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Higher symmetries of the Laplacian

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Abstract

We identify the symmetry algebra of the Laplacian on Euclidean space as an explicit quotient of the universal enveloping algebra of the Lie algebra of conformal motions. We construct analogues of these symmetries on a general conformal manifold.

1. Introduction

The space of smooth first order linear differential operators on $\mathbb{R}^n$ that preserve harmonic functions is closed under Lie bracket. For $n \geq 3$, it is finite-dimensional (of dimension $(n^2+3n+4)/2$). Its commutator subalgebra is isomorphic to $\mathfrak{so}(n+1,1)$, the Lie algebra of conformal motions of $\mathbb{R}^n$. Second order symmetries of the Laplacian on $\mathbb{R}^3$ were classified by Boyer, Kalnins, and Miller [6]. Commuting pairs of second order symmetries, as observed by Winternitz and Friš [52], correspond to separation of variables for the Laplacian. This leads to classical coördinate systems and special functions [6], [41].

General symmetries of the Laplacian on $\mathbb{R}^n$ give rise to an algebra, filtered by degree (see Definition 2 below). For $n \geq 3$, the filtering subspaces are finite-dimensional and closely related to the space of conformal Killing tensors as in Theorems 1 and 2 below. The main result of this article is an explicit algebraic description of this symmetry algebra (namely Theorem 3 and its Corollary 1). Most of this article is concerned with the Laplacian on $\mathbb{R}^n$. Its symmetries, however, admit conformally invariant analogues on a general Riemannian manifold. They are constructed in §5 and further discussed in §6.

The motivation for this article comes from physics, especially the recent theory of higher spin fields and their symmetries: see [40], [45], [48] and references therein. In particular, conformal Killing tensors arise explicitly in [40] and implicitly in [48] for similar reasons. Underlying this progress is the AdS/CFT correspondence [25], [38], [53]. Indeed, we shall use a version of

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this correspondence to prove Theorem 2 in §3 and to establish the algebraic structure of the symmetry algebra in §4.

Symmetry operators for the conformal Laplacian [31], Maxwell's equations [30], and the Dirac operator [39] have been much studied in general relativity. This is owing to the separation of variables that they induce. These matters are discussed further in §6.

This article is the result of questions and suggestions from Edward Witten. In particular, he suggested that Theorems 1 and 2 should be true and that they lead to an understanding of the symmetry algebra. For this, and other help, I am extremely grateful. I would also like to thank Erik van den Ban, David Calderbank, Andreas Čap, Rod Gover, Robin Graham, Keith Hannabuss, Bertram Kostant, Toshio Oshima, Paul Tod, Misha Vasiliev, and Joseph Wolf for useful conversations and communications. For detailed comments provided by the anonymous referee, I am much obliged.

2. Notation and statement of results

Sometimes we shall work on a Riemannian manifold, in which case ∇_a will denote the metric connection. Mostly, we shall be concerned with Euclidean space $\mathbb{R}^n$ and then $\nabla_a = \partial/\partial x^a$, differentiation in coördinates. In any case, we shall adopt the standard convention of raising and lowering indices with the metric g_{ab}. Thus, $\nabla^a = g^{ab}\nabla_b$ and $\Delta = \nabla^a\nabla_a$ is the Laplacian. Here and throughout, we employ the Einstein summation convention: repeated indices carry an implicit sum. The use of of indices does not refer to any particular choice of coördinates. Indices are merely markers, serving to identify the type of tensor under consideration. Formally, this is Penrose's abstract index notation [44].

We shall be working on Euclidean space $\mathbb{R}^n$ or on a Riemannian manifold of dimension n. We shall always suppose that $n \geq 3$ (ensuring that the space of conformal Killing vectors is finite-dimensional).

Kostant [36] considers first order linear differential operators $\mathcal{D}$ such that $[\mathcal{D}, \Delta] = h\Delta$ for some function h. We extend these considerations to higher order operators:

Definition 1. A symmetry of the Laplacian is a linear differential operator $\mathcal{D}$ so that $\Delta\mathcal{D} = \delta\Delta$ for some linear differential operator δ.

In particular, such a symmetry preserves harmonic functions. A rather trivial way in which $\mathcal{D}$ may be a symmetry of the Laplacian is if it is of the form $\mathcal{P}\Delta$ for some linear differential operator $\mathcal{P}$. Such an operator kills harmonic functions. In order to suppress such trivialities, we shall say that two symmetries of the Laplacian $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent if and only if $\mathcal{D}_1 - \mathcal{D}_2 =$

$\mathcal{P}\Delta$ for some $\mathcal{P}$. It is evident that symmetries of the Laplacian are closed under composition and that composition respects equivalence. Thus, we have an algebra:

Definition 2. The symmetry algebra $\mathcal{A}_n$ comprises symmetries of the Laplacian on $\mathbb{R}^n$, considered up to equivalence, with algebra operation induced by composition.

The aim of this article is to study this algebra. We shall also be able to say something about the corresponding algebra on a Riemannian manifold. The signature of the metric is irrelevant. All results have obvious counterparts in the pseudo-Riemannian setting. On Minkowski space, for example, these counterparts are concerned with symmetries of the wave operator.

Any linear differential operator on a Riemannian manifold may be written in the form

$$\mathcal{D} = V^{bc\cdots d}\nabla_b\nabla_c\cdots\nabla_d + \text{ lower order terms},$$

where $V^{bc\cdots d}$ is symmetric in its indices. This tensor is called the symbol of $\mathcal{D}$. We shall write $\phi^{(ab\cdots c)}$ for the symmetric part of a tensor $\phi^{ab\cdots c}$.

Definition 3. A conformal Killing tensor is a symmetric trace-free tensor field with s indices satisfying

$$\text{the trace-free part of } \nabla^{(a}V^{bc\cdots d)} = 0 \tag{1}$$

or, equivalently,

$$\nabla^{(a}V^{bc\cdots d)} = g^{(ab}\lambda^{c\cdots d)} \tag{2}$$

for some tensor field $\lambda^{c\cdots d}$ or, equivalently (by taking a trace),

$$\nabla^{(a}V^{bc\cdots d)} = \tfrac{s}{n+2s-2}g^{(ab}\nabla_e V^{c\cdots d)e}. \tag{3}$$

When $s = 1$, these equations define a conformal Killing vector.

THEOREM 1. *Any symmetry $\mathcal{D}$ of the Laplacian on a Riemannian manifold is canonically equivalent to one whose symbol is a conformal Killing tensor.*

Proof. Since

$$g^{(bc}\mu^{d\cdots e)}\nabla_b\nabla_c\nabla_d\cdots\nabla_e = \mu^{d\cdots e}\nabla_d\cdots\nabla_e\Delta + \text{ lower order terms},$$

any trace in the symbol of $\mathcal{D}$ may be canonically removed by using equivalence. Thus, let us suppose that

$$\mathcal{D} = V^{bcd\cdots e}\nabla_b\nabla_c\nabla_d\cdots\nabla_e + \text{ lower order terms}$$

is a symmetry of Δ and that $V^{bcd\cdots e}$ is trace-free symmetric. Then

$$\begin{aligned}\Delta\mathcal{D} \;&=\; V^{bcd\cdots e}\nabla_b\nabla_c\nabla_d\cdots\nabla_e\Delta + 2\nabla^{(a}V^{bcd\cdots e)}\nabla_a\nabla_b\nabla_c\nabla_d\cdots\nabla_e \\ &\qquad + \text{lower order terms}\end{aligned}$$

and the only way that the Laplacian can emerge from the sub-leading term is if (2) holds. □

THEOREM 2. *Suppose $V^{b\cdots c}$ is a conformal Killing tensor on $\mathbb{R}^n$ with s indices. Then there are canonically defined differential operators $\mathcal{D}_V$ and δ_V each having $V^{b\cdots c}$ as their symbol so that $\Delta\mathcal{D}_V = \delta_V\Delta$.*

We shall prove this theorem in the following section but here are some examples. When $s = 1$,

$$\mathcal{D}_V f = V^a\nabla_a f + \frac{n-2}{2n}(\nabla_a V^a)f \tag{4}$$

$$\delta_V f = V^a\nabla_a f + \frac{n+2}{2n}(\nabla_a V^a)f.$$

When $s = 2$,

$$\mathcal{D}_V f = V^{ab}\nabla_a\nabla_b f + \frac{n}{n+2}(\nabla_a V^{ab})\nabla_b f + \frac{n(n-2)}{4(n+2)(n+1)}(\nabla_a\nabla_b V^{ab})f \tag{5}$$

$$\delta_V f = V^{ab}\nabla_a\nabla_b f + \frac{n+4}{n+2}(\nabla_a V^{ab})\nabla_b f + \frac{n+4}{4(n+1)}(\nabla_a\nabla_b V^{ab})f.$$

On $\mathbb{R}^n$, we shall write down in §3 all solutions of the conformal Killing equation (2). For tensors with s indices, these solutions form a finite-dimensional vector space $\mathcal{K}_{n,s}$ of dimension

$$\frac{(n+s-3)!(n+s-2)!(n+2s-2)(n+2s-1)(n+2s)}{s!(s+1)!(n-2)!n!}. \tag{6}$$

Therefore, Theorem 2 shows the existence of many symmetries of the Laplacian on $\mathbb{R}^n$. Together with Theorem 1, it also allows us to put any symmetry into a canonical form. Specifically, if $\mathcal{D}$ is a symmetry operator of order s, then we may apply Theorem 1 to normalise its symbol $V^{b\cdots c}$ to be a conformal Killing tensor. Furthermore, the tensor field $V^{b\cdots c}$ is clearly determined solely by the equivalence class of $\mathcal{D}$. Now consider $\mathcal{D}-\mathcal{D}_V$ where $\mathcal{D}_V$ is from Theorem 2. By construction, this is a symmetry of the Laplacian order less than s. Continuing in this fashion we obtain a canonical form for $\mathcal{D}$ up to equivalence, namely

$$\mathcal{D}_{V_s} + \mathcal{D}_{V_{s-1}} + \cdots + \mathcal{D}_{V_2} + \mathcal{D}_{V_1} + V_0,$$

where V_t is a conformal Killing tensor with t indices (whence V_1 is a conformal Killing vector and V_0 is constant). As a vector space, therefore, Theorems 1 and 2 imply a canonical isomorphism

$$\mathcal{A}_n = \bigoplus_{s=0}^{\infty}\mathcal{K}_{n,s}\,.$$

In the following section, we shall identify $\mathcal{K}_{n,s}$ more explicitly. This will enable us, in §4, to prove the following theorem identifying the algebraic structure on $\mathcal{A}_n$. To state it, we need some notation. If we identify $\mathfrak{so}(n+1,1) = \bigwedge^2 \mathbb{R}^{n+2}$, then $V \wedge W$ is an irreducible component of the symmetric tensor product $V \odot W$, for $V, W \in \mathfrak{so}(n+1,1)$. Let $V \circledcirc W$ denote the trace-free part of $V \odot W - V \wedge W$.

THEOREM 3. *The algebra $\mathcal{A}_n$ is isomorphic to the tensor algebra*

$$\bigoplus_{s=0}^{\infty} \bigotimes^s \mathfrak{so}(n+1,1)$$

modulo the two-sided ideal generated by the elements

$$V \otimes W - V \circledcirc W - \frac{1}{2}[V,W] + \frac{n-2}{4(n+1)}\langle V,W\rangle \tag{7}$$

for $V, W \in \mathfrak{so}(n+1,1)$.

Here, $[V,W]$ denotes the Lie bracket of V and W and $\langle V,W\rangle$ their inner product with respect to the Killing form (as normalised in §4). We can rewrite Theorem 3 as saying that $\mathcal{A}_n$ is the associative algebra generated by $\mathfrak{so}(n+1,1)$ but subject to the relations:

$$VW - WV = [V,W] \quad \text{and} \quad VW + WV = 2V \circledcirc W - \frac{n-2}{2(n+1)}\langle V,W\rangle.$$

In other words, we have the following description of $\mathcal{A}_n$.

COROLLARY 1. *The algebra $\mathcal{A}_n$ is isomorphic to the enveloping algebra $\mathfrak{U}(\mathfrak{so}(n+1,1))$ modulo the two-sided ideal generated by the elements*

$$VW + WV - 2V \circledcirc W + \frac{n-2}{2(n+1)}\langle V,W\rangle$$

for $V, W \in \mathfrak{so}(n+1,1)$.

That $\mathcal{A}_n$ must be a quotient of $\mathfrak{U}(\mathfrak{so}(n+1,1))$ is already noted in [47] on general grounds. Corollary 1 describes the relevant ideal.

Note added in proof: Nolan Wallach has pointed out that this is the Joseph ideal.

In §5 we shall work on a general curved background and prove the following result.

THEOREM 4. *Suppose $V^{b\cdots c}$ is a trace-free symmetric tensor field with s indices on a conformal manifold. Then, for any $w \in \mathbb{R}$, there is a naturally defined, conformally invariant differential operator $\mathcal{D}_V$, taking densities of weight w to densities of the same weight w, and having $V^{b\cdots c}$ as its symbol. If the background metric is flat, $w = 1 - n/2$, and $V^{b\cdots c}$ is a conformal Killing tensor, then $\mathcal{D}_V$ agrees with the symmetry operator given in Theorem 2 and δ_V from Theorem 2 is given by the same formula but with $w = -1 - n/2$.*

When $s=2$, for example,

$$\mathcal{D}_V f = V^{ab}\nabla_a\nabla_b f - \frac{2(w-1)}{n+2}(\nabla_a V^{ab})\nabla_b f$$
$$+\frac{w(w-1)}{(n+2)(n+1)}(\nabla_a\nabla_b V^{ab})f + \frac{w(n+w)}{(n+1)(n-2)}R_{ab}V^{ab}f,$$

where R_{ab} is the Ricci tensor. This extends (5) to the curved setting.

3. Results in the flat case

The proof of Theorem 2 is best approached in the realm of conformal geometry. As detailed in [19, §2], $\mathbb{R}^n$ may be conformally compactified as the sphere $S^n \subset \mathbb{RP}_{n+1}$ of null directions of the indefinite quadratic form

$$\widetilde{g}_{AB}x^A x^B = 2x^0 x^\infty + g_{ab}x^a x^b \quad \text{for } x^A = (x^0, x^a, x^\infty) \tag{8}$$

on $\mathbb{R}^{n+2}$. Then, the conformal symmetries of S^n are induced by the action of $\mathrm{SO}(n+1,1)$ on $\mathbb{R}^{n+2}$ realised as those linear transformations preserving (8) and of unit determinant.

We need to incorporate the Laplacian into this picture. To do so, suppose F is a smooth function defined in a neighbourhood of the origin in $\mathbb{R}^n$. Then, for any $w \in \mathbb{R}$,

$$f(x^0, x^0x^a, -x^0x^ax_a/2) = (x^0)^w F(x^a) \quad \text{for } x^0 > 0$$

defines a smooth function f on a conical neighbourhood of $(1,0,0)$ in the null cone $\mathcal{N}$ of the quadratic form (8). This is a homogeneous function of degree w, namely $f(\lambda x^A) = \lambda^w f(x^A)$, for $\lambda > 0$. Conversely, F may be recovered from f by setting $x^0 = 1$. Hence, for fixed w, the functions F and f are equivalent. In the language of conformal differential geometry, w is the *conformal weight* of F when viewed on $\mathcal{N}$ in this way.

Following Fefferman and Graham [20], let us use the term 'ambient' to refer to objects defined on open subsets of $\mathbb{R}^{n+2}$. Let $\widetilde{\Delta}$ denote the ambient wave operator

$$\widetilde{\Delta} = \widetilde{g}^{AB}\frac{\partial^2}{\partial x^A \partial x^B}$$

where $\widetilde{g}^{AB}$ is the inverse of $\widetilde{g}_{AB}$. Let $r = \widetilde{g}_{AB}x^Ax^B$. Then $\mathcal{N} = \{r = 0\}$. Now consider f, homogeneous of degree w near $(1,0,0) \in \mathcal{N}$. Choose a smooth ambient extension $\widetilde{f}$ of f as a homogeneous function defined near $(1,0,0) \in \mathbb{R}^{n+2}$. Any other such extension will have the form $\widetilde{f} + rg$ where g is homogeneous of degree $w-2$. A simple calculation gives

$$\widetilde{\Delta}(rg) = r\widetilde{\Delta}g + 2(n+2w-2)g.$$

It follows immediately that, if $w = 1 - n/2$, then $\widetilde{\Delta}\widetilde{f}|_{\mathcal{N}}$ depends only on f. This defines a differential operator on $\mathbb{R}^n$ and, as detailed in [19], one may easily verify that it is the Laplacian. The main point of this construction is that it is manifestly invariant under the action of $\mathrm{SO}(n+1,1)$. We say that Δ is conformally invariant acting on conformal densities of weight $1-n/2$ on $\mathbb{R}^n$. It takes values in the conformal densities of weight $-1-n/2$.

This argument is due to Dirac [16]. It was rediscovered and extended to general massless fields by Hughston and Hurd [28]. Fefferman and Graham [20] significantly upgraded the construction to apply to general Riemannian manifolds, producing the conformal Laplacian or Yamabe operator

$$\square = \Delta - \frac{n-2}{4(n-1)} R, \tag{9}$$

where R is scalar curvature. Their construction is an early form of the AdS/CFT correspondence [38], [53]. Many other conformally invariant differential operators were constructed in this manner by Jenne [29]. Arbitrary powers of the Laplacian Δ^k are conformally invariant, in the flat case, when acting on densities of weight $k-n/2$. This is demonstrated in [19, Proposition 4.4] by an ambient argument.

Conformal Killing tensors have a simple ambient interpretation. This is to be expected since the equation (1) is conformally invariant. In fact, the differential operator that is the left-hand side of (1) is the first operator in a conformally invariant complex of operators known as the Bernstein-Gelfand-Gelfand complex [3], [5], [8], [13], [37]. This implies that the conformal Killing tensors on $\mathbb{R}^n$ form an irreducible representation of the conformal Lie algebra $\mathfrak{so}(n+1,1)$, namely

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & \cdots & & & \\ \hline & & & & \cdots & & & \\ \hline \end{array}}_{s \text{ boxes in each row}} \ \text{trace-free part}$$

as a Young tableau. This is the vector space that we earlier denoted by $\mathcal{K}_{n,s}$. The formula (6) for its dimension is easily obtained from [32]. It is convenient to adopt a realisation of this representation as tensors

$$V^{BQCR\cdots DS} \in \bigotimes^{2s} \mathbb{R}^{n+2}$$

that are skew in each pair of indices BQ, $CR, \ldots, DS$, are totally trace-free, and so that skewing over any three indices gives zero. (It follows that $V^{BQCR\cdots DS}$ is symmetric in the paired indices and that symmetrising over any $s+1$ indices gives zero.) When $s=1$, for example, we have

$$V^{BQ} \in \bigwedge^2 \mathbb{R}^{n+2} = \mathfrak{so}(n+1,1).$$

This is the well-known identification of conformal Killing vectors as elements of the conformal Lie algebra. More specifically, following the conventions of

[19], we have

$$V^B{}_Q = \begin{pmatrix} V^0{}_0 & V^0{}_q & V^0{}_\infty \\ V^b{}_0 & V^b{}_q & V^b{}_\infty \\ V^\infty{}_0 & V^\infty{}_q & V^\infty{}_\infty \end{pmatrix} = \begin{pmatrix} \lambda & r_q & 0 \\ s^b & m^b{}_q & -r^b \\ 0 & -s_q & -\lambda \end{pmatrix}$$

and corresponds to the conformal Killing vector

$$V^b = -s^b - m^b{}_q x^q + \lambda x^b + r_q x^q x^b - (1/2) x_q x^q r^b.$$

More succinctly, if we introduce

$$\Phi^B = \begin{pmatrix} 1 \\ x^b \\ -x^b x_b/2 \end{pmatrix} \quad \text{and} \quad \Psi^{bQ} = \begin{pmatrix} 0 \\ g^{bq} \\ -x^b \end{pmatrix},$$

then, using the ambient metric $\widetilde{g}_{AB}$ to lower indices,

$$V^{BQ} \mapsto V^b = \Phi_B V^{BQ} \Psi^b{}_Q$$

associates to the ambient skew tensor V^{BQ}, the corresponding Killing vector V^b. This formula immediately generalises:

$$V^{BQCR\cdots DS} \mapsto V^{bc\cdots d} = \Phi_B \Phi_C \cdots \Phi_D V^{BQCR\cdots DS} \Psi^b{}_Q \Psi^c{}_R \cdots \Psi^d{}_S.$$

It is readily verified that if $V^{BQCR\cdots DS}$ satisfies the symmetries listed above, then $V^{bc\cdots d}$ is trace-free symmetric and satisfies the conformal Killing equation (1).

Proposition 1. *This gives the general conformal Killing tensor.*

Proof. This is a special case of Lepowsky's generalisation [37] of the Bernstein-Gelfand-Gelfand resolution. A direct proof may be gleaned from [21]. The result is also noted in [34] and is proved in [15] assuming that the space of conformal Killing tensors is finite-dimensional. □

Proof of Theorem 2. We are now in a position to prove this theorem by ambient methods. Let ∂_A denote the ambient derivative $\partial/\partial x^A$ on $\mathbb{R}^{n+2}$ and for $V^{BQCR\cdots DS}$ as above, consider the differential operator

$$\mathfrak{D}_V = V^{BQCR\cdots DS} x_B x_C \cdots x_D \partial_Q \partial_R \cdots \partial_S$$

on $\mathbb{R}^{n+2}$. Evidently, $\mathfrak{D}_V$ preserves homogeneous functions. Recall that $r = x^A x_A$. Using $\partial_A r = 2x_A$, it follows that

$$\mathfrak{D}_V(rg) = r\mathfrak{D}_V g \quad \text{and} \quad \widetilde{\Delta}\mathfrak{D}_V = \mathfrak{D}_V \widetilde{\Delta}. \tag{10}$$

The first of these implies that $\mathfrak{D}_V$ induces differential operators on $\mathbb{R}^n$ for densities of any conformal weight: simply extend the corresponding homogeneous function on $\mathcal{N}$ into $\mathbb{R}^{n+2}$, apply $\mathfrak{D}_V$, and restrict back to $\mathcal{N}$. In particular, let us denote by $\mathcal{D}_V$ and δ_V the differential operators so induced on densities of weight $1 - n/2$ and $-1 - n/2$, respectively. Bearing in mind the ambient construction of the Laplacian, it follows immediately from the second equation of (10) that $\Delta \mathcal{D}_V = \delta_V \Delta$. It remains to calculate the symbols of $\mathcal{D}_V$ and δ_V.

To do this first note that, by construction, their order is at most s. For any such operator $\mathcal{D}$, the symbol at fixed $y \in \mathbb{R}^n$ is given by

$$\mathcal{D} \left. \frac{(x^b - y^b)(x^c - y^c) \cdots (x^d - y^d)}{s!} \right|_{x=y}.$$

This is easily computed. As a homogeneous function of degree w on $\mathcal{N}$, the function $x^b - y^b$ may be ambiently extended as

$$(x^0, x^a, x^\infty) \mapsto (x^0)^{w-1} x^b - (x^0)^w y^b.$$

Then,

$$\partial^Q((x^0)^{w-1}x^b - (x^0)^w y^b) = \begin{pmatrix} 0 \\ (x^0)^{w-1} g^{bq} \\ (w-1)(x^0)^{w-2} x^b - w(x^0)^{w-1} y^b \end{pmatrix}$$

and when $x^0 = 1$ and $x = y$, this becomes Ψ^{bQ} at y. Similarly, x^B becomes Φ^B and, in case $s = 1$, we obtain $\Phi_B V^{BQ} \Psi^b{}_Q$. In other words, the symbol is V^b no matter what is the weight. The case of general s is similar. $\square$

Notice that, not only have we proved Theorem 2, but also we have a very simple ambient construction of the symmetries $\mathcal{D}_V$. Explicit formulae for $\mathcal{D}_V$ are another matter. Such formulae can, of course, be derived from the ambient construction but an easier route, using conformal invariance, will be provided in §5.

4. The algebraic structure of $\mathcal{A}_n$

In view of Theorem 2, Proposition 1, and the discussion in §2, we have identified $\mathcal{A}_n$ as a vector space:

$$\mathcal{A}_n \cong \bigoplus_{s=0}^{\infty} \underbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \cdots & \; & \; & \; \\ \hline \; & \; & \; & \; & \cdots & \; & \; & \; \\ \hline \end{array}}_{s} \ \text{trace-free part} \tag{11}$$

but we have yet to identify $\mathcal{A}_n$ as an associative algebra. To do this, let us first consider the composition $\mathcal{D}_V \mathcal{D}_W$ in case $V, W \in \mathfrak{so}(n+1,1)$. As ambient tensors, V^{BQ} and W^{CR} are skew. From the proof of Theorem 2, the operators $\mathcal{D}_V$ and $\mathcal{D}_W$ on $\mathbb{R}^n$ are induced by the ambient operators

$$\mathfrak{D}_V = V^{BQ} x_B \partial_Q \quad \text{and} \quad \mathfrak{D}_W = W^{CR} x_C \partial_R,$$

respectively. Their composition is, therefore, induced by

$$\mathfrak{D}_V\mathfrak{D}_W = V^{BQ}W^{CR}x_Bx_C\partial_Q\partial_R + V^B{}_CW^{CR}x_B\partial_R. \tag{12}$$

If we write

$$V^{BQ}W^{CR} = T^{BQCR} + \widetilde{g}^{BC}U^{QR} + \widetilde{g}^{QR}U^{BC} - \widetilde{g}^{QC}U^{BR} - \widetilde{g}^{BR}U^{QC}$$

where

$$U^{BR} = -\frac{1}{n}V^B{}_CW^{CR} + \frac{1}{2n(n+1)}V^Q{}_CW^C{}_Q\widetilde{g}^{BR}, \tag{13}$$

then it is easy to verify that T^{BQCR} is totally trace-free. Now, from (12), we may rewrite

$$\begin{aligned}\mathfrak{D}_V\mathfrak{D}_W &= T^{BQCR}x_Bx_C\partial_Q\partial_R + rU^{QR}\partial_Q\partial_R + U^{BC}x_Bx_C\widetilde{\Delta}\\ &\quad - U^{BR}x_Bx^C\partial_C\partial_R - U^{QC}x_Cx^B\partial_B\partial_Q + V^B{}_CW^{CR}x_B\partial_R\end{aligned}$$

and, bearing in mind (13) and that $x^C\partial_C$ is the Euler operator, if f has homogeneity w, then

$$\begin{aligned}\mathfrak{D}_V\mathfrak{D}_Wf &= T^{BQCR}x_Bx_C\partial_Q\partial_Rf + rU^{QR}\partial_Q\partial_Rf + U^{BC}x_Bx_C\widetilde{\Delta}f\\ &\quad + \frac{w-1+n}{n}V^B{}_CW^{CR}x_B\partial_Rf + \frac{w-1}{n}V^R{}_CW^{CB}x_B\partial_Rf\\ &\quad - \frac{w(w-1)}{n(n+1)}V^R{}_CW^C{}_Rf.\end{aligned}$$

In particular, if $w = 1 - n/2$, then

$$\begin{aligned}\mathfrak{D}_V\mathfrak{D}_Wf =\,& T^{BQCR}x_Bx_C\partial_Q\partial_Rf + rU^{QR}\partial_Q\partial_Rf + U^{BC}x_Bx_C\widetilde{\Delta}f\\ &+\frac{1}{2}V^B{}_CW^{CR}x_B\partial_Rf - \frac{1}{2}V^R{}_CW^{CB}x_B\partial_Rf - \frac{n-2}{4(n+1)}V^R{}_CW^C{}_Rf.\end{aligned}$$

Noting that $x_Bx_C\partial_Q\partial_R$ is symmetric in BC and QR, we may rewrite the first term and obtain, upon restriction to $\mathcal{N}$,

$$\mathcal{D}_V\mathcal{D}_W \equiv \mathcal{D}_{(VW)_2} + \mathcal{D}_{(VW)_1} + \mathcal{D}_{(VW)_0} \mod \Delta, \tag{14}$$

as predicted by Theorems 1 and 2, where

$$\begin{aligned}(VW)_2{}^{BQCR} &= (T^{BQCR} + T^{CRBQ})/3\\ &\qquad + (T^{BCQR} + T^{QRBC} - T^{QCBR} - T^{BRQC})/6\\ (VW)_1{}^{BQ} &= (V^B{}_CW^{CQ} - V^Q{}_CW^{CB})/2\\ (VW)_0 &= -(n-2)V^R{}_CW^C{}_R/(4(n+1)).\end{aligned}$$

Each of these expressions has a simple interpretation as follows. The first of them is the highest weight part of $V \otimes W$. More specifically, $\bigotimes^2 \mathfrak{so}(n+1,1)$

decomposes into six irreducibles:

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{2,2}_{\,\circ} \oplus \ydiagram{2}_{\,\circ} \oplus \mathbb{R} \oplus \ydiagram{2,1,1}_{\,\circ} \oplus \ydiagram{1,1} \oplus \ydiagram{1,1,1,1} \tag{15}$$

where $\circ$ denotes the trace-free part. The projection of $V \otimes W$ into the first of these irreducibles is $V \circledcirc W$. (More generally, the highest weight part is known as the Cartan product [17, Supplement].) The projection

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \ni V^{BQ}W^{CR} \mapsto V^{B}{}_{C}W^{CQ} - V^{Q}{}_{C}W^{CB} \in \ydiagram{1,1}$$

is the Lie bracket $V \otimes W \mapsto [V, W]$ and the projection

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \ni V^{BQ}W^{CR} \mapsto V^{R}{}_{C}W^{C}{}_{R} \in \mathbb{R}$$

is the Killing form $V \otimes W \mapsto \langle V, W\rangle$. Thus, we may rewrite (14) as

$$\mathcal{D}_V\mathcal{D}_W \equiv \mathcal{D}_{V \circledcirc W} + \frac{1}{2}\mathcal{D}_{[V,W]} - \frac{n-2}{4(n+1)}\mathcal{D}_{\langle V,W\rangle} \mod \Delta. \tag{16}$$

In particular, the other irreducibles of (15) are mapped by $\mathcal{D}$ to zero.

Proof of Theorem 3. Define a map from the tensor algebra to $\mathcal{A}_n$ by

$$V_1 \otimes V_2 \otimes \cdots \otimes V_s \longmapsto \mathcal{D}_{V_1}\mathcal{D}_{V_2}\cdots\mathcal{D}_{V_s},$$

extended by linearity. From (16) it follows that the elements (7) are mapped to zero. To complete the proof, it suffices to consider the corresponding graded algebras. The graded algebra of $\mathcal{A}_n$ is (11) under Cartan product. We must show that the kernel of the mapping

$$\bigoplus_{s=0}^{\infty} \bigotimes^{s} \ydiagram{1,1} \longrightarrow \bigoplus_{s=0}^{\infty} \underbrace{\begin{ytableau} \ & \ & \ & \cdots & \ & \ & \ \\ \ & \ & \ & \cdots & \ & \ & \ \end{ytableau}}_{s}{}_{\circ} \tag{17}$$

is the two-sided ideal generated by $V \otimes W - V \circledcirc W$ for $V, W \in \ydiagram{1,1}$. Equivalently, let us group the decomposition (15) as

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{2,2}_{\,\circ} \oplus \mathcal{I}_2\,.$$

Then $\mathcal{I}_2$ is claimed to generate the kernel of (17). In degree $s = 2$, this is true by definition. In degree $s = 3$, we must show that

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{3,3}_{\,\circ} \oplus \Big(\Big(\mathcal{I}_2 \otimes \ydiagram{1,1}\Big) + \Big(\ydiagram{1,1} \otimes \mathcal{I}_2\Big)\Big). \tag{18}$$

To do this, we first check, by inspection, that

$$\ydiagram{3,3}_{\circ} = \left(\ydiagram{2,2}_{\circ} \otimes \ydiagram{1,1} \right) \cap \left(\ydiagram{1,1} \otimes \ydiagram{2,2}_{\circ} \right). \tag{19}$$

Specifically, the right-hand side consists of tensors T^{BQCRDS} that are skew in the pairs BQ, CR, and DS, that are trace-free in the indices $BQCR$ and $CRDS$, and so that skewing over any three indices from $BQCR$ or $CRDS$ gives zero. In particular, T^{BQCRDS} is symmetric in the paired indices and, therefore, totally trace-free. To characterise an element of the left-hand side, it remains to show that skewing over BCD gives zero:

$$\begin{aligned} &T^{BQCRDS} + T^{CQDRBS} + T^{DQBRCS} \\ &\quad -T^{CQBRDS} - T^{DQCRBS} - T^{BQDRCS} \\ &\qquad = T^{BCQRDS} + T^{CDQRBS} + T^{DBQRCS} \\ &\qquad\quad = T^{QRBCDS} + T^{QRCDBS} + T^{QRDBCS} = 0. \end{aligned}$$

To continue with (18), we may as well establish the corresponding statement for representations of $\mathrm{SO}(n{+}2)$ since explicit formulae for decomposing representations of the orthogonal group are independent of the signature. (Equivalently, we may complexify and employ Weyl's unitary trick.) Then, the left-hand side of (18) admits an invariant inner product with respect to which the projections onto

$$\ydiagram{2,2}_{\circ} \otimes \ydiagram{1,1} \qquad \text{and} \qquad \ydiagram{1,1} \otimes \ydiagram{2,2}_{\circ}$$

are orthogonal; let us denote them by P and Q, respectively. Then, (19) says that

$$\ydiagram{3,3}_{\circ} = \operatorname{im} P \cap \operatorname{im} Q$$

and we are required to show that

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \otimes \ydiagram{1,1} = \bigl(\operatorname{im} P \cap \operatorname{im} Q\bigr) \oplus \bigl(\ker P + \ker Q\bigr).$$

This is a fact concerning orthogonal projections. The composition QP preserves $(\operatorname{im} P \cap \operatorname{im} Q)^{\perp}$ and is norm-decreasing there. Hence, $\mathrm{Id} - QP$ is invertible on this subspace. Therefore, we may write

$$\begin{aligned} T &= (\mathrm{Id} - QP)(\mathrm{Id} - QP)^{-1} T \\ &= \bigl((\mathrm{Id} - P) + (\mathrm{Id} - Q)P\bigr)(\mathrm{Id} - QP)^{-1} T \end{aligned}$$

for $T \in (\operatorname{im} P \cap \operatorname{im} Q)^{\perp}$, an expression evidently in $\ker P + \ker Q$. This completes the proof of (18) and hence that $\mathcal{I}_3$, the degree $s = 3$ component of the kernel of (17), is generated by $\mathcal{I}_2$. Higher components are similarly dealt with by induction. $\square$

5. Explicit formulae and the curved case

The ambient construction of $\mathcal{D}_V$ given in the proof of Theorem 2 may be converted into explicit formulae on $\mathbb{R}^n$ using the coördinates (4.4) of [19]. It is more convenient, however, to derive these formulae from their conformal invariance since, at the same time, we shall obtain conformally invariant differential operators valid on any Riemannian manifold. We implicitly follow the 'Lie algebra cohomology' method of [4, §5]. An alternative approach would be to follow Gover [23], [24], generalising the ambient operator $x_B\partial_Q - x_Q\partial_B$ to the curved setting (see also [10], [11]). One could also work directly with the ambient metric construction of Fefferman and Graham [20].

Proof of Theorem 4. We shall follow the conventions of [2] concerning conformal geometry. On a conformal manifold, we must distinguish between covariant and contravariant tensors but we can always lower indices if we keep track of the conformal weight: if V^a has weight v, then $V_a = g_{ab}V^b$ has weight $v+2$. Unless otherwise stated, we shall suppose that all tensors in the following discussion are covariant. If σ and τ are symmetric trace-free tensors, we shall write $\sigma \circ \tau$ to mean the symmetric trace-free part of $\sigma \otimes \tau$. For example,

$$(\sigma \circ \tau)_{abc} = \sigma_{(a}\tau_{bc)} - \tfrac{2}{n+2} g_{(ab}\sigma^d\tau_{c)d}.$$

Suppose τ^k is a symmetric trace-free tensor having k indices and of conformal weight w. Write $\nabla \circ \tau^k$ for the symmetric trace free part of $\nabla\tau^k$. Then, under conformal change of metric $g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab}$,

$$\hat{\nabla} \circ \tau^k = \nabla \circ \tau^k + (w-2k)\Upsilon \circ \tau^k, \tag{20}$$

where $\Upsilon = d\log\Omega$. Writing Φ_{ab} for the trace-free part of $\frac{1}{n-2}R_{ab}$,

$$\hat{\Phi} = \Phi - \nabla \circ \Upsilon + \Upsilon \circ \Upsilon. \tag{21}$$

Suppose f is a density of conformal weight $w \notin \{0, 1, \ldots, s-1\}$ and consider the symmetric trace-free tensors τ^k with k indices and of weight w defined inductively by

$$\begin{aligned} \tau^0 &= s!w(w-1)\cdots(w-s+1)f, \\ \tau^1 &= s!(w-1)\cdots(w-s+1)\nabla f, \quad \text{and} \\ \tau^k &= \frac{1}{k(w-k+1)}\left(\nabla \circ \tau^{k-1} + \Phi \circ \tau^{k-2}\right) \quad \text{for } 2 \le k \le s. \end{aligned} \tag{22}$$

For example, if $s=3$, then

$$\begin{aligned}\tau^0&=6w(w-1)(w-2)f\\ \tau^1&=6(w-1)(w-2)\nabla f\\ \tau^2&=3(w-2)(\nabla\circ\nabla f+w\Phi\circ f)\\ \tau^3&=\nabla\circ\nabla\circ\nabla f+(3w-2)\Phi\circ\nabla f+w(\nabla\circ\Phi)f.\end{aligned}\tag{23}$$

These formulae also make sense for $w\in\{0,1,2\}$. More generally, τ^k are well-defined for all w: the apparently rational coefficients introduced in their inductive definition are, in fact, polynomial. From (20), (21), and (22), it now follows easily that

$$\hat\tau^k=\tau^k+\Upsilon\circ\tau^{k-1}+\tfrac{1}{2}\Upsilon\circ\Upsilon\circ\tau^{k-2}+\cdots+\tfrac{1}{k!}\underbrace{\Upsilon\circ\cdots\circ\Upsilon}_{k}\circ\tau^0.\tag{24}$$

If ρ and σ are symmetric trace-free tensors, write $\rho\lrcorner\sigma$ to mean the symmetric trace-free tensor obtained by natural contraction. For example,

$$(\rho\lrcorner\sigma)_{abc}=\rho^{de}\sigma_{abcde}.$$

Recall that $V^{bc\cdots d}$ is a trace-free symmetric tensor field with s indices. Denote $V_{bc\cdots d}$ by σ_s. It has conformal weight $2s$. Define

$$\sigma_{s-1}=-\tfrac{1}{n+2s-2}\nabla^bV_{bc\cdots d}=-\tfrac{1}{n+2s-2}\nabla\lrcorner\sigma_s$$

and continue with

$$\sigma_{s-k}=-\tfrac{1}{k(n+2s-k-1)}(\nabla\lrcorner\sigma_{s-k+1}-\Phi\lrcorner\sigma_{s-k+2})\quad\text{for } 2\le k\le s.\tag{25}$$

Under rescaling of the metric $g_{ab}\mapsto\hat g_{ab}=\Omega^2g_{ab}$,

$$\hat\nabla\lrcorner\sigma_{s-k}=\nabla\lrcorner\sigma_{s-k}+(n+2s-2k-2)\Upsilon\lrcorner\sigma_{s-k}$$

and it follows that

$$\begin{aligned}\hat\sigma_{s-k}=\sigma_{s-k}-\Upsilon\lrcorner\sigma_{s-k+1}+\tfrac{1}{2}(\Upsilon\circ\Upsilon)\lrcorner\sigma_{s-k+2}+\cdots\\ \cdots+(-1)^k\frac{1}{k!}\underbrace{(\Upsilon\circ\cdots\circ\Upsilon)}_{k}\lrcorner\sigma_s.\end{aligned}\tag{26}$$

From (24) and (26) we conclude that

$$\sigma_s\lrcorner\tau^s+\sigma_{s-1}\lrcorner\tau^{s-1}\cdots+\sigma_1\lrcorner\tau^1+\sigma_0\tau^0\tag{27}$$

is conformally invariant. By construction, the leading term in τ^s is the s^{th} trace-free symmetric covariant derivative of f. Therefore, the expression (27) has the form

$$V^{bc\cdots d}\nabla_b\nabla_c\cdots\nabla_df+\text{lower order terms, linear in } V \text{ and } f.$$

This is our definition of $\mathcal{D}_V f$. It is a conformally invariant bilinear differential pairing of V and f and is natural in the sense of [33]. It is easily verified that the formulae (22) are forced by linearity in f, naturality, and the simple conformal transformation (24). Similarly, for the collection $\sigma_s, \ldots, \sigma_0$, linear in V. We conclude that there is no choice in $\mathcal{D}_V f$ and, in the flat case, it must agree with the ambient construction in the proof of Theorem 2. $\square$

A side-effect of this proof is the construction of certain conformally invariant operators. The use of Ricci-corrected covariant derivatives in this regard, as in (23) and (25), has a long history (the idea appears in [55]; see also [12], [22]) and has recently been formalised in [9]. For example, if we set $w = s-1$ in the formulae for τ^k, only τ^s is nonzero and (24) implies that τ^s is conformally invariant. Thus, when f has weight $w \in \mathbb{Z}_{\geq 0}$, we obtain a well-known series [4], [12], [22] of conformally invariant operators of the form

$$f \mapsto \underbrace{\nabla \circ \nabla \circ \cdots \circ \nabla}_{w+1} f + \text{lower order terms.}$$

When $s = 3$, for example, (23) yields

$$\nabla \circ \nabla \circ \nabla f + 4\Phi \circ \nabla f + 2(\nabla \circ \Phi) f.$$

acting on f of weight 2. The operators in [18] may be constructed by similar means.

Invariant bilinear differential pairings also appear as the cup product of Calderbank and Diemer [8]. The pairing $(V, f) \mapsto \mathcal{D}_V f$ of Theorem 4 is evidently in the same vein but only when $w = s$ is it a special case (from the $\mathfrak{so}(n+1,1)$-invariant pairing $\mathcal{K}_{n,s} \otimes \circledcirc^s \mathbb{R}^{n+2} \to \circledcirc^s \mathbb{R}^{n+2}$).

The construction in the proof of Theorem 4 gives rise to a formula for $\mathcal{D}_V f$ in the flat case, namely

$$\sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} \tfrac{(w-s+1)\cdots(w-k)}{(n+2s-2)\cdots(n+s+k-1)} (\underbrace{\nabla \lrcorner \cdots \lrcorner \nabla}_{s-k} \lrcorner V) \lrcorner \underbrace{\nabla \cdots \nabla}_{k} f$$

Notice that, in this formula, there is no need explicitly to remove the trace from $\nabla \cdots \nabla f$ since it is contracted with $\nabla \lrcorner \cdots \lrcorner \nabla \lrcorner V$, which is trace-free. The formulae (4) and (5) are special cases.

Alternative proof of Theorem 2. The comment just made implies that we can write $\mathcal{D}_V f = \sum_{k=0}^{s} \sigma_k \lrcorner \bar{\tau}^k$ where, following (22) and (25),

$$\begin{aligned} (k+1)(w-k)\bar{\tau}^{k+1} &= \nabla \bar{\tau}^k \\ (s-k)(n+s+k-1)\sigma_k &= -\nabla \lrcorner \sigma_{k+1}. \end{aligned} \tag{28}$$

We need some differential consequences of V being a conformal Killing tensor on $\mathbb{R}^n$, specifically that

$$(s-k+1)(n+s+k-2)\underbrace{\nabla\lrcorner\cdots\lrcorner\nabla}_{s-k}\lrcorner\Delta V$$
$$=k\Big[(k-1)g\odot\underbrace{\nabla\lrcorner\cdots\lrcorner\nabla}_{s-k+2}\lrcorner V-(n+2k-4)\nabla\odot\underbrace{\nabla\lrcorner\cdots\lrcorner\nabla}_{s-k+1}\lrcorner V\Big]$$

where $\odot$ denotes symmetric tensor product. The case $k=s+1$ is (3) and the general case is obtained by taking $s-k+1$ divergences thereof. It now follows from (28) that

$$\Delta\sigma_k=k(n+2k-4)\nabla\odot\sigma_{k-1}+k(k-1)(s-k+2)(n+s+k-3)g\odot\sigma_{k-2}.$$

It is convenient to define $\sigma_{s+1}=0$ so that this equation also holds when $k=s+1$. At the same time let us define $\bar{\tau}^{s+1}$ by (28). Bearing in mind that $\nabla\bar{\tau}^k$ is automatically symmetric,

$$\Delta\mathcal{D}_V f=\sum_{k=0}^{s}\Delta(\sigma_k)\lrcorner\bar{\tau}^k+2(\nabla\odot\sigma_k)\lrcorner(\nabla\bar{\tau}^k)+\sigma_k\lrcorner\Delta\bar{\tau}^k$$

and substituting from (28) for $\nabla\bar{\tau}^k$, we obtain

$$\sum_{k=0}^{s}\Delta(\sigma_k)\lrcorner\bar{\tau}^k+2(k+1)(w-k)(\nabla\odot\sigma_k)\lrcorner(\bar{\tau}^{k+1})+\sigma_k\lrcorner\Delta\bar{\tau}^k$$
$$=\sum_{k=0}^{s+1}\Delta(\sigma_k)\lrcorner\bar{\tau}^k+2k(w-k+1)(\nabla\odot\sigma_{k-1})\lrcorner(\bar{\tau}^k)+\sigma_k\lrcorner\Delta\bar{\tau}^k$$

in which the cross terms cancel provided

$$k(n+2k-4)+2k(w-k+1)=0,\quad\forall k.$$

This is true when $w=1-n/2$, as it should be in the conformally invariant formulation of Theorem 2. With this value of w, we obtain from (28)

$$k(k-1)(n+2k-4)(n+2k-6)g\lrcorner\bar{\tau}^k=4g\lrcorner\nabla\nabla\bar{\tau}^{k-2}=4\Delta\bar{\tau}^{k-2}$$

and so

$$\Delta\mathcal{D}_V f=\sum_{k=2}^{s+1}4\frac{(s-k+2)(n+s+k-3)}{(n+2k-4)(n+2k-6)}\sigma_{k-2}\lrcorner\Delta\bar{\tau}^{k-2}+\sum_{k=0}^{s}\sigma_k\lrcorner\Delta\bar{\tau}^k \tag{29}$$
$$=\sum_{k=0}^{s}\left(1+4\frac{(s-k)(n+s+k-1)}{(n+2k)(n+2k-2)}\right)\sigma_k\lrcorner\Delta\bar{\tau}^k$$
$$=\sum_{k=0}^{s}\frac{(n+2s)(n+2s-2)}{(n+2k)(n+2k-2)}\sigma_k\lrcorner\Delta\bar{\tau}^k.$$

If $\Delta f = 0$, then $\Delta \bar{\tau}^k = 0$ for all k and so $\Delta \mathcal{D}_V f = 0$, as required. More precisely, the final expression of (29) is $\mathcal{D}_V$ applied to Δf, having conformal weight $-1 - n/2$. $\square$

This alternative proof, though direct, is a brute force calculation. The ambient proof given in §3 is more conceptual. This is typical of the AdS/CFT correspondence with effects more clearly visible 'in the bulk'.

Explicit formulae allow us, in principle, to calculate the composition $\mathcal{D}_V \mathcal{D}_W$ for conformal Killing tensors V and W, then to throw the result into canonical form in accordance with Theorems 1 and 2. In practise, this is difficult but if V and W are Killing vectors, we find

$$\mathcal{D}_V \mathcal{D}_W = \mathcal{D}_{V \circledcirc W} + \frac{1}{2}\mathcal{D}_{[V,W]} - \frac{n-2}{4(n+1)}\mathcal{D}_{\langle V,W \rangle} + \frac{1}{n} V^a W_a \Delta, \tag{30}$$

where

$$\begin{aligned}
(V \circledcirc W)^{ab} &= V^{(a} W^{b)} - \frac{1}{n} g^{ab} V^c W_c \\
[V, W]^a &= V^b \nabla_b W^a - W^b \nabla_b V^a \\
\langle V, W \rangle &= \frac{n+2}{n} (\nabla_b V^a)(\nabla_a W^b) - \frac{n+2}{n^2} (\nabla_a V^a)(\nabla_b W^b) \\
&\quad - \frac{n+2}{n^2} \big[V^a \nabla_a \nabla_b W^b + W^a \nabla_a \nabla_b V^b \big] + \frac{1}{n} \Delta (V_a W^a).
\end{aligned} \tag{31}$$

It is a differential consequence of the conformal Killing equation that $V \circledcirc W$ is a conformal Killing tensor, $[V, W]$ is a conformal Killing vector (the usual Lie bracket of vector fields), and $\langle V, W \rangle$ is constant. Of course, (30) is the form taken by (16) when written on $\mathbb{R}^n$. With the addition of 'curvature correction terms'

$$-\frac{2(n+2)}{n(n-2)} R_{ab} V^a W^b + \frac{2}{(n-1)(n-2)} R \, V_a W^a \,,$$

the differential pairing $\langle V, W \rangle$ turns out to be conformally invariant on a general Riemannian manifold and on arbitrary vector fields . This is in accordance with the cup product of [8].

6. Concluding remarks

Several questions remain unanswered, the most obvious of which are concerned with what happens in the curved setting. Though Theorem 1 is stated for the Laplacian, its proof is equally valid for the conformal Laplacian (9). The operators of Theorem 4 are conformally invariant and natural in the sense of [33]. But it is difficult to say whether they are symmetry operators of the conformal Laplacian. The conformal Killing equation (1) is overdetermined and generically has no solutions. Even when it has, the equation $\square \mathcal{D} = \delta \square$, in which $\square$ denotes the conformal Laplacian (9), might only hold up to curvature terms—as one easily sees from the alternative proof of Theorem 2. Separation of variables for the geodesic equation was discovered in the Kerr solution by Carter [14] and an explanation of this phenomenon in terms of (conformal) Killing tensors was provided by Walker and Penrose [51] (see also [54]). In particular, there are space-times with conformal Killing tensors not arising from conformal Killing vectors. These can lead to extra symmetries for the (conformal) Laplacian [31]. Nevertheless, the relationship to Theorem 4, if any, is unclear.

The algebraic definition of the product on the flat symmetry algebra $\mathcal{A}_n$ is perhaps not as explicit as it might be. According to (11), as a vector space

$$\mathcal{A}_n \cong \bigoplus_{k=0}^{\infty} \circledcirc^k(\mathfrak{so}(n+1,1))$$

where $\circledcirc$ denotes the Cartan product [17]. It is difficult to realise the algebra structure directly from this point of view. Rather, it is induced by realising $\mathcal{A}_n$ as a quotient of the universal enveloping algebra of the conformal algebra as in Corollary 1. But, similar comments apply to the universal enveloping algebra itself. The filtration of $\mathcal{A}_n$ by degree is induced by the usual filtration on $\mathfrak{U}(\mathfrak{so}(n+1,1))$ and, for any Lie algebra $\mathfrak{g}$, there is a canonical vector space isomorphism with the corresponding graded algebra $\mathfrak{U}(\mathfrak{g}) \cong \bigodot(\mathfrak{g})$, namely the symmetric tensor algebra of $\mathfrak{g}$. It is possible to transfer (as is done in [26]) the algebra structure on $\mathfrak{U}(\mathfrak{g})$ to $\bigodot(\mathfrak{g})$. The result is Kontsevich's $\star$-product [1], [35]. An analogous description of $\mathcal{A}_n$ has recently been given by Vasiliev [50] in terms of the Weyl (or Moyal) $\star$-product. One may also exploit the coincidences amongst low-dimensional Lie algebras further to simplify $\mathcal{A}_n$. For Lorentzian signature, this is done for $n = 3$ in [46] (using $\mathfrak{so}(3,2) \cong \mathfrak{sp}(2,\mathbb{R})$) and for $n = 4$ in [49] (using $\mathfrak{so}(4,2) \cong \mathfrak{su}(2,2)$, as in [43]).

The algebra structure on $\mathcal{A}_n$ may be expressed in terms of the cup products of Calderbank and Diemer [8]. In this context, these are conformally invariant bilinear differential pairings between symmetric trace-free tensor fields, which reduce to $\mathfrak{so}(n+1,1)$-invariant algebraic pairings on conformal Killing tensors in the flat case (cf. (31)). Even in the flat case, general explicit formulae are obscure, especially because the algebraic pairings from $\mathcal{A}_n$ are not simply characterised by their invariance. The leading term induces an algebraic pairing, namely the trace-free symmetric product. Also, the first order pairing is relatively simple: if V has s indices and W has t indices, then

$$V \otimes W \mapsto \text{trace-free part of } \left[sV^{a(b\cdots c}\nabla_a W^{d\cdots f)} - tW^{a(b\cdots d}\nabla_a V^{e\cdots f)} \right].$$

It is due to Nijenhuis [42] (with conformal invariance observed by Woodhouse [54]). Though the explicit formulae are rather difficult, Calderbank and Diemer establish a Leibniz rule for their cup product. In the flat case, this is stronger than just preserving conformal Killing tensors.

Finally, we remark that it is straightforward to extend the results of this article to the two-dimensional case when the manifold is endowed with a complex projective structure in the sense of Gunning [27] or a Möbius structure in the sense of Calderbank [7].

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