at the end of the proof of the preceding lemma and Theorem 6.7 on page 86 of [HPS]. Fix x and y and let us try to prove that their holonomy satisfies the required property. We omit the subindex xy whenever there is no confusion. Let us try to see what φ is. Define $\phi : E^c \times E^c \rightarrow E^c$ by

$$\phi(v, w) = v + (x - y)^c - w + \left[\gamma^u \left(j_x^c(v), (y + \gamma_y^c(w) - x - \gamma_x^c(v))^u \right) \right]^c.$$

Then it is not hard to see that P^u satisfies $\phi(v, P^u(v)) = 0$. So, using the implicit function theorem, we get that if ϕ is C^r and the derivative with respect to the second variable is invertible, then P^u is C^r . Moreover, it is not hard to see that if ϕ is close enough in the C^r -sup-norm to $v + (x - y) - w$, then this last property is satisfied and the C^r -sup-norm of φ will be small. So let us examine this last property. We see that the C^r -sup-norm in v and w of

$$(v, w) \rightarrow \left[\gamma^u \left(x + v + \gamma_x^c(v), (y + \gamma_y^c(w) - x - \gamma_x^c(v))^u \right) \right]^c$$

can be taken as small as wanted once the distance between x and y is fixed and f is close to A . To this end, first notice that the image of the map $(v, w) \rightarrow (y - x - \gamma_x^c(v) + \gamma_y^c(w))^u$ is contained in the ball $B_{(y-x)^u}^u(2\kappa)$. So, if the \mathcal{C}^r -sup-norms of γ_x^c and γ_y^c are small enough and the \mathcal{C}^r -sup-norm of $\gamma^u|_{W^c(x) \times B_{(y-x)^u}^u(2\kappa)}$ is small enough too, then, we get the desired property. That the \mathcal{C}^r -sup-norms of γ_x^c and γ_y^c are small follows from Lemma B.1 and Lemma B.3. Let us focus our attention on γ^u . Define $\hat{F} = H^s \circ F \circ (H^s)^{-1}$, $\hat{F} : W \rightarrow W$, where $W = \bigsqcup(p + E^{cu})$ is the disjoint union of the translate of the center-unstable space. It is a nonseparable $(c + u)$ -dimensional manifold. Moreover, H^s , looking like a diffeomorphism from $W' = \bigsqcup W^{cu}(p)$ onto W , is \mathcal{C}^r and moreover, if F is \mathcal{C}^r close to A , then \hat{F} is also \mathcal{C}^r close to A (looking at \hat{F} and A as diffeomorphisms from W onto W). Now, we build the graph transform over W essentially as we did in the proof of Proposition B.1 (changing G to \hat{G} , where \hat{G} is defined as G but changing E^{cs} to E^c ; we also change \mathbb{T}^N to W) and it turns out that this satisfies all the hypotheses of Theorem 6.7 of [HPS]. So, there is a \mathcal{C}^r section $\eta_{\hat{F}} : W \rightarrow \hat{G}$ depending continuously on \hat{F} . Using the properties of the norm on \hat{G} we get the desired property for $\gamma_{\hat{F}}^u$ and hence for γ^u using $H^s : W' \rightarrow W$. \square

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