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The causal structure of microlocalized rough Einstein metrics

By Sergiu Klainerman and Igor Rodnianski

Abstract

This is the second in a series of three papers in which we initiate the study of very rough solutions to the initial value problem for the Einstein vacuum equations expressed relative to wave coordinates. By very rough we mean solutions which cannot be constructed by the classical techniques of energy estimates and Sobolev inequalities. In this paper we develop the geometric analysis of the Eikonal equation for microlocalized rough Einstein metrics. This is a crucial step in the derivation of the decay estimates needed in the first paper.

1. Introduction

This is the second in a series of three papers in which we initiate the study of *very rough* solutions of the Einstein vacuum equations. By very rough we mean solutions which cannot be dealt with by the classical techniques of energy estimates and Sobolev inequalities. In fact in this work we develop and take advantage of Strichartz-type estimates. The result, stated in our first paper [Kl-Ro1], is in fact optimal with respect to the full potential of such estimates.[1] We recall below our main result:

Theorem 1.1 (Main Theorem). *Let* $\mathbf{g}$ *be a classical solution*[2] *of the Einstein equations*

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0 \tag{1}$$

expressed[3] *relative to wave coordinates* x^α,

$$\square_{\mathbf{g}} x^\alpha = \frac{1}{|\mathbf{g}|}\partial_\mu(\mathbf{g}^{\mu\nu}|\mathbf{g}|\partial_\nu)x^\alpha = 0. \tag{2}$$

[1]To go beyond our result will require the development of bilinear techniques for the Einstein equations; see the discussion in the introduction to [Kl-Ro1].

[2]We denote by $R_{\alpha\beta}$ the Ricci curvature of $\mathbf{g}$.

[3]In wave coordinates the Einstein equations take the reduced form $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{g}_{\mu\nu} = N_{\mu\nu}(\mathbf{g}, \partial\mathbf{g})$ with N quadratic in the first derivatives $\partial\mathbf{g}$ of the metric.

Assume that on the initial spacelike hyperplane Σ given by $t = x^0 = 0$,

$$\nabla \mathbf{g}_{\alpha\beta}(0) \in H^{s-1}(\Sigma)\,, \quad \partial_t \mathbf{g}_{\alpha\beta}(0) \in H^{s-1}(\Sigma)$$

with ∇ denoting the gradient with respect to the space coordinates x^i, $i = 1, 2, 3$ and H^s the standard Sobolev spaces. Also assume that $\mathbf{g}_{\alpha\beta}(0)$ is a continuous Lorentz metric and $\sup_{|x|=r} |\mathbf{g}_{\alpha\beta}(0) - \mathbf{m}_{\alpha\beta}| \longrightarrow 0 \quad as \quad r \longrightarrow \infty$, where $|x| = (\sum_{i=1}^{3} |x^i|^2)^{\frac{1}{2}}$ and $\mathbf{m}_{\alpha\beta}$ is the Minkowski metric.

Then[4] the time T of existence depends in fact only on the size of the norm $\|\partial \mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}_{(\Sigma)}} = \|\nabla \mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}_{(\Sigma)}} + \|\partial_t \mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}_{(\Sigma)}}$, for any fixed $s > 2$.

In [Kl-Ro1] we have given a detailed proof of the theorem by relying heavily on a result, which we have called the Asymptotics Theorem, concerning the geometric properties of the causal structure of appropriately microlocalized rough Einstein metrics. This result, which is the focus of this paper, is of independent interest as it requires the development of new geometric and analytic methods to deal with characteristic surfaces of the Einstein metrics.

More precisely we study the solutions, called optical functions, of the Eikonal equation

$$H^{\alpha\beta}_{(\lambda)} \partial_\alpha u \partial_\beta u = 0, \tag{3}$$

associated to the family of regularized Lorentz metrics $H_{(\lambda)}$, $\lambda \in 2^{\mathbb{N}}$, defined, starting with an $H^{2+\varepsilon}$ Einstein metric $\mathbf{g}$, by the formula

$$H_{(\lambda)} = P_{<\lambda} \mathbf{g}(\lambda^{-1} t, \lambda^{-1} x) \tag{4}$$

where[5] $P_{<\lambda}$ is an operator which cuts off all the frequencies above[6] λ.

The importance of the eikonal equation (3) in the study of solutions to wave equations on a background Lorentz metric H is well known. It is mainly used, in the geometric optics approximation, to construct parametrices associated to the corresponding linear operator $\square_H$. In particular it has played a fundamental role in the recent works of Smith[Sm], Bahouri-Chemin [Ba-Ch1], [Ba-Ch2] and Tataru [Ta1], [Ta2] concerning rough solutions to linear and nonlinear wave equations. Their work relies indeed on parametrices defined with the help of specific families of optical functions corresponding to null

[4] We assume however that T stays sufficiently small, e.g. $T \leq 1$. This a purely technical assumption which one should be able to remove.

[5] More precisely, for a given function of the spatial variables $x = x^1, x^2, x^3$, the Littlewood Paley projection $P_{<\lambda} f = \sum_{\mu < \frac{1}{2}\lambda} P_\mu f, \quad P_\mu f = \mathcal{F}^{-1}\big(\chi(\mu^{-1}\xi)\hat{f}(\xi)\big)$ with χ supported in the unit dyadic region $\frac{1}{2} \leq |\xi| \leq 2$.

[6] The definition of the projector $P_{<\lambda}$ in [Kl-Ro1] was slightly different from the one we are using in this paper. There $P_{<\lambda}$ removed all the frequencies above $2^{-M_0}\lambda$ for some sufficiently large constant M_0. It is clear that a simple rescaling can remedy this discrepancy.

hyperplanes. In [Kl], [Kl-Ro], and also [Kl-Ro1] which do not rely on specific parametrices, a special optical function, corresponding to null cones with vertices on a timelike geodesic, was used to construct an almost conformal Killing vectorfield.

The main message of our paper is that optical functions associated to Einstein metrics, or microlocalized versions of them, have better properties. This fact was already recognized in [Ch-Kl] where the construction of an optical function normalized at infinity played a crucial role in the proof of the global nonlinear stability of the Minkowski space. A similar construction, based on two optical functions, can be found in [Kl-Ni]. Here, we take the use of the special structure of the Einstein equations one step further by deriving unexpected regularity properties of optical functions which are essential in the proof of the Main Theorem. It was well known (see [Ch-Kl], [Kl], [Kl-Ro]) that the use of Codazzi equations combined with the Raychaudhuri equation for the $\mathrm{tr}\chi$, the trace of null second fundamental form χ, leads to the improved estimate for the first angular derivatives of the traceless part of χ. A similar observation holds for another null component of the Hessian of the optical function, η. The role of the Raychaudhuri equation is taken by the transport equation for the "mass aspect function" μ.

In this paper we show, using the structure of the curvature terms in the main equations, how to derive improved regularity estimates for the undifferentiated quantities $\hat{\chi}$ and η. In particular, in the case of the estimates for η we are led to introduce a new nonlocal quantity $\not{\mu}$ tied to μ via a Hodge system.

The properties of the optical function are given in detail in the statement of the Asymptotics Theorem. We shall give a precise statement of it in Section 2 after we introduce a few essential definitions. The paper is organized as follows:

In Section 2 we construct an optical function u, constant on null cones with vertices on a fixed timelike geodesic, and describe our basic geometric entities associated to it. We define the surfaces $S_{t,u}$, the canonical null pair $L, \underline{L}$ and the associated Ricci coefficients. This allows us to give a precise statement of our main result, the Asymptotic Theorem 2.5.

In Section 3 we derive the structure equations for the Ricci coefficients. These equations are a coupled system of the transport and Codazzi equations and are fundamental for the proof of Theorem 2.5.

In Section 4 we obtain some crucial properties of the components of the Riemann curvature tensor $\mathbf{R}_{\alpha\beta\gamma\delta}$.

The remaining sections are occupied with the proof of the Asymptotics Theorem. We give a detailed description of their content and the strategy of the proof in Section 5.

The paper is essentially self-contained. From the first paper in this series [Kl-Ro1] we only need the result of Proposition 2.4 (Background Estimates) which in any case can be easily derived from the the metric hypothesis (5), the

Ricci condition (1), and the definition (4). We do however rely on the following results:

- Isoperimetric and trace inequalities, see Proposition 6.16.
- Calderon-Zygmund type estimates, see Proposition 6.20.
- Theorem 8.1.

The proof of the important Theorem 8.1 is delayed to our third paper in the series [Kl-Ro2]. The first two ingredients are standard modifications of the classical isoperimetric and Calderon-Zygmund estimates; see [Kl-Ro].

We recall our metric hypothesis (referred in [Kl-Ro1, §2] as the bootstrap hypothesis) on the components of $\mathbf{g}$ relative to our wave coordinates x^α.

Metric Hypothesis.

$$\|\partial\mathbf{g}\|_{L^\infty_{[0,T]}H^{1+\gamma}} + \|\partial\mathbf{g}\|_{L^2_{[0,T]}L^\infty_x} \le B_0, \tag{5}$$

for some fixed $\gamma > 0$.

2. Geometric preliminaries

We start by recalling the basic geometric constructions associated with a Lorentz metric $H = H_{(\lambda)}$. Recall, see [Kl-Ro1, §2], that the parameters of the Σ_t foliation are given by n, v, the induced metric h and the second fundamental form k_{ij}, according to the decomposition,

$$H = -n^2dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt), \tag{6}$$

with h_{ij} the induced Riemannian metric on Σ_t, n the lapse and $v = v^i\partial_i$ the shift of H. Denoting by T the unit, future oriented, normal to Σ_t and k the second fundamental form $k_{ij} = -\langle \mathbf{D}_i T, \partial_j\rangle$ we find,

$$\begin{aligned} &\partial_t = nT + v, \qquad \langle \partial_t, v\rangle = 0,\\ &k_{ij} = -\frac{1}{2}\mathcal{L}_T H_{ij} = -\frac{1}{2}n^{-1}(\partial_t h_{ij} - \mathcal{L}_v h_{ij}) \end{aligned} \tag{7}$$

with $\mathcal{L}_X$ denoting the Lie derivative with respect to the vectorfield X. We also have the following; see [Kl-Ro1, §§2, 8]:

$$c|\xi|^2 \le h_{ij}\xi^i\xi^j \le c^{-1}|\xi|^2, \qquad c \le n^2 - |v|_h^2 \tag{8}$$

for some $c > 0$. Also $n, |v| \lesssim 1$.

The time axis is defined as the integral curve of the forward unit normal T to the hypersurfaces Σ_t. The point Γ_t is the intersection between Γ and Σ_t.

Definition 2.1. The optical function u is an outgoing solution of the Eikonal equation

$$H^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 \tag{9}$$

with initial conditions $u(\Gamma_t) = t$ on the time axis.

The level surfaces of u, denoted C_u, are outgoing null cones with vertices on the time axis. Clearly,

$$T(u) = |\nabla u|_h \tag{10}$$

where h is the induced metric on Σ_t, $|\nabla u|_h^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t.

We denote by $S_{t,u}$ the surfaces of intersection between Σ_t and C_u. They play a fundamental role in our discussion.

Definition 2.2 (Canonical null pair).

$$L = bL' = T + N, \qquad \underline{L} = 2T - L = T - N. \tag{11}$$

Here $L' = -H^{\alpha\beta}\partial_\beta u \partial_\alpha$ is the geodesic null generator of C_u, b is the *lapse of the null foliation* (or shortly null lapse)

$$b^{-1} = -\langle L', T\rangle = T(u), \tag{12}$$

and N is the exterior unit normal, along Σ_t, to the surfaces $S_{t,u}$.

Definition 2.3. A null frame e_1, e_2, e_3, e_4 at a point $p \in S_{t,u}$ consists, in addition to the null pair $e_3 = \underline{L}, e_4 = L$, of *arbitrary orthonormal* vectors e_1, e_2 tangent to $S_{t,u}$. All the estimates in this paper are in fact local and independent of the choice of a particular frame. We do not need to worry that these frames cannot be globally defined.

Definition 2.4 (Ricci coefficients). Let e_1, e_2, e_3, e_4 be a null frame on $S_{t,u}$ as above. The following tensors on $S_{t,u}$

$$\begin{aligned} \chi_{AB} &= \langle \mathbf{D}_A e_4, e_B\rangle, & \underline{\chi}_{AB} &= \langle \mathbf{D}_A e_3, e_B\rangle, \\ \eta_A &= \frac{1}{2}\langle \mathbf{D}_3 e_4, e_A\rangle, & \underline{\eta}_A &= \frac{1}{2}\langle \mathbf{D}_4 e_3, e_A\rangle, \\ \underline{\xi}_A &= \frac{1}{2}\langle \mathbf{D}_3 e_3, e_A\rangle \end{aligned} \tag{13}$$

are called the Ricci coefficients associated to our canonical null pair.

We decompose χ and $\underline{\chi}$ into their trace and traceless components.

$$\mathrm{tr}\chi = H^{AB}\chi_{AB}, \qquad \mathrm{tr}\underline{\chi} = H^{AB}\underline{\chi}_{AB}, \tag{14}$$

$$\hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2}\mathrm{tr}\chi H_{AB}, \qquad \underline{\hat{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2}\mathrm{tr}\underline{\chi} H_{AB}. \tag{15}$$

We define s to be the affine parameter of L, i.e. $L(s) = 1$ and $s = 0$ on the time axis Γ_t. In [Kl-Ro], where $n = 1$ we had $s = t - u$. Such a simple relation does not hold in this case; we have instead, along any fixed C_u,

$$\frac{dt}{ds} = n^{-1}. \tag{16}$$

We shall also introduce the area $A(t,u)$ of the 2-surface $S(t,u)$ and the radius $r(t,u)$ defined by

$$A = 4\pi r^2. \tag{17}$$

Along a given C_u we have[7]

$$\frac{\partial A}{\partial t} = \int_S n\mathrm{tr}\chi.$$

Therefore, along C_u,

$$\frac{dr}{dt} = \frac{r}{2}\overline{n\mathrm{tr}\chi} \tag{18}$$

where, given a function f, we denote by $\bar{f}(t,u)$ its average on $S_{t,u}$. Thus

$$\bar{f}(t,u) = \frac{1}{4\pi r^2}\int_{S_{t,u}} f.$$

The following *Ricci equations* can also be easily derived (see [Kl-Ro]). They express the covariant derivatives $\mathbf{D}$ of the null frame $(e_A)_{A=1,2}, e_3, e_4$ relative to itself.

$$\begin{aligned}
&\mathbf{D}_A e_4 = \chi_{AB} e_B - k_{AN} e_4, && \mathbf{D}_A e_3 = \underline{\chi}_{AB} e_B + k_{AN} e_3,\\
&\mathbf{D}_4 e_4 = -\bar{k}_{NN} e_4, && \mathbf{D}_4 e_3 = 2\underline{\eta}_A e_A + \bar{k}_{NN} e_3,\\
&\mathbf{D}_3 e_4 = 2\eta_A e_A + \bar{k}_{NN} e_4, && \mathbf{D}_3 e_3 = 2\underline{\xi}_A e_A - \bar{k}_{NN} e_3,\\
&\mathbf{D}_4 e_A = \not{\mathbf{D}}_4 e_A + \underline{\eta}_A e_4, && \mathbf{D}_3 e_A = \not{\mathbf{D}}_3 e_A + \eta_A e_3 + \underline{\xi}_A e_4,\\
&\mathbf{D}_B e_A = \not{\nabla}_B e_A + \frac{1}{2}\chi_{AB}\, e_3 + \frac{1}{2}\underline{\chi}_{AB}\, e_4
\end{aligned} \tag{19}$$

where, $\not{\mathbf{D}}_3$, $\not{\mathbf{D}}_4$ denote the projection on $S_{t,u}$ of $\mathbf{D}_3$ and $\mathbf{D}_4$, $\not{\nabla}$ denotes the induced covariant derivative on $S_{t,u}$ and, for every vector X tangent to Σ_t,

$$\bar{k}_{NX} = k_{NX} - n^{-1}\nabla_X n. \tag{20}$$

Thus $\bar{k}_{NN} = k_{NN} - n^{-1}N(n)$ and $\bar{k}_{AN} = k_{AN} - n^{-1}\nabla_A n$. Also,

$$\begin{aligned}
\underline{\chi}_{AB} &= -\chi_{AB} - 2k_{AB},\\
\underline{\eta}_A v &= -\bar{k}_{AN},\\
\underline{\xi}_A &= k_{AN} + n^{-1}\nabla_A n - \eta_A
\end{aligned} \tag{21}$$

[7]This follows by writing the metric on $S_{t,u}$ in the form $\gamma_{AB}(s(t,\theta),\theta)d\theta^a d\theta^B$, relative to angular coordinates θ^1, θ^2, and its area $A(t,u) = \int\sqrt{\gamma}d\theta^1\wedge d\theta^2$. Thus, $\frac{d}{dt}A = \int\frac{1}{2}\gamma^{AB}\frac{d}{dt}\gamma_{AB}\sqrt{\gamma}d\theta^1\wedge d\theta^2$. On the other hand $\frac{d}{ds}\gamma_{AB} = 2\chi_{AB}$ and $\frac{ds}{dt} = n$.

and,

$$\eta_A = b^{-1}\nabla\!\!\!/_A b + k_{AN}. \tag{22}$$

The formulas (19), (21) and (22) can be checked in precisely the same manner as (2.45–2.53) in [Kl-Ro]. The only difference occurs because $\mathbf{D}_T T$ no longer vanishes. We have in fact, relative to any orthonormal frame e_i on Σ_t,

$$\mathbf{D}_T T = n^{-1} e_i(n) e_i. \tag{23}$$

To check (23) observe that we can introduce new local coordinates $\bar{x}^i = \bar{x}^i(t, x)$ on Σ_t which preserve the lapse n while making the shift V to vanish identically. Thus $\partial_t = nT$ and therefore, for an arbitrary vectorfield X tangent to Σ_t, we easily calculate, $\langle \mathbf{D}_T T, X\rangle = n^{-2} X^i \langle \mathbf{D}_{\partial_t}\partial_t, \partial_i\rangle = -n^{-2} X^i \langle \partial_t, \mathbf{D}_{\partial_t}\partial_i\rangle = -n^{-2} X^i \langle \partial_t, \mathbf{D}_{\partial_i}\partial_t\rangle = -n^{-2} X^i \frac{1}{2}\partial_i \langle \partial_t, \partial_t\rangle = n^{-2} X^i \frac{1}{2}\partial_i(n^2) = n^{-1} X(n)$.

Equations (21) indicate that the only independent geometric quantities, besides n, v and k are $\mathrm{tr}\chi, \hat{\chi}, \eta$. We now state the main result of our paper giving the precise description of the Ricci coefficients. Note that a subset of these estimates was stated in Theorem 4.5 of [Kl-Ro1].

Theorem 2.5. *Let* $\mathbf{g}$ *be an Einstein metric obeying the Metric Hypothesis* (5) *and* $H = H_{(\lambda)}$ *be the family of the regularized Lorentz metrics defined according to* (4). *Fix a sufficiently large value of the dyadic parameter* λ *and consider, corresponding to* $H = H_{(\lambda)}$, *the optical function* u *defined above. Let* $\mathcal{I}_0^+$ *be the future domain of the origin on* Σ_0. *Then for any* $\varepsilon_0 > 0$, *such that* $5\varepsilon_0 < \gamma$ *with* γ *from* (5), *the optical function* u *can be extended throughout the region* $\mathcal{I}_0^+ \cap ([0, \lambda^{1-8\varepsilon_0}] \times \mathbb{R}^3)$ *and there the Ricci coefficients* $\mathrm{tr}\chi$, $\hat{\chi}$, *and* η *satisfy the following estimates*:

$$\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L_t^2 L_x^\infty} + \|\hat{\chi}\|_{L_t^2 L_x^\infty} + \|\eta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2} - 3\varepsilon_0}, \tag{24}$$

$$\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^q(S_{t,u})} + \|\hat{\chi}\|_{L^q(S_{t,u})} + \|\eta\|_{L^q(S_{t,u})} \lesssim \lambda^{-3\varepsilon_0}, \tag{25}$$

with $2 \le q \le 4$. *In the estimate* (118) *the function* $\frac{2}{r}$ *can be replaced with* $\frac{2}{n(t-u)}$. *In addition, in the exterior region* $r \ge t/2$,

$$\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-4\varepsilon_0}, \qquad \|\hat{\chi}\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-\varepsilon_0} + \|\partial H(t)\|_{L_x^\infty}, \tag{26}$$
$$\|\eta\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1} + \lambda^{-\varepsilon_0} t^{-1} + \lambda^{\varepsilon}\|\partial H(t)\|_{L_x^\infty}$$

where the last estimate holds for an arbitrary positive ε, $\varepsilon < \varepsilon_0$. *Also, there exist the following estimates for the derivatives of* $\mathrm{tr}\chi$:

$$\|\sup_{r\ge\frac{t}{2}}\|\underline{L}\left(\mathrm{tr}\chi-\frac{2}{r}\right)\|_{L^2(S_{t,u})}\|_{L^1_t} + \|\sup_{r\ge\frac{t}{2}}\|\underline{L}\left(\mathrm{tr}\chi-\frac{2}{n(t-u)}\right)\|_{L^2(S_{t,u})}\|_{L^1_t}\le\lambda^{-3\varepsilon_0}, \tag{27}$$

$$\|\sup_{r\ge\frac{t}{2}}\|\nabla\!\!\!/\,\mathrm{tr}\chi\|_{L^2(S_{t,u})}\|_{L^1_t} + \|\sup_{r\ge\frac{t}{2}}\|\nabla\!\!\!/\left(\mathrm{tr}\chi-\frac{2}{n(t-u)}\right)\|_{L^2(S_{t,u})}\|_{L^1_t}\le\lambda^{-3\varepsilon_0}. \tag{28}$$

In addition, there are weak estimates of the form,

$$\sup_{u\le\frac{t}{2}}\left\|(\nabla\!\!\!/,\underline{L})\left(\mathrm{tr}\chi-\frac{2}{n(t-u)}\right)\right\|_{L^\infty(S_{t,u})}\lesssim\lambda^C \tag{29}$$

for some large value of C.

The inequalities $\lesssim$ indicate that the bounds hold with some universal constants including the constant B_0 from (5).

3. Null structure equations

In the proof of Theorem 2.5 we rely on the system of equations satisfied by the Ricci coefficients χ, η. Below we write down our main structural equations. Their derivation proceeds in exactly the same way as in [Kl-Ro] (see Propositions 2.2 and 2.3) from the formulas (19) above.

PROPOSITION 3.1. *The components* $\mathrm{tr}\chi$, $\hat\chi$, η *and the lapse* b *verify the following equations*:[8]

$$L(b)=-b\,\bar{k}_{NN}, \tag{30}$$

$$L(\mathrm{tr}\chi)+\frac{1}{2}(\mathrm{tr}\chi)^2=-|\hat\chi|^2-\bar{k}_{NN}\mathrm{tr}\chi-\mathbf{R}_{44}, \tag{31}$$

$$D\!\!\!\!/_4\hat\chi_{AB}+\frac{1}{2}\mathrm{tr}\chi\hat\chi_{AB}=-\bar{k}_{NN}\hat\chi_{AB}-\hat\alpha_{AB}, \tag{32}$$

$$D\!\!\!\!/_4\eta_A+\frac{1}{2}(\mathrm{tr}\chi)\eta_A=-(k_{BN}+\eta_B)\hat\chi_{AB}-\frac{1}{2}\mathrm{tr}\chi k_{AN}-\frac{1}{2}\beta_A. \tag{33}$$

Here $\hat\alpha_{AB}=\mathbf{R}_{4A4B}-\frac{1}{2}\mathbf{R}_{44}\delta_{AB}$ *and* $\beta_A=\mathbf{R}_{4A34}$. *Also, when*

$$\mu=\underline{L}(\mathrm{tr}\chi)-\frac{1}{2}(\mathrm{tr}\chi)^2-\big(k_{NN}+n^{-1}\nabla_N n\big)\mathrm{tr}\chi, \tag{34}$$

---

[8] which can be interpreted as transport equations along the null geodesics generated by L. Indeed observe that if an S tangent tensorfield Π satisfies the homogeneous equation $D\!\!\!\!/_4\Pi=0$ then Π is parallel transported along null geodesics.

there is the equality

$$
\begin{aligned}
(35)\quad & L(\mu) + \mathrm{tr}\chi\mu \\
&= 2(\underline{\eta}_A - \eta_A)\nabla\!\!\!/_A(\mathrm{tr}\chi) - 2\hat{\chi}_{AB}\Big(2\nabla\!\!\!/_A\eta_B + 2\eta_A\eta_B \\
&\quad +\bar{k}_{NN}\hat{\chi}_{AB} + \mathrm{tr}\chi\hat{\chi}_{AB} + \hat{\chi}_{AC}\hat{\chi}_{CB} + 2k_{AC}\chi_{CB} + \mathbf{R}_{B43A}\Big) \\
&\quad -\underline{L}(\mathbf{R}_{44}) + (2k_{NN} - 4n^{-1}\nabla_N n))\big(\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2 - \bar{k}_{NN}\mathrm{tr}\chi - \mathbf{R}_{44}\big) \\
&\quad +4\bar{k}_{NN}^2\mathrm{tr}\chi + (\mathrm{tr}\chi + 4\bar{k}_{NN})(|\hat{\chi}|^2 + \mathbf{R}_{44}) \\
&\quad -\mathrm{tr}\chi\bigg(2(k_{AN} - \eta_A)n^{-1}\nabla_A n - .2|n^{-1}N(n)|^2 + \mathbf{R}_{4343} + 2k_{Nm}k_N^m\bigg).
\end{aligned}
$$

Remark 3.2. Equation (31) is known as the Raychaudhuri equation in the relativity literature; see e.g. [Ha-El].

Remark 3.3. Observe that our definition of μ differs from that in [Kl-Ro]. Indeed there we had, instead of μ,

$$\tilde{\mu} = \underline{L}(\mathrm{tr}\chi) - \frac{1}{2}(\mathrm{tr}\chi)^2 - 3\bar{k}_{NN}\mathrm{tr}\chi$$

and the corresponding transport equation:

$$
\begin{aligned}
(36)\quad L(\tilde{\mu}) + \mathrm{tr}\chi\tilde{\mu} &= 2(\underline{\eta}_A - \eta_A)\nabla\!\!\!/_A(\mathrm{tr}\chi) - 2\hat{\chi}_{AB}\Big(2\nabla\!\!\!/_A\eta_B + 2\eta_A\eta_B \\
&\quad +\bar{k}_{NN}\hat{\chi}_{AB} + \mathrm{tr}\chi\hat{\chi}_{AB} + \hat{\chi}_{AC}\hat{\chi}_{CB} + 2k_{AC}\chi_{CB} + \mathbf{R}_{B43A}\Big) \\
&\quad -\underline{L}(\mathbf{R}_{44}) - \underline{L}(\bar{k}_{NN})\mathrm{tr}\chi - 3L(\bar{k}_{NN})\mathrm{tr}\chi + 4\bar{k}_{NN}^2\mathrm{tr}\chi \\
&\quad +(\mathrm{tr}\chi + 4\bar{k}_{NN})(|\hat{\chi}|^2 + \mathbf{R}_{44}).
\end{aligned}
$$

We obtain (35) from (36) as follows: The second fundamental form k verifies the equation (see formula (1.0.3a) in [Ch-Kl]),

$$\mathcal{L}_{nT}k_{ij} = -\nabla_i\nabla_j n + n(\mathbf{R}_{iTjT} - k_{im}k_j^m).$$

In particular,

$$\mathcal{L}_{nT}k_{NN} = -\nabla_N^2 n + n(\mathbf{R}_{NTNT} - k_{Nm}k_N^m).$$

Exploiting the definition of the Lie derivative $\mathcal{L}_{nT}$, we obtain

$$T(k_{NN}) + 2k(\nabla_N T, N) = -n^{-1}\nabla_N^2 n + (\mathbf{R}_{NTNT} - k_{Nm}k_N^m).$$

It then follows that

$$
\begin{aligned}
&\frac{1}{2}\underline{L}(k_{NN}) + \frac{1}{2}L(k_{NN}) - 2(k_{NN})^2 - 2(k_{AN})^2 \\
&\qquad\qquad = -n^{-1}\nabla_N^2 n + (\mathbf{R}_{NTNT} - k_{Nm}k_N^m).
\end{aligned}
$$

Therefore, since $L+\underline{L}=2T$, $L-\underline{L}=2N$,

$$\begin{aligned}\frac{1}{2}\underline{L}(k_{NN})-\frac{1}{2}\underline{L}\big(n^{-1}N(n)\big)&\\ &=-\frac{1}{2}L(k_{NN})-\frac{1}{2}L\big(n^{-1}N(n)\big)\\ &\quad+(\mathbf{R}_{NTNT}+k_{Nm}k_N^m)+n^{-1}(\nabla_N N)n-n^{-2}|N(n)|^2.\end{aligned}$$

Recall that $\bar{k}_{NN}=k_{NN}-n^{-1}N(n)$ and $\langle\nabla_N N,e_A\rangle=k_{AN}-\eta_A$. Thus

$$\begin{aligned}\underline{L}(\bar{k}_{NN})=&-L\big(k_{NN}+n^{-1}N(n)\big)+2(k_{AN}-\eta_A)n^{-1}\nabla_A n\\ &-2|n^{-1}N(n)|^2+\mathbf{R}_{4343}+2k_{Nm}k_N^m.\end{aligned}$$

Therefore taking $\mu=\underline{L}(\mathrm{tr}\chi)-\frac{1}{2}(\mathrm{tr}\chi)^2-(k_{NN}+n^{-1}N(n))\mathrm{tr}\chi$ we derive the desired transport equation (35).

Proposition 3.4. *The expressions* $(\mathrm{d}\!\!/\mathrm{iv}\,\hat{\chi})_A=\nabla\!\!\!\!/^B\hat{\chi}_{AB}$, $\mathrm{d}\!\!/\mathrm{iv}\,\eta=\nabla\!\!\!\!/^B\eta_B$ *and* $(\mathrm{c}\!\!/\mathrm{url}\,\eta)_{AB}=\nabla\!\!\!\!/_A\eta_B-\nabla\!\!\!\!/_B\eta_A$ *verify the following equations*:

$$(\mathrm{d}\!\!/\mathrm{iv}\,\hat{\chi})_A+\hat{\chi}_{AB}k_{BN}=\frac{1}{2}(\nabla\!\!\!\!/_A\mathrm{tr}\chi+k_{AN}\mathrm{tr}\chi)-\mathbf{R}_{B4AB},\tag{37}$$

$$\mathrm{d}\!\!/\mathrm{iv}\,\eta=\frac{1}{2}\bigg(\mu+2n^{-1}N(n)\mathrm{tr}\chi-2|\eta|^2-|\hat{\chi}|^2-2k_{AB}\chi_{AB}\bigg)-\frac{1}{2}\mathbf{R}_{B43A},\tag{38}$$

$$\mathrm{c}\!\!/\mathrm{url}\,\eta=\frac{1}{2}\varepsilon^{AB}k_{AC}\hat{\chi}_{CB}-\frac{1}{2}\varepsilon^{AB}\mathbf{R}_{B43A}.\tag{39}$$

We also have the Gauss equation,

$$2K=\hat{\chi}_{AB}\hat{\underline{\chi}}_{AB}-\frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}+\mathbf{R}_{ABAB}.\tag{40}$$

We add two useful commutation formulas.

Lemma 3.5. *Let* $\Pi_{\underline{A}}$ *be an* m*-covariant tensor tangent to the surfaces* $S_{t,u}$. *Then*,

(41)
$$\begin{aligned}\nabla\!\!\!\!/_B\mathcal{D}\!\!\!\!/_4\Pi_{\underline{A}}-\mathcal{D}\!\!\!\!/_4\nabla\!\!\!\!/_B\Pi_{\underline{A}}=&\,\chi_{BC}\nabla\!\!\!\!/_C\Pi_{\underline{A}}-n^{-1}\nabla\!\!\!\!/_B n\mathcal{D}\!\!\!\!/_4\Pi_{\underline{A}}\\ &+\sum_i(\chi_{A_iB}\bar{k}_{CN}-\chi_{BC}\bar{k}_{A_iN}+\mathbf{R}_{CA_i4B})\Pi_{A_1..\check{C}..A_m}.\end{aligned}$$

Also, for a scalar function f,

(42)
$$\nabla\!\!\!\!/_N\nabla\!\!\!\!/_A f-\nabla\!\!\!\!/_A\nabla\!\!\!\!/_N f=-\frac{3}{2}k_{AN}\mathbf{D}_4 f-(\eta_A+k_{AN})\mathbf{D}_3 f-(\chi_{AB}-\underline{\chi}_{AB})\nabla\!\!\!\!/_B f.$$

Proof. For simplicity we only provide the proof of the identity (42). The derivation of (41) is only slightly more involved (see [Ch-Kl], [Kl-Ro]). We have

$$\nabla\!\!\!/_N \nabla\!\!\!/_A f - \nabla\!\!\!/_A \nabla\!\!\!/_N f = [N, e_A] f - (\nabla\!\!\!/_N e_A) f = (\mathbf{D}_N e_A - \nabla\!\!\!/_N e_A) f - (\mathbf{D}_A N) f.$$

Now using the identity $N = \frac{1}{2}(e_4 - e_3)$ and the Ricci equations (19) we can easily infer (42). □

4. Special structure of the curvature tensor R

In this section we describe some remarkable decompositions[9] of the curvature tensor of the metric H. Given a system of coordinates[10] x^α relative to which H is a nondegenerate Lorentz metric with bounded components $H_{\alpha\beta}$, we define the coordinate dependent norm

$$|\partial H| = \max_{\alpha,\beta,\gamma} |\partial_\gamma H_{\alpha\beta}|. \tag{43}$$

A frame e_a, e_b, e_c, e_d is bounded, with respect to our given coordinate system, if all components of $e_a = e_a^\alpha \partial_\alpha$ are bounded.

Consider an arbitrary bounded frame e_a, e_b, e_c, e_d and $\mathbf{R}_{abcd}$ the components of the curvature tensor relative to it. Relative to any system of coordinates,

$$\mathbf{R}_{abcd} = e_a^\alpha e_b^\beta e_c^\gamma e_d^\delta (\partial^2_{\alpha\gamma} H_{\beta\delta} + \partial^2_{\beta\delta} H_{\alpha\gamma} - \partial^2_{\beta\gamma} H_{\alpha\delta} - \partial^2_{\alpha\delta} H_{\beta\gamma}). \tag{44}$$

Using our given coordinates x^α we introduce the flat Minkowski metric $m_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. We denote by $\overset{\circ}{\mathbf{D}}$ the corresponding flat connection. Using $\overset{\circ}{\mathbf{D}}$ we define the following tensor:

$$\pi(X, Y, Z) = \overset{\circ}{\mathbf{D}}_Z H(X, Y).$$

Thus in our local coordinates x^α, $\pi_{\alpha\beta\gamma} = \partial_\gamma H_{\alpha\beta}$.

PROPOSITION 4.1. *Relative to an arbitrary bounded frame e_a, e_b, e_c, e_d there is the following decomposition*:

$$\mathbf{R}_{abcd} = \mathbf{D}_a \pi_{bdc} + \mathbf{D}_b \pi_{acd} - \mathbf{D}_a \pi_{bcd} - \mathbf{D}_b \pi_{dac} + E_{abcd} \tag{45}$$

where the components of the tensor E are bounded pointwise by the square of the first derivatives of H. More precisely, since $|E| = \max_{a,b,c,d} |E_{abcd}| \approx \max_{\alpha,\beta,\gamma,\delta} |E_{\alpha\beta\gamma\delta}|$,

$$|E| \lesssim |\partial H|^2. \tag{46}$$

[9]The results of this section apply to an arbitrary Lorentz metric H.
[10]This applies to the original wave coordinates x^α.

Remark 4.2. It will be clear from the proof below that we can interchange the indices a, c and b, d in the formula above and obtain similar decompositions.

We show that each term appearing in (44) can be expressed in terms of a corresponding derivative of π plus terms of type E.

Consider the term $R_1 = e_a^\alpha e_b^\beta e_c^\gamma e_d^\delta \partial^2_{\alpha\delta} H_{\beta\gamma}$. We show that it can be expressed in the form $\mathbf{D}_a \pi_{bcd}$ plus terms of type E. Indeed,

$$\begin{aligned}
\mathbf{D}_a \pi_{bcd} &= e_a(\pi_{bcd}) - \pi_{\mathbf{D}_a bcd} - \pi_{b\mathbf{D}_a cd} - \pi_{bc\mathbf{D}_a d} \\
&= e_a^\alpha \partial_\alpha (e_d^\delta e_b^\beta e_c^\gamma \partial_\delta H_{\beta\gamma}) - \pi_{\mathbf{D}_a bcd} - \pi_{b\mathbf{D}_a cd} - \pi_{bc\mathbf{D}_a d} \\
&= R_1 + e_a^\alpha \partial_\alpha (e_d^\delta e_b^\beta e_c^\gamma) \partial_\delta H_{\beta\gamma} - \pi_{\mathbf{D}_a bcd} - \pi_{b\mathbf{D}_a cd} - \pi_{bc\mathbf{D}_a d} \\
&= R_1 + e_d^\delta e_a^\alpha \partial_\alpha (e_b^\beta) e_c^\gamma \partial_\delta H_{\beta\gamma} - \pi_{\mathbf{D}_a bcd} - \dots .
\end{aligned}$$

Now,

$$\pi_{\mathbf{D}_a bcd} = \mathring{\mathbf{D}}_d H(\mathbf{D}_a e_b, e_c) = e_d^\delta (\mathbf{D}_a e_b)^\beta e_c^\gamma \partial_\delta H_{\beta\gamma}.$$

Thus,

$$\mathbf{D}_a \pi_{bcd} = R_1 + e_d^\delta e_c^\gamma \partial_\delta H_{\beta\gamma} \big(e_a^\alpha \partial_\alpha (e_b^\beta) - (\mathbf{D}_a e_b)^\beta \big).$$

On the other hand

$$\begin{aligned}
(\mathbf{D}_a e_b)^\beta &= \langle \mathbf{D}_a e_b, \partial_\mu \rangle H^{\beta\mu} \\
&= e_a^\alpha \partial_\alpha (e_b^\beta) - \langle e_b, \mathbf{D}_a \partial_\mu \rangle H^{\beta\mu} - \langle e_b, \partial_\mu \rangle e_a^\alpha \partial_\alpha (H^{\beta\mu}).
\end{aligned}$$

Henceforth, we infer that,

$$R^{(1)}_{abcd} = \mathbf{D}_a \pi_{bcd} + E^{(1)}_{abcd}$$

with

$$E^{(1)} = e_d^\delta e_c^\gamma \partial_\delta H_{\beta\gamma} \big(\langle e_b, \mathbf{D}_a \partial_\mu \rangle H^{\beta\mu} + \langle e_b, \partial_\mu \rangle e_a^\alpha \partial_\alpha (H^{\beta\mu}) \big).$$

Since $\mathbf{D}_a \partial_\mu$ can be expressed in terms of the first derivatives[11] of H we conclude that $|E^{(1)}| \lesssim |\partial H|^2$ as desired. The other terms in the formula (44) can be handled in precisely the same way.

Remark 4.3. We will apply Proposition 4.1 to our metric H, wave coordinates x^α and our canonical null frames. We remark that our wave coordinates are nondegenerate relative to H, see (8), and any canonical null frame $e_4 = (T + N), e_3 = (T - N)$, e_A is bounded relative to x^α.

COROLLARY 4.4. *Relative to an arbitrary frame e_A on $S_{t,u}$,*

$$\mathbf{R}_{ABCD} = \nabla\!\!\!\!/_A \pi_{BDC} + \nabla\!\!\!\!/_B \pi_{ACD} - \nabla\!\!\!\!/_A \pi_{BCD} - \nabla\!\!\!\!/_B \pi_{DAC} + E_{ABCD} \tag{47}$$

[11] Recall that $\mathbf{D}_\beta \partial_\mu = \Gamma^\gamma_{\beta\mu} \partial_\gamma$ with Γ the standard Christoffel symbols of H.

where E is an error term of the type,

$$|E| \lesssim (|\partial H|^2 + |\chi||\partial H|)$$

and

$$|\pi| \lesssim |\partial H|.$$

COROLLARY 4.5. *There exist a scalar π, an S-tangent 2-tensor π_{AB} and 1-form E_A such that, the component R_{B4AB} admits the decomposition*

$$R_{B4AB} = \nabla\!\!\!/_A \pi + \nabla\!\!\!/^B \pi_{AB} + E_A.$$

Moreover,

$$\begin{aligned} |\pi| &\lesssim |\partial H|, \\ |E| &\lesssim (|\partial H|^2 + |\chi||\partial H|). \end{aligned}$$

COROLLARY 4.6. *There exist an S-tangent vector π_A and scalar E such that*

$$\varepsilon^{AB}\mathbf{R}_{AB34} = \mathrm{c}\!\!/\mathrm{url}\,\pi + E$$

and

$$\begin{aligned} |\pi| &\lesssim |\partial H| \\ |E| &\lesssim (|\partial H|^2 + |\chi||\partial H|). \end{aligned}$$

COROLLARY 4.7. *There exist S-tangent vectors $\pi_A^{(1)}, \pi_A^{(2)}$ and scalars $E^{(1)}, E^{(2)}$ such that*

$$\begin{aligned} \delta^{AB}\mathbf{R}_{A43B} &= \mathrm{d}\!\!/\mathrm{iv}\,\pi^{(1)} + \mathbf{R} + \mathbf{R}_{34} + E^{(1)}, \\ \varepsilon^{AB}\mathbf{R}_{A43B} &= \mathrm{c}\!\!/\mathrm{url}\,\pi^{(2)} + E^{(2)}, \end{aligned}$$

where $\mathbf{R}$ is the scalar curvature. Moreover,

$$\begin{aligned} |\pi^{(1,2)}| &\lesssim |\partial H|, \\ |E^{(1,2)}| &\lesssim (|\partial H|^2 + |\chi||\partial H|). \end{aligned}$$

Proof. Observe that $\mathbf{R}_{AB} = H^{\mu\nu}\mathbf{R}_{A\mu B\nu} = -\frac{1}{2}\mathbf{R}_{A3B4} - \frac{1}{2}\mathbf{R}_{A4B3} - \delta^{CD}\mathbf{R}_{ACBD}$. Hence, since $\mathbf{R}_{A3B4} = \mathbf{R}_{B4A3}$, we have $\delta^{AB}\mathbf{R}_{AB} = -\delta^{AB}\mathbf{R}_{A4B3} - \delta^{AB}\delta^{CD}\mathbf{R}_{ACBD}$, and therefore,

$$\begin{aligned} \delta^{AB}\mathbf{R}_{A43B} &= \delta^{AB}\mathbf{R}_{AB} + \delta^{AB}\delta^{CD}\mathbf{R}_{ACBD} \\ &= \mathbf{R} + \mathbf{R}_{34} + \delta^{AB}\delta^{CD}\mathbf{R}_{ACBD}. \end{aligned}$$

We now appeal to Corollary 4.4 and express $\delta^{AB}\mathbf{R}_{A43B}$ in the form

$$\delta^{AB}\mathbf{R}_{A43B} = \mathrm{d}\!\!/\mathrm{iv}\,\pi^{(1)} + \mathbf{R} + \mathbf{R}_{34} + E^{(1)},$$

where

$$|\pi^{(1)}| \lesssim |\partial H|$$
$$|E^{(1)}| \lesssim (|\partial H|^2 + |\chi||\partial H|).$$

On the other hand since $\mathbf{R}_{A3B4} + \mathbf{R}_{AB43} + \mathbf{R}_{A43B} = 0$, we infer that $\mathbf{R}_{A3B4} - \mathbf{R}_{A4B3} = -\mathbf{R}_{AB43}$. Thus,

$$2\varepsilon^{AB}\mathbf{R}_{A43B} = -\varepsilon^{AB}\mathbf{R}_{AB43}.$$

In view of Corollary 4.6 we can therefore express $\varepsilon^{AB}\mathbf{R}_{A43B}$ in the form $\not{\mathrm{curl}}\,\pi^{(2)} + E^{(2)}$. □

5. Strategy of the proof of the Asymptotics Theorem

In this section we describe the main ideas in the proof of the Asymptotics Theorem.

(1) *Section* 6. We start by making some primitive assumptions, which we refer to as

- Bootstrap assumptions.

They concern the geometric properties of the C_u and $S_{t,u}$ foliations. Based on these assumptions we derive further important properties, such as

- Sharp comparisons between the functions u, r and s.
- Isoperimetric and Sobolev inequalities on $S_{t,u}$.
- Trace inequality; restriction of functions in $H^2(\Sigma_t)$ to $S_{t,u}$.
- Transport lemma
- Elliptic estimates on Hodge systems.

(2) *Section* 7. We recall the *background estimates* on $H = H_{(\lambda)}$ proved in [Kl-Ro1]. We establish further estimates of H related to the surfaces $S_{t,u}$ and null hypersurfaces C_u.

- $L^q(S_{t,u})$ estimates for ∂H and $\mathbf{Ric}(H)$.
- Energy estimates on C_u.
- Statement of the estimate for the derivatives of $\mathbf{Ric}_{44}(H)$.

(3) *Section* 8. Using the bootstrap assumptions and the results of Sections 6 and 7 we provide a detailed proof of the Asymptotics Theorem.

6. Bootstrap assumptions and Basic Consequences

Throughout this section we shall use only the following background property, see Proposition 2.4 in [Kl-Ro1], of the metric H in $[0,t_*]\times\mathbb{R}^3$:

$$\|\partial H\|_{L^2_tL^\infty_x}\lesssim\lambda^{-\frac12-4\varepsilon_0}. \tag{48}$$

By the Hölder inequality we also have,

$$\|\partial H\|_{L^1_tL^\infty_x}\lesssim\lambda^{-8\varepsilon_0}. \tag{49}$$

The maximal time t_* verifies the estimate $t_*\le\lambda^{1-8\varepsilon_0}$.

6.1. *Bootstrap assumptions.* We start by constructing the outgoing null geodesics originating from the axis Γ_t, $t\in[0,t_*]$. The geodesics emanating from the same points $\in\Gamma_t$ form the null cones C_u. We define $\Omega^*\subset[0,t_*]\times\mathbb{R}^3$ to be the largest set properly foliated by the null cones C_u with the following properties:

A1) Any point in Ω^* lies on a unique outgoing null geodesic segment initiated from Γ_t and contained in Ω^*.

A2) Along any fixed C_u, $\frac{r}{s}\to1$ as $s\to0$. Here s denotes the affine parameter along C_u, i.e. $L(s)=1$ and $s|_{\Gamma_t}=0$. Recall also that $r=r(t,u)$ denotes the radius of $S_{t,u}=C_u\cap\Sigma_t$.

Moreover, the following bootstrap assumptions are satisfied for some $q>2$, sufficiently close to 2 :

B1) $\|\mathrm{tr}\chi-\frac2r\|_{L^2_tL^\infty_x}\lesssim\lambda^{-\frac12-2\varepsilon_0}$, $\quad\|\hat\chi\|_{L^2_tL^\infty_x}\lesssim\lambda^{-\frac12-2\varepsilon_0}$, $\quad\|\eta\|_{L^2_tL^\infty_x}\lesssim\lambda^{-\frac12-2\varepsilon_0}$,

B2) $\|\mathrm{tr}\chi-\frac2r\|_{L^q(S_{t,u})}\lesssim\lambda^{-2\varepsilon_0}$, $\quad\|\hat\chi\|_{L^q(S_{t,u})}\lesssim\lambda^{-2\varepsilon_0}$, $\quad\|\eta\|_{L^q(S_{t,u})}\lesssim\lambda^{-2\varepsilon_0}$.

Remark 6.2. It is straightforward to check that B1) and B2) are verified in a small neighborhood of the time axis Γ_t. Indeed for each fixed λ our metrics H_λ are smooth and therefore we can find a sufficiently small neighborhood, whose size possibly depends on λ, where the assumptions B1) and B2) hold.

Remark 6.3. We shall often have to estimate functions f in Ω_* which verify equations of the form $\frac{df}{ds}=F$ with $f=f_0$ on the axis Γ_t. According to A1) we can express the value of f at every point $P\in\Omega_*$ by the formula,

$$f(P)=f_0(P_0)+\int_\gamma F$$

with γ the unique null geodesic in Ω_* connecting the point P with the time axis Γ_t and $P_0=\gamma\cap\Gamma_t$. For convenience we shall rewrite this formula, relative to the affine parameter s in the form

$$f(s) = f(0) + \int_0^s F(s')ds'.$$

It will be clear from the context that the integral with respect to s' denotes the integral along a corresponding null geodesic γ.

6.4. *Comparison results.* We start with a simple comparison[12] between the affine parameter s and $n(t-u)$.

LEMMA 6.5. *In the region* Ω_*

$$s \approx (t-u), \quad \text{i.e.,} \quad s \lesssim (t-u) \text{ and } (t-u) \lesssim s.$$

Proof. Observe that $\frac{dt}{ds} = L(t) = T(t) = n^{-1}$ and, since $u|_{\Gamma_t} = t$,

$$t - u = \int_\gamma n^{-1} = \int_0^s n^{-1}(s')ds'. \tag{50}$$

Thus, since n is bounded uniformly from below and above, we infer that s and $t-u$ are comparable, i.e. $s \approx t-u$. In particular $s \le \lambda^{1-4\varepsilon_0}$ everywhere in Ω_*. $\square$

Remark 6.6. The formula $\frac{ds}{dt} = n$ along γ together with the uniform boundedness of n, used in Lemma 6.5 above, allows us to estimate integrals along the null geodesics γ as follows:

$$\begin{aligned} |\int_\gamma F| = |\int_0^s F(s')ds'| = |\int_0^s F(t(s'), x(s'))ds'| \\ = |\int_0^t (nF)(t', x(s'(t'))dt'| \lesssim \|F\|_{L^1_t L^\infty_x}. \end{aligned}$$

We shall make a frequent use of this remark and refine the comparison between s and $t-u$.

LEMMA 6.7. *In the region* Ω_*,

$$n(t-u) = s\Big(1 + O(\lambda^{-4\varepsilon_0})\Big).$$

[12] In [Kl-Ro] we had in fact $n = 1$ and $s = t - u$. In our context this is no longer true due to the nontriviality of the lapse function n.

Proof. Consider $U = \big(n(t-u) - s\big)$ and proceed as in the lemma above by noticing that $\frac{du}{ds} = 0$. Therefore,

$$\frac{d}{ds}U = \frac{d}{ds}\Big(n(t-u) - s\Big) = n^{-1}L(n)n(t-u)$$
$$= n^{-1}L(n)s + n^{-1}L(n)\Big(n(t-u) - s\Big).$$

Integrating from the axis Γ_t we find,

$$U(s) = \int_\gamma s' n^{-1}L(n)ds' + \int_\gamma U(s')n^{-1}L(n)ds' \tag{51}$$

where γ is the null geodesic starting on the axis Γ_t and passing through a point P_0 corresponding to the value s. By Gronwall we find,

$$U(s) \lesssim \int_0^s s'|n^{-1}L(n)|ds' \exp \int_0^s |n^{-1}L(n)|ds'.$$

According to Remark 6.6, $\int_0^s n^{-1}|L(n)| \lesssim \|\partial H\|_{L_t^1 L_x^\infty}$. We can now make use of the inequality (49) and infer that

$$n(t-u) = s\Big(1 + O(\lambda^{-8\varepsilon_0})\Big). \qquad \square$$

LEMMA 6.8. *The null lapse function b, see Definition 2.2, satisfies the estimate*

$$|b(s) - n(s)| \lesssim \lambda^{-8\varepsilon_0} \tag{52}$$

throughout the region Ω_*.

Proof. Integrating the transport equation (30), $L(b) = -b\,\bar{k}_{NN}$, along the null geodesic $\gamma(s)$, we infer that,

$$b(s) = b(0)\exp\Big(-\int_0^s \bar{k}_{NN}\Big).$$

Since $|\bar{k}_{NN}| \lesssim |\partial H|$, the condition (49) gives $\int_0^s |\bar{k}_{NN}| \lesssim \lambda^{-8\varepsilon_0}$. According to our definition $b^{-1} = T(u)$ and $u|_{\Gamma_t} = t$. Thus $b^{-1}(0) = T(t) = n^{-1}(0)$ and therefore, $|b(s) - n(0)| \lesssim \lambda^{-8\varepsilon_0}$. To finish the proof it only remains to observe that $|n(s) - n(0)| \le \int_\gamma |L(n)| \lesssim \lambda^{-8\varepsilon_0}$. $\square$

Recall that the Hardy-Littlewood maximal function[13] $\mathcal{M}(f)(t)$ of $f(t)$ is defined by

$$\mathcal{M}(f)(t) = \sup_{t_0} \frac{1}{|t-t_0|}\int_{t_0}^t f(\tau)\,d\tau,$$

[13] restricted to the interval $[0, t_*]$

and that,

$$\|\mathcal{M}(f)\|_{L^p_t} \lesssim \|f\|_{L^p_t}$$

for any $1 < p < \infty$.

LEMMA 6.9. *Let a be a solution of the transport equation*

$$L(a) = F.$$

Then for any point $P \in \Omega_* \cap \Sigma_t \cap \gamma$, where γ is the null geodesic beginning on the axis Γ_t at the point $P_0 \in \Sigma_{t_0}$ and terminating at the point P,

$$|a(P) - a(P_0)| \lesssim s\mathcal{M}(\|F\|_{L^\infty_x})(t) \tag{53}$$

where s is the value of the affine parameter of γ corresponding to P.

Proof. Integrating the equation $L(a) = \frac{da}{ds} = F$ along γ we obtain

$$|a(P) - a(P_0)| = |\int_\gamma F| \lesssim \int_{t_0}^t \|F\|_{L^\infty_x(\Sigma_\tau)}\, d\tau \lesssim (t - t_0)\mathcal{M}(\|F\|_{L^\infty_x})(t).$$

It remains to observe that $t - t_0 = t - u$ and that according to Lemma 6.5, $|t - u| \lesssim s$. □

Using Lemma 6.9 we can now refine the conclusions of Lemmas 6.8, 6.7.

COROLLARY 6.10.

$$b = n + s\, O\big(\mathcal{M}(\partial H)(t)\big), \tag{54}$$

$$n(t-u) = s + s^2 O\big(\mathcal{M}(\partial H))(t)\big), \tag{55}$$

$$|\frac{1}{n(t-u)} - \frac{1}{s}| \lesssim \mathcal{M}(\partial H)(t), \tag{56}$$

$$\|\frac{1}{n(t-u)} - \frac{1}{s}\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0} \tag{57}$$

where $\mathcal{M}(\partial H)(t)$ is the maximal function of $\|\partial H(t)\|_{L^\infty_x}$.

Proof. The proof of (54) is straightforward since $L(b-n) = -b\bar{k}_{NN} - L(n)$. Now observe that the right-hand side $|b\bar{k}_{NN} + L(n)|| \lesssim |\partial H|$ and $(b-n)|_{\Gamma_t} = 0$.

Since, according to Lemma 6.7, $n(t-u) \le 2s$, the equation $L(n(t-u)-s) = n^{-1}L(n)n(t-u)$ can be written in the form

$$|\frac{d}{ds}\big(n(t-u) - s\big)| \lesssim s|\partial H|.$$

Thus with the help of Lemma 6.9 we obtain

$$|n(t-u) - s| \lesssim s^2\mathcal{M}(\partial H).$$

The inequality (56) is an immediate consequence of (55) and Lemma 6.7. The estimate (57) follows from (56), (48), and the L^2 estimate for the Hardy-Littlewood maximal function. □

We shall now compare the values of the parameters s and $r = \frac{1}{4\pi} A^{\frac{1}{2}}(S_{t,u})$ at a point $P \in S_{t,u}$.

LEMMA 6.11. *The identity*

$$r = s\Big(1 + O(\lambda^{-6\varepsilon_0})\Big)$$

holds throughout the region Ω_*. *In particular this implies that*

$$2\pi s^2 \le A(t,u) \le 8\pi s^2$$

with $A(t,u)$ *the area of* $S_{t,u}$.

Proof. Similarly to (18), we have

$$L(r) = \frac{r}{2}\overline{\mathrm{tr}\chi} = \frac{1}{8\pi r}\int_{S_{t,u}} \mathrm{tr}\chi.$$

Using the identity $A(S_{t,u}) = 4\pi r^2$, we obtain

$$\frac{dr}{ds} = 1 + \frac{1}{8\pi r}\int_{S_{t,u}} \Big(\mathrm{tr}\chi - \frac{2}{r}\Big). \tag{58}$$

Integrating along the null geodesic γ passing through the point $P = P(s)$ [14] we have

$$\begin{aligned} |r(P) - s| &\lesssim \int_\gamma \tfrac{1}{r}\int_{S_{t,u}} \Big(\mathrm{tr}\chi - \tfrac{2}{r}\Big) \le 4\pi \int_\gamma r\|\mathrm{tr}\chi - \tfrac{2}{r}\|_{L^\infty_x} \\ &\lesssim \int_\gamma (r - s')\|\mathrm{tr}\chi - \tfrac{2}{r}\|_{L^\infty_x} + \int_\gamma s'\|\mathrm{tr}\chi - \tfrac{2}{r}\|_{L^\infty_x}. \end{aligned} \tag{59}$$

Thus by Gronwall, and the bootstrap estimate B1),

$$\Big\|\mathrm{tr}\chi - \frac{2}{r}\Big|_{L^1_t L^\infty_x} \lesssim \lambda^{\frac{1}{2}-4\varepsilon_0}\Big\|\mathrm{tr}\chi - \frac{2}{r}\Big\|_{L^2_t L^\infty_x} \lesssim \lambda^{-6\varepsilon_0}$$

we infer that, $|r - s| \lesssim s\lambda^{-6\varepsilon_0}$. □

Having established that $r \approx s$ we shall now derive more refined comparison estimates involving $\mathrm{tr}\chi - \frac{2}{s}$ and its iterated maximal functions. These will be needed later on in Section 9.6 where $\mathrm{tr}\chi - \frac{2}{s}$ rather than $\mathrm{tr}\chi - \frac{2}{r}$ appears naturally.

COROLLARY 6.12.

$$|r - s| \lesssim s^2 \mathcal{M}^3\left(\|\mathrm{tr}\chi - \frac{2}{s}\|_{L^\infty_x}\right), \tag{60}$$

$$|r - s| \lesssim s^{\frac{3}{2}}\|\mathrm{tr}\chi - \frac{2}{s}\|_{L^2_t L^\infty_x}. \tag{61}$$

[14] Observe that according to A2), $(r - s) \to 0$ as $s \to 0$ along C_u.

Here, $\mathcal{M}^k$ *is the* k^{th} *maximal function. Moreover,*

$$\left|\mathrm{tr}\chi-\frac{2}{r}\right|\lesssim\left|\mathrm{tr}\chi-\frac{2}{s}\right|+\mathcal{M}^3\left(\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^\infty_x}\right), \tag{62}$$

$$\left\|\mathrm{tr}\chi-\frac{2}{r}\right\|_{L^2_tL^\infty_x}\lesssim\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^2_tL^\infty_x}, \tag{63}$$

$$\left\|\mathrm{tr}\chi-\frac{2}{r}\right\|_{L^q(S_{t,u})}\lesssim\left(1+r^{\frac{2}{q}-\frac{1}{2}}\right)\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^2_tL^\infty_x}, \tag{64}$$

$$\left\|\frac{2}{r}-\frac{2}{n(t-u)}\right\|_{L^2_tL^\infty_x}\lesssim\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^2_tL^\infty_x}+\lambda^{-\frac{1}{2}-4\varepsilon_0}. \tag{65}$$

Proof. We write the transport equation for r in the following form:

$$L(r)=\frac{1}{8\pi r}\int_{S_{t,u}}\left(\mathrm{tr}\chi-\frac{2}{s}\right)+\frac{1}{8\pi r}\int_{S_{t,u}}\frac{2}{s}. \tag{66}$$

Differentiating $\int_{S_{t,u}}\frac{2}{s}$ we obtain

$$L\left(\int_{S_{t,u}}\frac{2}{s}\right)=\int_{S_{t,u}}\left(\frac{2}{s}\mathrm{tr}\chi-\frac{2}{s^2}\right)=\int_{S_{t,u}}\frac{2}{s}\left(\mathrm{tr}\chi-\frac{2}{s}\right)+\int_{S_{t,u}}\frac{2}{s^2}. \tag{67}$$

Furthermore,

$$L\left(\int_{S_{t,u}}\frac{2}{s^2}\right)=2\int_{S_{t,u}}\frac{1}{s^2}\left(\mathrm{tr}\chi-\frac{2}{s}\right).$$

Since $s-r\to 0$ as $r\to 0$, we have $\int_{S_{t,u}}\frac{2}{s^2}\to 8\pi$. Using Lemmas 6.11 and 6.9 we infer that

$$\int_{S_{t,u}}\frac{2}{s^2}=8\pi+s\mathcal{M}\left(\|\mathrm{tr}\chi-\frac{2}{s}\|_{L^\infty_x}\right).$$

Integrating (67) and using Lemma 6.9 once more we obtain

$$\int_{S_{t,u}}\frac{2}{s}=8\pi s+s^2\mathcal{M}^2\left(\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^\infty_x}\right)+s^2\mathcal{M}\left(\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^\infty_x}\right).$$

Again, according to Lemma 6.11, $r\approx s$. Thus by (66)

$$L(r)=\frac{s}{r}+\frac{1}{8\pi r}\int_{S_{t,u}}\left(\mathrm{tr}\chi-\frac{2}{s}\right)+s\mathcal{M}^2\left(\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^\infty_x}\right)$$

or, equivalently,

$$L(r^2)=2s+\frac{1}{4\pi}\int_{S_{t,u}}\left(\mathrm{tr}\chi-\frac{2}{s}\right)+rs\mathcal{M}^2\left(\left\|\mathrm{tr}\chi-\frac{2}{s}\right\|_{L^\infty_x}\right).$$

Integrating with the help of Lemma 6.9 we infer that,

$$r^2 = s^2 + s^3\mathcal{M}^3\left(\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty_x}\right) + s^3\mathcal{M}\left(\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty_x}\right).$$

It then follows that

$$r = s + s^2\mathcal{M}^3\left(\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty_x}\right). \tag{68}$$

Observe that if during each integration along γ we used Hölder inequality instead of the bounds involving maximal functions, we would have the estimate

$$r = s + s^{\frac{3}{2}}\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^2_tL^\infty_x}. \tag{69}$$

This estimate can be used effectively to compare r and s on a single surface $S_{t,u}$ while (68) works well with the norms involving integration in time. Thus, we infer from from (68) that

$$\left|\frac{2}{r} - \frac{2}{s}\right| \lesssim \mathcal{M}^3\left(\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty_x}\right), \tag{70}$$

$$\left\|\frac{2}{r} - \frac{2}{s}\right\|_{L^2_tL^\infty_x} \lesssim \left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^2_tL^\infty_x}. \tag{71}$$

In addition, (69) implies that

$$\left\|\frac{2}{r} - \frac{2}{s}\right\|_{L^q(S_{t,u})} \lesssim r^{\frac{2}{q}-\frac{1}{2}}\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^2_tL^\infty_x}. \tag{72}$$

Inequalities (62)–(64) follow from the identity $\mathrm{tr}\chi - \frac{2}{r} = \mathrm{tr}\chi - \frac{2}{s} + \frac{2}{r} - \frac{2}{s}$ and (70)–(72). Finally, (65) follows from (70) and (57). □

Remark 6.13. Observe that equation (58) and Lemma 6.9 also give the estimate

$$|r - s| \lesssim s^2\mathcal{M}\left(\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^\infty_x}\right)(t).$$

Thus with the help of the bootstrap assumption B1) and the L^2 estimate for the maximal function we infer that,

$$\begin{aligned}\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^2_tL^\infty_x} &\lesssim \left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^2_tL^\infty_x} + \left\|\frac{2}{r} - \frac{2}{s}\right\|_{L^2_tL^\infty_x} \\ &\lesssim 2\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^2_tL^\infty_x} \lesssim \lambda^{-\frac{1}{2}-2\varepsilon_0}.\end{aligned} \tag{73}$$

Moreover, since $r \approx s$, equation (58), Hölder inequality and the bootstrap assumption B2) also imply that

$$|r - s| \lesssim \int_\gamma r^{1-\frac{2}{q}} \left\| \mathrm{tr}\chi - \frac{2}{r} \right\|_{L^q(S_{t,u})} \lesssim \lambda^{-2\varepsilon_0} s\, r^{1-\frac{2}{q}}.$$

Using the bootstrap assumption B2) once again we infer that

$$\begin{aligned} \|\mathrm{tr}\chi - \frac{2}{s}\|_{L^q(S_{t,u})} &\lesssim \left\| \mathrm{tr}\chi - \frac{2}{r} \right\|_{L^q(S_{t,u})} + \left\| \frac{2}{r} - \frac{2}{s} \right\|_{L^q(S_{t,u})} \\ &\lesssim \lambda^{-2\varepsilon_0} + \lambda^{-2\varepsilon_0} \|r^{-\frac{2}{q}}\|_{L^q(S_{t,u})} \lesssim \lambda^{-2\varepsilon_0}. \end{aligned} \tag{74}$$

Estimates (74), (73) indicate that the bootstrap assumptions B1), B2) also hold for $(\mathrm{tr}\chi - \frac{2}{s})$.

6.14. *Isoperimetric, Sobolev inequalities and the transport lemma.* We consider now the foliation induced by $S_{t,u}$ on $\Sigma_t \cap \Omega_*$. Relative to this foliation the induced metric h on Σ_t takes the form

$$h = b^2 du^2 + \gamma_{AB} d\phi^A d\phi^B$$

where ϕ^A are local coordinates on S^2. We state below a proposition concerning the trace and isoperimetric inequalities on $\Sigma_t \cap \Omega_*$. The proposition requires a very weak assumption on the metric h; in fact we only need

$$\big(\sup_{\Omega_*} r^{\frac{1}{2}\varepsilon}\big) \|\nabla^{\frac{3}{2}+\varepsilon} h\|_{L^2(\Sigma_t)} \le \Lambda_0^{-1} \tag{75}$$

for some large constant $\Lambda_0 > 0$ and an arbitrarily small $\varepsilon > 0$. In this and the following subsection we shall assume a slightly stronger property that

$$\big(\sup_{\Omega_*} r^{\frac{1}{2}\varepsilon}\big) \|\nabla^{\frac{1}{2}+\varepsilon} \partial H\|_{L^2(\Sigma_t)} \le \Lambda_0^{-1}. \tag{76}$$

Remark 6.15. The assumption (76) is easily satisfied by our families of metrics $H = H_{(\lambda)}$; see Remark 7.2.

PROPOSITION 6.16. *Let $S_{t,u}$ be a fixed surface in $\Sigma_t \cap \Omega_*$ with N the exterior unit normal on Σ_t. Under the assumption* (76) *the following estimates hold true with constants independent of $S_{t,u}$*:

i) *For any smooth function $f : S_{t,u} \to \mathbb{R}$, the following isoperimetric inequality holds*:

$$\left(\int_{S_{t,u}} |f|^2 \right)^{\frac{1}{2}} \lesssim \int_{S_{t,u}} \left(|\nabla\!\!\!/ f| + \frac{1}{r}|f| \right). \tag{77}$$

ii) *The following Sobolev inequality holds on* $S_{t,u}$: *for any* $\delta \in (0,1)$ *and* p *from the interval* $p \in (2,\infty]$,

$$\sup_{S_{t,u}} |f| \lesssim r^{\frac{\varepsilon(p-2)}{2p+\delta(p-2)}} \left(\int_{S_{t,u}} (|\nabla\!\!\!/ f|^2 + r^{-2}|f|^2) \right)^{\frac{1}{2} - \frac{\delta p}{2p+\delta(p-2)}} \cdot \left[\int_{S_{t,u}} (|\nabla\!\!\!/ f|^p + r^{-p}|f|^p) \right]^{\frac{2\delta}{2p+\delta(p-2)}}. \tag{78}$$

iii) *Consider an arbitrary function* $f : \Sigma_t \to \mathbf{R}$ *such that* $f \in H^{\frac{1}{2}+\varepsilon}(\mathbf{R}^3)$. *The following trace inequality holds true*:

$$\|f\|_{L^2(S_{t,u})} \lesssim \|\partial^{\frac{1}{2}+\varepsilon} f\|_{L^2(\Sigma_t)} + \|\partial^{\frac{1}{2}-\varepsilon} f\|_{L^2(\Sigma_t)}. \tag{79}$$

More generally, for any $q \in [2,\infty)$

$$\|f\|_{L^q(S_{t,u})} \lesssim \|\partial^{\frac{3}{2}-\frac{2}{q}+\varepsilon} f\|_{L^2(\Sigma_t)} + \|\partial^{\frac{3}{2}-\frac{2}{q}-\varepsilon} f\|_{L^2(\Sigma_t)}. \tag{80}$$

Also, consider the region $\Omega_*(\frac{1}{4}r, r) = \cup_{\frac{1}{4}r \le \rho \le r} S_{t,u(\rho)}$: *where* $r = r(t,u)$, *then,*

$$\|f\|^2_{L^2(S_{t,u})} \lesssim \|\nabla_N f\|_{L^2(\Omega_*(\frac{1}{4}r,r))} \|f\|_{L^2(\Omega_*(\frac{1}{4}r,r))} + \frac{1}{r}\|f\|^2_{L^2(\Omega_*(\frac{1}{4}r,r))}. \tag{81}$$

Proof. See [Kl-Ro]. □

Finally we state below,

LEMMA 6.17 (The transport lemma). *Let* $\Pi_{\underline{A}}$ *be an* S*-tangent tensorfield verifying the following transport equation with* $\sigma > 0$:

$$\not{D}_4 \Pi_{\underline{A}} + \sigma \mathrm{tr}\chi \Pi_{\underline{A}} = F_{\underline{A}}.$$

Assume that the point $(t,x) = (t,s,\omega)$ *belongs to the domain* Ω_*. *If* Π *satisfies the initial condition* $s^{2\sigma}\Pi_{\underline{A}}(s) \to 0$ *as* $s \to 0$, *then*

$$|\Pi(t,x)| \le 4\|F\|_{L^1_t L^\infty_x}. \tag{82}$$

In addition, if $\sigma \ge \frac{1}{q}$ *and* Π *satisfies the initial condition* $r^{2(\sigma-\frac{1}{q})}\|\Pi\|_{L^q(S_{t,u})} \to 0$ *as* $r \to 0$, *then on each surface* $S_{t,u} \subset \Omega_*$,

$$\|\Pi\|_{L^q(S_{t,u})} \lesssim \frac{1}{r(t)^{2(\sigma-\frac{1}{q})}} \int_u^t r(t')^{2(\sigma-\frac{1}{q})} \|F\|_{L^q(S_{t',u})}\, dt'. \tag{83}$$

Finally, if Π *is a solution of the transport equation*

$$\not{D}_4 \Pi_{\underline{A}} + \sigma \mathrm{tr}\chi \Pi_{\underline{A}} = \frac{1}{r} F_{\underline{A}},$$

verifying the initial condition $s^{2\sigma}\Pi_{\underline{A}}(s) \to 0$ *with some* $\sigma > \frac{1}{2}$, *then*

$$|\Pi(t,x)| \le 4\mathcal{M}(\|F\|_{L^\infty_x})(t). \tag{84}$$

Proof. The proof of (82) and (83) is straightforward. For a similar version see Lemma 5.2 in [Kl-Ro]. Estimate (84) can be proved in the same manner as (53) of Lemma 6.9. □

6.18. *Elliptic estimates.* Next we establish a proposition concerning the L^2 estimates of Hodge systems on the surfaces $S_{t,u}$. They are similar to the estimates of Lemma 5.5 in [Kl-Ro]. We need however to make an important modification based on Corollary 4.4.

PROPOSITION 6.19. *Let ξ be an $m+1$ covariant, totally symmetric tensor, a solution of the Hodge system on the surface $S_{t,u} \subset \Omega_*$; then*

$$\begin{aligned} \not{\mathrm{d}}\mathrm{iv}\,\xi &= F, \\ \not{\mathrm{c}}\mathrm{url}\,\xi &= G, \\ \mathrm{tr}\xi &= 0. \end{aligned}$$

Then ξ obeys the estimate

$$\int_{S_{t,u}} |\nabla\!\!\!/\xi|^2 + \frac{m+1}{2r^2}|\xi|^2 \le 2\int_{S_{t,u}} \{|F|^2 + |G|^2\}. \tag{85}$$

Proof. Using the standard Hodge theory, see Theorem 5.4 in [Kl-Ro] or Chapter 2 in [Ch-Kl], we have

$$\int_{S_{t,u}} |\nabla\!\!\!/\xi|^2 + (m+1)K|\xi|^2 = \int_{S_{t,u}} \{|F|^2 + |G|^2\}. \tag{86}$$

The Gauss curvature K of the 2-surface $S_{t,u}$ can be expressed as

$$K = \frac{1}{4}(\mathrm{tr}\chi)^2 + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}k + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} + \frac{1}{2}R_{ABAB}.$$

Thus it follows from Corollary 4.4 that

$$K - r^{-2} = \nabla\!\!\!/_A\Pi_A + E$$

where the tensor Π and the error term E, relative to the standard coordinates x^α, obey the pointwise estimates $|\Pi| \lesssim |\partial H|$ and $|E| \lesssim (|\partial H|^2 + |\hat{\chi}|^2 + |\chi||\partial H|)$. Then,

$$\int_{S_{t,u}} |\nabla\!\!\!/\xi|^2 + \frac{m+1}{r^2}|\xi|^2 \le \int_{S_{t,u}} \{|F|^2 + |G|^2 + (m+1)(\nabla\!\!\!/_A\Pi_A + E)|\xi|^2\}. \tag{87}$$

Integrating the term $\int_{S_{t,u}} \nabla\!\!\!/_A\Pi_A|\xi|^2$ by parts we obtain for all sufficiently large p, $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$,

$$\int_{S_{t,u}} \nabla\!\!\!/_A\Pi_A|\xi|^2 = -2\int_{S_{t,u}} \Pi_A\nabla\!\!\!/_A\xi\cdot\xi \lesssim \|\nabla\!\!\!/\xi\|_{L^2(S_{t,u})}\|\xi\|_{L^p(S_{t,u})}\|\Pi\|_{L^q(S_{t,u})}.$$

The isoperimetric inequality implies that for $2 \le p < \infty$

$$\|\xi\|_{L^p(S_{t,u})} \lesssim r^{\frac{2}{p}} \Big(\|\nabla\!\!\!/ \xi\|_{L^2(S_{t,u})} + r^{-1} \|\xi\|_{L^2(S_{t,u})} \Big).$$

We also deduce from the trace inequality that

$$\|\Pi\|_{L^q(S_{t,u})} \lesssim \|\partial H\|_{L^q(S_{t,u})} \lesssim \|\partial^{(\frac{3}{2}+1-\frac{2}{q}+\varepsilon)} H\|_{L^2(\Sigma_t)} + \|\partial^{(\frac{3}{2}+1-\frac{2}{q}-\varepsilon)} H\|_{L^2(\Sigma_t)}.$$

Thus the smallness condition

$$r^{1-\frac{2}{q}} \|\partial^{(\frac{3}{2}+1-\frac{2}{q}+\varepsilon)} H\|_{L^2(\Sigma_t)} \le \Lambda_0^{-1}$$

ensures that we can absorb the term $(m+1)\int_{S_{t,u}} \nabla\!\!\!/_A \Pi_A |\xi|^2$ on the left-hand side of (87). For large p the above condition coincides with (75).

It remains to estimate $\int_{S_{t,u}} E|\xi|^2$. The most dangerous term is $\int_{S_{t,u}} |\hat{\chi}|^2 |\xi|^2$. Applying the Hölder inequality we infer that,

$$\int_{S_{t,u}} |\hat{\chi}|^2 |\xi|^2 \lesssim \|\xi\|^2_{L^p(S_{t,u})} \|\hat{\chi}\|^2_{L^q(S_{t,u})}.$$

Using the isoperimetric inequality once more, we conclude that we need a smallness condition on $r^{1-\frac{2}{q}} \|\hat{\chi}\|_{L^q(S_{t,u})}$ for some $q > 2$. This is guaranteed by our bootstrap assumption B2). □

We shall next formulate versions of the Calderon-Zygmund theorem for the above types of Hodge systems; see also [B-W]. The proof is a straightforward modification of the standard approach.

PROPOSITION 6.20. *Let ξ be a 2 covariant, traceless, symmetric tensor, verifying the Hodge system on the surface $S_{t,u} \subset \Omega_*$,*

$$\mathrm{d}\!\!\!/\mathrm{iv}\, \xi = \nabla\!\!\!/ \nu + e$$

for some scalar ν and 1-form e. Then,

$$\|\xi\|_{L^q(S_{t,u})} \lesssim \|\nu\|_{L^q(S_{t,u})} + \|e\|_{L^p(S_{t,u})} \tag{88}$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$.

Also,[15]

$$\|\xi\|_{L^\infty(S_{t,u})} \lesssim \|\nu\|_{L^\infty(S_{t,u})} \log^+(r\|\nabla\!\!\!/ \nu\|_{L^\infty(S_{t,u})}) + r^{1-\frac{2}{p}} \|e\|_{L^p(S_{t,u})} \tag{89}$$

for any $p > 2$, where $\log^+ z = \log(2 + |z|)$.

Similar estimates hold in the case when ξ is a 1-form verifying the Hodge system

$$\mathrm{d}\!\!\!/\mathrm{iv}\, \xi = \mathrm{d}\!\!\!/\mathrm{iv}\, \nu_1 + e_1,$$
$$\mathrm{c}\!\!\!/\mathrm{url}\, \xi = \mathrm{c}\!\!\!/\mathrm{url}\, \nu_2 + e_2$$

for some 1-forms $\nu = (\nu_1, \nu_2)$ and scalars $e = (e_1, e_2)$.

[15] The term $\|r\nabla\!\!\!/ \nu\|_{L^\infty(S_{t,u})}$ can in fact be replaced by $\|r\nabla\!\!\!/ \nu\|_{L^r(S_{t,u})}$ for $r > 2$.

7. Properties of the metric H and its curvature tensor $\mathbf{R}$

7.1. *Background estimates.* We start by recalling the *background* estimates on the family of the Lorentz metrics $H = H_{(\lambda)}$ proved in [Kl-Ro1]; see Proposition 2.4.

The metric H admits the canonical decomposition

$$H = -n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt)$$

and satisfies the following estimates on the time interval $[0, t_*]$ with $t_* \le \lambda^{1-8\varepsilon_0}$:

$$c|\xi|^2 \le h_{ij}\xi^i\xi^j \le c^{-1}|\xi|^2, \quad n^2 - |v|_h^2 \ge c > 0, \quad |n|, |v| \le c^{-1} \tag{90}$$

$$\|\partial^{1+m} H\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-8\varepsilon_0}, \qquad m \ge 0 \tag{91}$$

$$\|\partial^{1+m} H\|_{L^2_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\frac12 - 4\varepsilon_0}, \qquad m \ge 0 \tag{92}$$

$$\|\partial^{1+m} H\|_{L^\infty_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\frac12 - 4\varepsilon_0}, \qquad m \ge 0 \tag{93}$$

$$\|\nabla^{\frac12 + m}(\partial H)\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{-m}, \qquad \text{for } -\frac12 \le m \le \frac12 + 4\varepsilon_0 \tag{94}$$

$$\|\nabla^{\frac12 + m}(\partial^2 H)\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{-\frac12 - 4\varepsilon_0}, \quad \text{for } -\frac12 + 4\varepsilon_0 \le m \tag{95}$$

$$\|\nabla^m\big(H^{\alpha\beta}\partial_\alpha\partial_\beta H\big)\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-1-8\varepsilon_0}, \qquad m \ge 0 \tag{96}$$

$$\|\nabla^m(\nabla^{\frac12}\mathbf{R}_{\alpha\beta}(H))\|_{L^2_x} \lesssim \lambda^{-1}, \qquad m \ge 0 \tag{97}$$

$$\|\nabla^m \mathbf{R}_{\alpha\beta}(H)\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-1-8\varepsilon_0}, \qquad m \ge 0. \tag{98}$$

Remark 7.2. The inequality (92) with $m = 0$ is consistent with the property (48), which we have used throughout Section 6. Moreover, since in the region Ω_* the radius r of the surfaces $S_{t,u}$ does not exceed $\lambda^{1-8\varepsilon_0}$, we have, according to (94),

$$r^{\frac12\varepsilon}\|\nabla^{\frac12+\varepsilon}(\partial H)\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{(\frac12 - 4\varepsilon_0)\varepsilon}\lambda^{-\varepsilon} \le \lambda^{-\frac12\varepsilon}.$$

This verifies condition (76).

7.3. $L^q(S_{t,u})$ *estimates.* The trace inequality (80) of Proposition 6.16 allows us to derive the $L^q(S_{t,u})$ estimates on the metric H from (94).

Proposition 7.4. *For any q in the interval* $2 \le q \le 4$

$$\|\partial H\|_{L^q(S_{t,u})} \lesssim \lambda^{\frac2q - 1 - 8(\frac2q - \frac12)\varepsilon_0}. \tag{99}$$

In addition,

$$\|\mathbf{Ric}(H)\|_{L^p(S_{t,u})} \lesssim \lambda^{\frac2p - 2 - 8(\frac2p - 1)\varepsilon_0} \tag{100}$$

for $p \in [1, 2]$.

Proof. Since $q \le 4$, by the Hölder inequality,

$$\|\partial H\|_{L^q(S_{t,u})} \lesssim r^{\frac{2}{q}-\frac{1}{2}} \|\partial H\|_{L^4(S_{t,u})} \lesssim \lambda^{(\frac{2}{q}-\frac{1}{2})(1-8\varepsilon_0)} \|\partial H\|_{L^4(S_{t,u})}.$$

Using the trace estimate (80) we infer that

$$\|\partial H\|_{L^q(S_{t,u})} \lesssim \lambda^{(\frac{2}{q}-\frac{1}{2})(1-8\varepsilon_0)} \|\partial H\|_{\dot{H}^1(\mathbf{R}^3)} \lesssim \lambda^{\frac{2}{q}-1-8(\frac{2}{q}-\frac{1}{2})\varepsilon_0}$$

where we have used $\|\partial H\|_{\dot{H}^1(\mathbf{R}^3)} \lesssim \lambda^{-\frac{1}{2}}$ from (94). The inequality (100) follows similarly from the trace theorem and (97). □

7.5. *Energy estimates on C_u.* In this subsection we shall derive energy estimates, along the null hypersurfaces C_u, for tangential derivatives of the first derivatives of the rescaled metric

$$G(t,x) = \mathbf{g}\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right). \tag{101}$$

Recall that the original space time Einstein metric $\mathbf{g}$ verifies $\mathbf{R}_{\mu\nu}(\mathbf{g}) = 0$. In addition, since our coordinates x^α satisfy the wave coordinate condition (2), the metric $\mathbf{g}$ satisfies the quasilinear wave equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta \mathbf{g}_{\mu\nu} = N_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}). \tag{102}$$

We have also defined the truncated metric $\mathbf{g}_{<\lambda} = \sum_{\mu<\frac{1}{2}\lambda} P_\mu \mathbf{g}$ and, by rescaling, our background metric

$$H(t,x) = \mathbf{g}_{<\lambda}\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right).$$

Similarly, for a dyadic $\mu \ge \frac{1}{2}$ we can define

$$G^{(\mu)}(t,x) = P_{\mu\lambda}\mathbf{g}\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right).$$

Observe that H has frequencies ≤ 1 and $G^{(\mu)}$ is localized to the frequencies of size μ which cannot fall below $\frac{1}{2}$.

We now formulate a basic energy estimate on the null cones C_u for H and $G^{(\mu)}$.

Definition 7.6. Given a scalar function F in Ω_* we denote by D_*F the C_u *tangential derivatives* of F. More precisely, $D_*F = (\nabla\!\!\!/ F, LF)$. We shall use this notation for the components of the metrics H and G relative to our fixed system of coordinates. We also use this notation applied to all components of the derivatives ∂H and ∂G. Thus $|D_*\partial H| = \sum_{\alpha,\beta,\gamma} |D_*\partial_\gamma H_{\alpha\beta}|$.

Proposition 7.7. *The following estimates hold in the region* Ω_*:

$$\|D_*\partial H\|_{L^2(C_u)} \lesssim \lambda^{-\frac{1}{2}}, \qquad \|D_* H\|_{L^2(C_u)} \lesssim \lambda^{\frac{1}{2}}. \tag{103}$$

In addition, for the functions $G^{(\mu)}$ defined above,

$$\|D_*\partial G^{(\mu)}\|_{L^2(C_u)} \lesssim \mu^{\frac{1}{2}-4\varepsilon_0}\lambda^{-\frac{1}{2}-4\varepsilon_0},$$
$$\|D_* G^{(\mu)}\|_{L^2(C_u)} \lesssim \max\{\mu^{-1-4\varepsilon_0}\lambda^{-\frac{1}{2}-4\varepsilon_0},\, \mu^{-\frac{1}{2}-4\varepsilon_0}\lambda^{-1-4\varepsilon_0}\}. \tag{104}$$

Proof of Proposition 7.7. Metric $\mathbf{g}$ is a $H^{2+\gamma}$ solution of the Einstein equation. Thus after rescaling and taking into account $\gamma > 5\varepsilon_0$, we infer that in addition to the estimates (91)–(96) for H, we also have

$$\|\partial^{1+m} G^{(\mu)}\|_{L^\infty_t L^2_x} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0}\mu^{m-1-4\varepsilon_0}, \quad \text{for} \quad m = 0, 1. \tag{105}$$

We shall make use of the rescaled version of Lemma 8.9 in [Kl-Ro1] to derive the equations for H and $G^{(\mu)}$.

$$H^{\alpha\beta}\partial_\alpha\partial_\beta H = F, \qquad H^{\alpha\beta}\partial_\alpha\partial_\beta G^{(\mu)} = F_\mu, \tag{106}$$

with the right-hand sides F, F_μ obeying the estimates

$$\|F\|_{L^1_t L^2_x} \lesssim \lambda^{\frac{1}{2}}, \qquad \|\partial F\|_{L^1_t L^2_x} \lesssim \lambda^{-\frac{1}{2}}, \tag{107}$$

$$\|F_\mu\|_{L^1_t L^2_x} \lesssim \mu^{-4\varepsilon_0}\lambda^{-\frac{1}{2}-4\varepsilon_0}, \qquad \|\partial F_\mu\|_{L^1_t L^2_x} \lesssim \mu^{1-4\varepsilon_0}\lambda^{-\frac{1}{2}-4\varepsilon_0}. \tag{108}$$

We shall use the generalized energy identity with the vectorfield T in the region $M_{t_0,t,u}$ bounded by the cone C_u and the time slices Σ_{t_0}, Σ_t intersecting C_u. The vectorfield L is orthogonal, in the sense of the Lorentzian metric H, to the cone C_u. Thus

$$\int_{C_u} Q[H](T,L) + \int_{\Sigma_{t_0}} Q[H](T,T) = \int_{\Sigma_{t_0}} Q[H](T,T) - \int_{M_{t_0,t,u}} \Big(Q^{\alpha\beta}[H]^T\pi_{\alpha\beta} + FT(H)\Big)$$

with the energy-momentum tensor

$$Q[f]_{\alpha\beta} = \partial_\alpha f\partial_\beta f - \frac{1}{2}H_{\alpha\beta}(\partial_\nu f\partial^\nu f)$$

and the deformation tensor ${}^{(T)}\pi_{\alpha\beta} = \mathcal{L}_T H$ of the vectorfield T. A similar identity also holds for G^μ. According to (7) and (23) the components of the deformation tensor ${}^T\pi$ can be described as follows:

$${}^{(T)}\pi_{ij} = -2k_{ij}, \qquad {}^{(T)}\pi_{i0} = n^{-1}\partial_i n, \qquad {}^{(T)}\pi_{00} = 0.$$

Thus the deformation tensor $|{}^{(T)}\pi| \lesssim |\partial H|$, and by (91) obeys the estimate

$$\|{}^{(T)}\pi\|_{L^1_t L^\infty_x} \lesssim \lambda^{-4\varepsilon_0}. \tag{109}$$

Observe that

$$Q[H](T,L) = \frac{1}{2}(LH)^2 + \frac{1}{2}|\nabla\!\!\!/ H|^2 = \frac{1}{2}|D_* H|^2,$$
$$Q[H](T,T) = \frac{1}{2}(TH)^2 + \frac{1}{2}|\nabla H|^2 = \frac{1}{2}|\partial H|^2.$$

In addition, $|Q_{\alpha\beta}(f)| \le 2|\partial f|^2$. Thus, using (94), (107), and (109), we obtain

$$\int_{C_u} |D_* H|^2 \le \int_{\Sigma_{t_0}} |\partial H|^2 + 4\int_{M_{t_0,t,u}} \Big(|{}^{(T)}\pi|\, |\partial H|^2 + |F|\, |\partial H| \Big)$$
$$\lesssim \|\partial H\|^2_{L^\infty_t L^2_x} + \|{}^{(T)}\pi\|_{L^1_t L^\infty_x} \|\partial H\|^2_{L^\infty_t L^2_x} + \|F\|_{L^1_t L^2_x} \|\partial H\|_{L^\infty_t L^2_x} \lesssim \lambda.$$

Similarly,

$$\int_{C_u} |D_* G^\mu|^2 \le \int_{\Sigma_{t_0}} |\partial G^\mu|^2 + 4\int_{M_{t_0,t,u}} \Big(|{}^{T}\pi|\, |\partial G^\mu|^2 + |F_\mu|\, |\partial G^\mu| \Big)$$
$$\lesssim \|\partial G^\mu\|^2_{L^\infty_t L^2_x} + \|{}^{T}\pi\|_{L^1_t L^\infty_x} \|\partial G^\mu\|^2_{L^\infty_t L^2_x} + \|F_\mu\|_{L^1_t L^2_x} \|\partial G^\mu\|_{L^\infty_t L^2_x}$$
$$\lesssim \max\{\mu^{-2-8\varepsilon_0}\lambda^{-1-8\varepsilon_0}, \mu^{-1-8\varepsilon_0}\lambda^{-2-8\varepsilon_0}\}.$$

To get the estimates for $D_*\partial H$ and $D_*\partial G^\mu$ we differentiate the equations (106). Commuting the derivative with the metric H we obtain,

$$H^{\alpha\beta}\partial_\alpha\partial_\beta\partial H = \partial F + (\partial H^{\alpha\beta})\partial_\alpha\partial_\beta\partial H = F^1,$$
$$H^{\alpha\beta}\partial_\alpha\partial_\beta\partial G^\mu = \partial F_\mu + (\partial H^{\alpha\beta})\partial_\alpha\partial_\beta\partial G^\mu = F^1_\mu.$$

Using (107), (108) and the inequality $\|\partial H\|_{L^1_t L^\infty_x} \lesssim \lambda^{-4\varepsilon_0}$ of (91), we infer that

$$\|F\|_{L^1_t L^2_x} \lesssim \lambda^{-\frac12}, \qquad \|F^1_\mu\|_{L^1_t L^2_x} \lesssim \mu^{1-4\varepsilon_0}\lambda^{-\frac12-4\varepsilon_0}.$$

Thus using the generalized energy identity for ∂H and ∂G^μ we will have

$$\int_{C_u} |D_*\partial H|^2 \lesssim \|\partial^2 H\|^2_{L^\infty_t L^2_x} + \|{}^{(T)}\pi\|_{L^1_t L^\infty_x} \|\partial^2 H\|^2_{L^\infty_t L^2_x}$$
$$+ \|F^1\|_{L^1_t L^2_x} \|\partial^2 H\|_{L^\infty_t L^2_x} \lesssim \lambda^{-1}.$$

Also,

$$\int_{C_u} |D_*\partial G^\mu|^2 \lesssim \|\partial^2 G^\mu\|^2_{L^\infty_t L^2_x} + \|{}^{(T)}\pi\|_{L^1_t L^\infty_x} \|\partial^2 G^\mu\|^2_{L^\infty_t L^2_x}$$
$$+ \|F^1_\mu\|_{L^1_t L^2_x} \|\partial^2 G^\mu\|_{L^\infty_t L^2_x} \lesssim \mu^{1-8\varepsilon_0}\lambda^{-1-8\varepsilon_0}. \qquad \square$$

The following result can be deduced from Propositions 7.7 and 4.1.

COROLLARY 7.8. *Any component of the curvature* $\mathbf{R}_{abcd} = \mathbf{R}(e_a, e_b, e_c, e_d)$, *of the metric* H, *with vectorfields* e_a, e_b, e_c *varying between* $L, e_A, A = 1, 2$, *obeys the energy estimates on* C_u:

$$\|\mathbf{R}_{abcd}\|_{L^2(C_u)} \lesssim \lambda^{-\frac12}.$$

In particular,

$$\|\mathbf{R}_*\|_{L^2(C_u)} := \sum_{A,B,C,D} \|\mathbf{R}_{ABCD}\|_{L^2(C_u)} + \|\mathbf{R}_{ABC4}\|_{L^2(C_u)} + \|\mathbf{R}_{B43A}\|_{L^2(C_u)}$$
$$\lesssim \lambda^{-\frac12}.$$

8. A remarkable property of $\mathbf{R}_{44}$

While the spacetime metric $\mathbf{g}$ verifies the Einstein equations $\mathbf{R}_{\mu\nu}(\mathbf{g}) = 0$ this is certainly not true for the effective metric $H = H_{(\lambda)}$. This could create serious problems in the proof of the asymptotics theorem as the Ricci curvature appears as a source term in the null structure equations. We have already established an improved estimate for $\mathbf{Ric}(H)$ in $L^1_t L^\infty_x$, see (98). This was done by comparing $\mathbf{R}_{\mu\nu}(H)$ with $\mathbf{R}_{\mu\nu}(G) = 0$ where $G = \mathbf{g}(\lambda^{-1}t, \lambda^{-1}x)$ is the rescaled Einstein metric. We need however a stronger estimate involving the derivatives of $\mathbf{R}_{44}(H)$ along the null cones C_u. To establish such an estimate we encounter an additional difficulty: the null cones C_u have been constructed relative to the approximate metric H. This leads to significant differences between the C_u energy estimates for the second derivatives of H, see (103) and the corresponding ones[16] for G; see (104) in Proposition 7.7. Using however the specific structure of the component $\mathbf{R}_{44}$ relative to the wave coordinates we can overcome this difficulty and prove the following:

THEOREM 8.1. *On any null hypersurface C_u,*

$$\int_u^t \|\nabla \mathbf{R}_{44}(H)\|_{L^2(S_{\tau,u})} d\tau \lesssim \lambda^{-1}. \tag{110}$$

Proof. The proof of the theorem requires a rather long and tedious argument which we present in our paper [Kl-Ro2]. □

9. Asymptotics Theorem

We start by recalling already established estimates for the metric related quantities which play a crucial role in what follows.

$$\|\partial H\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0}, \tag{111}$$

$$\|\partial H\|_{L^q(S_{t,u})} \lesssim \lambda^{\frac{2}{q}-1-8(\frac{2}{q}-\frac{1}{2})\varepsilon_0} \quad \text{for} \quad 2 \le q \le 4, \tag{112}$$

$$\|\mathbf{Ric}(H)\|_{L^1_t L^\infty_x} \lesssim \lambda^{-1-8\varepsilon_0}, \tag{113}$$

$$\|\mathbf{Ric}(H)\|_{L^p(S_{t,u})} \lesssim \lambda^{\frac{2}{p}-2-8(\frac{2}{p}-1)\varepsilon_0} \quad \text{for} \quad 1 \le p \le 2, \tag{114}$$

$$\|D_* \partial H\|_{L^2(C_u)} \lesssim \lambda^{-\frac{1}{2}}, \tag{115}$$

$$\int_0^s \|\nabla \mathbf{R}_{44}\|_{L^2(S_{t,u})} \lesssim \lambda^{-1-2\varepsilon_0}, \tag{116}$$

$$\|\mathbf{R}_*\|_{L^2(C_u)} \lesssim \lambda^{-\frac{1}{2}} \tag{117}$$

[16]The estimates for the second derivatives of the higher frequencies of G do in fact diverge badly.

where

$$\|\mathbf{R}_*\|_{L^2(C_u)} := \sum_{A,B,C,D} \|\mathbf{R}_{ABCD}\|_{L^2(C_u)} + \|\mathbf{R}_{ABC4}\|_{L^2(C_u)} + \|\mathbf{R}_{B43A}\|_{L^2(C_u)}.$$

Note that some of the above estimates hold only throughout the region Ω_*.

THEOREM 9.1. *Throughout the region* Ω_* *the quantities* $\mathrm{tr}\chi - \frac{2}{r}$, $\hat{\chi}$, *and* η *satisfy the following estimates*:

$$\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L_t^2 L_x^\infty} + \|\hat{\chi}\|_{L_t^2 L_x^\infty} + \|\eta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-3\varepsilon_0}, \tag{118}$$

$$\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^q(S_{t,u})} + \|\hat{\chi}\|_{L^q(S_{t,u})} + \|\eta\|_{L^q(S_{t,u})} \lesssim \lambda^{-3\varepsilon_0}. \tag{119}$$

In the estimate (118) *function* $\frac{2}{r}$ *can be replaced with* $\frac{2}{n(t-u)}$. *Also*, *the corresponding* L_t^1 *estimate follows by Hölder inequality*:

$$\left\|\mathrm{tr}\chi - \frac{2}{n(t-u)}\right\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-3\varepsilon_0}, \qquad \left\|\mathrm{tr}\chi - \frac{2}{n(t-u)}\right\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-3\varepsilon_0}. \tag{120}$$

In addition, *in the exterior region* $r \geq t/2$,

$$\begin{gathered}\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-4\varepsilon_0}, \qquad \|\hat{\chi}\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-\varepsilon_0} + \|\partial H(t)\|_{L_x^\infty}, \\ \|\eta\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1} + \lambda^{-\varepsilon_0} t^{-1} + \lambda^{\varepsilon}\|\partial(H)(t)\|_{L_x^\infty}\end{gathered} \tag{121}$$

where the last estimate holds for an arbitrary positive ε, $\varepsilon < \varepsilon_0$. *There are now the following estimates for the derivatives of* $\mathrm{tr}\chi$:

$$\begin{aligned}&\|\sup_{r\geq\frac{t}{2}}\|\underline{L}\left(\mathrm{tr}\chi - \frac{2}{r}\right)\|_{L^2(S_{t,u})}\|_{L_t^1} \\ &\qquad + \|\sup_{r\geq\frac{t}{2}}\|\underline{L}\left(\mathrm{tr}\chi - \frac{2}{n(t-u)}\right)\|_{L^2(S_{t,u})}\|_{L_t^1} \lesssim \lambda^{-3\varepsilon_0},\end{aligned} \tag{122}$$

$$\begin{aligned}&\|\sup_{r\geq\frac{t}{2}}\|\nabla\!\!\!/\,\mathrm{tr}\chi\|_{L^2(S_{t,u})}\|_{L_t^1} \\ &\qquad + \|\sup_{r\geq\frac{t}{2}}\|\nabla\!\!\!/\left(\mathrm{tr}\chi - \frac{2}{n(t-u)}\right)\|_{L^2(S_{t,u})}\|_{L_t^1} \lesssim \lambda^{-3\varepsilon_0}.\end{aligned} \tag{123}$$

In addition, *there are have weak estimates of the form*,

$$\sup_{u\leq\frac{t}{2}}\left\|(\nabla\!\!\!/, \underline{L})\left(\mathrm{tr}\chi - \frac{2}{n(t-u)}\right)\right\|_{L^\infty(S_{t,u})} \lesssim \lambda^C \tag{124}$$

for some large value of C.

Corollary 9.2. *The estimates of Theorem* 9.1 *can be extended to the whole region* $\mathcal{I}_0^+ \cap ([0,t_*]\times\mathbb{R}^3)$, *where* $\mathcal{I}_0^+$ *is the future domain of the origin on* Σ_0.

Remark 9.3. The proof of Corollary 9.2 requires an extension argument. The estimates of the Asymptotics Theorem, which are uniform with respect to the bootstrap region Ω_*, provide very good control of the foliations C_u and $S_{t,u}$. By the standard continuity argument this allows us to show that the estimates, in fact, hold in the maximal domain allowed by the background estimates (111)–(117) on the metric H, $\mathcal{I}_0^+ \cap ([0,t_*]\times\mathbb{R}^3)$.

Remark 9.4. Observe also that we can extend the results of (118)–(121) to a slightly larger domain $\mathcal{I}_{-1}^+ \cap ([0,t_*]\times\mathbb{R}^3)$. This is in fact needed to derive the first derivative estimates (122)–(123), in $\mathcal{I}_0^+ \cap ([0,t_*]\times\mathbb{R}^3)$, whose proof depends on Theorem 8.1. That theorem, to be proved in [Kl-Ro2], requires indeed the estimates for Θ, see definition below, in a slightly larger domain. The estimates for Θ however, i.e. (118)–(121), are independent of Theorem 8.1.

Proof. To simplify our calculations we start with the following definition.

Definition 9.5.

$$\Theta = \left|\mathrm{tr}\chi - \frac{2}{r}\right| + \left|\mathrm{tr}\chi - \frac{2}{s}\right| + |\hat\chi| + |\eta| + |\partial H|. \tag{125}$$

In our calculations below we shall often us the notation Θ but mean in fact $O(\Theta)$.

In view of our bootstrap assumptions B1), B2) (see Section 6.1), Remark 6.13, as well as the estimates (111), (112) for ∂H we can freely make use of the following:

$$\|\Theta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac12 - 2\varepsilon_0}, \qquad \|\Theta\|_{L^q(S_{t,u})} \lesssim \lambda^{-2\varepsilon_0} \tag{126}$$

inside the bootstrap region Ω_*.

9.6. *Estimates for* $\mathrm{tr}\chi, \hat\chi$. We start with estimates (118)–(121) for $\mathrm{tr}\chi$. Observe that in view of Corollary 6.12 it suffices to prove the desired estimates for $\mathrm{tr}\chi - \frac{2}{s}$.

Writing $y = (\mathrm{tr}\chi - \frac{2}{s})$ we have,

$$L(y) + \mathrm{tr}\chi y = -R_{44} - \frac{2}{s}\bar{k}_{NN} + \Theta^2. \tag{127}$$

Applying the transport Lemma 6.17 we infer that at any point $P \in \Omega_*$,

$$|s^2 y(P)| \lesssim \int_\gamma s^2\left(|\mathbf{R}_{44}| + \frac{1}{s}|\partial H| + \Theta^2\right)$$

where γ is the outgoing null geodesic starting on the time axis Γ_t, passing through P, and s is the corresponding value of the affine parameter s. Therefore,

$$|y(P)| \lesssim \|\mathbf{R}_{44}\|_{L^1_t L^\infty_x} + \frac{1}{s}\int_\gamma |\partial H| + \|\Theta\|^2_{L^2_t L^\infty_x}$$

and, in view of (126) and (113),

$$\|y(P)\|_{L^\infty} \lesssim \lambda^{-1-4\varepsilon_0} + \lambda^{-1-4\varepsilon_0} + \frac{1}{s}\int_\gamma |\partial H|. \tag{128}$$

In the exterior region $s \geq \frac{t}{2}$, using the condition (111), we infer that,

$$\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-4\varepsilon_0}, \tag{129}$$

which proves (121). On the other hand, see also the proof of Lemma 6.9, (128) leads to a global estimate,

$$\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty_x} \lesssim \lambda^{-1-4\varepsilon_0} + \mathcal{M}(\partial H)(t) \tag{130}$$

where $\mathcal{M}(\partial H)$ is the maximal function of $\|\partial H(t)\|_{L^\infty_x}$. The estimates (130) and (111) together with the corresponding maximal function estimates readily imply that

$$\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0} + \|\mathcal{M}(\partial H)(t)\|_{L^2_t} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0} + \|\partial H\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0}.$$

On the other hand, using the comparison results between r and s, see Section 6.3., $s \lesssim \lambda^{1-8\varepsilon_0} \lesssim \lambda$, and the Hölder inequalities

$$\begin{aligned}\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^q(S_{t,u})} &\lesssim r^{\frac{2}{q}}\|y\|_{L^\infty(S_{t,u})} \\ &\lesssim \lambda^{\frac{2}{q}}\lambda^{-1-4\varepsilon_0} + s^{\frac{2}{q}}s^{-\frac{1}{2}}\|\partial H\|_{L^2_t L^\infty_x} \lesssim \lambda^{\frac{2}{q}}\lambda^{-1-4\varepsilon_0} \lesssim \lambda^{-4\varepsilon_0}\end{aligned}$$

provided that $q > 2$ is chosen sufficiently close to 2. Using the comparison results between $\frac{2}{r}$ and $\frac{2}{s}$ of Corollary 6.12 we infer that,

$$\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0}, \tag{131}$$

$$\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^q(S_{t,u})} \lesssim \lambda^{-4\varepsilon_0}, \tag{132}$$

as desired in (118) and (119). Finally, (120) follows from (57) of Corollary 6.10.

We shall now estimate $\hat{\chi}$ from the Codazzi equations (37),

$$(\mathrm{d}\!\!/\mathrm{iv}\,\hat{\chi})_A + \hat{\chi}_{AB}k_{BN} = \frac{1}{2}(\nabla\!\!\!\!/_A\mathrm{tr}\chi + k_{AN}\mathrm{tr}\chi) - \mathbf{R}_{B4AB}. \tag{133}$$

Taking advantage of Corollary 4.5, with a different error term E, we rewrite (133) in the form,

$$(\not{\mathrm{d}}\mathrm{iv}\,\hat{\chi})_A = \frac{1}{2}\not\nabla_A\left(\mathrm{tr}\chi - \frac{2}{r}\right) + \not\nabla_A \pi + \not\nabla^B \pi_{AB} + E \tag{134}$$

with π and E obeying pointwise estimates

$$|\pi| \lesssim |\partial H|, \qquad |E| \lesssim |\Theta \cdot \partial H| + \frac{1}{r}|\partial H|.$$

We shall now take advantage of the elliptic estimate of Proposition 6.20 and write

$$\begin{aligned}\|\hat{\chi}\|_{L^\infty(S_{t,u})} \lesssim \lambda^\varepsilon & \left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^\infty(S_{t,u})} \\ & + \lambda^\varepsilon \|\pi\|_{L^\infty(S_{t,u})} + r^{1-\frac{2}{q}}\|E\|_{L^q(S_{t,u})}\end{aligned} \tag{135}$$

with $q > 2$.

Remark 9.7. In the application of the elliptic estimate (89) in the derivation of (135) we need some rough estimates for $\not\nabla\mathrm{tr}\chi$ of the type

$$\|r\not\nabla\mathrm{tr}\chi\|_{L^\infty(S_{t,u})} \lesssim \lambda^C$$

for some large constant $C > 0$. These weak estimates, consistent with (124), are a lot easier to derive and can be obtained directly from the transport equations (31), (32) for $\mathrm{tr}\chi$ and $\hat{\chi}$. We refer the reader to our paper [Kl-Ro] for more details.

Therefore, choosing $q = 2 + \varepsilon$ for sufficiently small $\varepsilon > 0$, and using the bootstrap assumptions B2) as well as assumptions (112) we infer that

$$\begin{aligned}\|\hat{\chi}\|_{L^\infty(S_{t,u})} & \lesssim \lambda^\varepsilon \left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^\infty(S_{t,u})} + \lambda^\varepsilon \|\partial H\|_{L^\infty_x} \\ & \quad + r^{1-\frac{2}{q}}\left(\|\Theta\|_{L^q(S_{t,u})}\|\partial H\|_{L^\infty_x} + r^{-1+\frac{2}{q}}\|\partial H\|_{L^\infty_x}\right) \\ & \lesssim \lambda^\varepsilon\left(\|\mathrm{tr}\chi - \frac{2}{r}\|_{L^\infty(S_{t,u})} + \|\partial H\|_{L^\infty_x}\right).\end{aligned} \tag{136}$$

Now we observe that the desired pointwise estimate (121) in the exterior region $r \ge t/2$ follows from (129) and the estimate $|\frac{2}{r} - \frac{2}{s}| \lesssim \lambda^{-\varepsilon_0}s^{-1} \lesssim \lambda^{-\varepsilon_0}t^{-1}$. Thus

$$\|\hat{\chi}\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-\varepsilon_0} + \|\partial H\|_{L^\infty_x}. \tag{137}$$

We can also add a global estimate following from Corollary 6.12[17] and (130)

$$\|\hat{\chi}\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1-4\varepsilon_0} + \partial H(t) + \mathcal{M}^4(\partial H)(t). \tag{138}$$

[17] Namely, the inequality $\|\mathrm{tr}\chi - \frac{2}{r}\|_{L^\infty_x} \lesssim \mathcal{M}^3(\|\mathrm{tr}\chi - \frac{2}{s}\|_{L^\infty_x})$.

Now squaring and integrating (136) in time we infer from (111) and the just proved estimate (131) for $\mathrm{tr}\chi - \frac{2}{r}$ that

$$\|\hat\chi\|_{L_t^2 L_x^\infty} \lesssim \lambda^{\varepsilon}\Big(\|\mathrm{tr}\chi - \frac{2}{r}\|_{L_t^2 L_x^\infty} + \|\partial H\|_{L_t^2 L_x^\infty}\Big) \lesssim \lambda^{-\frac{1}{2}-3\varepsilon_0}, \tag{139}$$

which is the estimate claimed in (118) of Theorem 9.1.

On the other hand, application of the elliptic estimate (88) of Proposition 6.20 to the equation (134) yields the following:

$$\|\hat\chi\|_{L^q(S_{t,u})} \lesssim \|\mathrm{tr}\chi - \frac{2}{r}\|_{L^q(S_{t,u})} + \|\partial H\|_{L^q(S_{t,u})} + \|E\|_{L^p(S_{t,u})}$$

for some $q \ge 2$, $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Choosing $q = 2 + \varepsilon$ as in bootstrap assumption B2) we infer, with the help of the estimate (132) for $\mathrm{tr}\chi - \frac{2}{r}$ and (112), that

$$\begin{aligned}\|\hat\chi\|_{L^q(S_{t,u})} &\lesssim \lambda^{-4\varepsilon_0} + \|\Theta\,\partial H\|_{L^p(S_{t,u})} + \frac{1}{r}\|\partial H\|_{L^p(S_{t,u})}\\ &\lesssim \lambda^{-4\varepsilon_0} + \|\partial H\|_{L^2(S_{t,u})}\|\Theta\|_{L^q(S_{t,u})} + \|\partial H\|_{L^q(S_{t,u})}\\ &\lesssim \lambda^{-4\varepsilon_0}.\end{aligned}$$

9.8. *Estimates for η.* We start with the Hodge system (38), (39):

$$\begin{aligned}\operatorname{d\!\!/iv}\eta &= \frac{1}{2}\Big(\mu + 2\bar k_{NN}\mathrm{tr}\chi - 2|\eta|^2 - |\hat\chi|^2 - 2k_{AB}\chi_{AB}\Big) - \frac{1}{2}\delta^{AB}\mathbf{R}_{A43B},\\ \operatorname{cu\!\!/rl}\eta &= \frac{1}{2}\varepsilon^{AB}k_{AC}\hat\chi_{CB} - \frac{1}{2}\varepsilon^{AB}\mathbf{R}_{A43B}\end{aligned}$$

with μ defined as in (34), $\mu = \underline{L}(\mathrm{tr}\chi) - \frac{1}{2}(\mathrm{tr}\chi)^2 - \big(k_{NN} + n^{-1}\nabla_N n\big)\mathrm{tr}\chi$, satisfying the transport equation (35),

$$\begin{aligned}L(\mu) + \mathrm{tr}\chi\mu &= 2(\underline\eta_A - \eta_A)\nabla\!\!\!/_A(\mathrm{tr}\chi)\\ &-2\hat\chi_{AB}\Big(2\nabla\!\!\!/_A\eta_B + 2\eta_A\eta_B + \bar k_{NN}\hat\chi_{AB}\\ &\qquad + \mathrm{tr}\chi\hat\chi_{AB} + \hat\chi_{AC}\hat\chi_{CB} + 2k_{AC}\chi_{CB} + \mathbf{R}_{B43A}\Big)\\ &-\underline{L}(\mathbf{R}_{44}) + (2k_{NN} - 4n^{-1}\nabla_N n)\left(\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat\chi|^2 - \bar k_{NN}\mathrm{tr}\chi - \mathbf{R}_{44}\right)\\ &+4\bar k_{NN}^2\mathrm{tr}\chi + (\mathrm{tr}\chi + 4\bar k_{NN})(|\hat\chi|^2 + \mathbf{R}_{44})\\ &-\mathrm{tr}\chi\Big(2(k_{AN} - \eta_A)n^{-1}\nabla_A n - 2|n^{-1}N(n)|^2 + \frac{1}{2}\mathbf{R}_{4343} + 2k_{Nm}k_N^m\Big).\end{aligned} \tag{140}$$

Observe that in view of Corollary 4.7 we can rewrite our div-curl system for η as follows:

(141)

$$\begin{aligned}\operatorname{d\!\!/iv}\eta &= \operatorname{d\!\!/iv}\pi^{(1)} + \frac{1}{2}\Big(\mu + 2\bar k_{NN}\mathrm{tr}\chi - 2|\eta|^2 - |\hat\chi|^2 - 2k_{AB}\chi_{AB}\Big) - \frac{1}{2}\mathbf{w} + E^{(1)},\\ \operatorname{cu\!\!/rl}\eta &= \operatorname{cu\!\!/rl}\pi^{(2)} + \frac{1}{2}\varepsilon^{AB}k_{AC}\hat\chi_{CB} + E^{(2)}\end{aligned}$$

where $\mathbf{w} = (\mathbf{R} + \mathbf{R}_{34})$ and

$$|\pi^{(1,2)}| \lesssim |\partial H|$$
$$|E^{(1,2)}| \lesssim (|\partial H|^2 + |\chi||\partial H|).$$

Remark 9.9. We would like to treat the system formed by the transport equation (140) coupled with the elliptic system (141) in the same manner as we have dealt with the system for $\mathrm{tr}\chi$ and $\hat{\chi}$. Indeed the Hodge system (141) is similar to the Hodge system (133). The transport equation for μ differs however significantly from the transport equation (127) for $\mathrm{tr}\chi$. Indeed the only curvature term on the right-hand side of (127) is $\mathbf{R}_{44}$ while the right-hand side of (140) exhibits the far more dangerous term $\underline{L}(\mathbf{R}_{44})$. In what follows we shall get around this difficulty by introducing a new covector $\not\mu$ through a Hodge system on the surfaces $S_{t,u}$. Using once more the special structure of the Einstein equations we shall derive a new transport equation for $\not\mu$ whose right-hand side exhibits only terms depending on $\mathbf{Ric}(H)$ and favorable components of the curvature tensor.

We define an auxiliary S-tangent co-vector $\not\mu_A$ as a solution of the Hodge system

$$\mathrm{d}\!\!/\mathrm{iv}\, \not\mu = \mu - \mathbf{w}, \tag{142}$$
$$\mathrm{cu}\!\!/\mathrm{rl}\, \not\mu = 0 \tag{143}$$

with $\mathbf{w} = \mathbf{R}_{43} + \mathbf{R}$. We now prove the following

PROPOSITION 9.10. (1) *The covector $\not\mu$ verifies the following*:

$$\begin{aligned}
\mathrm{d}\!\!/\mathrm{iv}\left(\not{D}_4\not\mu + \frac{1}{2}\mathrm{tr}\chi\not\mu - \hat{\chi}\cdot\not\mu\right) &= \partial H\cdot\not{D}_4\not\mu + \nabla\!\!\!/_A\left(2\mathbf{R}_{A4} + \frac{2}{r}\pi_A + \Theta\cdot\Theta\right)\\
&\quad -\frac{2}{r}(3\mathbf{R}_{34} + 2\mathbf{R}) + \Theta\mathbf{Ric} + \Theta\mathbf{R}_* + \Theta\cdot D_*\partial H\\
&\quad +\Theta\cdot\Theta\cdot\Theta + \frac{1}{r}\Theta\cdot\Theta + \frac{1}{r^2}\partial H,\\
\mathrm{cu}\!\!/\mathrm{rl}\left(\not{D}_4\not\mu + \frac{1}{2}\mathrm{tr}\chi\not\mu - \hat{\chi}\cdot\not\mu\right) &= \partial H\cdot\not{D}_4\not\mu + \nabla\!\!\!/(\Theta\cdot\Theta) + \mathbf{R}_*\cdot\Theta\\
&\quad +\frac{1}{r}\Theta\cdot\Theta + \Theta\cdot\Theta\cdot\Theta.
\end{aligned}$$

(2) *The covector $\not\mu$ verifies the following estimates*

$$\|\not\mu\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1} + \mathcal{M}(\partial\mathcal{H}), \tag{144}$$
$$\|\not\mu\|_{L^q(S_{t,u})} \lesssim \lambda^{-3\varepsilon_0}. \tag{145}$$

Proof of part 2 *of Proposition* 9.10.

Remark 9.11. For convenience we extend our bootstrap assumptions B1) and B2) to include $\not\mu$. Thus, throughout the proof below, we redefine Θ, see (125), as follows:

$$\Theta = O\left(\left|\mathrm{tr}\chi - \frac{2}{r}\right| + \left|\mathrm{tr}\chi - \frac{2}{s}\right| + |\hat{\chi}| + |\eta| + |\partial H| + |\not\mu|\right). \tag{146}$$

This is justified since our stated estimates are stronger than B1) and B2) for $\not\mu$.

Assuming the first part of Proposition 9.10 we now derive the estimates of part 2 and start by applying the elliptic estimates of Proposition 6.20 to the Hodge system of Proposition 9.10. Thus for some $q > 2$, with

$$M = \left(\not\mathcal{D}_4 \not\mu + \frac{1}{2}\mathrm{tr}\chi \not\mu - \hat{\chi} \cdot \not\mu\right),$$

we have,

$$\begin{aligned}
\|M\|_{L^\infty(S_{t,u})} \lesssim & \|\partial H\|_{L^q(S_{t,u})} \|M\|_{L^\infty(S_{t,u})} \\
& + \lambda^\varepsilon\left(\|\mathbf{Ric}(H)\|_{L^\infty(S_{t,u})} + \|\Theta\|^2_{L^\infty(S_{t,u})} + \frac{1}{r}\|\partial H\|_{L^\infty(S_{t,u})}\right) \\
& + r^{1-\frac{2}{q}}\Big(\|\Theta \mathbf{R}_*\|_{L^q(S_{t,u})} + \|\Theta \not\nabla(\partial H)\|_{L^q(S_{t,u})} \\
& \qquad + \|\Theta \mathbf{Ric}(H)\|_{L^q(S_{t,u})} + \frac{1}{r}\|\mathbf{Ric}(H)\|_{L^q(S_{t,u})} \\
& \qquad + \|\Theta^3\|_{L^q(S_{t,u})} + \frac{1}{r}\|\Theta^2\|_{L^q(S_{t,u})} + \frac{1}{r^2}\|\partial H\|_{L^q(S_{t,u})}\Big).
\end{aligned}$$

Remark 9.12. As in the case of the estimates for $\hat{\chi}$, the use of the elliptic estimates (89) of Proposition 6.20 for the Hodge system satisfied by the quantity M requires rough estimates of the type

$$\|r \not\nabla \mathbf{R}_{A4}\|_{L^\infty(S_{t,u})} + \|\not\nabla \pi\|_{L^\infty(S_{t,u})} + \|r\Theta \cdot \not\nabla \Theta\|_{L^q(S_{t,u})} \lesssim \lambda^C$$

for some $q > 2$. The estimate for the derivatives of the Ricci curvature and the metric H are contained in our background estimates (91)–(98). In addition to $\mathrm{tr}\chi$ and $\hat{\chi}$, for which we have already outlined the procedure of obtaining such weak estimates, the quantity Θ contains η and $\not\mu$. Once again, we can use the transport equation (33) for η and the Hodge system (142), (143) combined with the transport equation (140) for μ to handle these terms.

Taking q sufficiently close to $q = 2$, using the bootstrap assumption, $\|\Theta\|_{L^q(S_{t,u})} \lesssim \lambda^{-2\varepsilon_0} \lesssim 1$, and the estimate $\|\partial H\|_{L^q(S_{t,u})} \lesssim \lambda^{-2\varepsilon_0} < 1/2$ we can then conclude that

$$\|M\|_{L^\infty(S_{t,u})} \lesssim \lambda^\varepsilon \Big(\|\mathbf{Ric}(H)\|_{L^\infty(S_{t,u})} + \|\Theta\|^2_{L^\infty(S_{t,u})} + \frac{1}{r}\|\partial H\|_{L^\infty(S_{t,u})} + \|\Theta\|_{L^\infty(S_{t,u})}\|D_*\partial H\|_{L^q(S_{t,u})} + \|\Theta\|_{L^\infty(S_{t,u})}\|\mathbf{R}_*\|_{L^q(S_{t,u})}\Big).$$

Applying the transport Lemma 6.17 to the transport equation

$$\not{D}_4 \not{\mu} + \frac{1}{2}\mathrm{tr}\chi \not{\mu} = M + \hat{\chi}\cdot\not{\mu}, \tag{147}$$

we infer that at any point $P \in \Omega_*$,

$$|s\not{\mu}(P)| \lesssim \int_\gamma s\Big(|M| + \Theta^2\Big)$$

where γ is the outgoing null geodesic starting on the time axis Γ_t passing through P, and s is the corresponding value of the affine parameter s. Hence,

$$|\not{\mu}(P)| \lesssim \lambda^\varepsilon \int_\gamma \Big(\|\Theta\|^2_{L^\infty(S_{t,u})} + \|\Theta\|_{L^\infty(S_{t,u})}\|D_*\partial H\|_{L^q(S_{t,u})} + \|\Theta\|_{L^\infty(S_{t,u})}\|\mathbf{R}_*\|_{L^q(S_{t,u})}\Big) + \frac{1}{s}\int_\gamma \|\partial H\|_{L^\infty(S_{t,u})}.$$

Observe that by B1) and (115), we have

$$\begin{aligned}\int_\gamma \|\Theta\|_{L^\infty(S_{t,u})}\|D_*\partial H\|_{L^q(S_{t,u})} &\lesssim \|\Theta\|_{L^2_tL^\infty_x}\Big(\int_\gamma \|D_*\partial H\|^2_{L^q(S_{t,u})}\Big)^{\frac12} \\ &\lesssim \lambda^{-\frac12-2\varepsilon_0}\|\partial^2 H\|^{1-\frac2q}_{L^2_tL^\infty_x}\|D_*\partial H\|^{\frac2q}_{L^2(C_u)} \\ &\lesssim \lambda^{-\frac12-2\varepsilon_0}\lambda^{-(1+4\varepsilon_0)(\frac12-\frac1q)}\lambda^{-\frac1q} \\ &\lesssim \lambda^{-1-2\varepsilon_0}.\end{aligned}$$

A similar estimate, by (117), also holds for the term involving $\mathbf{R}_*$. Consequently,

$$\|\not{\mu}\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1} + \mathcal{M}(\partial\mathcal{H})$$

as desired.

Observe also that in the exterior region $r \geq \frac{t}{2}$,

$$\|\not{\mu}\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1} + r^{-1}\lambda^{-4\varepsilon_0}. \tag{148}$$

Going back to Proposition 9.10 and applying now the estimate (88) of Proposition 6.20 we deduce, for $\frac1p = \frac12 + \frac1q$,

$$\begin{aligned}\|M\|_{L^q(S_{t,u})} \lesssim &\|\partial H\|_{L^2(S_{t,u})}\|M\|_{L^q(S_{t,u})} + \|\mathbf{Ric}(H)\|_{L^q(S_{t,u})} \\ &+ \frac1r\|\partial H\|_{L^q(S_{t,u})} + \|\Theta\|^2_{L^{2q}(S_{t,u})} \\ &+ \|\Theta\|_{L^q(S_{t,u})}\Big(\|D_*\partial H\|_{L^2(S_{t,u})} + \|\mathbf{R}_*\|_{L^2(S_{t,u})}\Big) \\ &+ r^{\frac2p-1}\|\Theta\|^2_{L^\infty(S_{t,u})} + r^{\frac2p-2}\|\partial H\|_{L^\infty(S_{t,u})}.\end{aligned}$$

According to the estimates (112), $\|\partial H\|_{L^2(S_{t,u})} \lesssim \lambda^{-4\varepsilon_0} < 1$. Thus we can absorb the term with M into the left-hand side.

On the other hand, using the transport Lemma 6.17 applied to the transport equation (147) we infer,

$$\|\not{\mu}\|_{L^q(S_{t,u})} \lesssim \frac{1}{r(t)^{(1-\frac{2}{q})}} \int_u^t r(t')^{(1-\frac{2}{q})} \Big(\|M\|_{L^q(S_{t',u})} + \|\Theta\|^2_{L^{2q}(S_{t',u})} \Big) dt'.$$

Applying the bootstrap assumptions B1), B2), and (111)–(117) we infer that,

$$\begin{aligned}
\|\not{\mu}\|_{L^q(S_{t,u})} \lesssim & \lambda^{\varepsilon} \bigg(\lambda^{\frac12} \|\mathbf{Ric}(H)\|^{\frac12}_{L^1_t L^\infty_x} \|\mathbf{Ric}(H)\|^{\frac12}_{L^{\frac{q}{2}}(S_{t,u})} + \|\Theta\|_{L^1_t L^\infty_x} \|\Theta\|_{L^q(S_{t,u})} \\
& + \frac{1}{r^{(1-\frac{2}{q})}} \int_u^t r(t')^{(1-\frac{2}{q})} r(t')^{\frac{2}{q}-1} \|\partial H\|_{L^\infty_x(S_{t',u})} dt' \bigg) \\
& + r^{\frac12} \|\Theta\|_{L^q(S_{t,u})} \|D_* \partial H\|_{L^2(C_u)} + r^{1-\frac{2p}{q}} \|\Theta\|^2_{L^q(S_{t,u})} \|\Theta\|_{L^1_t L^\infty_x} \\
& + r^{\frac{2}{p}-1} \|\Theta\|^2_{L^2_t L^\infty_x} + \frac{1}{r^{(1-\frac{2}{q})}} \int_u^t r(t')^{(1-\frac{2}{q})} r(t')^{\frac{2}{p}-2} \|\partial H\|_{L^\infty_x(S_{t',u})} dt' \\
\lesssim & \lambda^{-3\varepsilon_0} + r^{\frac{2}{p}-\frac32} \|\partial H\|_{L^2_t L^\infty_x} \lesssim \lambda^{-3\varepsilon_0}
\end{aligned}$$

as desired. On the right-hand side of the last series of inequalities, for the sake of brevity, we have abused the notation using $\|f\|_{L^q(S_{t,u})}$ to denote $\sup_{t,u} \|f\|_{L^q(S_{t,u})}$.

Using the estimates (144) and (145) for $\not{\mu}$ we are now ready to return to the proof of the estimates for η. Now with the help of the established estimates for $\not{\mu}$ we shall derive the desired estimates for $\|\eta\|_{L^2_t L^\infty_x}$ and $\|\eta\|_{L^q(S_{t,u})}$. First observe that using the definition (142), (143) of $\not{\mu}$ the div-curl system (141) for η takes the form

$$\begin{aligned}
\not{\mathrm{div}} \left(\eta - \frac12 \not{\mu} \right) &= \not{\mathrm{div}}\, \pi^{(1)} + \frac{1}{r} \partial H + \Theta \cdot \Theta, \\
\not{\mathrm{curl}} \left(\eta - \frac12 \not{\mu} \right) &= \not{\mathrm{curl}}\, \pi^{(2)} + \frac{1}{r} \partial H + \Theta \cdot \Theta.
\end{aligned}$$

We are now ready to apply Proposition 6.20 to our Hodge system for $\eta - \frac12 \not{\mu}$. Thus, for some $q > 2$, sufficiently close to 2,

$$\begin{aligned}
\|\eta - \frac12 \not{\mu}\|_{L^\infty(S_{t,u})} &\lesssim \lambda^{\varepsilon} \|\partial H\|_{L^\infty(S_{t,u})} + r^{-\frac{2}{q}} \|\partial H\|_{L^q(S_{t,u})} + r^{1-\frac{2}{q}} \|\Theta^2\|_{L^q(S_{t,u})} \\
&\lesssim \lambda^{\varepsilon} \|\partial H\|_{L^\infty(S_{t,u})} + \lambda^{-\varepsilon_0} \|\Theta\|_{L^\infty(S_{t,u})},
\end{aligned}$$

where we have used the bootstrap estimate $\|\Theta\|_{L^q(S_{t,u})} \lesssim \lambda^{-2\varepsilon_0}$. Furthermore, we infer with the help of (144) that

$$\|\eta\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1} + \mathcal{M}(\partial \mathcal{H}) + \lambda^{\varepsilon} \|\partial H\|_{L^\infty(S_{t,u})} + \lambda^{-\varepsilon_0} \|\Theta\|_{L^\infty(S_{t,u})}. \tag{149}$$

The desired $L_t^2 L_x^\infty$ estimate follows immediately from the bootstrap assumption B1) and the estimates (111)–(117).

Consider also the exterior region $r \geq \frac{t}{2}$. Observe that, using the estimates (129), (137) and (148) for $\mathrm{tr}\chi - \frac{2}{r}$, $\hat{\chi}$, $\not{\mu}$ already established in the exterior region we infer that,

$$\|\eta\|_{L^\infty(S_{t,u})} \lesssim \lambda^{-1} + \lambda^{-\varepsilon_0} t^{-1} + \lambda^{\varepsilon} \|\partial H\|_{L_x^\infty}. \tag{150}$$

On the other hand, for $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$,

$$\left\| \eta - \frac{1}{2}\not{\mu} \right\|_{L^q(S_{t,u})} \lesssim \lambda^{\varepsilon} \|\partial H\|_{L^q(S_{t,u})} + \frac{2}{r} \|\partial H\|_{L^p(S_{t,u})} + \|\Theta^2\|_{L^p(S_{t,u})}.$$

Since $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$ and $q \geq 2$, we have $2p \leq q$ and the Hölder inequality gives

$$\|\Theta\|^2_{L^{2p}(S_{t,u})} \lesssim r^{2-\frac{4}{q}} \|\Theta\|^2_{L^q(S_{t,u})} \lesssim \lambda^{-3\varepsilon_0}$$

from the bootstrap assumption B2), provided that q is sufficiently close to 2. Thus with the help of (112) and the estimate (145) we obtain,

$$\|\eta\|_{L^q(S_{t,u})} \lesssim \|\not{\mu}\|_{L^q(S_{t,u})} + \lambda^{-3\varepsilon_0} \lesssim \lambda^{-3\varepsilon_0} \tag{151}$$

as desired.

Proof of part 1 *of Proposition* 9.10. We start by expressing the transport equation (140) for $\mu = \underline{L}(\mathrm{tr}\chi) - \frac{1}{2}(\mathrm{tr}\chi)^2 - \big(k_{NN} + n^{-1}\nabla_N n\big)\mathrm{tr}\chi$ in a more tractable form. The troublesome terms are $\underline{L}\mathbf{R}_{44}$ and $\mathrm{tr}\chi\mathbf{R}_{4343}$. We shall first eliminate $\underline{L}R_{44}$ in exchange for more favorable terms. We do this with the help of the twice contracted Bianchi identity:

$$D^\nu \left(\mathbf{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathbf{R} \right) = 0$$

with $\mathbf{R}$ the scalar curvature $= g^{\mu\nu}\mathbf{R}_{\mu\nu}$. Thus, relative to our canonical null frame,

$$D^3\mathbf{R}_{43} + D^4\mathbf{R}_{44} + D^A\mathbf{R}_{4A} = \frac{1}{2}L(\mathbf{R}),$$

or,

$$D_3\mathbf{R}_{44} = -D_4\mathbf{R}_{43} + 2D^A\mathbf{R}_{4A} - L(\mathbf{R}).$$

On the other hand,

$$\begin{aligned} D_3\mathbf{R}_{44} &= \underline{L}\mathbf{R}_{44} - 4\eta_A\mathbf{R}_{A4} - 2\bar{k}_{NN}\mathbf{R}_{44}, \\ D_4\mathbf{R}_{43} &= L\mathbf{R}_{43} - 2\underline{\eta}_A\mathbf{R}_{4A}, \\ D^A\mathbf{R}_{4A} &= \nabla\!\!\!/^A\mathbf{R}_{4A} - \chi_{AC}\mathbf{R}_{CA} + k_{AN}\mathbf{R}_{4A} - \frac{1}{2}\mathrm{tr}\chi\mathbf{R}_{43} - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathbf{R}_{44}. \end{aligned}$$

Therefore,

$$\begin{aligned} \underline{L}(\mathbf{R}_{44}) = &-L(\mathbf{R}_{43} + \mathbf{R}) + 2\nabla\!\!\!/^A\mathbf{R}_{4A} \\ &-(2\mathbf{R}_{34} + \mathbf{R} - \mathbf{R}_{44}) \cdot \mathrm{tr}\chi + \mathbf{Ric} \cdot (\hat{\chi}, k, \eta). \end{aligned}$$

Using this formula we can rewrite the transport equation for μ in the form:

$$\begin{aligned}L(\mu)+\mathrm{tr}\chi\mu = {} & L(\mathbf{w})+2\nabla\!\!\!/_A\mathbf{R}_{A4}+\mathrm{tr}\chi(2\mathbf{R}_{34}+\mathbf{R})-\mathrm{tr}\chi\mathbf{R}_{4343}\\ & +2(\underline{\eta}_A-\eta_A)\nabla\!\!\!/_A\mathrm{tr}\chi-4\hat{\chi}\cdot\nabla\!\!\!/\eta+\Theta\cdot\mathbf{R}_*\\ & +\Theta\cdot\Theta\cdot\Theta+\Theta\cdot\mathbf{Ric}+\frac{1}{r}\Theta\cdot\Theta+\frac{1}{r^2}\partial H,\end{aligned}$$

where $\mathbf{w}=\mathbf{R}_{43}+\mathbf{R}$. Thus

$$\begin{aligned}L(\mu-\mathbf{w})+\mathrm{tr}\chi(\mu-\mathbf{w}) = {} & 2\nabla\!\!\!/_A\mathbf{R}_{A4}+\mathrm{tr}\chi\mathbf{R}_{34}-\mathrm{tr}\chi\mathbf{R}_{4343}+\Theta\cdot\mathbf{Ric}\\ & +2(\underline{\eta}_A-\eta_A)\nabla\!\!\!/_A\mathrm{tr}\chi-4\hat{\chi}\cdot\nabla\!\!\!/\eta+\Theta\cdot\mathbf{R}_*\\ & +\Theta\cdot\Theta\cdot\Theta+\frac{1}{r}\Theta\cdot\Theta+\frac{1}{r^2}\partial H.\end{aligned}\tag{152}$$

Observe that $\mathbf{R}_{34}=H^{\alpha\beta}\mathbf{R}_{\alpha3\beta4}=\frac{1}{2}\mathbf{R}_{4343}-\delta^{AB}\mathbf{R}_{A34B}$. Also, $\mathbf{R}_{AB}=-\frac{1}{2}\mathbf{R}_{3A4B}-\frac{1}{2}\mathbf{R}_{4A3B}+\delta^{CD}\mathbf{R}_{CADB}$. Therefore,

$$\mathbf{R}_{3434}=2(\mathbf{R}_{34}+\delta^{AB}\mathbf{R}_{AB})+\delta^{CD}\delta^{AB}\mathbf{R}_{CADB}$$

or, since $\delta^{AB}\mathbf{R}_{AB}=\mathbf{R}_{34}+\mathbf{R}$, by Corollary 4.4 for $\mathbf{R}_{ABCD}$,

$$\mathbf{R}_{3434}=2(2\mathbf{R}_{34}+\mathbf{R})-\mathrm{d}\!\!\!/\mathrm{iv}\,\pi+E,$$

where

$$|\pi|\lesssim|\partial H|\qquad\text{and}\qquad|E|\lesssim|\partial H|^2+|\chi||\partial H|.$$

Using this we can rewrite (152) in the form,

(153)

$$\begin{aligned}L(\mu-\mathbf{w})+\mathrm{tr}\chi(\mu-\mathbf{w}) = {} & 2\nabla\!\!\!/_A\mathbf{R}_{A4}+\mathrm{tr}\chi\mathrm{d}\!\!\!/\mathrm{iv}\,\pi-\mathrm{tr}\chi(3\mathbf{R}_{34}+2\mathbf{R})\\ & +2(\underline{\eta}_A-\eta_A)\nabla\!\!\!/_A\mathrm{tr}\chi-4\hat{\chi}\cdot\nabla\!\!\!/\eta+\Theta\cdot\mathbf{Ric}+\Theta\cdot\mathbf{R}_*\\ & +\Theta\cdot\Theta\cdot\Theta+\frac{1}{r}\Theta\cdot\Theta+\frac{1}{r^2}\partial H.\end{aligned}$$

Recall that we defined an S-tangent co-vector $\mu\!\!\!/_A$ as a solution of the Hodge system

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\mu\!\!\!/=\mu-\mathbf{w},\tag{154}$$

$$\mathrm{c}\!\!\!/\mathrm{url}\,\mu\!\!\!/=0.\tag{155}$$

We shall now use the commutation formula of Lemma 3.5.

$$\begin{aligned}\mathrm{d}\!\!\!/\mathrm{iv}\,(\nabla\!\!\!/_4\mu\!\!\!/)-L(\mathrm{d}\!\!\!/\mathrm{iv}\,\mu\!\!\!/) = {} & \frac{1}{2}\mathrm{tr}\chi\mathrm{d}\!\!\!/\mathrm{iv}\,\mu\!\!\!/+\hat{\chi}\cdot\nabla\!\!\!/\mu\!\!\!/-n^{-1}\nabla\!\!\!/n\cdot\nabla\!\!\!/_4\mu\!\!\!/\\ & +\frac{1}{2}\mathrm{tr}\chi\bar{k}_{AN}\mu\!\!\!/_A-\hat{\chi}_{AB}\bar{k}_{BN}\mu\!\!\!/+\mathbf{R}_{AB4B}\mu\!\!\!/_A,\\ \mathrm{c}\!\!\!/\mathrm{url}\,(\nabla\!\!\!/_4\mu\!\!\!/)-L(\mathrm{c}\!\!\!/\mathrm{url}\,\mu\!\!\!/) = {} & \frac{1}{2}\mathrm{tr}\chi\mathrm{c}\!\!\!/\mathrm{url}\,\mu\!\!\!/+\varepsilon^{BA}\hat{\chi}_{BC}\nabla\!\!\!/_C\mu\!\!\!/_A-\varepsilon^{BA}n^{-1}\nabla\!\!\!/_Bn\nabla\!\!\!/_4\mu\!\!\!/_A\\ & -\varepsilon^{BC}\chi_{BA}\bar{k}_{CN}\mu\!\!\!/_A+\varepsilon^{BC}\mathbf{R}_{AC4B}\mu\!\!\!/_A.\end{aligned}$$

Using the transport equation (152) and commuting L with $\slashed{\mathrm{div}}$ and $\slashed{\mathrm{curl}}$ (see Lemma 3.5) we can derive the following Hodge system for $\slashed{\mathcal{D}}_4(\slashed{\mu})$:

$$\begin{aligned}
\slashed{\mathrm{div}}\,(\slashed{\mathcal{D}}_4\slashed{\mu}) = & -\frac{1}{2}\mathrm{tr}\chi\,\slashed{\mathrm{div}}\,\slashed{\mu} + \hat{\chi}\cdot\slashed{\nabla}\slashed{\mu} + \partial H\cdot\slashed{\mathcal{D}}_4\slashed{\mu} + \frac{2}{r}\slashed{\mathrm{div}}\,\pi \\
& +2(\underline{\eta}_A - \eta_A)\slashed{\nabla}_A\mathrm{tr}\chi - 4\hat{\chi}\cdot\slashed{\nabla}\eta + \mathbf{R}_{AB4B}\slashed{\mu}_A + 2\slashed{\nabla}_A\mathbf{R}_{A4} \\
& -\frac{2}{r}(3\mathbf{R}_{34} + 2\mathbf{R}) + \Theta\mathbf{Ric} + \Theta\mathbf{R}_* + \Theta\cdot D_*\partial H \\
& +\Theta\cdot\Theta\cdot\Theta + \frac{1}{r}\Theta\cdot\Theta + \frac{1}{r^2}\partial H, \\
\slashed{\mathrm{curl}}\,(\slashed{\mathcal{D}}_4\slashed{\mu}) = & \frac{1}{2}\mathrm{tr}\chi\,\slashed{\mathrm{curl}}\,\slashed{\mu} + \partial H\cdot\slashed{\mathcal{D}}_4\slashed{\mu} + \varepsilon^{BA}\slashed{\nabla}_C\mu_A\hat{\chi}_{BC} \\
& +\varepsilon^{CB}\mathbf{R}_{AB4C}\slashed{\mu}_A + \frac{1}{r}\Theta\cdot\Theta + \Theta\cdot\Theta\cdot\Theta.
\end{aligned}$$

Remark 9.13. We got rid of the dangerous term $\underline{L}(\mathbf{R}_{44})$. We still need to eliminate the terms of the form $\Theta\cdot\slashed{\nabla}\Theta$.

Observe that, according to the Codazzi equation,

$$\slashed{\mathrm{div}}\,\hat{\chi}_A - \frac{1}{2}\slashed{\nabla}_A\mathrm{tr}\chi = \frac{1}{2}k_{AN}\mathrm{tr}\chi - \hat{\chi}_{BN}k_{BN} - \mathbf{R}_{B4AB}.$$

Therefore,

$$\begin{aligned}
-\frac{1}{2}\mathrm{tr}\chi\,\slashed{\mathrm{div}}\,\slashed{\mu} + \hat{\chi}_{AB}\slashed{\nabla}_B\slashed{\mu}_A = & -\frac{1}{2}\slashed{\mathrm{div}}\,(\mathrm{tr}\chi\slashed{\mu}) + \slashed{\nabla}^B(\hat{\chi}_{AB}\slashed{\mu}_A) \\
& +\frac{1}{2}\slashed{\mu}\cdot\slashed{\nabla}\mathrm{tr}\chi - (\slashed{\nabla}^B\hat{\chi}_{AB})\slashed{\mu}_A \\
= & -\slashed{\nabla}^A\left(\frac{1}{2}\mathrm{tr}\chi\slashed{\mu}_A - \hat{\chi}_{AB}\slashed{\mu}_B\right) \\
& -\slashed{\mu}\left(\frac{1}{2}k_{AN}\mathrm{tr}\chi - \hat{\chi}_{BN}k_{BN} - \mathbf{R}_{B4AB}\right).
\end{aligned}$$

Thus

(156)
$$\begin{aligned}
\slashed{\mathrm{div}}\left(\slashed{\mathcal{D}}_4\slashed{\mu} + \frac{1}{2}\mathrm{tr}\chi\slashed{\mu}_A - \hat{\chi}_{AB}\slashed{\mu}_B\right) = & \,\partial H\cdot\slashed{\mathcal{D}}_4\slashed{\mu} + \frac{2}{r}\slashed{\mathrm{div}}\,\pi \\
& +2(\underline{\eta}_A - \eta_A)\slashed{\nabla}_A\mathrm{tr}\chi - 4\hat{\chi}\cdot\slashed{\nabla}\eta \\
& +2\slashed{\nabla}_A\mathbf{R}_{A4} - \frac{2}{r}(3\mathbf{R}_{34} + 2\mathbf{R}) + \Theta\mathbf{Ric} + \Theta\mathbf{R}_* \\
& +\Theta\cdot D_*\partial H + \Theta\cdot\Theta\cdot\Theta + \frac{1}{r}\Theta\cdot\Theta + \frac{1}{r^2}\partial H.
\end{aligned}$$

Also, since $\slashed{\mathrm{curl}}\,\slashed{\mu} = 0$,

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}\chi\,\slashed{\mathrm{curl}}\,\slashed{\mu} + \varepsilon^{BA}\slashed{\nabla}_C\slashed{\mu}_A\hat{\chi}_{BC} = & \,\varepsilon^{BA}\slashed{\nabla}_A\slashed{\mu}_C\hat{\chi}_{BC} \\
= & -\varepsilon^{AB}\slashed{\nabla}_A\,(\hat{\chi}_{BC}\slashed{\mu}_C) + \varepsilon^{AB}\slashed{\nabla}_A\hat{\chi}_{BC}\slashed{\mu}_C
\end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon^{AB} \nabla\!\!\!\!/_A \left(\frac{1}{2} \mathrm{tr}\chi \mu\!\!\!/_B \hat{\chi}_{BC} \mu\!\!\!/_C\right) \\
&\quad + \varepsilon^{AB} \nabla\!\!\!\!/_A (\mathrm{tr}\chi) \mu\!\!\!/_B + \varepsilon^{AB} (\nabla\!\!\!\!/_A \chi_{BC}) \mu\!\!\!/_C.
\end{aligned}
$$

On the other hand (see [Kl-Ro, §2]), $\nabla\!\!\!\!/_A \chi_{BC} = \nabla\!\!\!\!/_C \chi_{AB} - \mathbf{R}_{B4CA} + k \cdot \chi$. Therefore,

(157)

$$
\begin{aligned}
\mathrm{c{\!\!/}url} \left(\mathcal{D}\!\!\!\!/_4 \mu\!\!\!/ + \frac{1}{2} \mathrm{tr}\chi \mu\!\!\!/_A - \hat{\chi}_{AB} \mu\!\!\!/_B\right) &= \partial H \cdot \mathcal{D}\!\!\!\!/_4 \mu\!\!\!/ - 2 \mathrm{c{\!\!/}url} (\hat{\chi} \cdot \mu\!\!\!/) + \varepsilon^{AB} \nabla\!\!\!\!/_A (\mathrm{tr}\chi) \mu\!\!\!/_B \\
&\quad + \frac{1}{r} \Theta \cdot \Theta + \Theta \cdot \Theta \cdot \Theta.
\end{aligned}
$$

Observe also, in (156), by Codazzi,

$$
\begin{aligned}
-2\eta_A \nabla\!\!\!\!/_A \mathrm{tr}\chi - 4\hat{\chi}^{AB} \cdot \nabla\!\!\!\!/_B \eta_A &= -2\eta_A \nabla\!\!\!\!/_A \mathrm{tr}\chi - 4\nabla\!\!\!\!/^A (\hat{\chi}_{AB} \eta_B) + 4\nabla\!\!\!\!/^A \hat{\chi}_{AB} \eta_B \\
&= -4\nabla\!\!\!\!/^A (\hat{\chi}_{AB} \eta_B) + 4\eta_B \mathbf{R}_{A4BA} + \eta \cdot \chi \cdot k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{d{\!\!/}iv} \left(\mathcal{D}\!\!\!\!/_4 \mu\!\!\!/ + \frac{1}{2} \mathrm{tr}\chi \mu\!\!\!/_A - \hat{\chi}_{AB} \mu\!\!\!/_B\right) &= \partial H \cdot \mathcal{D}\!\!\!\!/_4 \mu\!\!\!/ + \nabla\!\!\!\!/_A \left(2\mathbf{R}_{A4} - 4\hat{\chi}_{AB} \eta_B + \frac{2}{r} \pi_A\right) \\
&\quad + 2\underline{\eta}_A \nabla\!\!\!\!/_A \mathrm{tr}\chi - \frac{2}{r} (3\mathbf{R}_{34} + 2\mathbf{R}) + \Theta \mathbf{Ric} + \Theta \mathbf{R}_* \\
&\quad + \Theta \cdot D_* \partial H + \Theta \cdot \Theta \cdot \Theta + \frac{1}{r} \Theta \cdot \Theta + \frac{1}{r^2} \partial H.
\end{aligned}
$$

In addition, since $\underline{\eta}_A = -\bar{k}_{AN}$,

$$
\begin{aligned}
\underline{\eta}_A \nabla\!\!\!\!/_A \mathrm{tr}\chi &= -\nabla\!\!\!\!/_A \left(\bar{k}_{AN} \left(\mathrm{tr}\chi - \frac{2}{r}\right)\right) + \left(\mathrm{tr}\chi - \frac{2}{r}\right) \nabla\!\!\!\!/_A (\bar{k}_{AN}) \\
&= -\nabla\!\!\!\!/_A \left(\bar{k}_{AN} \left(\mathrm{tr}\chi - \frac{2}{r}\right)\right) + \Theta \cdot D_* \partial H.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathrm{d{\!\!/}iv} \left(\mathcal{D}\!\!\!\!/_4 \mu\!\!\!/ + \frac{1}{2} \mathrm{tr}\chi \mu\!\!\!/_A - \hat{\chi}_{AB} \mu\!\!\!/_B\right) \\
&\quad = \partial H \cdot \mathcal{D}\!\!\!\!/_4 \mu\!\!\!/ + \nabla\!\!\!\!/_A \left(2\mathbf{R}_{44} - 4\hat{\chi}_{AB} \eta_B + \frac{2}{r} \pi - 2\bar{k}_{AN} \left(\mathrm{tr}\chi - \frac{2}{r}\right)\right) \\
&\qquad - \frac{2}{r} (3\mathbf{R}_{34} + 2\mathbf{R}) + \Theta \mathbf{Ric} + \Theta \mathbf{R}_* + \Theta \cdot D_* \partial H \\
&\qquad + \Theta \cdot \Theta \cdot \Theta + \frac{1}{r} \Theta \cdot \Theta + \frac{1}{r^2} \partial H.
\end{aligned}
$$

Since $\nabla\!\!\!/ r = 0$ and $\text{c}\nabla\!\!\!/\text{rl}\, \mu\!\!\!/ = 0$, the corresponding curl equation takes the following final form:

$$\begin{aligned}
&\text{c}\nabla\!\!\!/\text{rl}\left(\mathcal{D}\!\!\!/_4 \mu\!\!\!/ + \frac{1}{2}\text{tr}\chi \mu\!\!\!/_A + \hat{\chi}_{AB}\mu\!\!\!/_B\right)\\
&\qquad = \partial H \cdot \mathcal{D}\!\!\!/_4 \mu\!\!\!/ - 2\text{c}\nabla\!\!\!/\text{rl}\,(\hat{\chi}\cdot\mu\!\!\!/) + \varepsilon^{AB}\nabla\!\!\!/_A\left(\left(\text{tr}\chi - \frac{2}{r}\right)\mu\!\!\!/_B\right)\\
&\qquad\quad + \frac{1}{r}\Theta\cdot\Theta + \Theta\cdot\Theta\cdot\Theta.
\end{aligned}$$

9.14. *Estimate for* $\nabla\!\!\!/\text{tr}\chi$. To estimate $\nabla\!\!\!/\text{tr}\chi$ we commute (taking advantage of Lemma 3.5) the equation for $\text{tr}\chi$ with angular derivatives $\nabla\!\!\!/$. Therefore,

$$\begin{aligned}
\mathcal{D}\!\!\!/_4\nabla\!\!\!/\text{tr}\chi + \frac{3}{2}\text{tr}\chi\nabla\!\!\!/\text{tr}\chi &= -\nabla\!\!\!/ R_{44} - \text{tr}\chi\nabla\!\!\!/\bar{k}_{NN} - \bar{k}_{NN}\nabla\!\!\!/\text{tr}\chi - 2\nabla\!\!\!/\hat{\chi}\cdot\hat{\chi}\\
&\quad - \frac{1}{2}n^{-1}\nabla\!\!\!/ n\left(\frac{1}{2}\text{tr}\chi^2 + \bar{k}_{NN}\text{tr}\chi + \mathbf{R}_{44}\right).
\end{aligned}$$

Using the transport Lemma 6.17 we deduce

$$\begin{aligned}
\|\nabla\!\!\!/\text{tr}\chi\|_{L^2(S_{t,u})} \lesssim \frac{1}{r^2(t)}\int_u^t r(t')^2
&\cdot\Big(\|\nabla\!\!\!/\mathbf{R}_{44}\|_{L^2(S_{t',u})} + r(t')^{-1}\|\nabla\!\!\!/\partial H\|_{L^2(S_{t',u})}\\
&+ r(t')^{-2}\|\partial H\|_{L^2(S_{t',u})} + \|\hat{\chi}\cdot\nabla\!\!\!/\hat{\chi}\|_{L^2(S_{t',u})}\\
&+ r(t')^{-1}\|(\partial H)^2\|_{L^2(S_{t',u})} + \|\partial H\mathbf{Ric}(H)\|_{L^2(S_{t',u})}\Big)dt'.
\end{aligned}$$

Consider the most dangerous term $\frac{1}{r^2(t)}\int_u^t r(t')^2\|\nabla\!\!\!/\mathbf{R}_{44}\|_{L^2(S_{t',u})}dt'$. We estimate it with the help of estimate (116) and find,

$$\frac{1}{r^2(t)}\int_u^t r(t')^2\|\nabla\!\!\!/\mathbf{R}_{44}\|_{L^2(S_{t',u})}dt' \lesssim \int_u^t \|\nabla\!\!\!/\mathbf{R}_{44}\|_{L^2(S_{t',u})}dt' \lesssim \lambda^{-1-2\varepsilon_0}.$$

Also, with the help of (115),

$$\frac{1}{r^2(t)}\int_u^t r(t')\|\nabla\!\!\!/\partial H\|_{L^2(S_{t',u})}dt' \lesssim r^{-\frac{1}{2}}\|\nabla\!\!\!/\partial H\|_{L^2(C_u)} \lesssim r^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}.$$

All other terms are easier to treat. Therefore,

$$r^{\frac{1}{2}}\|\nabla\!\!\!/\text{tr}\chi\|_{L^2(S_{t,u})} \lesssim r^{\frac{1}{2}}\lambda^{-1-2\varepsilon_0} + \lambda^{-\frac{1}{2}} + \int_u^t r(t')^{\frac{1}{2}}\|\hat{\chi}\cdot\nabla\!\!\!/\hat{\chi}\|_{L^2(S_{t',u})}dt'. \tag{158}$$

It remains to estimate $\nabla\!\!\!/\chi$. We do this with the help of Proposition 6.19 applied to the Codazzi equation (37) written in the form (134). Thus

$$\int_{S_{t,u}} |\nabla\!\!\!/\hat{\chi}|^2 + \frac{1}{r^2}|\hat{\chi}|^2 \le \int_{S_{t,u}} |\nabla\!\!\!/\text{tr}\chi|^2 + |\nabla\!\!\!/\partial H|^2 + \frac{1}{r^2}|\partial H|^2 + |\Theta|^4.$$

Therefore,

$$\|\nabla\!\!\!/\hat{\chi}\|_{L^2(S_{t,u})} \le \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(S_{t,u})} + \|\nabla\!\!\!/\partial H\|_{L^2(S_{t,u})} + \|\partial H(t)\|_{L^\infty(S_{t,u})} + \|\Theta\|^{2-\frac{q}{2}}_{L^\infty_x}\|\Theta\|^{\frac{q}{2}}_{L^q(S_{t,u})} \tag{159}$$

for some $q > 2$. Observe that we can take q sufficiently close to 2 and use the already proved estimates (119) to obtain $\|\Theta\|^{\frac{q}{2}}_{L^q(S_{t,u})} \lesssim \lambda^{-3\varepsilon_0}$. In addition, observe that by Hölder inequality and (118)

$$\int_0^s \|\Theta\|^{4-q}_{L^\infty_x} \lesssim s^{\frac{q-2}{2}}\|\Theta\|^{4-q}_{L^2_tL^\infty_x} \lesssim \lambda^{-1-6\varepsilon_0} \tag{160}$$

for all values of q sufficiently close to 2.

Using (159) we estimate,

$$\begin{aligned}
&\int_u^t r(t')^{\frac12}\|\hat{\chi}\cdot\nabla\!\!\!/\hat{\chi}\|_{L^2(S_{t',u})}dt'\\
&\quad\lesssim \int_u^t r(t')^{\frac12}\|\hat{\chi}\|_{L^\infty(S_{t',u})}\|\nabla\!\!\!/\hat{\chi}\|_{L^2(S_{t',u})}dt'\\
&\quad\lesssim \int_u^t r(t')^{\frac12}\|\hat{\chi}\|_{L^\infty(S_{t',u})}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(S_{t',u})}dt'\\
&\qquad + r^{\frac12}(t)\|\hat{\chi}\|_{L^2_tL^\infty_x}\Big(\|\nabla\!\!\!/\partial H\|_{L^2(C_u)} + \|\partial H\|_{L^2_tL^\infty_x} + \lambda^{-\frac12-4\varepsilon_0}\Big)\\
&\quad\lesssim \int_u^t r(t')^{\frac12}\|\hat{\chi}\|_{L^\infty(S_{t',u})}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(S_{t',u})}\,dt' + r^{\frac12}(t)\lambda^{-1-4\varepsilon_0}.
\end{aligned}$$

Here we have used (111), (115), (118), (160), and the fact that $r(t') \le cr(t)$ for all $t' \le t$, which follows from the comparison $r(t') \approx t'-u$ and the monotonicity of $t'-u$ along the cone C_u. Therefore, returning to (158), we obtain,

$$r^{\frac12}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(S_{t,u})} \lesssim r^{\frac12}\lambda^{-1-2\varepsilon_0} + \lambda^{-\frac12} + \int_u^t r(t')^{\frac12}\|\hat{\chi}\|_{L^\infty(S_{t',u})}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(S_{t',u})}\,dt'.$$

Thus, by Gronwall inequality, and the fact that $\int_u^t \|\hat{\chi}\|_{L^\infty(S_{t',u})}\,dt' \lesssim \|\hat{\chi}\|_{L^1_tL^\infty_x} \lesssim \lambda^{-3\varepsilon_0}$, we infer that

$$r^{\frac12}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(S_{t,u})} \lesssim r^{\frac12}\lambda^{-1-2\varepsilon_0} + \lambda^{-\frac12}. \tag{161}$$

Consequently, since the time interval $[0, t_*]$ obeys $t_* \le \lambda^{1-8\varepsilon_0}$, we have

$$\|\sup_{r(t)\ge\frac{t}{2}}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(S_{t,u})}\|_{L^1_t} \le \lambda^{-3\varepsilon_0}. \tag{162}$$

This establishes the first part of the estimate (123).

9.15. *Estimates for* $\underline{L}(\mathrm{tr}\chi)$. Recall the relation between $\underline{L}(\mathrm{tr}\chi)$ and μ:

$$\mu = \underline{L}(\mathrm{tr}\chi) - \frac12(\mathrm{tr}\chi)^2 - \big(k_{NN} + n^{-1}N(n)\big)\mathrm{tr}\chi.$$

Observe also that $\underline{L}(r) = \frac{1}{8\pi r}\int_{S_{t,u}} \mathrm{tr}\underline{\chi}$. Thus

$$\underline{L}\left(\frac{2}{r}\right) = -\frac{1}{4\pi r^3}\int_{S_{t,u}} \mathrm{tr}\underline{\chi} = \frac{2}{r^2} + \frac{1}{4\pi r^3}\int_{S_{t,u}}\left(\mathrm{tr}\chi - \frac{2}{r} + 2\mathrm{tr}k\right) = \frac{2}{r^2} + \frac{1}{r}\Theta.$$

In addition, $|\frac{1}{2}(\mathrm{tr}\chi)^2 - \frac{2}{r^2}| \lesssim \frac{1}{r}\Theta$. Therefore,

(163)

$$\begin{aligned}\|\underline{L}\left(\mathrm{tr}\chi - \frac{2}{r}\right)\|_{L^2(S_{t,u})} &\lesssim \|\mu\|_{L^2(S_{t,u})} + \|\partial H\cdot\Theta\|_{L^2(S_{t,u})} + \frac{1}{r}\|\Theta\|_{L^2(S_{t,u})}\\ &\lesssim \|\Theta\|_{L^\infty(S_{t,u})} + \|\mu\|_{L^2(S_{t,u})}.\end{aligned}$$

Here we have used the Hölder inequality combined with the estimate (112):

$$\|\partial H\|_{L^2(S_{t,u})} \lesssim \lambda^{-4\varepsilon_0}.$$

It remains to estimate $\|\mu\|_{L^2(S_{t,u})}$. We obtain this estimate from the transport equation (35) for μ which combined with Corollary 4.7 can be written in the form:

(164)

$$\begin{aligned}L(\mu + \mathbf{R}_{44}) + \mathrm{tr}\chi(\mu + \mathbf{R}_{44}) = {}&\Theta\nabla\!\!\!/(\mathrm{tr}\chi) + \Theta\nabla\!\!\!/\eta + 2N(\mathbf{R}_{44}) + \Theta^3 + \frac{1}{r}\Theta^2\\ &+ \frac{1}{r^2}\partial H + \frac{1}{r}\mathbf{Ric}(H) + \Theta\mathbf{Ric}(H) + \Theta\mathbf{R}_* + \frac{1}{r}\mathbf{R}_*.\end{aligned}$$

Remark 9.16. In the derivation of (164) we have expressed $\underline{L}(\mathbf{R}_{44})$ in the form $L(\mathbf{R}_{44}) - 2N(\mathbf{R}_{44})$.

Using the transport lemma and the estimate (116), $\int_u^t \|\nabla\mathbf{R}_{44}\|_{L^2(S_{t',u})}\,dt' \lesssim \lambda^{-1-2\varepsilon_0}$, we infer that,

(165)

$$\begin{aligned}\|\mu\|_{L^2(S_{t,u})} \lesssim{}& \|\mathbf{R}_{44}\|_{L^2(S_{t,u})} + \frac{1}{r(t)}\int_u^t r(t')\|\Theta\|_{L^\infty(S_{t',u})}\|\nabla\!\!\!/\eta\|_{L^2(S_{t',u})}\,dt'\\ &+ \frac{1}{r(t)^a}\|\Theta\|_{L^2_tL^\infty_x}\|r(t')^a\nabla\!\!\!/(\mathrm{tr}\chi)\|_{L^2(C_u)} + \lambda^{-1-2\varepsilon_0}\\ &+ \|\Theta\|^2_{L^2_tL^\infty_x}\|\Theta\|_{L^2(S_{t,u})} + \|\Theta\|^2_{L^2_tL^\infty_x} + r(t)^{-\frac{1}{2}}\|\partial H\|_{L^2_tL^\infty_x}\\ &+ \|\mathbf{Ric}(H)\|_{L^1_tL^\infty_x} + \|\Theta\|_{L^2(S_{t,u})}\|\mathbf{Ric}(H)\|_{L^1_tL^\infty_x}\\ &+ \|\Theta\|_{L^2_tL^\infty_x}\|\mathbf{R}_*\|_{L^2(C_u)} + r(t)^{-\frac{1}{2}}\|\mathbf{R}_*\|_{L^2(C_u)}\\ \lesssim{}& \frac{1}{r(t)}\int_u^t r(t')\|\Theta\|_{L^\infty(S_{t',u})}\|\nabla\!\!\!/\eta\|_{L^2(S_{t',u})}\,dt' + \lambda^{-1} + r(t)^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}.\end{aligned}$$

Here we have repeatedly used the Hölder inequality, the assumptions on the metric (111)–(117), the already proved estimates (118), (119) for Θ, and the estimate[18]

$$\begin{aligned}\|r(t')^a \nabla\!\!\!/(\mathrm{tr}\chi)\|_{L^2(C_u)} &= \left(\int_u^t \|r(t')^a \nabla\!\!\!/ \mathrm{tr}\chi\|^2_{L^2(S_{t',u})}\, dt'\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_u^t r(t')^{2a}(\lambda^{-1-2\varepsilon_0} + r(t')^{-\frac{1}{2}}\lambda^{-\frac{1}{2}})^2\, dt'\right)^{\frac{1}{2}} \\ &\lesssim r(t)^{\frac{1}{2}+a}\lambda^{-1-2\varepsilon_0} + r(t)^a\lambda^{-\frac{1}{2}} \lesssim r(t)^a\lambda^{-\frac{1}{2}}\end{aligned}$$

following from the estimate for $\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{L^2(S_{t,u})}$, proved in (161).

On the other hand, η is the solution of the Hodge system (38), (39):

$$\begin{aligned}\mathrm{d}\!\!\!/\mathrm{iv}\ \eta &= \frac{1}{2}\left(\mu + 2\bar{k}_{NN}\mathrm{tr}\chi - 2|\eta|^2 - |\hat{\chi}|^2 - 2k_{AB}\chi_{AB}\right) - \frac{1}{2}\delta^{AB}\mathbf{R}_{A43B}, \\ \mathrm{c}\!\!\!/\mathrm{url}\ \eta &= \frac{1}{2}\varepsilon^{AB}k_{AC}\hat{\chi}_{CB} - \frac{1}{2}\varepsilon^{AB}\mathbf{R}_{A43B}.\end{aligned}$$

The elliptic estimate of Proposition 6.19 applied to this div-curl system gives us the bound

$$\begin{aligned}\|\nabla\!\!\!/\eta\|_{L^2(S_{t,u})} + \frac{1}{r}\|\eta\|_{L^2(S_{t,u})} &\lesssim \|\mu\|_{L^2(S_{t,u})} + \|\Theta\|_{L^\infty(S_{t,u})}\|\Theta\|_{L^2(S_{t,u})} \\ &\quad + \|\Theta\|_{L^\infty(S_{t,u})} + \|\mathbf{R}_{A43B}\|_{L^2(S_{t,u})}.\end{aligned} \tag{166}$$

Recall that according to (117), $\|\mathbf{R}_{A43B}\|_{L^2(C_u)} \lesssim \lambda^{-\frac{1}{2}}$. Thus substituting estimate (166) into (165) we obtain

$$\|\mu\|_{L^2(S_{t,u})} \lesssim \frac{1}{r(t)}\int_u^t r(t')\|\Theta\|_{L^\infty(S_{t',u})}\|\mu\|_{L^2(S_{t',u})}\, dt' + \lambda^{-1} + r(t)^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}.$$

We rewrite the above inequality in a more convenient form:

$$r(t)^{\frac{1}{2}}\|\mu\|_{L^2(S_{t,u})} \lesssim \int_u^t \|\Theta\|_{L^\infty(S_{t',u})} r(t')^{\frac{1}{2}}\, \|\mu\|_{L^2(S_{t',u})}\, dt' + r(t)^{\frac{1}{2}}\lambda^{-1} + \lambda^{-\frac{1}{2}}.$$

Since

$$\int_u^t \|\Theta\|_{L^\infty(S_{t',u})}\, dt' \le \int_0^t \|\Theta\|_{L^\infty_x}\, dt \lesssim t^{\frac{1}{2}}\|\Theta\|_{L^2_t L^\infty_x} \lesssim \lambda^{-4\varepsilon_0},$$

application of Gronwall's inequality yields the estimate

$$r(t)^{\frac{1}{2}}\|\mu\|_{L^2(S_{t,u})} \lesssim r(t)^{\frac{1}{2}}\lambda^{-1} + \lambda^{-\frac{1}{2}}.$$

Returning to (163) we obtain

$$\|\underline{L}\left(\mathrm{tr}\chi - \frac{2}{r}\right)\|_{L^2(S_{t,u})} \lesssim \lambda^{-1} + \|\Theta\|_{L^\infty_x} + r^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}.$$

[18]Constant a can be chosen arbitrarily from the interval $(0, 2)$. Its only purpose is to remove the logarithmic divergence at $\rho = 0$.

Similarly to (162) we then derive the following estimates in the exterior region:

$$\| \sup_{r \geq \frac{t}{2}} \| \underline{L}(\mathrm{tr}\chi - \frac{2}{r}) \|_{L^2(S_{t,u})} \|_{L^1_t} \leq \lambda^{-3\varepsilon_0}. \tag{167}$$

This proves the first part of the estimate (122).

To finish the proof of (122) and (123), we first recall that $\underline{L}\big(\frac{2}{r}\big) = \frac{2}{r^2} + \frac{1}{r}\Theta$. Observe also that

$$\begin{aligned} \underline{L}\big(n(t-u)\big) &= n^{-1}\underline{L}(n)n(t-u) + n(n^{-1} - 2b^{-1}) \\ &= -1 + 2n(b^{-1} - n^{-1}) + n^{-1}\underline{L}(n)n(t-u). \end{aligned}$$

According to Corollary 6.10 $|b - n| \lesssim s\mathcal{M}(\partial H)$. Since by Lemmas 6.7, 6.11 the quantities r, s, and $n(t-u)$ are comparable, we infer that

$$\begin{aligned} \underline{L}\left(\frac{2}{r}\right) - \underline{L}\left(\frac{2}{n(t-u)}\right) &= \frac{2}{r^2} - \frac{2}{n^2(t-u)^2} + \frac{1}{r}\mathcal{M}(\partial H) + \frac{1}{r}\Theta \\ &= 2\left(\frac{1}{r} + \frac{1}{n(t-u)}\right)\left(\frac{1}{r} - \frac{1}{n(t-u)}\right) \\ &\quad + \frac{1}{r}(\mathcal{M}(\partial H) + \Theta). \end{aligned}$$

Thus using Corollary 6.12, (111), and (118) together with the estimate for the maximal function we obtain

$$\begin{aligned} &\| \sup_{r \geq \frac{t}{2}} \| \underline{L}\left(\frac{2}{r}\right) - \underline{L}\left(\frac{2}{n(t-u)}\right) \|_{L^2(S_{t,u})} \|_{L^2_t} \\ &\qquad \lesssim \| \frac{1}{r} - \frac{1}{n(t-u)} \|_{L^2_t L^\infty_x} + \| \mathcal{M}(\partial H) \|_{L^2_t} + \| \Theta \|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2} - 4\varepsilon_0}. \end{aligned}$$

The above inequality followed by Hölder and (167) allow us to conclude that

$$\| \sup_{r \geq \frac{t}{2}} \| \underline{L}(\mathrm{tr}\chi - \frac{2}{n(t-u)}) \|_{L^2(S_{t,u})} \|_{L^1_t} \leq \lambda^{-3\varepsilon_0}. \tag{168}$$

Similarly,

$$\nabla\!\!\!/ \big(n(t-u)\big) = n^{-1}\nabla\!\!\!/(n)n(t-u)$$

and consequently,

$$\begin{aligned} \| \sup_{r \geq \frac{t}{2}} \| \nabla\!\!\!/ \big(\frac{2}{n(t-u)}\big) \|_{L^2(S_{t,u})} \|_{L^2_t} &\lesssim \| \sup_{r \geq \frac{t}{2}} \frac{1}{r} \| \partial H \|_{L^2(S_{t,u})} \|_{L^2_t} \\ &\lesssim \| \partial H \|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2} - 4\varepsilon_0}. \end{aligned}$$

Thus we can complement (162) with the estimate

$$\| \sup_{r \geq \frac{t}{2}} \| \nabla\!\!\!/ \big(\mathrm{tr}\chi - \frac{2}{n(t-u)}\big) \|_{L^2(S_{t,u})} \|_{L^1_t} \leq \lambda^{-3\varepsilon_0}. \tag{169}$$

It only remains to discuss the weak estimates (124). These are a lot easier to prove and can be derived directly from the transport equations for $\mathrm{tr}\chi$ and $\hat{\chi}$ (see Proposition 3.1), in the case of the tangential derivatives $\nabla\!\!\!/\,\mathrm{tr}\chi$, and from the transport equation for η (see Proposition 3.1), in the case of the $\underline{L}$ derivative[19]. □

Princeton University, Princeton, NJ
E-mail addresses: seri@math.princeton.edu
irod@math.princeton.edu

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[19] We can express $\underline{L}\mathrm{tr}\chi$ in terms of $\nabla\!\!\!/\,\eta$, see definition of μ, and estimate the latter with the help of the transport equation for η.