

Gröbner geometry of Schubert polynomials

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Abstract

Given a permutation $w \in S_n$, we consider a determinantal ideal I_w whose generators are certain minors in the generic $n \times n$ matrix (filled with independent variables). Using ‘multidegrees’ as simple algebraic substitutes for torus-equivariant cohomology classes on vector spaces, our main theorems describe, for each ideal I_w :

- variously graded multidegrees and Hilbert series in terms of ordinary and double Schubert and Grothendieck polynomials;
- a Gröbner basis consisting of minors in the generic $n \times n$ matrix;
- the Stanley–Reisner simplicial complex of the initial ideal in terms of known combinatorial diagrams [FK96], [BB93] associated to permutations in S_n ; and
- a procedure inductive on weak Bruhat order for listing the facets of this complex.

We show that the initial ideal is Cohen–Macaulay, by identifying the Stanley–Reisner complex as a special kind of “subword complex in S_n ”, which we define generally for arbitrary Coxeter groups, and prove to be shellable by giving an explicit vertex decomposition. We also prove geometrically a general positivity statement for multidegrees of subschemes.

Our main theorems provide a geometric explanation for the naturality of Schubert polynomials and their associated combinatorics. More precisely, we apply these theorems to:

- define a single geometric setting in which polynomial representatives for Schubert classes in the integral cohomology ring of the flag manifold are determined uniquely, and have positive coefficients for geometric reasons;

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- rederive from a topological perspective Fulton’s Schubert polynomial formula for universal cohomology classes of degeneracy loci of maps between flagged vector bundles;
- supply new proofs that Schubert and Grothendieck polynomials represent cohomology and K -theory classes on the flag manifold; and
- provide determinantal formulae for the multidegrees of ladder determinantal rings.

The proofs of the main theorems introduce the technique of “Bruhat induction”, consisting of a collection of geometric, algebraic, and combinatorial tools, based on divided and isobaric divided differences, that allow one to prove statements about determinantal ideals by induction on weak Bruhat order.

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Introduction

The manifold $\mathcal{F}\ell_n$ of complete flags (chains of vector subspaces) in the vector space \mathbb{C}^n over the complex numbers has historically been a focal point for a number of distinct fields within mathematics. By definition, $\mathcal{F}\ell_n$ is an object at the intersection of algebra and geometry. The fact that $\mathcal{F}\ell_n$ can be expressed as the quotient $B \backslash \mathrm{GL}_n$ of all invertible $n \times n$ matrices by its subgroup of lower triangular matrices places it within the realm of Lie group theory, and explains its appearance in representation theory. In topology, flag manifolds arise as fibers of certain bundles constructed universally from complex vector bundles, and in that context the cohomology ring $H^*(\mathcal{F}\ell_n) = H^*(\mathcal{F}\ell_n; \mathbb{Z})$ with integer coefficients \mathbb{Z} plays an important role. Combinatorics, especially related to permutations of a set of cardinality n , aids in understanding the topology of $\mathcal{F}\ell_n$ in a geometric manner.

To be more precise, the cohomology ring $H^*(\mathcal{F}\ell_n)$ equals—in a canonical way—the quotient of a polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ modulo the ideal generated by all nonconstant homogeneous functions invariant under permutation of the indices $1, \dots, n$ [Bor53]. This quotient is a free abelian group of rank $n!$ and has a basis given by monomials dividing $\prod_{i=1}^{n-1} x_i^{n-i}$. This algebraic basis does not reflect the geometry of flag manifolds as well as the basis of *Schubert classes*, which are the cohomology classes of Schubert varieties X_w , indexed by permutations $w \in S_n$ [Ehr34]. The *Schubert variety* X_w consists of flags $V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V_n$ whose intersections $V_i \cap \mathbb{C}^j$ have dimensions determined in a certain way by w , where \mathbb{C}^j is spanned by the first j basis vectors of \mathbb{C}^n .

A great deal of research has grown out of attempts to understand the connection between the algebraic basis of monomials and the geometric basis of Schubert classes $[X_w]$ in the cohomology ring $H^*(\mathcal{F}\ell_n)$. For this purpose, Lascoux and Schützenberger singled out Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \dots, x_n]$ as representatives for Schubert classes [LS82a], relying in large part on earlier work of Demazure [Dem74] and Bernstein–Gel’fand–Gel’fand [BGG73]. Lascoux and Schützenberger justified their choices with algebra and combinatorics, whereas the earlier work had been in the context of geometry. This paper bridges the algebra and combinatorics of Schubert polynomials on the one hand with the geometry of Schubert varieties on the other. In the process, it brings a new perspective to problems in commutative algebra concerning ideals generated by minors of generic matrices.

Combinatorialists have in fact recognized the intrinsic interest of Schubert polynomials \mathfrak{S}_w for some time, and have therefore produced a wealth of interpretations for their coefficients. For example, see [Ber92], [Mac91, App. Ch. IV, by N. Bergeron], [BJS93], [FK96], [FS94], [Koh91], and [Win99]. Geometers, on the other hand, who take for granted Schubert *classes* $[X_w]$ in cohomol-

ogy of flag manifold $\mathcal{F}\ell_n$, generally remain less convinced of the naturality of Schubert *polynomials*, even though these polynomials arise in certain universal geometric contexts [Ful92], and there are geometric proofs of positivity for their coefficients [BS02], [Kog00].

Our primary motivation for undertaking this project was to provide a geometric context in which both (i) polynomial representatives for Schubert classes $[X_w]$ in the integral cohomology ring $H^*(\mathcal{F}\ell_n)$ are uniquely singled out, with no choices other than a Borel subgroup of the general linear group $\mathrm{GL}_n\mathbb{C}$; and (ii) it is geometrically obvious that these representatives have nonnegative coefficients. That our polynomials turn out to be the Schubert polynomials is a testament to the naturality of Schubert polynomials; that our geometrically positive formulae turn out to reproduce known combinatorial structures is a testament to the naturality of the combinatorics previously unconvincing to geometers.

The kernel of our idea was to translate ordinary cohomological statements concerning Borel orbit closures on the flag manifold $\mathcal{F}\ell_n$ into equivariant-cohomological statements concerning double Borel orbit closures on the $n \times n$ matrices M_n . Briefly, the preimage $\tilde{X}_w \subseteq \mathrm{GL}_n$ of a Schubert variety $X_w \subseteq \mathcal{F}\ell_n = B \backslash \mathrm{GL}_n$ is an orbit closure for the action of $B \times B^+$, where B and B^+ are the lower and upper triangular Borel subgroups of GL_n acting by multiplication on the left and right. When $\bar{X}_w \subseteq M_n$ is the closure of \tilde{X}_w and T is the torus in B , the T -equivariant cohomology class $[\bar{X}_w]_T \in H_T^*(M_n) = \mathbb{Z}[x_1, \dots, x_n]$ is our polynomial representative. It has positive coefficients because there is a T -equivariant flat (Gröbner) degeneration $\bar{X}_w \rightsquigarrow \mathcal{L}_w$ to a union of coordinate subspaces $L \subseteq M_n$. Each subspace $L \subseteq \mathcal{L}_w$ has equivariant cohomology class $[L]_T \in H_T^*(M_n)$ that is a monomial in x_1, \dots, x_n , and the sum of these is $[\bar{X}_w]_T$. Our obviously positive formula is thus simply

$$(1) \quad [\bar{X}_w]_T = [\mathcal{L}_w]_T = \sum_{L \in \mathcal{L}_w} [L]_T.$$

In fact, one need not actually *produce* a degeneration of \bar{X}_w to a union of coordinate subspaces: the mere *existence* of such a degeneration is enough to conclude positivity of the cohomology class $[\bar{X}_w]_T$, although if the limit is nonreduced then subspaces must be counted according to their (positive) multiplicities. This positivity holds quite generally for sheaves on vector spaces with torus actions, because existence of degenerations is a standard consequence of Gröbner basis theory. That being said, in our main results we identify a particularly natural degeneration of the matrix Schubert variety \bar{X}_w , with reduced and Cohen–Macaulay limit \mathcal{L}_w , in which the subspaces have combinatorial interpretations, and (1) coincides with the known combinatorial formula [BJS93], [FS94] for Schubert polynomials.

The above argument, as presented, requires equivariant cohomology classes associated to closed subvarieties of noncompact spaces such as M_n , the subtleties of which might be considered unpalatable, and certainly require characteristic zero. Therefore we instead develop our theory in the context of *multidegrees*, which are algebraically defined substitutes. In this setting, equivariant considerations for *matrix Schubert varieties* $\overline{X}_w \subseteq M_n$ guide our path directly toward multigraded commutative algebra for the *Schubert determinantal ideals* I_w cutting out the varieties \overline{X}_w .

Example. Let $w = 2143$ be the permutation in the symmetric group S_4 sending $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4$ and $4 \mapsto 3$. The matrix Schubert variety \overline{X}_{2143} is the set of 4×4 matrices $Z = (z_{ij})$ whose upper-left entry is zero, and whose upper-left 3×3 block has rank at most two. The equations defining \overline{X}_{2143} are the vanishing of the determinants

$$\left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = -z_{13}z_{22}z_{31} + \dots \right\rangle.$$

When we Gröbner-degenerate the matrix Schubert variety to the scheme defined by the initial ideal $\langle z_{11}, -z_{13}z_{22}z_{31} \rangle$, we get a union \mathcal{L}_{2143} of three coordinate subspaces

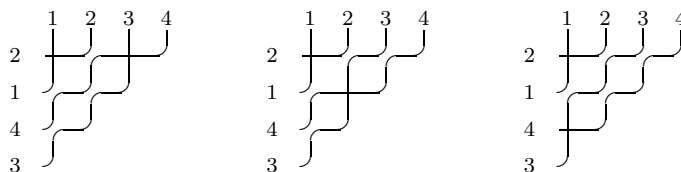
$L_{11,13}, L_{11,22}$, and $L_{11,31}$, with ideals $\langle z_{11}, z_{13} \rangle, \langle z_{11}, z_{22} \rangle$, and $\langle z_{11}, z_{31} \rangle$.

In the \mathbb{Z}^n -grading where z_{ij} has weight x_i , the multidegree of $L_{i_1j_1, i_2j_2}$ equals $x_{i_1}x_{i_2}$. Our “obviously positive” formula (1) for $\mathfrak{S}_{2143}(\mathbf{x})$ says that $[\overline{X}_{2143}]_T = x_1^2 + x_1x_2 + x_1x_3$.

Pictorially, we represent the subspaces $L_{11,13}, L_{11,22}$, and $L_{11,31}$ inside \mathcal{L}_{2143} as subsets

$$\langle z_{11}, z_{13} \rangle = \begin{array}{|c|c|c|c|} \hline + & & + & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}, \quad \langle z_{11}, z_{22} \rangle = \begin{array}{|c|c|c|c|} \hline + & & & \\ \hline & + & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}, \quad \langle z_{11}, z_{31} \rangle = \begin{array}{|c|c|c|c|} \hline + & & & \\ \hline & & & \\ \hline + & & & \\ \hline & & & \\ \hline \end{array}$$

of the 4×4 grid, or equivalently as “pipe dreams” with crosses $\begin{array}{c} \text{+} \\ \text{+} \end{array}$ and “elbow joints” $\begin{array}{c} \text{ } \\ \text{ } \end{array}$ instead of boxes with + or nothing, respectively (imagine $\begin{array}{c} \text{ } \\ \text{ } \end{array}$ filling the lower right corners):



These are the three “reduced pipe dreams”, or “planar histories”, for $w = 2143$ [FK96], and so we recover the combinatorial formula for $\mathfrak{S}_w(\mathbf{x})$ from [BJS93], [FS94].

Our main ‘Gröbner geometry’ theorems describe, for every matrix Schubert variety \overline{X}_w :

- its multidegree and Hilbert series, in terms of Schubert and Grothendieck polynomials (Theorem A);
- a Gröbner basis consisting of minors in its defining ideal I_w (Theorem B);
- the Stanley–Reisner complex \mathcal{L}_w of its initial ideal J_w , which we prove is Cohen–Macaulay, in terms of pipe dreams and combinatorics of S_n (Theorem B); and
- an inductive irredundant algorithm (‘mitosis’) on weak Bruhat order for listing the facets of \mathcal{L}_w (Theorem C).

Gröbner geometry of Schubert polynomials thereby provides a geometric explanation for the naturality of Schubert polynomials and their associated combinatorics.

The divided and isobaric divided differences used by Lascoux and Schützenberger to define Schubert and Grothendieck polynomials inductively [LS82a], [LS82b] were originally invented by virtue of their geometric interpretation by Demazure [Dem74] and Bernstein–Gel’fand–Gel’fand [BGG73]. The heart of our proof of the Gröbner geometry theorem for Schubert polynomials captures the divided and isobaric divided differences in their algebraic and combinatorial manifestations. Both manifestations are positive: one in terms of the generators of the initial ideal J_w and the monomials outside J_w , and the other in terms of certain combinatorial diagrams (reduced pipe dreams) associated to permutations by Fomin–Kirillov [FK96]. Taken together, the geometric, algebraic, and combinatorial interpretations provide a powerful inductive method, which we call *Bruhat induction*, for working with determinantal ideals and their initial ideals, as they relate to multigraded cohomological and combinatorial invariants. In particular, Bruhat induction applied to the facets of \mathcal{L}_w proves a geometrically motivated substitute for Kohnert’s conjecture [Koh91].

At present, “almost all of the approaches one can choose for the investigation of determinantal rings use standard bitableaux and the straightening law” [BC01, p. 3], and are thus intimately tied to the Robinson–Schensted–Knuth (RSK) correspondence. Although Bruhat induction as developed here may seem similar in spirit to RSK, in that both allow one to work directly with vector space bases in the quotient ring, Bruhat induction contrasts with methods based on RSK in that it compares standard monomials of *different ideals* inductively on weak Bruhat order, instead of comparing distinct bases associated to the same ideal, as RSK does. Consequently, Bruhat induction encompasses a substantially larger class of determinantal ideals.

Bruhat induction, as well as the derivation of the main theorems concerning Gröbner geometry of Schubert polynomials from it, relies on two general results concerning

- positivity of multidegrees—that is, positivity of torus-equivariant cohomology classes represented by subschemes or coherent sheaves on vector spaces (Theorem D); and
- shellability of certain simplicial complexes that reflect the nature of reduced subwords of words in Coxeter generators for Coxeter groups (Theorem E).

The latter of these allows us to approach the combinatorics of Schubert and Grothendieck polynomials from a new perspective, namely that of simplicial topology. More precisely, our proof of shellability for the initial complex \mathcal{L}_w draws on previously unknown combinatorial topological aspects of reduced expressions in symmetric groups, and more generally in arbitrary Coxeter groups. We touch relatively briefly on this aspect of the story here, only proving what is essential for the general picture in the present context, and refer the reader to [KnM04] for a complete treatment, including applications to Grothendieck polynomials.

Organization. Our main results, Theorems A, B, C, D, and E, appear in Sections 1.3, 1.5, 1.6, 1.7, and 1.8, respectively. The sections in Part 1 are almost entirely expository in nature, and serve not merely to define all objects appearing in the central theorems, but also to provide independent motivation and examples for the theories they describe. For each of Theorems A, B, C, and E, we develop before it just enough prerequisites to give a complete statement, while for Theorem D we first provide a crucial characterization of multidegrees, in Theorem 1.7.1.

Readers seeing this paper for the first time should note that Theorems A, B, and D are core results, not to be overlooked on a first pass through. Theorems C and E are less essential to understanding the main point as outlined in the introduction, but still fundamental for the combinatorics of Schubert polynomials as derived from geometry via Bruhat induction (which is used to prove Theorems A and B), and for substantiating the naturality of the degeneration in Theorem B.

The paper is structured logically as follows. There are no proofs in Sections 1.1–1.6 except for a few easy lemmas that serve the exposition. The complete proof of Theorems A, B, and C must wait until the last section of Part 3 (Section 3.9), because these results rely on Bruhat induction. Section 3.9 indicates which parts of the theorems from Part 1 imply the others, while gathering the results from Part 3 to prove those required parts. In con-

trast, the proofs of Theorems D and E in Sections 1.7 and 1.8 are completely self-contained, relying on nothing other than definitions. Results of Part 1 are used freely in Part 2 for applications to consequences not found or only briefly mentioned in Part 1. The development of Bruhat induction in Part 3 depends only on Section 1.7 and definitions from Part 1.

In terms of content, Sections 1.1, 1.2, and 1.4, as well as the first half of Section 1.3, review known definitions, while the other sections in Part 1 introduce topics appearing here for the first time. In more detail, Section 1.1 recalls the Schubert and Grothendieck polynomials of Lascoux and Schützenberger via divided differences and their isobaric relatives. Then Section 1.2 reviews K -polynomials and multidegrees, which are rephrased versions of the equivariant multiplicities in [BB82], [BB85], [Jos84], [Ros89]. We start Section 1.3 by introducing matrix Schubert varieties and Schubert determinantal ideals, which are due (in different language) to Fulton [Ful92]. This discussion culminates in the statement of Theorem A, giving the multidegrees and K -polynomials of matrix Schubert varieties.

We continue in Section 1.4 with some combinatorial diagrams that we call ‘reduced pipe dreams’, associated to permutations. These were invented by Fomin and Kirillov and studied by Bergeron and Billey, who called them ‘rc-graphs’. Section 1.5 begins with the definition of ‘antidiagonal’ squarefree monomial ideals, and proceeds to state Theorem B, which describes Gröbner bases and initial ideals for matrix Schubert varieties in terms of reduced pipe dreams. Section 1.6 defines our combinatorial ‘mitosis’ rule for manipulating subsets of the $n \times n$ grid, and describes in Theorem C how mitosis generates all reduced pipe dreams.

Section 1.7 works with multidegrees in the general context of a positive multigrading, proving the characterization Theorem 1.7.1 and then its consequence, the Positivity Theorem D. Also in a general setting—that of arbitrary Coxeter groups—we define ‘subword complexes’ in Section 1.8, and prove their vertex-decomposability in Theorem E.

Our most important application, in Section 2.1, consists of the geometrically positive formulae for Schubert polynomials that motivated this paper. Other applications include connections with Fulton’s theory of degeneracy loci in Section 2.2, relations between our multidegrees and K -polynomials on $n \times n$ matrices with classical cohomological theories on the flag manifold in Section 2.3, and comparisons in Section 2.4 with the commutative algebra literature on determinantal ideals.

Part 3 demonstrates how the method of Bruhat induction works geometrically, algebraically, and combinatorially to provide full proofs of Theorems A, B, and C. We postpone the detailed overview of Part 3 until Section 3.1, although we mention here that the geometric Section 3.2 has a rather different flavor from Sections 3.3–3.8, which deal mostly with the combinatorial

commutative algebra spawned by divided differences, and Section 3.9, which collects Part 3 into a coherent whole in order to prove Theorems A, B, and C. Generally speaking, the material in Part 3 is more technical than earlier parts.

We have tried to make the material here as accessible as possible to combinatorialists, geometers, and commutative algebraists alike. In particular, except for applications in Part 2, we have assumed no specific knowledge of the algebra, geometry, or combinatorics of flag manifolds, Schubert varieties, Schubert polynomials, Grothendieck polynomials, or determinantal ideals. Many of our examples interpret the same underlying data in varying contexts, to highlight and contrast common themes. In particular this is true of Examples 1.3.5, 1.4.2, 1.4.6, 1.5.3, 1.6.2, 1.6.3, 3.3.6, 3.3.7, 3.4.2, 3.4.7, 3.4.8, 3.7.4, 3.7.6, and 3.7.10.

Conventions. Throughout this paper, \mathbf{k} is an arbitrary field. In particular, we impose no restrictions on its characteristic. Furthermore, although some geometric statements or arguments may seem to require that \mathbf{k} be algebraically closed, this hypothesis could be dispensed with formally by resorting to sufficiently abstruse language.

We consciously chose our notational conventions (with considerable effort) to mesh with those of [Ful92], [LS82a], [FK94], [HT92], and [BB93] concerning permutations (w^T versus w), the indexing on (matrix) Schubert varieties and polynomials (open orbit corresponds to identity permutation and smallest orbit corresponds to long word), the placement of one-sided ladders (in the north-west corner as opposed to the southwest), and reduced pipe dreams. These conventions dictated our seemingly idiosyncratic choices of Borel subgroups as well as the identification $\mathcal{F}\ell_n \cong B \backslash GL_n$ as the set of right cosets, and resulted in our use of row vectors in \mathbf{k}^n instead of the usual column vectors. That there even existed consistent conventions came as a relieving surprise.

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Part 1. The Gröbner geometry theorems

1.1. Schubert and Grothendieck polynomials

We write all permutations in one-line (not cycle) notation, where $w = w_1 \dots w_n$ sends $i \mapsto w_i$. Set $w_0 = n \dots 321$ equal to the *long permutation* reversing the order of $1, \dots, n$.

Definition 1.1.1. Let R be a commutative ring, and $\mathbf{x} = x_1, \dots, x_n$ independent variables. The i^{th} *divided difference operator* ∂_i takes each polynomial $f \in R[\mathbf{x}]$ to

$$\partial_i f(x_1, x_2, \dots) = \frac{f(x_1, x_2, \dots) - f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots)}{x_i - x_{i+1}}.$$

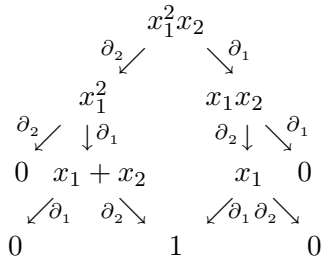
The *Schubert polynomial* for $w \in S_n$ is defined by the recursion

$$\mathfrak{S}_{ws_i}(\mathbf{x}) = \partial_i \mathfrak{S}_w(\mathbf{x})$$

whenever $\text{length}(ws_i) < \text{length}(w)$, and the initial condition $\mathfrak{S}_{w_0}(\mathbf{x}) = \prod_{i=1}^n x_i^{n-i} \in \mathbb{Z}[\mathbf{x}]$. The *double Schubert polynomials* $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ are defined by the same recursion, but starting from $\mathfrak{S}_{w_0}(\mathbf{x}, \mathbf{y}) = \prod_{i+j \leq n} (x_i - y_j) \in \mathbb{Z}[\mathbf{y}][\mathbf{x}]$.

In the definition of $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$, the operator ∂_i is to act only on the \mathbf{x} variables and not on the \mathbf{y} variables. Checking monomial by monomial verifies that $x_i - x_{i+1}$ divides the numerator of $\partial_i(f)$, and so $\partial_i(f)$ is again a polynomial, homogeneous of degree $d - 1$ if f is homogeneous of degree d .

Example 1.1.2. Here are all of the Schubert polynomials for permutations in S_3 , along with the rules for applying divided differences.



The recursion for both single and double Schubert polynomials can be summarized as

$$\mathfrak{S}_w = \partial_{i_k} \cdots \partial_{i_1} \mathfrak{S}_{w_0},$$

where $w_0 w = s_{i_1} \cdots s_{i_k}$ and $\text{length}(w_0 w) = k$. The condition $\text{length}(w_0 w) = k$ means by definition that k is minimal, so that $w_0 w = s_{i_1} \cdots s_{i_k}$ is a *reduced expression* for $w_0 w$. It is not immediately obvious from Definition 1.1.1 that \mathfrak{S}_w is well-defined, but it follows from the fact that divided differences satisfy the *Coxeter relations*, $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$ and $\partial_i \partial_{i'} = \partial_{i'} \partial_i$ when $|i - i'| \geq 2$.

Divided differences arose geometrically in work of Demazure [Dem74] and Bernstein–Gel’fand–Gel’fand [BGG73], where they reflected a ‘Bott–Samelson crank’: form a \mathbb{P}^1 bundle over a Schubert variety and smear it out onto the flag manifold $\mathcal{F}\ell_n$ to get a Schubert variety of dimension 1 greater than before. In their setting, the variables \mathbf{x} represented Chern classes of standard line bundles L_1, \dots, L_n on $\mathcal{F}\ell_n$, where the fiber of L_i over a flag $F_0 \subset \cdots \subset F_n$ is the dual

vector space $(F_i/F_{i-1})^*$. The divided differences acted on the cohomology ring $H^*(\mathcal{F}\ell_n)$, which is the quotient of $\mathbb{Z}[\mathbf{x}]$ modulo the ideal generated by symmetric functions with no constant term [Bor53]. The insight of Lascoux and Schützenberger in [LS82a] was to impose a stability condition on the collection of polynomials \mathfrak{S}_w that defines them uniquely among representatives for the cohomology classes of Schubert varieties. More precisely, although Definition 1.1.1 says that w lies in S_n , the number n in fact plays no role: if $w_N \in S_N$ for $n \geq N$ agrees with w on $1, \dots, n$ and fixes $n+1, \dots, N$, then $\mathfrak{S}_{w_N}(x_1, \dots, x_N) = \mathfrak{S}_w(x_1, \dots, x_n)$.

The ‘double’ versions represent Schubert classes in equivariant cohomology for the Borel group action on $\mathcal{F}\ell_n$. As the ordinary Schubert polynomials are much more common in the literature than double Schubert polynomials, we have phrased many of our coming results both in terms of Schubert polynomials as well as double Schubert polynomials. This choice has the advantage of demonstrating how the notation simplifies in the single case.

Schubert polynomials have their analogues in K -theory of $\mathcal{F}\ell_n$, where the recurrence uses a “homogenized” operator (sometimes called an *isobaric divided difference operator*):

Definition 1.1.3. Let R be a commutative ring. The i^{th} Demazure operator $\bar{\partial}_i : R[[\mathbf{x}]] \rightarrow R[[\mathbf{x}]]$ sends a power series $f(\mathbf{x})$ to

$$\frac{x_{i+1}f(x_1, \dots, x_n) - x_i f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)}{x_{i+1} - x_i} = -\bar{\partial}_i(x_{i+1}f).$$

The *Grothendieck polynomial* $\mathcal{G}_w(\mathbf{x})$ is obtained recursively from the “top” Grothendieck polynomial $\mathcal{G}_{w_0}(\mathbf{x}) := \prod_{i=1}^n (1 - x_i)^{n-i}$ via the recurrence

$$\mathcal{G}_{ws_i}(\mathbf{x}) = \bar{\partial}_i \mathcal{G}_w(\mathbf{x})$$

whenever $\text{length}(ws_i) < \text{length}(w)$. The *double Grothendieck polynomials* are defined by the same recurrence, but start from $\mathcal{G}_{w_0}(\mathbf{x}, \mathbf{y}) := \prod_{i+j \leq n} (1 - x_i y_j^{-1})$.

As with divided differences, one can check directly that Demazure operators $\bar{\partial}_i$ take power series to power series, and satisfy the Coxeter relations. Lascoux and Schützenberger [LS82b] showed that Grothendieck polynomials enjoy the same stability property as do Schubert polynomials; we shall rederive this fact directly from Theorem A in Section 2.3 (Lemma 2.3.2), where we also construct the bridge from Gröbner geometry of Schubert and Grothendieck polynomials to classical geometry on flag manifolds.

Schubert polynomials represent data that are leading terms for the richer structure encoded by Grothendieck polynomials.

LEMMA 1.1.4. *The Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ is the sum of all lowest-degree terms in $\mathcal{G}_w(\mathbf{1} - \mathbf{x})$, where $(\mathbf{1} - \mathbf{x}) = (1 - x_1, \dots, 1 - x_n)$. Similarly, the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ is the sum of all lowest-degree terms in $\mathcal{G}_w(\mathbf{1} - \mathbf{x}, \mathbf{1} - \mathbf{y})$.*

Proof. Assuming $f(\mathbf{1} - \mathbf{x})$ is homogeneous, plugging $\mathbf{1} - \mathbf{x}$ for \mathbf{x} into the first displayed equation in Definition 1.1.3 and taking the lowest degree terms yields $\partial_i f(\mathbf{1} - \mathbf{x})$. Since \mathfrak{S}_{w_0} is homogeneous, the result follows by induction on $\text{length}(w_0 w)$. \square

Although the Demazure operators are usually applied only to polynomials in \mathbf{x} , it will be crucial in our applications to use them on power series in \mathbf{x} . We shall also use the fact that, since the standard denominator $f(\mathbf{x}) = \prod_{i=1}^n (1 - x_i)^n$ for \mathbb{Z}^n -graded Hilbert series over $\mathbf{k}[\mathbf{z}]$ is symmetric in x_1, \dots, x_n , applying $\bar{\partial}_i$ to a Hilbert series g/f simplifies: $\bar{\partial}_i(g/f) = (\bar{\partial}_i g)/f$. This can easily be checked directly. The same comment applies when $f(\mathbf{x}) = \prod_{i,j=1}^n (1 - x_i/y_j)$ is the standard denominator for \mathbb{Z}^{2n} -graded Hilbert series.

1.2. Multidegrees and K -polynomials

Our first main theorem concerns cohomological and K -theoretic invariants of matrix Schubert varieties, which are given by multidegrees and K -polynomials, respectively. We work with these here in the setting of a polynomial ring $\mathbf{k}[\mathbf{z}]$ in m variables $\mathbf{z} = z_1, \dots, z_m$, with a grading by \mathbb{Z}^d in which each variable z_i has *exponential weight* $\text{wt}(z_i) = \mathbf{t}^{\mathbf{a}_i}$ for some vector $\mathbf{a}_i = (a_{i1}, \dots, a_{id}) \in \mathbb{Z}^d$, where $\mathbf{t} = t_1, \dots, t_d$. We call \mathbf{a}_i the *ordinary weight* of z_i , and sometimes write $\mathbf{a}_i = \text{deg}(z_i) = a_{i1}t_1 + \dots + a_{id}t_d$. It is useful to think of this as the logarithm of the Laurent monomial $\mathbf{t}^{\mathbf{a}_i}$.

Example 1.2.1. Our primary concern is the case $\mathbf{z} = (z_{ij})_{i,j=1}^n$ with various gradings, in which the different kinds of weights are:

grading	\mathbb{Z}	\mathbb{Z}^n	\mathbb{Z}^{2n}	\mathbb{Z}^{n^2}
exponential weight of z_{ij}	t	x_i	x_i/y_j	z_{ij}
ordinary weight of z_{ij}	t	x_i	$x_i - y_j$	z_{ij}

The exponential weights are Laurent monomials that we treat as elements in the group rings $\mathbb{Z}[t^{\pm 1}]$, $\mathbb{Z}[\mathbf{x}^{\pm 1}]$, $\mathbb{Z}[\mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}]$, $\mathbb{Z}[\mathbf{z}^{\pm 1}]$ of the grading groups. The ordinary weights are linear forms that we treat as elements in the integral symmetric algebras $\mathbb{Z}[t] = \text{Sym}_{\mathbb{Z}}^{\bullet}(\mathbb{Z})$, $\mathbb{Z}[\mathbf{x}] = \text{Sym}_{\mathbb{Z}}^{\bullet}(\mathbb{Z}^n)$, $\mathbb{Z}[\mathbf{x}, \mathbf{y}] = \text{Sym}_{\mathbb{Z}}^{\bullet}(\mathbb{Z}^{2n})$, $\mathbb{Z}[\mathbf{z}] = \text{Sym}_{\mathbb{Z}}^{\bullet}(\mathbb{Z}^{n^2})$ of the grading groups.

Every finitely generated \mathbb{Z}^d -graded module $\Gamma = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \Gamma_{\mathbf{a}}$ over $\mathbf{k}[\mathbf{z}]$ has a free resolution

$$\mathcal{E} : 0 \leftarrow \mathcal{E}_0 \leftarrow \mathcal{E}_1 \leftarrow \cdots \leftarrow \mathcal{E}_m \leftarrow 0, \quad \text{where} \quad \mathcal{E}_i = \bigoplus_{j=1}^{\beta_i} \mathbf{k}[\mathbf{z}](-\mathbf{b}_{ij})$$

is graded, with the j^{th} summand of \mathcal{E}_i generated in \mathbb{Z}^d -graded degree \mathbf{b}_{ij} .

Definition 1.2.2. The K -polynomial of Γ is $\mathcal{K}(\Gamma; \mathbf{t}) = \sum_i (-1)^i \sum_j \mathbf{t}^{\mathbf{b}_{ij}}$.

Geometrically, the K -polynomial of Γ represents the class of the sheaf $\tilde{\Gamma}$ on \mathbf{k}^m in equivariant K -theory for the action of the d -torus whose weight lattice is \mathbb{Z}^d . Algebraically, when the \mathbb{Z}^d -grading is *positive*, meaning that the ordinary weights $\mathbf{a}_1, \dots, \mathbf{a}_d$ lie in a single open half-space in \mathbb{Z}^d , the vector space dimensions $\dim_{\mathbf{k}}(\Gamma_{\mathbf{a}})$ are finite for all $\mathbf{a} \in \mathbb{Z}^d$, and the K -polynomial of Γ is the numerator of its \mathbb{Z}^d -graded Hilbert series $H(\Gamma; \mathbf{t})$:

$$H(\Gamma; \mathbf{t}) := \sum_{\mathbf{a} \in \mathbb{Z}^d} \dim_{\mathbf{k}}(\Gamma_{\mathbf{a}}) \cdot \mathbf{t}^{\mathbf{a}} = \frac{\mathcal{K}(\Gamma; \mathbf{t})}{\prod_{i=1}^m (1 - \text{wt}(z_i))}.$$

We shall only have a need to consider positive multigradings in this paper.

Given any Laurent monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$, the rational function $\prod_{j=1}^d (1 - t_j)^{a_j}$ can be expanded as a well-defined (that is, convergent in the \mathbf{t} -adic topology) formal power series $\prod_{j=1}^d (1 - a_j x_j + \cdots)$ in \mathbf{t} . Doing the same for each monomial in an arbitrary Laurent polynomial $\mathcal{K}(\mathbf{t})$ results in a power series denoted by $\mathcal{K}(\mathbf{1} - \mathbf{t})$.

Definition 1.2.3. The *multidegree* of a \mathbb{Z}^d -graded $\mathbf{k}[\mathbf{z}]$ -module Γ is the sum $\mathcal{C}(\Gamma; \mathbf{t})$ of the lowest degree terms in $\mathcal{K}(\Gamma; \mathbf{1} - \mathbf{t})$. If $\Gamma = \mathbf{k}[\mathbf{z}]/I$ is the coordinate ring of a subscheme $X \subseteq \mathbf{k}^m$, then we may also write $[X]_{\mathbb{Z}^d}$ or $\mathcal{C}(X; \mathbf{t})$ to mean $\mathcal{C}(\Gamma; \mathbf{t})$.

Geometrically, multidegrees are just an algebraic reformulation of torus-equivariant cohomology of affine space, or equivalently the equivariant Chow ring [Tot99], [EG98]. Multidegrees originated in [BB82], [BB85] as well as [Jos84], and are called *equivariant multiplicities* in [Ros89].

Example 1.2.4. Let $n = 2$ in Example 1.2.1, and set

$$\Gamma = \mathbf{k} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} / \langle z_{11}, z_{22} \rangle.$$

Then

$$\mathcal{K}(\Gamma; \mathbf{z}) = (1 - z_{11})(1 - z_{22}) \quad \text{and} \quad \mathcal{K}(\Gamma; \mathbf{x}, \mathbf{y}) = (1 - x_1/y_1)(1 - x_2/y_2)$$

because of the Koszul resolution. Thus $\mathcal{K}(\Gamma; \mathbf{1} - \mathbf{z}) = z_{11}z_{22} = \mathcal{C}(\Gamma; \mathbf{z})$, and

$$\mathcal{K}(\Gamma; \mathbf{1} - \mathbf{x}, \mathbf{1} - \mathbf{y}) = (x_1 - y_1 + x_1y_1 - y_1^2 + \cdots)(x_2 - y_2 + x_2y_2 - y_2^2 + \cdots),$$

whose sum of lowest degree terms is $\mathcal{C}(\Gamma; \mathbf{x}, \mathbf{y}) = (x_1 - y_1)(x_2 - y_2)$.

The letters \mathcal{C} and \mathcal{K} stand for ‘cohomology’ and ‘ K -theory’, the relation between them (‘take lowest degree terms’) reflecting the Grothendieck–Riemann–Roch transition from K -theory to its associated graded ring. When \mathbf{k} is the complex field \mathbb{C} , the (Laurent) polynomials denoted by \mathcal{C} and \mathcal{K} are honest torus-equivariant cohomology and K -classes on \mathbb{C}^m .

1.3. Matrix Schubert varieties

Let M_n be the variety of $n \times n$ matrices over \mathbf{k} , with coordinate ring $\mathbf{k}[\mathbf{z}]$ in indeterminates $\{z_{ij}\}_{i,j=1}^n$. Throughout the paper, q and p will be integers with $1 \leq q, p \leq n$, and Z will stand for an $n \times n$ matrix. Most often, Z will be the *generic matrix* of variables (z_{ij}) , although occasionally Z will be an element of M_n . Denote by $Z_{q \times p}$ the northwest $q \times p$ submatrix of Z . For instance, given a permutation $w \in S_n$, the permutation matrix w^T with ‘1’ entries in row i and column $w(i)$ has upper-left $q \times p$ submatrix with rank given by

$$\text{rank}(w_{q \times p}^T) = \#\{(i, j) \leq (q, p) \mid w(i) = j\},$$

the number of ‘1’ entries in the submatrix $w_{q \times p}^T$.

The class of determinantal ideals in the following definition was identified by Fulton in [Ful92], though in slightly different language.

Definition 1.3.1. Let $w \in S_n$ be a permutation. The *Schubert determinantal ideal* $I_w \subset \mathbf{k}[\mathbf{z}]$ is generated by all minors in $Z_{q \times p}$ of size $1 + \text{rank}(w_{q \times p}^T)$ for all q, p , where $Z = (z_{ij})$ is the matrix of variables.

The subvariety of M_n cut out by I_w is the central geometric object in this paper.

Definition 1.3.2. Let $w \in S_n$. The *matrix Schubert variety* $\overline{X}_w \subseteq M_n$ consists of the matrices $Z \in M_n$ such that $\text{rank}(Z_{q \times p}) \leq \text{rank}(w_{q \times p}^T)$ for all q, p .

Example 1.3.3. The smallest matrix Schubert variety is \overline{X}_{w_0} , where w_0 is the *long permutation* $n \cdots 21$ reversing the order of $1, \dots, n$. The variety \overline{X}_{w_0} is just the linear subspace of lower-right-triangular matrices; its ideal is $\langle z_{ij} \mid i + j \leq n \rangle$.

Example 1.3.4. Five of the six 3×3 matrix Schubert varieties are linear subspaces:

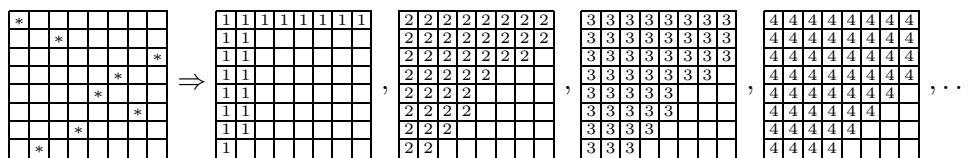
$$\begin{aligned}
 I_{123} &= 0 & \overline{X}_{123} &= M_3 \\
 I_{213} &= \langle z_{11} \rangle & \overline{X}_{213} &= \{Z \in M_3 \mid z_{11} = 0\} \\
 I_{231} &= \langle z_{11}, z_{12} \rangle & \overline{X}_{231} &= \{Z \in M_3 \mid z_{11} = z_{12} = 0\} \\
 I_{321} &= \langle z_{11}, z_{21} \rangle & \overline{X}_{312} &= \{Z \in M_3 \mid z_{11} = z_{21} = 0\} \\
 I_{321} &= \langle z_{11}, z_{12}, z_{21} \rangle & \overline{X}_{321} &= \{Z \in M_3 \mid z_{11} = z_{12} = z_{21} = 0\}.
 \end{aligned}$$

The remaining permutation, $w = 132$, has

$$I_{132} = \langle z_{11}z_{22} - z_{12}z_{21} \rangle, \quad \overline{X}_{132} = \{Z \in M_3 \mid \text{rank}(Z_{2 \times 2}) \leq 1\},$$

so that \overline{X}_{132} is the set of matrices whose upper-left 2×2 block is singular.

Example 1.3.5. Let $w = 13865742$, so that w^T is given by replacing each $*$ by 1 in the left matrix below.



Each matrix in $\overline{X}_w \subseteq M_n$ has the property that every rectangular submatrix contained in the region filled with 1's has rank ≤ 1 , and every rectangular submatrix contained in the region filled with 2's has rank ≤ 2 , and so on. The ideal I_w therefore contains the 21 minors of size 2×2 in the first region and the 144 minors of size 3×3 in the second region. These 165 minors in fact generate I_w , as can be checked either directly by Laplace expansion of each determinant in I_w along its last row(s) or column(s), or indirectly using Fulton's notion of 'essential set' [Ful92]. See also Example 1.5.3.

Our first main theorem provides a straightforward geometric explanation for the naturality of Schubert and Grothendieck polynomials. More precisely, our context automatically makes them well-defined as (Laurent) polynomials, as opposed to being identified as (particularly nice) representatives for classes in some quotient of a polynomial ring.

THEOREM A. *The Schubert determinantal ideal I_w is prime, so I_w is the ideal $I(\overline{X}_w)$ of the matrix Schubert variety \overline{X}_w . The \mathbb{Z}^n -graded and \mathbb{Z}^{2n} -graded K -polynomials of \overline{X}_w are the Grothendieck and double Grothendieck polynomials for w , respectively:*

$$\mathcal{K}(\overline{X}_w; \mathbf{x}) = \mathcal{G}_w(\mathbf{x}) \text{ and } \mathcal{K}(\overline{X}_w; \mathbf{x}, \mathbf{y}) = \mathcal{G}_w(\mathbf{x}, \mathbf{y}).$$

The \mathbb{Z}^n -graded and \mathbb{Z}^{2n} -graded multidegrees of \overline{X}_w are the Schubert and double Schubert polynomials for w , respectively:

$$[\overline{X}_w]_{\mathbb{Z}^n} = \mathfrak{S}_w(\mathbf{x}) \text{ and } [\overline{X}_w]_{\mathbb{Z}^{2n}} = \mathfrak{S}_w(\mathbf{x}, \mathbf{y}).$$

Primality of I_w was proved by Fulton [Ful92], but we shall not assume it in our proofs.

Example 1.3.6. Let $w = 2143$ as in the example from the introduction. Computing the K -polynomial of the complete intersection $\mathbf{k}[\mathbf{z}]/I_{2143}$ yields (in the \mathbb{Z}^n -grading for simplicity)

$$(1 - x_1)(1 - x_1x_2x_3) = \mathcal{G}_{2143}(\mathbf{x}) = \overline{\partial}_2\overline{\partial}_1\overline{\partial}_3\overline{\partial}_2((1 - x_1)^3(1 - x_2)^2(1 - x_3)),$$

the latter equality by Theorem A. Substituting $\mathbf{x} \mapsto \mathbf{1} - \mathbf{x}$ in $\mathcal{G}_{2143}(\mathbf{x})$ yields

$$\mathcal{G}_{2143}(\mathbf{1} - \mathbf{x}) = x_1(x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 + x_1x_2x_3),$$

whose sum of lowest degree terms equals the multidegree $\mathcal{C}(\overline{X}_{2143}; \mathbf{x})$ by definition. This agrees with the Schubert polynomial $\mathfrak{S}_{2143}(\mathbf{x}) = x_1^2 + x_1x_2 + x_1x_3$.

That Schubert and Grothendieck polynomials represent cohomology and K -theory classes of Schubert varieties in flag manifolds will be shown in Section 2.3 to follow from Theorem A.

1.4. Pipe dreams

In this section we introduce the set $\mathcal{RP}(w)$ of reduced pipe dreams¹ for a permutation $w \in S_n$. Each diagram $D \in \mathcal{RP}(w)$ is a subset of the $n \times n$ grid $[n]^2$ that represents an example of the curve diagrams invented by Fomin and Kirillov [FK96], though our notation follows Bergeron and Billey [BB93] in this regard.² Besides being attractive ways to draw permutations, reduced pipe dreams generalize to flag manifolds the semistandard Young tableaux for Grassmannians. Indeed, there is even a natural bijection between tableaux and reduced pipe dreams for Grassmannian permutations (see [Kog00], for instance).

Consider a square grid $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ extending infinitely south and east, with the box in row i and column j labeled (i, j) , as in an $\infty \times \infty$ matrix. If each box in the grid is covered with a square tile containing either + or └ , then one can think of the tiled grid as a network of pipes.

Definition 1.4.1. A *pipe dream* is a finite subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, identified as the set of crosses in a tiling by *crosses* + and *elbow joints* └ .

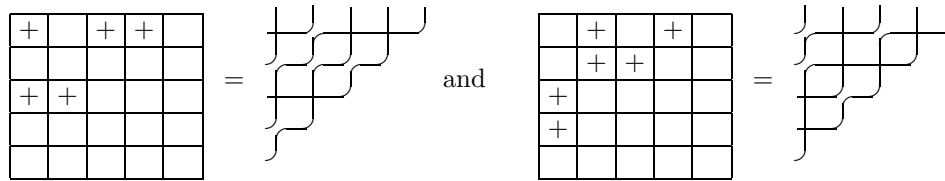
Whenever we draw pipe dreams, we fill the boxes with crossing tiles by ‘+’. However, we often leave the elbow tiles blank, or denote them by dots

¹In the game Pipe Dream, the player is supposed to guide water flowing out of a spigot at one edge of the game board to its destination at another edge by laying down given square tiles with pipes going through them; see Definition 1.4.1. The spigot placements and destinations are interpreted in Definition 1.4.3.

²The corresponding objects in [FK96] look like reduced pipe dreams rotated by 135°.

for ease of notation. The pipe dreams we consider all represent subsets of the pipe dream D_0 that has crosses in the triangular region strictly above the main antidiagonal (in spots (i, j) with $i + j \leq n$) and elbow joints elsewhere. Thus we can safely limit ourselves to drawing inside $n \times n$ grids.

Example 1.4.2. Here are two rather arbitrary pipe dreams with $n = 5$:



Another (slightly less arbitrary) example, with $n = 8$, is the pipe dream D in Figure 1 below. The first diagram represents D as a subset of $[8]^2$, whereas the second demonstrates how the tiles fit together. Since no cross in D occurs on or below the 8th antidiagonal, the pipe entering row i exits column $w_i = w(i)$ for some permutation $w \in S_8$. In this case, $w = 13865742$ is the permutation from Example 1.3.5. For clarity, we omit the square tile boundaries as well as the wavy “sea” of elbows below the main antidiagonal in the right pipe dream. We also use the thinner symbol w_i instead of $w(i)$ to make the column widths come out right.

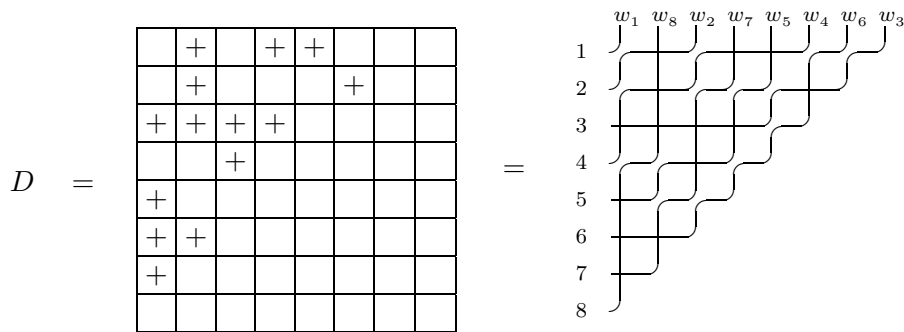


Figure 1: A pipe dream with $n = 8$

Definition 1.4.3. A pipe dream is *reduced* if each pair of pipes crosses at most once. The set $\mathcal{RP}(w)$ of reduced pipe dreams for the permutation $w \in S_n$ is the set of reduced pipe dreams D such that the pipe entering row i exits from column $w(i)$.

We shall give some idea of what it means for a pipe dream to be reduced, in Lemma 1.4.5, below. For notation, we say that a ‘+’ at (q, p) in a pipe

dream D sits on the i^{th} antidiagonal if $q + p - 1 = i$. Let $Q(D)$ be the ordered sequence of simple reflections s_i corresponding to the antidiagonals on which the crosses sit, starting from the northeast corner of D and reading *right to left* in each row, snaking down to the southwest corner.³

Example 1.4.4. The pipe dream D_0 corresponds to the ordered sequence

$$Q(D_0) = Q_0 := s_{n-1} \cdots s_2 s_1 s_{n-1} \cdots s_3 s_2 \cdots \cdots s_{n-1} s_{n-2} s_{n-1},$$

the *triangular* reduced expression for the long permutation $w_0 = n \cdots 321$. Thus $Q_0 = s_3 s_2 s_1 s_3 s_2 s_3$ when $n = 4$. For another example, the first pipe dream in Example 1.4.2 yields the ordered sequence $s_4 s_3 s_1 s_5 s_4$.

LEMMA 1.4.5. *If D is a pipe dream, then multiplying the reflections in $Q(D)$ yields the permutation w such that the pipe entering row i exits column $w(i)$. Furthermore, the number of crossing tiles in D is at least $\text{length}(w)$, with equality if and only if $D \in \mathcal{RP}(w)$.*

Proof. For the first statement, use induction on the number of crosses: adding a ‘+’ in the i^{th} antidiagonal at the end of the list switches the destinations of the pipes beginning in rows i and $i + 1$. Each inversion in w contributes at least one crossing in D , whence the number of crossing tiles is at least $\text{length}(w)$. The expression $Q(D)$ is reduced when D is reduced because each inversion in w contributes at most one crossing tile to D . \square

In other words, pipe dreams with no crossing tiles on or below the main antidiagonal in $[n]^2$ are naturally ‘subwords’ of $Q(D_0)$, while reduced pipe dreams are naturally *reduced* subwords. This point of view takes center stage in Section 1.8.

Example 1.4.6. The upper-left triangular pipe dream $D_0 \subset [n]^2$ is the unique pipe dream in $\mathcal{RP}(w_0)$. The 8×8 pipe dream D in Example 1.4.2 lies in $\mathcal{RP}(13865742)$.

1.5. Gröbner geometry

Using Gröbner bases, we next degenerate matrix Schubert varieties into unions of vector subspaces of M_n corresponding to reduced pipe dreams. A total order ‘>’ on monomials in $\mathbf{k}[\mathbf{z}]$ is a *term order* if $1 \leq m$ for all monomials $m \in \mathbf{k}[\mathbf{z}]$, and $m \cdot m' < m \cdot m''$ whenever $m' < m''$. When a term order ‘>’ is

³The term ‘rc-graph’ was used in [BB93] for what we call reduced pipe dreams. The letters ‘rc’ stand for “reduced-compatible”. The ordered list of row indices for the crosses in D , taken in the same order as before, is called in [BJS93] a “compatible sequence” for the expression $Q(D)$; we shall not need this concept.

fixed, the largest monomial $\text{in}(f)$ appearing with nonzero coefficient in a polynomial f is its *initial term*, and the *initial ideal* of a given ideal I is generated by the initial terms of all polynomials $f \in I$. A set $\{f_1, \dots, f_n\}$ is a *Gröbner basis* if $\text{in}(I) = \langle \text{in}(f_1), \dots, \text{in}(f_n) \rangle$. See [Eis95, Ch. 15] for background on term orders and Gröbner bases, including geometric interpretations in terms of flat families.

Definition 1.5.1. The *antidiagonal ideal* J_w is generated by the antidiagonals of the minors of $Z = (z_{ij})$ generating I_w . Here, the *antidiagonal* of a square matrix or a minor is the product of the entries on the main antidiagonal.

There exist numerous *antidiagonal term orders* on $\mathbf{k}[\mathbf{z}]$, which by definition pick off from each minor its antidiagonal term, including:

- the reverse lexicographic term order that snakes its way from the northwest corner to the southeast corner, $z_{11} > z_{12} > \dots > z_{1n} > z_{21} > \dots > z_{nn}$; and
- the lexicographic term order that snakes its way from the northeast corner to the southwest corner, $z_{1n} > \dots > z_{nn} > \dots > z_{2n} > z_{11} > \dots > z_{n1}$.

The initial ideal $\text{in}(I_w)$ for any antidiagonal term order contains J_w by definition, and our first point in Theorem B will be equality of these two monomial ideals.

Our remaining points in Theorem B concern the combinatorics of J_w . Being a squarefree monomial ideal, it is by definition the *Stanley–Reisner ideal* of some simplicial complex \mathcal{L}_w with vertex set $[n]^2 = \{(q, p) \mid 1 \leq q, p \leq n\}$. That is, \mathcal{L}_w consists of the subsets of $[n]^2$ containing no antidiagonal in J_w . Faces of \mathcal{L}_w (or any simplicial complex with $[n]^2$ for vertex set) may be identified with coordinate subspaces in M_n as follows. Let E_{qp} denote the elementary matrix whose only nonzero entry lies in row q and column p , and identify vertices in $[n]^2$ with variables z_{qp} in the generic matrix Z . When $D_L = [n]^2 \setminus L$ is the pipe dream complementary to L , each face L is identified with the coordinate subspace

$$L = \{z_{qp} = 0 \mid (q, p) \in D_L\} = \text{span}(E_{qp} \mid (q, p) \notin D_L).$$

Thus, with D_L a pipe dream, its crosses \perp lie in the spots where L is zero. For instance, the three pipe dreams in the example from the introduction are pipe dreams for the subspaces $L_{11,13}$, $L_{11,22}$, and $L_{11,31}$.

The term *facet* means ‘maximal face’, and Definition 1.8.5 gives the meaning of ‘shellable’.

THEOREM B. *The minors of size $1 + \text{rank}(w_{q \times p}^T)$ in $Z_{q \times p}$ for all q, p constitute a Gröbner basis for any antidiagonal term order; equivalently, $\text{in}(I_w) = J_w$ for any such term order. The Stanley–Reisner complex \mathcal{L}_w of J_w is shellable, and hence Cohen–Macaulay. In addition,*

$$\{D_L \mid L \text{ is a facet of } \mathcal{L}_w\} = \mathcal{RP}(w)$$

places the set of reduced pipe dreams for w in canonical bijection with the facets of \mathcal{L}_w .

The displayed equation is equivalent to J_w having the prime decomposition

$$J_w = \bigcap_{D \in \mathcal{RP}(w)} \langle z_{ij} \mid (i, j) \in D \rangle.$$

Geometrically, Theorem B says that the matrix Schubert variety \overline{X}_w has a flat degeneration whose limit is both reduced and Cohen–Macaulay, and whose components are in natural bijection with reduced pipe dreams. On its own, Theorem B therefore ascribes a truly geometric origin to reduced pipe dreams. Taken together with Theorem A, it provides in addition a natural geometric explanation for the combinatorial formulae with Schubert polynomials in terms of pipe dreams: interpret in equivariant cohomology the decomposition of \mathcal{L}_w into irreducible components. We carry out this procedure in Section 2.1 using multidegrees, for which the required technology is developed in Section 1.7. The analogous K -theoretic formula, which additionally involves *nonreduced* pipe dreams, requires more detailed analysis of subword complexes (Definition 1.8.1), and therefore appears in [KnM04].

Example 1.5.2. Let $w = 2143$ as in the example from the introduction and Example 1.3.6. The term orders that interest us pick out the antidiagonal term $-z_{13}z_{22}z_{31}$ from the northwest 3×3 minor. For I_{2143} , this causes the initial terms of its two generating minors to be relatively prime, so the minors form a Gröbner basis as in Theorem B. Observe that the minors generating I_w do not form a Gröbner basis with respect to term orders that pick out the *diagonal* term $z_{11}z_{22}z_{33}$ of the 3×3 minor, because z_{11} divides that.

The initial complex \mathcal{L}_{2143} is shellable, being a cone over the boundary of a triangle, and as mentioned in the introduction, its facets correspond to the reduced pipe dreams for 2143.

Example 1.5.3. A direct check reveals that every antidiagonal in J_w for $w = 13865742$ stipulated by Definition 1.3.1 is divisible by an antidiagonal of some 2- or 3-minor from Example 1.3.5. Hence the 165 minors of size 2×2 and 3×3 in I_w form a Gröbner basis for I_w .

Remark 1.5.4. M. Kogan also has a geometric interpretation for reduced pipe dreams, identifying them in [Kog00] as subsets of the flag manifold map-

ping to corresponding faces of the Gel'fand–Cetlin polytope. These subsets are not cycles, so they do not individually determine cohomology classes whose sum is the Schubert class; nonetheless, their union is a cycle, and its class is the Schubert class. See also [KoM03].

Remark 1.5.5. Theorem B says that every antidiagonal shares at least one cross with every reduced pipe dream, and moreover, that each antidiagonal and reduced pipe dream is minimal with this property. Loosely, antidiagonals and reduced pipe dreams ‘minimally poison’ each other. Our proof of this purely combinatorial statement in Sections 3.7 and 3.8 is indeed essentially combinatorial, but rather roundabout; we know of no simple reason for it.

Remark 1.5.6. The Gröbner basis in Theorem B defines a flat degeneration over any ring, because all of the coefficients of the minors in I_w are integers, and the leading coefficients are all ± 1 . Indeed, each loop of the division algorithm in Buchberger’s criterion [Eis95, Th. 15.8] works over \mathbb{Z} , and therefore over any ring.

1.6. Mitosis algorithm

Next we introduce a simple combinatorial rule, called ‘mitosis’,⁴ that creates from each pipe dream a number of new pipe dreams called its ‘offspring’. Mitosis serves as a geometrically motivated improvement on Kohnert’s rule [Koh91], [Mac91], [Win99], which acts on other subsets of $[n]^2$ derived from permutation matrices. In addition to its independent interest from a combinatorial standpoint, our forthcoming Theorem C falls out of Bruhat induction with no extra work, and in fact the mitosis operation plays a vital role in Bruhat induction, toward the end of Part 3.

Given a pipe dream in $[n] \times [n]$, define

$$(2) \quad \begin{aligned} \text{start}_i(D) &= \text{column index of leftmost empty box in row } i \\ &= \min(\{j \mid (i, j) \notin D\} \cup \{n + 1\}). \end{aligned}$$

Thus in the region to the left of $\text{start}_i(D)$, the i^{th} row of D is filled solidly with crosses. Let

$$\mathcal{J}_i(D) = \{\text{columns } j \text{ strictly to the left of } \text{start}_i(D) \mid (i + 1, j) \text{ has no cross in } D\}.$$

For $p \in \mathcal{J}_i(D)$, construct the *offspring* $D_p(i)$ as follows. First delete the cross at (i, p) from D . Then take all crosses in row i of $\mathcal{J}_i(D)$ that are to the left of column p , and move each one down to the empty box below it in row $i + 1$.

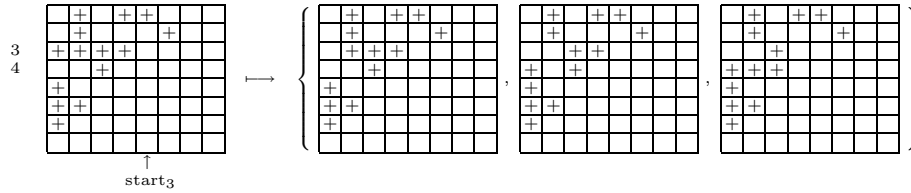
⁴The term *mitosis* is biological lingo for cell division in multicellular organisms.

Definition 1.6.1. The i^{th} mitosis operator sends a pipe dream D to

$$\text{mitosis}_i(D) = \{D_p(i) \mid p \in \mathcal{J}_i(D)\}.$$

Thus all the action takes place in rows i and $i + 1$, and $\text{mitosis}_i(D)$ is an empty set if $\mathcal{J}_i(D)$ is empty. Write $\text{mitosis}_i(\mathcal{P}) = \bigcup_{D \in \mathcal{P}} \text{mitosis}_i(D)$ whenever \mathcal{P} is a set of pipe dreams.

Example 1.6.2. The left diagram D below is the reduced pipe dream for $w = 13865742$ from Example 1.4.2 (the pipe dream in Fig. 1) and Example 1.4.6:



The set of three pipe dreams on the right is obtained by applying mitosis_3 , since $\mathcal{J}_3(D)$ consists of columns 1, 2, and 4.

THEOREM C. *If $\text{length}(ws_i) < \text{length}(w)$, then $\mathcal{RP}(ws_i)$ is equal to the disjoint union $\bigcup_{D \in \mathcal{RP}(w)} \text{mitosis}_i(D)$. Thus if $s_{i_1} \cdots s_{i_k}$ is a reduced expression for $w_0 w$, and D_0 is the unique reduced pipe dream for w_0 , in which every entry above the antidiagonal is a ‘+’, then*

$$\mathcal{RP}(w) = \text{mitosis}_{i_k} \cdots \text{mitosis}_{i_1}(D_0).$$

Readers wishing a simple and purely combinatorial proof that avoids Bruhat induction as in Part 3 should consult [Mil03]; the proof there uses only definitions and the statement of Corollary 2.1.3, below, which has elementary combinatorial proofs. However, granting Theorem C does not by itself simplify the arguments in Part 3 here: we still need the ‘lifted Demazure operators’ from Section 3.4, of which mitosis is a distilled residue.

Example 1.6.3. The left pipe dream in Example 1.6.2 lies in $\mathcal{RP}(13865742)$. Therefore the three diagrams on the right-hand side of Example 1.6.2 are reduced pipe dreams for $13685742 = 13865742 \cdot s_3$ by Theorem C, as can also be checked directly.

Like Kohnert’s rule, mitosis is inductive on weak Bruhat order, starts with subsets of $[n]^2$ naturally associated to the permutations in S_n , and produces more subsets of $[n]^2$. Unlike Kohnert’s rule, however, the offspring of mitosis still lie in the same natural set as the parent, and the algorithm in Theorem C for generating $\mathcal{RP}(w)$ is irredundant, in the sense that each reduced pipe dream appears exactly once in the implicit union on the right-hand side of the equation

in Theorem C. See [Mil03] for more on properties of the mitosis recursion and structures on the set of reduced pipe dreams, as well as background on other combinatorial algorithms for coefficients of Schubert polynomials.

1.7. Positivity of multidegrees

The key to our view of positivity, which we state in Theorem D, lies in three properties of multidegrees (Theorem 1.7.1) that characterize them uniquely among functions on multigraded modules. Since the multigradings considered here are *positive*, meaning that every graded piece of $\mathbf{k}[\mathbf{z}]$ (and hence every graded piece of every finitely generated graded module) has finite dimension as a vector space over the field \mathbf{k} , we are able to present short complete proofs of the required assertions.

In this section we resume the generality and notation concerning multigradings from Section 1.2. Given a (reduced and irreducible) variety X and a module Γ over $\mathbf{k}[\mathbf{z}]$, let $\text{mult}_X(\Gamma)$ denote the *multiplicity* of Γ along X , which by definition equals the length of the largest finite-length submodule in the localization of Γ at the prime ideal of X . The *support* of Γ consists of those points at which the localization of Γ is nonzero.

THEOREM 1.7.1. *The multidegree $\Gamma \mapsto \mathcal{C}(\Gamma; \mathbf{t})$ is uniquely characterized among functions from the class of finitely generated \mathbb{Z}^d -graded modules to $\mathbb{Z}[\mathbf{t}]$ by the following.*

- *Additivity: The (automatically \mathbb{Z}^d -graded) irreducible components X_1, \dots, X_r of maximal dimension in the support of a module Γ satisfy*

$$\mathcal{C}(\Gamma; \mathbf{t}) = \sum_{\ell=1}^r \text{mult}_{X_\ell}(\Gamma) \cdot \mathcal{C}(X_\ell, \mathbf{t}).$$

- *Degeneration: Let u be a variable of ordinary weight zero. If a finitely generated \mathbb{Z}^d -graded module over $\mathbf{k}[\mathbf{z}][u]$ is flat over $\mathbf{k}[u]$ and has $u = 1$ fiber isomorphic to Γ , then its $u = 0$ fiber Γ' has the same multidegree as Γ does:*

$$\mathcal{C}(\Gamma; \mathbf{t}) = \mathcal{C}(\Gamma'; \mathbf{t}).$$

- *Normalization: If $\Gamma = \mathbf{k}[\mathbf{z}]/\langle z_i \mid i \in D \rangle$ is the coordinate ring of a coordinate subspace of \mathbf{k}^m for some subset $D \subseteq \{1, \dots, m\}$, then*

$$\mathcal{C}(\Gamma; \mathbf{t}) = \prod_{i \in D} \left(\sum_{j=1}^d a_{ij} t_j \right)$$

is the corresponding product of ordinary weights in $\mathbb{Z}[\mathbf{t}] = \text{Sym}_{\mathbb{Z}}^{\bullet}(\mathbb{Z}^d)$.

Proof. For uniqueness, first observe that every finitely generated \mathbb{Z}^d -graded module Γ can be degenerated via Gröbner bases to a module Γ' supported on a union of coordinate subspaces [Eis95, Ch. 15]. By degeneration the module Γ' has the same multidegree; by additivity the multidegree of Γ' is determined by the multidegrees of coordinate subspaces; and by normalization the multidegrees of coordinate subspaces are fixed.

Now we must prove that multidegrees satisfy the three conditions. Degeneration is easy: since we have assumed the grading to be positive, \mathbb{Z}^d -graded modules have \mathbb{Z}^d -graded Hilbert series, which are constant in flat families of multigraded modules.

Normalization involves a bit of calculation. Using the Koszul complex, the K -polynomial of $\mathbf{k}[\mathbf{z}]/\langle z_i \mid i \in D \rangle$ is computed to be $\prod_{i \in D} (1 - \mathbf{t}^{\mathbf{a}_i})$. Thus it suffices to show that if $K(\mathbf{t}) = 1 - \mathbf{t}^{\mathbf{b}} = 1 - t_1^{b_1} \cdots t_d^{b_d}$, then substituting $1 - t_j$ for each occurrence of t_j yields $K(\mathbf{1} - \mathbf{t}) = b_1 t_1 + \cdots + b_d t_d + O(\mathbf{t}^2)$, where $O(\mathbf{t}^e)$ denotes a sum of terms each of which has total degree at least e . Indeed, then we can conclude that

$$\mathcal{K}(\mathbf{k}[\mathbf{z}]/\langle z_i \mid i \in D \rangle; \mathbf{1} - \mathbf{t}) = \left(\prod_{i \in D} \mathbf{a}_i \right) + O(\mathbf{t}^{r+1}),$$

where r is the size of D . Calculating $K(\mathbf{1} - \mathbf{t})$ yields

$$1 - \prod_{j=1}^d (1 - t_j)^{b_j} = 1 - \prod_{j=1}^d (1 - b_j t_j + O(t_j^2)),$$

from which we get the desired formula

$$1 - \left(1 - \sum_{j=1}^d (b_j t_j) + O(\mathbf{t}^2) \right) = \left(\sum_{j=1}^d b_j t_j \right) + O(\mathbf{t}^2).$$

All that remains is additivity. Every associated prime of Γ is \mathbb{Z}^d -graded by [Eis95, Exercise 3.5]. Choose by noetherian induction a filtration $\Gamma = \Gamma_\ell \supset \Gamma_{\ell-1} \supset \cdots \supset \Gamma_1 \supset \Gamma_0 = 0$ in which $\Gamma_j/\Gamma_{j-1} \cong (\mathbf{k}[\mathbf{z}]/\mathfrak{p}_j)(-\mathbf{b}_j)$ for multigraded primes \mathfrak{p}_j and vectors $\mathbf{b}_j \in \mathbb{Z}^d$. Additivity of K -polynomials on short exact sequences implies that $\mathcal{K}(\Gamma; \mathbf{t}) = \sum_{j=1}^\ell \mathcal{K}(\Gamma_j/\Gamma_{j-1}; \mathbf{t})$.

The variety of \mathfrak{p}_j is contained inside the support of Γ , and if \mathfrak{p} has dimension exactly $\dim(\Gamma)$, then \mathfrak{p} equals the prime ideal of some top-dimensional component $X \in \{X_1, \dots, X_r\}$ for exactly $\text{mult}_X(\Gamma)$ values of j (localize the filtration at \mathfrak{p} to see this).

Assume for the moment that Γ is a direct sum of multigraded shifts of quotients of $\mathbf{k}[\mathbf{z}]$ by monomial ideals. The filtration can be chosen so that all the primes \mathfrak{p}_j are of the form $\langle z_i \mid i \in D \rangle$. By normalization and the obvious equality $\mathcal{K}(\Gamma'(\mathbf{b}); \mathbf{t}) = \mathbf{t}^{\mathbf{b}} \mathcal{K}(\Gamma'; \mathbf{t})$ for any \mathbb{Z}^d -graded module Γ' , the only power series $\mathcal{K}(\Gamma_j/\Gamma_{j-1}; \mathbf{1} - \mathbf{t})$ contributing terms to $\mathcal{K}(\Gamma; \mathbf{1} - \mathbf{t})$ are those for which

Γ_j/Γ_{j-1} has maximal dimension. Therefore the theorem holds for direct sums of shifts of monomial quotients.

By Gröbner degeneration, a general module Γ of codimension r has the same multidegree as a direct sum of shifts of monomial quotients. Using the filtration for this general Γ , it follows from the previous paragraph that $\mathcal{K}(\Gamma_j/\Gamma_{j-1}; \mathbf{1} - \mathbf{t}) = \mathcal{C}(\Gamma_j/\Gamma_{j-1}; \mathbf{t}) + O(\mathbf{t}^{r+1})$. Therefore the last two sentences of the previous paragraph work also for the general module Γ . \square

Our general view of positivity proceeds thus: multidegrees, like ordinary degrees, are additive on unions of schemes with equal dimension and no common components. Additivity under unions becomes quite useful for monomial ideals, because their irreducible components are coordinate subspaces, whose multidegrees are simple. Explicit knowledge of the multidegrees of monomial subschemes of \mathbf{k}^m yields formulae for multidegrees of arbitrary subschemes because multidegrees are constant in flat families.

THEOREM D. *The multidegree of any module of dimension $m - r$ over a positively \mathbb{Z}^d -graded polynomial ring $\mathbf{k}[\mathbf{z}]$ in m variables is a positive sum of terms of the form $\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_r} \in \text{Sym}_{\mathbb{Z}}^r(\mathbb{Z}^d)$, where $i_1 < \cdots < i_r$.*

Proof. The special fiber of any Gröbner degeneration of the module Γ has support equal to a union of coordinate subspaces. Now use Theorem 1.7.1. \square

The products $\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_r}$ are all nonzero, and all lie in a single polyhedral cone containing no linear subspace (a semigroup with no units) inside $\text{Sym}_{\mathbb{Z}}^r(\mathbb{Z}^d)$, by positivity. Thus, when we say “positive sum” in Theorem D, we mean in particular that the sum is nonzero. A similar theorem occurs in [Jos97], applied to the special case of “orbital varieties”, where an induction based on hyperplane sections is used in place of our Gröbner geometry argument.

Although the indices on $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ are distinct, some of the weights themselves might be equal. This occurs when $\mathbf{k}^m = M_n$ and $\text{wt}(z_i) = x_i$, for example: any monomial of degree at most n in each x_i is attainable. Theorem D implies in this case that a polynomial expressible as the \mathbb{Z}^n -graded multidegree of some subscheme of M_n has positive coefficients. In fact, the coefficients count geometric objects, namely subspaces (with multiplicity) in *any* Gröbner degeneration. Therefore Theorem D completes our second goal (ii) from the introduction, that of proving positivity of Schubert polynomials in a natural geometric setting, in view of Theorem A, which completed the first goal (i).

The conditions in Theorem 1.7.1 overdetermine the multidegree function: there is usually no *single* best way to write a multidegree as a positive sum in Theorem D. It happens that antidiagonal degenerations of matrix Schubert varieties as in Theorem B give particularly nice multiplicity 1 formulae, where the geometric objects have combinatorial significance as in Theorems B and C. The details of this story are fleshed out in Section 2.1.

Example 1.7.2. Five of the six 3×3 matrix Schubert varieties in Example 1.3.4 have \mathbb{Z}^{2n} -graded multidegrees that are products of expressions having the form $x_i - y_j$ by the normalization condition in Theorem 1.7.1:

$$\begin{aligned} [\overline{X}_{123}]_{\mathbb{Z}^{2n}} &= 1, \\ [\overline{X}_{213}]_{\mathbb{Z}^{2n}} &= x_1 - y_1, \\ [\overline{X}_{231}]_{\mathbb{Z}^{2n}} &= (x_1 - y_1)(x_1 - y_2), \\ [\overline{X}_{312}]_{\mathbb{Z}^{2n}} &= (x_1 - y_1)(x_2 - y_1), \\ [\overline{X}_{321}]_{\mathbb{Z}^{2n}} &= (x_1 - y_1)(x_1 - y_2)(x_2 - y_1). \end{aligned}$$

The last one, \overline{X}_{132} , has multidegree

$$[\overline{X}_{132}]_{\mathbb{Z}^{2n}} = x_1 + x_2 - y_1 - y_2$$

that can be written as a sum of expressions $(x_i - y_j)$ in two different ways. To see how, pick term orders that choose different leading monomials for $z_{11}z_{22} - z_{12}z_{21}$. Geometrically, these degenerate \overline{X}_{132} to either the scheme defined by $z_{11}z_{22}$ or the scheme defined by $z_{12}z_{21}$, while preserving the multidegree in both cases. The degenerate limits break up as unions

$$\begin{aligned} \overline{X}'_{132} &= \{Z \in M_3 \mid z_{11} = 0\} \cup \{Z \in M_3 \mid z_{22} = 0\} = \{Z \in M_3 \mid z_{11}z_{22} = 0\}, \\ \overline{X}''_{132} &= \{Z \in M_3 \mid z_{12} = 0\} \cup \{Z \in M_3 \mid z_{21} = 0\} = \{Z \in M_3 \mid z_{12}z_{21} = 0\}, \end{aligned}$$

and therefore have multidegrees

$$\begin{aligned} [\overline{X}'_{132}]_{\mathbb{Z}^{2n}} &= (x_1 - y_1) + (x_2 - y_2), \\ [\overline{X}''_{132}]_{\mathbb{Z}^{2n}} &= (x_1 - y_2) + (x_2 - y_1). \end{aligned}$$

Either way calculates $[\overline{X}_{132}]_{\mathbb{Z}^{2n}}$ as in Theorem D. For most permutations $w \in S_n$, only antidiagonal degenerations (such as \overline{X}''_{132}) can be read off the minors generating I_w .

Multidegrees are functorial with respect to changes of grading, as the following proposition says. It holds for prime monomial quotients $\Gamma = \mathbf{k}[\mathbf{z}]/\langle z_i \mid i \in D \rangle$ by normalization, and generally by Gröbner degeneration along with additivity.

PROPOSITION 1.7.3. *If $\mathbb{Z}^d \rightarrow \mathbb{Z}^{d'}$ is a homomorphism of groups, then any \mathbb{Z}^d -graded module Γ is also $\mathbb{Z}^{d'}$ -graded. Furthermore, K -polynomials and multidegrees specialize naturally:*

1. *The \mathbb{Z}^d -graded \mathcal{K} -polynomial $\mathcal{K}(\Gamma; \mathbf{t})$ maps to the $\mathbb{Z}^{d'}$ -graded K -polynomial $\mathcal{K}(\Gamma; \mathbf{t}')$ under the natural homomorphism $\mathbb{Z}[\mathbb{Z}^d] \rightarrow \mathbb{Z}[\mathbb{Z}^{d'}]$ of group rings; and*
2. *The \mathbb{Z}^d -graded multidegree $\mathcal{C}(\Gamma; \mathbf{t})$ maps to the $\mathbb{Z}^{d'}$ -graded multidegree $\mathcal{C}(\Gamma; \mathbf{t}')$ under the natural homomorphism $\text{Sym}_{\mathbb{Z}}(\mathbb{Z}^d) \rightarrow \text{Sym}_{\mathbb{Z}}(\mathbb{Z}^{d'})$.*

Example 1.7.4. Changes between the gradings from Example 1.2.1 go as follows.

change of grading	$\mathbb{Z} \leftarrow \mathbb{Z}^n$	$\mathbb{Z}^n \leftarrow \mathbb{Z}^{2n}$	$\mathbb{Z}^{2n} \leftarrow \mathbb{Z}^{n^2}$
map on variables in K -polynomials	$t \leftarrow x_i$	$x_i \leftarrow x_i$ $1 \leftarrow y_j$	$x_i/y_j \leftarrow z_{ij}$
map on variables in multidegrees	$t \leftarrow x_i$	$x_i \leftarrow x_i$ $0 \leftarrow y_j$	$x_i - y_j \leftarrow z_{ij}$

We often call these maps *specialization*, or *coarsening the grading*. Setting all occurrences of y_j to zero in Example 1.7.2 yields \mathbb{Z}^n -graded multidegrees, for instance; compare these to the diagram in Example 1.1.2.

The connective tissue in our proofs of Theorems A, B, and C (Section 3.9) consists of the next observation. It appears in its \mathbb{Z} -graded form independently in [Mar02] (although Martin applies the ensuing conclusion that a candidate Gröbner basis actually is one to a different ideal). This will be applied with $I' = \text{in}(I)$ for some ideal I and some term order.

LEMMA 1.7.5. *Let $I' \subseteq \mathbf{k}[z_1, \dots, z_m]$ be an ideal, homogeneous for a positive \mathbb{Z}^d -grading. Suppose that J is an equidimensional radical ideal contained inside I' . If the zero schemes of I' and J have equal multidegrees, then $I' = J$.*

Proof. Let X and Y be the schemes defined by I' and J , respectively. The multidegree of $\mathbf{k}[\mathbf{z}]/J$ equals the sum of the multidegrees of the components of Y , by additivity. Since $J \subseteq I'$, each maximal dimensional irreducible component of X is contained in some component of Y , and hence is equal to it (and reduced) by comparing dimensions: equal multidegrees implies equal dimensions by Theorem D. Additivity says that the multidegree of X equals the sum of multidegrees of components of Y that happen also to be components of X . By hypothesis, the multidegrees of X and Y coincide, so the sum of multidegrees of the remaining components of Y is zero. This implies that no components remain, by Theorem D, so $X \supseteq Y$. Equivalently, $I' \subseteq J$, whence $I' = J$ by the hypothesis $J \subseteq I'$. □

1.8. Subword complexes in Coxeter groups

This section exploits the properties of reduced words in Coxeter groups to produce shellings of the initial complex \mathcal{L}_w from Theorem B. More precisely, we define a new class of simplicial complexes that generalizes to arbitrary Coxeter groups the construction in Section 1.4 of reduced pipe dreams for a permutation $w \in S_n$ from the triangular reduced expression for w_0 . The manner in which subword complexes characterize reduced pipe dreams is similar in spirit to [FK96]; however, even for reduced pipe dreams our topological perspective is new.

We felt it important to include the Cohen–Macaulayness of the initial scheme \mathcal{L}_w as part of our evidence for the naturality of Gröbner geometry for Schubert polynomials, and the generality of subword complexes allows our simple proof of their shellability. However, a more detailed analysis would take us too far afield, so that we have chosen to develop the theory of subword complexes in Coxeter groups more fully elsewhere [KnM04]. There, we show that subword complexes are balls or spheres, and calculate their Hilbert series for applications to Grothendieck polynomials. We also comment there on how our forthcoming Theorem E reflects topologically some of the fundamental properties of reduced (and nonreduced) expressions in Coxeter groups, and how Theorem E relates to known results on simplicial complexes constructed from Bruhat and weak orders.

Let (Π, Σ) be a Coxeter system, so that Π is a Coxeter group and Σ is a set of simple reflections, which generate Π . See [Hum90] for background and definitions; the applications to reduced pipe dreams concern only the case where $\Pi = S_n$ and Σ consists of the adjacent transpositions switching i and $i + 1$ for $1 \leq i \leq n - 1$.

Definition 1.8.1. A *word* of size m is an ordered sequence $Q = (\sigma_1, \dots, \sigma_m)$ of elements of Σ . An ordered subsequence P of Q is called a *subword* of Q .

1. P *represents* $\pi \in \Pi$ if the ordered product of the simple reflections in P is a reduced decomposition for π .
2. P *contains* $\pi \in \Pi$ if some subsequence of P represents π .

The *subword complex* $\Delta(Q, \pi)$ is the set of subwords $P \subseteq Q$ whose complements $Q \setminus P$ contain π .

Often we write Q as a string without parentheses or commas, and abuse notation by saying that Q is a word in Π . Note that Q need not itself be a reduced expression, but the facets of $\Delta(Q, \pi)$ are the complements of reduced subwords of Q . The word P contains π if and only if the product of P in the degenerate Hecke algebra is $\geq \pi$ in Bruhat order [FK96].

Example 1.8.2. Let $\Pi = S_4$, and consider the subword complex $\Delta = \Delta(s_3s_2s_3s_2s_3, 1432)$. Then $\pi = 1432$ has two reduced expressions, namely $s_3s_2s_3$ and $s_2s_3s_2$. Labeling the vertices of a pentagon with the reflections in $Q = s_3s_2s_3s_2s_3$ (in cyclic order), we find that the facets of Δ are the pairs of adjacent vertices. Therefore Δ is the pentagonal boundary.

Example 1.8.3. Let $\Pi = S_{2n}$ and let the *square word*

$$Q_{n \times n} = s_n s_{n-1} \dots s_2 s_1 \ s_{n+1} s_n \dots s_3 s_2 \ \dots \ s_{2n-1} s_{2n-2} \dots s_{n+1} s_n$$

be the ordered list constructed from the pipe dream whose crosses entirely fill the $n \times n$ grid. Reduced expressions for permutations $w \in S_n$ never involve reflections s_i with $i \geq n$. Therefore, if Q_0 is the triangular long word for S_n (not S_{2n}) in Example 1.4.4, then $\Delta(Q_{n \times n}, w)$ is the join of $\Delta(Q_0, w)$ with a simplex whose $\binom{n}{2}$ vertices correspond to the lower-right triangle of the $n \times n$ grid. Consequently, the facets of $\Delta(Q_{n \times n}, w)$ are precisely the complements in $[n] \times [n]$ of the reduced pipe dreams for w , by Lemma 1.4.5.

The following lemma is immediate from the definitions and the fact that all reduced expressions for $\pi \in \Pi$ have the same length.

LEMMA 1.8.4. $\Delta(Q, \pi)$ is a pure simplicial complex whose facets are the subwords $Q \setminus P$ such that $P \subseteq Q$ represents π .

Definition 1.8.5. Let Δ be a simplicial complex and $F \in \Delta$ a face.

1. The *deletion* of F from Δ is $\text{del}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset\}$.
2. The *link* of F in Δ is $\text{link}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}$.

Δ is *vertex-decomposable* if Δ is pure and either (1) $\Delta = \{\emptyset\}$, or (2) for some vertex $v \in \Delta$, both $\text{del}(v, \Delta)$ and $\text{link}(v, \Delta)$ are vertex-decomposable. A *shelling* of Δ is an ordered list F_1, F_2, \dots, F_t of its facets such that $\bigcup_{j < i} F_j \cap F_i$ is a union of codimension 1 faces of F_i for each $i \leq t$. We say Δ is *shellable* if it is pure and has a shelling.

Provan and Billera [BP79] introduced the notion of vertex-decomposability and proved that it implies shellability (proof: use induction on the number of vertices by first shelling $\text{del}(v, \Delta)$ and then shelling the cone from v over $\text{link}(v, \Delta)$ to get a shelling of Δ). It is well-known that shellability implies Cohen–Macaulayness [BH93, Th. 5.1.13]. Here, then, is our central observation concerning subword complexes.

THEOREM E. Any subword complex $\Delta(Q, \pi)$ is vertex-decomposable. In particular, subword complexes are shellable and therefore Cohen–Macaulay.

Proof. With $Q = (\sigma, \sigma_2, \sigma_3, \dots, \sigma_m)$, we show that both the link and the deletion of σ from $\Delta(Q, \pi)$ are subword complexes. By definition, both consist of subwords of $Q' = (\sigma_2, \dots, \sigma_m)$. The link is naturally identified with the subword complex $\Delta(Q', \pi)$. For the deletion, there are two cases. If $\sigma\pi$ is longer than π , then the deletion of σ equals its link because no reduced expression for π begins with σ . On the other hand, when $\sigma\pi$ is shorter than π , the deletion is $\Delta(Q', \sigma\pi)$. \square

Remark 1.8.6. The vertex decomposition that results for initial ideals of matrix Schubert varieties has direct analogues in the Gröbner degenerations and formulae for Schubert polynomials. Consider the sequence $>_1, >_2, \dots, >_{n^2}$ of partial term orders, where $>_i$ is lexicographic in the first i matrix entries snaking from northeast to southwest one row at a time, and treats all remaining variables equally. The order $>_{n^2}$ is a total order; this total order is antidiagonal, and hence degenerates \overline{X}_w to the subword complex by Theorem B and Example 1.8.3. Each $>_i$ gives a degeneration of \overline{X}_w to a union of components, every one of which degenerates at $>_{n^2}$ to its own subword complex.

If we study how a component at stage i degenerates into components at stage $i + 1$, by degenerating both using $>_{n^2}$, we recover the vertex decomposition for the corresponding subword complex.

Note that these components are *not* always matrix Schubert varieties; the set of rank conditions involved does not necessarily involve only upper-left submatrices. We do not know how general a class of determinantal ideals can be tackled by partial degeneration of matrix Schubert varieties, using antidiagonal partial term orders.

However, if we degenerate using the partial order $>_n$ (order just the first row of variables), then the components *are* matrix Schubert varieties, except that the minors involved are all shifted down one row. This gives a geometric interpretation of the inductive formula for Schubert polynomials appearing in Section 1.3 of [BJS93].

Part 2. Applications of the Gröbner geometry theorems

2.1. Positive formulae for Schubert polynomials

The original definition of Schubert polynomials by Lascoux and Schützenberger via the divided difference recursion involves negation, so it is not quite obvious from their formulation that the coefficients of $\mathfrak{S}_w(\mathbf{x})$ are in fact positive. Although Lascoux and Schützenberger did prove positivity using their ‘transition formula’, the first combinatorial proofs, showing what the coefficients count, appeared in [BJS93], [FS94]. More recently, [Kim00], [BS02], [Kog00] show that the coefficients are positive for geometric reasons.

Our approach has the advantage that it produces geometrically a uniquely determined polynomial representative for each Schubert class, and moreover, that it provides an obvious geometric reason why this representative has non-negative coefficients in the variables x_1, \dots, x_n (or $\{x_i - y_j\}_{i,j=1}^n$ in the double case). Only then do we identify the coefficients as counting known combinatorial objects; it is coincidence (or naturality of the combinatorics) that our positive formula for Schubert polynomials agrees with—and provides a new geometric proof of—the combinatorial formula of Billey, Jockusch, and Stanley [BJS93].

THEOREM 2.1.1. *There is a multidegree formula written*

$$\mathfrak{S}_w = [\overline{X}_w] = \sum_{L \in \mathcal{L}_w} [L]$$

as a sum over the facets L of the initial complex \mathcal{L}_w , thereby expressing the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ as a positive of monomials $[L]_{\mathbb{Z}^n} = \prod_{(i,j) \in D_L} x_i$ in the variables x_1, \dots, x_n , and the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ as a sum of expressions $[L]_{\mathbb{Z}^{2n}} = \prod_{(i,j) \in D_L} (x_i - y_j)$, which are themselves positive in the variables x_1, \dots, x_n and $-y_1, \dots, -y_n$.

Proof. By Theorem B and degeneration in Theorem 1.7.1, the multidegrees of \overline{X}_w and the zero set \mathcal{L}_w of J_w are equal in any grading. Since $[\overline{X}_w] = \mathfrak{S}_w$ by Theorem A, the formulae then follow from additivity and normalization in Theorem 1.7.1, given the ordinary weights in Example 1.7.4. \square

Remark 2.1.2. The version of this positivity in algebraic geometry is the notion of “effective homology class”, meaning “representable by a subscheme”. On the flag manifold, a homology class is effective exactly if it is a nonnegative combination of Schubert classes. (*Proof:* One direction is a tautology. For the other, if X is a subscheme of the flag manifold $\mathcal{F}\ell_n$, consider the induced action of the Borel group B on the Hilbert scheme for $\mathcal{F}\ell_n$. The closure of the B -orbit through the Hilbert point X will be projective because the Hilbert scheme is, so Borel’s theorem produces a fixed point, necessarily a union of Schubert varieties, perhaps nonreduced.) In particular the classes of monomials in the x_i (the first Chern classes of the standard line bundles; see Section 2.3) are not usually effective.

We work instead on M_n , where the standard line bundles become trivial, but not equivariantly, and a class is effective exactly if it is a nonnegative combination of monomials in the equivariant first Chern classes x_i . (*Proof:* Instead of using B to degenerate a subscheme X inside M_n , use a 1-parameter subgroup of the n^2 -dimensional torus. Algebraically, this amounts to picking a Gröbner basis.)

The next formula was our motivation for relating the antidiagonal complex \mathcal{L}_w to the set $\mathcal{RP}(w)$ of reduced pipe dreams. Although formulated here in language based on [FK96], its first proof (in transparently equivalent language) was by Billey, Jockusch, and Stanley, while Fomin and Stanley shortly thereafter gave a better combinatorial proof.

COROLLARY 2.1.3 ([BJS93], [FS94]).

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{D \in \mathcal{RP}(w)} \mathbf{x}^D, \quad \text{where } \mathbf{x}^D = \prod_{(i,j) \in D} x_i.$$

Proof. Apply Theorem B to the \mathbb{Z}^n -graded version of the formula in Theorem 2.1.1. \square

Example 2.1.4. As in the example from the introduction, Lemma 1.4.5 calculates the multidegree as

$$\begin{aligned} [\overline{X}_{2143}] &= [L_{11,13}] + [L_{11,22}] + [L_{11,31}] \\ &= \text{wt}(z_{11}z_{13}) + \text{wt}(z_{11}z_{22}) + \text{wt}(z_{11}z_{31}) \\ &= x_1^2 + x_1x_2 + x_1x_3 \end{aligned}$$

in $\mathbb{Z}[x_1, x_2, x_3, x_4]$, for the \mathbb{Z}^n -grading.

The double version of Corollary 2.1.3 has the same proof, using the \mathbb{Z}^{2n} -grading; deriving it directly from reduced pipe dreams here bypasses the “double rc-graphs” of [BB93].

$$\text{COROLLARY 2.1.5 ([FK96]). } \mathfrak{S}_w(\mathbf{x}, \mathbf{y}) = \sum_{D \in \mathcal{RP}(w)} \prod_{(i,j) \in D} (x_i - y_j).$$

Theorem 2.1.1 and Corollary 2.1.3 together are consequences of Theorem B and the multidegree part of Theorem A. There is a more subtle kind of positivity for Grothendieck polynomials, due to Fomin and Kirillov [FK94], that can be derived from Theorem B and the Hilbert series part of Theorem A, along with the Eagon–Reiner theorem from combinatorial commutative algebra [ER98]. In fact, this “positivity” was our chief evidence leading us to conjecture the Cohen–Macaulayness of J_w .

More precisely, the work of Fomin and Kirillov implies that for each d there is a homogeneous polynomial $\mathcal{G}_w^{(d)}(\mathbf{x})$ of degree d with nonnegative coefficients such that

$$(3) \quad \mathcal{G}_w(\mathbf{1} - \mathbf{x}) = \sum_{d \geq \ell} (-1)^{d-\ell} \mathcal{G}_w^{(d)}(\mathbf{x}),$$

where $\ell = \text{length}(w)$. In other words, the coefficients on each homogeneous piece of $\mathcal{G}_w(\mathbf{1} - \mathbf{x})$ all have the same sign. On the other hand, the Eagon–Reiner theorem states:

A simplicial complex Δ is Cohen–Macaulay if and only if the Alexander dual J_Δ^* of its Stanley–Reisner ideal has *linear free resolution*, meaning that the differential in its minimal \mathbb{Z} -graded free resolution over $\mathbf{k}[\mathbf{z}]$ can be expressed using matrices filled with linear forms.

The K -polynomial of any module with linear resolution alternates as in (3). But the Alexander inversion formula [KnM04] implies that $\mathcal{G}_w(\mathbf{1} - \mathbf{x})$ is the K -polynomial of J_w^* , given that $\mathcal{G}_w(\mathbf{x})$ is the K -polynomial of $\mathbf{k}[\mathbf{z}]/J_w$ as in

Theorem A. Therefore, $\mathcal{G}_w(\mathbf{1} - \mathbf{x})$ must alternate as in (3), if the Cohen-Macaulayness in Theorem B holds. It would take suspiciously fortuitous cancellation to have a squarefree monomial ideal J_w^* whose K -polynomial $\mathcal{G}_w(\mathbf{1} - \mathbf{x})$ behaves like (3) without the ideal J_w^* actually having linear resolution.

In fact, further investigation into the algebraic combinatorics of subword complexes can identify the coefficients of the homogeneous pieces of (double) Grothendieck polynomials. We carry out this program in [KnM04], recovering a formula of Fomin and Kirillov [FK94].

2.2. Degeneracy loci

We recall here Fulton’s theory of degeneracy loci, and explain its relation to equivariant cohomology. This was our initial interest in Gröbner geometry of double Schubert polynomials: to get universal formulae for the cohomology classes of degeneracy loci. However, since completing this work, we learned of the papers [FR02], [Kaz97] taking essentially the same viewpoint, and we refer to them for detail.

Given a flagged vector bundle $E. = (E_1 \hookrightarrow E_2 \hookrightarrow \dots \hookrightarrow E_n)$ and a co-flagged vector bundle $F. = (F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \dots \twoheadrightarrow F_1)$ over the same base X , a generic map $\sigma : E_n \rightarrow F_n$, and a permutation w , define the *degeneracy locus* Ω_w as the subset

$$\Omega_w = \{x \in X \mid \text{rank}(E_q \rightarrow E_n \xrightarrow{\sigma} F_n \twoheadrightarrow F_p) \leq \text{rank}(w_{q \times p}^T) \text{ for all } q, p\}.$$

The principal goal in Fulton’s paper [Ful92] was to provide “formulae for degeneracy loci” as polynomials in the Chern classes of the vector bundles. In terms of the Chern roots $\{c_1(E_p/E_{p-1}), c_1(\ker F_q \rightarrow F_{q-1})\}$, Fulton found that the desired polynomials were actually the double Schubert polynomials.

It is initially surprising that there is a single formula, for all X, E, F and not really depending on σ . This follows from a classifying space argument, when $\mathbf{k} = \mathbb{C}$, as follows.

The group of automorphisms of a flagged vector space consists of the invertible lower triangular matrices B , so that the classifying space BB of B -bundles carries a universal flagged vector bundle. The classifying space of interest to us is thus $BB \times BB_+$, which carries a pair of universal vector bundles \mathcal{E} and \mathcal{F} , the first flagged and the second co-flagged. We write $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ for the bundle whose fiber at $(x, y) \in BB \times BB_+$ equals $\text{Hom}(\mathcal{E}_x, \mathcal{F}_y)$.

Define the *universal degeneracy locus* $U_w \subseteq \mathcal{H}om(\mathcal{E}, \mathcal{F})$ as the subset

$$U_w = \{(x, y, \phi) \mid \text{rank}((\mathcal{E}_x)_q \xrightarrow{\phi} (\mathcal{F}_y)_p) \leq \text{rank}(w_{q \times p}^T) \text{ for all } q, p\},$$

where $x \in BB$, $y \in BB_+$, and $\phi : \mathcal{E}_x \rightarrow \mathcal{F}_y$. In other words, the homomorphisms in the fiber of U_w at (x, y) lie in the corresponding matrix Schubert variety.

The name is justified by the following. Recall that our setup is a space X , a flagged vector bundle E on it, a coflagged vector bundle F , and a ‘generic’ vector bundle map $\sigma : E \rightarrow F$; we will soon see what ‘generic’ means. Pick a classifying map $\chi : X \rightarrow BB \times BB_+$, which means that E, F are isomorphic to pullbacks of the universal bundles. (Classifying maps exist uniquely up to homotopy.) Over the target we have the universal Hom-bundle $\mathcal{H}om(\mathcal{E}, \mathcal{F})$, and the vector bundle map σ is a choice of a way to factor the map χ through a map $\tilde{\sigma} : X \rightarrow \mathcal{H}om(\mathcal{E}, \mathcal{F})$. The degeneracy locus Ω_w is then $\tilde{\sigma}^{-1}(U_w)$, and it is natural to request that $\tilde{\sigma}$ be transverse to each U_w —this will be the notion of σ being generic.

What does this say cohomologically? The closed subset U_w defines a class in Borel–Moore homology (and thus ordinary cohomology) of the bundle $\mathcal{H}om(\mathcal{E}, \mathcal{F})$. If σ is generic, then

$$[\Omega_w] = [\tilde{\sigma}^{-1}(U_w)] = \tilde{\sigma}^*([U_w])$$

and this is the sense in which there is a universal formula $[U_w] \in H^*(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$. The cohomology ring of this Hom-bundle is the same as that of the base $BB \times BB_+$ (to which it retracts), namely a polynomial ring in the $2n$ first Chern classes, so that one knows *a priori* that the universal formula should be expressible as a polynomial in these $2n$ variables.

We can rephrase this using Borel’s mixing space definition of equivariant cohomology. Given a space S carrying an action of a group G , and a contractible space EG upon which G acts freely, the equivariant cohomology $H_G^*(S)$ of S is defined as

$$H_G^*(S) := H^*((S \times EG)/G),$$

where the quotient is with respect to the diagonal action. Note that the Borel ‘mixing space’ $(S \times EG)/G$ is a bundle over $EG/G =: BG$, with fibers S . In particular $H_G^*(S)$ is automatically a module over $H^*(BG)$, thereby called the ‘base ring’ of G -equivariant cohomology.

For us, the relevant group is $B \times B_+$, and we have two spaces S : the space of matrices M_n under left and right multiplication, and inside it the matrix Schubert variety \overline{X}_w . Applying the mixing construction to the pair $M_n \supseteq \overline{X}_w$, it can be shown that we recover the bundles $\mathcal{H}om(\mathcal{E}, \mathcal{F}) \supseteq U_w$. As such, the universal formula $[U_w] \in H^*(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$ we seek can be viewed instead as the class defined in $(B \times B_+)$ -equivariant cohomology by \overline{X}_w inside M_n . As we prove in Theorem A (in the setting of multidegrees, although a direct equivariant cohomological version is possible), these are the double Schubert polynomials.

The main difference between this mixing space approach and that of Fulton in [Ful92] is that in the algebraic category, where Fulton worked, some pairs (E, F) of algebraic vector bundles may have no *algebraic* generic maps σ .

The derivation given above works more generally in the topological category, where no restriction on (E, F) is necessary.

In addition, we don't even need to know *a priori* which polynomials represent the cohomology classes of matrix Schubert varieties to show that these classes are the universal degeneracy locus classes. This contrasts with methods relying on divided differences.

2.3. Schubert classes in flag manifolds

Having in the main body of the exposition supplanted the topology of the flag manifold with multigraded commutative algebra, we would like now to connect back to the topological language. In particular, we recover a geometric result from our algebraic treatment of matrix Schubert varieties: the (double) Grothendieck polynomials represent the $(B_+$ -equivariant) K -classes of ordinary Schubert varieties in the flag manifold [LS82b], [Las90].

Our derivation of this result requires no prerequisites concerning the rationality of the singularities of Schubert varieties: the multidegree proof of the Hilbert series calculation is based on cohomological considerations that ignore phenomena at complex codimension 1 or more, and automatically produces the K -classes as numerators of Hilbert series. The material in this section actually formed the basis for our original proof of Theorem B over $\mathbf{k} = \mathbb{C}$, and therefore of Theorem 2.1.1, before we had available the technology of multidegrees.

We use standard facts about the flag variety and notions from (equivariant) algebraic K -theory, for which background material can be found in [Ful98]. In particular, we use freely the correspondence between T -equivariant sheaves on M_n and \mathbb{Z}^n -graded $\mathbf{k}[\mathbf{z}]$ -modules, where T is the torus of diagonal matrices acting by left multiplication on the left. Under this correspondence, the K -polynomial $\mathcal{K}(\Gamma; \mathbf{x})$ equals the T -equivariant K -class $[\tilde{\Gamma}]_T \in K_T^\circ(M_n) \cong \mathbb{Z}[\mathbf{x}^{\pm 1}]$ of the associated sheaf $\tilde{\Gamma}$ on M_n .

The K -cohomology ring $K^\circ(\mathcal{F}\ell_n)$ is the quotient of $\mathbb{Z}[\mathbf{x}]$ by the ideal

$$K_n = \langle e_d(\mathbf{x}) - \binom{n}{d} \mid d \leq n \rangle,$$

where e_d is the d^{th} elementary symmetric function. These relations hold in $K^\circ(\mathcal{F}\ell_n)$ because the exterior power $\bigwedge^d \mathbf{k}^n$ of the trivial rank n bundle is itself trivial of rank $\binom{n}{d}$, and there can be no more relations because $\mathbb{Z}[\mathbf{x}]/K_n$ is an abelian group of rank $n!$. Indeed, substituting $\tilde{x}_k = 1 - x_k$, we find that $\mathbb{Z}[\mathbf{x}]/K_n \cong \mathbb{Z}[\tilde{\mathbf{x}}]/\tilde{K}_n$, where $\tilde{K}_n = \langle e_d(\tilde{\mathbf{x}}) \mid d \leq n \rangle$, and this quotient has rank $n!$ because it is isomorphic to the familiar cohomology ring of $\mathcal{F}\ell_n$ [Bor53].

Thus it makes sense to say that a polynomial in $\mathbb{Z}[\mathbf{x}]$ “represents a class” in $K^\circ(\mathcal{F}\ell_n)$. Lascoux and Schützenberger, based on work of Bernstein–Gel'fand–Gel'fand [BGG73] and Demazure [Dem74], realized that the classes $[\mathcal{O}_{X_w}] \in \mathbb{Z}[\mathbf{x}]/K_n$ of (structure sheaves of) Schubert varieties could be represented in-

dependently of n . To make a precise statement, let $\mathcal{F}\ell_N = B \backslash \mathrm{GL}_N$ be the manifold of flags in \mathbf{k}^N for $N \geq n$, so that B is understood to consist of $N \times N$ lower triangular matrices. Let $X_w(N) \subseteq \mathcal{F}\ell_N$ be the *Schubert variety* for the permutation $w \in S_n$ considered as an element of S_N that fixes $n + 1, \dots, N$. In our conventions, $X_w = B \backslash (\mathrm{GL}_n \cap \overline{X}_w)$, and similarly for $N \geq n$.

COROLLARY 2.3.1 ([LS82b], [Las90]). *The Grothendieck polynomial $\mathcal{G}_w(\mathbf{x})$ represents the K -class $[\mathcal{O}_{X_w(N)}] \in K^\circ(\mathcal{F}\ell_N)$ for all $N \geq n$.*

This is almost a direct consequence of Theorem A, but we still need a lemma. Note that $\mathcal{G}_w(\mathbf{x})$ is expressed without reference to N ; here is the reason why.

LEMMA 2.3.2. *The n -variable Grothendieck polynomial $\mathcal{G}_w(\mathbf{x})$ equals the Grothendieck polynomial $\mathcal{G}_{w_N}(x_1, \dots, x_N)$, whenever w_N agrees with w on $1, \dots, n$ and fixes $n + 1, \dots, N$.*

Proof. The ideal I_{w_N} in the polynomial ring $\mathbf{k}[z_{ij} \mid i, j = 1, \dots, N]$ is extended from the ideal I_w in the multigraded polynomial subring $\mathbf{k}[\mathbf{z}] = \mathbf{k}[z_{ij} \mid i, j = 1, \dots, n]$. Therefore I_{w_N} has the same multigraded Betti numbers as I_w , and so their K -polynomials, which equal the Grothendieck polynomials \mathcal{G}_w and \mathcal{G}_{w_N} by Theorem A, are equal. \square

Proof of Corollary 2.3.1. In view of Lemma 2.3.2, we may as well assume $N = n$. Let us justify the following diagram:

$$\begin{array}{ccccc} X_w & & & & \overline{X}_w \\ \cap & & & & \cap \\ B \backslash \mathrm{GL}_n & \leftarrow & \mathrm{GL}_n & \hookrightarrow & M_n \\ K^\circ(B \backslash \mathrm{GL}_n) & \xrightarrow{\cong} & K_B^\circ(\mathrm{GL}_n) & \leftarrow & K_B^\circ(M_n). \end{array}$$

Pulling back vector bundles under the quotient map $B \backslash \mathrm{GL}_n \leftarrow \mathrm{GL}_n$ induces the isomorphism $K^\circ(B \backslash \mathrm{GL}_n) \rightarrow K_B^\circ(\mathrm{GL}_n)$. The inclusion $\mathrm{GL}_n \hookrightarrow M_n$ induces a surjection $K_B^\circ(\mathrm{GL}_n) \leftarrow K_B^\circ(M_n)$ because the classes of (structure sheaves of) algebraic cycles generate both of the equivariant K -homology groups $K_B^\circ(M_n)$ and $K_B^\circ(\mathrm{GL}_n)$.

Now let $\tilde{X}_w = \overline{X}_w \cap \mathrm{GL}_n$. Any B -equivariant resolution of $\mathcal{O}_{\overline{X}_w} = \mathbf{k}[\mathbf{z}]/I_w$ by vector bundles on M_n pulls back to a B -equivariant resolution E of $\mathcal{O}_{\tilde{X}_w}$ on GL_n . When a vector bundle on GL_n is viewed as a geometric object (i.e. as the scheme $E = \mathbf{Spec}(\mathrm{Sym} \cdot \mathcal{E}^\vee)$ rather than its sheaf of sections $\mathcal{E} = \Gamma(E)$), the quotient $B \backslash E$ is a resolution of \mathcal{O}_{X_w} by vector bundles on $B \backslash \mathrm{GL}_n$. Thus $[\mathcal{O}_{\overline{X}_w}]_B \in K_B^\circ(M_n)$ maps to $[\mathcal{O}_{X_w}] \in K^\circ(B \backslash \mathrm{GL}_n)$.

The corollary follows by identifying the B -equivariant class $[\mathcal{O}_{\overline{X}_w}]_B$ as the T -equivariant class $[\mathcal{O}_{\overline{X}_w}]_T$ under the natural isomorphism $K_B^\circ(M_n) \rightarrow$

$K_T^\circ(M_n)$, and identifying the T -equivariant class as the K -polynomial $\mathcal{K}(\mathbf{k}[\mathbf{z}]/I_w; \mathbf{x}) = \mathcal{G}_w(\mathbf{x})$ by Theorem A. □

Remark 2.3.3. The same line of reasoning recovers the double version of Corollary 2.3.1, in which $\mathcal{G}_w(\mathbf{x})$ and $K^\circ(\mathcal{F}\ell_N)$ are replaced by $\mathcal{G}_w(\mathbf{x}, \mathbf{y})$ and the equivariant K -group $K_{B^+}^\circ(\mathcal{F}\ell_N)$ for the action of the invertible upper triangular matrices B^+ by inverses on the right.

The above proof can be worked in reverse: by assuming Corollary 2.3.1 one can then *conclude* Theorem A. This was in fact the basis for our first proof of Theorem A. However, it requires substantially more prerequisites (such as rationality of singularities for Schubert varieties), and is no shorter because it fails to eliminate the inductive arguments in Part 3.

Remark 2.3.4. There exists technology to assign equivariant cohomology classes to complex subvarieties of noncompact spaces such as M_n in the cases that interest us (see [Kaz97], [FR02], for instance). Therefore the argument for Corollary 2.3.1 also works when $\mathcal{G}_w(\mathbf{x})$ is replaced by a Schubert or double Schubert polynomial, and $K^\circ(\mathcal{F}\ell_N)$ is replaced by the appropriate version of cohomology, either $H^*(\mathcal{F}\ell_N)$ or $H_{B^+}^*(\mathcal{F}\ell_N)$.

Just as in Remark 2.3.3, this argument can be reversed: by assuming the results of [LS82a] that characterize Schubert polynomials in terms of stability properties, one can then conclude the multidegree statement in Theorem A. Since this part of Theorem A is essential to proving the other main theorems from Part 1, what we actually do is give an independent proof of the multidegree part of Theorem A in Section 3.2, to avoid issues of direct translation between equivariant cohomology and multidegrees.

Remark 2.3.5. The substitution $\mathbf{x} \mapsto \mathbf{1} - \mathbf{x}$ in the definition of multidegree (Section 1.2) is the change of basis accompanying the Poincaré isomorphism from K -cohomology to K -homology. In general geometric terms, $c_1(L_i) \in H^*(\mathcal{F}\ell_n)$ is the cohomology class Poincaré dual to the divisor D_i of the i^{th} standard line bundle L_i on $\mathcal{F}\ell_n$. The exact sequence $0 \rightarrow L_i^\vee \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{D_i} \rightarrow 0$ implies that the K -homology class $[\mathcal{O}_{D_i}]$ corresponds to the K -cohomology class $1 - [L_i^\vee]$ under the Poincaré map. Thus $\mathcal{G}_w(\mathbf{x})$ indicates $[\mathcal{O}_{X_w}]$ as a polynomial in the Chern characters $x_i = e^{c_1(L_i)}$ of the line bundles L_i , whereas $\mathcal{G}_w(\mathbf{1} - \mathbf{x})$ indicates $[\mathcal{O}_{X_w}]$ as a polynomial in the expressions

$$1 - e^{c_1(L_i)} = 1 - e^{-c_1(L_i^\vee)} = c_1(L_i^\vee) - \frac{c_1(L_i^\vee)^2}{2!} + \frac{c_1(L_i^\vee)^3}{3!} - \frac{c_1(L_i^\vee)^4}{4!} + \dots,$$

whose lowest degree terms are the first Chern classes $c_1(L_i^\vee)$ of the dual bundles L_i^\vee . Forgetting the higher degree terms here, in Definition 1.2.3, and in Lemma 1.1.4 amounts to taking images in the associated graded ring of $K_\circ(B \backslash \text{GL}_n \mathbb{C})$, which is $H^*(B \backslash \text{GL}_n \mathbb{C})$. See [Ful98, Ch. 15] for details.

It is an often annoying quirk of history that we end up using the same variable x_i for both $c_1(L_i^\vee) \in H^*$ and $[L_i] \in K^\circ$. We tolerate (and sometimes even come to appreciate) this confusing abuse of notation because it can be helpful at times. In terms of algebra, it reinterprets the displayed equation as: the lowest degree term in $1 - e^{-x_i}$ is just x_i again.

Remark 2.3.6. Not only do the cohomological and K -theoretic statements in Theorem A descend to the flag manifold $\mathcal{F}\ell_n$, but so also does the degeneration of Theorem B [KoM03]. On $\mathcal{F}\ell_n$, the degeneration can be interpreted in representation theory, where it explains geometrically the construction of Gel'fand–Cetlin bases for GL_n representations [GC50], [GS83].

2.4. Ladder determinantal ideals

The importance of Gröbner bases in recent work on determinantal ideals and their relatives, such as their powers and symbolic powers, cannot be overstated. They are used in treatments of questions about Cohen–Macaulayness, rational singularities, multiplicity, dimension, a -invariants, and divisor class groups; see [CGG90], [Stu90], [HT92], [Con95], [MS96], [CH97], [BC98], [KP99], [GM00] for a small sample. Since determinantal ideals and their Gröbner bases also arise in the study of (partial) flag varieties and their Schubert varieties (see [Mul89], [GL97], [BL00], [GL00], [GM00], for instance), it is surprising to us that Gröbner bases for the determinantal ideals defining matrix Schubert varieties \overline{X}_w do not seem to be in the literature, even though the ideals themselves appeared in [Ful92].

Most of the papers above concern a class of determinantal ideals called ‘(one-sided) ladder determinantal ideals’, about which we now comment. Consider a sequence of boxes $(b_1, a_1), \dots, (b_k, a_k)$ in the $n \times n$ grid, with

$$a_1 \leq a_2 \leq \dots \leq a_k \quad \text{and} \quad b_1 \geq b_2 \geq \dots \geq b_k.$$

Fill the boxes (b_ℓ, a_ℓ) with nonnegative integers r_ℓ satisfying

$$(4) \quad 0 < a_1 - r_1 < a_2 - r_2 < \dots < a_k - r_k$$

and

$$b_1 - r_1 > b_2 - r_2 > \dots > b_k - r_k > 0.$$

The *ladder determinantal ideal* $I(\underline{a}, \underline{b}, \underline{r})$ is generated by the minors of size r_ℓ in the northwest $a_\ell \times b_\ell$ corner of Z for all $\ell \in 1, \dots, k$. Condition (4) simply ensures that the vanishing of the minors of size r_ℓ in the northwest $b_\ell \times a_\ell$ submatrix does not imply the vanishing of the minors of size $r_{\ell'}$ in the northwest $b_{\ell'} \times a_{\ell'}$ submatrix when $\ell \neq \ell'$. For example, the ladder determinantal ideals in [GL00] have ranks \underline{r} that weakly increase from southwest to northeast (in our language), while those treated in [GM00] have ranks such that no two labeled boxes lie in the same row or column (that is, $(b_\ell, a_\ell)_{\ell=1}^k$ is an antidiagonal).

Since ‘ladders’ are just another name for ‘partitions’ one might also like to call these ‘partition determinantal ideals’, but in fact, a better name is ‘vexillary determinantal ideals’. Indeed, Fulton identified ladder determinantal ideals as Schubert determinantal ideals I_w for *vexillary permutations* (also known as *2143-avoiding* and *single-shaped* permutations) [Ful92, Prop. 9.6]. Therefore Theorems A and B hold in full for ladder determinantal ideals. Note, however (as Fulton does), that the probability of a permutation being vexillary decreases exponentially to zero as n approaches infinity.

Our Gröbner bases are new even for the vexillary determinantal ideals we found in the literature, since previous authors seem always to use what in our notation are diagonal rather than antidiagonal term orders. The general phenomenon making the diagonal Gröbner basis fail as in Example 1.5.2 is precisely the fact that rank conditions are “nested” for every Schubert determinantal ideal that is not vexillary (this follows from Fulton’s essential set characterization [Ful92, §9]).

The multidegree formula in Theorem A becomes beautifully explicit for vexillary ideals.

COROLLARY 2.4.1. *The \mathbb{Z}^{2n} -graded multidegree of a ladder determinantal variety is a multi-Schur polynomial, and therefore has an explicit determinantal expression.*

This corollary is substantially more general than previous \mathbb{Z} -graded degree formulae, which held only for special kinds of vexillary ideals, and were after all only \mathbb{Z} -graded. Readers wishing to see the determinantal expression in its full glory can check [Ful92] for a brief introduction to multi-Schur polynomials, or [Mac91] for much more. It would be desirable to make the Hilbert series in Theorem B just as explicit in closed form, given that combinatorial formulae are known (see [KnM04] for details and references):

QUESTION 2.4.2. *Is there an analogously “nice” formula⁵ for vexillary double Grothendieck polynomials $\mathcal{G}_w(\mathbf{x}, \mathbf{y})$, or an ordinary version for $\mathcal{G}_w(\mathbf{x})$, or even for $\mathcal{G}_w(t, \dots, t)$?*

The answer is ‘yes’ for $\mathcal{G}_w(t, \dots, t)$ in certain vexillary cases; e.g. see [CH94], [KP99], [Gho02].

⁵In the introductions to [Abh88] and its second chapter, Abhyankar writes of formulae he first presented at a conference at the University of Nice, in France. Although his formulae enumerate certain kinds of tableaux, his results were used to obtain formulae for degrees and Hilbert series of determinantal ideals. Since then, some authors have been looking for “nice” (uncapitalized, and always in quotes) formulae for Hilbert series of determinantal ideals; cf. [HT92, p. 3] and [AK89, p. 55].

Theorem B provides a new proof that Schubert varieties $X_w \subseteq \mathcal{F}\ell_n$ in the flag manifold are Cohen–Macaulay. Instead of giving the quick derivation of this specific consequence, let us instead mention a more general local equivalence principle between Schubert varieties and matrix Schubert varieties, special cases of which have been applied numerous times in the literature. By a *local condition*, we mean a condition that holds for a variety whenever it holds on each subvariety in some open cover.

THEOREM 2.4.3. *Let \mathfrak{C} be a local condition that holds for a variety X whenever it holds for the product of X with any vector space. Then \mathfrak{C} holds for every Schubert variety in every flag variety if and only if \mathfrak{C} holds for all matrix Schubert varieties.*

Sketch of proof. The complete proof of one direction is easy: if \mathfrak{C} holds for the matrix Schubert variety $\overline{X}_w \subseteq M_n$, then it holds for $\tilde{X}_w = \overline{X}_w \cap \mathrm{GL}_n$. Therefore \mathfrak{C} holds for the Schubert variety $X_w \subseteq B \backslash \mathrm{GL}_n$, because \tilde{X}_w is locally isomorphic to the product of X_w with B , the latter being an open subset of a vector space.

On the other hand, if \mathfrak{C} holds for Schubert varieties, then it holds for matrix Schubert varieties because of the following, whose proof (which uses Fulton’s essential set [Ful92] and would require introducing a fair amount of notation) we omit. Given $w \in S_n$, consider w as an element of S_{2n} fixing $n+1, \dots, 2n$. The product $\overline{X}_w \times \mathbf{k}^{n^2-n}$ of the matrix Schubert variety $\overline{X}_w \subseteq M_n$ with a vector space of dimension $n^2 - n$ is isomorphic to the intersection of the Schubert variety $X_w \subseteq \mathcal{F}\ell_{2n}$ with the opposite big cell in $\mathcal{F}\ell_{2n}$. \square

Since rationality of singularities and normality are among such local statements, \overline{X}_w possesses these properties because Schubert varieties do [Ram85], [RR85]. However, these statements could just as easily have been derived by Fulton in [Ful92], although they are not mentioned there. Thus we know of no new results that can be proved using Theorem 2.4.3.

Part 3. Bruhat induction

3.1. Overview

With motivation coming from the statements of the main results in Part 1, we now introduce the details of *Bruhat induction* to combinatorial commutative algebra.

Geometric considerations occupy Section 3.2, where we start with large matrix Schubert varieties (associated to shorter permutations) and chop them rather bluntly with multihomogeneous functions (certain minors, actually). This basically yields matrix Schubert varieties that have dimension one less, which we understand already by Bruhat induction. However, the messy hyper-

surface section leaves some debris components, which get cleaned up using the technology of multidegrees (Sections 1.2 and 1.7). In particular, multidegrees allow us to ignore geometric phenomena in codimension ≥ 1 .

Beginning in Section 3.3 we switch to a different track, namely the combinatorial algebra of antidagonal ideals. Bruhat induction manifests itself here via Demazure operators, which we interpret as combinatorial rules for manipulating Hilbert series monomial by monomial. Thus, in Section 3.4 we justify certain \mathbb{Z}^{n^2} -graded lifts of Demazure operators that take monomials outside J_w as input and return sums of monomials outside J_{ws_i} . The resulting Theorem 3.4.11 is substantially stronger than the Hilbert series statement required for the main theorems, given that these operators really do “lift” the \mathbb{Z}^{2n} -graded Demazure operators to \mathbb{Z}^{n^2} . This lifting property is proved in Section 3.5 (Theorem 3.5.4) via a certain combinatorial duality on standard monomials of antidiagonal ideals.

The transition from arbitrary standard monomials to squarefree monomials, which coincide with faces of the Stanley–Reisner complex, starts with Section 3.6, where a rough interpretation of Bruhat induction on facets of \mathcal{L}_w already proves equidimensionality. More detailed (but less algebraic and more pleasantly combinatorial) analysis in Section 3.7 reduces monomial-by-monomial Demazure induction to facet-by-facet divided difference Bruhat induction. It culminates in mitosis for facets, which is the combinatorial residue of Bruhat induction for standard monomials of antidiagonal ideals. Further use of mitosis in Section 3.8 characterizes the facets of antidiagonal complexes as reduced pipe dreams.

The final Section 3.9 gathers the main results of Bruhat induction into a proof of the remaining unproved assertions from Part 1 (Theorems A, B, and C). Logically, the remainder of this paper depends only on the definitions in Part 1, and on Section 1.7.

Conventions for downward induction on Bruhat order. We shall repeatedly invoke the hypothesis $\text{length}(ws_i) < \text{length}(w)$. In terms of permutation matrices, this means that w^T differs from $(ws_i)^T$ only in rows i and $i + 1$, where they look heuristically like

$$(5) \quad \begin{array}{ccc} & & w(i) \\ & & \downarrow \dots \\ i & & \begin{array}{|c|c|c|c|c|} \hline & & & & 1 \\ \hline \end{array} \\ i + 1 & & \begin{array}{|c|c|c|c|c|} \hline 1 & & \dots & & \\ \hline \end{array} \\ \dots & \uparrow & \dots \\ & & w(i+1) \quad w^T \end{array} \quad \begin{array}{ccc} & & w(i+1) \\ & & \downarrow \dots \\ i & & \begin{array}{|c|c|c|c|c|} \hline 1 & & & & \\ \hline \end{array} \\ i + 1 & & \begin{array}{|c|c|c|c|c|} \hline & & \dots & & 1 \\ \hline \end{array} \\ \dots & \uparrow & \dots \\ & & (ws_i)^T \quad w(i) \end{array}$$

between columns $w(i + 1)$ and $w(i)$. We always write this hypothesis as $\text{length}(ws_i) < \text{length}(w)$, for consistency, even though we may actually use

one of the following equivalent formulations in any given lemma or proposition. We hope that collecting this list of standard statements (“shorter permutation \Leftrightarrow bigger variety”) will prevent the reader from stumbling on this as many times as we did. The string of characters ‘ $\text{length}(ws_i) < \text{length}(w)$ ’ can serve as a visual cue to this frequent assumption; we shall *never* assume the opposite inequality.

LEMMA 3.1.1. *The following are equivalent for a permutation $w \in S_n$.*

- | | |
|---|---|
| 1. $\text{length}(ws_i) < \text{length}(w)$. | 6. $\dim(\overline{X}_{ws_i}) > \dim(\overline{X}_w)$. |
| 2. $\text{length}(ws_i) = \text{length}(w) - 1$. | 7. $\dim(\overline{X}_{ws_i}) = \dim(\overline{X}_w) + 1$. |
| 3. $w(i) > w(i + 1)$. | 8. $s_i \overline{X}_w \neq \overline{X}_w$. |
| 4. $ws_i(i) < ws_i(i + 1)$. | 9. $s_i \overline{X}_{ws_i} = \overline{X}_{ws_i}$. |
| 5. $I(\overline{X}_{ws_i}) \subset I(\overline{X}_w)$. | 10. $\overline{X}_{ws_i} \supset \overline{X}_w$. |

Here, the transposition s_i acts on the left of M_n , switching rows i and $i + 1$.

Proof. The equivalence of 1–4 comes from Claim 3.2.2. The equivalence of these with 5–7 and 10 uses Proposition 3.2.5 and Lemma 3.2.3. Finally, 8–10 are equivalent by Lemma 3.2.4 and its proof. (We shall not apply Lemma 3.1.1 until the proof of Lemma 3.2.6.) \square

3.2. Multidegrees of matrix Schubert varieties

This section provides a proof of the divided difference recursion satisfied by the multidegrees of matrix Schubert varieties, in Theorem 3.2.8. Although many of the preliminary results can be deduced from a number of places in the literature, notably [Ful92], we believe it important to provide proofs (or at least sketches) so as to make our foundations explicit.

Matrix Schubert varieties are clearly stable under rescaling any row or column. Moreover, since we only impose rank conditions on submatrices that are as far north and west as possible, any operation that adds a multiple of some row to a row below it (“sweeping downward”), or that adds a multiple of some column to another column to its right (“sweeping to the right”) preserves every matrix Schubert variety.

In terms of group theory, let B denote the group of invertible *lower* triangular matrices and B_+ the invertible upper triangular matrices. The previous paragraph says exactly that each matrix Schubert variety \overline{X}_w is preserved by the action⁶ of $B \times B_+$ on M_n in which $(b, b_+) \cdot Z = bZb_+^{-1}$. Proposition 3.2.5 will say more.

⁶This is a left group action, in the sense that $(b, b_+) \cdot ((b', b'_+) \cdot Z)$ equals $((b, b_+) \cdot (b', b'_+)) \cdot Z$ instead of $((b', b'_+) \cdot (b, b_+)) \cdot Z$, even though—in fact because—the b_+ acts via its inverse on the *right*.

The next four results, numbered 3.2.1–3.2.4, are basically standard facts concerning Bruhat order for S_n , enhanced slightly for M_n instead of GL_n . They serve as prerequisites for Proposition 3.2.5, Lemma 3.2.6, and Lemma 3.2.7, which enter at key points in the proof of Theorem 3.2.8.

Call a matrix $Z \in M_n$ that is zero except for at most one 1 in each row and column a *partial permutation matrix*. These arise in the M_n analogue of Bruhat decomposition:

LEMMA 3.2.1. *In each $B \times B_+$ orbit on M_n lies a unique partial permutation matrix.*

Proof. By doing row and column operations that sweep down and to the right, we can get from an arbitrary matrix Z' to a partial permutation matrix Z . Such sweeping preserves the ranks of northwest $q \times p$ submatrices, and Z can be reconstructed uniquely by knowing only $\text{rank}(Z'_{q \times p})$ for $1 \leq q, p \leq n$. \square

Define the *length* of a partial permutation matrix Z as the number of zeros in Z that lie neither due south nor due east of a 1. In other words, for every 1 in Z , cross out all the boxes beneath it in the same column as well as to its right in the same row, and count the number of uncrossed-out boxes to get the length of Z . When $Z = w^T$ is a permutation matrix, $\text{length}(w)$ agrees with the length of $w^T \in M_n$. Write $Z \subseteq Z'$ for partial permutation matrices Z and Z' if the 1's in Z are a subset of the 1's in Z' . Finally, let $t_{i,i'} \in S_n$ be the transposition switching i and i' . The following claim is self-evident.

CLAIM 3.2.2. *Suppose Z is a partial permutation matrix with 1's at (i, j) and (i', j') , where $(i, j) \leq (i', j')$. Switching rows i and i' of Z creates a partial permutation matrix Z' satisfying $\text{length}(Z') = \text{length}(Z) + 1 + \text{twice the number of 1's strictly inside the rectangle enclosed by } (i, j) \text{ and } (i', j')$.* \square

LEMMA 3.2.3. *Fix a permutation $w \in S_n$ and a partial permutation matrix Z . If $Z \in \overline{X}_w$ and $\text{length}(Z) \geq \text{length}(w)$, then either $Z \subseteq w^T$, or there is a transposition $t_{i,i'}$ such that $v = wt_{i,i'}$ satisfies: $Z \in \overline{X}_v$ and $\text{length}(v) > \text{length}(w)$.*

Sketch of proof. Use reasoning similar to the case when Z is a permutation matrix: work by downward induction on the number of 1's shared by w^T and Z , using Claim 3.2.2. We omit the details. \square

LEMMA 3.2.4. *Let Z be a partial permutation matrix with orbit closure $\overline{\mathcal{O}}_Z$ in M_n . If $\text{length}(s_i Z) < \text{length}(Z)$, then*

$$n^2 - \text{length}(s_i Z) = \dim(\overline{\mathcal{O}}_{s_i Z}) = \dim(\overline{\mathcal{O}}_Z) + 1 = n^2 - \text{length}(Z) + 1,$$

and $s_i(\overline{\mathcal{O}}_{s_i Z}) = \overline{\mathcal{O}}_{s_i Z}$.

Proof. Let $P_i \subseteq \mathrm{GL}_n$ be the i^{th} parabolic subgroup containing B , in which the only nonzero entry outside the lower triangle may lie at $(i, i+1)$. Consider the image Y of the multiplication map $P_i \times \overline{\mathcal{O}}_Z \rightarrow M_n$ sending $(p, x) \mapsto p \cdot x$. This map factors through the quotient $B \backslash (P_i \times \overline{\mathcal{O}}_Z)$ by the diagonal action of B , which is an $\overline{\mathcal{O}}_Z$ -bundle over $P_i/B \cong \mathbb{P}^1$ and hence has dimension $\dim(\overline{\mathcal{O}}_Z) + 1$. Thus $\dim(Y) \leq \dim(\overline{\mathcal{O}}_Z) + 1$.

The variety Y is $B \times B_+$ -stable by construction. Since $B \times B_+$ has only finitely many orbits on M_n , the irreducibility of Y implies that \overline{Y} is an orbit closure $\overline{\mathcal{O}}_{Z'}$ for some partial permutation matrix Z' , by Lemma 3.2.1. Clearly $\overline{\mathcal{O}}_Z \subseteq Y$, and $s_i Z \in Y$, so that the dimension bound implies that $Z' = s_i Z$ because $Z \in \overline{\mathcal{O}}_{s_i Z}$, as can be checked directly. This \mathbb{P}^1 -bundle argument also shows that $\overline{\mathcal{O}}_{s_i Z} = \overline{Y}$ is stable under multiplication by P_i on the left, whence $s_i \in P_i$ takes $\overline{\mathcal{O}}_{s_i Z}$ to itself.

That $\dim(\overline{\mathcal{O}}_Z) = n^2 - \mathrm{length}(Z)$ follows by direct calculation whenever Z has nonzero entries only along the main diagonal, and then by downward induction on $\mathrm{length}(Z)$. \square

Fulton derived the next result in [Ful92] from the corresponding result on flag manifolds.

PROPOSITION 3.2.5. *The matrix Schubert variety \overline{X}_w is the closure $\overline{Bw^T B_+}$ of the $B \times B_+$ orbit on M_n through the permutation matrix w^T . Thus \overline{X}_w is irreducible of dimension $n^2 - \mathrm{length}(w)$, and w^T is a smooth point of it.*

Proof. The stability of \overline{X}_w under $B \times B_+$ means that \overline{X}_w is a union of orbits. By the obvious containment $\overline{\mathcal{O}}_{w^T} \subseteq \overline{X}_w$ (sweeping down and right preserves northwest ranks) and Lemma 3.2.1, it suffices to show that partial permutation matrices Z lying in \overline{X}_w lie also in $\overline{Bw^T B_+}$. Standard arguments analogous to the case where Z is a permutation matrix work here, by Lemma 3.2.3. The last sentence of the proposition is standard for orbit closures, except for the dimension count, which comes from Lemma 3.2.4. \square

LEMMA 3.2.6. *Let Z be a partial permutation matrix and $w \in S_n$. If the orbit closure $\overline{\mathcal{O}}_Z$ has codimension 1 inside \overline{X}_{ws_i} , then $\overline{\mathcal{O}}_Z$ is mapped to itself by s_i unless $Z = w$.*

Proof. First note that $\mathrm{length}(Z) = \mathrm{length}(ws_i) + 1$ by Lemma 3.2.4 and Proposition 3.2.5. Using Lemma 3.2.3 and Claim 3.2.2, we find that Z is obtained from $(ws_i)^T$ by switching some pairs of rows to make partial permutations of strictly longer length and then deleting some 1's. Since the length of Z is precisely one less than that of ws_i , we can switch exactly one pair of rows of $(ws_i)^T$, or we can delete a single 1 from $(ws_i)^T$.

Any 1 that we delete from $(ws_i)^T$ must have no 1's southeast of it, or else the length increases by more than one. Thus the 1 in row i of $(ws_i)^T$

cannot be deleted by statement 4 of Lemma 3.1.1, leaving us in the situation of Lemma 3.2.4 with $Z = w^T$, and completing the case where a 1 has been deleted from $(ws_i)^T$ to get Z .

Suppose now that switching rows q and q' of $(ws_i)^T$ results in the matrix $Z = v^T$ for some permutation v , and assume that $s_i(\overline{\mathcal{O}}_Z) \neq \overline{\mathcal{O}}_Z$. Since $s_i(\overline{\mathcal{O}}_Z) = \overline{\mathcal{O}}_Z$ unless v satisfies $v(i) > v(i + 1)$, by Lemma 3.2.4, we find that $v(i) > v(i + 1)$. At least one of q and q' must lie in $\{i, i + 1\}$ by part 4 of Corollary 3.1.1, which says that moving neither row q nor row q' of $(ws_i)^T$ leaves $v(i) < v(i + 1)$. On the other hand, it is impossible for exactly one of q and q' to lie in $\{i, i + 1\}$; indeed, switching rows q and q' increases length, so that either the 1 at $(i, w(i + 1))$ or the 1 at $(i + 1, w(i))$ would lie inside the rectangle formed by the switched 1's, making $\overline{\mathcal{O}}_Z$ have codimension more than 1 by Claim 3.2.2 and Proposition 3.2.5. Thus $\{q, q'\} = \{i, i + 1\}$ and $v = w$, completing the proof. \square

LEMMA 3.2.7. *If $\text{length}(ws_i) < \text{length}(w)$, and \mathfrak{m}_{ws_i} is the maximal ideal in the local ring of $(ws_i)^T \in \overline{X}_{ws_i}$, then the variable $z_{i+1, w(i+1)}$ maps to a regular parameter in \mathfrak{m}_{ws_i} . In other words $z_{i+1, w(i+1)}$ lies in $\mathfrak{m}_{ws_i} \setminus \mathfrak{m}_{ws_i}^2$.*

Proof. Let $v = ws_i$, and consider the map $B \times B_+ \rightarrow M_n$ sending $(b, b^+) \mapsto b \cdot v^T \cdot b^+$. The image of this map is contained in \overline{X}_v by Proposition 3.2.5, and the identity $\text{id} := (\text{id}_B, \text{id}_{B_+})$ maps to v^T . The induced map of local rings the other way thus takes \mathfrak{m}_v to the maximal ideal

$$\mathfrak{m}_{\text{id}} := \langle b_{ii} - 1, b_{ii}^+ - 1 \mid 1 \leq i \leq n \rangle + \langle b_{ij}, b_{ji}^+ \mid i > j \rangle$$

in the local ring at the identity $\text{id} \in B \times B_+$. It is enough to demonstrate that the image of $z_{i+1, w(i+1)}$ lies in $\mathfrak{m}_{\text{id}} \setminus \mathfrak{m}_{\text{id}}^2$.

Direct calculation shows that $z_{i+1, w(i+1)}$ maps to

$$b_{i+1, i} b_{w(i+1), w(i+1)}^+ + \sum_{q \in Q} b_{i+1, q} b_{q, w(i+1)}^+ \quad \text{where } Q = \{q < i \mid w(q) < w(i + 1)\}.$$

In particular, all of the summands $b_{i+1, q} b_{q, w(i+1)}^+$ lie in $\mathfrak{m}_{\text{id}}^2$. On the other hand, $b_{w(i+1), w(i+1)}^+$ is a unit near the identity, and so $b_{i+1, i} b_{w(i+1), w(i+1)}^+$ lies in $\mathfrak{m}_{\text{id}} \setminus \mathfrak{m}_{\text{id}}^2$. \square

The argument forming the proof of the previous lemma is an alternative way to calculate the dimension as in Lemma 3.2.4.

THEOREM 3.2.8. *If the permutation w satisfies $\text{length}(ws_i) < \text{length}(w)$, then*

$$[\overline{X}_{ws_i}] = \partial_i[\overline{X}_w]$$

holds for both the \mathbb{Z}^{2n} -graded and \mathbb{Z}^n -graded multidegrees.

Proof. The proof here works for the \mathbb{Z}^n -grading as well as the \mathbb{Z}^{2n} -grading, simply if we ignore all occurrences of \mathbf{y} , or set them to zero.

Let $j = w(i) - 1$, and suppose $\text{rank}(w_{i \times j}^T) = r - 1$. Then the permutation matrix $(ws_i)^T$ has r entries equal to 1 in the submatrix $(ws_i)_{i \times j}^T$. Consider the $r \times r$ minor Δ using the rows and columns in which $(ws_i)_{i \times j}^T$ has 1's. Thus Δ is not the zero function on \overline{X}_{ws_i} ; in fact, Δ is nonzero everywhere on its interior $B(ws_i)^T B_+$. Therefore the subscheme X_Δ defined by Δ inside \overline{X}_{ws_i} is supported on a union of orbit closures \overline{O}_Z contained in \overline{X}_{ws_i} with codimension 1. Now we compare the subscheme X_Δ to its image $s_i X_\Delta = X_{s_i \Delta}$ under switching rows i and $i + 1$.

CLAIM 3.2.9. *Every irreducible component of X_Δ other than \overline{X}_w has the same multiplicity in $s_i X_\Delta$, and \overline{X}_w has multiplicity 1 in X_Δ .*

Proof. Lemma 3.2.6 says that s_i induces an automorphism of the local ring at the generic point (i.e. the prime ideal) of \overline{O}_Z inside \overline{X}_{ws_i} , for every irreducible component \overline{O}_Z of X_Δ other than \overline{X}_w . This automorphism takes Δ to $s_i \Delta$, so these two functions have the same multiplicity along \overline{O}_Z . The only remaining codimension 1 irreducible component of X_Δ is \overline{X}_w , and we shall now verify that the multiplicity equals 1 there. As a consequence, the multiplicity of $s_i X_\Delta$ along $s_i \overline{X}_w$ also equals 1.

By Proposition 3.2.5, the local ring of $(ws_i)^T$ in \overline{X}_{ws_i} is regular. Since s_i is an automorphism of \overline{X}_{ws_i} , we find that the local ring of $w^T \in \overline{X}_{ws_i}$ is also regular. In a neighborhood of w^T , the variables z_{qp} corresponding to the locations of the 1's in $w_{i \times j}^T$ are units. This implies that the coefficient of $z_{i,w(i+1)}$ in Δ is a unit in the local ring of $w^T \in \overline{X}_{ws_i}$. On the other hand, the set of variables in spots where w^T has zeros generates the maximal ideal in the local ring at $w^T \in \overline{X}_{ws_i}$. Therefore, all terms of Δ lie in the square of this maximal ideal, except for the unit times $z_{i,w(i+1)}$ term produced above. Hence, to prove multiplicity one, it is enough to prove that $z_{i,w(i+1)}$ itself is a regular parameter at $w^T \in \overline{X}_{ws_i}$, or equivalently (after applying s_i) that $z_{i+1,w(i+1)}$ is a regular parameter at $(ws_i)^T \in \overline{X}_{ws_i}$. This is Lemma 3.2.7. \square

Now we use a multidegree trick. Consider Δ and $s_i \Delta$ as elements not in $\mathbf{k}[\mathbf{z}]$, but in the ring $\mathbf{k}[\mathbf{z}, u]$ with $n^2 + 1$ variables, where the ordinary weight of the new variable u is $x_i - x_{i+1}$. Denote by $M_n \times \mathbb{A}^1$ the spectrum of $\mathbf{k}[\mathbf{z}, u]$. Then Δ and the product $us_i \Delta$ in $\mathbf{k}[\mathbf{z}, u]$ have the same ordinary weight $f := f(\mathbf{x}, \mathbf{y})$. Since the affine coordinate ring of \overline{X}_{ws_i} is a domain, neither Δ nor $s_i \Delta$ vanishes on \overline{X}_{ws_i} , and so we get two short exact sequences

$$(6) \quad 0 \rightarrow \mathbf{k}[\mathbf{z}, u]/I(\overline{X}_{ws_i})\mathbf{k}[\mathbf{z}, u](-f) \xrightarrow{\Theta} \mathbf{k}[\mathbf{z}, u]/I(\overline{X}_{ws_i})\mathbf{k}[\mathbf{z}, u] \rightarrow Q(\Theta) \rightarrow 0,$$

in which Θ equals either Δ or $us_i\Delta$. The quotients $Q(\Delta)$ and $Q(us_i\Delta)$ therefore have equal \mathbb{Z}^{2n} -graded Hilbert series, and hence equal multidegrees.

Note that $Q(\Delta)$ is the coordinate ring of $X_\Delta \times \mathbb{A}^1$, while $Q(us_i\Delta)$ is the coordinate ring of $(s_iX_\Delta \times \mathbb{A}^1) \cup (\overline{X}_{ws_i} \times \{0\})$, the latter component being the zero scheme of u in $\mathbf{k}[\mathbf{z}, u]/I(\overline{X}_{ws_i})\mathbf{k}[\mathbf{z}, u]$. Breaking up the multidegrees of $Q(\Delta)$ and $Q(us_i\Delta)$ into sums over irreducible components by additivity in Theorem 1.7.1, Claim 3.2.9 says that almost all terms in the equation

$$[X_\Delta \times \mathbb{A}^1] = [s_iX_\Delta \times \mathbb{A}^1] + [\overline{X}_{ws_i} \times \{0\}]$$

cancel, leaving us only with

$$(7) \quad [\overline{X}_w \times \mathbb{A}^1] = [s_i\overline{X}_w \times \mathbb{A}^1] + [\overline{X}_{ws_i} \times \{0\}].$$

The brackets in these equations denote multidegrees over $\mathbf{k}[\mathbf{z}, u]$. However, the ideals in $\mathbf{k}[\mathbf{z}, u]$ of $\overline{X}_w \times \mathbb{A}^1$ and $s_i\overline{X}_w \times \mathbb{A}^1$ are extended from the ideals in $\mathbf{k}[\mathbf{z}]$ of \overline{X}_w and $s_i\overline{X}_w$. Therefore their K -polynomials agree with those of \overline{X}_w and $s_i\overline{X}_w$, respectively, whence $[\overline{X}_w \times \mathbb{A}^1] = [\overline{X}_w]$ and $[s_i\overline{X}_w \times \mathbb{A}^1] = [s_i\overline{X}_w]$ as polynomials in \mathbf{x} and \mathbf{y} . The same argument shows that $[\overline{X}_{ws_i} \times \mathbb{A}^1] = [\overline{X}_{ws_i}]$. The coordinate ring of $[\overline{X}_{ws_i} \times \{0\}]$, on the other hand, is the right-hand term of the exact sequence that results after replacing f by $x_i - x_{i+1}$ and Θ by u in (6). We therefore find that

$$[\overline{X}_{ws_i} \times \{0\}] = (x_i - x_{i+1})[\overline{X}_{ws_i} \times \mathbb{A}^1] = (x_i - x_{i+1})[\overline{X}_{ws_i}]$$

as polynomials in \mathbf{x} and \mathbf{y} . Substituting back into (7) yields the equation on multidegrees $[\overline{X}_w] = [s_i\overline{X}_w] + (x_i - x_{i+1})[\overline{X}_{ws_i}]$, which produces the desired result after moving the $[s_i\overline{X}_w]$ to the left and dividing through by $x_i - x_{i+1}$. \square

Remark 3.2.10. This proof, although translated into the language of multigraded commutative algebra, is actually derived from a standard proof of divided difference formulae by localization in equivariant cohomology, when $\mathbf{k} = \mathbb{C}$. The connection is our two functions Δ and $s_i\Delta$, which yield a map $\overline{X}_{ws_i} \rightarrow \mathbb{C}^2$. The preimage of one axis is \overline{X}_w union some junk components, and the preimage of the other axis is $s_i\overline{X}_w$ union the same junk components. Therefore all of the unwanted (canceling) contributions map to the point $(0, 0) \in \mathbb{C}^2$. Essentially, the standard equivariant localization proof makes the map to \mathbb{C}^2 into a map to \mathbb{CP}^1 , thus avoiding the extra components, and pulls back the localization formula on \mathbb{CP}^1 to a formula on whatever \overline{X}_{ws_i} has become (a Schubert variety).

3.3. Antidiagonals and mutation

In this section we begin investigating the combinatorial properties of the monomials outside J_w and the antidiagonals generating J_w . For the rest of this

section, fix a permutation w and a transposition s_i satisfying $\text{length}(ws_i) < \text{length}(w)$.

Define the *rank matrix* $\text{rk}(w)$ to have (q, p) entry equal to $\text{rank}(w_{q \times p}^T)$. There are two standard facts we need concerning rank matrices, both proved simply by looking at the picture of $(ws_i)^T$ in (5).

LEMMA 3.3.1. *Suppose the (i, j) entry of $\text{rk}(ws_i)$ is r .*

1. *If $j \geq w(i + 1)$ then the $(i - 1, j)$ entry of $\text{rk}(ws_i)$ is $r - 1$.*
2. *If $j < w(i + 1)$ then the $(i + 1, j)$ entry of $\text{rk}(ws_i)$ is r .*

In what follows, a *rank condition* refers to a statement requiring “ $\text{rank}(Z_{q \times p}) \leq r$ ” for some $r \geq 0$. Most often, r will be either $\text{rank}(w_{q \times p}^T)$ or $\text{rank}((ws_i)_{q \times p}^T)$, thereby making the entries of $\text{rk}(w)$ and $\text{rk}(ws_i)$ into rank conditions. We say that a rank condition $\text{rank}(Z_{q \times p}) \leq r$ *causes* an antidiagonal a of the generic matrix Z if $Z_{q \times p}$ contains a and the number of variables in a is strictly larger than r . For instance, when the rank condition is in $\text{rk}(w)$, the antidiagonals it causes include those $a \in J_w$ that are contained in $Z_{q \times p}$ but no smaller northwest submatrix. Although antidiagonals in Z (that is, antidiagonals of square submatrices of the generic matrix Z) are by definition monomials, we routinely identify each antidiagonal with its *support*: the subset of the variables in $\mathbf{k}[\mathbf{z}]$ dividing it

LEMMA 3.3.2. *Antidiagonals in $J_w \setminus J_{ws_i}$ are subsets of $Z_{i \times w(i)}$ and intersect row i .*

Proof. If an antidiagonal in J_w is either contained in $Z_{i-1 \times w(i)}$ or not contained in $Z_{i \times w(i)}$, then some rank condition causing it is in both $\text{rk}(w)$ and $\text{rk}(ws_i)$. Indeed, it is easy to check that the rank matrices $\text{rk}(ws_i)$ and $\text{rk}(w)$ differ only in row i between the columns $w(i + 1)$ and $w(i) - 1$, inclusive. \square

Though simple, the next lemma is the key combinatorial observation. Note that the permutation w there is arbitrary; in particular, we will frequently apply the lemma in the context of antidiagonals for a permutation called ws_i .

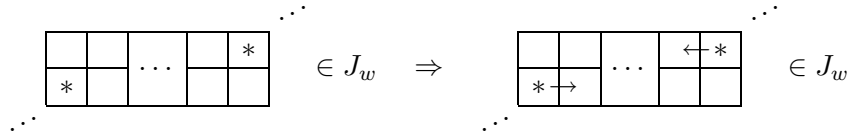
LEMMA 3.3.3. *Suppose $a \in J_w$ is an antidiagonal and $a' \subset Z$ is another antidiagonal.*

- (W) *If a' is obtained by moving west one or more of the variables in a , then $a' \in J_w$.*
- (E) *If $a' \in \mathbf{k}[\mathbf{z}]$ is obtained by moving east any variable viz. the northeast one in a , then $a' \in J_w$.*
- (N) *If a' is obtained by moving north one or more of the variables in a , then $a' \in J_w$.*

(S) If $a' \in \mathbf{k}[\mathbf{z}]$ is obtained by moving south any variable viz. the southwest one in a , then $a' \in J_w$.

Proof. Every rank condition causing a also causes all of the antidiagonals a' . □

Example 3.3.4. Parts (W) and (E) of Lemma 3.3.3 together imply that the type of motion depicted in the following diagram preserves the property of an antidiagonal being in J_w . The presence of the northeast $*$ justifies moving the southwest $*$ east.



The two rows could also be separated by some other rows—possibly themselves containing elements of the original antidiagonal—as long as the indicated motion preserves the fact that we have an antidiagonal.

Definition 3.3.5. Let \mathbf{b} be an array $\mathbf{b} = (b_{rs})$ of nonnegative integers, that is, \mathbf{b} is an *exponent array* for a monomial $\mathbf{z}^{\mathbf{b}} \in \mathbf{k}[\mathbf{z}]$. Let

$$\text{west}_q(\mathbf{b}) := \min(\{p \mid b_{qp} \neq 0\} \cup \{\infty\})$$

be the column of the leftmost (most “western”) nonzero entry in row q . Define the *mutation* of \mathbf{b} in rows i and $i + 1$ by

$$\mu_i(\mathbf{b}) := \text{the exponent array of } (z_{i,p}/z_{i+1,p})\mathbf{z}^{\mathbf{b}} \text{ for } p = \text{west}_{i+1}(\mathbf{b}).$$

For ease of notation, we write $\mu_i(\mathbf{z}^{\mathbf{b}})$ for $\mathbf{z}^{\mu_i(\mathbf{b})}$.

If one thinks of the array \mathbf{b} as a chessboard with some coins stacked in each square, then μ_i is performed by taking a coin off the western stack in row $i + 1$ and putting it onto the stack due north of it in row i .

Example 3.3.6. Suppose that \mathbf{b} is the left array in Figure 2, and that $i = 3$. We list in Figure 2 (reading left to right as usual) 7 mutations of \mathbf{b} , namely $\mathbf{b} = (\mu_3)^0(\mathbf{b})$ through $(\mu_3)^6(\mathbf{b})$ (after that it involves the dots we left unspecified). Here, the empty boxes denote entries equal to 0, and the nonzero mutated entries at each step are in boxes. To make things easier to look at, the entries on or below the main antidiagonal are represented by dots, each of which may be zero or not (independently of the others). The 3 and 4 at left are labels for rows $i = 3$ and $i + 1 = 4$.

With our chosen permutation w and row index i , various entries of a given $\mathbf{b} \in \mathbb{Z}^{n^2}$ play special roles. To begin with, we call the union of rows i and $i + 1$

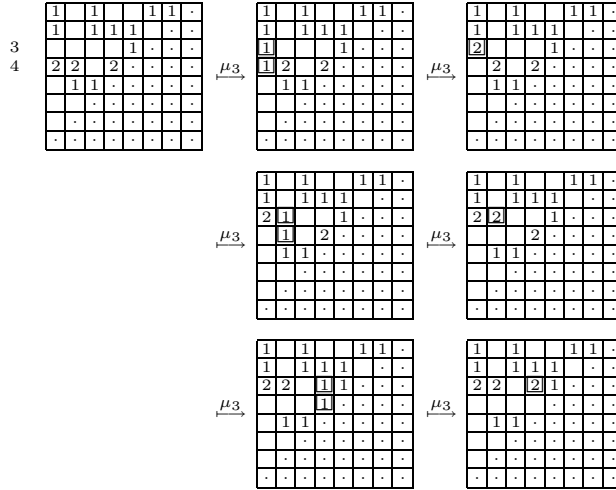


Figure 2: Mutation

the *gene*⁷ of \mathbf{b} . For exponent arrays \mathbf{b} such that $\mathbf{z}^{\mathbf{b}} \notin J_w$, the spot in row i and column

$$(8) \quad \text{start}_i(\mathbf{b}) := \min\{p \mid z_{ip}\mathbf{z}^{\mathbf{b}} \notin J_w\}$$

is called the *start codon* of \mathbf{b} . The minimum defining $\text{start}_i(\mathbf{b})$ is taken over a nonempty set because $J_w \subseteq J_{w_0} = I_{w_0} = \langle z_{qp} \mid q + p \leq n \rangle$, so that $z_{in}\mathbf{z}^{\mathbf{b}}$ remains outside of J_w . Of course, $z_{ip}\mathbf{z}^{\mathbf{b}} \notin J_w$ whenever z_{ip} divides $\mathbf{z}^{\mathbf{b}} \notin J_w$ because J_w is generated by squarefree monomials. Therefore,

$$(9) \quad \text{start}_i(\mathbf{b}) \leq \text{west}_i(\mathbf{b}).$$

For completeness, set $\text{start}_i(\mathbf{b}) = 0$ if $\mathbf{z}^{\mathbf{b}} \in J_w$.

Also of special importance is the *promoter* $\text{prom}(\mathbf{b})$, consisting of the rectangular $2 \times (\text{start}_i(\mathbf{b}) - 1)$ array of locations in the gene of \mathbf{b} that are strictly west of $\text{start}_i(\mathbf{b})$. Again, we omit the explicit reference to i and w in the notation because these are fixed for the discussion. The sum of all entries in the promoter of \mathbf{b} is

$$(10) \quad |\text{prom}(\mathbf{b})| = \sum_{j < \text{start}_i(\mathbf{b})} b_{i+1,j}.$$

Example 3.3.7. Let \mathbf{b} be the left array in Figure 2, $i = 3$, and $w = 13865742$, the permutation displayed in Example 1.3.5. Then the gene of \mathbf{b} consists of rows $i = 3$ and $i + 1 = 4$, and we claim $\text{start}_i(\mathbf{b}) = 5$.

⁷All of the unusual terminology in what follows comes from genetics. Superficially, our diagrams with two rows of boxes look like geneticists' schematic diagrams of the DNA double helix; but there is a much more apt analogy that will become clear only in Section 3.5, where the biological meanings of the terms can be found in another footnote.

To begin with, we have six choices for an antidiagonal $a \in J_w$ dividing $z_{31}\mathbf{z}^{\mathbf{b}}$: we must have $z_{31} \in a$, but other than that we are free to choose one element of $\{z_{23}, z_{24}, z_{25}\}$ and one element of $\{z_{16}, z_{17}\}$. (This gives an example of the a produced in the first paragraph of the proof of Lemma 3.4.3, below.) Even more varied choices are available for $z_{32}\mathbf{z}^{\mathbf{b}}$, such as $z_{41}z_{32}z_{23}$ or $z_{41}z_{32}z_{13}$. We can similarly find lots of antidiagonals in J_w dividing $z_{33}\mathbf{z}^{\mathbf{b}}$, and $z_{34}\mathbf{z}^{\mathbf{b}}$. On the other hand, z_{35} already divides $\mathbf{z}^{\mathbf{b}}$, and one can verify that $\mathbf{z}^{\mathbf{b}}$ is not divisible by the antidiagonals of any of the 2×2 or 3×3 minors defining I_w (see Example 1.3.5). Therefore $z_{35}\mathbf{z}^{\mathbf{b}} \notin J_w$, and so $\text{start}_i(\mathbf{b}) = 5$.

The promoter $\text{prom}(\mathbf{b})$ consists of the 2×4 block

$$\begin{array}{c} 3 \\ 4 \end{array} \begin{array}{|c|c|c|c|} \hline & & & \\ \hline 2 & 2 & & 2 \\ \hline \end{array}$$

at the western end. In particular, $|\text{prom}(\mathbf{b})| = 6$.

Nothing in this example depends on the values chosen for the dots on or below the main antidiagonal.

3.4. Lifting Demazure operators

Now we need to understand the Hilbert series of $\mathbf{k}[\mathbf{z}]/J_w$ for varying w . Since J_w is a monomial ideal, its \mathbb{Z}^{n^2} -graded Hilbert series $H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{z})$ is simply the sum of all monomials outside J_w . Using the combinatorics of the previous section, we construct operators ε_i^w defined on monomials and taking the power series $H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{z})$ to $H(\mathbf{k}[\mathbf{z}]/J_{ws_i}; \mathbf{z})$ whenever $\text{length}(ws_i) < \text{length}(w)$. In other words, the sum of all monomials outside J_{ws_i} is obtained from the sum of monomials outside J_w by replacing $\mathbf{z}^{\mathbf{b}} \notin J_w$ with $\varepsilon_i^w(\mathbf{z}^{\mathbf{b}})$. It is worth keeping in mind that we shall eventually show (in Section 3.5) how ε_i^w refines the usual \mathbb{Z}^n -graded Demazure operators $\bar{\partial}_i$, when these operators are applied to the variously graded Hilbert series of $\mathbf{k}[\mathbf{z}]/J_w$.

Again, fix for the duration of this section a permutation w and a transposition s_i satisfying $\text{length}(ws_i) < \text{length}(w)$.

Definition 3.4.1. The *lifted Demazure operator* corresponding to w and i is a map of abelian groups $\varepsilon_i^w : \mathbb{Z}[[\mathbf{z}]] \rightarrow \mathbb{Z}[[\mathbf{z}]]$ determined by its action on monomials:

$$\varepsilon_i^w(\mathbf{z}^{\mathbf{b}}) := \sum_{d=0}^{|\text{prom}(\mathbf{b})|} \mu_i^d(\mathbf{z}^{\mathbf{b}}).$$

Here, μ_i^d means take the result of applying μ_i a total of d times, and $\mu_i^0(\mathbf{b}) = \mathbf{b}$.

Example 3.4.2. If \mathbf{b} is the array in Examples 3.3.6 and 3.3.7, then $\varepsilon_3^w(\mathbf{z}^{\mathbf{b}})$ is the sum of the seven monomials whose exponent arrays are displayed in Figure 2.

Observe that ε_i^w replaces each monomial by a homogeneous polynomial of the same total degree, so that the result of applying ε_i^w to a power series is actually a power series.

In preparation for Theorem 3.4.11, we need a few lemmas detailing the effects of mutation on monomials and their genes. The first of these implies that ε_i^w takes monomials outside J_w to sums of monomials outside J_{ws_i} , given that $\mu_i^0(\mathbf{b}) = \mathbf{b}$.

LEMMA 3.4.3. *If $\mathbf{z}^{\mathbf{b}} \notin J_w$ and $1 \leq d \leq |\text{prom}(\mathbf{b})|$ then $\mu_i^d(\mathbf{z}^{\mathbf{b}}) \in J_w \setminus J_{ws_i}$.*

Proof. We may as well assume $|\text{prom}(\mathbf{b})| \geq 1$, or else the statement is vacuous. By definition of $\text{prom}(\mathbf{b})$ and $\text{start}_i(\mathbf{b})$, some antidiagonal $a \in J_w$ divides $z_{ip}\mathbf{z}^{\mathbf{b}}$, where here (and for the remainder of this proof) $p = \text{west}_{i+1}(\mathbf{b})$. Since a does not divide $\mathbf{z}^{\mathbf{b}}$, we find that $z_{ip} \in a$, whence a cannot intersect row $i + 1$, which is zero to the west of z_{ip} . Thus a also divides $\mu_i(\mathbf{z}^{\mathbf{b}})$, and hence $\mu_i^d(\mathbf{z}^{\mathbf{b}})$ for all d (including $d > |\text{prom}(\mathbf{b})|$, but we won't need this).

It remains to show that $\mu_i^d(\mathbf{z}^{\mathbf{b}}) \notin J_{ws_i}$ when $d \leq |\text{prom}(\mathbf{b})|$. We start with $d \leq b_{i+1,p}$. Any antidiagonal a dividing $\mu_i^d(\mathbf{z}^{\mathbf{b}})$ does not continue southwest of z_{ip} ; this is by Lemma 3.3.3(S) and the fact that $\mathbf{z}^{\mathbf{b}} \notin J_w$ (we could move z_{ip} south). Suppose for contradiction that $a \in J_{ws_i}$, and consider the smallest northwest submatrix $Z_{i \times j(a)}$ containing a . If $j(a) \geq w(i + 1)$ then the antidiagonal $a' = a/z_{ip}$ obtained by omitting z_{ip} from a is still in J_{ws_i} , which is caused by the entry of $\text{rk}(ws_i)$ at $(i - 1, j(a))$ as per Lemma 3.3.1.1. On the other hand, if $j(a) < w(i + 1)$, then $a'' = (z_{i+1,p}/z_{ip})a$ is still in J_{ws_i} , which is caused by the entry of $\text{rk}(ws_i)$ at $(i + 1, j(a))$ as per Lemma 3.3.1.2. Since both a' and a'' divide $\mathbf{z}^{\mathbf{b}}$ by construction, we find that $\mathbf{z}^{\mathbf{b}} \in J_{ws_i} \subset J_w$, the desired contradiction. It follows that $\mu_i^d(\mathbf{z}^{\mathbf{b}}) \notin J_{ws_i}$ for $d \leq b_{i+1,p}$.

Assuming the result for $d \leq \sum_{j=p}^{p'} b_{i+1,j}$, where $p' < \text{start}_i(\mathbf{b}) - 1$, we now demonstrate the result for $d \leq \sum_{j=p}^{p'+1} b_{i+1,j}$. Again, any antidiagonal $a \in J_w$ dividing $\mu_i^d(\mathbf{z}^{\mathbf{b}})$ must end at row i , for the same reason as in the previous paragraph. But now if $a \in J_{ws_i}$, then moving its southwest variable to z_{ip} creates an antidiagonal that is in J_{ws_i} (by Lemma 3.3.3(W)) and divides $\mu_i(\mathbf{z}^{\mathbf{b}})$, which we have seen is impossible. \square

Now we show that mutation of monomials outside J_w cannot produce the same monomial more than once, as long we stop after $|\text{prom}|$ many steps.

LEMMA 3.4.4. *Suppose $\mathbf{z}^{\mathbf{b}}, \mathbf{z}^{\mathbf{b}'} \notin J_w$ and that $d, d' \in \mathbb{Z}$ satisfy $1 \leq d \leq |\text{prom}(\mathbf{b})|$ and $1 \leq d' \leq |\text{prom}(\mathbf{b}')|$. If $\mathbf{b} \neq \mathbf{b}'$ then $\mu_i^d(\mathbf{b}) \neq \mu_i^{d'}(\mathbf{b}')$.*

Proof. The inequality $d \leq |\text{prom}(\mathbf{b})|$ guarantees that the mutations of \mathbf{b} only alter the promoter of \mathbf{b} , which is west of $\text{west}_i(\mathbf{b})$ by (9). Therefore, assuming (by switching \mathbf{b} and \mathbf{b}' if necessary) that $\text{west}_i(\mathbf{b}') \leq \text{west}_i(\mathbf{b})$, we reduce to the case where \mathbf{b} and \mathbf{b}' differ only in their genes, in columns strictly west of $\text{west}_i(\mathbf{b})$.

Let $\mathbf{c} = \mu_i^d(\mathbf{b})$ and $\mathbf{c}' = \mu_i^{d'}(\mathbf{b}')$. Mutating preserves the sums

$$b_{i+1,j} = c_{ij} + c_{i+1,j} \quad \text{and} \quad c'_{ij} + c'_{i+1,j} = b_{ij} + b_{i+1,j}$$

for $j < \text{west}_i(\mathbf{b})$, and we may as well assume these are equal for every j , or else $\mathbf{c} \neq \mathbf{c}'$ is clear. The westernmost column where \mathbf{b} and \mathbf{b}' disagree is now necessarily $p = \text{west}_i(\mathbf{b}')$. It follows that $z_{ip}\mathbf{z}^{\mathbf{b}} \notin J_w$, because \mathbf{b} agrees with \mathbf{b}' strictly to the west of column p as well as strictly to the north of row i , and any antidiagonal $a \in J_w$ dividing $z_{ip}\mathbf{z}^{\mathbf{b}}$ must be contained in this region (since it contains z_{ip}). In particular, $\text{start}_i(\mathbf{b}) \leq p$. We conclude that mutating \mathbf{b} and \mathbf{b}' fewer than $|\text{prom}(\mathbf{b})|$ or $|\text{prom}(\mathbf{b}')|$ times cannot alter the column p where \mathbf{b} and \mathbf{b}' differ. Thus \mathbf{c} differs from \mathbf{c}' in column p . \square

Example 3.4.5. If we apply μ_i more than $|\text{prom}(\mathbf{b})|$ times to some array \mathbf{b} , it is possible to reach $\mu_i^{d'}(\mathbf{b}')$ for some $\mathbf{b}' \neq \mathbf{b}$ and $d' \leq |\text{prom}(\mathbf{b}')|$. Take \mathbf{b} , i , and w as in Examples 3.3.6, 3.3.7, and 3.4.2, and set the dot in \mathbf{b} at position $(4, 5)$ equal to 3. If $\mathbf{z}^{\mathbf{b}'} = (z_{35}/z_{45})\mathbf{z}^{\mathbf{b}}$, then we have $|\text{prom}(\mathbf{b})| = |\text{prom}(\mathbf{b}')| = 6$, but the entries of \mathbf{b} and \mathbf{b}' in column 5 of their genes are $\frac{1}{3}$ and $\frac{2}{2}$, respectively. Mutating \mathbf{b} and \mathbf{b}' up to six times yields seven arrays each, all distinct because of the $\frac{1}{3}$ and $\frac{2}{2}$ in column 5. However, mutating \mathbf{b} to an eighth array $(\mu_3)^7(\mathbf{b})$ changes the $\frac{1}{3}$ to $\frac{2}{2}$, and outputs $(\mu_3)^6(\mathbf{b}')$.

If \mathbf{c} is the result of applying μ_i to \mathbf{b} some number of times, we can recover \mathbf{b} from \mathbf{c} by *reverting* certain entries of \mathbf{c} from row i back to row $i+1$. Formally, reverting an entry c_{ij} of \mathbf{c} means making a new array that agrees with \mathbf{c} except at (i, j) and $(i+1, j)$. In those spots, the new array has (i, j) entry 0 and $(i+1, j)$ entry $c_{ij} + c_{i+1,j}$. (In terms of the stacks-of-coins picture, we revert only entire stacks of coins, not single coins.) Even if we are just given \mathbf{c} without knowing \mathbf{b} , we still have a criterion to determine when a certain reversion of \mathbf{c} yields a monomial $\mathbf{z}^{\mathbf{b}} \notin J_w$.

CLAIM 3.4.6. *Suppose $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$. If \mathbf{b} is obtained from \mathbf{c} by reverting all entries of \mathbf{c} in row i that are west of or at column $\text{west}_{i+1}(\mathbf{c})$, then $\mathbf{z}^{\mathbf{b}} \notin J_w$.*

Proof. Suppose $\mathbf{z}^{\mathbf{b}} \in J_w$. Let us try to produce an antidiagonal witness $a \in J_w$ dividing it. Either a ends at row i , or not. In the first case, a divides $\mathbf{z}^{\mathbf{c}}$, because the nonzero entries in row i of \mathbf{b} are the same as the corresponding

entries of \mathbf{c} . Thus we can replace a by the result a' of tacking on $z_{i+1,p}$ to a , where $p = \text{west}_{i+1}(\mathbf{c})$. This new a' is in J_w because $a \in J_w$ divides a' . It follows from Lemma 3.3.2 that $a' \in J_{ws_i}$. Furthermore, a' divides $\mathbf{z}^{\mathbf{c}}$ by construction, and thus contradicts our assumption that $\mathbf{z}^{\mathbf{c}} \notin J_{ws_i}$. Therefore, we may assume for the remainder of the proof of this lemma that a does not end at row i , and so $a \in J_{ws_i}$ by Lemma 3.3.2.

We now prove that $\mathbf{z}^{\mathbf{b}} \notin J_w$ by showing that if $a \in J_{ws_i}$ and a divides $\mathbf{z}^{\mathbf{b}}$, then from a we can synthesize $a' \in J_{ws_i}$ dividing $\mathbf{z}^{\mathbf{c}} \in J_{ws_i}$, again contradicting our running assumption $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$. There are three possibilities (an illustration for (ii) is described in Example 3.4.7):

- (i) The antidiagonal $a \in J_{ws_i}$ intersects row i but does not end there.
- (ii) The antidiagonal $a \in J_{ws_i}$ skips row i but intersects row $i + 1$.
- (iii) The antidiagonal $a \in J_{ws_i}$ skips both row i as well as row $i + 1$.

In case (i) either a already divides $\mathbf{z}^{\mathbf{c}}$ or we can move east the row $i + 1$ variable in a , into the location $(i + 1, \text{west}_{i+1}(\mathbf{c}))$. The resulting antidiagonal a' divides \mathbf{c} by construction and is in J_{ws_i} by Lemma 3.3.3(E). In case (iii) the antidiagonal already divides $\mathbf{z}^{\mathbf{c}}$ because \mathbf{b} agrees with \mathbf{c} outside of their genes.

This leaves case (ii). If a does not already divide $\mathbf{z}^{\mathbf{c}}$, then the intersection $z_{i+1,j}$ of a with row $i + 1$ is strictly west of $\text{west}_{i+1}(\mathbf{c})$. The antidiagonal $a' = (z_{ij}/z_{i+1,j})a$ then divides $\mathbf{z}^{\mathbf{c}}$ by construction, and is in J_{ws_i} by Lemma 3.3.3(N). □

Example 3.4.7. Here is an instance of what occurs in case (ii) from the proof of Claim 3.4.6. Let \mathbf{b} and \mathbf{c} be the first and last arrays from Example 3.3.6, and consider what happens when we fiddle with their $(5, 1)$ entries. The antidiagonals $z_{51}z_{42}z_{23}$ and $z_{51}z_{42}z_{24} \in J_{ws_i}$ both divide $z_{51}\mathbf{z}^{\mathbf{b}}$. Using Lemma 3.3.3(N) we can move z_{42} north to z_{32} to get $a' \in \{z_{51}z_{32}z_{23}, z_{51}z_{32}z_{24}\}$ in J_{ws_i} dividing $z_{51}\mathbf{z}^{\mathbf{c}}$. (It almost goes without saying, of course, that $z_{51}\mathbf{z}^{\mathbf{c}}$ is no longer in $J_w \setminus J_{ws_i}$, and so it does not satisfy the hypothesis of Claim 3.4.6; we were, after all, looking for a contradiction.)

Any array \mathbf{c} whose row i begins west of its row $i + 1$ can be expressed as a mutation of *some* array \mathbf{b} . By Claim 3.4.6, we even know how to make sure $\mathbf{z}^{\mathbf{b}} \notin J_w$ whenever $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$. But we also want each $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$ to appear in $\varepsilon_i^w(\mathbf{z}^{\mathbf{b}})$ for some $\mathbf{z}^{\mathbf{b}} \notin J_w$, and this involves making sure $\text{start}_i(\mathbf{b})$ is not too far west.

Example 3.4.8. If $\text{west}_i(\mathbf{c})$ is sufficiently smaller than $\text{west}_{i+1}(\mathbf{c})$, then it might be hard to determine which entries in row i of \mathbf{c} to revert while still assuring that $\mathbf{z}^{\mathbf{c}}$ appears in $\varepsilon_i^w(\mathbf{z}^{\mathbf{b}})$. For example, let \mathbf{c} be the last array in Example 3.3.6, that is, $\mathbf{c} = (\mu_3)^6(\mathbf{b})$. Suppose further that the dot at $(4, 5)$,

is really blank—i.e. zero. Without *a priori* knowing \mathbf{b} , how are we to know *not* to revert the 1 in position (3, 5)? Well, suppose we did revert this entry, along with all of the entries west of it in row 3. Then we would end up with an array \mathbf{b}' such that $\mathbf{z}^{\mathbf{b}'} \notin J_w$ all right, as per Claim 3.4.6, but also such that $z_{35}\mathbf{z}^{\mathbf{b}'} \notin J_w$. This latter condition is intolerable, since $5 \geq \text{start}_i(\mathbf{b})$ implies that our original $\mathbf{z}^{\mathbf{c}}$ will not end up in the sum $\varepsilon_3^w(\mathbf{z}^{\mathbf{b}'})$.

Thus the problem with trying to revert the 1 in position (3, 5) is that it is too far east. On the other hand, we might also try reverting only the row 3 entries in columns 1 and 2, but with dire consequences: we end up with an array \mathbf{b}'' such that $\mathbf{z}^{\mathbf{b}''}$ is divisible by $z_{42}z_{34}z_{25} \in J_w$. (This is an example of the antidiagonal a to be produced after the displayed equation in the proof of Lemma 3.4.10.) We are left with only one choice: revert the boldface $\mathbf{2}$ in position (3, 4) and all of its more western brethren.

In general, as in the previous example, the *critical column* for $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$ is

$$\text{crit}(\mathbf{c}) := \min(p \leq \text{west}_{i+1}(\mathbf{c}) \mid z_{ip} \text{ divides } \mathbf{z}^{\mathbf{c}} \text{ and } z_{i+1,p}\mathbf{z}^{\mathbf{c}} \notin J_{ws_i}).$$

CLAIM 3.4.9. *If $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$ then:*

1. *The set used to define $\text{crit}(\mathbf{c})$ is nonempty;*
2. *Reverting $c_{i,\text{crit}(\mathbf{c})}$ creates an array \mathbf{c}' such that $\mathbf{z}^{\mathbf{c}'} \notin J_{ws_i}$; and*
3. *If $\text{west}_i(\mathbf{c}) < \text{crit}(\mathbf{c})$, then the monomial $\mathbf{z}^{\mathbf{c}'}$ from statement 2 remains in J_w .*

Proof. Claim 3.4.6 implies $\text{west}_i(\mathbf{c}) \leq \text{west}_{i+1}(\mathbf{c})$, so that

$$p' = \max(p \leq \text{west}_{i+1}(\mathbf{c}) \mid c_{ip} \neq 0)$$

is well-defined. If a is an antidiagonal dividing the monomial whose exponent array is the result of reverting $c_{ip'}$, then a divides either $\mathbf{z}^{\mathbf{c}}$ or the monomial $\mathbf{z}^{\mathbf{b}}$ from Claim 3.4.6, and neither of these is in J_{ws_i} . Thus $a \notin J_{ws_i}$ and statement 1 is proved. Part 2 is by definition, and statement 3 follows from it by Lemmas 3.3.2 and 3.3.3(W). □

LEMMA 3.4.10. *Suppose $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$ and that \mathbf{b} is obtained by reverting all row i entries of \mathbf{c} west of or at $\text{crit}(\mathbf{c})$. Then $\mathbf{z}^{\mathbf{b}} \notin J_w$, and $\text{crit}(\mathbf{c}) < \text{start}_i(\mathbf{b})$.*

Proof. The proof that this $\mathbf{z}^{\mathbf{b}}$ is not in J_w has two cases. In the first we have $\text{crit}(\mathbf{c}) = \text{west}_{i+1}(\mathbf{c})$, and Claim 3.4.6 immediately implies the result. In the second case we have $\text{crit}(\mathbf{c}) < \text{west}_{i+1}(\mathbf{c})$, and we can apply Claim 3.4.6 to the monomial $\mathbf{z}^{\mathbf{c}'} \in J_w \setminus J_{ws_i}$ from Claim 3.4.9.

Now we need to show $z_{ip}\mathbf{z}^{\mathbf{b}} \in J_w$ for two kinds of p : for $p \leq \text{west}_i(\mathbf{c})$ and $\text{west}_i(\mathbf{c}) < p \leq \text{crit}(\mathbf{c})$. (Of course, when $\text{west}_i(\mathbf{c}) = \text{crit}(\mathbf{c})$ the second of these cases is vacuous.) The case $p \leq \text{west}_i(\mathbf{c})$ is a little easier, so we treat it first.

There is some antidiagonal in J_w ending on row i and dividing $\mathbf{z}^{\mathbf{c}}$, by Lemma 3.3.2. When $p \leq \text{west}_i(\mathbf{c})$, we get the desired result by appealing to Lemma 3.3.3(W).

Next we treat $\text{west}_i(\mathbf{c}) < p \leq \text{crit}(\mathbf{c})$. These inequalities mean precisely that

$$j = \max\{p' < p \mid c_{ip'} \neq 0\}$$

is well-defined, and that $z_{i+1,j}\mathbf{z}^{\mathbf{c}} \in J_{ws_i}$. Any antidiagonal $a \in J_{ws_i}$ dividing $z_{i+1,j}\mathbf{z}^{\mathbf{c}}$ must contain $z_{i+1,j}$ because a does not divide $\mathbf{z}^{\mathbf{c}}$, and the fact that $\mathbf{z}^{\mathbf{b}} \notin J_w$ implies that a also does not divide $\mathbf{z}^{\mathbf{b}}$. It follows that a intersects row i at some spot in which \mathbf{c} is nonzero strictly east of column j . This spot is necessarily east of or at (i, p) by construction. Without changing whether $a \in J_{ws_i}$, Lemma 3.3.3(W) says that we may assume a contains z_{ip} itself. This a divides $z_{ip}\mathbf{z}^{\mathbf{b}}$, whence $z_{ip}\mathbf{z}^{\mathbf{b}} \in J_{ws_i}$. \square

The next theorem is the main result of Section 3.4. It pinpoints, at the level of individual standard monomials, the relation between J_w and J_{ws_i} .

THEOREM 3.4.11. $H(\mathbf{k}[\mathbf{z}]/J_{ws_i}; \mathbf{z}) = \varepsilon_i^w H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{z})$ if $\text{length}(ws_i) < \text{length}(w)$.

Proof. We need the sum $H(\mathbf{k}[\mathbf{z}]/J_{ws_i}; \mathbf{z})$ of monomials outside J_{ws_i} to be obtained from the sum of monomials outside J_w by replacing $\mathbf{z}^{\mathbf{b}} \notin J_w$ with $\varepsilon_i^w(\mathbf{z}^{\mathbf{b}})$. We know by Lemma 3.4.3 that $\varepsilon_i^w H(\mathbf{k}[\mathbf{z}]/J_w)$ is a sum of monomials outside J_{ws_i} . Furthermore, no monomial $\mathbf{z}^{\mathbf{c}}$ is repeated in this sum: if $\mathbf{z}^{\mathbf{c}} \notin J_w$ appears in $\varepsilon_i^w(\mathbf{b})$, then \mathbf{b} must equal $\mathbf{c} = \mu_i^0(\mathbf{b})$ by Lemma 3.4.3; and if $\mathbf{z}^{\mathbf{c}} \in J_w$ then Lemma 3.4.4 applies.

It remains to demonstrate that each monomial $\mathbf{z}^{\mathbf{c}} \notin J_{ws_i}$ is equal to $\mu_i^d(\mathbf{z}^{\mathbf{b}})$ for some monomial $\mathbf{z}^{\mathbf{b}} \notin J_w$ and $d \leq |\text{prom}(\mathbf{b})|$. This is easy if $\mathbf{z}^{\mathbf{c}}$ is not even in J_w : we take $\mathbf{z}^{\mathbf{b}} = \mu_i^0(\mathbf{z}^{\mathbf{b}}) = \mathbf{z}^{\mathbf{c}}$. Since we can now assume $\mathbf{z}^{\mathbf{c}} \in J_w \setminus J_{ws_i}$, the result follows from Lemma 3.4.10, once we notice that the inequality $\text{crit}(\mathbf{c}) < \text{start}_i(\mathbf{b})$ there is equivalent to the inequality $d \leq |\text{prom}(\mathbf{b})|$. \square

3.5. Coarsening the grading

As in Section 3.4, fix a permutation w and an index i such that $\text{length}(ws_i) < \text{length}(w)$. Our goal in this section is to prove (in Theorem 3.5.4) that the set of \mathbb{Z}^n -graded Hilbert series $H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x})$ for varying w is closed under Demazure operators. The idea is to combine lifted Demazure operators ε_i^w with

the specialization $\mathcal{X} : \mathbb{Z}[[\mathbf{z}]] \rightarrow \mathbb{Z}[[\mathbf{x}]]$ sending $z_{qp} \mapsto x_q$; see Example 1.7.4. We present Lemma 3.5.1 and the proof of Proposition 3.5.3 in “single” language, for ease of notation, but indicate at the end of the proof of Proposition 3.5.3 which changes of notation make the arguments work for the \mathbb{Z}^{2n} -graded Hilbert series $H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x}, \mathbf{y})$, with the specialization $\mathcal{X}_y : \mathbb{Z}[[\mathbf{z}]] \rightarrow \mathbb{Z}[[\mathbf{x}, \mathbf{y}^{-1}]]$ sending $z_{qp} \mapsto x_q/y_p$.

At the outset, we could hope that $\mathcal{X} \circ \varepsilon_i^w = \bar{\partial}_i \circ \mathcal{X}$ monomial by monomial. However, although this works in some cases (see (11), below) *it fails in general*. The next lemma will be used to take care of the general case. Its proof is somewhat involved (but fun) and irrelevant to its application, so we postpone the proof until after Theorem 3.5.4. Denote by $\text{std}(J_w)$ the set of *standard exponent arrays*: the exponents on monomials not in J_w .

LEMMA 3.5.1. *There is an involution $\tau : \text{std}(J_w) \rightarrow \text{std}(J_w)$ such that $\tau^2 = 1$ and:*

1. $\tau \mathbf{b}$ agrees with \mathbf{b} outside their genes;
2. $\text{prom}(\tau \mathbf{b}) = \text{prom}(\mathbf{b})$;
3. if $\mathcal{X}(\mathbf{z}^{\mathbf{b}}) = x_{i+1}^\ell \mathbf{x}^{\mathbf{a}}$ with $\ell = |\text{prom}(\mathbf{b})|$, then $\mathcal{X}(\mathbf{z}^{\tau \mathbf{b}}) = x_{i+1}^\ell s_i(\mathbf{x}^{\mathbf{a}})$; and
4. τ preserves column sums. In other words, if $\mathbf{b}' = \tau \mathbf{b}$, then $\sum_q b_{qp} = \sum_q b'_{qp}$ for any fixed column index p .

In particular, $\mathcal{X}(\varepsilon_i^w \mathbf{z}^{\tau \mathbf{b}}) = \bar{\partial}_i(x_{i+1}^\ell)(s_i \mathbf{x}^{\mathbf{a}})$.

Remark 3.5.2. The squarefree monomials outside J_w for a Grassmannian permutation v (that is, a permutation having a unique descent) are in natural bijection with the semistandard Young tableaux of the appropriate shape and content. (This follows from Definition 1.4.3 and Theorem 3.8.5, below, along with the bijection in [Kog00] between reduced pipe dreams and semistandard Young tableaux.) Under this natural bijection, intron mutation (which we will introduce in Definition 3.5.6) reduces to an operation that arises in a well-known combinatorial proof of the symmetry of the Schur function associated to v , which equals \mathfrak{S}_v .

Our next result justifies the term ‘lifted Demazure operator’ for ε_i^w .

PROPOSITION 3.5.3. *Specializing \mathbf{z} to \mathbf{x} in $\varepsilon_i^w H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{z})$ yields*

$$\bar{\partial}_i H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x}).$$

More generally, specializing z_{qp} to x_q/y_p in $\varepsilon_i^w H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{z})$ yields

$$\bar{\partial}_i H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x}, \mathbf{y}).$$

Proof. Suppose $\mathbf{z}^{\mathbf{b}} \notin J_w$ specializes to $\mathcal{X}(\mathbf{z}^{\mathbf{b}}) = x_{i+1}^\ell \mathbf{x}^{\mathbf{a}}$, where $\ell = |\text{prom}(\mathbf{b})|$. The definition of $\varepsilon_i^w \mathbf{z}^{\mathbf{b}}$ implies that

$$\begin{aligned} \mathcal{X}(\varepsilon_i^w \mathbf{z}^{\mathbf{b}}) &= \sum_{d=0}^\ell x_i^d x_{i+1}^{\ell-d} \mathbf{x}^{\mathbf{a}} \\ &= \frac{x_{i+1}^{\ell+1} - x_i^{\ell+1}}{x_{i+1} - x_i} \mathbf{x}^{\mathbf{a}} \\ &= \bar{\partial}_i(x_{i+1}^\ell) \mathbf{x}^{\mathbf{a}}. \end{aligned}$$

If it happens that $s_i \mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{a}}$, so that $\mathbf{x}^{\mathbf{a}}$ is symmetric in x_i and x_{i+1} , then

$$(11) \quad \mathcal{X}(\varepsilon_i^w \mathbf{z}^{\mathbf{b}}) = \bar{\partial}_i(x_{i+1}^\ell) \mathbf{x}^{\mathbf{a}} = \bar{\partial}_i(x_{i+1}^\ell \mathbf{x}^{\mathbf{a}}) = \bar{\partial}_i \mathcal{X}(\mathbf{z}^{\mathbf{b}}).$$

Of course, there will in general be lots of $\mathbf{z}^{\mathbf{b}} \notin J_w$ whose $\mathbf{x}^{\mathbf{a}}$ is not fixed by s_i . We overcome this difficulty using Lemma 3.5.1, which tells how to pair each $\mathbf{z}^{\mathbf{b}} \notin J_w$ with a partner so that their corresponding $\mathcal{X} \circ \varepsilon_i^w$ sums add up nicely. Using the notation of the lemma, notice that if $\tau \mathbf{b} = \mathbf{b}$, then $s_i \mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{a}}$ and $\mathcal{X}(\varepsilon_i^w \mathbf{z}^{\mathbf{b}}) = \bar{\partial}_i \mathcal{X}(\mathbf{z}^{\mathbf{b}})$, as in (11). On the other hand, if $\tau \mathbf{b} \neq \mathbf{b}$, then the lemma implies

$$\begin{aligned} \mathcal{X}(\varepsilon_i^w(\mathbf{z}^{\mathbf{b}} + \mathbf{z}^{\tau \mathbf{b}})) &= \bar{\partial}_i(x_{i+1}^\ell)(\mathbf{x}^{\mathbf{a}} + s_i \mathbf{x}^{\mathbf{a}}) \\ &= \bar{\partial}_i(x_{i+1}^\ell(\mathbf{x}^{\mathbf{a}} + s_i \mathbf{x}^{\mathbf{a}})) \\ &= \varepsilon_i^w(\mathcal{X}(\mathbf{z}^{\mathbf{b}} + \mathbf{z}^{\tau \mathbf{b}})) \end{aligned}$$

because $\mathbf{x}^{\mathbf{a}} + s_i \mathbf{x}^{\mathbf{a}}$ is symmetric in x_i and x_{i+1} . This proves the \mathbb{Z}^n -graded statement.

The \mathbb{Z}^{2n} -graded version of the argument works *mutatis mutandis* by the preservation of column sums under mutation (Definition 3.3.5 and statement 4 of Lemma 3.5.1), which allows us to replace \mathcal{X} by $\mathcal{X}_{\mathbf{y}}$ and $\mathbf{x}^{\mathbf{a}}$ by a monomial in the \mathbf{x} variables and the inverses of the \mathbf{y} variables. □

Now we come to a result that will be crucial in proving the main theorems of Part 1.

THEOREM 3.5.4. $H(\mathbf{k}[\mathbf{z}]/J_{ws_i}; \mathbf{x}) = \bar{\partial}_i H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x})$ if $\text{length}(ws_i) < \text{length}(w)$. More generally, $H(\mathbf{k}[\mathbf{z}]/J_{ws_i}; \mathbf{x}, \mathbf{y}) = \bar{\partial}_i H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x}, \mathbf{y})$ if $\text{length}(ws_i) < \text{length}(w)$.

Proof. Theorem 3.4.11 and Proposition 3.5.3. □

Before constructing this magic involution τ , we introduce some necessary notation and provide examples. Recall that the union of rows i and $i + 1$ is the *gene* of \mathbf{b} (we view the row index i as being fixed for the discussion). Order the boxes in columns east of $\text{start}_i(\mathbf{b})$ in the gene of \mathbf{b} as in the diagram, using

the notation $\text{start}_i(\mathbf{b})$ from (8) in Section 3.3:

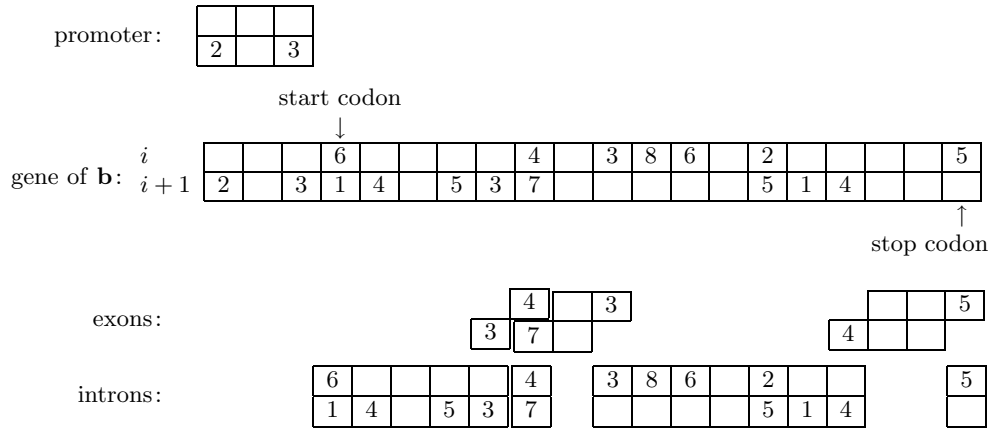
$$\begin{array}{c} \text{start}_i(\mathbf{b}) \\ \downarrow \\ \begin{array}{cccccc} & & 1 & 3 & 5 & 7 & \\ i & \cdots & \boxed{1} & \boxed{3} & \boxed{5} & \boxed{7} & \cdots \\ i+1 & & \boxed{2} & \boxed{4} & \boxed{6} & \boxed{8} & \end{array} \end{array}$$

Now define five different kinds of blocks in the gene of \mathbf{b} , called the promoter, the start codon, exons, introns, and the stop codon.⁸ In the following, $k, \ell \in \mathbb{N}$.

- *promoter*: the rectangle consisting of unnumbered boxes at the left end
- *start codon*: the box numbered 1, which lies at $(i, \text{start}_i(\mathbf{b}))$
- *stop codon*: the last numbered box, which lies at $(i + 1, n)$
- *exon*: any sequence $2k, \dots, 2\ell + 1$ (with $k \leq \ell$) of consecutive boxes satisfying:
 1. the entries of \mathbf{b} in the boxes corresponding to $2k + 1, \dots, 2\ell$ are all zero;
 2. either box $2k + 1$ is the start codon, or box $2k$ has a nonzero entry in \mathbf{b} ; and
 3. either box 2ℓ is the stop codon, or box $2\ell + 1$ has a nonzero entry in \mathbf{b}
- *intron*: any rectangle of consecutive boxes $2\ell + 1, \dots, 2k$ (with $\ell < k$) satisfying:
 1. the rectangle contains no exons;
 2. box $2\ell + 1$ is either the start codon or the last box in an exon; and
 3. box $2k$ is either the stop codon or the first box in an exon.

Roughly speaking, the nonzero entries in $\text{gene}(\mathbf{b})$ are partitioned into the promoter and introns, the latter being contiguous rectangles having nonzero entries in their northwest and southeast corners. Exons connect adjacent introns via bridges of zeros.

⁸All of these are terms from genetics. The DNA sequence for a single gene is not necessarily contiguous. Instead, it sometimes comes in blocks called *exons*. The intervening DNA sequences whose data are excised are called *introns* (note that the structure of the gene of an exponent array is determined by its exons, not its introns). The *promoter* is a medium-length region somewhat before the gene that signals the transcriptase enzyme where to attach to the DNA, so that it may begin transcribing the DNA into RNA. The *start codon* is a short sequence signaling the beginning of the actual gene; the *stop codon* is a similar sequence signaling the end of the gene.



For ease of comparison, we dissect here the mutated gene $\tau\mathbf{b}$ of \mathbf{b} .

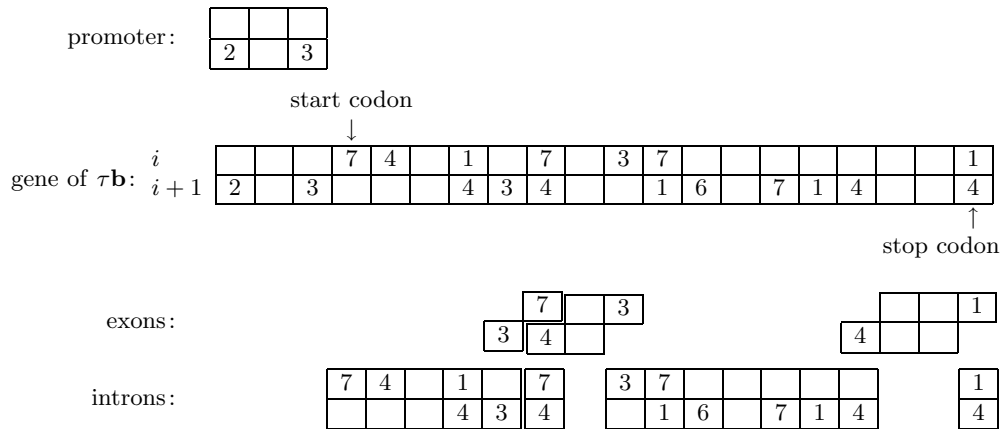


Figure 3: Intron mutation

Example 3.5.5. Suppose we are given a permutation w , an array \mathbf{b} such that $\mathbf{z}^{\mathbf{b}} \notin J_w$, and a row index i such that $\text{start}_i(\mathbf{b}) = 4$ and \mathbf{b} has the gene in Figure 3. The gene of \mathbf{b} breaks up into promoter, start codon, exons, introns, and stop codon as indicated. We shall say something more about the mutated gene $\tau\mathbf{b}$ in Example 3.5.7.

If \mathbf{c} is an array having two rows filled with nonnegative integers, then let $\bar{\mathbf{c}}$ be the rectangle obtained by rotating \mathbf{c} through an angle of 180° . For purposes of applying the mutation operator μ_i (Definition 3.3.5), we identify the rows of an intron \mathbf{c} as rows i and $i+1$ in a gene, and we view \mathbf{c} as an $n \times n$ array that happens to be zero outside of its $2 \times k$ rectangle.

Definition 3.5.6 (Intron mutation). Let c_i and c_{i+1} be the sums of the entries in the top and bottom nonzero rows of an intron \mathbf{c} , and set $d = |c_i - c_{i+1}|$. Then

$$\tau \mathbf{c} = \begin{cases} \overline{\mu_i^d(\overline{\mathbf{c}})} & \text{if } c_i > c_{i+1} \\ \mu_i^d(\mathbf{c}) & \text{if } c_i < c_{i+1} \end{cases}$$

is the *mutation* of \mathbf{c} . Define the *intron mutation* $\tau \mathbf{b}$ of an exponent array \mathbf{b} by

- adding 1 to the start and stop codons of \mathbf{b} ;
- mutating every intron in the gene of the resulting exponent array; and then
- subtracting 1 from the boxes that were the start and stop codons of \mathbf{b} .

Intron mutation pushes the entries of each intron either upward from left to right or downward from right to left—whichever initially brings the row sums in that particular intron closer to agreement.

Example 3.5.7. Although the “look” of \mathbf{b} in Example 3.5.5 completely changes when it is mutated into $\tau \mathbf{b}$, the columns of $\tau \mathbf{b}$ containing a nonzero entry are exactly the same as those in \mathbf{b} , and the column sums are preserved. Note that mutating the gene of $\tau \mathbf{b}$ yields back the gene of \mathbf{b} , as long as $\mathbf{z}^{\mathbf{c}} \notin J_w$ and the location of the start codon has not changed. The proof of Lemma 3.5.1 shows why τ always works this way.

LEMMA 3.5.8. *Intron mutation puts out an exponent array (that is, the entries are nonnegative). Assume, for the purpose of defining exons in $\tau \mathbf{b}$, that the start codon of $\tau \mathbf{b}$ lies at the same location as the start codon in \mathbf{b} . The boxes occupied by exons of $\tau \mathbf{b}$ thus defined coincide with the boxes occupied by exons of \mathbf{b} itself.*

Proof. The definitions ensure that any intron not containing the start or stop codon has nonzero northwest and southeast corners. After we add 1 to the start and stop codons, every intron has this property. Mutation of such an intron leaves strictly positive entries in the northwest and southeast corners (this is crucial—it explains why we have to add and subtract the 1’s from the codons), so that subtracting 1 preserves nonnegativity. Furthermore, intron mutation does not introduce any new exons, because the nonzero entries in an intron both before and after mutation follow a snake pattern that drops from row i to row $i + 1$ precisely once. \square

Proof of Lemma 3.5.1. First we show that $\tau \mathbf{b} \in \text{std}(J_w)$, or equivalently that $\mathbf{z}^{\tau \mathbf{b}} \in J_w \Rightarrow \mathbf{z}^{\mathbf{b}} \in J_w$. Observe that $\mathbf{z}^{\mathbf{b}} \in J_w$ if and only if $z_{ip} z_{i+1,n} \mathbf{z}^{\mathbf{b}} \in J_w$, where $p = \text{start}_i(\mathbf{b})$, by definition of $\text{start}_i(\mathbf{b})$ and the fact that $z_{i+1,n}$ is a

nonzero divisor modulo J_w for all w . Therefore, it suffices to demonstrate how an antidiagonal $a \in J_w$ dividing $\mathbf{z}^{\tau\mathbf{b}}$ gives rise to a possibly different antidiagonal $a' \in J_w$ dividing $z_{ip}z_{i+1,n}\mathbf{z}^{\mathbf{b}}$, where $p = \text{start}_i(\mathbf{b})$. There are five cases:

- (i) a intersects neither row i nor row $i + 1$;
- (ii) the southwest variable in a is in row i ;
- (iii) a intersects row i and continues south, but skips row $i + 1$;
- (iv) a intersects both row i and row $i + 1$; or
- (v) a skips row i but intersects row $i + 1$.

In each of these cases, a' is constructed as follows. Outside the gene of the generic matrix Z , the new a' will agree with a in all five cases, since \mathbf{b} and $\tau\mathbf{b}$ agree outside of their genes. Inside their genes, we may need some adjustments.

- (i) Leave $a' = a$ as is.
- (ii) Move the variable in row i west to z_{ip} , using Lemma 3.3.3(W).
- (iii) The gene of \mathbf{b} has nonzero entries in precisely the same columns as the gene of $\tau\mathbf{b}$, by definition. Either a already divides $z_{ip}\mathbf{z}^{\mathbf{b}}$, or moving the variable in row i due south to row $i + 1$ yields a' by Lemma 3.3.3(S).
- (iv) Use Example 3.3.4 to make a' contain the nonzero entries in some exon of \mathbf{b} (see Lemma 3.5.8).
- (v) Same as (iii), except that either a already divides $z_{i+1,n}\mathbf{z}^{\mathbf{b}}$ or by Lemma 3.3.3(N) we can move the variable due north from row $i + 1$ to row i .

Now that we know $\tau\mathbf{b} \in \text{std}(J_w)$, we find that

$$\text{start}_i(\tau\mathbf{b}) = \text{start}_i(\mathbf{b}).$$

Indeed, when $j \leq \text{start}_i(\mathbf{b})$, we have $z_{ij}\mathbf{z}^{\tau\mathbf{b}} \in J_w$ if and only if $z_{ij}\mathbf{z}^{\mathbf{b}} \in J_w$, because any antidiagonal containing z_{ij} interacts with $\tau\mathbf{b}$ north of row i and west of column $\text{start}_i(\mathbf{b})$, where $\tau\mathbf{b}$ agrees with \mathbf{b} . It follows that $\text{prom}(\tau\mathbf{b}) = \text{prom}(\mathbf{b})$, so exons of $\tau\mathbf{b}$ occupy the same boxes as those of \mathbf{b} by Lemma 3.5.8. We conclude that $\tau\mathbf{b}$ also has introns in the same boxes as the introns of \mathbf{b} . The statement $\tau^2 = \text{identity}$ holds because the partitions of the genes of \mathbf{b} and $\tau\mathbf{b}$ into promoter and introns are the same, and mutation on each of these blocks in the partition has order 1 or 2. Part 3 follows intron by intron, except that the added and subtracted 1's in the first and last introns cancel. \square

3.6. Equidimensionality

We demonstrate here that the facets of \mathcal{L}_w all have the same dimension.

The monomials $\mathbf{z}^{\mathbf{b}}$ that are nonzero in $\mathbf{k}[\mathbf{z}]/J_w$ are the so-called *standard monomials* for J_w , and are precisely those with support sets

$$\text{supp}(\mathbf{z}^{\mathbf{b}}) := \{z_{qp} \in Z \mid z_{qp} \text{ divides } \mathbf{z}^{\mathbf{b}}\}$$

in the complex \mathcal{L}_w . The maximal support sets of standard monomials are the facets of \mathcal{L}_w . The following lemma says that maximal support monomials for J_{ws_i} can only be mutations of maximal support monomials for J_w .

LEMMA 3.6.1. *If $\mathbf{z}^{\mathbf{b}}$ divides $\mathbf{z}^{\mathbf{c}} \notin J_w$ and $d \leq |\text{prom}(\mathbf{b})|$, then $\mu_i^d(\mathbf{z}^{\mathbf{b}})$ divides $\mu_i^e(\mathbf{z}^{\mathbf{c}})$ for some $e \leq |\text{prom}(\mathbf{c})|$. If $\text{supp}(\mathbf{z}^{\mathbf{b}}) \in \mathcal{L}_w$ is not a facet and $d \leq |\text{prom}(\mathbf{b})|$, then $\text{supp}(\mu_i^d(\mathbf{z}^{\mathbf{b}})) \in \mathcal{L}_{ws_i}$ is not a facet.*

Proof. If $\mathbf{z}^{\mathbf{b}}$ divides $\mathbf{z}^{\mathbf{c}}$, then $\text{start}_i(\mathbf{b}) \leq \text{start}_i(\mathbf{c})$ by definition ($z_{ip}\mathbf{z}^{\mathbf{b}} \in J_w \Rightarrow z_{ip}\mathbf{z}^{\mathbf{c}} \in J_w$). Therefore, if the d^{th} mutation of \mathbf{b} is the k^{th} occurring in column $j < \text{start}_i(\mathbf{b})$, we can choose e so that the e^{th} mutation of \mathbf{c} is also the k^{th} occurring in column j . If, in addition, $z_{qp} \in \text{supp}(\mathbf{z}^{\mathbf{c}}) \setminus \text{supp}(\mathbf{z}^{\mathbf{b}})$, then either z_{qp} or $z_{q-1,p}$ ends up in $\text{supp}(\mu_i^e(\mathbf{z}^{\mathbf{c}})) \setminus \text{supp}(\mu_i^d(\mathbf{z}^{\mathbf{b}}))$, depending on whether or not $(q,p) \in \text{prom}(\mathbf{c})$ and $p < j$. Note that $\mu_i^d(\mathbf{z}^{\mathbf{b}})$ and $\mu_i^e(\mathbf{z}^{\mathbf{c}})$ are not in J_{ws_i} by Theorem 3.4.11, so their supports are in \mathcal{L}_{ws_i} . \square

Recall that D_0 is the pipe dream with crosses in the strict upper-left triangle, that is, all locations (q,p) such that $q+p \leq n$. Number the crosses in D_0 as follows, where $N = \binom{n}{2}$:

N				
	...			
		...		
			...	
				...

For notation, given an $n \times n$ exponent array \mathbf{b} , let $D(\mathbf{b}) = [n]^2 \setminus \text{supp}(\mathbf{z}^{\mathbf{b}})$.

LEMMA 3.6.2. *Consider the following condition on a permutation w : there is a fixed $\alpha \geq 1$ such that for every facet L of \mathcal{L}_w , the associated pipe dream D_L has crosses in boxes marked $\geq \alpha$ and an elbow joint in the box due south of the box marked α . Given this condition, it follows that if α sits in row i , then for every facet L' of \mathcal{L}_{ws_i} , the associated pipe dream $D_{L'}$ has crosses in boxes marked $\geq \alpha + 1$ and an elbow joint at α . Moreover, if α sits in column j and $\text{supp}(\mathbf{z}^{\mathbf{b}})$ is a facet of \mathcal{L}_w , then $\text{start}_i(\mathbf{b}) > j$.*

Proof. If α sits in column j , then every variable other than $z_{i+1,j}$ in $Z_{i+1 \times j}$ lies in J_w , by the hypothesis on α . It follows that $w(i+1) = j$ and

$\text{length}(ws_i) < \text{length}(w)$, because $\text{rank}(w_{i+1 \times j}^T) = 1$ while $\text{rank}(w_{q \times p}^T) = 0$ whenever $z_{i+1,j} \neq z_{qp} \in Z_{i+1 \times j}$.

By Lemma 3.6.1, every facet $L' \in \mathcal{L}_{ws_i}$ can be expressed as $\text{supp}(\mu_i^d(\mathbf{z}^{\mathbf{b}}))$ for some monomial $\mathbf{z}^{\mathbf{b}}$ such that $\text{supp}(\mathbf{z}^{\mathbf{b}}) \in \mathcal{L}_w$ is a facet. The maximality of $\text{supp}(\mathbf{z}^{\mathbf{b}})$ implies $\text{start}_i(\mathbf{b}) = \text{west}_i(\mathbf{b})$; but $\text{west}_i(\mathbf{b}) > j$ because $D(\mathbf{b})$ has crosses in $Z_{i \times j}$. The result follows because all mutations of $\mathbf{z}^{\mathbf{b}}$ (except $\mathbf{z}^{\mathbf{b}}$ itself) are therefore divisible by z_{ij} . \square

In the hypothesis of Lemma 3.6.2, note that the box due south of α will not be marked $\alpha - 1$ when $\alpha - 1$ lies in the top row. The next result allows induction on α .

LEMMA 3.6.3. *Given a permutation $v \in S_n$ with $v \neq w_0$, there exists an integer $\alpha \geq 1$ such that w satisfies the hypothesis of Lemma 3.6.2 and $v = ws_i$. In particular, there exists a unique $\alpha \in \{1, \dots, \binom{n}{2}\}$ such that for every facet L of \mathcal{L}_v , the associated pipe dream D_L has crosses in boxes marked $\geq \alpha + 1$ and an elbow joint at α .*

Proof. For each $\alpha \geq 0$, let $\nu(\alpha)$ be the number of permutations ws_i satisfying the conclusion of Lemma 3.6.2 for that choice of α , where we declare $\nu(0) = 1$ to take care of $ws_i = w_0$. Lemma 3.6.2 implies by induction on α that

- $\nu(\alpha) \geq \nu(\alpha - 1)$ if $\alpha - 1$ does not lie in the top row; and
- $\nu(\alpha) \geq \sum_{\beta \leq \alpha} \nu(\beta)$ if $\alpha - 1$ lies in the top row.

Let $\nu_j = \sum \nu(\beta)$ be the sum over all β in column j , and set $\sigma_{j+1} = \sum_{j' \geq j+1} \nu_{j'}$. The itemized claims above imply that if α lies in column j , then $\nu(\alpha) \geq \sigma_{j+1}$. Assume by downward induction on j that $\sigma_{j+1} \geq (n - j)!$. Then $\nu_j \geq (n - j)(n - j)!$ because there are $n - j$ marked boxes in column j . It follows that $\sigma_j \geq (n - j)! + (n - j)(n - j)! = (n - j + 1)!$. In particular, $\sigma_1 \geq n!$, whence $\sigma_1 = n!$. \square

If we knew *a priori* that J_w were the initial ideal of $I(\overline{X}_w)$, then the following proposition would follow from [KS95] (except for embedded components). Since we are working in the opposite direction, we have to prove it ourselves.

PROPOSITION 3.6.4. *The simplicial complex \mathcal{L}_w is pure, each facet having $\dim \overline{X}_w = n^2 - \text{length}(w)$ vertices; i.e. $\text{Spec}(\mathbf{k}[\mathbf{z}]/J_w)$ is equidimensional of dimension $\dim \overline{X}_w$.*

Proof. The result is obvious for \mathcal{L}_{w_0} . Taking w and ws_i as in Lemma 3.6.2 by Lemma 3.6.3, we prove the result for ws_i by assuming it for w . Theorem 3.4.11 implies that the facets of \mathcal{L}_{ws_i} are supports of monomials $\mu_i^d(\mathbf{z}^{\mathbf{b}})$

for $\mathbf{z}^{\mathbf{b}} \notin J_w$ and $d \leq |\text{prom}(\mathbf{b})|$. By Lemma 3.6.1, we may restrict our attention to square monomials $\mathbf{z}^{2\mathbf{b}}$ with maximal support.

Repeated mutation increases the support size of a monomial by 0 or 1 over the original. Hence it suffices to show that if the cardinalities of $\text{supp}(\mu_i^d(\mathbf{z}^{2\mathbf{b}}))$ and $\text{supp}(\mathbf{z}^{2\mathbf{b}})$ are equal for some $d \leq |\text{prom}(2\mathbf{b})|$, then $\text{supp}(\mu_i^d(\mathbf{z}^{2\mathbf{b}})) \subsetneq \text{supp}(\mu_i^e(\mathbf{z}^{2\mathbf{b}}))$ for some $e \leq |\text{prom}(2\mathbf{b})|$. By Lemma 3.6.2 we may take $e = 1$ if $d = 0$. If $d > 0$, then mutation will have just barely pushed a row $i + 1$ entry of $2\mathbf{b}$ up to row i , and we may take $e = d - 1$. Containment uses that every entry b_{ij} is at least 2; strict containment is automatic, by checking cardinalities. \square

Example 3.6.5. The ideal J_{1432} is generated by the antidiagonals of the five 2×2 minors contained in the union of the northwest 2×3 and 3×2 submatrices of (z_{ij}) :

$$\begin{aligned} J_{1432} &= \langle z_{12}z_{21}, z_{13}z_{21}, z_{13}z_{22}, z_{12}z_{31}, z_{22}z_{31} \rangle \\ &= \langle z_{12}, z_{13}, z_{22} \rangle \\ &\quad \cap \langle z_{12}, z_{21}, z_{22} \rangle \cap \langle z_{21}, z_{22}, z_{31} \rangle \cap \langle z_{13}, z_{21}, z_{31} \rangle \cap \langle z_{12}, z_{13}, z_{31} \rangle. \end{aligned}$$

\mathcal{L}_{1432} is the join of a pentagon with a simplex having 11 vertices $\{z_{11}\} \cup \{z_{rs} \mid r + s \geq 5\}$ (note $n = 4$ here). Each facet of \mathcal{L}_w therefore has $13 = 4^2 - \text{length}(1432)$ vertices.

3.7. Mitosis on facets

This section and the next translate the combinatorics of lifted Demazure operators into a language compatible with reduced pipe dreams. The present section concerns the relation between mutation and mitosis. We again think of the row index i as being fixed, as we did in Sections 3.3–3.5.

Recall the definition of pipe dream from Section 1.4. The similarity between $\text{start}_i(D)$ for pipe dreams D introduced there and $\text{start}_i(\mathbf{b})$ for arrays in Section 3.4 is apparent. It is explained precisely in the following lemma, whose proof is immediate from the definitions.

LEMMA 3.7.1. *If $\mathbf{z}^{\mathbf{b}} \notin J_w$ has maximal support and $D(\mathbf{b}) = [n]^2 \setminus \text{supp}(\mathbf{z}^{\mathbf{b}})$, then*

$$\text{start}_i(\mathbf{b}) = \text{west}_i(\mathbf{b}) = \text{start}_i(D(\mathbf{b})).$$

Proposition 3.7.3 will present a verbal description of mitosis that is different from the one in Section 1.6. The new description is a little more algorithmic, using a certain local transformation on pipe dreams that was discovered by Bergeron and Billey.

Definition 3.7.2 ([BB93]). A *chutable rectangle* is a connected $2 \times k$ rectangle C inside a pipe dream D such that $k \geq 2$ and all but the following three locations in C are crosses: the northwest, southwest, and southeast corners. Applying a *chute move*⁹ to D is accomplished by placing a ‘+’ in the southwest corner of a chutable rectangle C and removing the ‘+’ from the northeast corner of the same C .

Heuristically, a chute move therefore looks like:



PROPOSITION 3.7.3. *Let D be a pipe dream, and suppose j is the smallest column index such that $(i + 1, j) \notin D$ and $(i, p) \in D$ for all $p \leq j$. Then $D_p(i) \in \text{mitosis}_i(D)$ is obtained from D by*

1. *removing (i, j) , and then*
2. *performing chute moves from row i to row $i + 1$, each one as far west as possible, so that (i, p) is the last ‘+’ removed.*

Proof. Immediate from Definitions 3.7.2 and 1.6.1. □

Example 3.7.4. The offspring in Example 1.6.2 are listed in the order they are born via the algorithmic ‘chute’ form of mitosis in Proposition 3.7.3, with $i = 3$.

Proposition 3.7.3 for mitosis has the following analogue for mutation.

CLAIM 3.7.5. *Suppose that $\text{length}(ws_i) < \text{length}(w)$, and let $\mathbf{z}^{\mathbf{b}} \notin J_w$ be a squarefree monomial of maximal support. If $D = D(\mathbf{b})$ then*

$$|\text{prom}(\mathbf{b})| = |\mathcal{J}(D)|.$$

If $0 \leq d < |\text{prom}(\mathbf{b})|$, then $D(\mu_i^{2d+1}(2\mathbf{b}))$ is obtained from D by

1. *removing (i, j) , where j is as in Proposition 3.7.3, and then*
2. *performing d chute moves from row i to row $i + 1$, each as far west as possible.*

Proof. By Lemma 3.7.1, the columns in $\mathcal{J}(D)$ are in bijection with the nonzero entries in the promoter of \mathbf{b} , each of which is a 1 in row $i + 1$. The final statement follows easily from the definitions. □

⁹The transpose of a chute move is called a *ladder move* in [BB93].

Example 3.7.6. The left array \mathbf{b} in Figure 2 has maximal support among exponent arrays on monomials not in J_w , where $w = 13865742$ as in Examples 1.3.5, 3.3.6, and 3.3.7. Substituting ‘+’ for each blank space and then removing the numbers and dots yield the left pipe dream Example 1.6.2. Applying the same makeover to the middle column of Figure 2 results in the offspring $D(\mu_3^1(\mathbf{b}))$, $D(\mu_3^3(\mathbf{b}))$, and $D(\mu_3^5(\mathbf{b}))$.

LEMMA 3.7.7. *If $\text{length}(ws_i) < \text{length}(w)$ and $\mathbf{z}^{\mathbf{b}} \notin J_w$ is squarefree of maximal support, then $\text{mitosis}_i(D(\mathbf{b})) = \{D(\mu_i^{2d+1}(2\mathbf{b})) \mid 0 \leq d < |\text{prom}(\mathbf{b})|\}$.*

Proof. Compare Claim 3.7.5 with Proposition 3.7.3. □

Theorem 3.7.9 will conclude the translation of mutation on monomials into mitosis on facets, but the translation requires an intermediate result.

LEMMA 3.7.8. *If $\text{length}(ws_i) < \text{length}(w)$, then $\{D_L \mid L \text{ is a facet of } \mathcal{L}_{ws_i}\}$ is the set of pipe dreams $D(\mu_i^{2d+1}(2\mathbf{b}))$ such that $\mathbf{z}^{\mathbf{b}} \notin J_w$ is squarefree of maximal support and $0 \leq d < |\text{prom}(\mathbf{b})|$.*

Proof. By Theorem 3.4.11 and Lemma 3.6.1, every facet of \mathcal{L}_{ws_i} is the support of a mutation $\mu_i^d(\mathbf{z}^{2\mathbf{b}})$ for some monomial $\mathbf{z}^{2\mathbf{b}} \notin J_w$ and $d \leq |\text{prom}(2\mathbf{b})|$. Furthermore, it is clear from Lemma 3.6.1 and the definition of mutation that we may assume $\mathbf{z}^{\mathbf{b}}$ is a squarefree monomial (so the entries of $2\mathbf{b}$ are all 0 or 2). In this case, the supports of odd mutations $\mu_i^{2d+1}(\mathbf{z}^{2\mathbf{b}})$ for $0 \leq d < |\text{prom}(\mathbf{b})|$ are facets of \mathcal{L}_{ws_i} by Proposition 3.6.4 because they each have cardinality $n^2 - \text{length}(w) + 1 = n^2 - \text{length}(ws_i)$, while the supports of even mutations $\mu_i^{2d}(\mathbf{z}^{2\mathbf{b}})$ for $d \leq |\text{prom}(\mathbf{b})|$ are not facets, each having cardinality $n^2 - \text{length}(w)$. □

THEOREM 3.7.9. *If $\text{length}(ws_i) < \text{length}(w)$, then*

$$\{D_L \mid L \text{ is a facet of } \mathcal{L}_{ws_i}\} = \text{mitosis}_i(\{D_L \mid L \text{ is a facet of } \mathcal{L}_w\}).$$

Moreover, $\text{mitosis}_i(D_L) \cap \text{mitosis}_i(D_{L'}) = \emptyset$ if $L \neq L'$ are facets of \mathcal{L}_w .

Proof. The displayed equation is a consequence of Lemma 3.7.8 and Lemma 3.7.7, and so we concentrate on the final statement. Let $\mathbf{z}^{\mathbf{b}} \notin J_w$ be a squarefree monomial of maximal support $L = \text{supp}(\mathbf{b})$, and let $0 \leq d < |\text{prom}(\mathbf{b})|$. The entries of the array $\mu_i^{2d+1}(2\mathbf{b})$ are all either 0 or 2, except for precisely two 1’s, both in the same column p (the boldface entries in Figure 2, middle column). By Lemma 3.7.1, p is the westernmost column of $D(\mu_i^{2d+1}(2\mathbf{b}))$ in which neither row i nor row $i + 1$ has a cross.

Now suppose $\mathbf{z}^{\mathbf{b}'}$ $\notin J_w$ is another squarefree monomial of maximal support L' , and let $0 \leq d' < |\text{prom}(\mathbf{b}')|$. If $D(\mu_i^{2d+1}(2\mathbf{b})) = D(\mu_i^{2d'+1}(2\mathbf{b}'))$, then the argument in the first paragraph of the proof implies that $\mu_i^{2d+1}(2\mathbf{b}) =$

$\mu_i^{2d'+1}(2\mathbf{b}')$, since they have the same entries equal to 1 as well as the same support, and all of their other nonzero entries equal 2. We conclude that $\mathbf{b} = \mathbf{b}'$ by Lemma 3.4.4. Using Lemma 3.7.8, we have proved that $\text{mitosis}_i(D_L) \cap \text{mitosis}_i(D_{L'}) \neq \emptyset$ implies $L = L'$. \square

Example 3.7.10. The pipe dream D in Example 3.7.6 and the left side of Example 1.6.2 is D_L for a facet of $\mathcal{L}_{13865742}$. By Theorem 3.7.9, the three pipe dreams at the right of Example 1.6.2 can be expressed as $D_{L'}$ for facets $L' \in \mathcal{L}_{13685742}$, where $13685742 = 13865742 \cdot s_3$.

3.8. Facets and reduced pipe dreams

To derive the connection between the initial complex \mathcal{L}_w and reduced pipe dreams, in Theorem 3.8.5, we need the next result, whose proof connects chuting with antidiagonals.

LEMMA 3.8.1. *The set $\{D_L \mid L \in \text{facets}(\mathcal{L}_w)\}$ is closed under chute moves.*

Proof. A pipe dream D is equal to D_L for some (not necessarily maximal) $L \in \mathcal{L}_w$ if and only if D meets every antidiagonal in J_w , which by definition of \mathcal{L}_w equals $\bigcap_{L \in \mathcal{L}_w} \langle z_{qp} \mid (q,p) \in D_L \rangle$. Suppose that C is a chutable rectangle in D_L for $L \in \mathcal{L}_w$. It is enough to show that the intersection $a \cap D_L$ of any antidiagonal $a \in J_w$ with D_L does not consist entirely of the single cross in the northeast corner of C , unless a also contains the southwest corner of C . Indeed, the purity of \mathcal{L}_w (Proposition 3.6.4) will then imply that chuting D_L in C yields $D_{L'}$ for some *facet* L' whenever $L \in \text{facets}(\mathcal{L}_w)$.

To prove the claim concerning $a \cap D_L$, we may assume a contains the cross in the northeast corner (q,p) of C , but not the cross in the southwest corner of C , and splits into cases:

- (i) a does not continue south of row q .
- (ii) a continues south of row q but skips row $q + 1$.
- (iii) a intersects row $q + 1$, but strictly east of the southwest corner of C .
- (iv) a intersects row $q + 1$, but strictly west of the southwest corner of C .

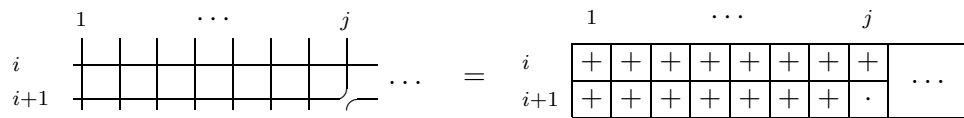
Letting $(q + 1, t)$ be the southwest corner of C , construct new antidiagonals a' that are in J_w (and hence intersect D_L) by replacing the cross at (q,p) with a cross at:

- (i) (q, t) , using Lemma 3.3.3(W);
- (ii) $(q + 1, p)$, using Lemma 3.3.3(S);

- (iii) (q, p) , so $a = a'$ trivially; or
- (iv) (q, t) , using Lemma 3.3.3(W).

Observe that in case (iii), a already shares a box in row $q + 1$ where D_L has a cross. Each of the other antidiagonals a' intersects both a and D_L in some box that is not (q, p) , since the location of $a' \setminus a$ has been constructed not to be a cross in D_L . □

Lemma 1.4.5 implies the following criterion for when removing a ‘+’ from a pipe dream $D \in \mathcal{RP}(w)$ yields a pipe dream in $\mathcal{RP}(ws_i)$. Specifically, it concerns the removal of a cross at (i, j) from configurations that look like



at the west end of rows i and $i + 1$ in D .

LEMMA 3.8.2. *Let $D \in \mathcal{RP}(w)$ and j be a fixed column index with $(i + 1, j) \notin D$, but $(i, p) \in D$ for all $p \leq j$, and $(i + 1, p) \in D$ for all $p < j$. Then $\text{length}(ws_i) < \text{length}(w)$, and if $D' = D \setminus (i, j)$ then $D' \in \mathcal{RP}(ws_i)$.*

Proof. Removing (i, j) only switches the exit points of the two pipes starting in rows i and $i + 1$, so the pipe starting in row k of D' exits out of column $ws_i(k)$ for each k . The result follows from Lemma 1.4.5. □

The connection between the complexes \mathcal{L}_w and reduced pipe dreams requires certain facts proved by Bergeron and Billey [BB93]. The next lemma consists mostly of the combinatorial parts (a), (b), and (c) of [BB93, Th. 3.7], their main result. Its proof there relies exclusively on elementary properties of reduced pipe dreams.

LEMMA 3.8.3 ([BB93]).

1. *The set $\mathcal{RP}(w)$ of reduced pipe dreams for w is closed under chute operations.*
2. *There is a unique top reduced pipe dream for w such that every cross not in the first row has a cross due north of it.*
3. *Every reduced pipe dream for w can be obtained by applying a sequence of chute moves to the top reduced pipe dream for w .*

LEMMA 3.8.4. *The top reduced pipe dream for w is D_L for a facet $L \in \mathcal{L}_w$.*

Proof. The unique reduced pipe dream for w_0 , whose crosses lie at $\{(q, p) \mid q + p \leq n\}$, is also D_L for the unique facet of \mathcal{L}_{w_0} . By Lemma 3.6.3, take w and ws_i as in Lemma 3.6.2 and assume the result for w . By Lemma 3.6.3 again, the top pipe dream $D \in \mathcal{RP}(w)$ satisfies the conditions of Lemma 3.8.2, where α lies at $(i + 1, j)$. Now combine Lemma 3.8.2 with the last sentence of Lemma 3.6.2 to prove the desired result for ws_i . \square

THEOREM 3.8.5. $\mathcal{RP}(w) = \{D_L \mid L \text{ is a facet of } \mathcal{L}_w\}$, where $D_L = [n]^2 \setminus L$. In other words, reduced pipe dreams for w are complements of maximal supports of monomials $\notin J_w$.

Proof. Lemma 3.8.4 and Lemma 3.8.1 imply that $\mathcal{RP}(w) \subseteq \{D_L \mid L \in \text{facets}(\mathcal{L}_w)\}$, given Lemma 3.8.3. Since the opposite containment $\{D_L \mid L \in \text{facets}(\mathcal{L}_v)\} \subseteq \mathcal{RP}(v)$ is obvious for $v = w_0$, it suffices to prove it for $v = ws_i$ by assuming it for $v = w$.

A pair $(i + 1, j)$ as in Lemma 3.8.2 exists in D if and only if the set $\mathcal{J}(D)$ in Definition 1.6.1 is nonempty, which occurs if and only if $\text{mitosis}_i(D) \neq \emptyset$. In this case, the first offspring of D under mitosis_i (as in Proposition 3.7.3) is $D' = D \setminus (i, j) \in \mathcal{RP}(ws_i)$. The desired containment follows from Theorem 3.7.9 and statement 1 of Lemma 3.8.3. \square

3.9. Proofs of Theorems A, B, and C

In this section, we tie up loose ends, completing the proofs of Theorems A, B, and C. All statements in these theorems are straightforward for the long permutation $w = w_0$, so we may prove the rest by Bruhat induction—that is, by downward induction on $\text{length}(w)$.

The multidegrees of $\{\overline{X}_w\}_{w \in S_n}$ of matrix Schubert varieties satisfy the divided difference recursion defining double Schubert polynomials by Theorem 3.2.8. Separately, the Hilbert series of $\{\mathbf{k}[\mathbf{z}]/J_w\}_{w \in S_n}$ satisfy the Demazure recursion defining the double Grothendieck polynomials by Theorem 3.5.4. It follows by Lemma 1.1.4 that the multidegrees $\{\mathcal{L}_w\}_{w \in S_n}$ satisfy the divided difference recursion.

The subspace arrangement \mathcal{L}_w is equidimensional by Proposition 3.6.4, and the multidegree of \overline{X}_w equals that of $\mathbf{k}[\mathbf{z}]/\text{in}(I(\overline{X}_w))$ under any term order, by degeneration in Theorem 1.7.1. Moreover, if the term order is antidiagonal, then $J_w \subseteq \text{in}(I_w) \subseteq \text{in}(I(\overline{X}_w))$, whence $J_w = \text{in}(I_w) = \text{in}(I(\overline{X}_w))$ by Lemma 1.7.5 with $I' = \text{in}(I(\overline{X}_w))$ and $J = J_w$. The primality of I_w in Theorem A follows because $\text{in}(I_w) = \text{in}(I(\overline{X}_w))$ and $I_w \subseteq I(\overline{X}_w)$. The Gröbner basis statement in Theorem B is an immediate consequence. Hilbert series are preserved under taking initial ideals, so the K -polynomials and multidegrees of

$\{\mathbf{k}[\mathbf{z}]/I_w\}_{w \in S_n}$ satisfy the Demazure and divided difference recursions, proving Theorem A.

The last sentence of Theorem B is Theorem 3.8.5. Using that, we conclude the remaining statement in Theorem B, namely shellability and Cohen–Macaulayness, by applying Theorem E to Example 1.8.3. Mitosis in Theorem C comes from applying Theorem 3.8.5 to the mitosis in Theorem 3.7.9.

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REFERENCES

- [Abh88] S. S. ABHYANKAR, *Enumerative Combinatorics of Young Tableaux*, Marcel Dekker Inc., New York, 1988.
- [AK89] S. ABHYANKAR and D. M. KULKARNI, On Hilbertian ideals, *Linear Algebra Appl.* **116** (1989), 53–79.
- [BB82] W. BORHO and J.-L. BRYLINSKI, Differential operators on homogeneous spaces. I. Irreducibility of the associated variety for annihilators of induced modules, *Invent. Math.* **69** (1982), 437–476.
- [BB85] ———, Differential operators on homogeneous spaces. III. Characteristic varieties of Harish–Chandra modules and of primitive ideals, *Invent. Math.* **80** (1985), 1–68.
- [BB93] N. BERGERON and S. BILLEY, RC-graphs and Schubert polynomials, *Experimental Math.* **2** (1993), 257–269.
- [BC98] W. BRUNS and A. CONCA, KRS and powers of determinantal ideals, *Compositio Math.* **111** (1998), 111–122.
- [BC01] ———, KRS and determinantal ideals, in *Geometric and Combinatorial Aspects of Commutative Algebra* (Messina, 1999), 67–87, *Lecture Notes in Pure and Appl. Math.* **217**, Dekker, New York, 2001.
- [Ber92] N. BERGERON, A combinatorial construction of the Schubert polynomials, *J. Combin. Theory Ser. A* **60** (1992), 168–182.
- [BGG73] I. N. BERNŠTEĪN, I. M. GEL’FAND, and S. I. GEL’FAND, Schubert cells, and the cohomology of the spaces G/P , *Uspehi Mat. Nauk* **28** (1973), 3–26.
- [BH93] W. BRUNS and J. HERZOG, *Cohen-Macaulay Rings*, *Cambridge Studies in Advanced Math.* **39**, Cambridge Univ. Press, Cambridge, 1993.
- [BJS93] S. C. BILLEY, W. JOCKUSCH, and R. P. STANLEY, Some combinatorial properties of Schubert polynomials, *J. Algebraic Combin.* **2** (1993), 345–374.
- [BKL85] A. BJÖRNER, B. KORTE, and L. LOVÁSZ, Homotopy properties of greedoids, *Adv. in Appl. Math.* **6** (1985), 447–494.
- [BL00] S. BILLEY and V. LAKSHMIBAI, *Singular Loci of Schubert Varieties*, Birkhäuser Boston Inc., Boston, MA, 2000.
- [Bor53] A. BOREL, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. of Math.* **57** (1953), 115–207.

- [BP79] L. J. BILLERA and J. S. PROVAN, A decomposition property for simplicial complexes and its relation to diameters and shellings, *Second International Conference on Combinatorial Mathematics* (New York, 1978), New York Acad. Sci., New York, 1979, pp. 82–85.
- [BS02] N. BERGERON and F. SOTTILE, A Pieri-type formula for isotropic flag manifolds, *Trans. Amer. Math. Soc.* **354** (2002), 2659–2705 (electronic).
- [BW82] A. BJÖRNER and M. WACHS, Bruhat order of Coxeter groups and shellability, *Adv. in Math.* **43** (1982), 87–100.
- [CGG90] L. CANIGLIA, J. A. GUCCIONE, and J. J. GUCCIONE, Ideals of generic minors, *Comm. Algebra* **18** (1990), 2633–2640.
- [CH94] A. CONCA and J. HERZOG, On the Hilbert function of determinantal rings and their canonical module, *Proc. Amer. Math. Soc.* **122** (1994), 677–681.
- [CH97] ———, Ladder determinantal rings have rational singularities, *Adv. Math.* **132** (1997), 120–147.
- [Con95] A. CONCA, Ladder determinantal rings, *J. Pure Appl. Algebra* **98** (1995), 119–134.
- [Dem74] M. DEMAZURE, Désingularisation des variétés de Schubert généralisées, *Ann. Sci. École Norm. Sup.* **7** (1974), 53–88.
- [EG98] D. EDIDIN and W. GRAHAM, Equivariant intersection theory, *Invent. Math.* **131** (1998), 595–634.
- [Ehr34] C. EHRESMANN, Sur la topologie de certains espaces homogènes, *Ann. of Math.* **35** (1934), 396–443.
- [Eis95] D. EISENBUD, Commutative algebra, with a view toward algebraic geometry, *Graduate Texts in Math.* **150**, Springer-Verlag, New York, 1995.
- [ER98] J. A. EAGON and V. REINER, Resolutions of Stanley-Reisner rings and Alexander duality, *J. Pure Appl. Algebra* **130** (1998), 265–275.
- [FK94] S. FOMIN and A. N. KIRILLOV, Grothendieck polynomials and the Yang-Baxter equation, *Proc. Sixth Conference in Formal Power Series and Algebraic Combinatorics*, DIMACS, 1994, 183–190.
- [FK96] ———, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, *Discrete Math.* **153** (1996), 123–143, *Proc. Fifth Conf. Formal Power Series and Algebraic Combinatorics* (Florence, 1993).
- [FR02] L. FEHÉR and R. RIMÁNYI, Schur and Schubert polynomials as Thom polynomials–cohomology of moduli spaces, *Cent. Eur. J. Math.* **1** (2003), 418–434 (electronic).
- [FS94] S. FOMIN and R. P. STANLEY, Schubert polynomials and the nil-Coxeter algebra, *Adv. Math.* **103** (1994), 196–207.
- [Ful92] W. FULTON, Flags, Schubert polynomials, degeneracy loci, and determinantal formulas, *Duke Math. J.* **65** (1992), 381–420.
- [Ful98] ———, *Intersection Theory*, second ed., Springer-Verlag, New York, 1998.
- [GC50] I. M. GEL'FAND and M. L. CETLIN, Finite-dimensional representations of the group of unimodular matrices, *Dokl. Akad. Nauk SSSR* **71** (1950), 825–828.
- [Gho02] S. R. GHORPADE, Hilbert functions of ladder determinantal varieties, *Discrete Math.* **246** (2002), 131–175, Formal power series and algebraic combinatorics (Barcelona, 1999).
- [GL97] N. GONCIULEA and V. LAKSHMIBAI, Schubert varieties, toric varieties, and ladder determinantal varieties, *Ann. Inst. Fourier (Grenoble)* **47** (1997), 1013–1064.

- [GL00] N. GONCIULEA and V. LAKSHMIBAI, Singular loci of ladder determinantal varieties and Schubert varieties, *J. Algebra* **229** (2000), 463–497.
- [GM00] N. GONCIULEA and C. MILLER, Mixed ladder determinantal varieties, *J. Algebra* **231** (2000), 104–137.
- [GS83] V. GUILLEMIN and S. STERNBERG, The Gel’fand-Cetlin system and quantization of the complex flag manifolds, *J. Funct. Anal.* **52** (1983), 106–128.
- [HT92] J. HERZOG and N. TRUNG, Gröbner bases and multiplicity of determinantal and Pfaffian ideals, *Adv. Math.* **96** (1992), 1–37.
- [Hum90] J. E. HUMPHREYS, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.
- [Jos84] A. JOSEPH, On the variety of a highest weight module, *J. Algebra* **88** (1984), 238–278.
- [Jos97] ———, Orbital varieties, Goldie rank polynomials and unitary highest weight modules, in *Algebraic and Analytic Methods in Representation Theory* (Sonderborg, 1994), 53–98, *Perspect. Math.* **17**, Academic Press, San Diego, CA, 1997.
- [Kaz97] M. É. KAZARIAN, Characteristic classes of singularity theory, *The Arnold-Gelfand Mathematical Seminars*, Birkhäuser Boston, Boston, MA, 1997, pp. 325–340.
- [KiM00] A. N. KIRILLOV and T. MAENO, Quantum double Schubert polynomials, quantum Schubert polynomials and Vafa–Intriligator formula, *Discrete Math.* **217** (2000), 191–223, Formal power series and algebraic combinatorics (Vienna, 1997).
- [KnM04] A. KNUTSON and E. MILLER, Subword complexes in Coxeter groups, *Adv. in Math.* **184** (2004), 161–176.
- [Kog00] M. KOGAN, Schubert geometry of flag varieties and Gel’fand-Cetlin theory, Ph.D. thesis, Massachusetts Institute of Technology, 2000.
- [KoM03] M. KOGAN and E. MILLER, Toric degeneration of Schubert varieties and Gel’fand–Tsetlin polytopes, *Adv. in Math.*, in press; [arXiv:math.AG/0303208](https://arxiv.org/abs/math/0303208)
- [Koh91] A. KOHNERT, Weintrauben, Polynome, Tableaux, Bayreuth. Math. Schr. (1991), no. 38, 1–97, Dissertation, Universität Bayreuth, Bayreuth, 1990.
- [KP99] C. KRATTENTHALER and M. PROHASKA, A remarkable formula for counting nonintersecting lattice paths in a ladder with respect to turns, *Trans. Amer. Math. Soc.* **351** (1999), 1015–1042.
- [KS95] M. KALKBRENER and B. STURMFELS, Initial complexes of prime ideals, *Adv. Math.* **116** (1995), 365–376.
- [Las90] A. LASCoux, Anneau de Grothendieck de la variété de drapeaux, in *The Grothendieck Festschrift*, Vol. III, Birkhäuser Boston, Boston, MA, 1990, pp. 1–34.
- [LS82a] A. LASCoux and M.-P. SCHÜTZENBERGER, Polynômes de Schubert, *C. R. Acad. Sci. Paris Sér. I Math.* **294** (1982), 447–450.
- [LS82b] ———, Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux, *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982), 629–633.
- [Mac91] I. G. MACDONALD, *Notes on Schubert Polynomials*, Publications du LACIM, Université du Québec à Montréal, 1991.
- [Mar02] J. MARTIN, Graph varieties, Ph.D. thesis, University of California at San Diego, 2002.
- [Mil03] E. MILLER, Mitosis recursion for coefficients of Schubert polynomials, *J. Combin. Theory Ser. A* **101** (2003), 223–235.

- [MS96] J. V. MOTWANI and M. A. SOHONI, Divisor class groups of ladder determinantal varieties, *J. Algebra* **186** (1996), 338–367.
- [Mul89] S. B. MULAY, Determinantal loci and the flag variety, *Adv. Math.* **74** (1989), 1–30.
- [Ram85] A. RAMANATHAN, Schubert varieties are arithmetically Cohen-Macaulay, *Invent. Math.* **80** (1985), 283–294.
- [Ros89] W. ROSSMANN, Equivariant multiplicities on complex varieties, Orbites unipotentes et représentations, III, *Astérisque* **11**, No. 173–174 (1989), 313–330.
- [RR85] S. RAMANAN and A. RAMANATHAN, Projective normality of flag varieties and Schubert varieties, *Invent. Math.* **79** (1985), 217–224.
- [Stu90] B. STURMFELS, Gröbner bases and Stanley decompositions of determinantal rings, *Math. Z.* **205** (1990), 137–144.
- [Tot99] B. TOTARO, The Chow ring of a classifying space, in *Algebraic K-theory* (Seattle, WA, 1997), Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.
- [Win99] R. WINKEL, Diagram rules for the generation of Schubert polynomials, *J. Combin. Theory Ser. A* **86** (1999), 14–48.

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