Annals of Mathematics, **161** (2005), 1143–1193

Rough solutions of the Einstein-vacuum equations

By Sergiu Klainerman and Igor Rodnianski

To Y. Choquet-Bruhat in honour of the 50^{th} *anniversary of her fundamental paper* [Br] *on the Cauchy problem in General Relativity*

Abstract

This is the first in a series of papers in which we initiate the study of very rough solutions to the initial value problem for the Einstein-vacuum equations expressed relative to wave coordinates. By very rough we mean solutions which cannot be constructed by the classical techniques of energy estimates and Sobolev inequalities. Following [Kl-Ro] we develop new analytic methods based on Strichartz-type inequalities which result in a gain of half a derivative relative to the classical result. Our methods blend paradifferential techniques with a geometric approach to the derivation of decay estimates. The latter allows us to take full advantage of the specific structure of the Einstein equations.

1. Introduction

We consider the Einstein-vacuum equations,

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0 \tag{1}$$

where $\mathbf{g}$ is a four-dimensional Lorentz metric and $\mathbf{R}_{\alpha\beta}$ its Ricci curvature tensor. In wave coordinates x^α,

$$\square_{\mathbf{g}} x^\alpha = \frac{1}{|\mathbf{g}|}\partial_\mu(\mathbf{g}^{\mu\nu}|\mathbf{g}|\partial_\nu)x^\alpha = 0, \tag{2}$$

the Einstein-vacuum equations take the reduced form; see [Br], [H-K-M].

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{g}_{\mu\nu} = N_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}) \tag{3}$$

with N quadratic in the first derivatives $\partial\mathbf{g}$ of the metric. We consider the initial value problem along the spacelike hyperplane Σ given by $t = x^0 = 0$,

$$\nabla\mathbf{g}_{\alpha\beta}(0) \in H^{s-1}(\Sigma)\,, \quad \partial_t\mathbf{g}_{\alpha\beta}(0) \in H^{s-1}(\Sigma) \tag{4}$$

with ∇ denoting the gradient with respect to the space coordinates x^i, $i = 1, 2, 3$ and H^s the standard Sobolev spaces. We also assume that $\mathbf{g}_{\alpha\beta}(0)$ is a continuous Lorentz metric and

$$\sup_{|x|=r} |\mathbf{g}_{\alpha\beta}(0) - \mathbf{m}_{\alpha\beta}| \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty, \tag{5}$$

where $|x| = (\sum_{i=1}^{3} |x^i|^2)^{\frac{1}{2}}$ and $\mathbf{m}_{\alpha\beta}$ is the Minkowski metric.

The following local existence and uniqueness result (well-posedness) is well known (see [H-K-M] and the previous result of Ch. Bruhat [Br] for $s \geq 4$).

THEOREM 1.1. *Consider the reduced equation* (3) *subject to the initial conditions* (4) *and* (5) *for some* $s > 5/2$. *Then there exists a time interval* $[0, T]$ *and unique* (*Lorentz metric*) *solution* $\mathbf{g} \in C^0([0, T] \times \mathbb{R}^3)$, $\partial \mathbf{g}_{\mu\nu} \in C^0([0, T]; H^{s-1})$ *with* T *depending only on the size of the norm* $\|\partial \mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}}$. *In addition, condition* (5) *remains true on any spacelike hypersurface* Σ_t, *i.e. any level hypersurface of the time function* $t = x^0$.

We establish a significant improvement of this result bearing on the issue of minimal regularity of the initial conditions:

MAIN THEOREM. *Consider a classical solution of the equations* (3) *for which* (1) *also holds*[1]. *The time* T *of existence*[2] *depends in fact only on the size of the norm* $\|\partial \mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}}$, *for any fixed* $s > 2$.

Remark 1.2. Theorem 1.1 implies the classical local existence result of [H-K-M] for asymptotically flat initial data sets Σ, g, k with $\nabla g, k \in H^{s-1}(\Sigma)$ and $s > \frac{5}{2}$, relative to a fixed system of coordinates. Uniqueness can be proved for additional regularity $s > 1 + \frac{5}{2}$. We recall that an initial data set (Σ, g, k) consists of a three-dimensional complete Riemannian manifold (Σ, g), a 2-covariant symmetric tensor k on Σ verifying the constraint equations:

$$\nabla^j k_{ij} - \nabla_i \operatorname{tr} k = 0,$$
$$R - |k|^2 + (\operatorname{tr} k)^2 = 0,$$

where ∇ is the covariant derivative, R the scalar curvature of (Σ, g). An initial data set is said to be asymptotically flat (AF) if there exists a system of

[1] In other words for any solution of the reduced equations (3) whose initial data satisfy the constraint equations, see [Br] or [H-K-M]. The fact that our solutions verify (1) plays a fundamental role in our analysis.

[2] We assume however that T stays sufficiently small, e.g. $T \leq 1$. This a purely technical assumption which one should be able to remove.

coordinates (x^1, x^2, x^3) defined in a neighborhood of infinity[3] on Σ relative to which the metric g approaches the Euclidean metric and k approaches zero.[4]

Remark 1.3. The Main Theorem ought to imply existence and uniqueness[5] for initial conditions with H^s, $s > 2$, regularity. To achieve this we only need to approximate a given H^s initial data set (i.e. $\nabla g \in H^{s-1}(\Sigma)$, $k \in H^{s-1}(\Sigma)$, $s > 2$) for the Einstein vacuum equations by classical initial data sets, i.e. $H^{s'}$ data sets with $s' > \frac{5}{2}$, for which Theorem 1.1 holds. The Main Theorem allows us to pass to the limit and derive existence of solutions for the given, rough, initial data set. We do not know however if such an approximation result for the constraint equations exists in the literature.

For convenience we shall also write the reduced equations (3) in the form

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\phi = N(\phi, \partial\phi) \tag{6}$$

where $\phi = (\mathbf{g}_{\mu\nu})$, $N = N_{\mu\nu}$ and $\mathbf{g}^{\alpha\beta} = \mathbf{g}^{\alpha\beta}(\phi)$.

Expressed relative to the wave coordinates x^α the spacetime metric $\mathbf{g}$ takes the form:

$$\mathbf{g} = -\mathbf{n}^2 dt^2 + g_{ij}(dx^i + \mathbf{v}^i dt)(dx^j + \mathbf{v}^j dt) \tag{7}$$

where g_{ij} is a Riemannian metric on the slices Σ_t, given by the level hypersurfaces of the time function $t = x^0$, $\mathbf{n}$ is the lapse function of the time foliation, and $\mathbf{v}$ is a vector-valued shift function. The components of the inverse metric $\mathbf{g}^{\alpha\beta}$ can be found as follows:

$$\mathbf{g}^{00} = -\mathbf{n}^{-2}, \quad \mathbf{g}^{0i} = \mathbf{n}^{-2}\mathbf{v}^i, \quad \mathbf{g}^{ij} = g^{ij} - \mathbf{n}^{-2}\mathbf{v}^i\mathbf{v}^j.$$

In view of the Lorentzian character of $\mathbf{g}$ and the spacelike character of the hypersurfaces Σ_t,

$$c|\xi|^2 \le \mathbf{g}_{ij}\xi^i\xi^j \le c^{-1}|\xi|^2, \qquad c \le \mathbf{n}^2 - |\mathbf{v}|_g^2 \tag{8}$$

for some $c > 0$.

The classical local existence result for systems of wave equations of type (6) is based on energy estimates and the standard $H^s \subset L^\infty$ Sobolev inequality.

[3]We assume, for simplicity, that Σ has only one end. A neighborhood of infinity means the complement of a sufficiently large compact set on Σ.

[4]Because of the constraint equations the asymptotic behavior cannot be arbitrarily prescribed. A precise definition of asymptotic flatness has to involve the ADM mass of (Σ, g). Taking the mass into account we write $g_{ij} = (1 + \frac{2M}{r})\delta_{ij} + o(r^{-1})$ as $r = \sqrt{(x^1)^2 + (x^2)^2 + (x_3)^2} \to \infty$. According to the positive mass theorem $M \ge 0$ and $M = 0$ implies that the initial data set is flat. Because of the mass term we cannot assume that $g - e \in L^2(\Sigma)$, with e the 3D Euclidean metric.

[5]Properly speaking uniqueness holds, with $s > 2$, only for the reduced equations. Uniqueness for the actual Einstein equations requires one more derivative; see [H-K-M].

Indeed using energy estimates and simple commutation inequalities one can show that,

$$\|\partial\phi(t)\|_{H^{s-1}} \le E\|\partial\phi(0)\|_{H^{s-1}} \tag{9}$$

with a constant E,

$$E = \exp\left(C\int_0^t \|\partial\phi(\tau)\|_{L^\infty_x} d\tau\right). \tag{10}$$

By the classical Sobolev inequality,

$$E \le \exp\left(Ct \sup_{0\le\tau\le t} \|\partial\phi(\tau)\|_{H^{s-1}} d\tau\right)$$

provided that $s > \frac{5}{2}$. The classical local existence result follows by combining this last estimate, for a small time interval, with the energy estimates (9).

This scheme is very wasteful. To do better one would like to take advantage of the mixed $L^1_t L^\infty_x$ norm appearing on the right-hand side of (10). Unfortunately there are no good estimates for such norms even when ϕ is simply a solution of the standard wave equation

$$\Box\phi = 0 \tag{11}$$

in Minkowski space. There exist however improved regularity estimates for solutions of (11) in the mixed $L^2_t L^\infty_x$ norm . More precisely, if ϕ is a solution of (11) and $\epsilon > 0$ arbitrarily small,

$$\|\partial\phi\|_{L^2_t L^\infty_x([0,T]\times\mathbb{R}^3)} \le CT^\epsilon \|\partial\phi(0)\|_{H^{1+\epsilon}}. \tag{12}$$

Based on this fact it was reasonable to hope that one can improve the Sobolev exponent in the classical local existence theorem from $s > \frac{5}{2}$ to $s > 2$. This can be easily done for solutions of semilinear equations; see [Po-Si]. In the quasilinear case, however, the situation is far more difficult. One can no longer rely on the Strichartz inequality (12) for the flat D'Alembertian in (11); we need instead its extension to the operator $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$ appearing in (6). Moreover, since the metric $\mathbf{g}^{\alpha\beta}$ depends on the solution ϕ, it can have only as much regularity as ϕ itself. This means that we have to confront the issue of proving Strichartz estimates for wave operators $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$ with *very rough coefficients* $\mathbf{g}^{\alpha\beta}$. This issue was recently addressed in the pioneering works of Smith[Sm], Bahouri-Chemin [Ba-Ch1], [Ba-Ch2] and Tataru [Ta1], [Ta2], we refer to the introduction in [Kl1] and [Kl-Ro] for a more thorough discussion of their important contributions.

The results of Bahouri-Chemin and Tataru are based on establishing a Strichartz type inequality, *with a loss*, for wave operators with *very rough* *coefficients.*[6] The optimal result[7] in this regard, due to Tataru, see [Ta2], requires a loss of $\sigma = \frac{1}{6}$. This leads to a proof of local well-posedness for systems of type (6) with $s > 2 + \frac{1}{6}$.

To do better than that one needs to take into account the nonlinear structure of the equations. In [Kl-Ro] we were able to improve the result of Tataru by taking into account not only the expected regularity properties of the coefficients $\mathbf{g}^{\alpha\beta}$ in (6) but also the fact that they are themselves solutions to a similar system of equations. This allowed us to improve the exponent s, needed in the proof of well-posedness of equations of type (6),[8] to $s > 2 + \frac{2-\sqrt{3}}{2}$. Our approach was based on a combination of the paradifferential calculus ideas, initiated in [Ba-Ch1] and [Ta2], with a geometric treatment of the actual equations introduced in [Kl1]. The main improvement was due to a gain of conormal differentiability for solutions to the Eikonal equations

$$H^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 \tag{13}$$

where the background metric H is a properly microlocalized and rescaled version of the metric $\mathbf{g}^{\alpha\beta}$ in (6). That gain could be traced to the fact that a certain component of the Ricci curvature of H has a special form. More precisely denoting by L' the null geodesic vectorfield associated to u, $L' = -H^{\alpha\beta}\partial_\beta u \partial_\alpha$, and rescaling it in an appropriate fashion,[9] $L = bL'$, we found that the null Ricci component $\mathbf{R}_{LL} = \mathbf{Ric}(H)(L, L)$, verifies the remarkable identity:

$$\mathbf{R}_{LL} = L(z) - \frac{1}{2} L^\mu L^\nu (H^{\alpha\beta}\partial_\alpha\partial_\beta H_{\mu\nu}) + \mathrm{e} \tag{14}$$

where $z \le O(|\partial H|)$ and $e \le O(|\partial H|^2)$. Thus, apart from $L(z)$ which is to be integrated along the null geodesic flow generated by L, the only terms which depend on the second derivatives of H appear in $H^{\alpha\beta}\partial_\alpha\partial_\beta H$ and can therefore be eliminated with the help of the equations (6).

In this paper we develop the ideas of [Kl-Ro] further by taking full advantage of the Einstein equations (1) in wave coordinates (6). An important aspect of our analysis here is that the term $L(z)$ appearing on the right-hand side of (14) vanishes identically. We make use of both the vanishing of the Ricci curvature of $\mathbf{g}$ and the wave coordinate condition (2). The other important new features are the use of energy estimates along the null hypersurfaces

---

[6] The derivatives of the coefficients $\mathbf{g}$ are required to be bounded in $L_t^\infty H_x^{s-1}$ and $L_t^2 L_x^\infty$ norms, with s compatible with the regularity required on the right-hand side of the Strichartz inequality one wants to prove.

[7] Recently Smith-Tataru [Sm-Ta] have shown that the result of Tataru is indeed sharp.

[8] The result in [Kl-Ro] applies to general equations of type (6) not necessarily tied to (1). In [Kl-Ro] we have also made the simplifying assumptions $\mathbf{n} = 1$ and $\mathbf{v} = 0$.

[9] such that $\langle L, T\rangle_H = 1$ where T is the unit normal to the level hypersurfaces Σ_t associated to the time function t.

generated by the optical function u and a deeper analysis of the conormal properties of the null structure equations.

Our work is divided in three parts. In this paper we give all the details in the proof of the Main Theorem with the exception of those results which concern the asymptotic properties of the Ricci coefficients (the Asymptotics Theorem), and the straightforward modifications of the standard isoperimetric and trace inequalities on 2-surfaces. We give precise statements of these results in Section 4. Our second paper [Kl-Ro2] is dedicated to the proof of the Asymptotics Theorem which relies on an important result concerning the *Ricci defect* $\mathbf{Ric}(H)$. This result is proved in our third paper [Kl-Ro3].

We strongly believe that the result of our main theorem is not sharp. The critical Sobolev exponent for the Einstein equations is $s_c = \frac{3}{2}$. A proof of well-posedness for $s = s_c$ will provide a much stronger version of the *global stability of Minkowski space* than that of [Ch-Kl]. This is completely out of reach at the present time. A more reasonable goal now is to prove the L^2- *curvature conjecture*, see [Kl2], corresponding to the exponent $s = 2$.

2. Reduction to decay estimates

The proof of the Main Theorem can be reduced to a microlocal decay estimate. The reduction is standard;[10] we quickly review here the main steps. The precise statements and their proofs are given in Section 8.

- *Energy estimates.* Assuming that ϕ is a solution[11] of (6) on $[0,T]\times\mathbb{R}^3$ we have the *a priori* energy estimate:

$$\|\partial\phi\|_{L^\infty_{[0,T]}\dot{H}^{s-1}} \le C\|\partial\phi(0)\|_{\dot{H}^{s-1}} \tag{15}$$

  with a constant C depending only on $\|\phi\|_{L^\infty_{[0,T]}L^\infty_x}$ and $\|\partial\phi\|_{L^1_{[0,T]}L^\infty_x}$.

- *The Strichartz estimate.* To prove our Main Theorem we need, in addition to (15) an estimate of the form:

$$\|\partial\phi\|_{L^1_{[0,T]}L^\infty_x} \le C\|\partial\phi(0)\|_{H^{s-1}}$$

  for any $s > 2$. We accomplish it by establishing a Strichartz type inequality of the form,

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le C\|\partial\phi(0)\|_{H^{1+\gamma}} \tag{16}$$

  with any fixed $\gamma > 0$. We achieve this with the help of a bootstrap argument. More precisely we make the assumption,

[10] See [Kl-Ro] and the references therein.
[11] i.e., a classical solution according to Theorem 1.1.

Bootstrap Assumption.

$$\|\partial\phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} + \|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le B_0, \tag{17}$$

and use it to prove the better estimate:

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le C(B_0)\, T^\delta \tag{18}$$

for some $\delta > 0$. Thus, for sufficiently small $T > 0$, we find that (16) holds true.

- *Proof of the Main Theorem.* This can be done easily by combining the energy estimates with the Strichartz estimate stated above.

- *The Dyadic Strichartz Estimate.* The proof of the Strichartz estimate can be reduced to a dyadic version for each $\phi^\lambda = P_\lambda\phi$, λ sufficiently large,[12] where P_λ is the Littlewood-Paley projection on the space frequencies of size $\lambda \in 2^{\mathbb{Z}}$,
$$\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} \le C(B_0)\, c_\lambda T^\delta \|\partial\phi\|_{H^{1+\gamma}},$$
with $\sum_\lambda c_\lambda \le 1$.

- *Dyadic linearization and time restriction.* Consider the new metric $\mathbf{g}_{<\lambda} = P_{<\lambda}\mathbf{g} = \sum_{\mu \le 2^{-M_0}\lambda} P_\mu \mathbf{g}$, for some sufficiently large constant $M_0 > 0$, restricted to a subinterval I of $[0,T]$ of size $|I| \approx T\lambda^{-8\epsilon_0}$ with $\epsilon_0 > 0$ fixed such that $\gamma > 5\epsilon_0$. Without loss of generality[13] we can assume that $I = [0,\bar{T}]$, $\bar{T} \approx T\lambda^{-8\epsilon_0}$. Using an appropriate (now standard, see [Ba-Ch1], [Ta2], [Kl1], [Kl-Ro]) paradifferential linearization together with the Duhamel principle we can reduce the proof of the dyadic Strichartz estimate mentioned above to a homogeneous Strichartz estimate for the equation
$$\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\psi = 0,$$
with initial conditions at $t = 0$ verifying,
$$(2^{-10}\lambda)^m \le \|\nabla^m\partial\psi(0)\|_{L^2_x} \le (2^{10}\lambda)^m\|\partial\psi(0)\|_{L^2_x}.$$
There exists a sufficiently small $\delta > 0$, $5\epsilon_0 + \delta < \gamma$, such that

$$\|P_\lambda\, \partial\psi\|_{L^2_I L^\infty_x} \le C(B_0)\, \bar{T}^\delta \|\partial\psi(0)\|_{\dot{H}^{1+\delta}}. \tag{19}$$

- *Rescaling.* Introduce the rescaled metric[14]
$$H_{(\lambda)}(t,x) = \mathbf{g}_{<\lambda}(\lambda^{-1}t, \lambda^{-1}x)$$

[12] The low frequencies are much easier to treat.

[13] In view of the translation invariance of our estimates.

[14] $H_{(\lambda)}$ is a Lorentz metric for $\lambda \ge \Lambda$ with Λ sufficiently large. See the discussion following (133) in Section 8.

and consider the rescaled equation

$$H_{(\lambda)}^{\alpha\beta}\partial_\alpha\partial_\beta\psi = 0$$

in the region $[0, t_*] \times \mathbb{R}^3$ with $t_* \le \lambda^{1-8\epsilon_0}$. Then, with $P = P_1$,

$$\|P\,\partial\psi\|_{L_t^2 L_x^\infty} \le C(B_0)\, t_*^\delta \|\partial\psi(0)\|_{L^2}$$

would imply the estimate (19).

- *Reduction to an $L^1 - L^\infty$ decay estimate.* The standard way to prove a Strichartz inequality of the type discussed above is to reduce it, by a TT^* type argument, to an $L^1 - L^\infty$ dispersive type inequality. The inequality we need, concerning the initial value problem

$$\Box_{H_{(\lambda)}}\psi = \frac{1}{\sqrt{|H_{(\lambda)}|}}\partial_\alpha\big(H_{(\lambda)}^{\alpha\beta}\sqrt{|H_{(\lambda)}|}\,\partial_\beta\psi\big) = 0,$$

with data at $t = t_0$ has the form,

$$\|P\,\partial\psi(t)\|_{L_x^\infty} \le C(B_0)\bigg(\frac{1}{(1+|t-t_0|)^{1-\delta}} + d(t)\bigg)\sum_{k=0}^m \|\nabla^k\partial\psi(t_0)\|_{L_x^1}$$

for some integer $m \ge 0$.

- *Final reduction to a localized $L^2 - L^\infty$ decay estimate.* We state this as the following theorem:

THEOREM 2.1. *Let ψ be a solution of the equation,*

$$\Box_{H_{(\lambda)}}\psi = 0 \tag{20}$$

on the time interval $[0, t_*]$ with $t_* \le \lambda^{1-8\epsilon_0}$. Assume that the initial data are given at $t = t_0 \in [0, t_*]$, supported in the ball $B_{\frac{1}{2}}(0)$ of radius $\frac{1}{2}$ centered at the origin. We fix a large constant $\Lambda > 0$ and consider only the frequencies $\lambda \ge \Lambda$. There exist a function $d(t)$, with $t_*^{\frac{1}{q}}\|d\|_{L^q([0,t_*])} \le 1$ for some $q > 2$ sufficiently close to 2, an arbitrarily small $\delta > 0$ and a sufficiently large integer $m > 0$ such that for all $t \in [0, t_*]$,

$$\|P\,\partial\psi(t)\|_{L_x^\infty} \le C(B_0)\bigg(\frac{1}{(1+|t-t_0|)^{1-\delta}} + d(t)\bigg)\sum_{k=0}^m \|\nabla^k\partial\psi(t_0)\|_{L_x^2}. \tag{21}$$

Remark 2.2. In view of the proof of the Main Theorem presented above, which relies on the final estimate (18), we can in what follows treat the bootstrap constant B_0 as a universal constant and bury the dependence on it in the notation $\lesssim$ introduced below.

Definition 2.3. We use the notation $A \lesssim B$ to express the inequality $A \leq CB$ with a universal constant, which may depend on B_0 and various other parameters depending only on B_0 introduced in the proof.

The proof of Theorem 2.1 relies on a generalized Morawetz-type energy estimate which will be presented in the next section. We shall in fact construct a vectorfield, analogous to the Morawetz vectorfield in the Minkowski space, which depends heavily on the "background metric" $H = H_{(\lambda)}$. In the next proposition we display most of the main properties of the metric H which will be used in the following section.

PROPOSITION 2.4 (Background estimates). *Fix the region* $[0, t_*] \times \mathbb{R}^3$, *with* $t_* \leq \lambda^{1-8\epsilon_0}$, *where the original Einstein metric*[15] $\mathbf{g} = \mathbf{g}(\phi)$ *verifies the bootstrap assumption* (17). *The metric*

$$H(t,x) = H_{(\lambda)}(t,x) = (P_{<\lambda}\mathbf{g})(\lambda^{-1}t, \lambda^{-1}x) \tag{22}$$

can be decomposed relative to our spacetime coordinates

$$H = -n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt) \tag{23}$$

where n *and* v *are related to* $\mathbf{n}$, $\mathbf{v}$ *according to the rule* (22). *The metric components* n, v, *and* h *satisfy the conditions*

$$c|\xi|^2 \leq h_{ij}\xi^i\xi^j \leq c^{-1}|\xi|^2, \quad n^2 - |v|_h^2 \geq c > 0, \quad |n|, |v| \leq c^{-1}. \tag{24}$$

In addition, the derivatives of the metric H *verify the following*:

$$\|\partial^{1+m} H\|_{L^1_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-8\epsilon_0}, \qquad m \geq 0 \tag{25}$$

$$\|\partial^{1+m} H\|_{L^2_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \qquad m \geq 0 \tag{26}$$

$$\|\partial^{1+m} H\|_{L^\infty_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \qquad m \geq 0 \tag{27}$$

$$\|\nabla^{\frac{1}{2}+m}(\partial H)\|_{L^\infty_{[0,t_*]} L^2_x} \lesssim \lambda^{-m}, \qquad -\frac{1}{2} \leq m \leq \frac{1}{2} + 4\epsilon_0 \tag{28}$$

$$\|\nabla^{\frac{1}{2}+m}(\partial^2 H)\|_{L^\infty_{[0,t_*]} L^2_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \qquad -\frac{1}{2} + 4\epsilon_0 \leq m \tag{29}$$

$$\|\nabla^m \big(H^{\alpha\beta}\partial_\alpha\partial_\beta H\big)\|_{L^1_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}, \qquad m \geq 0 \tag{30}$$

$$\|\nabla^m \big(\nabla^{\frac{1}{2}}\mathbf{Ric}(H)\big)\|_{L^\infty_{[0,t_*]} L^2_x} \lesssim \lambda^{-1}, \qquad m \geq 0 \tag{31}$$

$$\|\nabla^m \mathbf{Ric}(H)\|_{L^1_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}, \qquad m \geq 0. \tag{32}$$

[15]Recall that in fact $\mathbf{g}$ is ϕ^{-1}. Thus, in view of the nondegenerate Lorentzian character of $\mathbf{g}$ the bootstrap assumption for ϕ reads as an assumption for $\mathbf{g}$.

Proof. This follows from Proposition 8.22 and a straightforward rescaling argument. □

Remark 2.5. Among the long list of estimates above, the main ones reflect that ∂H is controlled in $L^2_t L^\infty_x$, $\partial^{2+} H$ is in $L^\infty_t L^2_x$, and that $\mathbf{Ric}(H) \approx (\partial H)^2$. The remaining estimates follow from these by rescaling, Sobolev, and frequency localization.

3. Generalized energy estimates and the Boundedness Theorem

Consider the Lorentz metric $H = H_{(\lambda)}$, as in (22), verifying, in particular, the properties of Proposition 2.4 in the region $[0, t_*] \times \mathbb{R}^3$, $t_* \le \lambda^{1-8\epsilon_0}$. We denote by D the compatible covariant derivative and by ∇ the induced covariant differentiation on Σ_t. We denote by T the future oriented unit normal to Σ_t and by k the second fundamental form.

Associated to H we have the energy momentum tensor of $\square_H$,

$$Q_{\mu\nu} = Q[\psi]_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} H_{\mu\nu} (H^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi). \tag{33}$$

The energy density associated to an arbitrary timelike vectorfield K is given by $Q(K, T)$. We consider also the modified energy density,

$$\bar{Q}(K, T) = \bar{Q}[\psi](K, T) = Q[\psi](K, T) + 2t\psi T(\psi) - \psi^2 T(t), \tag{34}$$

and the total conformal energy,

$$\mathcal{Q}[\psi](t) = \int_{\Sigma_t} \bar{Q}[\psi](K, T). \tag{35}$$

We recall below the statement of the main generalized energy estimate upon which we rely; see [Kl-Ro].

Proposition 3.1. *Let K be an arbitrary vectorfield with deformation tensor*

$$^{(K)}\pi_{\mu\nu} = \mathcal{L}_K H_{\mu\nu} = D_\mu K_\nu + D_\nu K_\mu$$

and ψ a solution of $\square_H \psi = 0$. Then

$$\mathcal{Q}[\psi](t) = \mathcal{Q}[\psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbb{R}^3} Q^{\alpha\beta} \, {}^{(K)}\bar{\pi}_{\alpha\beta} + \frac{1}{4} \int_{[t_0, t] \times \mathbb{R}^3} \psi^2 \square_H \Omega \tag{36}$$

where

$$^{(K)}\bar{\pi} = {}^{(K)}\pi - \Omega H \tag{37}$$

and Ω is an arbitrary function.

Remark 3.2. In the particular case of the Minkowski spacetime we can choose K to be the conformal time-like Killing vectorfield

$$K = \frac{1}{2}\Big((t+r)^2(\partial_t + \partial_r) + (t-r)^2(\partial_t - \partial_r)\Big).$$

In this case we can choose $\Omega = 4t$ and obtain the total conservation law,

$$\mathcal{Q}[\psi](t) = \mathcal{Q}[\psi](t_0).$$

This conservation law can be used to get the desired decay estimate for the free wave equation; see [Kl1].

As in [Kl-Ro] we construct a special vectorfield K whose modified deformation tensor ${}^{(K)}\bar{\pi}$ is such that we can control the error terms

$$\int_{[t_0,t]\times\mathbb{R}^3} Q^{\alpha\beta}\ {}^{(K)}\bar{\pi}_{\alpha\beta} + \frac{1}{4}\int_{[t_0,t]\times\mathbb{R}^3} \psi^2 \Box_H \Omega.$$

As in [Kl-Ro] we set[16]

$$K = \frac{1}{2} n(u^2 \underline{L} + \underline{u}^2 L) \tag{38}$$

with $u, \underline{u}, L, \underline{L}$ defined as follows:

- *Optical function u.* This is an outgoing solution of the Eikonal equation

$$H^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 \tag{39}$$

with initial conditions $u(\Gamma_t) = t$ on the time axis. The *time axis* is defined as the integral curve, originating from zero on Σ_0, of the forward unit normal T to the hypersurfaces Σ_t. The point Γ_t is the intersection between Γ and Σ_t. The level surfaces of u, denoted C_u are *outgoing null cones* with vertices on the time axis. Clearly,

$$T(u) = |\nabla u|_h \tag{40}$$

where h is the metric induced by H on Σ_t, $|\nabla u|_h^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t.

- *Canonical null pair $L, \underline{L}$.*

$$L = bL' = T + N, \qquad \underline{L} = 2T - L = T - N \tag{41}$$

with $L' = -H^{\alpha\beta}\partial_\beta u \partial_\alpha$ the geodesic null generator of C_u, b the lapse of the null foliation(or shortly null lapse) defined by

$$b^{-1} = -\langle L', T\rangle = T(u), \tag{42}$$

[16] Observe that this definition of K differs from the one in [Kl-Ro] by an important factor of n.

and N the exterior unit normal, along Σ_t, to the surfaces $S_{t,u}$, i.e. the surfaces of intersection between Σ_t and C_u. We shall also use the notation

$$e_3 = \underline{L}, \qquad e_4 = L.$$

- *The function* $\underline{u} = -u + 2t$.

- *The* $S_{t,u}$ *foliation.* The intersection between the level hypersurfaces[17] and u form compact 2- Riemannian surfaces denoted by $S_{t,u}$. We define $r(t,u)$ by the formula Area($S_{t,u}$)= $4\pi r^2$. We denote by $\nabla\!\!\!/$ the induced covariant derivative on $S_{t,u}$. A vectorfield X is called S-tangent if it is tangent to $S_{t,u}$ at every point. Given an S-tangent vectorfield X we denote by $\nabla\!\!\!/_N X$ the projection on $S_{t,u}$ of $\nabla_N X$.

Remark 3.3. Observe that in Minkowski space $u = t - r$, $r = |x|$, $L = \partial_t + \partial_r$, $S_{t,u}$ are the 2 spheres centered at $t, 0$ and radius $r = t - u$.

With the help of these constructions the proof of the $L^2 - L^\infty$ decay estimate stated in Theorem 2.1 can be reduced to the following:

THEOREM 3.4 (Boundedness Theorem). *Consider the Lorentz metric* $H = H_{(\lambda)}$ *as in* (22) *verifying, in particular, the properties of Proposition* 2.4 *in the region* $[0, t_*] \times \mathbb{R}^3$, $t_* \le \lambda^{1-8\epsilon_0}$. *Let* ψ *be a solution of the wave equation*

$$\square_H \psi = \frac{1}{\sqrt{|H|}} \partial_\alpha \big(H^{\alpha\beta} \sqrt{|H|} \partial_\beta \psi \big) = 0 \tag{43}$$

with initial data $\psi[t_0]$, *at* $t = t_0 > 2$, *supported in the geodesic ball* $B_{\frac{1}{2}}(0)$. *Let* $\mathcal{D}_{u'}$ *be the region determined by* $u > u'$ *in the slab* $[0, t_*] \times \mathbf{R}^3$. *For all* $t_0 \le t \le t_*$, $\psi(t)$ *is supported in* $\mathcal{D}_{t_0 - 1} \subset \mathcal{D}_0$ *and*

$$\mathcal{Q}[\psi](t) \lesssim \mathcal{Q}[\psi](t_0).$$

Proof. See Section 5. $\square$

We consider also the auxiliary energy type quantity,

$$\mathcal{E}[\psi](t) = \mathcal{E}^{(i)}[\psi](t) + \mathcal{E}^{(e)}[\psi](t) \tag{44}$$

where,

$$\mathcal{E}^{(i)}[\psi](t) = \int_{\Sigma_t} (1 - \zeta)(t^2 |\partial\psi|^2 + \psi^2)$$
$$\mathcal{E}^{(e)}[\psi](t) = \int_{\Sigma_t} \zeta \, (\underline{u}^2 (L\psi)^2 + u^2 (\underline{L}\psi)^2 + \underline{u}^2 |\nabla\!\!\!/ \psi|^2 + \psi^2).$$

with ζ a smooth cut-off function equal to 1 in the wave zone region $u \le \frac{t}{2}$.

[17]The level hypersurfaces of u are outgoing null cones C_u with vertices on the time axis Γ_t.

In the proof of Theorem 3.4 we need the following comparison between the quantity $\mathcal{Q}(t)$ and the auxiliary norm $\mathcal{E}(t) = \mathcal{E}[\psi](t)$.

THEOREM 3.5 (The comparison theorem). *Under the same assumptions as in Theorem* 3.4, *for any* $1 \le t \le t_*$,

$$\mathcal{E}[\psi](t) \lesssim \mathcal{Q}[\psi](t).$$

Proof. See Section 6. □

4. The Asymptotics Theorem and other geometric tools

In this section we record the crucial properties of all the important geometric objects associated to our spacetime foliations Σ_t, C_u and $S_{t,u}$ introduced above. Most of the results of this section will be proved only in the second part of this work.

We start with some simple facts concerning the parameters of the foliation Σ_t relative to the spacetime geometry associated to the metric $H = H_\lambda$.

The Σ_t foliation. Recall, see (23), that the parameters of the Σ_t foliation are given by n, v, the induced metric h and the second fundamental form k_{ij}, according to the decomposition,

$$H = -n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt), \tag{45}$$

with h_{ij} the induced Riemannian metric on Σ_t, n the lapse and $v = v^i\partial_i$ the shift of H. Denoting by T the unit, future oriented, normal to Σ_t and k the second fundamental form $k_{ij} = -\langle \mathbf{D}_i T, \partial_j\rangle$ we find,

$$\begin{gathered}\partial_t = nT + v, \qquad \langle \partial_t, v\rangle = 0,\\ k_{ij} = -\frac{1}{2}\mathcal{L}_T H_{ij} = -12n^{-1}(\partial_t h_{ij} - \mathcal{L}_v h_{ij})\end{gathered} \tag{46}$$

with $\mathcal{L}_X$ denoting the Lie derivative with respect to the vectorfield X. We also have the following (see (8), (24), and (135) in Section 8):

$$c|\xi|^2 \le h_{ij}\xi^i\xi^j \le c^{-1}|\xi|^2, \qquad c \le n^2 - |v|_h^2 \tag{47}$$

for some $c > 0$. Also

$$n, |v| \lesssim 1, \tag{48}$$

$$|\partial n| + |\partial v| + |\partial h| + |k| \lesssim |\partial H|. \tag{49}$$

$S_{t,u}$- foliation. We define the Ricci coefficients associated to the $S_{t,u}$ foliation and null pair $L, \underline{L}$.

Definition 4.1. Using an arbitrary orthonormal frame $(e_A)_{A=1,2}$ on $S_{t,u}$ we define the following tensors on the surfaces $S_{t,u}$:

$$
\begin{aligned}
\chi_{AB} &= \langle \mathbf{D}_A e_4, e_B \rangle, \qquad \underline{\chi}_{AB} = \langle \mathbf{D}_A e_3, e_B \rangle, \\
\eta_A &= \frac{1}{2} \langle \mathbf{D_3} e_4, e_A \rangle, \qquad \underline{\eta}_A = \frac{1}{2} \langle \mathbf{D_4} e_3, e_A \rangle, \\
\underline{\xi}_A &= \frac{1}{2} \langle \mathbf{D_3} e_3, e_A \rangle.
\end{aligned} \tag{50}
$$

Using the parameters n, v, k of the Σ_t foliation we find(see [Kl-Ro2] and [Kl-Ro]),

$$
\begin{aligned}
\underline{\chi}_{AB} &= -\chi_{AB} - 2k_{AB}, \\
\underline{\eta}_A &= -k_{AN} + n^{-1} \nabla\!\!\!\!/_A n, \\
\underline{\xi}_A &= k_{AN} - \eta_A + n^{-1} \nabla\!\!\!\!/_A n, \\
\eta_A &= b^{-1} \nabla\!\!\!\!/_A b + k_{AN}.
\end{aligned}
$$

Thus all the Ricci coefficients can be expressed in terms of k_{ij}, n, the scalar function b and, most important, the Ricci coefficients χ and η. Recall that χ decomposes into its trace $\operatorname{tr}\chi$ and traceless part $\hat{\chi}$; see [Kl-Ro].

We shall also denote by $\theta_{AB} = \langle \nabla\!\!\!\!/_A N, e_B \rangle$ the second fundamental form of $S_{t,u}$ relative to Σ_t. It is easy to check that

$$\chi_{AB} = -k_{AB} + \theta_{AB}.$$

We consider the parameters b, $\operatorname{tr}\chi$, $\hat{\chi}$ and η associated to the $S_{t,u}$ foliation according to (42) and (50). For convenience we shall introduce the quantity:

$$\Theta = |\operatorname{tr}\chi - \frac{2}{r}| + |\operatorname{tr}\chi - \frac{2}{n(t-u)}| + |\hat{\chi}| + |\eta|. \tag{51}$$

Remark 4.2. Strictly speaking we need only one of the two quantities $|\operatorname{tr}\chi - \frac{2}{r}|$, $|\operatorname{tr}\chi - \frac{2}{n(t-u)}|$ in the expression above. Indeed we show in [Kl-Ro2] that these two are comparable.

Remark 4.3. Simple calculations based on Definition 4.1, see also Ricci equations in Section 2 of [Kl-Ro2], allow us to derive the following:

$$|DL|, |D\underline{L}|, |\nabla N| \lesssim r^{-1} + \Theta + |\partial H|. \tag{52}$$

Remark 4.4. We shall make use of the following simple commutation estimates; see Lemma 3.5 in [Kl-Ro2],

$$|(\nabla\!\!\!\!/_N \nabla\!\!\!\!/ - \nabla\!\!\!\!/ \nabla\!\!\!\!/_N) f| \lesssim \big(r^{-1} + \Theta + |\partial H|\big) |\nabla f|. \tag{53}$$

We state below the crucial theorem which establishes the desired asymptotic behavior of these quantities relative to λ.

THEOREM 4.5 (The asymptotics theorem). *In the spacetime region* $\mathcal{D}_0$ (*see Theorem* 3.4) *the quantities* b, Θ *satisfy the following estimates*:

$$|b-n| \lesssim \lambda^{-4\epsilon_0}, \tag{54}$$

$$\|\Theta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}, \tag{55}$$

$$\|\Theta\|_{L^q(S_{t,u})} \lesssim \lambda^{-3\epsilon_0}, \qquad 2 \le q \le 4. \tag{56}$$

In addition, *in the exterior region* $u \le t/2$,

$$\|\Theta\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-\epsilon_0} + \lambda^{\epsilon}\|\partial H(t)\|_{L_x^\infty} \tag{57}$$

for an arbitrarily small $\epsilon > 0$.

The following estimates hold for the derivatives of $\operatorname{tr}\chi$:

(58)

$$\|\sup_{u\le\frac{t}{2}}\|\underline{L}\Big(\operatorname{tr}\chi-\frac{2}{r}\Big)\|_{L^2(S_{t,u})}\|_{L_t^1}+\|\sup_{u\le\frac{t}{2}}\|\underline{L}\Big(\operatorname{tr}\chi-\frac{2}{n(t-u)}\Big)\|_{L^2(S_{t,u})}\|_{L_t^1}\le\lambda^{-3\epsilon_0};$$

(59)

$$\|\sup_{u\le\frac{t}{2}}\|\nabla\!\!\!/\operatorname{tr}\chi\|_{L^2(S_{t,u})}\|_{L_t^1}+\|\sup_{u\le\frac{t}{2}}\|\nabla\!\!\!/\left(\operatorname{tr}\chi-\frac{2}{n(t-u)}\right)\|_{L^2(S_{t,u})}\|_{L_t^1}\le\lambda^{-3\epsilon_0}.$$

In addition, *there are weak estimates of the form*,

$$\sup_{u\le\frac{t}{2}}\|(\nabla\!\!\!/,\underline{L})\left(\operatorname{tr}\chi-\frac{2}{n(t-u)}\right)\|_{L^\infty(S_{t,u})}\lesssim\lambda^C \tag{60}$$

for some large value of C.

There also is the following comparison between the functions r *and* $t-u$,

$$c^{-1}\le\frac{r}{t-u}\le c. \tag{61}$$

The proof of the Asymptotics Theorem is truly at the heart of this work and it is quite involved. Our second paper [Kl-Ro2] is almost entirely dedicated to it.

Remark 4.6. Observe that the estimate (55) holds true also for ∂H. We shall show, see [Kl-Ro2, Prop. 7.4], that ∂H also verifies the estimate (56). Thus we can incorporate the term $|\partial H|$ in the definition (51) of Θ.

$$\Theta=|\operatorname{tr}\chi-\frac{2}{r}|+|\operatorname{tr}\chi-\frac{2}{n(t-u)}|+|\hat\chi|+|\eta|+|\partial H|. \tag{62}$$

For convenience we shall also often use Θ to denote $O(\Theta)$. We shall do this freely throughout this paper.

The proof of the next proposition will be delayed to [Kl-Ro3]; see also [Kl-Ro].

Proposition 4.7. *Let $S_{t,u}$ be a fixed surface in $\Sigma_t \cap \mathcal{D}_0$.*

i) Isoperimetric inequality. *For any smooth function $f : S_{t,u} \to \mathbb{R}$ there is the following isoperimetric inequality*:

$$\Big(\int_{S_{t,u}} |f|^2\Big)^{\frac{1}{2}} \lesssim \int_{S_{t,u}} (|\nabla\!\!\!\!/ f| + |\mathrm{tr}\theta||f|). \tag{63}$$

ii) Sobolev Inequality. *For any $\delta \in (0,1)$ and p from the interval $p \in (2,\infty]$,*

$$\sup_{S_{t,u}} |f| \lesssim r^{\frac{\epsilon(p-2)}{2p+\delta(p-2)}} \left(\int_{S_{t,u}} (|\nabla\!\!\!\!/ f|^2 + r^{-2}|f|^2)\right)^{\frac{1}{2}-\frac{\delta p}{2p+\delta(p-2)}} \cdot \left[\int_{S_{t,u}} (|\nabla\!\!\!\!/ f|^p + r^{-p}|f|^p)\right]^{\frac{2\delta}{2p+\delta(p-2)}}. \tag{64}$$

iii) Trace Inequality. *For an arbitrary function $f : \Sigma_t \to \mathbf{R}$ such that $f \in H^{\frac{1}{2}+\epsilon}(\mathbf{R}^3)$,*

$$\|f\|_{L^2(S_{t,u})} \le \|\partial^{\frac{1}{2}+\epsilon} f\|_{L^2(\Sigma_t)} + \|\partial^{\frac{1}{2}-\epsilon} f\|_{L^2(\Sigma_t)}. \tag{65}$$

More generally, for any $q \in [2,\infty)$

$$\|f\|_{L^q(S_{t,u})} \le \|\partial^{\frac{3}{2}-\frac{2}{q}+\epsilon} f\|_{L^2(\Sigma_t)} + \|\partial^{\frac{3}{2}-\frac{2}{q}-\epsilon} f\|_{L^2(\Sigma_t)}. \tag{66}$$

Also, considering the region $\mathrm{Ext}_t = \Sigma_t \cap \{0 \le u \le \frac{t}{2}\}$, *we have the following*:

$$\|f\|^2_{L^2(S_{t,u})} \le \|N(f)\|_{L^2(\mathrm{Ext}_t)} \|f\|_{L^2(\mathrm{Ext}_t)} + \frac{1}{t}\|f\|_{L^2(\mathrm{Ext}_t)}. \tag{67}$$

We shall make use of the following; see Lemma 6.3 in [Kl-Ro].

Proposition 4.8. *The following inequality holds for all $t \in [1, t_*]$ and $2 < p < \infty$:*

$$\int_{\Sigma_t} V^2 w^2 \le t^{\frac{2}{p}} \sup_u \|V\|^2_{L^{2p'}(S_{t,u})} \int_{\Sigma_t} \big(|\nabla\!\!\!\!/ w|^2 + r^{-2}|w|^2\big), \tag{68}$$

where p' is the exponent dual to p.

We shall also make use of the variant,

$$\int_{\Sigma_t} V^2 w^2 \le t^{\frac{2}{p}} \|V\|^{\frac{2}{p}}_{L^\infty_x} \sup_u \|V\|^{\frac{2}{p'}}_{L^2(S_{t,u})} \int_{\Sigma_t} \big(|\nabla\!\!\!\!/ w|^2 + r^{-2}|w|^2\big). \tag{69}$$

In particular, if $\|V\|_{L^\infty_x}$ is bounded by some positive power of λ, and we restrict ourselves to the exterior region Ext_t, *we deduce that for every $\varepsilon > 0$ and some constant C,*

$$\int_{\mathrm{Ext}_t} V^2 w^2 \le t^{-2} \lambda^{C\varepsilon} \sup_{0 \le u \le t/2} \|V\|^{2-\varepsilon}_{L^2(S_{t,u})} \mathcal{E}[w](t). \tag{70}$$

Proof. The proof is straightforward and relies only on the isoperimetric inequality (63); see also 6.1. in [Kl-Ro]. ∎

5. Proof of the Boundedness Theorem

We first calculate the components of the modified[18] deformation tensor $\bar{\pi} = {}^{(K)}\bar{\pi} = {}^{(K)}\pi - 4tH$ of our vectorfield $K = \frac{1}{2}n(\underline{u}^2 L + u^2 \underline{L})$. Recall that $\underline{u} = 2t - u$ and $\underline{L} = -L + 2T$; thus

$$\begin{aligned}
\underline{L}\,(u^2) &= 4u\,b^{-1},\\
L(\,\underline{u}^2) &= 4\underline{u}\,n^{-1},\\
\underline{L}(\,\underline{u}^2) &= 4\underline{u}\,(n^{-1} - b^{-1}).
\end{aligned}$$

We proceed as in Section 6.1 of [Kl-Ro] to calculate the null components of $\bar{\pi} = {}^{(K)}\bar{\pi}$ relative[19] to $e_4 = L, e_3 = \underline{L}$ and $(e_A)_{A=1,2}$ an arbitrary orthonormal frame on $S_{t,u}$,

(71)

$$\begin{aligned}
\bar{\pi}_{\mathbf{44}} &= 2u^2\, n(\bar{k}_{NN} - n^{-1}e_4(n)),\\
\bar{\pi}_{\mathbf{34}} &= 4un(n^{-1} - b^{-1}) + \underline{u}^2 n\big(\bar{k}_{NN} - n^{-1}e_4(n)\big) + u^2 n\big(\bar{k}_{NN} - n^{-1}e_3(n)\big),\\
\bar{\pi}_{\mathbf{33}} &= -8\underline{u}\,n(n^{-1} - b^{-1}) - 2\underline{u}^2\, n\big(\bar{k}_{NN} + n^{-1}e_3(n)\big),\\
\bar{\pi}_{\mathbf{3}A} &= \underline{u}^2\, n(\eta_A + k_{AN} - n^{-1}\nabla\!\!\!/_A n) + u^2 n\underline{\xi}_A,\\
\bar{\pi}_{\mathbf{4}A} &= u^2\, n(\underline{\eta}_A - k_{AN} - n^{-1}\nabla\!\!\!/_A n),\\
\bar{\pi}_{AB} &= 2tn(t-u)\big(\mathrm{tr}\,\chi - \frac{2}{n(t-u)}\big)\delta_{AB} + 4tn(t-u)\hat{\chi}_{AB} - 2u^2 n k_{AB}
\end{aligned}$$

where $\bar{k}_{NN} = k_{NN} - n^{-1}\nabla_N n$.

The following proposition concerning the behavior of the null components of $\bar{\pi}$ is an immediate consequence of the above formulae and the Asymptotics Theorem stated above.

Proposition 5.1.

$$\begin{aligned}
&\|u^{-2}\bar{\pi}_{44}\|_{L^1_t L^\infty_x} \lesssim \lambda^{-3\epsilon_0}, && \|(\underline{u})^{-2}\bar{\pi}_{34}\|_{L^1_t L^\infty_x} \lesssim \lambda^{-3\epsilon_0},\\
&\|(\underline{u})^{-2}\bar{\pi}_{33}\|_{L^1_t L^\infty_x} \lesssim \lambda^{-3\epsilon_0}, && \|(\underline{u})^{-2}\bar{\pi}_{3A}\|_{L^1_t L^\infty_x} \lesssim \lambda^{-3\epsilon_0},\\
&\|(u)^{-2}\bar{\pi}_{4A}\|_{L^1_t L^\infty_x} \lesssim \lambda^{-3\epsilon_0}, && \|(\underline{u})^{-2}\bar{\pi}_{AB}\|_{L^1_t L^\infty_x} \lesssim \lambda^{-3\epsilon_0}.
\end{aligned}$$

The proof of the boundedness theorem relies on the generalized energy identity (36) with $K = \frac{1}{2}n\big(u^2\underline{L} + \underline{u}^2 L\big)$ and $\Omega = 4t$. Thus,

$$\begin{aligned}
\text{(72)}\qquad \mathcal{Q}[\psi](t) &= \mathcal{Q}[\psi](t_0) - \frac{1}{2}\int_{[t_0,t]\times\mathbb{R}^3} Q^{\alpha\beta\ (K)}\bar{\pi}_{\alpha\beta} + \int_{[t_0,t]\times\mathbb{R}^3}\psi^2\Box_H t\\
&= \mathcal{Q}[\psi](t_0) - \frac{1}{2}\mathcal{J} + \mathcal{Y}.
\end{aligned}$$

[18] corresponding to the choice $\Omega = 4t$.

[19] We say that $(e_i)_{1=1,2,3,4}$ forms a null frame.

Observe that we can decompose:

$$\begin{aligned}\mathcal{J} &= \int_{[t_0,t]\times\mathbf{R}^3} Q^{\alpha\beta}[\psi]\bar{\pi}_{\alpha\beta}\\ &= \int_{[t_0,t]\times\mathbf{R}^3} \Big(\frac{1}{4}\bar{\pi}_{33}(L\psi)^2 + \frac{1}{4}\bar{\pi}_{44}(\underline{L}\psi)^2 + \frac{1}{2}\bar{\pi}_{34}|\nabla\!\!\!/\psi|^2 - \bar{\pi}_{4A}\underline{L}\psi\nabla\!\!\!/_A\psi\\ &\qquad - \bar{\pi}_{3A}L\psi\nabla\!\!\!/_A\psi + \bar{\pi}_{AB}\nabla\!\!\!/_A\psi\nabla\!\!\!/_B\psi + \mathrm{tr}\bar{\pi}(\frac{1}{2}\underline{L}\psi L\psi - |\nabla\!\!\!/\psi|^2)\Big).\end{aligned}$$

Consider, for example, $I = |\int_{[t_0,t]\times\mathbf{R}^3}\bar{\pi}_{4A}\underline{L}\psi\nabla\!\!\!/_A\psi|$. We can estimate it as follows:

$$\begin{aligned}I &\le \frac{1}{2}\int_{[t_0,t]\times\mathbf{R}^3} |(\underline{u}u)^{-1}\bar{\pi}_{4A}|\Big(u^2(\underline{L}\psi)^2 + \underline{u}^2(\nabla\!\!\!/_A\psi)^2\Big)\\ &\lesssim \int_{t_0}^t \|(\underline{u}u)^{-1}\bar{\pi}_{4A}\|_{L^\infty_x}\mathcal{E}[\psi](\tau)\,d\tau.\end{aligned}$$

Making use of the comparison theorem and the estimate $\|(\underline{u}u)^{-1}\bar{\pi}_{4A}\|_{L^1_tL^\infty_x} \lesssim \lambda^{-3\epsilon_0}$ we infer that,

$$I \lesssim \int_{t_0}^t \|(\underline{u}u)^{-1}\bar{\pi}_{4A}\|_{L^\infty_x}\mathcal{Q}[\psi](\tau)\,d\tau \lesssim \lambda^{-3\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau).$$

We can proceed in the same manner with all the terms of $\mathcal{J}$ with the exception of $\int_{[t_0,t]\times\mathbf{R}^3}\mathrm{tr}\bar{\pi}\,\underline{L}\psi\,L\psi$. Observe that[20]

$$\mathrm{tr}\bar{\pi} = \delta^{AB}\bar{\pi}_{AB} = 2tn(t-u)\left(\mathrm{tr}\,\chi - \frac{2}{n(t-u)}\right) - 2u^2n\mathrm{tr}k,$$

$$\begin{aligned}\int_{[t_0,t]\times\mathbf{R}^3}|u^2n\mathrm{tr}kL\psi\underline{L}\psi| &\le \frac{1}{2}\int_{[t_0,t]\times\mathbf{R}^3}|\mathrm{tr}k|\big(u^2(L\psi)^2 + u^2(\underline{L}\psi)^2\big)\\ &\lesssim \int_{t_0}^t \|\partial H\|_{L^\infty_x}\mathcal{E}[\psi](\tau)\,d\tau.\end{aligned}$$

Since $\|\partial H\|_{L^1_tL^\infty_x} \lesssim \lambda^{-4\epsilon_0}$, this term can be treated in the same manner as I. We are thus left with the integral

$$\mathcal{B} = \int_{[t_0,t]\times\mathbf{R}^3} 2tn(t-u)\left(\mathrm{tr}\,\chi - \frac{2}{n(t-u)}\right)\underline{L}\psi L\psi.$$

All other terms $\mathcal{J}-\mathcal{B}$ can be estimated in precisely the same manner, using the comparison theorem and the estimates of Proposition 5.1, by

$$\mathcal{J} - \mathcal{B} \lesssim \lambda^{-3\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau). \tag{73}$$

[20]We use tr here to denote the trace relative to the surfaces $S_{t,u}$. Thus $\mathrm{tr}k = \delta^{AB}k_{AB}$. We use $\mathrm{Tr}k = h^{ij}k_{ij}$ to denote the usual trace of k with respect to Σ_t.

To estimate the remaining term $\mathcal{B}$ requires a more involved argument. In fact we shall need more information concerning the geometry of the null cones C_u and surfaces $S_{t,u}$.

Denote Ext_t the exterior region $\mathrm{Ext}_t = \{0 \le u \le t/2\}$. Let ζ be a smooth cut-off function with support in Ext_t. Observe that

$$\int_{\Sigma_t} \big(t^2(\partial\psi)^2 + \psi^2\big)(1-\zeta) \lesssim \int_{\Sigma_t} (1-\zeta)\bar{Q}[\psi](t). \tag{74}$$

We can split the remaining integral

$$\begin{aligned}
\mathcal{B} &= \mathcal{B}^i + \mathcal{B}^e,\\
\mathcal{B}^i &= \int_{[t_0,t]\times\mathbf{R}^3} 2tn(t-u)\big(\mathrm{tr}\,\chi - \frac{2}{n(t-u)}\big) L\psi \underline{L}\psi\,(1-\zeta),\\
\mathcal{B}^e &= \int_{[t_0,t]\times\mathbf{R}^3} 2tn(t-u)\big(\mathrm{tr}\,\chi - \frac{2}{n(t-u)}\big) L\psi \underline{L}\psi\,\zeta.
\end{aligned}$$

With the help of (74) the first integral can be estimated as follows:

$$\begin{aligned}
|\mathcal{B}^i| &\lesssim \int_{[t_0,t]\times\mathbf{R}^3} |\mathrm{tr}\,\chi - \frac{2}{n(\tau-u)}|\ \tau^2\,(\partial\psi)^2(1-\zeta)\\
&\lesssim \int_{t_0}^{t} \|\mathrm{tr}\,\chi - \frac{2}{n(\tau-u)}\|_{L^\infty_x}\ \bar{Q}[\psi](\tau)\,d\tau.
\end{aligned}$$

In view of the estimate $\|\mathrm{tr}\,\chi - \frac{2}{n(t-u)}\|_{L^1_t L^\infty_x} \lesssim \lambda^{-3\epsilon_0}$, given by the Asymptotics Theorem (4.5) we infer that,

$$|\mathcal{B}^i| \lesssim \lambda^{-3\epsilon_0} \sup_{[t_0,t]} \mathcal{Q}[\psi](\tau).$$

Therefore, it remains to estimate $\mathcal{B}^e$.

According to the Asymptotics Theorem the quantity $z = \mathrm{tr}\,\chi - \frac{2}{n(t-u)}$ verifies the following estimates:

$$\|z\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac12 - 3\epsilon_0}, \qquad \|z\|_{L^2(S_{t,u})} \lesssim \lambda^{-2\epsilon_0}, \tag{75}$$

$$\|\sup_{u\le\frac{t}{2}} \|\nabla\!\!\!\!/\, z\|_{L^2(S_{t,u})}\|_{L^2_t} \lesssim \lambda^{-\frac12 - 3\epsilon_0}, \qquad \|\sup_{u\le\frac{t}{2}} \|\underline{L}\ z\|_{L^2(S_{t,u})}\|_{L^2_t} \lesssim \lambda^{-\frac12 - 3\epsilon_0}. \tag{76}$$

Remark 5.2. The same estimates hold true if we replace $\mathrm{tr}\,\chi - \frac{2}{n(t-u)}$ by $\mathrm{tr}\,\chi - \frac{2}{r}$.

It would therefore suffice to prove the following result. Using the estimates (75), (76) we shall prove that:

$$\mathcal{B}^e = \int_{[t_0,t]\times\mathbf{R}^3} 2tn(t-u) z L\psi \underline{L}\psi\,\zeta \lesssim \lambda^{-\epsilon_0} \sup_{[t_0,t]} \mathcal{Q}[\psi](\tau). \tag{77}$$

To prove (77) we need to rely on the fact that ψ is a solution of the wave equation $\Box_H \psi = 0$. We shall also make use of the following standard integration by parts formulae[21],

$$\int_{\Sigma_t} F N(G) = -\int_{\Sigma_t} \Big(N(F) + \big(\mathrm{tr}\theta + n^{-1} N(n)\big) F \Big) G, \tag{78}$$

where N is the unit normal to $S_{t,u}$.

If Y is a vectorfield in $T\Sigma_t$ tangent to $S_{t,u}$ then

$$\int_{\Sigma_t} F \,\mathrm{d\!\!/iv}\, Y = -\int_{\Sigma_t} \Big(\nabla\!\!\!/ F + (b^{-1} \nabla\!\!\!/ b + n^{-1} \nabla\!\!\!/ n) F \Big) \cdot Y. \tag{79}$$

It is also not difficult to verify that

$$\int_{[t_0,t]\times\mathbf{R}^3} F T(G) = -\int_{[t_0,t]\times\mathbf{R}^3} (T(F) + Trk + \mathrm{div} v) G + \int_{\Sigma_t} FG - \int_{\Sigma_{t_0}} FG. \tag{80}$$

Writing $\underline{L} = T - N$ we integrate by parts and express the integral $\mathcal{B}^e$ in the form,

$$\begin{aligned}
\frac{1}{2}\mathcal{B}^e &= -I_1 + I_2 + I_3 - I_4, \\
I_1 &= \int_{[t_0,t]\times\mathbf{R}^3} \zeta\, nt(t-u) z \,(\underline{L} L\psi)\,\psi, \\
I_2 &= \int_{[t_0,t]\times\mathbf{R}^3} \Big(-\underline{L}(\zeta\, nt(t-u)z) \\
&\qquad + (\mathrm{tr}\theta + n^{-1}N(n) - \mathrm{Tr}\, k - \mathrm{div} v) \zeta\, nt(t-u) z \Big) L\psi\,\psi, \\
I_3 &= \int_{\Sigma_t} \zeta\, nt(t-u) z\, L\psi\psi, \\
I_4 &= \int_{\Sigma_{t_0}} \zeta\, nt(t-u) z\, L\psi\psi.
\end{aligned} \tag{81}$$

We first handle the boundary terms I_3, I_4. With the help of Proposition 4.8 (which we can apply in view of the estimates (57) for Θ as well as the estimate (26) for ∂H) we have

$$\|n(t-u)z\psi\|_{L^2(Ext_t)} \lesssim \lambda^{C\epsilon} \sup_{u \le \frac{t}{2}} \|nz\|^{1-\epsilon/2}_{L^2(S_{t,u})} \mathcal{E}^{\frac{1}{2}}[\psi](t).$$

[21] These are simple adaptations of the formulae in Lemma 6.2, [Kl-Ro].

Therefore,

$$
\begin{aligned}
|I_3| \le \int_{\Sigma_t} |\zeta\, nt(t-u)z\, L\psi\psi| &\lesssim \int_{\mathrm{Ext}_t} |n(t-u)z\, tL\psi\psi| \\
&\lesssim \|t\, L\psi\|_{L^2(\Sigma_t)} \|n(t-u)z\psi\|_{L^2(Ext_t)} \\
&\lesssim \|n(t-u)\, z\psi\|_{L^2(Ext_t)} \mathcal{E}^{\frac12}[\psi](t) \\
&\lesssim \lambda^{C\epsilon} \sup_{s\ge \frac{t}{2}} \|\, nz\|^{1-\epsilon/2}_{L^2(S_{t,u})} \mathcal{E}[\psi](t) \lesssim \lambda^{-\epsilon_0}\mathcal{E}[\psi](t).
\end{aligned}
$$

The last inequality followed from the boundness of n and (75). A similar estimate holds for the second boundary term I_4.

To estimate I_2 we observe that, as an immediate consequence of Theorem 4.5, we have

$$
|\underline{L}(t)|,\ |\underline{L}(t-u)| \lesssim 1, \qquad |\underline{L}(\zeta)| \lesssim t^{-1}.
$$

Denoting

$$
\Theta(t,x) = |\mathrm{tr}\,\chi - \frac{2}{n(t-u)}| + |\hat{\chi}| + |\eta| + |\partial H|,
$$

we easily find

$$
|I_2| \lesssim \int_{t_0}^t \int_{\mathrm{Ext}_\tau} \Big(\tau^2 |\underline{L}(z)| + \tau|z| + \tau^2\Theta|z|\Big)|L\psi\,\psi|\, d\tau.
$$

To treat the term involving $\underline{L}(z)$ we proceed as in the case of I_3; we estimate $\int_{\mathrm{Ext}_\tau} \tau^2|\underline{L}(z)||L\psi\,\psi|\,d\tau$ by Cauchy-Schwartz followed by an application of Proposition 4.8. The space integral of the other two terms can be estimated as follows:

$$
\int_{\mathrm{Ext}_\tau} \big(\tau|z| + \tau^2\Theta|z|\big)|L\psi\,\psi|\,d\tau \le \big(\|z\|_{L^\infty_x} + \tau\|\Theta\|_{L^\infty_x}\|z\|_{L^\infty_x}\big)\mathcal{E}[\psi](\tau).
$$

Consequently, using the inequalities (75), (76) for z (as well as the weak estimate (60)) and the estimates for Θ from the Asymptotics Theorem 4.5 we obtain

$$
\begin{aligned}
|I_2| \lesssim \int_{t_0}^t \Bigg(&\lambda^{C\epsilon} \sup_{u\le\frac{\tau}{2}} \|\underline{L}(z)\|^{1-\epsilon/2}_{L^2(S_{t,u})} \\
&+ \|z\|_{L^\infty(\Sigma_\tau)} + \tau\|\Theta\|_{L^\infty(\Sigma_\tau)}\|z\|_{L^\infty(\Sigma_\tau)}\Bigg)\mathcal{E}[\psi](\tau)\,d\tau \\
\lesssim \lambda^{C'\epsilon}\Bigg(&\| \sup_{u\le\frac{\tau}{2}} \|\underline{L}(z)\|_{L^2(S_{t,u})}\|^{1-\epsilon/2}_{L^1_t} \\
&+ \|z\|_{L^1_t L^\infty_x} + \lambda\|\Theta\|_{L^2_t L^\infty_x}\|z\|_{L^2_t L^\infty_x}\Bigg) \sup_{[t_0,t]} \mathcal{Q}[\psi](\tau) \\
\lesssim \lambda^{-\epsilon_0} &\sup_{[t_0,t]} \mathcal{Q}[\psi](\tau)
\end{aligned}
$$

as desired.

It remains therefore to consider I_1. We shall make use of the fact that ψ is a solution of the wave equation. This allows us to express the $\underline{L}L(\psi)$ in terms of the angular laplacian[22] $\triangle\!\!\!\!/$ and lower order terms. Expressed relative to a null frame the wave operator $\Box_H\psi$ takes the form

$$\Box_H\psi = H^{\alpha\beta}\psi_{;\alpha\beta} = -\psi_{;\mathbf{43}} + \psi_{;AA},$$

where $\psi_{;e_ie_j} = e_j(e_i(\psi)) - \mathbf{D}_{e_i}e_j(\psi)$. We use the Ricci formulas: $\mathbf{D_3}e_4 = 2\eta_A e_A + \bar{k}_{NN}e_4$, and $\mathbf{D}_B e_A = \nabla\!\!\!\!/_B e_A + \frac{1}{2}\chi_{AB}e_3 + \frac{1}{2}\underline{\chi}_{AB}e_4$ to derive

$$\Box_H\psi = -\underline{L}L\psi + \triangle\!\!\!\!/\psi + 2\eta_A\nabla\!\!\!\!/_A\psi + \frac{1}{2}\mathrm{tr}\,\chi\underline{L}\psi + (\frac{1}{2}\mathrm{tr}\,\underline{\chi} + \bar{k}_{NN})L\psi. \tag{82}$$

As a result of this calculation

$$\begin{aligned} I_1 = &\int_{[t_0,t]\times\mathbf{R}^3} \zeta\, n\tau(\tau-u)z\,\underline{L}L\psi\,\psi = \int_{[t_0,t]\times\mathbf{R}^3}\zeta\, n\tau(\tau-u)z\,\triangle\!\!\!\!/\psi\psi \\ &+\frac{1}{2}\int_{[t_0,t]\times\mathbf{R}^3}\zeta\, n\tau(\tau-u)z\,\mathrm{tr}\,\chi(\underline{L}\psi)\psi \\ &+\int_{[t_0,t]\times\mathbf{R}^3}\zeta\, n\tau(\tau-u)z\,\left(2\eta_A\nabla\!\!\!\!/_A\psi + \left(\frac{1}{2}\mathrm{tr}\,\underline{\chi} + \bar{k}_{NN}\right)L\psi\right)\,\psi \\ = &I_{11} + I_{12} + I_{13}. \end{aligned} \tag{83}$$

Consider first I_{13}. Since $t-u \ge \frac{t}{2}$,

$$\begin{aligned} |I_{13}| &\lesssim \int_{t_0}^t\int_{\mathrm{Ext}_\tau}\tau^2|z|\Big(\Theta\nabla\!\!\!\!/\psi + (\frac{1}{\tau}+\Theta)L\psi\Big)\psi \\ &\lesssim \int_{t_0}^t\Big(\tau\|z\|_{L^\infty(\Sigma_\tau)}\|\Theta\|_{L^\infty(\Sigma_\tau)} + \|z\|_{L^\infty(\Sigma_\tau)}\Big)\mathcal{E}[\psi](\tau)\,d\tau \\ &\lesssim \lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau) \end{aligned} \tag{84}$$

as before.

To estimate I_{12} we need first to integrate once more by parts.

$$\begin{aligned} I_{12} = &\frac{1}{4}\int_{[t_0,t]\times\mathbf{R}^3}\Big(-\underline{L}(\zeta\, n\tau(\tau-u)z\,\mathrm{tr}\,\chi) \\ &+ (\mathrm{tr}\theta + n^{-1}N(n) - \mathrm{Tr}\,k - \mathrm{div}v)\,\zeta\, n\tau(\tau-u)z\mathrm{tr}\,\chi\Big)\psi^2 \\ &+\frac{1}{4}\int_{\Sigma_t}\zeta\, n\tau(\tau-u)z\,\mathrm{tr}\,\chi(\psi)^2 - \frac{1}{4}\int_{\Sigma_{t_0}}\zeta\, n\tau(\tau-u)z\,\mathrm{tr}\,\chi(\psi)^2. \end{aligned}$$

All terms can be treated as above. Take, for example, the worst term involving $\underline{L}(\mathrm{tr}\,\chi)$. Recall that

$$\underline{L}(\mathrm{tr}\,\chi) = \underline{L}\left(\mathrm{tr}\,\chi - \frac{2}{r}\right) + \underline{L}\left(\frac{2}{r}\right) \lesssim \underline{L}\left(\mathrm{tr}\,\chi - \frac{2}{r}\right) + \frac{2}{r^2} + \frac{1}{r}\Theta.$$

[22] the Laplace-Beltrami operator on $S_{t,u}$.

Thus

$$\begin{aligned}
\int_{[t_0,t]\times\mathbf{R}^3} &|\zeta\, nt(t-u)z\underline{L}(\operatorname{tr}\chi)(\psi^2)|\\
&\lesssim \int_{t_0}^t\int_{Ext_\tau}\tau^2|z|\left(|\underline{L}\left(\operatorname{tr}\chi-\frac{2}{r}\right)|+\frac{1}{\tau^2}+\frac{1}{\tau}\Theta\right)\psi^2\\
&\lesssim \int_{t_0}^t\int_{\mathrm{Ext}_\tau}\tau^2\,|z|\,|\underline{L}\left(\operatorname{tr}\chi-\frac{2}{r}\right)|\,\psi^2\\
&\quad+\int_{t_0}^t\left(\|z\|_{L^\infty(\Sigma_\tau)}+\tau\|z\|_{L^\infty(\Sigma_\tau)}\|\Theta\|_{L^\infty(\Sigma_\tau)}\right)\mathcal{E}[\psi](\tau)\,d\tau.
\end{aligned}$$

The second term has already been treated above; see (83). To estimate the first we apply first Cauchy-Schwartz and then make use of Proposition 4.8,

$$\begin{aligned}
\int_{t_0}^t\int_{\mathrm{Ext}_\tau}&\tau^2|z||\underline{L}\left(\operatorname{tr}\chi-\frac{2}{r}\right)|\psi^2\\
&\lesssim\int_{t_0}^t\|\,\tau^2\,|z|\,|\underline{L}\left(\operatorname{tr}\chi-\frac{2}{r}\right)|\,\psi\|_{L^2(Ext_\tau)}\;\mathcal{E}^{\frac12}[\psi](\tau)\,d\tau\\
&\lesssim\int_{t_0}^t\lambda^{C\epsilon}\sup_{u\le\frac{\tau}{2}}\|\tau|z||\underline{L}\left(\operatorname{tr}\chi-\frac{2}{r}\right)|\|^{1-\epsilon/2}_{L^2(S_{t,u})}\mathcal{E}[\psi](\tau)\,d\tau.
\end{aligned}$$

Taking into account the estimates in (75), (76) and Remark 5.2 we deduce,

$$\begin{aligned}
\lambda^{C\epsilon}&\int_{t_0}^t\sup_{u\le\frac{\tau}{2}}\|\tau z\underline{L}\left(\operatorname{tr}\chi-\frac{2}{r}\right)\|^{1-\epsilon/2}_{L^2(S_{t,u})}\\
&\lesssim\lambda^{C'\epsilon}\left(t\|z\|_{L^2_tL^\infty_x}\|\sup_{u\le\frac{t}{2}}\|\underline{L}\left(\operatorname{tr}\chi-\frac{2}{r}\right)|\|_{L^2(S_{t,u})}\|_{L^2_t}\right)^{1-\epsilon/2}\lesssim\lambda^{-\epsilon_0}.
\end{aligned}$$

Therefore,

$$|I_{12}|\lesssim\lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau). \tag{85}$$

Finally we estimate $I_{11}=\int_{[t_0,t]\times\mathbf{R}^3}\zeta\, nt(t-u)z\triangle\!\!\!\!/\,\psi\,\psi$ by integrating once more by parts as follows:

$$\begin{aligned}
I_{11}=&-\int_{[t_0,t]\times\mathbf{R}^3}\zeta\, nt(t-u)z\,|\nabla\!\!\!\!/\,\psi|^2\\
&-\int_{[t_0,t]\times\mathbf{R}^3}n^{-1}b^{-1}\nabla\!\!\!\!/_A(bn\,\zeta\, nt(t-u)z)\nabla\!\!\!\!/_A\psi\,\psi.
\end{aligned}$$

The first integral on the right can easily be estimated

$$\begin{aligned}
|\int_{[t_0,t]\times\mathbf{R}^3}\zeta\, nt(t-u)z\,|\nabla\!\!\!\!/\,\psi|^2|&\lesssim\int_{t_0}^t\|z\|_{L^\infty_x}\mathcal{E}[\psi](\tau)\,d\tau\\
&\lesssim\|z\|_{L^1_tL^\infty_x}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau)\\
&\lesssim\lambda^{-3\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau).
\end{aligned}\tag{86}$$

To estimate the second we write schematically

$$\begin{aligned}\nabla\!\!\!\!/\,(bn^2\,\zeta\, t(t-u)z) &\approx t(t-u)(\nabla\!\!\!\!/\, b)z + t(t-u)\nabla\!\!\!\!/\, z + t(t-u)z\Theta\\ &= t(t-u)\nabla\!\!\!\!/\, z + t(t-u)z\Theta\end{aligned}$$

since $\nabla\!\!\!\!/_A b = b(\eta_A - k_{AN})$. Thus with the help of Proposition 4.8 (using also the weak estimate (60)), we have

$$\begin{aligned}&\int_{[t_0,t]\times\mathbf{R}^3} |n^{-1}b^{-1}\nabla\!\!\!\!/_A(bn^2\,\zeta\,\tau(\tau-u)z)\nabla\!\!\!\!/_A\psi\,\psi|\\ &\quad\lesssim \int_{t_0}^t\int_{\mathrm{Ext}_\tau}\big(\tau|\nabla\!\!\!\!/\, z| + \tau|z||\Theta|\big)|\tau\nabla\!\!\!\!/_A\psi|\,|\psi|\\ &\quad\lesssim \int_{t_0}^t\left(\lambda^{C\epsilon}\sup_{u\le\frac{\tau}{2}}\|\nabla\!\!\!\!/\, z\|^{1-\epsilon/2}_{L^2(S_{t,u})} + \tau\|z\|_{L^\infty(\Sigma_\tau)}\|\Theta\|_{L^\infty(\Sigma_\tau)}\right)\mathcal{E}[\psi](\tau)\,d\tau.\end{aligned}$$

Using (76) once more we have,

$$\begin{aligned}&\int_{t_0}^t\left(\lambda^{C\epsilon}\sup_{u\le\frac{\tau}{2}}\|\nabla\!\!\!\!/\, z\|^{1-\epsilon/2}_{L^2(S_{t,u})} + \tau\|z\|_{L^\infty(\Sigma_\tau)}\|\Theta\|_{L^\infty(\Sigma_\tau)}\right)d\tau\\ &\qquad\lesssim \lambda^{C'\epsilon}\|\sup_{u\le\frac{t}{2}}\|\nabla\!\!\!\!/\, z\|_{L^2(S_{t,u})}\|^{1-\epsilon/2}_{L^1_t} + t\|z\|_{L^2_tL^\infty_x}\|\Theta\|_{L^2_tL^\infty_x}\lesssim \lambda^{-\epsilon_0}.\end{aligned}$$

Therefore, combining this with (86) we infer that,

$$|I_{11}| \lesssim \lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau). \tag{87}$$

Recalling also (85) and (84) we conclude that

$$|I_1| \lesssim \lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau). \tag{88}$$

Since I_2, I_3, I_4 and $\mathcal{B}^i$ have already been estimated we finally derive,

$$|\mathcal{B}| \lesssim \lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau) \tag{89}$$

as desired. This combined with (73) yields,

$$|\mathcal{J}| \lesssim \lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau). \tag{90}$$

Going back to the identity (72) we still have to estimate $\mathcal{Y}$. For this we only need to observe that $\Box_H t$ depends only on the first derivatives of H. Thus also

$$|\mathcal{Y} \lesssim \lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau). \tag{91}$$

Therefore,

$$\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau) \le \mathcal{Q}[\psi](t_0) + \lambda^{-\epsilon_0}\sup_{[t_0,t]}\mathcal{Q}[\psi](\tau)$$

which implies the boundedness theorem.

6. Proof of the comparison theorem

We proceed precisely as in [Kl-Ro, §6.1]. The purpose of the calculation below is to estimate the potentially negative term $\int_{\Sigma_t}(2t\psi T\psi - n^{-1}\psi^2)$, appearing in the integral of $\bar{Q}(K,T)[\psi]$, via integration by parts in terms of the positive terms of the Morawetz energy.[23] Define S and $\underline{S}$:

$$S = \frac{1}{2}(\underline{u}L + u\underline{L}), \quad \underline{S} = \frac{1}{2}(\underline{u}L - u\underline{L}). \tag{92}$$

Since $\underline{u} = -u + 2t$, $\underline{L} = T - N$, $L = T + N$,

$$tT = \frac{1}{4}(u+\underline{u})(L+\underline{L}) = S - \frac{1}{4}(u-\underline{u})(L-\underline{L}) = S - (t-u)N,$$
$$tT = \frac{1}{2}t(\underline{L}+L) = \frac{t}{t-u}\underline{S} - \frac{t^2}{t-u}N.$$

Therefore, with the help of the identities (78), and $N(t) = 0$, $N(u) = -b^{-1}$,

$$\begin{aligned}
2\int_{\Sigma_t}\psi tT(\psi) &= 2\int_{\Sigma_t}(\psi(S\psi) - \frac{1}{2}(t-u)N(\psi^2))\\
&= 2\int_{\Sigma_t}\psi(S\psi) + \int_{\Sigma_t}\Big(b^{-1} + (t-u)\big(\mathrm{tr}\theta + n^{-1}N(n)\big)\Big)\psi^2,\\
2\int_{\Sigma_t}\psi tT(\psi) &= 2\int_{\Sigma_t}\Big(\psi\frac{t}{t-u}(\underline{S}\psi) - \frac{1}{2}\frac{t^2}{t-u}N(\psi^2)\Big)\\
&= 2\int_{\Sigma_t}\psi\frac{t}{t-u}(\underline{S}\psi)\\
&\quad + \int_{\Sigma_t}\frac{t^2}{(t-u)^2}\Big(-b^{-1} + (t-u)\big(\mathrm{tr}\theta + n^{-1}N(n)\big)\Big)\psi^2.
\end{aligned}$$

Recall that $\theta_{AB} = \chi_{AB} + k_{AB}$. Recall also that Θ was defined in (62).

$$\Theta(t,x) = |\mathrm{tr}\,\chi - \frac{2}{r}| + |\mathrm{tr}\,\chi - \frac{2}{n(t-u)}| + |\hat{\chi}| + |\eta| + |\partial H|.$$

Thus,

$$\begin{aligned}
2\int_{\Sigma_t}\psi tT(\psi) &= 2\int_{\Sigma_t}\psi(S\psi) + \int_{\Sigma_t}\Big(b^{-1} + \frac{2}{n} + (t-u)\Theta\Big)\psi^2,\\
2\int_{\Sigma_t}\psi tT(\psi) &= 2\int_{\Sigma_t}\psi\frac{t}{t-u}(\underline{S}\psi) + \int_{\Sigma_t}\frac{t^2}{(t-u)^2}\big(-b^{-1} + \frac{2}{n} + (t-u)\Theta\big)\psi^2.
\end{aligned}$$

Recall, from the Asymptotics Theorem 4.5,

$$|b-n| \lesssim \lambda^{-4\epsilon_0}.$$

[23] In flat space this was first done in [Kl3]; see also [Kl2]. It is also related to the well known Morawetz calculation in [Mor].

Also, since n is bounded away from zero so is b. Therefore,

$$2\int_{\Sigma_t}\psi tT(\psi) = 2\int_{\Sigma_t}\psi(S\psi) + \int_{\Sigma_t}\Big(\frac{3}{n} + (t-u)\Theta + \lambda^{-4\epsilon_0}\Big)\psi^2,$$

$$2\int_{\Sigma_t}\psi tT(\psi) = 2\int_{\Sigma_t}\psi\frac{t}{t-u}(\underline{S}\psi) + \int_{\Sigma_t}\frac{t^2}{(t-u)^2}\left(\frac{1}{n} + (t-u)\Theta + \lambda^{-4\epsilon_0}\right)\psi^2.$$

Since

$$\bar{Q}(K,T)[\psi] = \frac{n}{4}\big(\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2 + (\underline{u}^2+u^2)|\nabla\psi|^2\big) + 2t\psi T\psi - n^{-1}\psi^2,$$

and

$$\frac{1}{4}(\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2) = \frac{1}{2}\big((S\psi)^2 + (\underline{S}\psi)^2\big)$$

we can introduce positive constants $A, B:\ A+B=2$ such that

$$\begin{aligned}\mathcal{Q}[\psi](t) =& \frac{1}{2}\int_{\Sigma_t}\Big(n(S\psi)^2 + 2A\psi(S\psi) + (\frac{3}{n}A - \frac{2}{n})\psi^2 + \big((t-u)\Theta + \lambda^{-4\epsilon_0}\big)\psi^2\Big)\\ &+\frac{1}{2}\int_{\Sigma_t}\Bigg(n(\underline{S}\psi)^2 + 2B\psi\frac{t}{t-u}(\underline{S}\psi)\\ &\qquad + \left(\frac{1}{n}\frac{t^2}{(t-u)^2}B\psi^2 + \frac{t^2}{(t-u)^2}\big((t-u)\Theta + \lambda^{-4\epsilon_0}\big)\psi^2\right)\Bigg)\\ &+\frac{1}{4}\int_{\Sigma_t}n(\underline{u}^2+u^2)|\nabla\psi|^2.\end{aligned}$$

For any values of A, B such that $1 < A < 2$ and $0 < B < 1$ it is possible to find positive constants c_1, c_2 such that

$$n(S\psi)^2 + 2A\psi(S\psi) + \frac{1}{n}(3A-2)\psi^2 \geq c_1\big((S\psi)^2 + \psi^2\big),$$

$$n(\underline{S}\psi)^2 + 2B\psi\frac{t}{t-u}(\underline{S}\psi) + \frac{1}{n}B\frac{t^2}{(t-u)^2}\psi^2 \geq c_2\left((\underline{S}\psi)^2 + \frac{t^2}{(t-u)^2}\psi^2\right).$$

Therefore,

$$\begin{aligned}\mathcal{Q}[\psi](t) \gtrsim& \int_{\Sigma_t}(\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2 + (u^2+\underline{u}^2)|\nabla\psi|^2 + \left(1 + \frac{t^2}{(t-u)^2}\right)\psi^2\\ &-\int_{\Sigma_t}\left(1 + \frac{t^2}{(t-u)^2}\right)\Big((t-u)\Theta + \lambda^{-4\epsilon_0}\Big)\psi^2,\end{aligned}$$

$$\begin{aligned}\mathcal{Q}[\psi](t) \gtrsim& \int_{\Sigma_t}(\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2 + (u^2+\underline{u}^2)|\nabla\psi|^2 + \left(1 + \frac{t^2}{(t-u)^2}\right)\psi^2\\ &-\int_{\Sigma_t}\left(1 + \frac{t^2}{(t-u)^2}\right)(t-u)\Theta\psi^2.\end{aligned}$$

Therefore it suffices to show that

(93)

$$\int_{\Sigma_t}\left(1 + \frac{t^2}{(t-u)^2}\right)(t-u)\Theta\psi^2 \leq \lambda^{-\epsilon_0}\int_{\Sigma_t}t^2|\nabla\psi|^2 + \left(1 + \frac{t^2}{(t-u)^2}\right)\psi^2.$$

Consider the worst term

$$\int_{\Sigma_t} \frac{t^2}{(t-u)}\Theta\psi^2 \lesssim \left(\int_{\Sigma_t} t^2\Theta^2\psi^2\right)^{\frac{1}{2}}\left(\int_{\Sigma_t} \frac{t^2}{(t-u)^2}\psi^2\right)^{\frac{1}{2}}. \tag{94}$$

According to the estimate (68) of Proposition 4.8, applied to exponent p such that $2p' = q$,

$$\int_{\Sigma_t} t^2\Theta^2\psi^2 \lesssim t^{2-\frac{4}{q}} \sup_u \|\Theta\|^2_{L^q(S_{t,u})} t^2 \int_{\Sigma_t} \big(|\nabla\!\!\!/\psi|^2 + r^{-2}|\psi|^2\big).$$

Or, since according to (61), $c^{-1} \le \frac{r}{(t-u)} \le c$, and with the help of the estimate (57) for Θ with $q > 2$ sufficiently close to 2,

$$\int_{\Sigma_t} t^2\Theta^2\psi^2 \lesssim \lambda^{-5\epsilon_0} \int_{\Sigma_t} \left(t^2|\nabla\!\!\!/\psi|^2 + \frac{t^2}{(t-u)^2}|\psi|^2\right).$$

Thus, back to (94),

$$\int_{\Sigma_t} \frac{t^2}{(t-u)}\Theta\psi^2 \lesssim \lambda^{-2\epsilon_0} \int_{\Sigma_t} \left(t^2|\nabla\!\!\!/\psi|^2 + \frac{t^2}{(t-u)^2}|\psi|^2\right) \tag{95}$$

as desired in the proof of (93). The remaining term on the left-hand side of (93) is easier to treat.

7. Proof of the $L^2 - L^\infty$ decay estimate; Theorem 2.1

In this section we rely on the Boundedness Theorem 3.4 to prove the crucial Theorem 2.1.

Recall that $\mathcal{E}[\psi] = \mathcal{E}^i[\psi] + \mathcal{E}^e[\psi]$, where

$$\mathcal{E}^i[\psi](t) = \int_{\Sigma_t} \Big(t^2|\partial\psi|^2 + |\psi|^2\Big)(1-\zeta),$$
$$\mathcal{E}^e[\psi](t) = \int_{\Sigma_t} \Big(\underline{u}^2|L\psi|^2 + \underline{u}^2|\nabla\!\!\!/\psi|^2) + u^2|\underline{L}\psi|^2 + |\psi|^2\Big)\,\zeta$$

with a cut-off function ζ equal to 1 in the region $u \le \frac{t}{2}$.

Estimate for $(1-\zeta)P\psi$. Observe that since the projector P is an averaging operator on the scale of size 1 and $(1-\zeta)$ is a cut-off function with the scale of size $t \ge 1$, we can essentially write that $(1-\zeta)P\psi \approx P(\psi(1-\zeta))$. Thus the Bernstein inequality, followed by the facts $\|(1-\zeta)\nabla\psi)\|_{L^2(\Sigma_t)} \le t^{-1}\mathcal{E}^{\frac{1}{2}}[\psi](t)$ and $|\nabla\zeta| \lesssim t^{-1}$, imply that

$$\|P(\psi(t))(1-\zeta)\|_{L^\infty_x} \lesssim \|\nabla\big(\psi(1-\zeta)\big)\|_{L^2(\Sigma_t)} \le t^{-1}\mathcal{E}^{\frac{1}{2}}[\psi](t) \tag{96}$$

as desired.

Estimate for $\zeta P\psi$ *.* It clearly suffices to establish the estimate for $P\psi(t,x)$ at any point (t,x) with $0 \le u \le \frac{t}{4}$. According to the Sobolev inequality (64), with $p=4$, of Proposition 4.7 we have for any positive $\delta < 1$,

$$\sup_{S_{t,u}} |P\psi|^2 \lesssim t^{\frac{4\delta}{4+\delta}} \left(\int_{S_{t,u}} \left(|\nabla\!\!\!/ P\psi|^2 + \frac{1}{t^2}|P\psi|^2\right)\right)^{1-\frac{4\delta}{4+\delta}} \left[\int_{S_{t,u}} (|\nabla\!\!\!/ P\psi|^4 + \frac{1}{t^4}|P\psi|^4)\right]^{\frac{2\delta}{4+\delta}}.$$

By the isoperimetric inequality (63) applied to $(P\psi)^2$ and $|\nabla\!\!\!/ P\psi|^2$,

$$\Big(\int_{S_{t,u}} |P\psi|^4\Big)^{\frac12} \lesssim \Big(\int_{S_{t,u}} |\nabla\!\!\!/ P\psi|^2\Big)^{\frac12}\Big(\int_{S_{t,u}} |P\psi|^2\Big)^{\frac12} + \frac{1}{t}\int_{S_{t,u}} |P\psi|^2,$$
$$\Big(\int_{S_{t,u}} |\nabla\!\!\!/ P\psi|^4\Big)^{\frac12} \lesssim \Big(\int_{S_{t,u}} |\nabla\!\!\!/^2 P\psi|^2\Big)^{\frac12}\Big(\int_{S_{t,u}} |\nabla\!\!\!/ P\psi|^2\Big)^{\frac12} + \frac{1}{t}\int_{S_{t,u}} |\nabla\!\!\!/ P\psi|^2.$$

In addition, by the trace inequality (65), [24]

$$\int_{S_{t,u}} |f|^2 \lesssim \Big(\int_{\mathrm{Ext}\,t} |N(f)|^2\Big)^{\frac12}\Big(\int_{\mathrm{Ext}\,t} |f|^2\Big)^{\frac12} + \frac{1}{t}\int_{\mathrm{Ext}\,t} |f|^2.$$

Here, $\mathrm{Ext}\,_t = \Sigma_t \cap \{0 \le u \le \frac{t}{2}\}$ and N is the vectorfield of the unit normals to $S_{t,t-\rho}$.

Thus, setting $\varepsilon = \frac{4\delta}{4+\delta}$, using the fact that $t \ge 1$, and applying the Hölder inequality, we obtain

(97)
$$\sup_{S_{t,u}} |P\psi|^2 \lesssim t^{\varepsilon}\Big(\int_{\mathrm{Ext}\,t} |\nabla\!\!\!/_N \nabla\!\!\!/ P\psi|^2 + |\nabla\!\!\!/ P\psi|^2 + \frac{1}{t^2}\big(|N(P\psi)|^2 + |P\psi|^2\big)\Big)^{1-\varepsilon}\cdot \mathcal{I}^{\varepsilon},$$
$$\mathcal{I} = \int_{\Sigma_t} |\nabla\!\!\!/_N \nabla\!\!\!/^2 P\psi|^2 + |\nabla\!\!\!/^2 P\psi|^2 + |\nabla\!\!\!/_N \nabla\!\!\!/ P\psi|^2 + |\nabla\!\!\!/ P\psi|^2 + \frac{1}{t^4}(|N(P\psi)|^2 + |P\psi|^2).$$

Note that we can always replace the outside N derivative with a generic derivative ∂. More precisely, $|N(f)|^2 \lesssim \sum_i |\partial_i f|^2$.

We make the following three observations:

1) The derivatives in the second factor $\mathcal{I}$ can be ignored in view of the presence of the projection P. Thus we can crudely bound it by $\mathcal{I} \lesssim \int_{\Sigma_t} |\psi|^2 \le \mathcal{E}[\psi](t)$.

2) The terms $\frac{1}{t^2}\int_{\mathrm{Ext}\,t} \big(|N(P\psi)|^2 + |P\psi|^2\big)$ are easily estimated by $t^{-2}\mathcal{E}[\psi](t)$.

[24]The tensor version of the estimate requires the covariant $\nabla\!\!\!/_N$ derivative. Recall that $\nabla\!\!\!/_N$ denotes the projection on $S_{t,u}$ of the covariant derivative ∇_N.

3) It remains to handle the terms

$$\int_{\mathrm{Ext}_t} |\nabla\!\!\!/_N \nabla\!\!\!/ P\psi|^2 + |\nabla\!\!\!/ P\psi|^2.$$

Consider first the integral $\int_{\mathrm{Ext}_t} |\nabla\!\!\!/ P\psi|^2$. Let ζ be a cut-off function of the exterior region Ext_t such that $\zeta|_{\mathrm{Ext}_t} = 1$ and $|\nabla\zeta| \lesssim t^{-1}$. We introduce the angular vectorfields $A_i = \zeta\,(\partial_i - <\partial_i, N> N)$. Clearly, for any scalar function f, $|\nabla\!\!\!/ f|^2 \approx \sum_{i=1}^3 |A_i f|^2$ in the exterior region Ext_t. Now,

$$\begin{aligned}
\int_{\mathrm{Ext}_t} |\nabla\!\!\!/ P\psi|^2 &\approx \sum_{i=1}^3 \int_{\mathrm{Ext}_t} |A_i P\psi|^2 \\
&\lesssim \sum_{i=1}^3 \int_{\mathrm{Ext}_t} |PA_i\psi|^2 + \sum_{i=1}^3 \int_{\mathrm{Ext}_t} |[P, A_i]\psi|^2 \\
&\lesssim \sum_{i=1}^3 \int_{\Sigma_t} |PA_i\psi|^2 + \text{error} \\
&\lesssim \int_{\Sigma_t} |\nabla\!\!\!/ \psi|^2 + \text{Error}.
\end{aligned}$$

We estimate the error term $\int_{\mathrm{Ext}_t} \sum_i |[P, A_i]\psi|^2$ with the help of the following:

LEMMA 7.1. *Consider a vectorfield* $X = \sum_i X^i \partial_i$ *vanishing on the complement of the exterior region* Ext_t *of* Σ_t *and* P *the standard Littlewood-Paley projection on frequencies of size* 1. *Then, for arbitrary scalar functions* f *there is the inequality*:[25]

$$\|[P, X]f\|_{L^2(\mathrm{Ext}_t)} \lesssim \sup_{i,j} \|\partial_i X^j\|_{L^\infty(\mathrm{Ext}_t)} \|f\|_{L^2(\Sigma_t)}.$$

Proof. We postpone the proof until the end of Section 8; see Lemma 8.38. $\square$

We apply the above lemma to the vectorfields $A_k = \zeta\,(\delta_k^j - N_k N^j)\partial_j$. Observe that the components A_k^j are bounded and $|\nabla\zeta| \lesssim t^{-1}$. Thus

$$\text{Error} \lesssim \left(t^{-2} + \|\nabla N\|^2_{L^\infty(\mathrm{Ext}_t)}\right) \|\psi\|^2_{L^2(\Sigma_t)}.$$

Recall the expression, see (62), $\Theta = |\mathrm{tr}\,\chi - \frac{2}{r}| + |\hat\chi| + |\eta| + |\partial H|$ and the inequality (52) $|\nabla N| \lesssim \frac{1}{r} + \Theta$. Observe also that in the exterior region Ext_t,

[25] In fact the exterior region on the right-hand side of the inequality should be somewhat enlarged (by size one). Since this enlargement does not affect our arguments we prefer to ignore it.

$\frac{1}{r} \le \frac{2}{t}$. Therefore,

$$|\mathrm{Error}| \lesssim \big(t^{-1} + \|\Theta\|_{L^\infty(\mathrm{Ext}\,_t)}\big)^2 \|\psi\|^2_{L^2(\Sigma_t)}.$$

We can finally conclude that

$$\begin{aligned}\int_{\mathrm{Ext}\,_t} |\nabla\!\!\!/ P\psi|^2 &\lesssim \int_{\Sigma_t} |\nabla\!\!\!/ \psi|^2 + \big(t^{-1} + \|\Theta\|_{L^\infty(\mathrm{Ext}\,_t)}\big)^2 \int_{\Sigma_t} |\psi|^2 \\ &\lesssim \big(t^{-2} + \|\Theta\|^2_{L^\infty(\mathrm{Ext}\,_t)}\big)\mathcal{E}[\psi](t).\end{aligned} \tag{98}$$

We now consider $\int_{\mathrm{Ext}\,_t} |\nabla\!\!\!/_N \nabla\!\!\!/ P\psi|^2$. In view of the simple commutation estimates (53) we can write:

$$\begin{aligned}\int_{\mathrm{Ext}\,_t} |\nabla\!\!\!/_N \nabla\!\!\!/ P\psi|^2 &\lesssim \int_{\mathrm{Ext}\,_t} |\nabla\!\!\!/ (NP\psi)|^2 + \int_{\mathrm{Ext}\,_t} \big(r^{-1} + \Theta\big)^2 |\nabla P\psi|^2 \\ &\approx \sum_{i=1}^3 \int_{\mathrm{Ext}\,_t} |A_i(NP\psi)|^2 + \int_{\mathrm{Ext}\,_t} \big(r^{-1} + \Theta\big)^2 |\nabla P\psi|^2.\end{aligned}$$

Observe that

$$\begin{aligned}A_i(NP\psi) &= A_i\, N^j \partial_j(P\psi) = N^j A_i \partial_j(P\psi) + [A_i, N^j]\partial_j(P\psi) \\ &= NP(A_i\psi) + N^j [A_i, \partial_j P]\psi + [A_i, N^j]\partial_j(P\psi).\end{aligned}$$

Therefore, by Lemma 7.1, with P replaced by ∇P, as well as the estimates (52)

$$\int_{\mathrm{Ext}\,_t} |A_i(NP\psi)|^2 \lesssim \int_{\mathrm{Ext}\,_t} |A_i\psi|^2 + \big(t^{-1} + \|\Theta\|_{L^\infty(\mathrm{Ext}\,_t)}\big)^2 \int_{\Sigma_t} |\psi|^2$$

and finally,

$$\int_{\mathrm{Ext}\,_t} |\nabla\!\!\!/_N \nabla\!\!\!/ P\psi|^2 \lesssim \big(t^{-2} + \|\Theta\|^2_{L^\infty(\mathrm{Ext}\,_t)}\big)\mathcal{E}[\psi](t). \tag{99}$$

Substituting (98), (99) back into (97) we infer that in the exterior region,

$$\begin{aligned}\sup_{S_{t,u}} |P\psi|^2 &\lesssim t^\varepsilon \big(t^{-2} + \|\Theta\|^2_{L^\infty(\mathrm{Ext}\,_t)}\big)^{1-\varepsilon} \mathcal{E}^{1-\varepsilon}[\psi](t) \cdot \mathcal{I}^\varepsilon \\ &\lesssim t^\varepsilon \big(t^{-2} + \|\Theta\|^2_{L^\infty(\mathrm{Ext}\,_t)}\big)^{1-\varepsilon} \mathcal{E}[\psi](t).\end{aligned}$$

Finally, together with the interior estimates (96) this implies that

$$\|P\psi(t)\|_{L^\infty_x} \lesssim \left(\frac{1}{(1+t)^{1-2\varepsilon}} + t^\varepsilon \|\Theta\|^{1-\varepsilon}_{L^\infty(\mathrm{Ext}\,_t)}\right) \mathcal{E}^{\frac{1}{2}}[\psi](t). \tag{100}$$

Observe that according to (57) of the Asymptotics Theorem, Θ obeys the following estimate in the exterior region:

$$\|\Theta(t)\|_{L^\infty(\mathrm{Ext}\,_t)} \lesssim t^{-1}\lambda^{-\varepsilon_0} + \lambda^\varepsilon \|\partial H(t)\|_{L^\infty_x}.$$

Define

$$d(t) = t^\varepsilon \big(\lambda^\varepsilon \|\partial H(t)\|_{L^\infty_x}\big)^{1-\varepsilon}.$$

Therefore,

$$\|P\psi(t)\|_{L^\infty_x} \lesssim \left(\frac{1}{(1+t)^{1-2\varepsilon}} + d(t)\right)\mathcal{E}^{\frac{1}{2}}[\psi](t).$$

To prove the desired $L^2 - L^\infty$ decay estimate it remains to check that for some $q > 2$,[26]

$$t_*^{\frac{1}{q}}\|d\|_{L^q_{[0,t_*]}} \lesssim 1.$$

Since $t_* \le \lambda^{1-4\varepsilon_0}$ it clearly suffices to show that $\|d\|_{L^q_{[0,t_*]}} \lesssim \lambda^{-\frac{1}{2}}$. In view of the estimates, see Proposition 2.4,

$$\|\partial H\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0}, \qquad \|\partial H\|_{L^\infty_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}+\varepsilon_0},$$

we infer that

$$\|d\|_{L^q_{[0,t_*]}} \lesssim t_*^{\varepsilon}\lambda^{\varepsilon}\|\partial H\|^{1-\varepsilon}_{L_t^{q(1-\varepsilon)}L^\infty_x} \lesssim t_*^{\varepsilon}\lambda^{\varepsilon}\|\partial H\|^{1-\frac{2}{q}-\varepsilon}_{L^\infty_t L^\infty_x}\cdot\|\partial H\|^{\frac{2}{q}}_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}},$$

as desired.

8. Proof of the reduction steps

In this section we give precise statements and proofs for the reduction steps discussed in Section 2. Recall the equation (3), written in the form (6),

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\phi = N(\phi,\partial\phi) \tag{101}$$

where $\phi = (\mathbf{g}_{\mu\nu})$, $N = N_{\mu\nu}$ and $\mathbf{g}^{\alpha\beta} = \mathbf{g}^{\alpha\beta}(\phi)$. In fact $(\mathbf{g}^{\alpha\beta}) = \phi^{-1}$. We consider solutions ϕ of (101) such that the components of both ϕ and ϕ^{-1} are uniformly bounded. Moreover $\mathbf{g}_{\mu\nu}$ approaches the Minkowski metric $\mathbf{m}_{\mu\nu}$ at infinity according to (5). To avoid repeating this statement in what follows we introduce the following notation:

Definition 8.1. We say that $f \in \mathbf{H}^s = \mathbf{H}^s(\mathbb{R}^3)$ if $\nabla f \in H^{s-1}$, f is continuous and tends to zero as $|x| \to \infty$. Observe that $\mathbf{H}^s$, with $s > \frac{3}{2}$, is the closure of C_0^∞ in the norm $\|\nabla f\|_{H^{s-1}}$. Given a solution ϕ of (101) we say that $\phi = (\mathbf{g}_{\mu\nu}) \in C([0,T];\mathbf{m}+\mathbf{H}^s)$ if, for every $t \in [0,T]$, $(\mathbf{g}_{\mu\nu}(t) - \mathbf{m}_{\mu\nu}) \in \mathbf{H}^s(\Sigma_t)$ and $\partial_t\phi \in H^{s-1}(\Sigma_t)$.

Throughout the section we shall use the following notation:

Definition 8.2. For any function f on $\Sigma_t = \mathbb{R}^3$, $P_\lambda f = \mathcal{F}^{-1}\big(\chi(\lambda^{-1}\xi)\hat{f}(\xi)\big)$ with χ supported in the unit dyadic region $\frac{1}{2} \le |\xi| \le 2$ and $\lambda \in 2^{\mathbb{Z}}$ an integer power of 2. Also $f = \sum_{\lambda\in 2^{\mathbb{Z}}} P_\lambda f$. We shall denote $f_{\le\lambda} = P_{\le\lambda} f = \sum_{\mu\le\lambda} f^\mu$. We shall also use the notation $f_{<\lambda} = P_{<\lambda} f = \sum_{\mu<2^{-M_0}\lambda} f^\mu$, for a fixed, sufficiently large constant M_0, such as 100. We shall often, improperly, refer to

[26] We can assume that $\frac{2}{1-\varepsilon} < q < 2 + 10^{-1}\epsilon_0$.

the P_λ' s as Littlewood-Paley projections. They are in fact only approximate projections or rather cut-off operators.

Remark 8.3. Observe that if f is continuous, approaches a constant c at infinity, i.e. $\sup_{|x|=r}|f(x)-c| \to 0$ as $r \to \infty$, and $\nabla f \in H^{s-1}$, $s > \frac{3}{2}$, then $P_\lambda f \in H^s$.[27]

8.4. *Energy estimates.* We start with the following well known statement:

PROPOSITION 8.5 (Energy estimate). *Let $\phi \in C([0,T]; m + \mathbf{H}^s)$ be a solution of* (101) *on the time interval $[0,T]$ for some $s > \frac{3}{2}$ such that*

$$\|\phi, \phi^{-1}\|_{L^\infty_{[0,T]} L^\infty_x} \le \Lambda_0.$$

Then ϕ verifies the following energy estimate.

$$\|\partial\phi\|_{L^\infty_{[0,T]} \dot{H}^{s-1}} \le C(\|\partial\phi\|_{L^1_{[0,T]} L^\infty_x}, \Lambda_0)\|\partial\phi(0)\|_{\dot{H}^{s-1}}. \tag{102}$$

Remark 8.6. Throughout this section we shall often ignore the dependence on Λ_0 and the constant M_0 involved in the definition of $P_{<\lambda}$.

Proof. The proof of Proposition 8.5 can be easily reduced to the following lemma.

LEMMA 8.7. *Let ϕ satisfy the conditions of Proposition* 8.5. *Then for each dyadic $\lambda \in 2^{\mathbb{Z}}$, $\phi^\lambda = P_\lambda \phi$ verifies the equation*

$$-\partial_t^2 \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{0i})_{<\lambda}(\phi)\partial_t \partial_i \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{ij})_{<\lambda}(\phi)\partial_i \partial_j \phi^\lambda = R_\lambda, \tag{103}$$

where for any $s > 1$ and $t \in [0,T]$ the right-hand side R_λ has Fourier support in $\{\xi : \frac{1}{4}\lambda \le |\xi| \le 4\lambda\}$ and obeys the estimate

$$\Big(\sum_\lambda \|R_\lambda(t)\|^2_{\dot{H}^{s-1}}\Big)^{\frac{1}{2}} \le C\|\partial\phi(t)\|_{L^\infty_x} \cdot \|\partial\phi(t)\|_{\dot{H}^{s-1}} \tag{104}$$

with C a constant depending only on Λ_0. Moreover ϕ^λ also satisfies the equation

$$\mathbf{g}^{\alpha\beta}_{<\lambda} \partial_\alpha \partial_\beta \phi^\lambda = R_\lambda \tag{105}$$

with a different R_λ which verifies the same estimate (104) *and the frequency property.*

[27] This can be proved easily by a density argument.

Proof. The proof is based on the technique of the paradifferential calculus and is now standard.[28] For a detailed treatment see [Ba-Ch1], or [Kl-Ro] for notation similar to that used here.

Remark 8.8. In the subsequent paper we shall also need the following more general result concerning other dyadic projections of our equation.

LEMMA 8.9. *Under the assumptions of Lemma* 8.7,

$$\mathbf{g}_{<\lambda}^{\alpha\beta}\partial_\alpha\partial_\beta\phi_{<\lambda} = F_\lambda.$$

The function F_λ *obeys the estimates*

$$\|F_\lambda\|_{L_t^1L_x^2} \le C\|\partial\phi\|_{L_t^1L_x^\infty}\|\partial\phi\|_{L_t^\infty L_x^2}, \qquad \|F_\lambda\|_{L_t^1\dot H^1} \le C\|\partial\phi\|_{L_t^1L_x^\infty}\|\partial\phi\|_{L_t^\infty \dot H_x^1}.$$

In addition, *for any dyadic* $\mu \ge 1$,

$$\mathbf{g}_{<\lambda}^{\alpha\beta}\partial_\alpha\partial_\beta P_{\lambda\mu}\phi = F_{\lambda,\mu},$$

where $F_{\lambda,\mu}$ *verifies*

$$\begin{aligned}\|F_{\lambda,\mu}\|_{L_t^1L_x^2} &\le C(\lambda\mu)^{-\gamma}\lambda^{-1}\|\partial\phi\|_{L_t^1L_x^\infty}\|\partial\phi\|_{L_t^\infty\dot H^{1+\gamma}},\\ \|F_{\lambda,\mu}\|_{L_t^1\dot H^1} &\le C\lambda^{-\gamma}\mu^{1-\gamma}\|\partial\phi\|_{L_t^1L_x^\infty}\|\partial\phi\|_{L_t^\infty\dot H^{1+\gamma}}.\end{aligned}$$

The components of the metric $\mathbf{g}$ *satisfy similar equations.*

The proof of Lemma 8.9 proceeds in the same manner as the proof of Lemma 8.7 after we apply the respective projections $P_{<\lambda}$ and $P_{\lambda\mu}$.

To finish the proof of Proposition 8.5 we choose a large parameter Λ in such a way that for any $\lambda \ge \Lambda$ the metric $(\mathbf{n}^2\mathbf{g}^{ij})_{<\lambda}$ is uniformly elliptic. This is always possible since $P_{<\lambda}$ is an approximation of the identity and the original metric $(\mathbf{n}^2\mathbf{g}^{ij})$ is uniformly elliptic in $[0,T]$.

For the values of the dyadic parameter $\lambda \le \Lambda$ rewrite the equation for ϕ^λ in the form

$$-\partial_t^2\phi^\lambda + (\mathbf{n}^2\mathbf{g}^{0i})_{<\lambda}\partial_t\partial_i\phi^\lambda + (\mathbf{n}^2\mathbf{g}^{ij})_{<\lambda}\partial_i\partial_j\phi^\lambda = R'_\lambda$$

noting that the change of the metric introduces the error term of type E_2.

For $\lambda \ge \Lambda$ we keep the form of the equation as in Lemma 8.7,

$$-\partial_t^2\phi^\lambda + (\mathbf{n}^2\mathbf{g}^{0i})_{<\lambda}\partial_t\partial_i\phi^\lambda + (\mathbf{n}^2\mathbf{g}^{ij})_{<\lambda}\partial_i\partial_j\phi^\lambda = R_\lambda.$$

In either case, the standard H^1 energy estimate for the wave equation yields

$$\|\partial\phi^\lambda\|_{L^\infty_{[0,T]}L^2} \le C(\Lambda_0)(\|\partial\phi^\lambda(0)\|_{L^2} + \|R_\lambda\|_{L^1_{[0,T]}L_x^2}).$$

[28]The equations discussed in the literature are somewhat different from the one treated here because of the nontriviality of the components $\mathbf{g}^{00}$ and $\mathbf{g}^{0i}$ of the metric. This adds only minor technical complications.

Using Lemma 8.7 and the Gronwall inequality we immediately obtain for $s > 1$,

$$\|\partial\phi\|_{L^\infty_{[0,T]}\dot{H}^{s-1}} \lesssim \exp\left(\|\partial\phi\|_{L^1_{[0,T]}L^\infty_x}\right)\|\partial\phi(0)\|_{\dot{H}^{s-1}}.$$

The estimate for $s = 1$ follows by standard energy estimates without the paradifferential decomposition.

8.10. *Reduction to the Strichartz-type estimates.* As discussed in Section 2 we need to prove the Strichartz type inequality (16). This is achieved by the following:

Theorem 8.11 (A1). *Let $\phi \in C([0,T]; \mathbf{m}+\mathbf{H}^{1+\gamma})$ be a solution of* (101) *on the time interval $[0,T]$, $T \le 1$. Assume that*

$$\|\partial\phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} + \|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le B_0. \tag{106}$$

There exists a small positive exponent $\delta = \delta(B_0)$ such that ϕ satisfies the following local in time Strichartz-type estimate,

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le C(B_0)\, T^\delta. \tag{107}$$

Remark 8.12. In view of Remark 2.2 and Definition 2.3 we shall treat B_0 as a universal constant in what follows and hide the dependence on it in the notation $\lesssim$.

8.13. *The dyadic version of the Strichartz-type estimate.* Fix a large frequency parameter Λ. It easily follows from the triangle inequality that for $p \in [1, \infty]$,

$$\|\partial\phi\|_{L^p_x} \le \|\partial\phi_{\le\Lambda}\|_{L^p_x} + \sum_{\lambda>\Lambda}\|\partial\phi^\lambda\|_{L^p_x}.$$

Thus, Theorem 8.11 follows from the following dyadic version of the Strichartz-type estimates for $\phi^\lambda = P_\lambda\phi$.

Theorem 8.14 (A2). *Let ϕ be as in Theorem* 8.11. *There exists a small positive exponent $\delta = \delta(B_0)$ such that for each $\lambda \ge \Lambda$, the function ϕ^λ satisfies the Strichartz-type estimate*

$$\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} \lesssim c_\lambda T^\delta \tag{108}$$

with constants c_λ such that $\sum_\lambda c_\lambda \le 1$.

Remark 8.15. The corresponding estimate for small frequencies, i.e. for $\phi_{<\lambda}$, follows trivially from the Sobolev inequality,

$$\|\partial\phi_{<\lambda}\|_{L^2_{[0,T]}L^\infty_x} \lesssim T^{\frac12}\|\partial\phi_{<\lambda}\|_{L^\infty_{[0,T]}H^{\frac32+\gamma}} \lesssim \Lambda^{\frac12}T^{\frac12}\|\partial\phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} \lesssim \Lambda^{\frac12}T^{\frac12}.$$

Since Λ is a fixed large parameter, which could depend only upon B_0, we have the desired bound for the low frequency part of ϕ.

Remark 8.16. We shall need the following version of the estimate (104) for R_λ and any $s < 2+\gamma$:

$$\|R_\lambda(t)\|_{\dot H^{s-1}} \lesssim c_\lambda \,\|\partial\phi\|_{L^\infty_x}\|\partial\phi\|_{H^{1+\gamma}} \tag{109}$$

with constants c_λ: $\sum_\lambda c_\lambda \le 1$. The estimate (109) can be easily obtained from (104) by making use of the fact that the Fourier support of R_λ is localized on the set $\{\xi :\ \lambda \le |\xi| \le 4\lambda\}$. As a consequence, using the bootstrap assumption (106), we also have the estimate

$$\|R_\lambda(t)\|_{L^1_{[0,T]}\dot H^{s-1}} \lesssim c_\lambda T^{\frac12}\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x}\|\partial\phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} \lesssim c_\lambda. \tag{110}$$

8.17. *Dyadic linearization and time restriction.* This step reduces Theorem 8.14 to a Strichartz-type estimate for the linearized equation $\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\psi = 0$ on smaller subintervals of $[0,T]$. We partition $[0,T]$ by the intervals $I_k = [t_k, t_{k+1}]$, $k = 0,..,\lambda^{8\epsilon_0}$ with the properties $|I_k| \le T\lambda^{-8\epsilon_0}$ and $\|\partial\phi\|_{L^2_{I_k}L^\infty_x} \le \lambda^{-4\epsilon_0}B_0$. The existence of such a partition is insured by the bootstrap condition (106).

THEOREM 8.18 (A3). *Fix $\lambda \ge \Lambda$ and $k \in \mathbb{Z}\cap[0,\lambda^{8\epsilon_0}]$ and let ψ be a solution of the linear wave equation*

$$\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\psi = 0$$

on the interval $I_k = [t_k, t_{k+1}]$, verifying,

$$(2^{-10}\lambda)^m\|\partial\psi(t_k)\|_{L^2_x} \le \|\nabla^m\partial\psi(t_k)\|_{L^2_x} \le (2^{10}\lambda)^m\|\partial\psi(t_k)\|_{L^2_x} \tag{111}$$

for every $m \ge 0$. Then there exists a sufficiently small exponent $\delta > 0$ such that:

$$\|P_\lambda\,\partial\psi\|_{L^2_{I_k}L^\infty_x} \lesssim |I_k|^\delta\|\partial\psi(t_k)\|_{\dot H^{1+\delta}}. \tag{112}$$

The size of δ depends only on ϵ_0, B_0. In particular, for any $\epsilon_0 > 0$, δ can be chosen such that $\delta < 10^{-1}\gamma$.

Remark 8.19. The condition (111) implies that, modulo a negligible "tail", the Fourier support of $\partial\psi(t_k)$ belongs to the set $\{\xi :\ 2^{-10}\lambda \le |\xi| \le 2^{10}\lambda\}$. In general, we shall say that function f obeys the property $(113)_M$ if

$$(2^{-M}\lambda)^m\|f\|_{L^2_x} \le \|\nabla^m f\|_{L^2_x} \le (2^M\lambda)^m\|f\|_{L^2_x}. \tag{113}$$

LEMMA 8.20. (1) *Assume f in $\mathbb{R}^3$ is a function whose frequency is localized to the region $|\xi| \le 2^{-M_0}\lambda$ and $c \le f \le c^{-1}$ for some positive number c. Then $u = f^{-1}$ verifies,*

$$\|\nabla^m u\|_{L^\infty} \lesssim (2^{-M_0}\lambda)^m. \tag{114}$$

(2) *Assume*[29] *that* u *verifies* (114) *and* $c \le u \le c^{-1}$. *Let* v *be another function verifying the condition* $(113)_5$. *Then*[30] $u \cdot v$ *verifies* $(113)_{10}$.

Proof. The proof of (1) is based on the trivial identity $f \cdot f^{-1} = 1$. Differentiating it and applying the Leibnitz rule we conclude that, although the Fourier support of f^{-1} does not belong to the set $\{\xi : |\xi| \le 2^{-M_0}\lambda\}$, we still have the property,

$$\|\nabla^m (f^{-1})\|_{L^\infty_x} \lesssim (2^{-M_0}\lambda)^m.$$

The proof of (2) is once again an exercise using the Leibnitz rule. In particular, for $m = 1$ we have

$$\begin{aligned}\|\nabla(u\cdot v)\|_{L^2_x} &\lesssim \|\nabla u\|_{L^\infty_x}\|v\|_{L^2_x} + \|u\|_{L^\infty_x}\|\nabla v\|_{L^2_x}\\ &\lesssim 2^{-M_0}\lambda\|v\|_{L^2_x} + 2^5\lambda\|v\|_{L^2_x} \lesssim 2^{10}\lambda\|u\cdot v\|_{L^2_x}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\nabla(u\cdot v)\|_{L^2_x} &\gtrsim \|u\|_{L^\infty_x}\|\nabla v\|_{L^2_x} - \|\nabla u\|_{L^\infty_x}\|v\|_{L^2_x}\\ &\gtrsim 2^{-5}\lambda\|v\|_{L^2_x} - 2^{-M_0}\lambda\|v\|_{L^2_x} \gtrsim 2^{-10}\lambda\|u\cdot v\|_{L^2_x}.\end{aligned}$$ □

Proof of the implication Theorem (A3) $\to$ *Theorem* (A2). We shall first prove an inhomogeneous version of the Strichartz estimate (112) for solutions of the equation $\mathbf{g}^{\alpha\beta}_{<\lambda}\psi = F$, with the right-hand side F verifying $(113)_5$. Recall that $\mathbf{g}^{\alpha\beta}_{<\lambda} = P_{\le 2^{-M_0}\lambda}\mathbf{g}^{\alpha\beta}$. The Duhamel formula on the interval I_k for the inhomogeneous equation $\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\psi = F$ takes the form

$$\psi(t) = [W(t,0)]\psi[t_k] + \int_0^t W(t,s)\Big((\mathbf{g}^{00}_{<\lambda})^{-1}F(s)\Big)\,ds \tag{115}$$

with $\psi[t]$ denoting the vector $\big(\psi(t), \partial_t\psi(t)\big)$. Here $[W(t,s)]$ is the solution operator of the homogeneous equation acting on the pair of initial data (w_0, w_1) at time s, and $W(t,s)$ is a solution operator corresponding to the special type of the initial data $(0, w_1)$. We need to check that $(\mathbf{g}^{00}_{<\lambda})^{-1}F(s)$ verifies the same conditions (113) as F.

Recall $-\mathbf{g}^{00} = \mathbf{n}^{-2}$. Since F verifies $(113)_5$, using 1. and 2. of Lemma 8.20, we conclude that $\big[(\mathbf{n}^{-2})_{<\lambda}\big]^{-1}F$ verifies $(113)_{10}$.

We now apply Theorem 8.18 to (115), assuming also that the initial data $\partial\psi(t_k)$ verify the assumption $(113)_{10}$,

$$\|P_\lambda\,\partial\psi\|_{L^2_{I_k}L^\infty_x} \lesssim |I_k|^\delta\Big(\|\partial\psi(t_k)\|_{\dot H^{1+\delta}} + \|F\|_{L^1_{I_k}\dot H^{1+\delta}}\Big). \tag{116}$$

[29]Recall that M_0 is a large positive constant

[30]This property is analogous to the standard paraproduct rule concerning the multiplication of functions u, v where the frequency of v dominates.

Fix a sufficiently small ϵ_0 such that $5\epsilon_0 + \delta < \gamma$. Consider the λ-dyadic piece ϕ^λ of ϕ, solution of the equation (101), as in Theorem (A2). We know that ϕ^λ verifies the equation $\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\phi^\lambda = R_\lambda$ on $[0,T]$ and the Fourier support of R_λ belongs to the set $\{\xi : \frac{1}{4}\lambda \le |\xi| \le 4\lambda\}$, thus automatically satisfying property $(113)_5$. We can therefore apply (116) to ϕ^λ on each I_k to obtain:

$$
\begin{aligned}
\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} &= \Big(\sum_{k=0}^{\lambda^{8\epsilon_0}-1} \|\partial\phi^\lambda\|^2_{L^2_{I_k}L^\infty_x}\Big)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{k=0}^{\lambda^{8\epsilon_0}-1} |I_k|^{2\delta}\Big(\|\partial\phi^\lambda(t_k)\|_{\dot H^{1+\delta}} + \|R_\lambda\|_{L^1_{[0,T]}\dot H^{1+\delta}}\Big)^2\right)^{\frac{1}{2}} \\
&\lesssim |T|^\delta\lambda^{4\epsilon_0}\Big(\|\partial\phi^\lambda\|_{L^\infty_{[0,T]}\dot H^{1+\delta}} + \|R_\lambda\|_{L^1_{[0,T]}\dot H^{1+\delta}}\Big) \\
&\lesssim |T|^\delta\Big(\|\partial\phi^\lambda\|_{L^\infty_{[0,T]}H^{1+\gamma}} + \|R_\lambda\|_{L^1_{[0,T]}\dot H^{1+4\epsilon_0+\delta}}\Big) \\
&\lesssim |T|^\delta c_\lambda.
\end{aligned}
$$

The last two inequalities follow from the inequality $\delta+5\epsilon_0 < \gamma$ and the estimate (110).

8.21. *Properties of the metric* $\mathbf{g}_{<\lambda}$. Recall that $\mathbf{g}^{\mu\nu}_{<\lambda} = P_{\le 2^{-M_0}\lambda}(\mathbf{g}^{\mu\nu})$ where $\mathbf{g}^{\mu\nu}$ is the inverse of the Lorentz metric $\mathbf{g}_{\mu\nu} = \phi$. We shall use the notation $\mathbf{g}_{<\lambda}$ to denote the inverse of $\mathbf{g}^{\mu\nu}_{<\lambda}$. Observe that, in view of our assumption $\lambda \ge \Lambda$, $\mathbf{g}_{<\lambda}$ defines a Lorentz metric in our spacetime region $[0,T]\times\mathbb{R}^3$. It clearly depends on the solution ϕ of the quasilinear problem (101). In the next proposition we state the properties of the family $\mathbf{g}_{<\lambda}$ which follow from the bootstrap condition (106) on ϕ. We denote by $\mathbf{R}_{\alpha\beta}(\mathbf{g}_{<\lambda})$ the components of Ricci curvature of the metric $\mathbf{g}_{<\lambda}$.

PROPOSITION 8.22. *Let* $\phi \in C([0,T];\mathbf{m}+\mathbf{H}^{1+\gamma})$ *be a solution of* (101) *on* $[0,T]$, $T \le 1$. *Assume that* ϕ *verifies the assumption* (106) *of Theorem* 8.11. *Then the family of metrics* $\mathbf{g}_{<\lambda}$ *obeys the following conditions on each interval* I_k *such that* $|I_k| \le T\lambda^{-8\epsilon_0}$, *and* $\|\partial\phi\|_{L^2_{I_k}L^\infty_x} \le \lambda^{-4\epsilon_0}$:

$$\|\partial^{1+m}\,\mathbf{g}_{<\lambda}\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0+m}, \tag{117}$$

$$\|\partial^{1+m}\,\mathbf{g}_{<\lambda}\|_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-4\epsilon_0+m}, \tag{118}$$

$$\|\partial^{1+m}\,\mathbf{g}_{<\lambda}\|_{L^\infty_{I_k}L^\infty_x} \lesssim \lambda^{\frac{1}{2}-4\epsilon_0+m}, \tag{119}$$

$$\|\nabla^{\frac{1}{2}+m}(\partial\mathbf{g}_{<\lambda})\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{\frac{1}{2}+m} \quad \textit{for} \quad 0 \le m \le \frac{1}{2}+4\epsilon_0, \tag{120}$$

$$\|\nabla^{\frac{1}{2}+m}(\partial^2\mathbf{g}_{<\lambda})\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{\frac{1}{2}+m-4\epsilon_0} \quad \textit{for} \quad -\frac{1}{2}+4\epsilon_0 \le m, \tag{121}$$

$$\|\nabla^m\,\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\mathbf{g}_{<\lambda}\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0+m}, \tag{122}$$

$$\|\nabla^m(\nabla^{\frac{1}{2}}\mathbf{R}_{\alpha\beta}(\mathbf{g}_{<\lambda}))\|_{L^\infty_{I_k}L^2_x} \lesssim \lambda^m, \tag{123}$$

$$\|\nabla^m\mathbf{R}_{\alpha\beta}(\mathbf{g}_{<\lambda})\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0+m}. \tag{124}$$

Remark 8.23. It suffices to prove the above estimates for the inverse metric $\mathbf{g}^{\mu\nu}_{<\lambda} = P_{<\lambda}(\mathbf{g}^{\mu\nu})$. This can easily be seen by the Leibnitz rule and the nondegeneracy of $\mathbf{g}_{<\lambda}$. On the other hand, due to the explicit presence of P_λ, the estimates for $\mathbf{g}^{\mu\nu}_{<\lambda}$ can be immediately reduced to $m = 0$.

To be precise, the argument above works only for the spatial derivatives ∇, since $P_{<\lambda}$ truncates the frequencies of $\mathbf{g}^{\mu\nu}$ only with respect to the space variable x. However, using the fact that $\mathbf{g}_{\mu\nu} = \phi$ is a solution of the wave equation, one can recover the corresponding estimates for the time derivatives. Let us illustrate this by proving the estimate (117)[31] with $m = 1$. We assume that we have already proved (117)–(122) for $m = 0$. Then, clearly the derivatives $\nabla^2\mathbf{g}_{<\lambda}$ and $\nabla\partial_t\mathbf{g}_{<\lambda}$ can be estimated with an additional factor of λ. It remains to address the derivative $\partial_t^2\,\mathbf{g}_{<\lambda}$. Observe that

$$\mathbf{g}^{00}_{<\lambda}\partial_t^2 = \mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta + \sum_{\alpha=0,..,3,\beta=1,..,3} \mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta.$$

The desired estimate follows from the condition (122) with $m = 0$ and the fact that the second term in the previous formula contains at least one spatial derivative.

In view of the above remark we shall make no distinction between $\mathbf{g}_{<\lambda}$ and $\mathbf{g}^{-1}_{<\lambda}$ in what follows.

Proof of (117)–(124) *for* $m = 0$. The proof of inequality (118) follows immediately from the definition of I_k, since

$$\|\partial\mathbf{g}_{<\lambda}\|_{L^2_{I_k}L^\infty_x} \lesssim \|\partial\phi\|_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-4\epsilon_0}.$$

Moreover, we have an even stronger estimate,

$$\|\partial\mathbf{g}\|_{L^2_{I_k}L^\infty_x} \lesssim \|\partial\phi\|_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-4\epsilon_0}. \tag{125}$$

The Hölder inequality yields (117) from (118).

The estimates (119), (120), and (121) follow by a simple application of the Sobolev inequality, the composition properties of Sobolev spaces and the condition $\gamma > 4\epsilon_0$.

$$\begin{aligned} \|\partial(P_{<\lambda}\,\mathbf{g}(\phi))\|_{L^\infty_{I_k}L^\infty_x} &\lesssim \|\partial(P_{<\lambda}\,\mathbf{g}(\phi))\|_{L^\infty_{I_k}H^{\frac{3}{2}+\epsilon}} \\ &\lesssim \lambda^{\frac{1}{2}-4\epsilon_0}\|\partial\phi\|_{L^\infty_{I_k}H^{1+\gamma}} \lesssim \lambda^{\frac{1}{2}-4\epsilon_0}. \end{aligned} \tag{126}$$

[31] This is one of the few estimates with $m \neq 0$ which we shall actually use.

The most interesting part of the proposition are the estimates (122), (124). Recall that the original metric $\mathbf{g}$ satisfied the Einstein equation, $\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0$. In addition, since $(\mathbf{g}^{\mu\nu}) = \phi^{-1}$ and $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\phi = N$, each component of $\mathbf{g}^{\mu\nu}$ satisfies the equation which can be written schematically as $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\,\mathbf{g}^{\mu\nu} = |\partial\phi|^2$. Thus,

$$\|\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\,\mathbf{g}\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0}. \tag{127}$$

On the other hand we recall the expression for $\mathbf{R}_{\alpha\beta}(\mathbf{g})$ relative to arbitrary coordinates,

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = \frac{1}{2}\,\mathbf{g}^{\mu\nu}(\partial^2_{\mu\beta}\,\mathbf{g}_{\alpha\nu} + \partial^2_{\alpha\nu}\,\mathbf{g}_{\mu\beta} - \partial^2_{\alpha\beta}\,\mathbf{g}_{\mu\nu} - \partial^2_{\mu\nu}\,\mathbf{g}_{\alpha\beta}) + \mathbf{g}_{\gamma\delta}(\Gamma^\gamma_{\mu\beta}\Gamma^\delta_{\alpha\nu} - \Gamma^\gamma_{\mu\nu}\Gamma^\delta_{\alpha\beta}).$$

Here $\Gamma^\gamma_{\mu\beta}$ are the Christoffel symbols of the metric $\mathbf{g}$. It is then easy to see that the equation $\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0$ also implies that

$$\|\mathbf{g}^{\mu\nu}(\partial^2_{\mu\beta}\,\mathbf{g}_{\alpha\nu} + \partial^2_{\alpha\nu}\,\mathbf{g}_{\mu\beta} - \partial^2_{\alpha\beta}\,\mathbf{g}_{\mu\nu} - \partial^2_{\mu\nu}\,\mathbf{g}_{\alpha\beta})\|_{L^1_{I_k}L^\infty_x} \lesssim \|\partial\mathbf{g}\|^2_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0} \tag{128}$$

and

$$\|\mathbf{g}^{\mu\nu}(\partial^2_{\mu\beta}\,\mathbf{g}_{\alpha\nu} + \partial^2_{\alpha\nu}\,\mathbf{g}_{\mu\beta} - \partial^2_{\alpha\beta}\,\mathbf{g}_{\mu\nu} - \partial^2_{\mu\nu}\,\mathbf{g}_{\alpha\beta})\|_{L^\infty_{I_k}\dot H^{\frac12}} \lesssim \|\partial\mathbf{g}\cdot\partial\mathbf{g}\|_{L^\infty_{I_k}\dot H^{\frac12}} \lesssim 1. \tag{129}$$

The last inequality follows from the generalized Leibnitz rule and the fact that $\partial\mathbf{g} \in H^{1+\gamma}$.

To derive the desired estimates (122)-(124) we simply need to apply the following lemma to the estimates (127) and (128).[32]

LEMMA 8.24. *Let $\mathbf{A} = (A^{\alpha\beta\mu\nu}_{\gamma\delta})$ be a fixed constant tensor. Denote $\mathbf{g}\cdot\mathbf{A}\cdot\partial^2\,\mathbf{g} = \mathbf{g}^{\gamma\delta}A^{\alpha\beta\mu\nu}_{\gamma\delta}\partial_\alpha\partial_\beta\,\mathbf{g}_{\mu\nu}$. Assume that the linear combination $\mathbf{g}\cdot\mathbf{A}\cdot\partial^2\,\mathbf{g}$ of the second derivatives of the metric $\mathbf{g}$ satisfies the estimate $\|\mathbf{g}\cdot\mathbf{A}\cdot\partial^2\,\mathbf{g}\|_{L^1_{I_k}L^\infty_x} \le c(B_0)\lambda^{-8\epsilon_0}$. Then the same estimate holds for the linear combination associated with the metric $\mathbf{g}_{<\lambda}$:*

$$\|\mathbf{g}_{<\lambda}\cdot\mathbf{A}\cdot\partial^2\,\mathbf{g}_{<\lambda}\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0}, \qquad \|\mathbf{g}_{<\lambda}\cdot\mathbf{A}\cdot\partial^2\,\mathbf{g}_{<\lambda}\|_{L^\infty_{I_k}\dot H^{\frac12}} \lesssim 1. \tag{130}$$

Proof. Recall that $\mathbf{g}_{<\lambda} = P_{<\lambda}\mathbf{g}$. Clearly,

$$\|\,\mathbf{g}_{<\lambda} - \mathbf{g}\|_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-1}\|\nabla\,\mathbf{g}\|_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-1-4\epsilon_0}. \tag{131}$$

[32]The estimates (123) and (124) also require the following obvious estimates:

$$\|\partial\mathbf{g}_{<\lambda}\|^2_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0}, \qquad \|\partial\mathbf{g}_{<\lambda}\cdot\partial\mathbf{g}_{<\lambda}\|_{L^\infty_{I_k}\dot H^{\frac12}} \lesssim 1.$$

Then

$$\|\Big(\mathbf{g}_{<\lambda} - \mathbf{g}\Big) \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}_{<\lambda}\|_{L^1_{I_k} L^\infty_x} \le \| \mathbf{g}_{<\lambda} - \mathbf{g}\|_{L^2_{I_k} L^\infty_x} \|\partial^2 \mathbf{g}_{<\lambda}\|_{L^2_{I_k} L^\infty_x} \lesssim \lambda^{-1-4\epsilon_0} \lambda \|\partial \mathbf{g}_{<\lambda}\|_{L^2_{I_k} L^\infty_x} \lesssim \lambda^{-8\epsilon_0}. \tag{132}$$

We can now consider the term $\mathbf{g} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}_{<\lambda}$. We have

$$\mathbf{g}\cdot\mathbf{A}\cdot\partial^2 P_{<\lambda}\mathbf{g} = \mathbf{g} P_{<\lambda}\partial\cdot\mathbf{A}\cdot\partial\,\mathbf{g} = [\,\mathbf{g}, P_{<\lambda}\partial]\cdot\mathbf{A}\cdot\partial\,\mathbf{g} + P_{<\lambda}\Big(\,\mathbf{g}\cdot\mathbf{A}\cdot\partial^2\mathbf{g} + \partial\mathbf{g}\cdot\mathbf{A}\cdot\partial\,\mathbf{g}\Big).$$

The commutator term can be estimated:

$$\|\Big([\,\mathbf{g}, P_{<\lambda}\partial]\Big) f\|_{L^2_{I_k} L^\infty_x} \lesssim \|\partial\mathbf{g}\|_{L^2_{I_k} L^\infty_x} \|f\|_{L^\infty_x} \lesssim \lambda^{-4\epsilon_0} \|f\|_{L^\infty_x}.$$

It then follows that

$$\|\Big([\,\mathbf{g}, P_{<\lambda}\partial]\Big) \cdot \mathbf{A} \cdot \partial\,\mathbf{g}\|_{L^1_{I_k} L^\infty_x} \lesssim \lambda^{-8\epsilon_0}.$$

The remaining term satisfies the desired estimate by the assumptions of the lemma. The proof of the $\dot{H}^{\frac{1}{2}}$ estimate in (130) is similar. □

8.25. *Rescaling.* According to Theorem 8.18 we need to prove a Strichartz estimate for any solution of the problem $\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\psi = 0$ on the interval $I_k = [t_k, t_{k+1}]$, with initial data $\psi[t_k] = (\psi(t_k), \partial_t\psi(t_k))$ obeying condition (111), uniformly in λ, k.

It is convenient to replace the above problem by its rescaled version, so that the initial data satisfy condition (111) with $\lambda = 1$ and the rescaled time interval I with length $\le \lambda^{1-8\epsilon_0}$.

Introduce the family of the rescaled metrics[33]

$$H_{(\lambda)}(t,x) = g_{<\lambda}(\lambda^{-1}(t-t_k), \lambda^{-1}x). \tag{133}$$

We decompose the Lorentz metric $H = H_{(\lambda)}$ relative to our spacetime coordinates;

$$-n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt), \tag{134}$$

where n and v are related to $\mathbf{n}$, $\mathbf{v}$ according to the rule (133). In view of our choice of $\lambda \ge \Lambda$ and (8) it easily follows that H is indeed a Lorentz metric and

$$c|\xi|^2 \le h_{ij}\xi^i\xi^j \le c^{-1}|\xi|^2, \quad n^2 - |v|^2_h \ge c > 0, \quad |n|, |v| \le c^{-1}. \tag{135}$$

Proposition 8.22 implies that $H = H_{(\lambda)}$ obeys the following estimates on the time interval $I = [0, t_*]$ with $t_* \le \lambda^{1-8\epsilon_0}$:

Background estimates (*see Proposition* 2.4).

$$\|\partial^{1+m} H\|_{L^1_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-8\epsilon_0}, \tag{136}$$

[33] Just as for $g_{<\lambda}$ we make no distinction between $H_{(\lambda)}$, as the Lorentz metric and its inverse.

$$\|\partial^{1+m} H\|_{L^2_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \tag{137}$$

$$\|\partial^{1+m} H\|_{L^\infty_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \tag{138}$$

$$\|\nabla^{\frac{1}{2}+m}(\partial H)\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{-m} \qquad \text{for} \quad -\frac{1}{2} \le m \le \frac{1}{2} + 4\epsilon_0, \tag{139}$$

$$\|\nabla^{\frac{1}{2}+m}(\partial^2 H)\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \qquad \text{for} \quad -\frac{1}{2} + 4\epsilon_0 \le m, \tag{140}$$

$$\|\partial^m\big(H^{\alpha\beta}\partial_\alpha\partial_\beta H\big)\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}, \tag{141}$$

$$\|\nabla^m\big(\nabla^{\frac{1}{2}}\mathbf{Ric}(H)\big)\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{-1}, \tag{142}$$

$$\|\partial^m \mathbf{R}_{\alpha\beta}(H)\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}. \tag{143}$$

We now formulate the rescaled version of the desired Strichartz estimate.

Theorem 8.26 (A4). *Let ψ be a solution of the linear wave equation*

$$H^{\alpha\beta}\partial_\alpha\partial_\beta\psi = 0, \tag{144}$$

on the time interval $[0, t_*]$ with $t_* \le \lambda^{1-8\epsilon_0}$. Assume that the parameter $\lambda \ge \Lambda$ for a sufficiently large constant Λ and that the metric H verifies (136)–(143) *with a sufficiently small $\epsilon_0 > 0$. Let P be the operator of projection on the set $\{\xi : 1 \le |\xi| \le 2\}$ in Fourier space. Then there exists a small constant $\delta = \delta(\epsilon_0) > 0$ such that*

$$\|P\,\partial\psi\|_{L^2_{[0,t_*]}L^\infty_x} \lesssim |t_*|^\delta \|\partial\psi(0)\|_{L^2_x}. \tag{145}$$

Remark. Note that Theorem (A4) does not contain any assumptions on the Fourier support of the initial data $\psi[0]$.

8.27. *Decay estimates.* A variation of the standard TT^* type argument, see [Kl1], allows us to reduce the Strichartz estimate (145) to a corresponding dispersive inequality; see (146). In the process we replace[34] the equation $H^{\alpha\beta}\partial_\alpha\partial_\beta\psi = 0$ by the *geometric* wave equation

$$\square_H\psi = \frac{1}{\sqrt{|H|}}\partial_\alpha(H^{\alpha\beta}\sqrt{|H|}\,\partial_\beta\psi) = 0.$$

Theorem 8.28 (A5). *Let ψ be a solution of the linear wave equation*

$$\square_H\psi = 0, \tag{146}$$
$$\psi|_{t_0} = \psi_0, \quad \partial_t\psi|_{t_0} = \psi_1$$

[34]The two wave operators differ only by lower order terms insofar as the Strichartz estimates are concerned.

on the time interval $[0,t_*]$ *with* $t_* \le \lambda^{1-4\epsilon_0}$ *and with initial data* $\psi[t_0] = (\psi(t_0), \partial_t\psi(t_0))$. *We consider only large values of the parameter* $\lambda \ge \Lambda$. *Assume that the metric* H *verifies* (136)–(143). *Then there exists a function* $d(t)$ *obeying the condition*

$$t_*^{\frac{1}{q}} \|d\|_{L^q_{[0,t_*]}} \le 1, \quad \text{for some } q > 2 \text{ sufficiently close to } 2, \tag{147}$$

such that for all $t_0 \le t \le t_*$, *a fixed arbitrary small* $\epsilon > 0$, *and a sufficiently large integer* m,

$$\|P\,\partial\psi(t)\|_{L^\infty_x} \lesssim \left(\frac{1}{(1+|t-t_0|)^{1-\epsilon}} + d(t)\right) \sum_{k=0}^m \|\nabla^k\psi[t_0]\|_{L^1_x}. \tag{148}$$

We make the final reduction by decomposing the initial data $\psi[t_0]$ in the physical space into a sum of functions with essentially disjoint supports contained in balls of radius $\frac{1}{2}$. Using the additivity of the L^1 norm and the standard Sobolev inequality we can reduce the dispersive inequality (148) to an $L^2 - L^\infty$ decay estimate.

Theorem 8.29 ($L^2 - L^\infty$ decay). *Let* ψ *be a solution of the linear wave equation* (146) *on the time interval* $[0,t_*]$ *with* $t_* \le \lambda_0^\epsilon$ *and with initial data* $\psi[t_0]$ *supported in the ball* $B_{\frac{1}{2}}(0)$ *of radius* $\frac{1}{2}$ *centered at the origin in the physical space. Fix a big constant* Λ *and consider only large values of the parameter* $\lambda \ge \Lambda$. *Assume that the metric* H *verifies* (136)–(143). *Then there exists a function* $d(t)$ *obeying the condition* (147) *such that for all* $t_0 \le t \le t_*$, *an arbitrary small* $\epsilon > 0$, *and a sufficiently large integer* $m > 0$,

$$\|P\,\partial\psi(t)\|_{L^\infty_x} \lesssim \left(\frac{1}{(1+|t-t_0|)^{1-\epsilon}} + d(t)\right) \sum_{k=0}^m \|\nabla^k\psi[t_0]\|_{L^2_x}. \tag{149}$$

8.30. *Proof of the implication Theorem* (A5) $\to$ *Theorem* (A4); *Decay* $\to$ *Strichartz.* On this step of the reduction we assume that the family of metrics $H = H_{(\lambda)}$ satisfies conditions (136)–(143) and that any solution of the geometric wave equation $\Box_H\psi = 0$ obeys the decay estimate

$$\|P\,\partial\psi(t)\|_{L^\infty_x} \lesssim \left(\frac{1}{(1+|t-t_0|)^{1-\epsilon}} + d(t)\right) \sum_{k=0}^m \|\nabla^k\psi[t_0]\|_{L^1_x}.$$

We need to show that under these assumptions any solution[35] of the wave equation $H^{\alpha\beta}\partial_\alpha\partial_\beta\phi = 0$ satisfies the Strichartz estimate $\|P\,\partial\phi\|_{L^2_{[0,t_*]}L^\infty_x} \lesssim |t_*|^\delta \|\psi[0]\|_{L^2_x}$.

[35]Note that we do not require any assumptions on the initial data. This is due to the presence of the projection P in the estimate.

First, observe that it suffices to prove the following estimate:

$$\|P\,\partial\phi\|_{L^q_{[0,t_*]}L^\infty_x} \lesssim \|\phi[0]\|_{L^2_x} \tag{150}$$

with $\delta = 1 - \frac{2}{q} > 0$ arbitrarily small. Observe also that the solutions of either the geometric wave equation $\Box_H \psi = F$ or the equation $H^{\alpha\beta}_\lambda \partial_\alpha \partial_\beta \psi = F$ obey the following energy inequality for any $t, t_0 \in [t_0, t_*]$:

$$\begin{aligned}\|\partial\psi(t)\|_{L^2_x} &\le \exp(C\,\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x})\Big(\|\partial\psi(t_0)\|_{L^2_x} + \|F\|_{L^1_{[0,t_*]}L^2_x}\Big)\\ &\le 2\Big(\|\partial\psi(t_0)\|_{L^2_x} + \|F\|_{L^1_{[0,t_*]}L^2_x}\Big),\end{aligned} \tag{151}$$

where the last inequality follows[36] from the condition (136) on the metric H.

Furthermore, since

$$\Box_H = H^{\alpha\beta}\partial_\alpha\partial_\beta + \frac{1}{\sqrt{|H|}}\partial_\alpha(\sqrt{|H|}H^{\alpha\beta})\partial_\beta,$$

it is easy to show[37] that it suffices to establish (150) for a solution of the geometric wave equation. We shall now prove a stronger result.

Proposition 8.31. *Let ϕ verify the wave equation $\Box_H \phi = 0$. Assume that the metric H is Lorentzian*[38] *and satisfies the condition*

$$C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x} \le \frac{1}{2} \tag{152}$$

for some sufficiently large positive constant C. We also assume that the conclusions of Theorem (A5) *hold true. Then, for any $q > 2$,*

$$\|P\,\partial\phi\|_{L^q_{[0,t_*]}L^\infty_x} \lesssim \|\partial\phi(0)\|_{L^2_x}. \tag{153}$$

Proof. As in [Kl1], [Kl-Ro] we start by observing that our desired estimate

$$\|P\,\partial\phi\|_{L^q_{[0,t_*]}L^\infty_x} \le M\|\partial\phi(0)\|_{L^2_x}, \tag{154}$$

is trivially true with a constant $M > 0$ which may depend on λ. Thus we only need to prove that the constant M is in fact independent of λ.

Remark 8.32. We shall first prove the estimate (153) for $P\,\partial_t\phi$.

[36] Recall that we consider $\lambda \ge \Lambda$ for a sufficiently large constant Λ.

[37] By the Duhamel Principle we would obtain

$$\|P\,\partial\phi\|_{L^q_{[0,t_*]}L^\infty_x} \le M(\|\phi[0]\|_{L^2_x} + \|\partial H\|_{L^1_{[0,t_*]}L^\infty_x}\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x})$$

and the condition (136) together with the energy inequality for ϕ would imply (150).

[38] For simplicity we can assume that the ellipticity constant of the restrictions of the metric H to the time slices Σ_t is 2.

Definition 8.33. Setting $(w_0, w_1) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$, $w = (w_0, w_1)$ we denote by $\Phi(t, s; w)$ the vector $(\phi, \partial_t \phi)$, where $\phi(t, s; w)$ is the solution at time t of the homogeneous equation $\Box_H \phi = 0$ subject to the initial data at time s, $\phi(s, s; w) = w_0$, $\partial_t \phi(s, s; w) = w_1$.

By a standard uniqueness argument [39] we can easily prove the following:

$$\Phi\Big(t, s; \Phi(s, t_0; w)\Big) = \Phi(t, t_0; w). \tag{155}$$

Definition 8.34. Denote by $\mathcal{H}$ the set of vector functions $w = (w_0, w_1)$ with $(w_0, w_1) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$. The scalar product in $\mathcal{H}$ is defined by

$$\langle w, v \rangle = \int_{\Sigma_0} \Big(- H^{00} w_1 \cdot v_1 + H^{ij} \partial_i w_0 \cdot \partial_j v_0 \Big).$$

Remark 8.35. Observe that the above scalar product is positive definite. Indeed H^{00} is strictly negative and H^{ij} is positive definite. To see the last assertion let h_{ij} denote the metric induced by H on Σ_t. In fact the metric H is given by $-n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt)$. Thus $H^{ij} = h^{ij} - n^{-2} v^i v^j$. Observe first[40] that $H^{ij} v_i v_j > c|v|_h^2$. This follows easily from $n^2 - |v|_h^2 > 0$; see (135). On the other hand, denoting by $T_v = \{\omega / h_{ij} \omega^i v^j = 0\}$ the orthogonal complement to v, we easily check that $H^{ij} \omega_i \omega_j > c|\omega|^2$. This follows from the positivity of h; see (135). Finally $H^{ij} \omega_i v_j = 0$.

Let $X = L^q_{[0,t_*]} L^\infty_x$ and its dual $X' = L^{q'}_{[0,t_*]} L^1_x$. Let $\mathcal{T}$ be the operator from $\mathcal{H}$ to X defined by:

$$\mathcal{T}(w) = -P\, \partial_t \phi(t, 0; w) \tag{156}$$

with ϕ defined according to Definition 8.33.

The adjoint $\mathcal{T}^*$ is defined from X' to $\mathcal{H}$. To prove the estimate (153) it suffices to check that $\mathcal{T} \cdot \mathcal{T}^*$ is a bounded operator from X' to X. In view of (154) we have[41] $\|\mathcal{T}\|_{\mathcal{H} \to \mathcal{X}} = M$ where $\|\mathcal{T}\|_{\mathcal{H} \to \mathcal{X}}$ denotes the operator norm of $\mathcal{T}$. Thus,

$$\|\mathcal{T} \cdot \mathcal{T}^*\|_{X' \to X} = M^2.$$

To calculate $\mathcal{T}^*$ we write,

$$\langle \mathcal{T}^* f, w \rangle := \langle f, \mathcal{T}(w) \rangle = - \int_{[0,t_*] \times \mathbf{R}^3} \partial_t \phi P f \, dt dx = \int_{[0,t_*] \times \mathbf{R}^3} \partial_t \phi \, \bar{\Box}_H \psi,$$

[39] which follows from the energy estimate (151), which still holds under assumption (152) on the metric H.

[40] Here $v_i = h_{ij} v^j$.

[41] We may assume that M is the smallest constant for which (154) holds true.

where ψ is the unique solution to the equation

$$\bar{\Box}_H\psi = \partial_\alpha(H^{\alpha\beta}\partial_\beta\psi) = -Pf,$$
$$\phi(t_*) = \partial_t\phi(t_*) = 0. \tag{157}$$

Consequently, integrating by parts, we obtain

$$\langle \mathcal{T}^* f, w\rangle = -\int_{\Sigma_0}\Big(\partial_t\phi H^{0\beta}\partial_\beta\psi - H^{0\beta}\partial_\beta\partial_t\phi\,\psi\Big) + \int_{[0,t_*]\times\mathbf{R}^3}\bar{\Box}_H\partial_t\phi\,\psi.$$

Observe that

$$\bar{\Box}_H\partial_t\phi = \partial_t\bar{\Box}_H\phi - \partial_\alpha(\partial_t(H^{\alpha\beta})\partial_\beta\phi).$$

Therefore, integrating by parts once more, we have

$$\begin{aligned}\langle \mathcal{T}^* f, w\rangle = &-\int_{\Sigma_0}\Big(\partial_t\phi H^{0\beta}\partial_\beta\psi - H^{0\beta}\partial_\beta\partial_t\phi\,\psi \\ &+ \bar{\Box}_H\phi\,\psi - \partial_t(H^{0\beta})\partial_\beta\phi\,\psi\Big) \\ &- \int_{[0,t_*]\times\mathbf{R}^3}\Big(\bar{\Box}_H\phi\,\partial_t\psi - \partial_t(H^{\alpha\beta})\partial_\beta\phi\,\partial_\alpha\psi\Big).\end{aligned}$$

Further, note that $-H^{0\beta}\partial_\beta\partial_t\phi + \bar{\Box}_H\phi - \partial_t(H^{0\beta})\partial_\beta\phi = \partial_i(H^{i\beta}\partial_\beta\phi)$, and therefore,

$$\begin{aligned}\int_{\Sigma_0}&\Big(\partial_t\phi H^{0\beta}\partial_\beta\psi - H^{0\beta}\partial_\beta\partial_t\phi\,\psi + \bar{\Box}_H\phi\,\psi - \partial_t(H^{0\beta})\partial_\beta\phi\,\psi\Big) \\ &= \int_{\Sigma_0}\Big(\partial_t\phi H^{0\beta}\partial_\beta\psi - H^{i\beta}\partial_\beta\phi\partial_i\psi\Big) \\ &= \int_{\Sigma_0}\Big(H^{00}\partial_t\phi\partial_t\psi - H^{ij}\partial_i\phi\partial_j\psi\Big).\end{aligned}$$

Thus, since

$$\Box_H = \bar{\Box}_H + H^{\alpha\beta}\frac{\partial_\alpha\sqrt{|H|}}{\sqrt{|H|}}\partial_\beta$$

and $\Box_H\phi = 0$,

$$\begin{aligned}\langle \mathcal{T}^* f, w\rangle &= \int_{\Sigma_0}\Big(-H^{00}\partial_t\phi\partial_t\psi + H^{ij}\partial_i\phi\partial_j\psi\Big) \\ &\quad - \int_{[0,t_*]\times\mathbf{R}^3}\Big(\bar{\Box}_H\phi\,\partial_t\psi - \partial_t(H^{\alpha\beta})\partial_\beta\phi\,\partial_\alpha\psi\Big) \\ &= \int_{\Sigma_0}\Big(-H^{00}\partial_t\phi\partial_t\psi + H^{ij}\partial_i\phi\partial_j\psi\Big) \\ &\quad + \int_{[0,t_*]\times\mathbf{R}^3}\Big(H^{\alpha\beta}\frac{\partial_\alpha\sqrt{|H|}}{\sqrt{|H|}}\partial_\beta\phi\,\partial_t\psi + \partial_t(H^{\alpha\beta})\partial_\beta\phi\,\partial_\alpha\psi\Big).\end{aligned}$$

Thus, since $\phi[0] = w$ and by the definition of $\langle\, ,\, \rangle_{\mathcal{H}}$

$$\langle \mathcal{T}^* f, w\rangle = \langle \psi[0]\, , w\rangle + \langle R(f), w\rangle$$

with $R(f)$ the linear operator from X' to $\mathcal{H}$ defined by the formula,

$$\langle R(f)\, ,\, w\rangle = \int_{[0,t_*]\times\mathbf{R}^3} \left(H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta \phi\, \partial_t \psi + \partial_t (H^{\alpha\beta}) \partial_\beta \phi\, \partial_\alpha \psi \right). \tag{158}$$

Therefore,

$$\mathcal{T}^* f = \psi[0] + R(f) \tag{159}$$

with $\psi[0] = (\psi(0), \partial_t \psi(0))$.

Henceforth,

$$\mathcal{T}\mathcal{T}^* f = \mathcal{T}\psi[0] + \mathcal{T}R(f). \tag{160}$$

Observe that $\Box_H \psi = -Pf + e$ with $e = H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta \psi$. Thus we can write $\psi = -\psi_1 + \psi_2$ with,

$$\Box_H \psi_1 = Pf,$$
$$\Box_H \psi_2 = e$$

with both ψ_1, ψ_2 verifying zero initial conditions at $t = t_*$ as in (157). Now $\mathcal{T}\psi[0] = -\mathcal{T}\psi_1[0] + \mathcal{T}\psi_2[0]$ and, by Definition 8.33, $\mathcal{T}\psi_1[0] = -P\partial_t \phi\big(t, 0; \psi_1[0]\big)$. According to the Duhamel principle, as in (115) we have, with $\psi[t] = \big(\psi(t), \partial_t \psi(t)\big)$,

$$\psi_1[t] = \int_{t_*}^{t} \Phi(t, s; F(s)) ds$$

with $F(s) = (0, (H^{00})^{-1} Pf(s)) = (0, -n^{-2} Pf(s))$ and therefore,

$$\psi_1[0] = -\int_0^{t_*} \Phi(0, s; F(s)) ds.$$

Now, in view of (155),

$$\mathcal{T}\psi_1[0] = P\partial_t \phi\left(t, 0; \int_0^{t_*} \Phi(0, s; F(s)) ds\right) = P \int_0^{t_*} \partial_t \phi(t, s; F(s)) ds.$$

We are now in a position to apply the dispersive inequality of Theorem (A6).

$$\|P\, \partial_t \phi(t, s; F(s))\|_{L^\infty} \le C\bigg((1 + |t - s|)^{-1+\epsilon} + d(t)\bigg) \sum_{k=0}^{m} \|\nabla^k (n^{-2} Pf(s))\|_{L^1}.$$

In view of (135) and (138), we have $\|\nabla^k n^{-2}\|_{L^\infty} \lesssim 1$. Thus, since P is the projection on the frequencies of size 1, we infer that

$$\|P\, \partial_t \phi(t, s; F(s))\|_{L^\infty} \le C\bigg((1 + |t - s|)^{-1+\epsilon} + d(t)\bigg) \|f(s))\|_{L^1}.$$

Therefore, by the Hardy-Littlewood-Sobolev inequality,

$$\|\mathcal{T}\psi_1[0]\|_{L^q_{[0,t_*]}L^\infty_x} \le C\|f\|_{L^{q'}_{[0,t_*]}L^1_x} + \|\int_0^{t_*} d(t)\|f(s)\|_{L^1}\,ds\|_{L^q_{[0,t_*]}}.$$

We can now make use of the assumption (147) of Theorem ($A5$) and infer that

$$\|\int_0^{t_*} d(t)\|Pf(s)\|_{L^1}\,ds\|_{L^q_{[0,t_*]}} \le Ct_*^{\frac{1}{q}}\|d\|_{L^q_{[0,t_*]}}\|f\|_{L^{q'}_{[0,t_*]}L^1_x} \le C\|f\|_{L^{q'}_{[0,t_*]}L^1_x}.$$

Thus

$$\|\mathcal{T}\psi_1[0]\|_{L^q_{[0,t_*]}L^\infty_x} \le C\|f\|_{L^{q'}_{[0,t_*]}L^1_x} \tag{161}$$

with C a constant, independent of λ.

To estimate $\mathcal{T}\psi_2[0]$ we apply the Strichartz inequality with a bound M; see (154),

$$\|\mathcal{T}\psi_2[0]\|_{L^q_{[0,t_*]}L^\infty_x} \le M\|\psi_2[0]\|_{\mathcal{H}}$$

where,

$$\|\psi_2[0]\|_{\mathcal{H}} = \sup_{\|w\|_{\mathcal{H}}\le 1} < w, \psi_2[0] >_{\mathcal{H}} \le C\|\partial\psi_2(0)\|_{L^2}.$$

We shall now make use of the energy estimate (151) for ψ_2 verifying the equation $\Box_H\psi_2 = e$, subject to the initial conditions $\psi_2(t_*) = \partial_t\psi_2(t_*) = 0$,

$$\|\partial\psi_2(0)\|_{L^2_x} \le C\|e\|_{L^1_{[0,t_*]}L^2_x} \le C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x}\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}.$$

Therefore, with the help of condition (152), we have

$$\|\mathcal{T}\psi_2[0]\|_{L^q_{[0,t_*]}L^\infty_x} \le \frac{1}{4}M\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}. \tag{162}$$

We shall now estimate the other error term $\mathcal{T}Rf$. Since the operator norm of $\mathcal{T}$ is bounded by M,

$$\|\mathcal{T}R(f)\|_{L^q_{[0,t_*]}L^\infty_x} \le M\|R(f)\|_{\mathcal{H}}.$$

On the other hand,

$$\begin{aligned}\|R(f)\|_{\mathcal{H}} &= \sup_{\|w\|_{\mathcal{H}}\le 1} < w, R(f) >_{\mathcal{H}} \\ &= -\sup_{\|w\|_{\mathcal{H}}\le 1}\int_{[0,t_*]\times\mathbf{R}^3}\Big(H^{\alpha\beta}\frac{\partial_\alpha\sqrt{|H|}}{\sqrt{|H|}}\partial_\beta\phi\,\partial_t\psi + \partial_t(H^{\alpha\beta})\partial_\beta\phi\,\partial_\alpha\psi\Big).\end{aligned}$$

Estimating in a straightforward manner we derive,

$$\|R(f)\|_{\mathcal{H}} \le C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x}\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x}\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}.$$

We use the energy inequality (151) to estimate $\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x}$. Since the initial data $\|w\|_{\mathcal{H}} \le 1$ we infer that $\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x} \le C$. Therefore, with the help of (152), we have

$$\|\mathcal{T}R(f)\|_{L^q_{[0,t_*]}L^\infty_x} \le \frac{1}{4}M\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}. \tag{163}$$

To estimate $\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}$ we rely on the following:

LEMMA 8.36. *The solution ψ of the equation $\bar{\Box}_H\psi = -Pf$, $\psi(t_*) = \partial_t\psi(t_*) = 0$, verifies the estimate,*

$$\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x} \le 2M\|f\|_{L^{q'}_{[0,t_*]}L^1_x}. \tag{164}$$

Gathering together (161), (162), (163) and (164) we infer that

$$\|\mathcal{T}\mathcal{T}^*f\|_X = \|\mathcal{T}(\psi_1[0] + \psi_2[0] + R(f))\|_{L^q_{[0,t_*]}L^\infty_x} \le (C + \frac{1}{2}M^2)\|f\|_{L^{q'}_{[0,t_*]}L^1_x}.$$

Therefore, in view of (160),

$$M^2 = \|\mathcal{T}\mathcal{T}^*\|_{X'\to X} \le (C + \frac{1}{2}M^2).$$

Thus we infer that M is a universal constant, as desired.

It only remains to prove Lemma 8.36. We proceed as follows. Let t be fixed in the interval $[0, t_*]$. We rewrite the equation $\Box_H\phi = 0$ in the form,

$$\bar{\Box}_H\phi = F = -H^{\alpha\beta}\frac{\partial_\alpha\sqrt{|H|}}{\sqrt{|H|}}\partial_\beta\phi \tag{165}$$

with initial data $\phi(t) = w_0, \partial_t\phi(t) = w_1$, and $(w_0, w_1) = w \in \mathcal{H}_t$, $\|w\|_{\mathcal{H}_t} \le 1$. Here, the space $\mathcal{H}_t$ is defined by the scalar product $\langle w, v\rangle_{\mathcal{H}_t} = \int_{\Sigma_t} -H^{00}w_1\, v_1 + H^{ij}\partial_i w_0\, \partial_j v_0$. We also recall that by (157),

$$\bar{\Box}_H\psi = -Pf \tag{166}$$

with initial data $\psi_1(t_*) = \partial_t\psi_1(t_*) = 0$. As in [Kl1] and [Kl-Ro] we multiply (165) by $\partial_t\psi$ and (166) by $\partial_t\phi$ after which we sum and integrate on our spacetime slab $[t, t_*] \times \mathbb{R}^3$. Observe that,

$$\begin{aligned}
\partial_\alpha(H^{\alpha\beta}\partial_\beta\psi) &= (\partial_\alpha H^{\alpha\beta})\partial_\beta\psi + H^{\alpha\beta}\partial_\alpha\partial_\beta\psi \\
H^{\alpha\beta}\partial_\alpha\partial_\beta\psi\, \partial_t\phi + H^{\alpha\beta}\partial_\alpha\partial_\beta\phi\, \partial_t\psi &= H^{\alpha\beta}\partial_\alpha(\partial_t\phi\partial_\beta\psi) + H^{\alpha\beta}\partial_\beta(\partial_t\psi\partial_\alpha\phi) \\
&\quad -H^{\alpha\beta}(\partial_\alpha\partial_t\phi\partial_\beta\psi) - H^{\alpha\beta}(\partial_\beta\partial_t\psi\partial_\alpha\phi) \\
&= H^{\alpha\beta}\partial_\alpha(\partial_t\phi\partial_\beta\psi) + H^{\alpha\beta}\partial_\beta(\partial_t\psi\partial_\alpha\phi) \\
&\quad -H^{\alpha\beta}\partial_t(\partial_\alpha\phi\partial_\beta\psi).
\end{aligned}$$

Thus

$$\begin{aligned}
\partial_\alpha(H^{\alpha\beta}\partial_\beta\psi)\partial_t\phi + \partial_\alpha(H^{\alpha\beta}\partial_\beta\phi)\partial_t\psi =& \partial_\alpha(H^{\alpha\beta}\partial_t\phi\partial_\beta\psi + \partial_t\psi\partial_\beta\phi) \\
&- \partial_t(H^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi) + (\partial_t H^{\alpha\beta})\partial_\alpha\phi\partial_\beta\psi \\
=& \partial_i(H^{i\beta}\partial_t\phi\partial_\beta\psi + \partial_t\psi\partial_\beta\phi) \\
&+ \partial_t(-H^{00}\partial_t\phi\partial_t\psi + H^{ij}\partial_i\phi\partial_j\psi) \\
&+ (\partial_t H^{\alpha\beta})\partial_\alpha\phi\partial_\beta\psi.
\end{aligned}$$

Integrating in the region $[t, t_*] \times \mathbf{R}^n$ we derive the identity,

$$\int_{\Sigma_t}\Big(-H^{00}\partial_t\phi\partial_t\psi+H^{ij}\partial_i\phi\partial_j\psi\Big)$$
$$=-\int_t^{t_*}\int_{\Sigma_\tau}\Big(-\partial_t\phi\,Pf+\partial_t\psi\,F+\partial_t(H^{\alpha\beta})\partial_\alpha\phi\,\partial_\beta\psi\Big).$$

Therefore,

$$\|\partial\psi(t)\|_{L^2}\le\|P\,\partial_t\phi\|_{L^q_{[0,t_*]}L^\infty_x}\|f\|_{L^{q'}_{[0,t_*]}L^1_x}$$
$$+C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x}\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x}\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}.$$

We recall that according to our assumption $\|P\,\partial_t\phi\|_{L^q_{[0,t_*]}L^\infty_x}\le M\|w\|_{\mathcal{H}_t}\le M$. Also according to the energy estimate, $\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x}\le 2\|w\|_{\mathcal{H}_t}\le 2$. Therefore,

$$\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}\le M\|f\|_{L^{q'}_{[0,t_*]}L^1_x}+C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x}\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}$$

and, since $C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x}\le\frac12$, we conclude that,

$$\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}\le 2M\|f\|_{L^{q'}_{[0,t_*]}L^1_x}$$

as desired.

To prove the Strichartz estimate for the spatial derivatives we rely on the proof, given above, for $P\,\partial_t\phi$. We thus assume that the estimate (8.36) holds true for $P\,\partial_t\phi$ with a universal constant M.

To estimate $\|P\,\partial_k\phi\|_{L^q_{[0,t_*]}L^\infty_x}$ it suffices to estimate the integral,

$$\mathcal{I}=\int_{[0,t_*]\times\mathbf{R}^3}P\,\partial_k\phi\,f\,dtdx$$

for functions f with $\|f\|_{L^{q'}_{[0,t_*]}L^\infty_x}\le 1$. Let ψ verify the equation $\bar\Box_H\psi=Pf$ with $\psi(t_*)=\partial_t\psi(t_*)=0$. Integrating by parts as before we infer that

$$\mathcal{I}=\int_{[0,t_*]\times\mathbf{R}^3}\partial_k\phi\,\bar\Box_H\psi=\int_{\Sigma_0}H^{0\beta}\Big(\partial_k\phi\,\partial_\beta\psi+\partial_k\psi\,\partial_\beta\phi\Big)$$
$$-\int_{[0,t_*]\times\mathbf{R}^3}\Big(\bar\Box_H\phi\,\partial_k\psi-(\partial_kH^{\alpha\beta})\partial_\alpha\phi\partial_\beta\psi\Big).$$

Once again

$$\Big|\int_{[0,t_*]\times\mathbf{R}^3}\bar\Box_H\phi\,\partial_k\psi\Big|\le C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x}\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x}\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}.$$

Also,

$$\int_{\Sigma_0}H^{0\beta}\Big(\partial_k\phi\,\partial_\beta\psi+\partial_k\psi\,\partial_\beta\phi\Big)\le\|\partial\phi(0)\|_{L^2}\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}.$$

The energy estimate (151) gives $\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x}\le 2\|\partial\phi(0)\|_{L^2}$. According to Lemma 8.36 we have,

$$\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}\le 2M\|f\|_{L^{q'}_{[0,t_*]}L^1_x}.$$

Observe that the M in Lemma 8.36 depends only on the Strichartz estimate (153) for $P\,\partial_t\phi$ which we have already proved. Therefore,

$$|\mathcal{I}| \le CM\|\partial\phi(0)\|_{L^2}(1+\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x})\|f\|_{L^{q'}_{[0,t_*]}L^1_x} \le CM\|\partial\phi(0)\|_{L^2}$$

which implies $\|P\,\partial_a\phi\|_{L^q_{[0,t_*]}L^\infty_x} \le CM\|\partial\phi(0)\|_{L^2}$ as desired. □

8.37. *Commutator lemma.* We conclude this section by presenting the proof of Lemma 7.1 from Section 2.1. Recall that the definition of the exterior region $\mathrm{Ext}_t = \{u \le t/2\}$.

LEMMA 8.38. *Consider a vectorfield $X = \sum_i X^i\partial_i$ vanishing on the complement of the exterior region Ext_t of Σ_t and P the standard Littlewood-Paley projection on frequencies of size* 1. *Then, for arbitrary scalar functions f there is the inequality*:

$$\|[P,X]f\|_{L^2(\mathrm{Ext}_t)} \lesssim \sup_{i,j}\|\partial_i X^j\|_{L^\infty(\mathrm{Ext}_t)}\|f\|_{L^2(\Sigma_t)}. \tag{167}$$

Proof. First observe, by expanding $X = X^j\partial_j$ relative to our system of coordinates on Σ_t, that $[P,X] = [P\partial_j, X^j] - P(\partial_j X^j)$. We shall denote $P_j = P\partial_j$, the modified cut-off of the unit frequencies. In what follows, the roles of P and P_j are identical. The convolution kernels of P, P_j are represented by the smooth functions $P(x), P_j(x)$ verifying the condition that $|P(x)|, |P_j(x)| \lesssim |x|^{-k}$ for any $k>0$ and $|x|\ge 1$. In particular, for any functions w, v

$$\begin{aligned} v &= \int_{\Sigma_t} P(x-y)\big(w(y)-w(x)\big)v(y)\,dy \\ &= -\int_0^1\int_{\Sigma_t} P(x-y)(x-y)^i\partial_i w(\tau x + (1-\tau)y)v(y)\,dy\,d\tau. \end{aligned}$$

As a consequence,

$$\|[P,w]v\|_{L^2(\Sigma_t)} \lesssim \|\nabla w\|_{L^\infty(\mathrm{Ext}_t)}\|v\|_{L^2(\Sigma_t)}. \tag{168}$$

A similar inequality also holds for P_j.

We shall show that

$$\|[P_j, X^j]f\|_{L^2(\Sigma_t)} + \|P\big((\partial_j X^j)f\big)\|_{L^2(\Sigma_t)} \lesssim \sup_{i,j}\|\partial_i X^j\|_{L^\infty(\mathrm{Ext}_t)}\|f\|_{L^2(\Sigma_t)}.$$

Since all X^j vanish outside of Ext_t and P is a bounded operator on $L^2(\Sigma_t)$, we can easily estimate the second term,

$$\|P\big((\partial_j X^j)f\big)\|_{L^2(\Sigma_t)} \lesssim \sup_{i,j}\|\partial_i X^j\|_{L^\infty(\mathrm{Ext}_t)}\|f\|_{L^2(\Sigma_t)}.$$

According to (168) we also have

$$\|[P_j, X^j]f\|_{L^2(\Sigma_t)} \lesssim \sup_{i,j}\|\partial_i X^j\|_{L^\infty(\mathrm{Ext}_t)}\|f\|_{L^2(\Sigma_t)}.$$

Princeton University, Princeton, NJ
E-mail addresses: seri@math.princeton.edu
irod@math.princeton.edu

References

[Ba-Ch1] H. Bahouri and J.-Y. Chemin, Équations d'ondes quasilinéaires et estimation de Strichartz, *Amer. J. Math.* **121** (1999), 1337–1377.

[Ba-Ch2] ———, Équations d'ondes quasilinéaires et effet dispersif, *Internat. Math. Res. Notices*, 1999, No. 21, 1141–1178.

[Br] Y. Choquét-Bruhat, Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires, *Acta Math.* **88** (1952), 141–225.

[Ch-Kl] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space*, *Princeton Math. Series* **41**, Princeton Univ. Press, 1993.

[H-K-M] T. Hughes, T. Kato, and J. Marsden, Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity, *Arch. Rational Mech. Anal.* **63** (1976), 273–294.

[Kl1] S. Klainerman, A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations, *Internat. Math. Res. Notices* **2001**, No. 5, 221–274.

[Kl2] ———, PDE as a unified subject, in *GAFA* 2000 (Tel Aviv, 1999), *Geom. Funct. Anal.* **2000**, 279–315.

[Kl3] ———, The null condition and global existence to nonlinear wave equations, in *Nonlinear Systems of Partial Differential Equations in Applied Mathematics*, *Part* 1 (Santa Fe, N.M., 1984), *Lectures in Applied Mathematics* **23** (1986), 293–326.

[Kl-Ro] S. Klainerman and I. Rodnianski, Improved local well-posedness for quasilinear wave equations in dimension three, *Duke Math. J.* **117** (2003), 1–124.

[Kl-Ro2] ———, The causal structure of microlocalized, rough, Einstein metrics, *Ann. of Math.* **161** (2005), 1195–1243.

[Kl-Ro3] ———, Ricci defects of microlocalized Einstein metrics, *J. Hyperbolic Differerential Equations* **1** (2004), 85–113.

[Li] H. Lindblad, Counterexamples to local existence for semi-linear wave equations, *Amer. J. Math.* **118** (1996), 1–16.

[Mor] C. Morawetz, The decay of solutions of the exterior initial-boundary value problem for the wave equation, *Comm. Pure Appl. Math.* **14** (1961), 561–568.

[Po-Si] G. Ponce and T. Sideris, Local regularity of nonlinear wave equations in three space dimensions, *Comm. Partial Differential Equations* **18** (1993), 169–177.

[Sm] H. Smith, A parametrix construction for wave equations with $C^{1,1}$ coefficients, *Ann. Inst. Fourier* **48** (1998), 797–835.

[Sm-So] H. Smith and C. Sogge, On Strichartz and eigenfunction estimates for low regularity metrics, *Math. Res. Lett.* **1** (1994), 729–737.

[Sm-Ta] H. Smith and D. Tataru, Sharp counterexamples for Strichartz estimates for low regularity metrics, *Math. Res. Lett.* **9** (2002), 199–204.

[Ta1] D. Tataru, Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation, *Amer. J. Math.* **122** (2000), 349–376.

[Ta2] ———, Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III, *J. Amer. Math. Soc.* **15** (2002), 419–442.

(Received October 24, 2001)
(Revised September 29, 2003)