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Moduli space of principal sheaves over projective varieties

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To A. Ramanathan, in memoriam

Abstract

Let G be a connected reductive group. The late Ramanathan gave a notion of (semi)stable principal G-bundle on a Riemann surface and constructed a projective moduli space of such objects. We generalize Ramanathan's notion and construction to higher dimension, allowing also objects which we call semistable principal G-sheaves, in order to obtain a projective moduli space: a principal G-sheaf on a projective variety X is a triple (P, E, ψ), where E is a torsion free sheaf on X, P is a principal G-bundle on the open set U where E is locally free and ψ is an isomorphism between $E|_U$ and the vector bundle associated to P by the adjoint representation.

We say it is (semi)stable if all filtrations $E_\bullet$ of E as sheaf of (Killing) orthogonal algebras, i.e. filtrations with $E_i^\perp = E_{-i-1}$ and $[E_i, E_j] \subset {E_{i+j}}^{\vee\vee}$, have

$$\sum (P_{E_i} \operatorname{rk} E - P_E \operatorname{rk} E_i) \,(\preceq)\, 0,$$

where P_{E_i} is the Hilbert polynomial of E_i. After fixing the Chern classes of E and of the line bundles associated to the principal bundle P and characters of G, we obtain a projective moduli space of semistable principal G-sheaves. We prove that, in case $\dim X = 1$, our notion of (semi)stability is equivalent to Ramanathan's notion.

Introduction

Let X be a smooth projective variety of dimension n over $\mathbb{C}$, with a very ample line bundle $\mathcal{O}_X(1)$, and let G be a connected algebraic reductive group. A principal $\mathrm{GL}(R, \mathbb{C})$-bundle over X is equivalent to a vector bundle of rank R. If X is a curve, the moduli space was constructed by Narasimhan and Seshadri [N-S], [Sesh]. If $\dim X > 1$, to obtain a projective moduli space we have to consider also torsion free sheaves, and this was done by Gieseker, Maruyama and Simpson [Gi], [Ma], [Si]. Ramanathan [Ra1], [Ra2], [Ra3] defined a notion of stability for principal G-bundles, and constructed the projective moduli space of semistable principal bundles on a curve.

We equivalently reformulate in terms of filtrations of the associated adjoint bundle of (Killing) orthogonal algebras the Ramanathan's notion of (semi)-stability, which is essentially of slope type (negativity of the degree of some associated line bundles), so when we generalize principal bundles to higher dimension by allowing their adjoints to be torsion free sheaves we are able to just switch degrees by Hilbert polynomials as definition of (semi)stability. We then construct a projective coarse moduli space of such semistable principal G-sheaves. Our construction proceeds by reductions to intermediate groups, as in [Ra3], although starting the chain higher, namely in a moduli of semistable tensors (as constructed in [G-S1]). In performing these reductions we have switched the technique, in particular studying the non-abelian étale cohomology sets with values in the groups involved, which provides a simpler proof also in Ramanathan's case $\dim X = 1$. However, for the proof of properness we have been able to just generalize the idea of [Ra3].

In order to make more precise these notions and results, let $G' = [G, G]$ be the commutator subgroup, and let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ be the Lie algebra of G, where $\mathfrak{g}'$ is the semisimple part and $\mathfrak{z}$ is the center. As a notion of principal G-sheaf, it seems natural to consider a rational principal G-bundle P, i.e. a principal G-bundle on an open set U with $\operatorname{codim} X \setminus U \geq 2$, and a torsion free extension of the form $\mathfrak{z}_X \oplus E$, to the whole of X, of the vector bundle $P(\mathfrak{g}) = P(\mathfrak{z} \oplus \mathfrak{g}') = \mathfrak{z}_U \oplus P(\mathfrak{g}')$ associated to P by the adjoint representation of G in $\mathfrak{g}$. This clearly amounts to the following

Definition 0.1. A principal G-sheaf $\mathcal{P}$ over X is a triple $\mathcal{P} = (P, E, \psi)$ consisting of a torsion free sheaf E on X, a principal G-bundle P on the maximal open set U_E where E is locally free, and an isomorphism of vector bundles

$$\psi : P(\mathfrak{g}') \xrightarrow{\cong} E|_{U_E} .$$

Recall that the algebra structure of $\mathfrak{g}'$ given by the Lie bracket provides $\mathfrak{g}'$ an orthogonal (Killing) structure, i.e. $\kappa : \mathfrak{g}' \otimes \mathfrak{g}' \to \mathbb{C}$ inducing an isomorphism $\mathfrak{g}' \cong \mathfrak{g}'^{\vee}$. Correspondingly, the adjoint vector bundle $P(\mathfrak{g}')$ on U has a Lie algebra structure $P(\mathfrak{g}') \otimes P(\mathfrak{g}') \to P(\mathfrak{g}')$ and an orthogonal structure, i.e. $\kappa : P(\mathfrak{g}') \otimes P(\mathfrak{g}') \to \mathcal{O}_U$ inducing an isomorphism $P(\mathfrak{g}') \cong P(\mathfrak{g}')^{\vee}$. In Lemma 0.25 it is shown that the Lie algebra structure uniquely extends to a homomorphism

$$[,] : E \otimes E \longrightarrow E^{\vee\vee} ,$$

where we have to take $E^{\vee\vee}$ in the target because an extension $E \otimes E \to E$ does not always exist (so the above definition of a principal G-sheaf is equivalent to the one given in our announcement of results [G-S2]). Analogously, the Killing form extends uniquely to

$$\kappa : E \otimes E \longrightarrow \mathcal{O}_X$$

inducing an inclusion $E \hookrightarrow E^\vee$. This form assigns an orthogonal $F^\perp = \ker(E \hookrightarrow E^\vee \twoheadrightarrow F^\vee)$ to each subsheaf $F \subset E$.

Definition 0.2. An orthogonal algebra filtration of E is a filtration

$$0 \subsetneq E_{-l} \subset E_{-l+1} \subset \cdots \subset E_l = E \tag{0.1}$$

with

$$(1) \quad E_i^\perp = E_{-i-1} \quad \text{and} \quad (2) \quad [E_i, E_i] \subset E_{i+j}{}^{\vee\vee}$$

for all i, j.

We will see that, if U is an open set with $\operatorname{codim} X \setminus U \geq 2$ such that $E|_U$ is locally free, a reduction of structure group of the principal bundle $P|_U$ to a parabolic subgroup Q together with a dominant character of Q produces a filtration of E, and the filtrations arising in this way are precisely the orthogonal algebra filtrations of E (Lemma 5.4 and Corollary 5.10). We define the Hilbert polynomial $P_{E_\bullet}$ of a filtration $E_\bullet \subset E$ as

$$P_{E_\bullet} = \sum (rP_{E_i} - r_i P_E)$$

where P_E, r, P_{E_i}, r_i always denote the Hilbert polynomials with respect to $\mathcal{O}_X(1)$ and ranks of E and E_i. If P is a polynomial, we write $P \prec 0$ if $P(m) < 0$ for $m \gg 0$, and analogously for "$\preceq$" and "$\leq$". We also use the usual convention: whenever "(semi)stable" and "($\preceq$)" appear in a sentence, two statements should be read: one with "semistable" and "$\preceq$" and another with "stable" and "$\prec$".

Definition 0.3 (See equivalent definition in Lemma 0.26). A principal G-sheaf $\mathcal{P} = (P, E, \psi)$ is said to be (semi)stable if all orthogonal algebra filtrations $E_\bullet \subset E$ have

$$P_{E_\bullet} (\preceq) 0 \,.$$

In Proposition 1.5 we prove that this is equivalent to the condition that the associated tensor

$$(E, \phi : E \otimes E \otimes \wedge^{r-1} E \longrightarrow \mathcal{O}_X)$$

is (semi)stable (in the sense of [G-S1]).

To grasp the meaning of this definition, recall that suppressing conditions (1) and (2) in Definitions 0.2 and 0.3 amounts to the (semi)stability of E as a torsion free sheaf, while just requiring condition (1) amounts to the (semi)stability of E as an orthogonal sheaf (cf. [G-S2]). Now, demanding (1) and (2) is having into account both the orthogonal and the algebra structure of the sheaf E, i.e. considering its (semi)stability as orthogonal algebra. By

Corollary 0.26, this definition coincides with the one given in the announcement of results [G-S2].

Replacing the Hilbert polynomials P_E and P_{E_i} by degrees we obtain the notion of *slope-(semi)stability*, which in Section 5 will be shown to be *equivalent to the Ramanathan's notion of (semi)stability* [Ra2], [Ra3] *of the rational principal G-bundle P* (this has been written at the end just to avoid interruption of the main argument of the article, and in fact we refer sometimes to Section 5 as a sort of appendix). Clearly

$$\text{slope-stable} \Longrightarrow \text{stable} \Longrightarrow \text{semistable} \Longrightarrow \text{slope-semistable}.$$

Since $G/G' \cong \mathbb{C}^{*q}$, given a principal G-sheaf, the principal bundle $P(G/G')$ obtained by extension of structure group provides q line bundles on U, and since $\operatorname{codim} X \setminus U \geq 2$, these line bundles extend uniquely to line bundles on X. Let $d_1, \ldots, d_q \in H^2(X;\mathbb{C})$ be their Chern classes. The rank r of E is clearly the dimension of $\mathfrak{g}'$. Let c_i be the Chern classes of E.

Definition 0.4 (Numerical invariants). We call the data $\tau = (d_1, \ldots, d_q, c_i)$ the numerical invariants of the principal G-sheaf (P, E, ψ).

Definition 0.5 (Family of semistable principal G-sheaves). A family of (semi)stable principal G-sheaves parametrized by a complex scheme S is a triple (P_S, E_S, ψ_S), with E_S a coherent sheaf on $X \times S$, flat over S and such that for every point s of S, $E_S \otimes k(s)$ is torsion free, P_S a principal G-bundle on the open set U_{E_S} where E_S is locally free, and $\psi : P_S(\mathfrak{g}') \to E_S|_{U_{E_S}}$ an isomorphism of vector bundles, such that for all closed points $s \in S$ the corresponding principal G-sheaf is (semi)stable with numerical invariants τ.

An isomorphism between two such families (P_S, E_S, ψ_S) and (P'_S, E'_S, ψ'_S) is a pair

$$(\beta : P_S \xrightarrow{\cong} P'_S, \gamma : E_S \xrightarrow{\cong} E'_S)$$

such that the following diagram is commutative

$$\begin{tikzcd}
P_S(\mathfrak{g}') \arrow[r, "\psi"] \arrow[d, "\beta(\mathfrak{g}')"'] & E_S|_{U_{E_S}} \arrow[d, "\gamma|_{U_{E_S}}"] \\
P'_S(\mathfrak{g}') \arrow[r, "\psi'"] & E'_S|_{U_{E_S}}
\end{tikzcd}$$

where $\beta(\mathfrak{g}')$ is the isomorphism of vector bundles induced by β. Given an S-family $\mathcal{P}_S = (P_S, E_S, \psi_S)$ and a morphism $f : S' \to S$, the pullback is defined as $\widetilde{f}^*\mathcal{P}_S = (\widetilde{f}^* P_S, \overline{f}^* E_S, \widetilde{f}^*\psi_S)$, where $\overline{f} = \mathrm{id}_X \times f : X \times S \to X \times S'$ and $\widetilde{f} = i^*(\overline{f}) : U_{\overline{f}^* E_S} \to U_{E_S}$, denoting $i : U_{E_S} \to X \times S$ the inclusion of the open set where E_S is locally free.

Definition 0.6. The functor $\widetilde{F}^{\tau}_{G}$ is the sheafification of the functor

$$F^{\tau}_{G} : (\mathrm{Sch}\,/\mathbb{C}) \longrightarrow (\mathrm{Sets})$$

sending a complex scheme S, locally of finite type, to the set of isomorphism classes of families of semistable principal G-sheaves with numerical invariants τ, and it is defined on morphisms as pullback.

Let $\mathcal{P} = (P, E, \psi)$ be a semistable principal G-sheaf on X. An orthogonal algebra filtration $E_\bullet$ of E which is *admissible*, i.e. having $P_{E_\bullet} = 0$, provides a reduction P^Q of $P|_U$ to a parabolic subgroup $Q \subset G$ (Lemma 5.4) on the open set U where it is a bundle filtration. Let $Q \twoheadrightarrow L$ be its Levi quotient, and $L \hookrightarrow Q \subset G$ a splitting. We call the semistable principal G-sheaf

$$\big(P^Q(Q \twoheadrightarrow L \hookrightarrow G), \oplus E_i/E_{i-1}, \psi'\big)$$

the associated *admissible deformation* of $\mathcal{P}$, where ψ' is the natural isomorphism between $P^Q(Q \twoheadrightarrow L \hookrightarrow G)(\mathfrak{g}')$ and $\oplus E_i/E_{i-1}|_U$. This principal G-sheaf is semistable. If we iterate this process, it stops after a finite number of steps, i.e. a semistable G-sheaf $\operatorname{grad}\mathcal{P}$ (only depending on $\mathcal{P}$) is obtained such that all its admissible deformations are isomorphic to itself (cf. Proposition 4.3).

Definition 0.7. Two semistable G-sheaves $\mathcal{P}$ and $\mathcal{P}'$ are said S-equivalent if $\operatorname{grad}\mathcal{P} \cong \operatorname{grad}\mathcal{P}'$.

When $\dim X = 1$ this is just Ramanathan's notion of S-equivalence of semistable principal G-bundles. Our main result generalizes Ramanathan's [Ra3] to arbitrary dimension:

THEOREM 0.8. *For a polarized complex smooth projective variety X there is a coarse projective moduli space of S-equivalence classes of semistable G-sheaves on X with fixed numerical invariants.*

Principal $\mathrm{GL}(R)$-sheaves are not objects equivalent to torsion free sheaves of rank R, but only in the case of bundles. As we remark at the end of Section 5, even in this case, the (semi)stability of both objects do not coincide. The philosophy is that, just as Gieseker changed in the theory of stable vector bundles both the objects (torsion free sheaves instead of vector bundles) and the condition of (semi)stability (involving Hilbert polynomials instead of degrees) in order to make $\dim X$ a parameter of the theory, it is now needed to change again the objects (principal sheaves) and the condition of (semi)stability (as that of the adjoint sheaf of orthogonal algebras) in order to make the group G a parameter of the theory (such variations of the conditions of stability and semistability are in both generalizations very slight, as these are implied by slope stability and imply slope semistability, and the slope conditions do not vary). The deep reason is that what we intend to do is not generalizing the notion of vector bundle of rank R (which was the task of Gieseker and Maruyama), but that of principal $\mathrm{GL}(R)$-bundle, and although both notions happen to be extensionally the same, i.e. happen to define equivalent objects, they are essentially different. This subtle fact is recognized by the very sensitive condition of existence of a moduli space, i.e. by (semi)stability.

The results of this article where announced in [G-S2]. There is independent work by Hyeon [Hy], who constructs, for higher dimensional varieties, the moduli space of principal bundles whose associated adjoint is a Mumford stable vector bundle, using the techniques of Ramanathan [Ra3], and also by Schmitt [Sch] who chooses a faithful representation of G in order to obtain and compactify a moduli space of principal G-bundles.

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Preliminaries

Notation. We denote by $(\mathrm{Sch}/\mathbb{C})$ the category of schemes over $\operatorname{Spec}\mathbb{C}$, locally of finite type. All schemes considered will belong to this category. If $f: Y \to Y'$ is a morphism, we denote $\overline{f} = \mathrm{id}_X \times f : X \times Y \to X \times Y'$. If E_S is a coherent sheaf on $X \times S$, we denote $E_S(m) := E_S \otimes p_X^* \mathcal{O}_X(m)$. An open set $U \subset Y$ of a scheme Y will be called *big* if $\operatorname{codim} Y \setminus U \geq 2$. Recall that in the étale topology, an open covering of a scheme U is a finite collection of morphisms $\{f_i : U_i \to U\}_{i\in I}$ such that each f_i is étale, and U is the union of the images of the f_i.

Given a principal G-bundle $P \to Y$ and a left action σ of G in a scheme F, we denote

$$P(\sigma, F) \;:=\; P \times_G F \;=\; (P \times F)/G,$$

the associated fiber bundle. If the action σ is clear from the context, we will write $P(F)$. In particular, for a representation ρ of G in a vector space V, $P(V)$ is a vector bundle on Y, this justifying the notation $P(\mathfrak{g}')$ in the introduction (understanding the adjoint representation of G in $\mathfrak{g}'$) and associating a line bundle $P(\sigma)$ on Y to any character σ of G. If $\rho : G \to H$ is a group homomorphism, let σ be the action of G on H defined by left multiplication $h \mapsto \rho(g)h$. Then the associated fiber bundle is a principal H-bundle, and it is denoted $\rho_* P$.

Let $\rho : H \to G$ be a homomorphism of groups, and let P be a principal G-bundle on a scheme Y. A reduction of structure group of P to H is a pair (P^H, ζ), where P^H is a principal H-bundle on Y and ζ is an isomorphism between $\rho_* P^H$ and P. Two reductions (P^H, ζ) and (Q^H, θ) are isomorphic if there is an isomorphism α giving a commutative diagram

$$\begin{tikzcd} P^H \arrow[d, "\alpha", "\cong"'] & \rho_* P^H \arrow[r, "\zeta"] \arrow[d, "\rho_*\alpha"] & P \arrow[d, equal] \\ Q^H & \rho_* Q^H \arrow[r, "\theta"] & P\,. \end{tikzcd} \tag{0.2}$$

Let $p : Y \to S$ be a morphism of schemes, and let P_S be a principal G-bundle on the scheme Y. Define the functor of families of reductions

$$\begin{aligned} \Gamma(\rho, P_S) : (\mathrm{Sch}/S) &\longrightarrow (\mathrm{Sets}) \\ (t : T \longrightarrow S) &\longmapsto \big\{(P_T^H, \zeta_T)\big\}/\text{isomorphism} \end{aligned}$$

where (P_T^H, ζ_T) is a reduction of structure group of $P_T := P_S \times_S T$ to H.

If ρ is injective, then $\Gamma(\rho, P_S)$ is a sheaf, and it is in fact representable by a scheme $S' \to S$, locally of finite type [Ra3, Lemma 4.8.1]. If ρ is not injective, this functor is not necessarily a sheaf, and we denote by $\widetilde{\Gamma}(\rho, P_S)$ its sheafification with respect to the étale topology on (Sch/S).

Lemma 0.9. *Let Y be a scheme, and let $f : \mathcal{K} \to \mathcal{F}$ be a homomorphism of sheaves on $X \times Y$. Assume that $\mathcal{F}$ is flat over Y. Then there is a unique closed subscheme Z satisfying the following universal property: given a Cartesian diagram*

$$\begin{tikzcd} X \times S \arrow[r, "\overline{h}"] \arrow[d, "p_S"] & X \times Y \arrow[d, "p"] \\ S \arrow[r, "h"] & Y \end{tikzcd}$$

it is $\overline{h}^* f = 0$ if and only if h factors through Z.

Proof. Uniqueness is clear. Recall that, if $\mathcal{G}$ is a coherent sheaf on $X \times Y$, we denote $\mathcal{G}(m) = \mathcal{G} \otimes p_X^* \mathcal{O}_X(m)$. Since $\mathcal{F}$ is Y-flat, taking m' large enough, $p_* \mathcal{F}(m')$ is locally free. The question is local on Y, so we can assume, shrinking Y if necessary, that $Y = \operatorname{Spec} A$ and $p_* \mathcal{F}(m')$ is given by a free A-module. Now, since Y is affine, the homomorphism

$$p_* f(m') : p_* \mathcal{K}(m') \longrightarrow p_* \mathcal{F}(m')$$

of sheaves on Y is equivalent to a homomorphism of A-modules

$$M \stackrel{(f_1,\dots,f_n)}{\longrightarrow} A \oplus \cdots \oplus A\,.$$

The zero locus of f_i is defined by the ideal $I_i \subset A$ image of f_i, thus the zero scheme $Z'_{m'}$ of $(f_1, \dots, f_n)$ is the closed subscheme defined by the ideal $I = \sum I_i$.

Since $\mathcal{O}_X(1)$ is very ample, if $m'' > m'$ we have an injection $p_* \mathcal{F}(m') \hookrightarrow p_* \mathcal{F}(m'')$ (and analogously for $\mathcal{K}$), hence $Z_{m''} \subset Z_{m'}$, and since Y is noetherian, there exists N' such that, if $m' > N'$, we get a scheme Z independent of m'.

We show now that if $\overline{h}^* f = 0$ then h factors through Z. Since the question is local on S, we can take $S = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$, and the morphism h is locally given by a ring homomorphism $A \to B$. Since $\mathcal{F}$ is flat over Y, for m' large enough the natural homomorphism $\alpha : h^* p_* \mathcal{F}(m') \to p_{S*} \overline{h}^* \mathcal{F}(m')$ (defined as in [Ha, Th. III 9.3.1]) is an isomorphism. This is a consequence of the equivalence between a) and d) of the base change theorem of [EGA III, 7.7.5 II]. For the reader more familiar with [Ha], we provide the following proof: For m' sufficiently large, $H^i(X, \mathcal{F}_y(m')) = 0$ and $H^i(X, \overline{h}^* (\mathcal{F}(m'))_s) = 0$ for all closed points $y \in Y$, $s \in S$ and $i > 0$, and since $\mathcal{F}$ is flat, this implies that $h^* p_* \mathcal{F}(m')$ and $p_{S*} \overline{h}^* \mathcal{F}(m')$ are locally free. Therefore, in order to prove that the homomorphism α is an isomorphism, it is enough to prove it at the fiber of every closed point $s \in S$, but this follows from [Ha, Th. III 12.11] or [Mu2, II §5, Cor. 3], hence proving the claim.

Hence the commutativity of the diagram

$$\begin{tikzcd}
p_{S*}\overline{h}^*\mathcal{K}(m') \arrow[r, "p_{S*}\overline{h}^* f(m')=0"] & p_{S*}\overline{h}^*\mathcal{F}(m') \\
h^* p_* \mathcal{K}(m') \arrow[u] \arrow[r, "h^* p_* f(m')"] & h^* p_* \mathcal{F}(m') \arrow[u, "\cong"']
\end{tikzcd}$$

implies that $h^* p_* f(m') = 0$. This means that for all i, in the diagram

$$\begin{tikzcd}
M \arrow[r, "f_i"] \arrow[d] & A \arrow[r] \arrow[d] & A/I_i \arrow[r] \arrow[d] & 0 \\
M \otimes_A B \arrow[r, "f_i \otimes B"] & B \arrow[r] & A/I_i \otimes_A B \arrow[r] & 0
\end{tikzcd}$$

it is $f_i \otimes B = 0$. Hence the image I_i of f_i is in the kernel J of $A \to B$. Therefore $I \subset J$, hence $A \to B$ factors through $A \to A/I$, which means that $h : S \to Y$ factors through Z.

Now we show that if we take $S = Z$ and $h : Z \hookrightarrow Y$ the inclusion, then $\overline{h}^* f = 0$. By definition of Z, we have $h^* p_* f(m') = 0$ for any m' with $m' > N'$. Showing that $\overline{h}^* f = 0$ is equivalent to showing that

$$\overline{h}^* f(m') : \overline{h}^* \mathcal{K}(m') \longrightarrow \overline{h}^* \mathcal{F}(m')$$

is zero for some m'. Take m' large enough so that ev : $p^* p_* \mathcal{K}(m') \to \mathcal{K}(m')$ is surjective. By the right exactness of $\overline{h}^*$, the homomorphism $\overline{h}^*$ev is still surjective. The commutative diagram

$$\begin{tikzcd}
\overline{h}^* \mathcal{K}(m') \arrow[rr, "\overline{h}^* f(m')"] & & \overline{h}^* \mathcal{F}(m') \\
\overline{h}^* p^* p_* \mathcal{K}(m') \arrow[u, two heads, "\overline{h}^* \mathrm{ev}"] \arrow[rr, "\overline{h}^* p^* p_* f(m')"] \arrow[d, equal] & & \overline{h}^* p^* p_* \mathcal{F}(m') \arrow[u] \arrow[d, equal] \\
p_S^* h^* p_* \mathcal{K}(m') \arrow[rr, "p_S^* h^* p_* f(m') = 0"] & & p_S^* h^* p_* \mathcal{F}(m')
\end{tikzcd}$$

implies $\overline{h}^* f(m') = 0$, as wanted. □

The following easy lemmas and corollary will help to relate the three main objects that will be introduced in this section.

Lemma 0.10. *Let E and F be coherent sheaves on a scheme Y, and L a locally free sheaf on Y. There is a natural isomorphism*

$$\mathrm{Hom}(E \otimes F, L) \cong \mathrm{Hom}(E, \mathcal{H}om(F, L)) \cong \mathrm{Hom}(E, F^\vee \otimes L)\,.$$

Lemma 0.11. *Let $f : Y \to S$ be a flat morphism of noetherian schemes such that, for every point s of S, the fiber Y_s is normal. Let E be a coherent sheaf on Y.*

(1) *If $i : U \hookrightarrow Y$ is the immersion of a relatively big open set of Y (i.e. an open set whose complement intersects the fibers in codimension at least* 2) *and $E|_U$ is locally free, then the natural homomorphism $E^\vee \to i_*(E^\vee|_U)$ is isomorphic.*

(2) *If E is S-flat, and $E \otimes k(s)$ is torsion free for every point s of S, then the maximal open set $U = U_E$ where E is locally free is relatively big, and the natural homomorphism $E^{\vee\vee} \to i_*(E|_U)$ is isomorphic, the natural homomorphism $E \to E^{\vee\vee}$ being just the natural $E \to i_*(E|_U)$.*

Proof. The fact that U is relatively big is equivalent to having $\dim \mathcal{O}_{Y_s,z} \geq 2$ for all points $z \in Z$. This, together with the fact that Y_s is normal, implies that $\operatorname{depth} \mathcal{O}_{Y_s,z} \geq 2$. Since f is flat, we see that $\operatorname{depth} \mathcal{O}_{Y,z} \geq 2$ by [EGA IV, 6.3.1]. From the exact sequence of $\mathcal{O}_{Y,z}$-modules

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{Y,z}^{\oplus r} \longrightarrow E_z \longrightarrow 0$$

we obtain another sequence

$$0 \longrightarrow E_z^\vee \longrightarrow \mathcal{O}_{Y,z}^{\oplus r} \longrightarrow Q \longrightarrow 0$$

where G is an $\mathcal{O}_{Y,z}$-submodule of $K^\vee$. We make now and elementary observation based on the fact that depth is at least n if an only if local cohomology of order at most $n-1$ vanishes: since $\operatorname{depth} K^\vee \geq 1$, also $\operatorname{depth} Q \geq 1$, and this, together with the fact that $\operatorname{depth} \mathcal{O}_{Y,z} \geq 2$, imply, by taking local cohomology in the last exact sequence, that $\operatorname{depth} E_z^\vee \geq 2$. Therefore $E^\vee$ is Z-close by [EGA IV, 5.10.5], that is, the map in (1) is bijective.

To prove (2), observe that U is relatively big because its intersection $U_E \cap Y_s$ with each fiber Y_s is, by S-flatness of E, the big open set where the torsion free sheaf $E \otimes k(s)$ is locally free (this follows, for instance, from [H-L, Lemma 2.1.7]). Note that natural homomorphism $E \to E^{\vee\vee}$ is an isomorphism on U. Therefore (2) follows from (1). ☐

Lemma 0.12. *If E is a coherent sheaf of rank r as in the hypothesis of Lemma* 0.11(2), *then there is a canonical isomorphism*

$$(\bigwedge{}^{r-1} E)^\vee \otimes \det E \xrightarrow{\cong} E^{\vee\vee}\,.$$

Proof. This is clearly true if we restrict to the maximal open set $U = U_E$ where E is locally free:

$$(\bigwedge{}^{r-1} E)^\vee|_U \otimes \det E|_U \xrightarrow{\cong} E|_U\,.$$

Therefore, taking i_* and applying Lemma 0.11(1) to $(\bigwedge{}^{r-1} E)^\vee$, we obtain

$$(\bigwedge{}^{r-1} E)^\vee \otimes \det E \,\cong\, i_*\big((\bigwedge{}^{r-1} E)^\vee|_U \otimes \det E|_U\big) \xrightarrow{\cong} i_*(E|_U) \,\cong\, E^{\vee\vee}\,,$$

where the last isomorphism is provided by Lemma 0.11(2). ☐

Combining Lemmas 0.10 and 0.12 we obtain the following

Corollary 0.13. *Let E be a coherent sheaf of rank r as in the hypothesis of Lemma* 0.11(2), *and L a line bundle on Y. Giving a homomorphism*

$$\eta : E \otimes E \otimes E^{\otimes r-1} \;= E^{\otimes r+1} \longrightarrow \det E \otimes L$$

which is skew-symmetric in the last $r-1$ entries, i.e. which factors through $E \otimes E \otimes \bigwedge^{r-1} E$, is equivalent to giving a homomorphism

$$\varphi : E \otimes E \longrightarrow E^{\vee\vee} \otimes L\,.$$

Proof. Lemma 0.10 associates to η a homomorphism

$$\varphi : E\otimes E \longrightarrow (\bigwedge^{r-1}E)^\vee \otimes \det E\otimes L \xrightarrow{\cong} E^{\vee\vee}\otimes L$$

where the isomorphism is given by Lemma 0.12. Conversely, given a homomorphism such as φ, Lemma 0.12 provides the desired homomorphism. □

Now we introduce the three progressively richer concepts of a Lie tensor, a $\mathfrak{g}'$-sheaf, and a principal G-sheaf, all relative to a scheme S. As usual, if no mention to the base scheme S is made, it will be understood $S=\operatorname{Spec}\mathbb{C}$. For each of these three concepts we give compatible notions of (semi)stability, leading in each case to a projective coarse moduli space.

Definition 0.14 (Lie tensor). A family of tensors parametrized by a scheme S is a triple (F_S,ϕ_S,N_S) consisting of an S-flat coherent sheaf F_S on $X\times S$, such that for every point s of S, $F_S\otimes k(s)$ is torsion free with trivial determinant (i.e., $\det F_S = p_S^*L$ for a line bundle L on S) and fixed Hilbert polynomial P, a line bundle N_S on S, and a homomorphism ϕ_S

$$\phi_S : {F_S}^{\otimes a} \longrightarrow p_S^* N_S\,. \tag{0.3}$$

A tensor is called a Lie tensor if $a=r+1$ for r the rank of F_S, and

(1) ϕ_S is skew-symmetric in the last $r-1$ entries, i.e. it factorizes through $F_S\otimes F_S\otimes \bigwedge^{r-1}F_S$,

(2) the homomorphism $\widetilde{\phi}_S : F_S\otimes F_S\to F_S^{\vee\vee}\otimes \det F_S^\vee \otimes p_S^*N_S$ associated to ϕ_S by Corollary 0.13 is antisymmetric,

(3) $\widetilde{\phi}_S$ satisfies the Jacobi identity.

To give a precise definition of the Jacobi identity, first define a homomorphism

$$[[\cdot,\cdot],\cdot] : F_S\otimes F_S\otimes F_S \xrightarrow{\widetilde{\phi}_S\otimes F_S} F_S^{\vee\vee}\otimes(\det F_S^\vee\otimes p_S^*N_S)\otimes F_S \xrightarrow{F_S^{\vee\vee}\otimes(\det F_S^\vee\otimes p_S^*N_S)\otimes\widetilde{\phi}_S}$$
$$F_S^{\vee\vee}\otimes F_S^\vee\otimes F_S^{\vee\vee}\otimes(\det F_S^\vee\otimes p_S^*N_S)^2 \longrightarrow F_S^{\vee\vee}\otimes(\det F_S^\vee\otimes p_S^*N_S)^2$$

where the last homomorphism comes from the natural pairing of the first two factors. Then define

$$\begin{aligned} J : F_S\otimes F_S\otimes F_S &\longrightarrow F_S^{\vee\vee}\otimes(\det F_S^\vee\otimes p_S^*N_S)^2 \\ (u,v,w) &\longmapsto [[u,v],w]+[[v,w],u]+[[w,u],v] \end{aligned} \tag{0.4}$$

and require $J=0$.

An isomorphism between two families of tensors (F_S,ϕ_S,N_S) and (F'_S,ϕ'_S,N'_S) parametrized by S is a pair of isomorphisms $\alpha : F_S\to F'_S$ and

$\beta : N_S \to N'_S$ such that the induced diagram

$$\begin{tikzcd} F_S^{\otimes a} \arrow[r, "\phi_S"] \arrow[d, "\alpha^{\otimes a}"] & p_S^* N_S \arrow[d, "p_S^*\beta"] \\ F_S'^{\otimes a} \arrow[r, "\phi'_S"] & p_S^* N'_S \end{tikzcd}$$

commutes. In particular, (F, ϕ) and $(F, \lambda\phi)$ are isomorphic for $\lambda \in \mathbb{C}^*$. Given an S-family of tensors (F_S, ϕ_S, N_S) and a morphism $f : S' \to S$, the pullback is the S'-family defined as $(\overline{f}^* F_S, \overline{f}^* \phi_S, f^* N_S)$.

Since we will work with GIT (Geometric Invariant Theory, [Mu1]), the notion of filtration $F_\bullet$ of a sheaf is going to be essential for us. By this we always understand a $\mathbb{Z}$-indexed filtration

$$\cdots \subset F_{i-1} \subset F_i \subset F_{i+1} \subset \ldots$$

starting with 0 and ending with F. If the filtration is *saturated* (i.e. with all F_i/F_{i-1} being torsion free), only a finite number of inclusions can be strict

$$0 \subsetneq F_{\lambda_1} \subsetneq F_{\lambda_2} \subsetneq \ \ldots \ \subsetneq F_{\lambda_t} \subsetneq F_{\lambda_{t+1}} = F \qquad \lambda_1 < \cdots < \lambda_{t+1}$$

where we have deleted, from 0 onward, all the non-strict inclusions. Reciprocally, from a saturated $F_{\lambda_\bullet}$ we recover the saturated $F_\bullet$ by defining $F_m = F_{\lambda_{i(m)}}$, *where* $i(m)$ *is the maximum index with* $\lambda_{i(m)} \leq m$*.*

Definition 0.15 (Balanced filtration). A saturated filtration $F_\bullet \subset F$ of a torsion free sheaf F is called a balanced filtration if $\sum i \operatorname{rk} F^i = 0$ for $F^i = F_i/F_{i-1}$. In terms of $F_{\lambda_\bullet}$, this is $\sum_{i=1}^{t+1} \lambda_i \operatorname{rk}(F^{\lambda_i}) = 0$ for $F^{\lambda_i} = F_{\lambda_i}/F_{\lambda_{i-1}}$.

Remark 0.16. The notion of balanced filtration appeared naturally in the Gieseker-Maruyama construction of the moduli space of (semi)stable sheaves, the condition of (semi)stability of a sheaf F being that all balanced filtrations of F have negative (nonpositive) Hilbert polynomial. In this case the condition "balanced" could be suppressed, since $P_{F_\bullet} = P_{F_{\bullet+l}}$ for any shift l in the indexing (and furthermore it is enough to consider filtrations of one element, i.e. just subsheaves).

Let $\mathcal{I}_a = \{1, \ldots, t+1\}^{\times a}$ be the set of all multi-indexes $I = (i_1, \ldots, i_a)$ of cardinality a. Define

$$\mu_{\text{tens}}(\phi, F_{\lambda_\bullet}) = \min_{I \in \mathcal{I}_a} \left\{ \lambda_{i_1} + \cdots + \lambda_{i_a} : \phi|_{F_{\lambda_{i_1}} \otimes \cdots \otimes F_{\lambda_{i_a}}} \neq 0 \right\}. \tag{0.5}$$

Definition 0.17 (Stability for tensors). Let δ be a polynomial of degree at most $n-1$ and positive leading coefficient. We say that (F, ϕ) is δ-(semi)stable if ϕ is not identically zero and for all balanced filtrations $F_{\lambda_\bullet}$ of F, it is

$$\Big(\sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)\big(rP_{F_{\lambda_i}}-r_{\lambda_i}P\big)\Big)+\mu_{\mathrm{tens}}(\phi,F_{\lambda_\bullet})\,\delta\;(\preceq)\;0\,. \tag{0.6}$$

It was proved in [G-S1] that there is a coarse moduli space of δ-semistable tensors.

Now we go to our second main concept, that of a $\mathfrak{g}'$-sheaf. It will appear as a particular case of Lie algebra sheaf, so this we define first. A family of *Lie algebra sheaves*, parametrized by S, is a pair

$$\big(E_S,\ \varphi_S : E_S\otimes E_S \longrightarrow \det E_S^{\vee\vee}\big)$$

where E_S is a coherent sheaf on $X\times S$, flat over S, such that for every point s of S, $E_S\otimes k(s)$ is torsion free, and the homomorphism φ_S, which is also denoted $[,]$, is antisymmetric and satisfies the Jacobi identity. Therefore, it gives a Lie algebra structure on the fibers of E_S where it is locally free.

The precise definition of the Jacobi identity is as in Definition 0.14, but with $\mathcal{O}_{X\times S}$ instead of $\det F_S^\vee\otimes p_S^* N_S$. An isomorphism between two families is an isomorphism $\alpha: E_S\to E'_S$ with

$$\begin{tikzcd}
E_S\otimes E_S \arrow[r, "\varphi_S"] \arrow[d, "\alpha\otimes\alpha"] & E_S^{\vee\vee} \arrow[d, "\alpha^{\vee\vee}"] \\
E'_S\otimes E'_S \arrow[r, "\varphi'_S"] & E_S'^{\vee\vee}\,.
\end{tikzcd}$$

Note that, since the conditions of being antisymmetric and satistying the Jacobi identity are closed, in order to have them for an S-family, it is not enough to check that they are satisfied for all closed points of S, because S could be nonreduced.

Definition 0.18. The Killing form κ_S associated to a Lie algebra sheaf (E_S,φ_S) is the composition

$$E_S\otimes E_S \stackrel{\varphi_S\otimes\varphi_S}{\longrightarrow} E_S^\vee\otimes E_S^{\vee\vee}\otimes E_S^\vee\otimes E_S^{\vee\vee}\longrightarrow E_S^\vee\otimes E_S^{\vee\vee}\longrightarrow \mathcal{O}_{X\times S}$$

where φ_S also denotes its own transpose (Corollary 0.13).

If the Lie algebra is semisimple, in the sense that the induced homomorphism $E_S^{\vee\vee}\to E_S^\vee$ is an isomorphism, the fiber of E_S over a closed point $(x,s)\in X\times S$ where E_S is locally free has the structure of a semisimple Lie algebra, which, because of the rigidity of semisimple Lie algebras, must be constant on connected components of S. This justifies the following

Definition 0.19 ($\mathfrak{g}'$-sheaf). A family of $\mathfrak{g}'$-sheaves is a family of Lie algebra sheaves where the Lie algebra associated to each connected component of the parameter space S is $\mathfrak{g}'$.

The following is the sheaf version of the well-known notion of Lie algebra filtration (see [J] for instance, recalled in Section 5).

Definition 0.20 (Algebra filtration). A filtration $E_\bullet \subset E$ of a Lie algebra sheaf $(E,[,])$ is called an algebra filtration if for all i, j,

$$[E_i, E_j] \subset {E_{i+j}}^{\vee\vee}.$$

In terms of $E_{\lambda_\bullet}$, this is

$$[E_{\lambda_i}, E_{\lambda_j}] \subset {E_{\lambda_{k-1}}}^{\vee\vee}$$

for all λ_i, λ_j, λ_k with $\lambda_i + \lambda_j < \lambda_k$.

Definition 0.21. A $\mathfrak{g}'$-sheaf is (semi)stable if for all balanced algebra filtrations $E_\bullet$ it is

$$\sum_{i=1}^{t} \big(rP_{E_i} - r_i P_E\big) \;(\preceq)\; 0$$

or, in terms of $E_{\lambda_\bullet}$,

$$\sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i} P_E\big) \;(\preceq)\; 0\,. \tag{0.7}$$

Remark 0.22. We will see in Corollary 5.10 that for an algebra filtration of a $\mathfrak{g}'$-sheaf, the fact of being balanced is equivalent to being orthogonal, i.e. $E_{-i-1} = E_i^\perp = \ker(E \hookrightarrow E^{\vee\vee} \overset{\kappa}{\cong} E^\vee \to E_i^\vee)$. Thus, in the previous definition we can change "balanced algebra filtration" by "orthogonal algebra filtration."

Remark 0.23. Observe that the condition "balanced" cannot be suppressed in this case, as it was in Remark 0.16, because a shifted filtration $E_{\bullet+l}$ of an algebra filtration is no longer an algebra filtration.

Construction 0.24 (Correspondence between Lie algebra sheaves and Lie tensors). Consider a Lie tensor

$$(F_S, \phi_S : {F_S}^{\otimes r+1} \longrightarrow p_S^* N_S, N_S)\,.$$

Corollary 0.13 gives

$$\big(F_S, \widetilde{\phi}_S : F_S \otimes F_S \longrightarrow F_S^{\vee\vee} \otimes (\det F_S^\vee \otimes p_S^* N_S), N_S\big)\,.$$

If we define $E_S = F_S \otimes (\det F_S \otimes p_S^* N_S^{-1})$, and $\varphi_S = \widetilde{\phi}_S \otimes (\det F_S \otimes p_S^* N_S^{-1})^2$ we obtain a Lie algebra sheaf

$$(E_S, \varphi_S : E_S \otimes E_S \longrightarrow E_S^{\vee\vee})\,. \tag{0.8}$$

Conversely, given a Lie algebra sheaf as in (0.8), if we define $F_S = E_S$ and $N_S = L_S$ where L_S is the line bundle on S such that $\det E_S = p_S^* L_S$, then Corollary 0.13 gives a Lie tensor

$$(F_S, \phi_S : F_S^{\otimes r+1} \longrightarrow p_S^* N_S, N_S).$$

Note that the notion of a Lie algebra sheaf is similar but not the same as that of a Lie tensor. The difference is that an isomorphism of Lie tensors is a pair (α, β), whereas an isomorphism of Lie algebra sheaves is just α (this is the reason why Lie tensors take values on a line bundle $p_S^* N_S$ with N_S arbitrary, whereas Lie algebra sheaves take values in $\det E_S$). In particular, the automorphism group of a Lie tensor is not the same as that of the associated Lie algebra sheaf. If $S = \operatorname{Spec} \mathbb{C}$, Construction 0.24 gives a bijection of isomorphism classes, but not for arbitrary S, because E_S is not in general isomorphic to F_S. They are only locally isomorphic, in the sense that we can cover S with open sets S_i (where the line bundles L_S and N_S are trivial), so that the objects restricted to S_i are isomorphic, which provides an *isomorphism between the sheafified functors. We will show that*, *for a* $\mathfrak{g}'$*-sheaf*, *its* (*semi*)*stability is equivalent to that of the corresponding tensor.* This is the key initial point of this article, allowing us to use in Section 1 the results in [G-S1] in order to construct the moduli space of $\mathfrak{g}'$-sheaves, then that of principal sheaves in Sections 2, 3 and 4.

Recall, from the introduction, the notion of a principal G-sheaf $\mathcal{P} = (P_S, E_S, \psi_S)$ for a reductive connected group G and its notion of (semi)stability. Let $\mathfrak{g}'$ be the semisimple part of its Lie algebra. We associate now to $\mathcal{P}$ a $\mathfrak{g}'$-sheaf (E_S, φ_S) by the following

Lemma 0.25. *Let* $\mathcal{U} = U_{E_S}$ *be the open set where* E_S *is locally free. The homomorphism* $\varphi_{\mathcal{U}} : E_S|_{\mathcal{U}} \otimes E_S|_{\mathcal{U}} \to E_S|_{\mathcal{U}}$, *given by the Lie algebra structure of* $P_S(\mathfrak{g}')$ *and the isomorphism* ψ_S, *extends uniquely to a homomorphism*

$$\varphi_S : E_S \otimes E_S \longrightarrow E_S^{\vee\vee}.$$

Proof. Let $i : \mathcal{U} \hookrightarrow X \times S$ be the natural open immersion. The homomorphism φ_S is defined as the composition

$$\varphi_S : E_S \otimes E_S \longrightarrow i_*(E_S|_{\mathcal{U}} \otimes E_S|_{\mathcal{U}}) \longrightarrow i_*(E_S|_{\mathcal{U}}) \xrightarrow{\cong} E_S^{\vee\vee}$$

the last homomorphism being an isomorphism by Lemma 0.11. □

The following corollary of Remark 0.22 provides thus an equivalent definition of (semi)stability

Corollary 0.26. *A principal* G*-sheaf* $\mathcal{P} = (P, E, \psi)$ *is* (*semi*)*stable* (*Definition* 0.3) *if and only if the associated* $\mathfrak{g}'$*-sheaf* (E, φ) *is* (*semi*)*stable (Definition* 0.21).

Remark 0.27. Lemma 0.25 implies that there is a natural bijection between the isomorphism classes of families of $\mathfrak{g}'$-sheaves and those of principal $\mathrm{Aut}(\mathfrak{g}')$-sheaves.

LEMMA 0.28. *Let G be a connected reductive algebraic group. Let P be a principal G-bundle on X and let $E = P(\mathfrak{g}')$ be the vector bundle associated to P by the adjoint representation of G on the semisimple part $\mathfrak{g}'$ of its Lie algebra. Then* $\det E \cong \mathcal{O}_X$.

Proof. We have $\mathrm{Aut}(\mathfrak{g}') \subset \mathrm{O}(\mathfrak{g}')$, where the orthogonal structure on $\mathfrak{g}'$ is given by its nondegenerate Killing form. Note that $P(\mathfrak{g}')$ is obtained by extension of structure group using the composition

$$\rho : G \longrightarrow \mathrm{Aut}(\mathfrak{g}') \hookrightarrow \mathrm{O}(\mathfrak{g}') \hookrightarrow \mathrm{GL}(\mathfrak{g}').$$

Since G is connected, the image of G in $\mathrm{O}(\mathfrak{g}')$ lies in the connected component of identity, i.e. in $\mathrm{SO}(\mathfrak{g}')$. Hence $P(\mathfrak{g}')$ admits a reduction of structure group to $\mathrm{SO}(\mathfrak{g}')$, and thus $\det P(\mathfrak{g}') \cong \mathcal{O}_X$. □

We end this section by extending to principal sheaves some well-known definitions and properties of principal bundles and by recalling some notions of GIT [Mu1] to be used later. Let $m : H \times R \to R$ be an action of an algebraic group H on a scheme R, and let $p_R : H \times R \to R$ be the projection to the second factor. If $h : S \to H$ and $t : S \to R$ and S-valued points of H and R, denote by $h[t]$ the S-valued point produced using the action, i.e. $h[t] : m \circ (h, t) : S \to R$.

Definition 0.29 (Universal family). Let $\mathcal{P}_R$ be a family of principal G-sheaves parametrized by R. Assume there is a lifting of the action of H to $\mathcal{P}_R$, i.e. there is an isomorphism

$$\Lambda : \overline{m}^* \mathcal{P}_R \xrightarrow{\cong} \overline{p}_R^* \mathcal{P}_R \,.$$

Assume that:

(1) Given a family $\mathcal{P}_S$ parametrized by S and a closed point $s \in S$, there is an open étale neighborhood $i : S_0 \to S$ of s and a morphism $t : S_0 \to R$ such that $\overline{i}^* \mathcal{P}_S \;\cong\; \overline{t}^* \mathcal{P}_R$.

(2) Given two morphisms t_1, $t_2 : S \to R$ and an isomorphism $\beta : \overline{t_2}^* \mathcal{P} \to \overline{t_1}^* \mathcal{P}$, there is a unique $h : S \to H$ such that $t_2 = h[t_1]$ and $\overline{(h, t_1)}^* \Lambda = \beta$.

Then we say that $\mathcal{P}_R$ is a universal family with group H for the functor $\widetilde{F}^{\tau}_G$.

Definition 0.30 (Universal space). Let $F : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ be a functor. Let $\underline{R}/\underline{H}$ be the sheaf on $(\mathrm{Sch}/\mathbb{C})$ associated to the presheaf $S \mapsto \mathrm{Mor}(S, R)/\mathrm{Mor}(S, H)$. We say that R is a universal space with group H for the functor F if F is isomorphic to $\underline{R}/\underline{H}$.

The difference between these two notions can be understood as follows. Recall that a groupoid is a category all whose morphisms are isomorphisms. Given a stack $\mathcal{M} : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Groupoids})$ we denote by $\overline{\mathcal{M}} : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ the functor associated by replacing each groupoid by the set of isomorphism classes of its objects. Let $[R/H]$ be the quotient stack and let $\mathcal{F}$ be the stack of semistable principal G-sheaves. Then R is a universal space with group H if $\overline{[R/H]} \cong \overline{\mathcal{F}}$, whereas it is a universal family if $[R/H] \cong \mathcal{F}$, i.e. if the isomorphism holds at the level of stacks, without taking isomorphism classes.

Definition 0.31 (Categorical quotient). A morphism $f : R \to Y$ of schemes is a categorical quotient for an action of an algebraic group H on R if:

(1) It is H-equivariant when we provide Y with the trivial action.

(2) If $f' : R \longrightarrow Y'$ is another morphism satisfying (1), then there is a unique morphism $g : Y \to Y'$ such that $f' = g \circ f$.

Definition 0.32 (Good quotient). A morphism $f : R \to Y$ of schemes is a good quotient for an action of an algebraic group H on R if:

(1) f is surjective, affine and H-equivariant, when we provide Y with the trivial action.

(2) $f_*(\mathcal{O}_R^H) = \mathcal{O}_Y$, where $\mathcal{O}_R^H$ is the sheaf of H-invariant functions on R.

(3) If Z is a closed H-invariant subset of R, then $p(Z)$ is closed in Y. Furthermore, if Z_1 and Z_2 are two closed H-invariant subsets of R with $Z_1 \cap Z_2 = \emptyset$, then $f(Z_1) \cap f(Z_2) = \emptyset$.

Definition 0.33 (Geometric quotient). A geometric quotient $f : R \to Y$ is a good quotient such that $f(x_1) = f(x_2)$ if and only if the orbit of x_1 is equal to the orbit of x_2.

Clearly, geometric quotients are good quotients, and good quotients are categorical quotients. Assume that R is projective, H is reductive, and the action of H on R has a linearization on an ample line bundle $\mathcal{O}_R(1)$. A closed point $y \in R$ is called GIT-semistable if, for some $m > 0$, there is an H-invariant section s of $\mathcal{O}_R(m)$ such that $s(y) \neq 0$. If, moreover, the orbit of y is closed in the open set of all GIT-semistable points, and has the same dimension as H, i.e. y has finite stabilizer, then y is called a GIT-stable point. We will use the following characterization in [Mu1] of GIT-(semi)stability: let $\lambda : \mathbb{C}^* \to H$ be a one-parameter subgroup, and $y \in R$. Then $\lim_{t\to 0} \lambda(t) \cdot y = y_0$ exists, and y_0 is fixed by λ. Let $t \mapsto t^a$ be the character by which λ acts on the fiber of $\mathcal{O}_R(1)$. Defining $\mu(y, \lambda) = a$, Mumford proves that y is GIT-(semi)stable if and only if, for all one-parameter subgroups, it is $\mu(y, \lambda)(\leq)0$.

Proposition 0.34. *Let R^{ss} (respectively R^s) be the open subset of* GIT-*semistable points (respectively* GIT-*stable). Then there is a good quotient $R^{ss} \to R/\!\!/H$, and the restriction $R^s \to R^s/\!\!/H$ is a geometric quotient. Furthermore, $R/\!\!/H$ is projective and $R^s/\!\!/H$ is an open subset.*

Definition 0.35. A scheme Y corepresents a functor $F : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ if

(1) There exists a natural transformation $f : F \to \underline{Y}$ (where $\underline{Y} = \mathrm{Mor}(\cdot, Y)$ is the functor of points represented by Y).

(2) For every scheme Y' and natural transformation $f' : F \to \underline{Y'}$, there exists a unique $g : \underline{Y} \to \underline{Y'}$ such that f' factors through f.

Remark 0.36. Let R be a universal space with group H for F, and let $f : R \to Y$ be a categorical quotient. It follows from the definitions that Y corepresents F.

Proposition 0.37. *Let $\mathcal{P}_R = (P_R, E_R, \psi_R)$ be a universal family with group H for the functor $\widetilde{F}^{\tau}_{G_1}$. Let $\rho : G_2 \to G_1$ be a homomorphism of groups, such that the center Z_{G_2} of G_2 is mapped to the center Z_{G_1} of G_1 and the induced homomorphism*

$$\mathrm{Lie}(G_2/Z_{G_2}) \longrightarrow \mathrm{Lie}(G_1/Z_{G_1})$$

is an isomorphism. Assume that the functor $\widetilde{\Gamma}(\rho, P_R)$ is represented by a scheme M. Then

(1) *There is a natural action of H on M, making it a universal space with group M for the functor $\widetilde{F}^{\tau}_{G_2}$.*

(2) *Moreover, if ρ is injective (so that $\Gamma(\rho, P_R)$ itself is representable by M), then the action of H lifts to the family $\mathcal{P}_M$ given by $\Gamma(\rho, P_R)$, and then $\mathcal{P}_M$ becomes a universal family with group H for the functor $\widetilde{F}^{\tau}_{G_2}$.*

Proof. Analogous to [Ra3, Lemma 4.10]. □

1. Construction of R and R_1

In this section we find a group acted projective scheme R_1 parametrizing based semistable $\mathfrak{g}'$-sheaves.

Given a principal G-bundle, we obtain a pair $(E, \varphi : E \otimes E \to E)$, where $E = P(\mathfrak{g}')$ is the vector bundle associated to the adjoint representation of G on the semisimple part $\mathfrak{g}'$ of the Lie algebra of G, and φ is given by the Lie algebra structure. To obtain a projective moduli space we have to allow E to become a torsion free sheaf. For technical reasons, when E is not locally free, we make φ take values in $E^{\vee\vee}$.

The first step to construct the moduli space is the construction of a scheme parametrizing semistable based $\mathfrak{g}'$-sheaves, i.e. triples $(q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E, E, \varphi : E \otimes E \to E^{\vee\vee})$, where (E,φ) is a semistable $\mathfrak{g}'$-sheaf, having E the given numerical invariants, m is a suitable large integer depending only on these numerical invariants, and V is a fixed vector space of dimension $P_E(m)$, thus depending only on the invariants. We have already seen that a $\mathfrak{g}'$-sheaf can be described as a tensor in the sense of [G-S1], where a notion of (semi)stability for tensors is given, depending on a polynomial δ of degree at most $n-1$ and positive leading coefficient. *In this article we will always assume that δ has degree $n-1$.* Recall that to a Lie tensor (F,ϕ) we associate a Lie algebra sheaf (E,φ) with $E = F \otimes \det F$ (cf. Construction 0.24 with $S = \operatorname{Spec}\mathbb{C}$). Since $\det F \cong \mathcal{O}_X$, choosing an isomorphism we will identify E and F (a different choice gives an isomorphic object). Now we will prove, after some lemmas, that the (semi)stability of the $\mathfrak{g}'$-sheaf coincides with the δ-(semi)stability of the corresponding tensor (in particular for the tensors associated to $\mathfrak{g}'$-sheaves, its δ-(semi)stability does not depend on δ, as long as $\deg(\delta) = n-1$), so that we can apply the results of [G-S1].

Given a $\mathfrak{g}'$-sheaf (E,φ) and a balanced filtration $E_{\lambda_\bullet}$, define

$$\begin{aligned}(1.1)\quad \mu(\varphi, E_{\lambda_\bullet}) &= \min\left\{\lambda_i + \lambda_j - \lambda_k \ :\ 0 \neq \varphi : E_{\lambda_i} \otimes E_{\lambda_j} \longrightarrow E^{\vee\vee}/E_{\lambda_{k-1}}{}^{\vee\vee}\right\} \\ &= \min\left\{\lambda_i + \lambda_j - \lambda_k \ :\ [E_{\lambda_i}, E_{\lambda_j}] \not\subset E_{\lambda_{k-1}}{}^{\vee\vee}\right\}.\end{aligned}$$

Lemma 1.1. *If (E,ϕ) is the associated tensor, then $\mu(\varphi, E_{\lambda_\bullet})$ in (1.1) is equal to $\mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet})$ in (0.17).*

Proof. For a general $x \in X$, let $e_1, \ldots, e_r$ be a basis adapted to the flag $E_{\lambda_\bullet}(x)$, thus giving a splitting $E(x) = \oplus E^{\lambda_i}(x)$. Writing $r^{\lambda_i} = \dim E^{\lambda_i}(x)$,

$$\begin{aligned}
&\mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet}) \\
&\quad = \min\Big\{\lambda_i + \lambda_j + \lambda_1 r^{\lambda_1} + \cdots + \lambda_k (r^{\lambda_k} - 1) + \cdots + \lambda_{t+1} r^{\lambda_{t+1}} \ :\\
&\qquad\qquad e_1 \wedge e_2 \wedge \ldots \wedge e_{k'-1} \wedge \varphi_x(e_{i'} \otimes e_{j'}) \wedge e_{k'+1} \wedge \ldots \wedge e_r \neq 0 \ \text{ for some}\\
&\qquad\qquad\qquad e_{i'} \in E^{\lambda_i}(x),\ e_{j'} \in E^{\lambda_j}(x),\ 1 \le k' \le r\Big\} \\
&\quad = \min\Big\{\lambda_i + \lambda_j - \lambda_k \ :\ \varphi_x(E^{\lambda_i}(x), E^{\lambda_j}(x)) \not\subset E_{\lambda_{k-1}}(x)\\
&\qquad\qquad \text{and } \varphi_x(E^{\lambda_i}(x), E^{\lambda_j}(x)) \subset E_{\lambda_k}(x)\Big\} \\
&\quad = \min\Big\{\lambda_i + \lambda_j - \lambda_k \ :\ [E_{\lambda_i}, E_{\lambda_j}] \not\subset E_{\lambda_{k-1}}{}^{\vee\vee}\\
&\qquad\qquad \text{and } [E_{\lambda_i}, E_{\lambda_j}] \subset E_{\lambda_k}{}^{\vee\vee}\Big\} \\
&\quad = \mu(\varphi, E_{\lambda_\bullet}).
\end{aligned}$$

□

We will need the following result, due to Ramanathan [Ra3, Lemma 5.5.1], whose proof we recall for convenience of the reader.

LEMMA 1.2. *Let W be a vector space and let $p \in \mathbb{P}(W^\vee \otimes W^\vee \otimes W)$ be the point corresponding to a Lie algebra structure on W. If the Lie algebra is semisimple, this point is* GIT-*semistable for the natural action of* $\mathrm{SL}(W)$ *and linearization in $\mathcal{O}(1)$ on $\mathbb{P}(W^\vee \otimes W^\vee \otimes W)$.*

Proof. Define the $\mathrm{SL}(W)$-equivariant homomorphism

$$\begin{aligned} g:\, (W^\vee \otimes W^\vee \otimes W) = \mathrm{Hom}(W, \mathrm{End}\, W) &\longrightarrow (W \otimes W)^\vee \\ f &\mapsto g(f)(\cdot \otimes \cdot) = \mathrm{tr}(f(\cdot) \circ f(\cdot))\,. \end{aligned}$$

Choose an arbitrary linear space isomorphism between W and $W^\vee$. This gives an isomorphism $(W \otimes W)^\vee \cong \mathrm{End}(W)$. Define the determinant map $\det :$ $(W \otimes W)^\vee \cong \mathrm{End}(W) \to \mathbb{C}$. Then $\det \circ g$ is an $\mathrm{SL}(W)$-invariant homogeneous polynomial on $W^\vee \otimes W^\vee \otimes W$ and it is nonzero when evaluated on the point f corresponding to a semisimple Lie algebra, because it is the determinant of the Killing form. Hence this point is GIT-semistable. $\square$

LEMMA 1.3. *Let (E, φ) be a Lie algebra sheaf, and $E_{\lambda_\bullet}$ a balanced filtration.*

(1) *If (E, φ) is furthermore a $\mathfrak{g}'$-sheaf, then $\mu(\varphi, E_{\lambda_\bullet}) \le 0$.*

(2) *$E_{\lambda_\bullet}$ is an algebra filtration if and only if $\mu(\varphi, E_{\lambda_\bullet}) \ge 0$.*

Proof. To prove item (1) assume (E, φ) is a $\mathfrak{g}'$-sheaf, i.e. the Lie algebra structure is semisimple. Since $E^{\vee\vee}$ is torsion free, the formula (1.1) is equivalent to

$$\mu(\varphi, E_{\lambda_\bullet}) = \min\left\{\lambda_i + \lambda_j - \lambda_k \;:\; [E_{\lambda_i}(x), E_{\lambda_j}(x)] \not\subset {E_{\lambda_{k-1}}}^{\vee\vee}(x)\right\} \tag{1.2}$$

where x is a general point of X, so that $E_{\lambda_\bullet}$ is a vector bundle filtration near x. Fixing a Lie algebra isomorphism between the fiber $E(x)$ and $\mathfrak{g}'$, the filtration $E_{\lambda_\bullet}$ induces a filtration on $\mathfrak{g}'$. Consider a vector space splitting $\mathfrak{g}' = \oplus {\mathfrak{g}'}^{\lambda_i}$ of this filtration and a basis e_l of $\mathfrak{g}'$ such that $e_l \in {\mathfrak{g}'}^{i(l)}$, in order to define a monoparametric subgroup of $\mathrm{SL}(\mathfrak{g}')$ given by $e_l \mapsto t^{\lambda_{i(l)}} e_l$ for all $t \in \mathbb{C}^*$ (cf. notation $i(l)$ introduced for Definition 0.15). The Lie algebra structure on $\mathfrak{g}'$ gives a point $\langle \varphi_{\mathfrak{g}'} \rangle \in \mathbb{P}({\mathfrak{g}'}^\vee \otimes {\mathfrak{g}'}^\vee \otimes \mathfrak{g}')$. Let a^n_{lm} be the homogeneous coordinates of this point, i.e. $[e_l, e_m] = \sum_n a^n_{lm} e_n$. The monoparametric subgroup acts as $t^{\lambda_{i(l)} + \lambda_{i(m)} - \lambda_{i(n)}} a^n_{lm}$ on the coordinates a^n_{lm}. Hence (1.2) is equivalent to

$$\mu(\varphi, E_{\lambda_\bullet}) = \min\left\{\lambda_{i(l)} + \lambda_{i(m)} - \lambda_{i(n)} \;:\; a^n_{lm} \neq 0\right\}.$$

By Lemma 1.2, the point $\varphi_{\mathfrak{g}'}$ is GIT-semistable under the $\mathrm{SL}(\mathfrak{g}')$ action because it corresponds to a semisimple Lie algebra, hence, by the Mumford criterion of GIT-semistability, $\mu(\varphi, E_{\lambda_\bullet}) \le 0$.

To prove item (2), assume that $\mu(\varphi, E_{\lambda_\bullet}) \geq 0$. If $\lambda_i + \lambda_j - \lambda_k < 0$, it follows from (1.1) that

$$[E_{\lambda_i}, E_{\lambda_j}] \subset E_{\lambda_{k-1}}{}^{\vee\vee},$$

i.e. $E_{\lambda_\bullet}$ is an algebra filtration of E.

Conversely, assume $E_{\lambda_\bullet}$ is an algebra filtration of E. For example, if

$$[E_{\lambda_i}, E_{\lambda_j}] \not\subset E_{\lambda_{k-1}}{}^{\vee\vee},$$

then $\lambda_i + \lambda_j \geq 0$. Therefore, the definition of μ (formula (1.1)) implies $\mu(\varphi, E_{\lambda_\bullet}) \geq 0$. □

Lemma 1.4. *Let $(E, \varphi : E \otimes E \to E^{\vee\vee})$ be a Lie algebra sheaf, and let $(E, \phi : E^{\otimes r+1} \to \mathcal{O}_X)$ be the associated Lie tensor. Assume that one of the following conditions is satisfied*:

(1) *(E, φ) is a semistable $\mathfrak{g}'$-sheaf* (*Definition* 0.21).

(2) *(E, ϕ) is a δ-semistable tensor* (*Definition* 0.17).

Then E is a Mumford semistable sheaf.

Proof. Assume E is not Mumford semistable. Consider its Harder-Narasimhan filtration, i.e. the saturated filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_t \subsetneq E_{t+1} = E \tag{1.3}$$

such that $E^i = E_i/E_{i-1}$ is Mumford semistable for all $i = 1, \ldots, t+1$, and

$$\mu_{\max}(E) := \mu(E^1) > \mu(E^2) > \cdots > \mu(E^{t+1}) =: \mu_{\min}(E), \tag{1.4}$$

where $\mu(F) := \deg(F)/\operatorname{rk}(F)$ denotes the slope of a sheaf F. Define

$$\lambda_i = -r!\mu(E^i) \tag{1.5}$$

(the factor $r!$ is used to make sure that λ_i is integer). Replacing the indexes i by λ_i, the Harder-Narasimhan filtration becomes

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E\,.$$

Since $\deg(E) = 0$ (by Lemma 0.28), it follows that this filtration is balanced (Definition 0.15). Now we will check that it is an algebra filtration. Given a triple $(\lambda_i, \lambda_j, \lambda_k)$, with $\lambda_i + \lambda_j < \lambda_k$, we have to show that

$$[E_{\lambda_i}, E_{\lambda_j}] \subset E_{\lambda_{k-1}}{}^{\vee\vee}.$$

Let k' be the minimum integer for which

$$[E_{\lambda_i}, E_{\lambda_j}] \subset E_{\lambda_{k'-1}}{}^{\vee\vee}.$$

We have to show that $k' \leq k$. By definition of k', the following composition is nonzero

$$E_{\lambda_i} \otimes E_{\lambda_j} \xrightarrow{[\cdot,\cdot]} E_{\lambda_{k'-1}}{}^{\vee\vee} \longrightarrow E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}\,.$$

It is well known that, if a homomorphism $F_1 \to F_2$ between two torsion free sheaves is nonzero, then $\mu_{\min}(F_1) \le \mu_{\max}(F_2)$; hence

$$\mu_{\min}(E_{\lambda_i} \otimes E_{\lambda_j}) \le \mu_{\max}(E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee})\,. \tag{1.6}$$

Using (1.5) and the fact that $\mu_{\min}(E_{\lambda_1} \otimes E_{\lambda_2}) = \mu_{\min}(E_{\lambda_1}) + \mu_{\min}(E_{\lambda_2})$ [A-B, Prop. 2.9]), we see that the left-hand side is

$$\mu_{\min}(E_{\lambda_i} \otimes E_{\lambda_j}) = \frac{-1}{r!}(\lambda_i + \lambda_j)\,.$$

Since the quotient $E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}$ is Mumford semistable, the right-hand side is

$$\mu_{\max}(E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}) = \mu(E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}) = \frac{-1}{r!}\lambda_{k'-1}\,.$$

Hence the inequality (1.6) becomes

$$\lambda_i + \lambda_j \ge \lambda_{k'-1},$$

so that $\lambda_{k'-1} < \lambda_k$, hence $k' \le k$, and we conclude that $E_{\lambda_\bullet}$ is a balanced algebra filtration.

If we plot the points $(r_{\lambda_i}, d_{\lambda_i}) = (\operatorname{rk} E_{\lambda_i}, \deg E_{\lambda_i})$, $1 \le i \le t{+}1$ in the plane $\mathbb{Z} \oplus \mathbb{Z}$ we get a polygon, called the Harder-Narasimhan polygon. Condition (1.4) means that this polygon is (strictly) convex. Since $d = 0$ (and $d_{\lambda_1} > 0$), this implies that $d_{\lambda_i} > 0$ for $1 \le i \le t$, and then

$$\sum_{i=1}^{t} r!\big(\mu(E^i) - \mu(E^{i+1})\big)(r d_{\lambda_i} - r_{\lambda_i} d) > 0. \tag{1.7}$$

Therefore

$$\sum_{i=1}^{t}(\lambda_{i+1} - \lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i}P_E\big) \succ 0 \tag{1.8}$$

because the leading coefficient of (1.8) is (1.7), and thus (E, φ) cannot be a semistable $\mathfrak{g}'$-sheaf, proving item (1).

Now, since $E_{\lambda_\bullet}$ is an algebra filtration, it is, by Lemma 1.3(2), $\mu(\varphi, E_{\lambda_\bullet}) \ge 0$. Now, Lemma 1.1 implies $\mu_{\text{tens}}(\phi, E_{\lambda_\bullet}) \ge 0$, hence

$$\sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i}P_E\big) + \mu(\phi, E_{\lambda_\bullet})\delta \ge \sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i}P_E\big) \succ 0$$

and therefore, by (1.8), (E, ϕ) cannot be a δ-semistable tensor, thus proving item (2). □

Proposition 1.5. *Let $(E, \varphi : E \otimes E \to E^{\vee\vee})$ be a $\mathfrak{g}'$-sheaf and let $(E, \phi : E^{\otimes r+1} \to \mathcal{O}_X)$ be the associated tensor. The following conditions are equivalent*:

(1) (E,ϕ) *is a* δ-*(semi)stable tensor.*

(2) (E,φ) *is a (semi)stable* $\mathfrak{g}'$-*sheaf.*

Proof. Assume that (E,ϕ) is δ-(semi)stable. Let $E_{\lambda_\bullet}$ be a balanced algebra filtration. Then $\mu_{\rm tens}(\phi, E_{\lambda_\bullet}) = \mu(\varphi, E_{\lambda_\bullet}) = 0$ (Lemmas 1.1, 1.3), hence inequality (0.6) in Definition 0.17 becomes (0.7) in Definition 0.21.

Conversely, assume that the $\mathfrak{g}'$-sheaf (E,φ) is (semi)stable, thus E is Mumford semistable by Lemma 1.4(1). Consider a balanced filtration $E_{\lambda_\bullet}$ of E. We must show that (0.6) is satisfied. If this is an algebra filtration, then $\mu(\varphi, E_{\lambda_\bullet}) = 0$ by Lemma 1.3, hence (0.6) holds. If it is not an algebra filtration, then $\mu(\varphi, E_{\lambda_\bullet}) < 0$ (again by Lemma 1.3). Since E is Mumford semistable, it is $rd_{\lambda_i} - r_{\lambda_i}d \le 0$ for all i. Denote by $\tau/(n-1)!$ the positive leading coefficient of δ. Then the leading coefficient of the polynomial of (0.6) becomes

$$\Big(\sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)\big(rd_{\lambda_i} - r_{\lambda_i}d\big)\Big) + \tau\mu(\varphi, E_{\lambda_\bullet}) < 0,$$

and thus (0.6) holds. □

Now, let us recall briefly how the moduli space of tensors was constructed in [G-S1]. Start with a δ-semistable tensor

$$(F, \phi : F^{\otimes a} \longrightarrow \mathcal{O}_X)$$

with $\operatorname{rk} F = r$ (i.e. $\dim \mathfrak{g}'$), fixed Chern classes and $\det F \cong \mathcal{O}_X$. Let m be a large integer (depending only on the polarization and numerical invariants of F) and an isomorphism g between $H^0(F(m))$ and a fixed vector space V of dimension $h^0(F(m))$. This gives a quotient

$$q : V \otimes \mathcal{O}_X(-m) \longrightarrow F$$

and hence a point in the Hilbert scheme $\mathcal{H}$ of quotients of $V\otimes\mathcal{O}_X(-m)$ with Hilbert polynomial P. Let $l > m$ be an integer, and $W = H^0(\mathcal{O}_X(l-m))$. The quotient q induces homomorphisms

$$\begin{array}{rcccl} q : & V\otimes\mathcal{O}_X(l-m) & \twoheadrightarrow & F(l) & \\ & V\otimes W & \to & H^0(F(l)) & \\ & \bigwedge^{P(l)}(V\otimes W) & \to & \bigwedge^{P(l)} H^0(F(l)) & \text{of dim } 1. \end{array}$$

If l is large enough, these homomorphisms are surjective, and they yield the Grothendieck embedding

$$\mathcal{H} \hookrightarrow \mathbb{P}\big(\bigwedge^{P(l)}(V^\vee \otimes W^\vee)\big),$$

and hence a restricted very ample line bundle $\mathcal{O}_{\mathcal{H}}(1)$ on $\mathcal{H}$ (depending on m and l). The isomorphism $g : V \xrightarrow{\cong} H^0(F(m))$ and ϕ induces a linear map

$$\Phi : V^{\otimes a} \longrightarrow H^0(F(m)^{\otimes a}) \longrightarrow H^0(\mathcal{O}_X(am)) =: B,$$

and so the tensor ϕ and the isomorphism g give a point in

$$\mathbb{P}\big(\bigwedge^{P(l)}(V^\vee \otimes W^\vee)\big) \times \mathbb{P}\big((V^{\otimes a})^\vee \otimes B\big) \;=\; \mathbb{P} \times \mathbb{P}' .$$

Let Z be the closure of the points associated to δ-semistable tensors. We give Z a polarization $\mathcal{O}_Z(1)$, by restricting a polarization $\mathcal{O}_{\mathbb{P}\times\mathbb{P}'}(b,b')$ of the ambient space, where the ratio between b and b' depends on the polynomial δ and the integers m and l as

$$\frac{b'}{b} = \frac{P(l)\delta(m) - \delta(l)P(m)}{P(m) - a\delta(m)} .$$

There is a tautological family of tensors on X parametrized by Z

$$\phi_Z : F_Z^{\otimes r+1} \longrightarrow p_{\mathbb{P}'}^* \mathcal{O}_{\mathbb{P}'}(1) . \tag{1.9}$$

The scheme Z has an open dense subscheme Z^{ss} representing the sheafification of the functor

$$F^b : (\mathrm{Sch}/\mathbb{C}) \longrightarrow (\mathrm{Sets}) \tag{1.10}$$

associating to a scheme S the set of equivalence classes of families of δ-semistable "based" tensors

$$\big(q_S : V \otimes \mathcal{O}_{X\times S}(-m) \to F_S,\ F_S,\ \phi_S : F_S^{\otimes a} \to p_S^* N_S,\ N_S\big) ,$$

where q_S is a surjection inducing an isomorphism

$$g_S = p_{S*}(q_S(m)) : V \otimes \mathcal{O}_S \to p_{S*}(F_S(m))$$

and (F_S, ϕ_S, N_S) is a family of δ-semistable tensors (Definition 0.17) with fixed rank r, Chern classes and trivial determinant. In particular,

$$\det(F_S) \cong p_S^* L, \tag{1.11}$$

where L is a line bundle on S. From now on, we will assume $a = r+1$.

Proposition 1.6. *There is a closed subscheme R of Z^{ss} representing the sheafification $\widetilde{F}^b_{\mathrm{Lie}}$ of the subfunctor of* (1.10)

$$\begin{aligned} F^b_{\mathrm{Lie}} : (\mathrm{Sch}/\mathbb{C}) &\longrightarrow (\mathrm{Sets}) \\ S &\longmapsto F^b_{\mathrm{Lie}}(S) \subset F^b(S) , \end{aligned} \tag{1.12}$$

where $F^b_{\mathrm{Lie}}(S) \subset F^b(S)$ is the subset of families of based δ-semistable Lie tensors.

A point of the closure $\overline{R}$ of R in Z is GIT-(*semi*)*stable with respect to the natural* $\mathrm{SL}(V)$-*action and linearization on $\mathcal{O}_{\overline{R}}(1) = \mathcal{O}_Z(1)|_{\overline{R}}$ (see* [G-S1]) *if and only if the corresponding tensor is δ-*(*semi*)*stable and q induces an isomorphism $V \cong H^0(E(m))$. In particular the open subset of* GIT-*semistable points of $\overline{R}$ is R.*

Proof. Let $(q_{Z^{ss}}, F_{Z^{ss}}, \phi_{Z^{ss}} : {F_{Z^{ss}}}^{\otimes r+1} \to p^*_{Z^{ss}} N_{Z^{ss}}, N_{Z^{ss}})$ be the tautological family on Z^{ss} coming from (1.9). For each pair (i,j) with $1 \leq i < j \leq r+1$, let

$$\sigma_{ij}(\phi_{Z^{ss}}) : {F_{Z^{ss}}}^{\otimes r+1} \longrightarrow p^*_{Z^{ss}} N_{Z^{ss}}$$

be the homomorphism obtained from $\phi_{Z^{ss}}$ by interchanging the factors i and j. Let $Z_{ij} \subset Z^{ss}$ be the zero subscheme defined by $\phi_{Z^{ss}} + \sigma_{ij}(\phi_{Z^{ss}})$, using Lemma 0.9. Finally, define

$$Z_{\text{skew}} = \bigcap_{3 \leq i < j \leq r+1} Z_{ij}.$$

From the universal property of Z_{ij} (Lemma 0.9) it follows that, for a family satisfying condition (1) of Definition 0.14, the classifying morphism into Z^{ss} factors through Z_{skew}. Furthermore, the restriction of the tautological family to Z_{skew} satisfies condition (1), hence by Corollary 0.13 we have a family parametrized by Z_{skew}

$$\text{(1.13)} \quad \begin{aligned}(q_{Z_{\text{skew}}}, F_{Z_{\text{skew}}}, \varphi_{Z_{\text{skew}}} : F_{Z_{\text{skew}}} \otimes F_{Z_{\text{skew}}} \\ \longrightarrow {F_{Z_{\text{skew}}}}^{\vee\vee} \otimes \det F^{\vee}_{Z_{\text{skew}}} \otimes p^*_{Z_{\text{skew}}} N_{\text{skew}}, N_{\text{skew}})\,.\end{aligned}$$

The closed subscheme ("antisymmetric locus") $Z_{\text{asym}} \subset Z_{\text{skew}}$ is defined as the zero subscheme of $\varphi_{Z_{\text{skew}}} + \sigma_{12}(\varphi_{Z_{\text{skew}}})$ given by Lemma 0.9. It follows that if a family satisfies conditions (1) and (2) of Definition 0.14, then its classifying morphism factors through Z_{asym}, and furthermore the restriction of the tautological family to Z_{asym} satisfies conditions (1) and (2).

Let J be the homomorphism defined as in (0.4) of Definition 0.14, using the tautological family parametrized by Z_{asym}. Note that this homomorphism is zero if and only if the associated homomorphism (Lemma 0.10)

$$\widetilde{J} : F_{Z_{\text{asym}}} \otimes F_{Z_{\text{asym}}} \otimes F_{Z_{\text{asym}}} \otimes F^{\vee}_{Z_{\text{asym}}} \longrightarrow (\det F^{\vee}_{Z_{\text{asym}}} \otimes p^*_{Z_{\text{asym}}} N_{\text{asym}})^2$$

is zero. Finally, let $R \subset Z_{\text{asym}}$ be the zero closed subscheme of $\widetilde{J}$ given in Lemma 0.9. If a family satisfies conditions (1) to (3) of Definition 0.14, then its classifying morphism will factor through R, and furthermore the restriction of the tautological family to R satisfies conditions (1) to (3).

The equivalence of δ-(semi)stability and GIT-(semi)stability is proved in [G-S1]. □

Recall that a $\mathfrak{g}'$-sheaf is (semi)stable if and only if the associated Lie tensor is δ-semistable (Proposition 1.5).

Proposition 1.7. *There is a subscheme $R_1 \subset R$ representing the sheafification $\widetilde{F}^b_{\mathfrak{g}'}$ of the subfunctor of* (1.12)

$$\text{(1.14)} \qquad \begin{aligned} F^b_{\mathfrak{g}'} : (\mathrm{Sch}/\mathbb{C}) &\longrightarrow (\mathrm{Sets}) \\ S &\longmapsto F^b_{\mathfrak{g}'}(S) \subset F^b_{\mathrm{Lie}}(S)\,, \end{aligned}$$

where $F^b_{\mathfrak{g}'}(S) \subset F^b_{\mathrm{Lie}}(S)$ is the subset of S-families of based δ-semistable Lie tensors such that the homomorphism associated by Construction 0.24 *provides a family of based semistable $\mathfrak{g}'$-sheaves with fixed numerical invariants τ.*

Furthermore, R_1 is a union of connected components of R, hence the inclusion $R_1 \hookrightarrow R$ is proper.

Proof. Consider the tautological family parametrized by R

$$(q_R, F_R, \phi_R : F_R^{\otimes r+1} \longrightarrow p_R^* N_R, N_R)$$

and the associated family obtained as in Construction 0.24

$$(q_R, E_R, \varphi_R : E_R \otimes E_R \to E_R^{\vee\vee})\,. \tag{1.15}$$

Let κ be the Killing form (Definition 0.18)

$$\kappa : E_R \otimes E_R \longrightarrow \mathcal{O}_{X\times R}.$$

This induces a homomorphism $\det\kappa' : \det E_R \to \det E_R^\vee$. Recall from (1.11) that $\det F_R$ is the pullback of a line bundle from R, hence the same holds for $\det E_R$, and then $\det\kappa'$ is constant along the fibers of $\pi : X\times R \to R$. Hence $\det\kappa'$ is nonzero on an open set of the form $X \times W$, where $W \subset R$ is an open set.

A point $(q, E, \varphi) \in R$ belongs to W if and only if for all $x \in U_E$ the Lie algebra $(E(x), \varphi(x))$ is semisimple, because the Killing form is nondegenerate if and only if the Lie algebra is semisimple.

Now we show that the open set W is in fact equal to R. Let $(q, E, \varphi : E\otimes E \to E^{\vee\vee})$ be a based Lie algebra sheaf corresponding to a point in $R\setminus W$. Then its Killing form $\kappa : E\otimes E \to \mathcal{O}_X$ is degenerate. Let E_1 be the kernel of the homomorphism induced by κ

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E^\vee.$$

By Lemma 1.4(2), E is Mumford semistable, thus $E^\vee$ is Mumford semistable, and, being both of degree 0, the sheaf E_1 is also of degree 0 and Mumford semistable. Note that E_1 is a solvable ideal of E, i.e. the fibers of E_1 are solvable ideals of the fibers of E (at closed points where both sheaves are locally free) [Se2, proof of Th. 2.1 in Chap. VI]. Since $E_1 \otimes E_1$ (modulo torsion) and $E_1^{\vee\vee}$ are Mumford semistable of degree zero, the image $E_2' = [E_1, E_1]$ of the Lie bracket homomorphism $\varphi : E_1 \otimes E_1 \to E_1^{\vee\vee}$, is a Mumford semistable subsheaf of $E_1^{\vee\vee}$ of degree zero. Define $E_2 = E_2' \cap E$. It is a Mumford semistable subsheaf of E of degree zero. Similarly $E_3' = [E_2, E_2]$, E_3, etc... are all Mumford semistable sheaves of degree zero. Since E_1 is solvable, we arrive eventually to a non-zero sheaf E' of degree zero, which is an abelian ideal of E.

For $\lambda_1 = \mathrm{rk}\, E' - r$ and $\lambda_2 = \mathrm{rk}\, E'$, let $E_{\lambda_1} \subsetneq E_{\lambda_2}$ be the balanced filtration having as E_{λ_1} the saturation of E' in E, and as E_{λ_2} the sheaf E itself. We claim that this filtration contradicts the δ-semistability of the tensor (E,φ) associated to (E,ϕ) by Construction 0.24 .

To prove this we need to calculate $\mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet})$ (cf. formula (0.5)). By Lemma 1.1 this is equal to $\mu(\varphi, E_{\lambda_\bullet})$ (cf. (1.1)). We need to estimate which triples (i,j,k) are relevant to calculate the minimum, i.e. which triples have $[E_{\lambda_i}, E_{\lambda_j}] \not\subset E_{\lambda_{k-1}}{}^{\vee\vee}$. Since E' is abelian, it is $[E',E'] = 0$, so $(1,1,k)$ is not relevant. Since E' is an ideal, we have $[E',E] \subset E'^{\vee\vee}$. If E' is in the center, then this bracket is zero, hence $(1,2,k)$ is not relevant. If, on the contrary, E' is not in the center, then $[E',E] \neq 0$, hence $(1,2,1)$ is relevant, and corresponds $\lambda_1 + \lambda_2 - \lambda_1 = \operatorname{rk} E' > 0$. Since E is not abelian, it is $[E,E] \neq 0$. Then there are two possibilities: if $[E,E] \subset E'^{\vee\vee}$, then $(2,2,1)$ is relevant and $\lambda_2 + \lambda_2 - \lambda_1 = \operatorname{rk} E' + \operatorname{rk} E > 0$. Otherwise $(2,2,2)$ is relevant, and $\lambda_2 + \lambda_2 - \lambda_2 = \operatorname{rk} E' > 0$. Summing up, we obtain

$$\mu(\varphi, E_{\lambda_\bullet}) > 0.$$

Since $\deg E' = \deg E = 0$, the leading coefficient of

$$r(rP_{E'} - rk(E')P_E) + \mu(\varphi, E_{\lambda_\bullet})\delta$$

is positive, hence (E,ϕ) is not δ-semistable (and by Proposition 1.5, (E,φ) is not semistable), contradicting the assumption, so we have proved that $W = R$.

Now assume that we have two based $\mathfrak{g}'$-sheaves (q,E,φ) and (q',E',φ') belonging to the same connected component of R, and $x \in U_E$, $x' \in U_{E'}$. Then we have

$$(E(x), \varphi(x)) \cong (E'(x'), \varphi'(x'))$$

as Lie algebras, because of the well-known rigidity of semisimple Lie algebras (see [Ri], for instance). Hence R_1 is the union of the connected components of R with $(E(x),\varphi(x)) \cong \mathfrak{g}'$ at the general closed point $x \in X$. $\square$

We will denote by $\mathcal{E}_{R_1}$ the tautological family of $\mathfrak{g}'$-sheaves which R_1 parametrizes, i.e. the one obtained by restricting (1.15) and ignoring the basing q_{R_1}

$$\mathcal{E}_{R_1} = (E_{R_1}, \varphi_{R_1}) \,. \tag{1.16}$$

Giving a family of (semi)stable $\mathfrak{g}'$-sheaves is equivalent to giving a family of (semi)stable principal $\mathrm{Aut}(\mathfrak{g}')$-sheaves. Furthermore, by Lemma 0.26, the (semi)stability conditions for a $\mathfrak{g}'$-sheaf and the corresponding principal $\mathrm{Aut}(\mathfrak{g}')$-sheaf coincide, hence (E_{R_1}, φ_{R_1}) can also be seen as a family of semistable principal $\mathrm{Aut}(\mathfrak{g}')$-sheaves.

Recall that $\mathcal{H}$ is the Hilbert scheme classifying quotients $V \otimes \mathcal{O}_X(-m) \to F$ (of fixed rank and Chern classes), $\mathbb{P}' = \mathbb{P}\big((V^{\otimes r+1})^\vee \otimes H^0(\mathcal{O}_X((r+1)m))\big)$ and, by the Construction 0.24 , it is $E_{R_1} = F_{R_1} \otimes \det F_{R_1} \otimes p^*\mathcal{O}_{\mathbb{P}'}(-1)$, where F_{R_1} is the restriction of (1.9) to R_1, and p is

$$p : R_1 \hookrightarrow \mathbb{P} \times \mathbb{P}' \to \mathbb{P}' \,.$$

Let $\xi : V \otimes \mathcal{O}_{\mathrm{GL}(V)} \to V \otimes \mathcal{O}_{\mathrm{GL}(V)}$ be the universal automorphism and π_1, π_2 be the projections to the two factors of $\mathrm{GL}(V) \times R_1$. The group $\mathrm{GL}(V)$ acts on R_1, and this action lifts to an action Λ_F on F_{R_1} ([H-L, §4.3, p. 90]), and to an action $\mathcal{B}$ on $p^*\mathcal{O}_{\mathbb{P}'}(-1)$, giving an isomorphism $(\Lambda_F, \mathcal{B})$

$$\begin{tikzcd} V \otimes \mathcal{O}_{X\times R_1}(-m) \arrow[r, two heads, "\overline{\sigma}^* q_{R_1}"] \arrow[d, "\pi_1^*\xi"'] & \overline{\sigma}^* F_{R_1} \arrow[d, "\Lambda_F", "\cong"'] \\ V \otimes \mathcal{O}_{X\times R_1}(-m) \arrow[r, two heads, "\overline{\pi_2}^* q_{R_1}"] & \overline{\pi_2}^* F_{R_1} \end{tikzcd} \qquad \begin{tikzcd} \overline{\sigma}^* F_{R_1}^{\otimes r+1} \arrow[r, "\overline{\sigma}^* \phi_{R_1}"] \arrow[d, "\Lambda_F{}^{\otimes r+1}", "\cong"'] & \overline{\sigma}^* p_{R_1}^* N_{R_1} \arrow[d, "p_{R_1}^*\mathcal{B}", "\cong"'] \\ \overline{\pi_2}^* F_{R_1}^{\otimes r+1} \arrow[r, "\overline{\pi_2}^* \phi_{R_1}"] & \overline{\pi_2}^* p_{R_1}^* N_{R_1} \end{tikzcd} \tag{1.17}$$

between the pullbacks of the family of Lie tensors $(F_{R_1}, \phi_{R_1}, N_{R_1})$, by the action $\sigma : \mathrm{GL}(V) \times R_1 \to R_1$ and by the projection π_2 to the second factor.

Since the action σ on R_1 lifts to the actions Λ_F on F_{R_1} and $\mathcal{B}$ on $p^*\mathcal{O}_{\mathbb{P}'}(-1)$, it also lifts to an action Λ on E_{R_1}. An element λ in the center of $\mathrm{GL}(V)$ acts trivially on R_1, hence the action σ factors through an action action $m : \mathrm{PGL}(V) \times R_1 \to R_1$ of $\mathrm{PGL}(V)$ on R_1. The element λ acts as multiplication by λ on F_{R_1} and as multiplication by λ^{-r-1} on $\mathcal{O}_{\mathbb{P}'}(-1)$, hence it acts trivially on E_{R_1}. Therefore the lifted action of $\mathrm{GL}(V)$ on E_{R_1} factors through $\mathrm{PGL}(V)$

$$\begin{tikzcd} \overline{m}^* E_{R_1} \arrow[d, "\Lambda", "\cong"'] \\ \overline{p_2}^* E_{R_1} \end{tikzcd} \qquad \begin{tikzcd} \overline{m}^* E_{R_1} \otimes \overline{m}^* E_{R_1} \arrow[r, "\overline{m}^* \varphi_{R_1}"] \arrow[d, "\Lambda\otimes\Lambda"] & \overline{m}^* E_{R_1}^{\vee\vee} \arrow[d, "\Lambda^{\vee\vee}"] \\ \overline{p_2}^* E_{R_1} \otimes \overline{p}_2^* E_{R_1} \arrow[r, "\overline{p_2}^* \varphi_{R_1}"] & \overline{p_2}^* E_{R_1}^{\vee\vee} \end{tikzcd} \tag{1.18}$$

where p_2 is the projection of $\mathrm{PGL}(V) \times R_1$ to the second factor. *This gives a lift* Λ *of the* $\mathrm{PGL}(V)$ *action on* R_1 *to the family* $\mathcal{E}_{R_1}$.

PROPOSITION 1.8. *With this action* σ *and lift* Λ, (E_{R_1}, φ_{R_1}) *becomes a universal family with group* $\mathrm{PGL}(V)$ *for the functor* $\widetilde{F}^{\tau}_{\mathrm{Aut}(\mathfrak{g}')}$ (*cf. Remark* 0.27).

Proof. Let (E_S, φ_S) be a family of semistable $\mathfrak{g}'$-sheaves. Shrink S if necessary, so that $\det E_S \cong \mathcal{O}_{X\times S}$. Using this isomorphism and Construction 0.24 we obtain a family of δ-semistable s

$$(F_S, \phi_S : F_S^{\otimes r+1} \to p_S^* N_S, N_S) \tag{1.19}$$

with $F_S = E_S$ and $N_S \cong \mathcal{O}_S$. Let $(F_{R_1}, \phi_{R_1}, N_{R_1})$ be the tautological family of tensors parametrized by R_1. By Proposition 1.7, after shrinking S if necessary, there is a morphism $f : S \to R_1$ such that the pullback $(\overline{f}^* F_{R_1}, \overline{f}^* \phi_{R_1}, f^* N_{R_1})$ of this family is isomorphic to the family (1.19), hence the families of $\mathfrak{g}'$-sheaves associated by Construction 0.24 to both of them are isomorphic.

Now we are going to check the second condition in the definition of universal family. Let $t_1, t_2 : S \to R_1$ be two morphisms, and let $\alpha : E_2 \to E_1$ be an isomorphism between the two pullbacks, (E_1, φ_1) and (E_2, φ_2), of (E_{R_1}, φ_{R_1}), by t_1 and t_2. We must find a morphism $h : S \to \mathrm{PGL}(V)$ such that $t_2 = h[t_1]$ and $\overline{(h, t_1)}^* \Lambda = \alpha$. Since the question is local on S, we may shrink S when needed along the proof.

By pulling back the family $(q_{R_1}, F_{R_1}, \phi_{R_1}, N_{R_1})$, these morphisms also give two families of semistable based Lie tensors (q_1, F_1, ϕ_1, N_1) and (q_2, F_2, ϕ_2, N_2). By definition of E_{R_1}, it is $E_i = F_i \otimes \det F_i \otimes p_S^* N_i^{-1}$, $i = 1, 2$. Shrinking S if necessary, there are isomorphisms $a_i : \det F_i \otimes p_S^* N_i^{-1} \to \mathcal{O}_{X \times S}$. Define α' by

$$\begin{tikzcd} E_2 \arrow[r, equal] \arrow[d, "\alpha"'] & F_2 \otimes \det F_2 \otimes p_S^* N_2^{-1} \arrow[r, "F_2 \otimes a_2"] & F_2 \arrow[d, "\alpha'"] \\ E_1 \arrow[r, equal] & F_1 \otimes \det F_1 \otimes p_S^* N_1^{-1} \arrow[r, "F_1 \otimes a_1"] & F_1 \end{tikzcd}$$

and hence $\alpha = \alpha' \otimes (a_1^{-1} \circ a_2)$. Given any isomorphism $\beta : N_2^{-1} \to N_1^{-1}$, we obtain an isomorphism

$$\alpha' \otimes \det \alpha' \otimes p_S^* \beta : E_2 = F_2 \otimes \det F_2 \otimes p_S^* N_2^{-1} \longrightarrow E_1 = F_1 \otimes \det F_1 \otimes p_S^* N_1^{-1} .$$

Choose β so that $\alpha' \otimes (a_1^{-1} \circ a_2) = \alpha' \otimes \det \alpha' \otimes p_S^* \beta$. Since $\alpha = \alpha' \otimes \det \alpha' \otimes p_S^* \beta$, the commutativity of

$$\begin{tikzcd} E_2 \otimes E_2 \arrow[r, "\varphi_2"] \arrow[d, "\alpha \otimes \alpha"'] & E_2^{\vee\vee} \arrow[d, "\alpha^{\vee\vee}"] \\ E_1 \otimes E_1 \arrow[r, "\varphi_1"] & E_1^{\vee\vee} \end{tikzcd}$$

implies the commutativity of

$$\begin{tikzcd} F_2^{\otimes r+1} \arrow[r, "\phi_2"] \arrow[d, "\alpha'^{\otimes r+1}"'] & p_S^* N_2 \arrow[d, "p_S^* \beta"] \\ F_1^{\otimes r+1} \arrow[r, "\phi_1"] & p_S^* N_1 \end{tikzcd}$$

and hence the pair (α', β) gives an isomorphism between (F_1, ϕ_1, N_1) and (F_2, ϕ_2, N_2). Using the based Lie tensors (q_1, F_1, ϕ_1, N_1) and (q_2, F_2, ϕ_2, N_2), let $g_i = p_{S*}(q_i(m))$, $i = 1, 2$, and define the isomorphism h'

$$\begin{tikzcd} V \otimes \mathcal{O}_S \arrow[r, "g_2", "\cong"'] \arrow[d, "h'"'] & p_{S*}(F_2(m)) \arrow[d, "\cong"', "p_{S*}(\alpha'(m))"] \\ V \otimes \mathcal{O}_S \arrow[r, "g_1", "\cong"'] & p_{S*}(F_1(m)) . \end{tikzcd}$$

This isomorphism can be seen as a morphism $h' : S \to \mathrm{GL}(V)$. By construction, it is $t_2 = h'[t_1]$, and (α', β) is the pullback of the isomorphism (1.17) by $\overline{(h', t_1)}$. Denote by $h : S \to \mathrm{PGL}(V)$ the composition of h' with projection to

$\mathrm{PGL}(V)$. Then we have $t_2 = h[t_1]$, and α is the pullback of the left arrow in (1.18) by $\overline{(h,t_1)}$.

Finally, we have to check that these two properties determine h uniquely. Let $h_1, h_2 : S \to \mathrm{PGL}(V)$ be two such morphisms. Define $h_3 = h_1 h_2^{-1}$. Then $h_3[t_1] = t_1$, and the pullback $\overline{(h_3,t_1)}^*\Lambda$ is the identity automorphism. Replacing S by an étale cover if necessary, we can lift h_3 to a morphism $h_3' : S \to \mathrm{GL}(V)$, and this induces an automorphism $\alpha_3' = \overline{(h_3',t_1')}^*\Lambda$ of $F_S = \overline{t_1}^* F_{R_1}$

$$\begin{array}{ccc} V \otimes \mathcal{O}_{X\times S} & \xrightarrow{\overline{t_1}^* q_{R_1}} & F_S(m) \\ {\scriptstyle p_S^* h_3'}\downarrow & & \downarrow{\scriptstyle \alpha_3'(m)} \\ V \otimes \mathcal{O}_{X\times S} & \xrightarrow{\overline{t_1}^* q_{R_1}} & F_S(m)\,. \end{array} \tag{1.20}$$

Applying p_{S*} to (1.20), we obtain

$$\begin{array}{ccc} V \otimes \mathcal{O}_S & \xrightarrow[\cong]{H^0(q_1(m))} & p_{S*} F_S(m) \\ {\scriptstyle h_3'}\downarrow & & \downarrow{\scriptstyle p_{S*}(\alpha_3'(m))} \\ V \otimes \mathcal{O}_S & \xrightarrow[\cong]{H^0(q_1(m))} & p_{S*} F_S(m)\,. \end{array}$$

Since $\overline{(h_3,t_1)}^*\Lambda = \mathrm{id}$, the automorphism α_3' is a family of homotethies, i.e. $p_{S*}\alpha_3'$ can be seen as a morphism $S \to \mathbb{C}^*$, and, using the previous diagram, h_3' can also be seen as a morphism from S to $\mathbb{C}^*$, the center of $\mathrm{GL}(V)$, hence h_3 is the identity morphism from S to $\mathrm{PGL}(V)$. □

2. Construction of R_2

In this section we construct a scheme $R_2 \to R_1$, finite and étale, parametrizing reductions to G/Z.

Recall that *all schemes considered in this article are locally of finite type over* $\operatorname{Spec}\mathbb{C}$. In this section and the following we are going to make use of the category of complex analytic spaces. For a scheme Y, we denote by Y^{an} the associated complex analytic space ([SGA1, XII], [Ha, App. B]), and given a morphism f in the category of schemes, we denote by f^{an} the corresponding morphism in the category of analytic spaces. Recall that the underlying set of Y^{an} is the set of closed points of Y, and it is endowed with the analytic topology.

LEMMA 2.1. *Let S be a scheme* (*not necessarily smooth*!). *Let $\mathcal{Z} \subset X \times S$ be a closed subscheme with* $\operatorname{codim}_{\mathbb{R}}(\mathcal{Z}_s^{\mathrm{an}}, X^{\mathrm{an}} \times s) \geq m$ *for all closed points $s \in S$, and let $\mathcal{U} \subset X \times S$ be its complement. Let M be a compact real* *manifold with boundary and having* $\dim_{\mathbb{R}}(M) \leq m-1$, *and let*

$$f = (f_X, f_S) : M \longrightarrow X^{\mathrm{an}} \times S^{\mathrm{an}}$$

be a continuous map such that the image of the boundary lies in $\mathcal{U}^{\mathrm{an}}$. *Then* f *can be modified by a homotopy, relative to its boundary to a continuous map* $\widetilde{f}$ *whose image lies in* $\mathcal{U}^{\mathrm{an}}$.

Proof. Consider the cartesian product (in the category of topological spaces and continuous maps)

$$\begin{tikzcd} \mathcal{Z}_M^{\mathrm{an}} \arrow[r] \arrow[d] & \mathcal{Z}^{\mathrm{an}} \arrow[d] \\ M \arrow[r, "f_S"] & S^{\mathrm{an}}. \end{tikzcd}$$

The map f factors as

$$\begin{tikzcd} M \arrow[r, "{(f_X,\mathrm{id})}"] \arrow[rr, bend left, "f"] & X^{\mathrm{an}} \times M \arrow[r, "\mathrm{id}\times f_S"] & X^{\mathrm{an}} \times S^{\mathrm{an}} \\ & \mathcal{Z}_M^{\mathrm{an}} \arrow[u, hook] \arrow[r] & \mathcal{Z}^{\mathrm{an}}. \arrow[u, hook] \end{tikzcd}$$

By hypothesis $\mathrm{codim}_{\mathbb{R}}(\mathcal{Z}_s^{\mathrm{an}}, X^{\mathrm{an}} \times s) \geq m$ for all $s \in S$, so $\mathrm{codim}_{\mathbb{R}}(\mathcal{Z}_M^{\mathrm{an}}, X^{\mathrm{an}} \times M) \geq m$, and, since $\dim_{\mathbb{R}}(M) \leq m-1$ and $X^{\mathrm{an}} \times M$ is smooth, we can modify (f_X, id) homotopically, relative to its boundary, to a map $\widetilde{f_1}$ whose image does not intersect $\mathcal{Z}_M^{\mathrm{an}}$. Then the image of $\widetilde{f} = (\mathrm{id}, f_S) \circ \widetilde{f_1}$ lies in $\mathcal{U}$. □

Lemma 2.2. *For a scheme* S, *let* $\mathcal{Z} \subset X \times S$ *be a closed subscheme such that* $\mathrm{codim}_{\mathbb{R}}(\mathcal{Z}_s^{\mathrm{an}}, X^{\mathrm{an}} \times s) \geq 4$ *for all closed points* $s \in S$. *Let* $\mathcal{U}$ *be the complement of* $\mathcal{Z}$, *and let* $x \in \mathcal{U} \subset X \times S$ *be a closed point. Then the inclusion* $i : \mathcal{U} \hookrightarrow X \times S$ *induces an isomorphism of topological fundamental groups*

$$\pi_1(i^{\mathrm{an}}, x) : \pi_1(\mathcal{U}^{\mathrm{an}}, x) \xrightarrow{\cong} \pi_1(X^{\mathrm{an}} \times S^{\mathrm{an}}, x).$$

Proof. To check that $\pi_1(i^{\mathrm{an}})$ is injective, let $f : \mathbb{S}^1 \to \mathcal{U}^{\mathrm{an}}$ be a continuous based loop (i.e. a continuous map from the unit interval $[0,1]$ sending 0 and 1 to the base point x) mapping to zero in $\pi_1(X^{\mathrm{an}} \times S^{\mathrm{an}}, x)$. So there is a continuous map g fitting into a commutative diagram

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "f"] \arrow[d, hook] & \mathcal{U}^{\mathrm{an}} \arrow[r, hook] & X^{\mathrm{an}} \times S^{\mathrm{an}} \arrow[d, equal] \\ \mathbb{D} \arrow[rr, "g"] & & X^{\mathrm{an}} \times S^{\mathrm{an}} \end{tikzcd}$$

where $\mathbb{D}$ denotes the unit disk (whose boundary is $\mathbb{S}^1$). By Lemma 2.1 we can change g by a homotopy relative to its boundary to a map whose image is in $\mathcal{U}^{\mathrm{an}}$, hence $[f] \in \pi_1(\mathcal{U}^{\mathrm{an}}, x)$ is zero.

To check that $\pi_1(i^{\mathrm{an}})$ is surjective, let

$$f : \mathbb{S}^1 \longrightarrow X^{\mathrm{an}} \times S^{\mathrm{an}}$$

be a continuous based loop. Applying Lemma 2.1 we can change f, by a homotopy relative to the endpoints of the interval, to be a based loop in $\mathcal{U}^{\mathrm{an}}$. □

Corollary 2.3. *With the same notation and hypothesis as in Lemma* 2.2, *the inclusion i induces an isomorphism of algebraic fundamental groups*

$$\pi^{\mathrm{alg}}(i, x) : \pi^{\mathrm{alg}}(\mathcal{U}, x) \xrightarrow{\cong} \pi^{\mathrm{alg}}(X \times S, x).$$

Proof. The algebraic fundamental group is canonically isomorphic to the completion of the topological fundamental group with respect to the topology of finite index subgroups (cf. [SGA1, XII, Cor. 5.2]), hence the result follows from Lemma 2.2. □

The monomorphism $\rho_2 : G/Z \hookrightarrow \mathrm{Aut}(\mathfrak{g}')$ is the inclusion of the connected component of the identity of $\mathrm{Aut}(\mathfrak{g}')$, thus $F = \mathrm{Aut}(\mathfrak{g}')/(G/Z)$ is a finite group.

Recall that the tautological family (1.16) parametrized by R_1 is denoted

$$\mathcal{E}_{R_1} = (E_{R_1}, \varphi_{R_1}) .$$

Let $\mathcal{U}_{R_1} \subset X \times R_1$ be the open set where E_{R_1} is locally free. Then $\mathcal{E}_{R_1}$ gives a principal $\mathrm{Aut}(\mathfrak{g}')$-bundle P_{R_1} on $\mathcal{U}_{R_1}$. Consider the functor $\Gamma(\rho_2, P_{R_1})$ of reductions defined as in (0.3).

Proposition 2.4. *The functor $\Gamma(\rho_2, P_{R_1})$ is represented by a scheme $R_2 \to R_1$, étale and finite over R_1, so that there is a tautological family parametrized by R_2*

$$(q_{R_2}, P^{G/Z}_{R_2}, E_{R_2}, \psi_{R_2}) . \tag{2.1}$$

Proof. For a scheme $S \to R_1$, the set of isomorphism classes of S-families of ρ_2-reductions is bijective to the set

$$\mathrm{Mor}_{\mathcal{U}_S}(\mathcal{U}_S, P_S(F)) \tag{2.2}$$

of sections of the principal F-bundle $P_S(F) \to \mathcal{U}_S$ pulled back from $P_{R_1}(F)$ by $\mathcal{U}_S \to \mathcal{U}_{R_1}$. Since F is a finite group, giving the principal F-bundle $p : P_{R_1}(F) \to \mathcal{U}_{R_1}$ is equivalent to giving a representation of the algebraic fundamental group $\pi^{\mathrm{alg}}(\mathcal{U}_{R_1}, x)$ in F ([SGA1, V §7]). By Lemma 2.2 this fundamental group is isomorphic to $\pi^{\mathrm{alg}}(X \times R_1, x)$, so there is a unique principal

F-bundle $\overline{P_{R_1}(F)}$ on $X \times R_1$ whose restriction to $\mathcal{U}_{R_1}$ is isomorphic to $P_{R_1}(F)$. We claim that the set (2.2) is bijective to

$$\mathrm{Mor}_{X\times S}(X \times S, \overline{P_{R_1}(F)}_S). \tag{2.3}$$

Indeed, an element of the set (2.2) corresponds to a trivialization of the principal bundle $P_S(F) \to \mathcal{U}_S$, and this in turn corresponds biunivocally to a trivialization of the principal bundle $\overline{P_{R_1}(F)}_S \to X \times S$, i.e. to an element of (2.3), thus proving the claim.

Finally, the morphism $X \times R_1 \to R_1$ is projective and faithfully flat, $\overline{P_{R_1}(F)} \to X \times R_1$ is an étale and surjective, and thus the composition $\overline{P_{R_1}(F)} \to R_1$ is projective. These three facts allow us to use [Ra3, Lemma 4.14.1] in order to conclude that the functor $\Gamma(\rho_2, P_{R_1})$ is representable by a scheme $R_2 \to R_1$ which is étale and finite over R_1. □

From this proposition, together with Proposition 0.37, we obtain the following

COROLLARY 2.5. *The family* $\mathcal{P}_{R_2} = (P^{G/Z}_{R_2}, E_{R_2}, \psi_{R_2})$ *is a universal family with group* PGL(V) *for the functor* $\widetilde{F}^{\tau}_{G/Z}$.

Recall $G' = [G, G]$ denotes the commutator subgroup. Clearly $G/G' \cong \mathbb{C}^{*q}$, and giving a principal G/G'-bundle is equivalent to giving q line bundles. Note that $G/Z \times G/G' = G/Z'$, where Z' is the center of G'. Denote the projection to the first factor by

$$\rho'_2 : G/Z' \to G/Z.$$

Let $d_1, \ldots, d_q$ be q fixed elements of $H^2(X, \mathbb{C})$. Define

$$R'_2 = J^{d_1}(X) \times \cdots \times J^{d_q}(X) \times R_2,$$

where $J^{d_i}(X)$ is the Jacobian variety parametrizing line bundles on X with first Chern class equal to $d_i \in H^2(X, \mathbb{C})$. Choosing a Poincaré line bundle on $J^{d_i}(X) \times X$, we construct a tautological family parametrized by R'_2

$$(q_{R'_2}, P^{G/Z'}_{R'_2}, E_{R'_2}, \psi_{R'_2}) \tag{2.4}$$

where the principal G/Z'-bundle $P^{G/Z'}_{R'_2}$ is the product of the pullback, by $X \times R'_2 \to X \times R_2$, of the principal G/Z-bundle $P^{G/Z}_{R_2}$ of the family (2.1), and the principal $\mathbb{C}^*$-bundles associated to the line bundles on $X \times R'_2$ pulled back from the chosen Poincaré line bundles on $X \times J^{d_i}$.

LEMMA 2.6. *The scheme* R'_2 *over* R_2 *represents the functor* $\Gamma(\rho'_2, P_{R_2})$.

Proof. It follows easily from the construction of R'_2. □

There is a lift of the trivial $\mathbb{C}^*$ action on the Jacobian $J(X)$ to the chosen Poincaré line bundle, providing it with a structure of a universal family with group $\mathbb{C}^*$. Using this action, we obtain from Lemma 2.6 and Proposition 0.37 the following

Corollary 2.7. *There is a natural action of* $G/G' \times \mathrm{PGL}(V)$ *on the family of principal* G/Z'*-sheaves* $\mathcal{P}^{G/Z'}_{R'_2} = (P^{G/Z'}_{R'_2}, E_{R'_2}, \psi_{R'_2})$, *providing it with a structure of universal family with group* $G/G' \times \mathrm{PGL}(V)$ *for the functor* $\widetilde{F}^{\tau}_{G/Z'}$.

3. Construction of R_3

In this section we construct a scheme $R_3 \to \mathrm{Jac}^{d_1}(X)\times\cdots\times\mathrm{Jac}^{d_q}(X)\times R_2$, finite and étale, parametrizing reductions to G.

We first recall some facts about nonabelian cohomology. For a scheme Y and an algebraic group H, we denote by $\underline{H}_Y$ the sheaf, in the étale topology of Y, of sections of the trivial H-bundle $Y\times H$. Given a morphism $p: Y\to S$, let $R^i p_*(\underline{H}_Y)$ be the sheaf, in the étale topology of S, generated by the presheaf

$$(u: U\to S) \longmapsto \check{H}^i_{\mathrm{et}}(Y_U, \underline{H}_Y),$$

where $\check{H}^i_{\mathrm{et}}$ denotes the Czech cohomology set with respect to the étale topology, and $Y_U = Y\times_S U$. For a finite abelian group F, let $H^i(Y^{\mathrm{an}}; F)$ be the singular cohomology of Y^{an} with coefficients in F. We will need the following comparison.

Theorem 3.1. *Let* F *be a finite abelian group, and* Y *scheme, locally of finite type. Then there is a canonical isomorphism*

$$\check{H}^i_{\mathrm{et}}(Y, \underline{F}) \;\cong\; H^i(Y^{\mathrm{an}}; F)\,.$$

Proof. The proof follows from $\check{H}^i_{\mathrm{et}}(Y,\underline{F}) \cong H^i(Y^{\mathrm{an}}; \underline{F})$ (proved in [SGA4, XVI Th. 4.1]) and from the fact that étale cohomology can be calculated using Czech cohomology. □

Recall that Z' denotes the center of the commutator subgroup $G' = [G,G]$. It is a finite abelian group.

Lemma 3.2. *Let* $p: \mathcal{U}_{R'_2} \to R'_2$ *be the projection to* R'_2 *of the big open set of the principal* G/Z'*-sheaf in Corollary* 2.7. *Then, for* $i\le 2$,

$$R^i p_* \underline{Z'} = \underline{H^i(X^{\mathrm{an}}; Z')}\,.$$

Proof. Let $U\to R'_2$ be an étale open set of R'_2, and let $\mathcal{U}_U = \mathcal{U}_{R'_2}\times_{R'_2} U$. The isomorphism of the homotopy groups in Lemma 2.2 provides an isomorphism of the singular homology groups

$$H_1({\mathcal{U}_U}^{\mathrm{an}}; \mathbb{Z}) \xrightarrow{\cong} H_1(X^{\mathrm{an}}\times U^{\mathrm{an}}; \mathbb{Z})\,.$$

Now we show that

$$H_2(\mathcal{U}_U{}^{\mathrm{an}};\mathbb{Z}) \longrightarrow H_2(X^{\mathrm{an}} \times U^{\mathrm{an}};\mathbb{Z})$$

is also an isomorphism. To check that it is injective, consider a class α in $H_2(\mathcal{U}_U{}^{\mathrm{an}};\mathbb{Z})$ mapping to zero. This class is represented by a 2-dimensional singular cycle, i.e. a sum $\sum n_i f_i$, with integer coefficients, where $f_i : M_i^2 \to \mathcal{U}_U{}^{\mathrm{an}}$ are continuous maps from a polyhedron M_i^2 of real dimension 2. Since it maps to zero, there is a 3-dimensional singular chain β in $X^{\mathrm{an}} \times U^{\mathrm{an}}$, represented by a sum $\sum m_j g_j$ with integer coefficients and $g_j : M_j^3 \to X^{\mathrm{an}} \times U^{\mathrm{an}}$ are continuous maps from a polyhedron M_j^3 of real dimension 3. We can assume that the boundary of M_j^3 is mapped to the union of the images of f_i. In particular, the image of this boundary is in $\mathcal{U}_U{}^{\mathrm{an}}$.

By Lemma 2.1, each map g_j can be changed by a homotopy, relative to its boundary, to a map $\widetilde{g}_j$ whose image also lies in $\mathcal{U}_U{}^{\mathrm{an}}$. Then $\sum m_j \widetilde{g}_j$ is a cycle in $\mathcal{U}_U{}^{\mathrm{an}}$ whose boundary is $\sum n_i f_i$, hence α is zero.

To check surjectivity, note that a singular cocycle in $X^{\mathrm{an}} \times U^{\mathrm{an}}$ can be represented by a sum $\sum n_i f_i$ where, for each i,

$$f_i : M_i^2 \longrightarrow X^{\mathrm{an}} \times U^{\mathrm{an}}$$

is a continuous map from M_i^2, now denoting a closed manifold with real dimension 2 with a triangulation. By Lemma 2.1 the maps f_i can be modified by a homotopy to maps $\widetilde{f}_i$ whose image lie in $\mathcal{U}_U{}^{\mathrm{an}}$. These modifications do not change the homology classes, so this proves surjectivity.

The inclusion $j : \mathcal{U}_U{}^{\mathrm{an}} \hookrightarrow X^{\mathrm{an}} \times U^{\mathrm{an}}$ induces an isomorphism

$$j^* : H^i(X^{\mathrm{an}} \times U^{\mathrm{an}}; Z') \xrightarrow{\cong} H^i(\mathcal{U}_U{}^{\mathrm{an}}; Z')$$

for $i = 1$ or 2. Indeed, denoting $\mathcal{U} = \mathcal{U}_U{}^{\mathrm{an}}$ and $\mathcal{M} = X^{\mathrm{an}} \times U^{\mathrm{an}}$, we see that the inclusion induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(H_{i-1}(\mathcal{M};\mathbb{Z}), Z') \arrow[r] \arrow[d, "\cong"] & H^i(\mathcal{M}; Z') \arrow[r] \arrow[d, "j^*"] & \mathrm{Hom}(H_i(\mathcal{M};\mathbb{Z}), Z') \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1(H_{i-1}(\mathcal{U};\mathbb{Z}), Z') \arrow[r] & H^i(\mathcal{U}; Z') \arrow[r] & \mathrm{Hom}(H_i(\mathcal{U};\mathbb{Z}), Z') \arrow[r] & 0\,,
\end{tikzcd}$$

where the exact rows are given by the universal coefficient theorem for singular cohomology ([Sp, Ch. 5 §5]), and then j^* is an isomorphism.

By Theorem 3.1, étale cohomology coincides with singular cohomology, thus we obtain

$$R^i p_* \underline{Z'} \xrightarrow{\cong} \underline{H^i(X; Z')}. \qquad \square$$

Given a scheme Y there is an exact sequence of pointed sets [Se1], [F-M]

$$\check{H}^1_{\mathrm{et}}(Y,\underline{G}) \longrightarrow \check{H}^1_{\mathrm{et}}(Y,\underline{G/Z'}) \longrightarrow \check{H}^2_{\mathrm{et}}(Y,\underline{Z'})$$

where the distinguished element for each set corresponds to the trivial cocycle (and exactness means that the inverse image of the distinguished element of the last set is equal to the image of the first map).

This exact sequence implies its own relative version for a morphism $p : Y \to S$ of schemes, i.e. the exact sequence of sheaves of sets, in the étale topology of S,

$$R^1p_*\underline{G} \longrightarrow R^1p_*\underline{G/Z'} \longrightarrow R^2p_*\underline{Z'}\,. \tag{3.1}$$

Lemma 3.3. *The set of algebraic isomorphism classes of reductions to G of an algebraic principal G/Z'-bundle P on a scheme Y is an $H^1(Y^{\mathrm{an}};Z')$-torsor, if nonempty.*

Proof. Recall that this means that $H^1(Y^{\mathrm{an}};Z')$ acts simply transitively on this set, and hence, for each reduction (P^G,ζ), there is a natural bijection between $H^1(Y^{\mathrm{an}};Z')$ and the set of isomorphism classes of reductions, sending the zero element of $H^1(Y^{\mathrm{an}};Z')$ to (P^G,ζ).

Since Z' is finite and abelian, $H^1(Y^{\mathrm{an}};Z') = \check{H}^1_{\mathrm{et}}(Y,\underline{Z'})$ (Theorem 3.1). The action of this group on the set of reductions is defined as follows. Let (P^G,ζ) be an analytic reduction, and $\alpha \in \check{H}^1_{\mathrm{et}}(Y,\underline{Z'})$. Let $\{g_{ij}\}$ be a $\underline{G}$-cocycle representing the isomorphism class of P^G, and let $\{z'_{ij}\}$ be a Z'-cocycle representing α. Then $\{g_{ij}z'_{ij}\}$ defines a principal G-bundle $\hat{P}^G$, and, using ζ, an isomorphism $\hat{\zeta} : \rho_{3*}(\hat{P}^G) \cong P$. The action is

$$(P^G,\zeta)\cdot\alpha = (\hat{P}^G,\hat{\zeta}).$$

It is easy to check that this is well defined on the set of isomorphism classes of reductions, and that the action is simply transitively. □

Remark 3.4. In the previous proof we have used the fact that Z' is in the center of G. In general the set of reductions is bijective to a cohomology set with twisted coefficients.

The relative version of the last lemma is the

Lemma 3.5. *Let $p : Y \to S$ be a morphism of schemes and P_S a principal G/Z'-bundle on Y. If there is a reduction $(P^G_S, \rho_{3*}P^G_S \cong P_S)$ of P_S by the projection $\rho_3 : G \to G/Z'$, it determines a bijection, for all étale open sets $U \to S$,*

$$\widetilde{\Gamma}(\rho_3,P_S)(U) = R^1p_*\underline{Z'}(U) \tag{3.2}$$

where $\widetilde{\Gamma}(\rho_3,P_S)$ is the sheaf of such reductions (as defined in the preliminaries).

Proof. Lemma 3.3 gives a bijection (depending only on the choice of P^G_S)

$$\Gamma(\rho_3, P_S)(U) = \check{H}^1_{\mathrm{et}}(Y_U, \underline{Z'})\,.$$

Sheafifying sides, we obtain the result. □

PROPOSITION 3.6. *The functor $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$ is representable by a scheme étale and finite over R'_2.*

Proof. The strategy of the proof is as follows. First we see that the subscheme $\hat{R}'_2 \subset R'_2$ corresponding to principal bundles that admit a reduction of structure group to G is a union of connected components of R'_2. Then we show that the functor $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$ is a principal space over $\hat{R}'_2$, and the structure group of this principal space is the finite group $H^1(X^{\mathrm{an}}; Z')$, hence affine, and then it follows from descent theory that the functor is representable by a principal $H^1(X^{\mathrm{an}}; Z')$-bundles over R'_2 [EGA].

Recall that a space over S is just a contravariant functor $(\mathrm{Sch}\,/S) \to (\mathrm{Sets})$. An action of an algebraic group H on a functor $F : (\mathrm{Sch}\,/S) \to (\mathrm{Sets})$ is a natural transformation $F \times \hat{H} \to F$ (where $\hat{H}$ is the functor represented by $H \times S \to S$) satisfying the axioms of a right action. We say it gives F the structure of a principal H-space when, locally in the étale topology of $(\mathrm{Sch}\,/S)$, the functor F is represented by $U \times H$, so that the action by H becomes just multiplication on the right.

The principal G/Z'-bundle $P^{G/Z'}_{R'_2} \to \mathcal{U}_{R'_2}$ (cf. 2.4) gives a section σ' of $R^1p_*\underline{G/Z'}$ over R'_2, and using (3.1) we obtain a section of $R^2p_*\underline{Z'}$. The principal G/Z'-bundle corresponding to a point in R'_2 can be lifted to G if and only if this section is zero at this point. By Lemma 3.2 this sheaf is $\underline{H^2(X^{\mathrm{an}}; Z')}$, and, being $H^2(X^{\mathrm{an}}; Z')$ a finite group, the section is locally constant, thus vanishes in a subscheme $\hat{R}'_2 \subset R'_2$, which is a union of connected components of R'_2.

By the exactness of the sequence (3.1), we can cover $\hat{R}'_2$ with open sets U_i (in the étale topology) so that the restricted sections $\sigma'_i := \sigma'|_{U_i}$ of $R^1p_*\underline{G/Z'}$ over U_i lift to sections σ_i of $R^1p_*\underline{G}$. Refining the cover U_i if necessary, we can assume

$$\sigma_i \in H^1(\mathcal{U}_{U_i}, \underline{G}).$$

This means that there are principal G-bundles $P^G_i \to \mathcal{U}_{U_i}$ such that $\rho_{3*}P^G_i \cong P^{G/Z'}_{U_i}$. The action of $H^1(X^{\mathrm{an}}; Z')$ described in the proof of Lemma 3.3 gives an action Θ on the functor of reductions $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$. By Lemma 3.5, after restricting to U_i, we have an equality of functors

$$\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{U_i}) = R^1p_*\underline{Z'}|_{U_i} : (\mathrm{Sch}\,/U_i) \longrightarrow (\mathrm{Sets})\,.$$

By Lemma 3.2, $R^1p_*\underline{Z'}$ is the sheaf of sections of $R'_2 \times H^1(X^{\mathrm{an}}; Z') \to R'_2$, and thus $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{U_i})$ is represented by the scheme $U_i \times H^1(X^{\mathrm{an}}; Z')$, the action

Θ becoming just multiplication on the right, i.e. the functor $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$ is a principal space with group $H^1(X^{\mathrm{an}}; Z')$. Since this group is affine, it follows by descent theory that it is represented by a principal $H^1(X^{\mathrm{an}}; Z')$-bundle over $\hat{R}'_2$, and the result follows. $\square$

Let R_3 be the union of components of the scheme obtained in this proposition, corresponding to principal G-sheaves with fixed numerical invariants τ. The morphism $R_3 \to R'_2$ is also finite. Thus, Proposition 3.6 together with Corollary 2.7 and Proposition 0.37, and the fact that the action of G/G' on R_2 is trivial, allows us to conclude the following

Corollary 3.7. *The scheme R_3 is a universal space with group* $\mathrm{PGL}(V)$ *for the functor $\widetilde{F}_G$.*

4. Construction of a quotient

In this section we construct the projective moduli space of semistable principal G-sheaves. We will use the following ([Ra3, Lemma 5.1])

Lemma 4.1 (Ramanathan). *If $f : T \to S$ is an affine H-equivariant morphism and $p : S \to \hat{S}$ is a good quotient for the action of a reductive algebraic group H, then there is a good quotient $q : T \to \hat{T}$ by H, and the induced morphism $\hat{f} : \hat{T} \to \hat{S}$ is affine.*

Furthermore, if f is finite, then $\hat{f}$ is finite. When f is finite and $p : S \to \hat{S}$ is a geometric quotient, then $q : T \to \hat{T}$ is also a geometric quotient.

Theorem 4.2. *There is a projective scheme $\mathfrak{M}^{\tau}_G$ corepresenting the functor $\widetilde{F}_G$ of families of semistable principal G-sheaves with numerical invariants τ. There is an open subscheme $\mathfrak{M}^{\tau,s}_G$ whose closed points are in bijection with isomorphism classes of stable principal G-sheaves.*

Proof. We use the notation of Proposition 1.6. Using geometric invariant theory, it is proved in [G-S1] that there is a good quotient for the action of $\mathrm{SL}(V)$ on the scheme of based δ-semistable tensors, so this is also true for the subscheme R of those which are Lie tensors

$$p_R : R \longrightarrow R /\!\!/ \mathrm{SL}(V) = \overline{R} /\!\!/ \mathrm{SL}(V),$$

where $\overline{R}$ is the closure of R in Z defined in Proposition 1.6, and thus $\overline{R} /\!\!/ \mathrm{SL}(V)$ is a projective scheme, and that it is a geometric quotient on the open subscheme R^s of based δ-semistable Lie tensors. By Proposition 1.7, the inclusion of based semistable $\mathfrak{g}'$-sheaves $R_1 \hookrightarrow R$ is proper, hence the restriction of p_R

$$p_{R_1} : R_1 \longrightarrow R_1/\,\mathrm{SL}(V) = \mathfrak{M}_1,$$

is also a good quotient onto a projective scheme, and it is a geometric quotient on the open set R_1^s corresponding to based stable $\mathfrak{g}'$-sheaves. Since the center of $\mathrm{SL}(V)$ acts trivially on R_1, this is also a quotient by $\mathrm{PGL}(V)$.

For the scheme R_3 of based semistable principal G-sheaves, i.e. pairs $(q,\mathcal{P})$ where $\mathcal{P}=(P,E,\psi)$ is a semistable principal G-sheaf and $q: V\otimes\mathcal{O}_X(-m)\twoheadrightarrow E$ is a surjection inducing an isomorphism $V\cong H^0(E(m))$, the composition

$$f: R_3 \longrightarrow R_2' = J^{\underline{d}}\times R_2 \longrightarrow J^{\underline{d}}\times R_1$$

is a finite morphism, where $J^{\underline{d}} = J^{d_1}(X)\times\cdots\times J^{d_q}(X)$. Let $\mathrm{PGL}(V)$ act trivially on $J^{\underline{d}}$. Then

$$p: J^{\underline{d}}\times R_1 \longrightarrow J^{\underline{d}}\times R_1/\,\mathrm{SL}(V)$$

is a good quotient by $\mathrm{PGL}(V)$, whose restriction to $J^{\underline{d}}\times R_1^s$ is a geometric quotient. Therefore, by Lemma 4.1, there exists a good quotient by $\mathrm{PGL}(V)$

$$q: R_3 \longrightarrow \mathfrak{M}_G^{\tau}$$

which is a geometric quotient on the subscheme R_3^s of based stable principal G-sheaves. Furthermore, the induced morphism $\overline{f}:\mathfrak{M}_G^{\tau}\to J^{\underline{d}}\times\mathfrak{M}_1$ is finite; hence $\mathfrak{M}_G^{\tau}$ is projective.

Since the scheme R_3 is a universal space with group $\mathrm{PGL}(V)$ for the functor $\widetilde{F}_G$ (cf. Corollary 3.7) the projective scheme $\mathfrak{M}_G^{\tau}$ corepresents the functor $\widetilde{F}_G$ (by Remark 0.36).

The last statement follows also from Ramanathan's lemma, because f is finite. $\square$

Two semistable principal sheaves are called GIT-*equivalent if they correspond to the same point in the moduli space.* Now we will show that this amounts to the notion of S-equivalence given in the introduction (Definition 0.7).

Let $\mathcal{P}=(P,E,\psi)$ be a semistable principal sheaf. If it is not stable, let $E_\bullet$, or $E_{\lambda_\bullet}$, be an *admissible filtration*, i.e. a balanced algebra filtration with

$$\sum_{i\in\mathbb{Z}}\big(rP_{E_i}-r_iP_E\big) \;=\; \sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)\big(rP_{E_{\lambda_i}}-r_{\lambda_i}P_E\big) \;=\; 0\,. \tag{4.1}$$

Since it is saturated, the open set $U'\subset X$ where it is a vector bundle filtration is big. By Lemma 5.4 this amounts to a reduction P^Q of $P|_{U'}$ to a parabolic subgroup $Q\subset G$ together with an integer dominant character χ of the Lie algebra of Q. Let $Q\twoheadrightarrow L$ be its Levi quotient, and $L\hookrightarrow Q\subset G$ a splitting. In the introduction we called the principal G-sheaf

$$\big(P^Q(Q\twoheadrightarrow L\hookrightarrow G),\oplus E^i,\psi'\big)$$

the *admissible deformation of* $\mathcal{P}$ *associated to* $E_\bullet$, whose associated $\mathfrak{g}'$-sheaf structure on $\oplus E^i$ (cf. Lemma 0.25 and Corollary 0.26) is the direct sum $\oplus[,]^{i,j}: E^i\otimes E^j\to E^{i+j\,\vee\vee}$ of the obvious homomorphisms.

PROPOSITION 4.3. *Any admissible deformation of a semistable principal G-sheaf $\mathcal{P}$ is semistable. After a finite number of admissible deformations, a principal G-sheaf is obtained such that any further admissible deformation is isomorphic to itself. This principal G-sheaf depends only on $\mathcal{P}$, and we denote it* $\operatorname{grad}\mathcal{P}$ (*and* $\operatorname{grad}\mathcal{P} := \mathcal{P}$ *if* $\mathcal{P}$ *is stable*).

Two principal sheaves $\mathcal{P}$ and $\mathcal{P}'$ are GIT-*equivalent if and only if they are S-equivalent in the sense that* $\operatorname{grad}\mathcal{P} \cong \operatorname{grad}\mathcal{P}'$.

Proof. Let $z \in R_3$ and let $\overline{\mathrm{SL}(V)\cdot z}$ be the closure of its orbit. It is a union of orbits, and by definition of good quotient, it has a unique closed orbit $B_3(z)$, which is characterized as the unique orbit in $\overline{\mathrm{SL}(V)\cdot z}$ with minimal dimension. Thus, two points z and z' in R_3 are GIT-equivalent (i.e. mapped to the same point in the moduli space) if and only if $B_3(z) = B_3(z')$.

Recall that there is a finite $\mathrm{SL}(V)$ equivariant morphism

$$R_3 \xrightarrow{f} J^{\underline{d}} \times R_1 \subset J^{\underline{d}} \times \overline{R_1}$$

where $\overline{R_1}$ is the closure of R_1 in the projective variety $\overline{R}$ defined in Proposition 1.6.

Claim. If $\mathrm{SL}(V)\cdot z$ is not closed, then there exists a one-parameter subgroup λ of $\mathrm{SL}(V)$ with $\mu(f(z),\lambda)=0$ such that the limit $z_0 = \lim_{t\to 0}\lambda(t)\cdot z$ is in $\overline{\mathrm{SL}(V)\cdot z}\setminus \mathrm{SL}(V)\cdot z$.

To prove the claim, note first that $J^{\underline{d}} \times R_1$ is the open subscheme of semistable points of the projective variety $J^{\underline{d}} \times \overline{R_1}$. Since z is not in $B_3(z)$, the point $f(z)$ is not in $B(f(z))$ (the closed orbit in the closure of $\mathrm{SL}(V)\cdot f(z) \subset J^{\underline{d}} \times R_1$), because the morphism f sends orbits into orbits, and $\dim(f(\mathrm{SL}(V)\cdot z)) = \dim(\mathrm{SL}(V)\cdot f(z))$ since f is equivariant and finite. By [Si, Lemma 1.25], since $B(f(z))$ is the closed orbit in the closure of the orbit of $f(z)$, there is a one parameter subgroup λ of $\mathrm{SL}(V)$ such that $\overline{f(z)} := \lim_{t\to 0}\lambda(t)\cdot f(z) \in B(f(z))$. Since $f(z)$ is semistable, $\mu(f(z),\lambda)\le 0$. If this inequality were strict, then $\mu(\overline{f(z)},\lambda^{-1})>0$, which is impossible because $\overline{f(z)}$ is a semistable point. Therefore $\mu(f(z),\lambda)=0$. Since f is proper, $\lim_{t\to 0}\lambda(t)\cdot z$ exists, and it belongs to $B(z)\subset \overline{\mathrm{SL}(V)\cdot z}\setminus \mathrm{SL}(V)\cdot z$, thus proving our claim.

For any one-parameter subgroup with $\mu(f(z),\lambda)=0$, $\lim_{t\to 0}\lambda(t)\cdot f(z)$ exists and is semistable [G-S, Prop. 2.14], and since f is proper, $\lim_{t\to 0}\lambda(t)\cdot z$ also exists in R_3.

Claim. There is a one-to-one correspondence between one-parameter subgroups of $\mathrm{SL}(V)$ with $\mu(f(z),\lambda)=0$, and admissible ($P_{E_{\lambda_\bullet}}=0$) filtrations $E_{\lambda_\bullet}$ of E together with a splitting of the induced filtration $H^0(E_{\lambda_\bullet}(m))$ in V.

Indeed, we established in [G-S1] a one-to-one correspondence between one-parameter subgroups of $\mathrm{SL}(V)$ with $\mu(f(z),\lambda)=0$ and balanced filtrations $E_{\lambda_\bullet}$ such that

$$P_{E_{\lambda_\bullet}} + \mu_{\mathrm{tens}}(E_{\lambda_\bullet},\phi)\delta = 0$$

for the δ-semistable tensor (E,ϕ) corresponding to the point $f(z)$, together with a splitting of the induced filtration $H^0(E_{\lambda_\bullet}(m))$ in V (the δ-semistability of this tensor implies that the filtration $E_{\lambda_\bullet}$ is saturated, since the left-hand side of the former equality becomes bigger when replaced by the saturation). The leading coefficient is

$$\sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)(\deg E_{\lambda_i}\operatorname{rk}E - \operatorname{rk}E_{\lambda_i}\deg E) + \mu_{\mathrm{tens}}(E_{\lambda_\bullet},\phi)\tau = 0\,. \tag{4.2}$$

By Lemma 0.28, $\deg E = 0$. Lemma 1.4(2) implies $\deg E_{\lambda_i}\le 0$, and recall $\tau > 0$. Therefore Lemmas 1.1 and 1.3(1) imply $\mu_{\mathrm{tens}}(E_{\lambda_\bullet},\phi)=\mu(E_{\lambda_\bullet},\varphi)\le 0$. Since we have equality in (4.2), it must be $\mu(E_{\lambda_\bullet},\varphi)=0$. Hence, by Lemma 1.3(2), the filtration $E_{\lambda_\bullet}$ is an algebra filtration, thus proving the claim.

Now, let $\mathcal{P}=(P,E,\psi)$ be a semistable principal G-sheaf. Choose a quotient $q: V\otimes\mathcal{O}_X(-m)\to E$, and let $z\in R_3$ be the point corresponding to the based principal G-sheaf $(q,\mathcal{P})$. Let $E_{\lambda_\bullet}$ be an admissible filtration, and choose a splitting of the filtration $H^0(E_{\lambda_\bullet}(m))$. Let $\lambda:\mathbb{C}^*\to\mathrm{SL}(V)$ be the one-parameter subgroup thus associated by the claim. The action of λ on the point z defines a morphism $\mathbb{C}^*\to R_3$ that extends to

$$h: T=\mathbb{C}\longrightarrow R_3\,,$$

with $h(t)=\lambda(t)\cdot z$ for $t\neq 0$ and $h(0)=\lim_{t\to 0}\lambda(t)\cdot z = z_0$. In the rest of this section we shall show that the point z_0 corresponds to the admissible deformation associated to $E_{\lambda_\bullet}$. Then it will follow that the limit z_0 fails to be in the orbit of z if and only if the associated admissible deformation fails to be isomorphic to $\mathcal{P}$.

If z_0 is not in the orbit of z, since $\mathrm{SL}(V)\cdot z_0\subset\overline{\mathrm{SL}(V)\cdot z}\setminus \mathrm{SL}(V)\cdot z$, it is $\dim\mathrm{SL}(V)\cdot z_0<\dim\mathrm{SL}(V)\cdot z$, so if we iterate this process (with z_0 and another one-parameter subgroup as before) we get a sequence of points z_0, z_0', z_0'',... that must stop giving a point in $B(z)$. Hence, the corresponding principal G-sheaf $\operatorname{grad}\mathcal{P}$ only depends –up to isomorphism– on $\mathcal{P}$, because there is only one closed orbit in $\overline{\mathrm{SL}(V)\cdot z}$.

To finish the proof of the proposition *it only remains to show that the point z_0 corresponds to the associated admissible deformation.* This will be done constructing a based family $(q_T,\mathcal{P}_T)=(q_T,P_T,E_T,\psi_T)$ such that $(q_t,\mathcal{P}_t)$ corresponds to the point $h(t)\in R_3$ when $t\neq 0$ and $\mathcal{P}_0$ is the associated admissible deformation. Since R_3 is separated, it will follow that $(q_0,\mathcal{P}_0)=(q_0,P_0,E_0,\psi_0)$ corresponds to z_0.

First we define a based family (q_T, E_T, φ_T) of $\mathfrak{g}'$-sheaves. For any $n \in \mathbb{Z}$, define $E_n = E_{\lambda_{i(n)}}$, where, recall, $i(n)$ is the maximum index with $\lambda_{i(n)} \le n$. Let $-N$ be a negative integer such that $E_n = 0$ for $n \le -N$. Using the quotient $q : V \otimes \mathcal{O}_X(m) \to E$, we can identify V with $H^0(E(m))$, so we can define a filtration $V_n = H^0(E_n(m))$ of V. The chosen splitting of the filtration $H^0(E_{\lambda_\bullet}(m))$ gives a splitting $V = \oplus V^n$. Borrowing the formalism from [H-L, §4.4], we define

$$E_T = \bigoplus_{n\in\mathbb{Z}} E_n \otimes t^n \subset E \otimes_{\mathbb{C}} t^{-N}\mathbb{C}[t] \subset E \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]\,,$$

$$\begin{array}{rcccl} q_T : V \otimes \mathcal{O}_X(-m) \otimes \mathbb{C}[t] & \stackrel{\gamma}{\longrightarrow} & \bigoplus_{n\in\mathbb{Z}} V_n \otimes \mathcal{O}_X(-m) \otimes t^n & \longrightarrow & E_T \\ v^n \otimes 1 & \longmapsto & v^n \otimes t^n & \longmapsto & q(v^n) \otimes t^n\,, \end{array}$$

$$\begin{aligned} \varphi_T : (\bigoplus_{n\in\mathbb{Z}} E_n \otimes t^n) \otimes (\bigoplus_{n\in\mathbb{Z}} E_n \otimes t^n) &\longrightarrow (\bigoplus_{n\in\mathbb{Z}} E_n \otimes t^n)^{\vee\vee} \\ w_n \otimes t^n \otimes w_{n'} \otimes t^{n'} &\longmapsto [w_n, w_{n'}] \otimes t^{n+n'}\,, \end{aligned}$$

where $v^n \in V^n$, and w_n, $w_{n'}$ are local sections of E_n and $E_{n'}$. Then (q_t, E_t, φ_t) corresponds, as in [H-L, §4.4], to $f(h(t))$ (in particular, if $t \neq 0$, then (E_t, φ_t) is canonically isomorphic to (E, φ)), and (E_0, φ_0) is the admissible deformation associated to $E_{\lambda_\bullet}$.

Now we will define the family of principal G-bundles P_T. The balanced algebra filtration $E_{\lambda_\bullet}$ provides, by Lemma 5.4, a reduction P^Q of $P|_{U'}$ to a parabolic subgroup Q on the open set U' where $E_{\lambda_\bullet}$ is a bundle filtration, together with an integer dominant character χ of $\mathfrak{q} = \mathrm{Lie}(Q)$. Let $Q = LU$ be a Levi decomposition of the parabolic subgroup Q, and denote $\mathfrak{l} = \mathrm{Lie}(L)$, $\mathfrak{u} = \mathrm{Lie}(U)$. Let $\mathfrak{h} \subset \mathfrak{l}$ be a Cartan algebra. Let $v \in \mathfrak{z}_{\mathfrak{l}} \subset \mathfrak{l}$ be the element associated to χ by Lemma 5.3, We can associate to v, without loss of generality, a one-parameter subgroup

$$\Psi : \mathbb{C}^* \to Z_L$$

of Z_L, the center of the Levi factor L, such that $d\Psi(1) = v$. Indeed, on the one hand, an integer multiple av provides such a subgroup (Lemma 5.5), and, on the other hand, if we replace the indexes λ_i by $a\lambda_i$, the associated one-parameter subgroup $\lambda(t)$ is replaced by $\lambda(t^a)$, and $h(t)$ is replaced by $h(t^a)$, and v by av, but this does not change the limit z_0.

The adjoint action of $\Psi(t)$ on any $x \in \mathfrak{u}$ has zero limit as $t = e^\tau \in \mathbb{C}^*$ goes to zero, since, using the $\mathfrak{z}_{\mathfrak{l}}$-root decomposition $x = \sum_{\alpha \in R^+(\mathfrak{z}_{\mathfrak{l}})} x_\alpha$, this action is

$$\Psi(t) \cdot x = \sum \Psi(t) \cdot x_\alpha = \sum e^{\tau v} \cdot x_\alpha = \sum e^{\tau\alpha(v)} x_\alpha = \sum t^{\alpha(v)} x_\alpha$$

and the limit is zero because $\alpha \in R^+(\mathfrak{z}_{\mathfrak{l}})$ means $\alpha(v) > 0$. Therefore, since the exponential map is G-equivariant with respect to the adjoint action, for any element $u = e^x \in U$, it is

$$\lim_{t\to 0} \Psi(t) \cdot e^x = \lim_{t\to 0} e^{\Psi(t)\cdot x} = 1\,.$$

Thus, since $\Psi(t)$ lies in the center Z_L of L, the adjoint action $\Psi(t) \cdot lu = \Psi(t)^{-1} lu \Psi(t)$ on any $lu \in LU = Q$ has limit

$$\lim_{t\to 0} \Psi(t) \cdot lu = l \lim_{t\to 0} \Psi(t) \cdot u = l\,. \tag{4.3}$$

Let $\{g_{\alpha\beta} : U'_{\alpha\beta} \to Q \subset G\}$ be a 1-cocycle on U' describing $P^Q|_{U'}$. Denote by P_T the principal G-bundle on $U' \times T$ described by

$$\{\Psi(t)^{-1} g_{\alpha\beta} \Psi(t) : U'_{\alpha\beta} \times T \to Q \subset G\}\,.$$

Note that Ψ is defined only on values $t \in \mathbb{C}^*$, but the previous observation (4.3) shows that this cocycle can be extended to $t = 0$, and for this special value it describes the principal G-bundle $P^Q(Q \twoheadrightarrow L \hookrightarrow G)$, thus admitting a reduction of structure group to L. Remark also that, for $t \neq 0$, there is a canonical isomorphism between the principal G-bundle P_t on U' and $P|_{U'}$, hence P_T extends canonically to a principal G-bundle on $U_{E_T} \subset X \times T$ which we still denote P_T.

It remains to construct an isomorphism of vector bundles $\psi_T : P_T(\mathfrak{g}') \to E_T|_{U' \times T}$. Let $\mathcal{W} = \mathcal{O}_X^{\oplus r}$, and let $\mathcal{W}_n \subset \mathcal{W}$ be the trivial subbundle defined as the direct sum of the first $\operatorname{rk} E_n$ summands, and let $\mathcal{W}^n \subset \mathcal{W}_n$ be the direct sum of the summands in $\mathcal{W}_n$ which are not contained in $\mathcal{W}_{n-1}$. Take a covering $\{U'_\alpha\}$ of U' with trivializations $\psi_\alpha : \mathcal{W}|_{U'_\alpha} \to E|_{U'_\alpha}$ preserving the filtration on E, i.e. such that ψ_α restricts to an isomorphism between $\mathcal{W}_n|_{U'_\alpha}$ and $E_n|_{U'_\alpha}$. Consider the $\mathfrak{g}'$-sheaf isomorphism

$$\begin{aligned} \gamma : \mathcal{W}|_{U'_\alpha} \otimes \mathbb{C}[t] &\longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n|_{U'_\alpha} \otimes t^n \\ v^n \otimes 1 &\longmapsto v^n \otimes t^n \end{aligned}$$

where v^n is a local section of $\mathcal{W}^n$. The transition functions $h_{\alpha\beta} : U'_{\alpha\beta} \to \operatorname{Aut}(\mathfrak{g}') \subset \operatorname{GL}(\mathfrak{g}')$ of $E|_{U'}$ can be chosen to be block-upper triangular matrices

$$h_{\alpha\beta} = \left\{ \begin{array}{cccc} M_{\lambda_1\lambda_1} & M_{\lambda_1\lambda_2} & \dots & M_{\lambda_1\lambda_{t+1}} \\ 0 & M_{\lambda_2\lambda_2} & \dots & M_{\lambda_2\lambda_{t+1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{\lambda_{t+1}\lambda_{t+1}} \end{array} \right\}$$

where $M_{\lambda_i\lambda_j}$ is a matrix of dimension $\operatorname{rk} E^{\lambda_i} \times \operatorname{rk} E^{\lambda_j}$. The commutativity of the diagram

$$\begin{tikzcd}
\mathcal{W}|_{U'_{\alpha\beta}} \otimes \mathbb{C}[t] \arrow[r, "\gamma", "\cong"'] \arrow[dd, "\gamma^{-1}(t)h_{\alpha\beta}\gamma(t)"'] & \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n|_{U'_{\alpha\beta}} \otimes t^{nr} \arrow[rr] \arrow[d, "\psi\otimes\mathrm{id}", "\cong"'] & & \mathcal{W}|_{U'_{\alpha\beta}} \otimes t^{-N}\mathbb{C}[t] \arrow[dl, "\cong"', "\psi\otimes\mathrm{id}"] \arrow[dd, "h_{\alpha\beta}\otimes\mathrm{id}", "\cong"'] \\
 & \bigoplus_{n\in\mathbb{Z}} E_n|_{U'_{\alpha\beta}} \otimes t^{nr} \arrow[r] & E|_{U'_{\alpha\beta}} \otimes t^{-N}\mathbb{C}[t] & \\
\mathcal{W}|_{U'_{\alpha\beta}} \otimes \mathbb{C}[t] \arrow[r, "\gamma", "\cong"'] & \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n|_{U'_{\alpha\beta}} \otimes t^{nr} \arrow[rr] \arrow[u, "\psi\otimes\mathrm{id}"', "\cong"] & & \mathcal{W}|_{U'_{\alpha\beta}} \otimes t^{-N}\mathbb{C}[t] \arrow[ul, "\psi\otimes\mathrm{id}", "\cong"']
\end{tikzcd}$$

shows that the transition functions of $E_T|_{U'\times T}$ are $\gamma^{-1}(t)h_{\alpha\beta}\gamma(t) : U'_{\alpha\beta} \times T \to \operatorname{Aut}(\mathfrak{g}') \subset \operatorname{GL}(\mathfrak{g}')$, i.e.

$$\gamma^{-1}(t)h_{\alpha\beta}\gamma(t) = \left\{ \begin{array}{cccc} M_{\lambda_1\lambda_1} & M_{\lambda_1\lambda_2}t^{\lambda_2-\lambda_1} & \dots & M_{\lambda_1\lambda_{t+1}}t^{\lambda_{t+1}-\lambda_1} \\ 0 & M_{\lambda_2\lambda_2} & \dots & M_{\lambda_2\lambda_{t+1}}t^{\lambda_{t+1}-\lambda_2} \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & M_{\lambda_{t+1}\lambda_{t+1}} \end{array} \right\}$$

This is well defined for $t = 0$ since all $\lambda_{i+1} - \lambda_i > 0$:

$$\lim_{t\to 0} \gamma^{-1}(t)h_{\alpha\beta}\gamma(t) = \operatorname{diag}(M_{\lambda_1\lambda_1}, M_{\lambda_2\lambda_2}, \dots, M_{\lambda_{t+1}\lambda_{t+1}})\,. \tag{4.4}$$

Since the adjoint action of $\Psi(t)$ on $h_{\alpha\beta}$ is precisely $\Psi(t)\cdot h_{\alpha\beta} = \gamma^{-1}(t)h_{\alpha\beta}\gamma(t)$, we obtain an isomorphism $\psi_T : P_T(\mathfrak{g}')|_{U'\times T} \to E_T|_{U'\times T}$, hence a family $\mathcal{P}_T = (P_T, E_T, \psi_T)$. Note that, for $t \neq 0$, using the canonical isomorphisms $E_t \cong E$ and $P_t \cong P|_{U'}$, the isomorphism ψ_t becomes ψ, hence ψ_T extends to an isomorphism $P_T(\mathfrak{g}') \to E_T|_{U_{E_T}}$, which we still denote ψ_T. Finally, it is easy to check that $(q_t, \mathcal{P}_t)$ corresponds to $h(t)$ and $\mathcal{P}_0 \cong \operatorname{grad} \mathcal{P}$ by (4.4). $\square$

5. Slope (semi)stability as Ramanathan (semi)stability

This section is kind of an appendix where we prove some results on Lie algebras which have already been used, and show that our slope notion of (semi)stability is just Ramanathan's (semi)stability.

In [Ra2], Ramanathan defines a *rational principal bundle* on X as a principal bundle P over a big open set $U \subset X$, and gives a notion of (semi)stability, which is a direct generalization of his notion of (semi)stability in [Ra3] for $\dim X = 1$.

Definition 5.1 (Ramanathan). A rational principal G bundle $P \to U \subset X$ is (semi)stable if for any reduction P^Q to a parabolic subgroup Q over a big open set $U' \subset U$, and for any dominant character χ of Q, it is

$$\deg P^Q(\chi)\ (\leq)\ 0.$$

Let $\mathcal{P}=(P,E,\psi)$ be a principal G-sheaf and let U be the open set where E is locally free. We will show in this section that $\mathcal{P}$ is slope-(semi)stable if and only if the rational bundle P is (semi)stable in the sense of Ramanathan. In particular, we will obtain that, if X is a curve, our notion of (semi)stability for principal bundles coincides with that of Ramanathan.

Recall (from [J], for instance) the well-known notions of filtration and graduation of a Lie algebra $\mathfrak{g}$. An *algebra filtration* $\mathfrak{g}_\bullet$ is a sequence

$$\cdots \subset \mathfrak{g}_{i-1} \subset \mathfrak{g}_i \subset \mathfrak{g}_{i+1} \cdots$$

starting by 0 and ending by $\mathfrak{g}$, such that

$$[\mathfrak{g}_i,\mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \text{for all } i,\, j \in \mathbb{Z}\,,$$

or, deleting (from 0 onward) all nonstrict inclusions, it is $\mathfrak{g}_{\lambda_\bullet}$

$$0 \subsetneq \mathfrak{g}_{\lambda_1} \subsetneq \mathfrak{g}_{\lambda_2} \subsetneq \ \ldots\ \subsetneq \mathfrak{g}_{\lambda_{t+1}} = \mathfrak{g}, \quad (\lambda_1<\cdots<\lambda_{t+1})\,,$$

with

$$[\mathfrak{g}_{\lambda_i},\mathfrak{g}_{\lambda_j}] \subset \mathfrak{g}_{\lambda_{k-1}} \quad \text{if } \lambda_i+\lambda_j<\lambda_k.$$

A graded algebra structure $\mathfrak{g}^\bullet$ is a decomposition

$$\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i \quad \text{with} \quad [\mathfrak{g}^i,\mathfrak{g}^j]\subset \mathfrak{g}^{i+j} \quad \text{for all } i,\, j\in\mathbb{Z}$$

or, deleting all zero summands,

$$\mathfrak{g}=\bigoplus_{i=1}^{t+1} \mathfrak{g}^{\lambda_i} \quad (\lambda_1<\cdots<\lambda_{t+1})\,,$$

with

$$[\mathfrak{g}^{\lambda_i},\mathfrak{g}^{\lambda_j}] \subset \begin{cases} \mathfrak{g}^{\lambda_k} & \text{if there is } k \text{ with } \lambda_k=\lambda_i+\lambda_j \\ 0 & \text{otherwise}\,. \end{cases}$$

To a graded algebra $\mathfrak{g}^\bullet$ it is associated a filtered algebra $\mathfrak{g}_\bullet$ with

$$\mathfrak{g}_i=\bigoplus_{j\le i} \mathfrak{g}^j$$

and reciprocally, to a filtered algebra $\mathfrak{g}_\bullet$ it is associated a graded algebra

$$(\operatorname{gr}\mathfrak{g})^i=\mathfrak{g}_i/\mathfrak{g}_{i-1}$$

with Lie algebra structure

$$[\overline{v},\overline{w}]=[v,w] \mod \mathfrak{g}_{i+j-1}$$

for $v\in\mathfrak{g}_i\setminus\mathfrak{g}_{i-1}$ and $w\in\mathfrak{g}_j\setminus\mathfrak{g}_{j-1}$.

A graded algebra $\mathfrak{g}^\bullet$ is called *balanced* if $\sum i\dim\mathfrak{g}^i=0$. In terms of $\mathfrak{g}^{\lambda_\bullet}$, this is $\sum\lambda_i\dim\mathfrak{g}^{\lambda_i}=0$. A filtered algebra is called *balanced* if the associated graded algebra is so. We start this appendix proving the following:

Lemma 5.2. *Let $\mathfrak{g}'_\bullet$ be a balanced algebra filtration of a semisimple Lie algebra $\mathfrak{g}'$. There is a Lie algebra isomorphism between $\mathfrak{g}'$ and the associated Lie algebra* $\operatorname{gr}(\mathfrak{g}'_\bullet)$.

Proof. Let W be the vector space underlying the Lie algebra $\mathfrak{g}'$. Choose a basis e_l of W adapted to the filtration $\mathfrak{g}'_{\lambda_\bullet}$. Associate the one-parameter subgroup $\lambda(t)$ of $\mathrm{GL}(W)$ expressed as $\operatorname{diag}(t^{\lambda_\bullet})$ in this basis. Since the filtration is balanced, this is in fact a one-parameter subgroup of $\mathrm{SL}(W)$. The Lie algebra structure of W is a point $v = \sum a^n_{lm} e^l \otimes e^m \otimes e_n$ in the linear space $W^\vee \otimes W^\vee \otimes W$. The action of the one-parameter subgroup is

$$a^n_{lm} \longmapsto t^{\lambda_{i(l)}+\lambda_{i(m)}-\lambda_{i(n)}} a^n_{lm},$$

where $i(l)$ is the minimum integer for which $e_l \in \mathfrak{g}'_{\lambda_{i(l)}}$. The point $\overline{v} \in \mathbb{P}(W^\vee \otimes W^\vee \otimes W)$ is GIT-semistable with respect to the induced action of $\mathrm{SL}(W)$ on this projective space and on its polarization line bundle $\mathcal{O}_{\mathbb{P}}(1)$ (by Lemma 1.2), hence the Hilbert-Mumford criterion implies

$$\mu := \min\left\{\lambda_{i(l)} + \lambda_{i(m)} - \lambda_{i(n)} \ : \ a^n_{lm} \neq 0\right\} \leq 0\,.$$

Furthermore, $\mu = 0$ because $\lambda_\bullet$ is an algebra filtration. Indeed, if $\mu < 0$ then for some triple $(\lambda_i, \lambda_j, \lambda_k)$ with $\lambda_i + \lambda_j < \lambda_k$ it would be $[\mathfrak{g}'_{\lambda_i}, \mathfrak{g}'_{\lambda_j}] \not\subset \mathfrak{g}'_{\lambda_{k-1}}$, contradicting the fact that $\mathfrak{g}'_{\lambda_\bullet}$ is algebra filtration.

Since $\mu = 0$, the following limit exists and is nonzero

$$v_0 := \lim_{t\to 0} \lambda(t)\cdot v \ \in \ W^\vee \otimes W^\vee \otimes W\,.$$

Since the subset of points of $W^\vee \otimes W^\vee \otimes W - \{0\}$ giving W a Lie algebra structure is closed, the point v_0 itself provides W with a Lie algebra structure. By construction, the coordinates b^n_{lm} of (W, v_0) are

$$b^n_{lm} = \begin{cases} a^n_{lm}\,, & \lambda_{i(l)} + \lambda_{i(m)} - \lambda_{i(n)} = 0 \\ 0\,, & \lambda_{i(l)} + \lambda_{i(m)} - \lambda_{i(n)} \neq 0\,. \end{cases}$$

In other words, $(W, v_0) \cong \operatorname{gr}(\mathfrak{g}'_{\lambda_\bullet})$. Let $k(t) : W \otimes W \to \mathbb{C}$ be the Killing form of $\lambda(t)\cdot v$. Since $\lambda(t) \in \mathrm{SL}(W)$,

$$\det(k(t)) = \det(\lambda(t)^t k(1)\lambda(t)) = \det(k(1)) \neq 0 \quad \text{for all } t \in \mathbb{C}^*,$$

thus also for $t = 0$. Since this determinant is nonzero, (W, v_0) is semisimple. By the rigidity of semisimple Lie algebras, $(W, v_0) \cong (W, v) = \mathfrak{g}'$. □

Let $\mathfrak{a}$ be a *toral algebra* $\mathfrak{a} \subset \mathfrak{g}$, i.e. an algebra consisting of semisimple elements, thus abelian [Hum, §8.1], not necessarily maximal. Following [B-T, §3], we can define the set $R(\mathfrak{a}) \subset \mathfrak{a}^\vee$ of $\mathfrak{a}$-roots in the following way. For $\alpha \in \mathfrak{a}^\vee$, write

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : [s, x] = \alpha(s)x, \text{ for all } s \in \mathfrak{a}\}\,. \tag{5.1}$$

Then $R(\mathfrak{a}) = \{\alpha \in \mathfrak{a}^\vee \setminus 0 : \mathfrak{g}^\alpha \neq 0\}$ For a maximal toral algebra $\mathfrak{h}$ (i.e. Cartan algebra) containing $\mathfrak{a}$, the $\mathfrak{a}$-roots can be thought of as classes of $\mathfrak{h}$-roots by considering two $\mathfrak{h}$-roots equivalent when their restrictions to $\mathfrak{a}$ coincide. Let $R(\mathfrak{h}) = R^+(\mathfrak{h}) \cup R^-(\mathfrak{h})$ be a decomposition into positive and negative $\mathfrak{h}$-roots. If $\beta \sim \beta' \nsim 0$, then β is positive if and only if β' is positive, hence there is an induced decomposition $R(\mathfrak{a}) = R^+(\mathfrak{a}) \cup R^-(\mathfrak{a})$. In particular, this gives a partial ordering among $\mathfrak{a}$-roots: $\alpha < \alpha'$ when $\alpha' - \alpha$ is a sum of positive $\mathfrak{a}$-roots.

LEMMA 5.3. *Let $\mathfrak{q}$ be a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}'$ and $\chi : \mathfrak{q} \to \mathbb{C}$ a character of $\mathfrak{q}$. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a Levi decomposition, and $\mathfrak{z}_\mathfrak{l}$ the center of the Levi subalgebra $\mathfrak{l}$. Then there is an element $v \in \mathfrak{z}_\mathfrak{l}$ such that*

$$\chi(\cdot) = (v, \cdot) : \mathfrak{q} \longrightarrow \mathbb{C}$$

where $(\cdot, \cdot)$ is the Killing form of $\mathfrak{g}'$.

Proof. Let $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$ be the commutator subalgebra. The decomposition $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{z}_\mathfrak{l}$ is orthogonal with respect to the Killing form $\kappa = (\cdot, \cdot)$ on $\mathfrak{g}'$. Indeed, since κ is $\mathfrak{g}'$-invariant, if l_1, $l_2 \in \mathfrak{l}$ and $z \in \mathfrak{z}_\mathfrak{l}$, then

$$([l_1, l_2], z) = (l_1, [l_2, z]) = 0.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}'$ containing $\mathfrak{z}_\mathfrak{l}$ and contained in $\mathfrak{l}$. The given decomposition of $\mathfrak{l}$ induces a decomposition $\mathfrak{h} = (\mathfrak{l}' \cap \mathfrak{h}) \oplus \mathfrak{z}_\mathfrak{l}$ which is also κ-orthogonal. Let $v \in \mathfrak{h}$ be the element in $\mathfrak{h}$, κ-dual to $\chi|_\mathfrak{h}$, i.e. $\chi(\cdot)$ and $(v, \cdot)$ coincide on $\mathfrak{h}$. We must show that both coincide also on $\mathfrak{q}$. Since both of them are characters of $\mathfrak{q}$, it is enough to show that they agree on the center $\mathfrak{z}_\mathfrak{q}$ of $\mathfrak{q}$, but this holds because $\mathfrak{z}_\mathfrak{q} \subset \mathfrak{z}_\mathfrak{l} \subset \mathfrak{h}$. Finally, the restriction $\chi|_{\mathfrak{l}' \cap \mathfrak{h}}$ is zero because $\mathfrak{l}'$ is semisimple, hence $v \in (\mathfrak{l}' \cap \mathfrak{h})^\perp = \mathfrak{z}_\mathfrak{l}$. □

For a parabolic subalgebra $\mathfrak{q}$ and splitting $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, let $R(\mathfrak{z}_\mathfrak{l}) = R^+(\mathfrak{z}_\mathfrak{l}) \cup R^-(\mathfrak{z}_\mathfrak{l})$ be the decomposition such that $\mathfrak{g}'^\alpha \subset \mathfrak{q}$ when $\alpha \in R^+(\mathfrak{z}_\mathfrak{l})$. Recall that a character χ of $\mathfrak{q}$ is called dominant if $2(\chi, \alpha)/(\alpha, \alpha)$ is a positive integer for all positive $\mathfrak{a}$-roots α. We call it integer if (χ, α) is integer for all $\mathfrak{a}$-roots α. Analogously, if Ξ is a character of a parabolic subgroup Q, then we say that it is dominant (respectively, integer) if the associated character χ of $\mathfrak{q} = \mathrm{Lie}(\mathfrak{q})$ is dominant (respectively, integer).

LEMMA 5.4. *Let G' be a semisimple group. Let P be a principal G'-bundle over a scheme Y (not necessarily proper). There is a canonical bijection between the following sets:*

(1) *Isomorphism classes of reductions to a parabolic subgroup Q on a big open set $U \subset Y$, together with an integer dominant character χ of $\mathfrak{q} = \mathrm{Lie}(Q)$.*

(2) *Isomorphism classes of balanced algebra filtrations*

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \ \dots \ \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E \tag{5.2}$$

of the bundle of algebras $E = P(\mathfrak{g}')$ associated to P by the adjoint representation of G'.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a Levi decomposition, and $v \in \mathfrak{z}_{\mathfrak{l}}$ the element associated by Lemma 5.3 to the character χ in (1). The set of integers $\{\lambda_i\}_{i=1,\dots,t+1}$ in (2) is then just the set $\{\alpha(v)\}_{\alpha \in R(\mathfrak{a}) \cup \{0\}}$

Proof. We start with a filtration (5.2). Take a point x of Y where the filtration is a bundle filtration. Fix an isomorphism between the fiber of E at this point and $\mathfrak{g}'$. We obtain a balanced algebra filtration ${\mathfrak{g}'}_{\lambda_\bullet}$ of $\mathfrak{g}'$. By Lemma 5.2, the associated graded Lie algebra $gr(\mathfrak{g}_{\lambda_\bullet})$ is isomorphic to $\mathfrak{g}'$, and using this isomorphism we obtain a decomposition, giving $\mathfrak{g}'$ the structure of a graded Lie algebra,

$$\mathfrak{g}' = \bigoplus_{i=1}^{t+1} {\mathfrak{g}'}^{\lambda_i}, \tag{5.3}$$

such that

$${\mathfrak{g}'}_{\lambda_i} = \bigoplus_{j=1}^{i} {\mathfrak{g}'}^{\lambda_j} . \tag{5.4}$$

Define a linear endomorphism of $\mathfrak{g}'$ (a key idea we thank to J. M. Marco)

$$\begin{aligned} f : \bigoplus_{i=1}^{t+1} {\mathfrak{g}'}^{\lambda_i} &\longrightarrow \bigoplus_{i=1}^{t+1} {\mathfrak{g}'}^{\lambda_i} \\ v \in {\mathfrak{g}'}^{\lambda_i} &\longmapsto -\lambda_i v \, . \end{aligned}$$

If $v_i \in {\mathfrak{g}'}^{\lambda_i}$ and $v_j \in {\mathfrak{g}'}^{\lambda_j}$, then $[v_i, v_j] \in {\mathfrak{g}'}^{\lambda_i + \lambda_j}$; thus

$$f([v_i, v_j]) = [f(v_i), v_j] + [v_i, f(v_j)] \, ,$$

i.e. f is a derivation. Thus, since $\mathfrak{g}'$ is semisimple, a semisimple element $v \in \mathfrak{g}'$ exists such that $f(\cdot) = [v, \cdot]$. Let $\mathfrak{z}_v$ be the center of the centralizer $\mathfrak{c}_v$ of v. It is a toral algebra. Consider the $\mathfrak{z}_v$-root decomposition (see (5.1) or [B-T, §3])

$$\mathfrak{g}' = \bigoplus_{\alpha \in R(\mathfrak{z}_v) \cup \{0\}} {\mathfrak{g}'}^{\alpha} . \tag{5.5}$$

Note that ${\mathfrak{g}'}^{\alpha=0}$ is just the centralizer $\mathfrak{c}_v$ of v. This decomposition is a refinement of (5.3):

$${\mathfrak{g}'}^{\lambda_i} = \bigoplus_{\alpha(v) = -\lambda_i} {\mathfrak{g}'}^{\alpha} . \tag{5.6}$$

Claim. The direct summand ${\mathfrak{g}'}^{\alpha=0}$ in decomposition (5.5) is equal to the direct summand ${\mathfrak{g}'}^{\lambda_i=0}$ in decomposition (5.3).

To prove this claim, let a $\mathfrak{z}_v$-root α be such that $\alpha(v)=0$. For $x\in{\mathfrak{g}'}^{\alpha}$ it is $[v,x]=\alpha(v)x=0$, i.e. x is in the centralizer $\mathfrak{c}_v$ of v. Thus, by definition of $\mathfrak{z}_v$, $[w,x]=0$ for all $w\in\mathfrak{z}_v$, proving the claim.

As a consequence, for all $\mathfrak{z}_v$-roots α, it is $\alpha(v)\neq 0$, and thus $\alpha(v)>0$ gives a set of positive $\mathfrak{z}_v$-roots $R^+(\mathfrak{z}_v)$. Using (5.4), (5.6) and the claim, we obtain for ${\mathfrak{g}'}_0$ in (5.4)

$$
{\mathfrak{g}'}_0=\bigoplus_{\beta\in R^+(\mathfrak{z}_v)\cup\{0\}}{\mathfrak{g}'}^{\beta};
$$

hence ${\mathfrak{g}'}_0\subset\mathfrak{g}'$ is a parabolic subalgebra ([B-T, §4]). Let U be the big open set where $E_{\lambda_\bullet}$ is a bundle filtration. The inclusion $E_0|_U\subset E|_U$ gives a reduction of structure group P^Q of the principal G'-bundle $P|_U$ to the parabolic subgroup $Q\subset G'$ corresponding to ${\mathfrak{g}'}_0\subset\mathfrak{g}'$, because the stabilizer (under the adjoint action of a connected group) of a parabolic subalgebra is the corresponding parabolic subgroup.

Finally, the character $\chi(\cdot)=(v,\cdot)$ of the parabolic ${\mathfrak{g}'}_0$ is dominant, because $(\chi,\alpha)=\alpha(v)$ is a positive integer for all positive $\mathfrak{z}_v$-roots.

Reciprocally, assume we are given a reduction P^Q of P to a parabolic subgroup Q on a big open set $U\subset Y$ and an integer dominant character χ of $\mathfrak{q}=\mathrm{Lie}(Q)$. Choose a decomposition $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ into a Levi and a unipotent subalgebras, and let $\mathfrak{z}_{\mathfrak{l}}$ be the center of $\mathfrak{l}$. Let $v\in\mathfrak{z}_{\mathfrak{l}}$ be the element associated to χ by Lemma 5.3. Consider the $\mathfrak{z}_{\mathfrak{l}}$-root decomposition of $\mathfrak{g}'$ (see (5.1) or [B-T, §3])

$$
\mathfrak{g}'=\bigoplus_{\alpha\in R(\mathfrak{z}_{\mathfrak{l}})\cup\{0\}}{\mathfrak{g}'}^{\alpha}.
$$

By hypothesis $\alpha(v)=(\chi,\alpha)$ is an integer for all $\mathfrak{z}_{\mathfrak{l}}$-roots α. Define a filtration ${\mathfrak{g}'}_{\lambda_\bullet}$ of $\mathfrak{g}'$ by

$$
{\mathfrak{g}'}_{\lambda_i}=\bigoplus_{-\alpha(v)\leq\lambda_i}{\mathfrak{g}'}^{\alpha}. \tag{5.7}
$$

This is a balanced algebra filtration of $\mathfrak{g}'$, because $\dim{\mathfrak{g}'}^{\alpha}=\dim{\mathfrak{g}'}^{-\alpha}$ and $[{\mathfrak{g}'}^{\alpha},{\mathfrak{g}'}^{\beta}]\subset{\mathfrak{g}'}^{\alpha+\beta}$. Clearly $\mathfrak{q}\subset{\mathfrak{g}'}_0$, and in fact $\mathfrak{q}={\mathfrak{g}'}_0$ because the character χ of $\mathfrak{q}$ is dominant. It is also clear that $\mathfrak{l}\subset\mathfrak{c}_v$, the centralizer of v, and, since χ is dominant, it is $\mathfrak{l}=\mathfrak{c}_v$, hence the center $\mathfrak{z}_{\mathfrak{l}}$ of $\mathfrak{l}$ is the center $\mathfrak{z}_v$ of $\mathfrak{c}_v$.

For adjoint action of Q on $\mathfrak{g}'$ it is

$$
Q\cdot{\mathfrak{g}'}^{\alpha}\subset\bigoplus_{\beta\geq\alpha}{\mathfrak{g}'}^{\beta}.
$$

Thus the filtration (5.7) is preserved by this action: $Q\cdot{\mathfrak{g}'}_{\lambda_i}\subset{\mathfrak{g}'}_{\lambda_i}$. Since P has a reduction to Q on $U\subset Y$, this produces a vector bundle filtration of $E|_U$, and it extends uniquely to a saturated filtration on Y, each vector bundle on U extending to the intersection, inside $E^{\vee\vee}$, of the torsion free sheaf E with its extension to a reflexive sheaf on the whole of X (cf. [Ha, II, Ex. 5.15]).

It is easy to check that the two constructions are inverse to each other. By construction, $\{\lambda_i\}_{i=1,\dots,t+1} = \{\alpha(v)\}_{\alpha\in R(\mathfrak{z}_\mathfrak{l})\cup\{0\}}$. □

LEMMA 5.5. *With the same hypothesis (and notation) as in Lemma 5.4, there are positive integers a and b such that av corresponds to a one-parameter subgroup of Z_L (i.e. its differential is av) and $b\chi$ corresponds to a character of the group Q.*

Proof. Let $\mathfrak{h}$ be a Cartan algebra of $\mathfrak{g}'$ with $\mathfrak{z}_\mathfrak{l} \subset \mathfrak{h} \subset \mathfrak{l}$ and let H be the maximal torus of the connected group G corresponding to $\mathfrak{h}$. Let $R(\mathfrak{h})$ be the set of roots with respect to $\mathfrak{h}$. The element $v \in \mathfrak{z}_\mathfrak{l} \subset \mathfrak{h}$ is in the coweight lattice $\mathbb{Z}(W^\vee)$, because any $\mathfrak{h}$-root gives an integer when evaluated on v. Indeed, the $\mathfrak{z}_\mathfrak{l}$-roots $\alpha : \mathfrak{z}_\mathfrak{l} \to \mathbb{C}$ are obtained by restricting the $\mathfrak{h}$-roots $\beta : \mathfrak{h} \to \mathbb{C}$ to $\mathfrak{z}_\mathfrak{l}$, but by hypothesis, $\alpha(v) \in \mathbb{Z}$ for all $\alpha \in R(\mathfrak{z}_\mathfrak{l})$. Let $X^\vee(H)$ be the lattice of one-parameter subgroups of H. Sending an element of $X^\vee(H)$ to its differential gives an embedding $X^\vee(H) \hookrightarrow \mathbb{Z}(W^\vee)$ with finite quotient, hence there is an integer a such that av corresponds to a one-parameter subgroup of H which can be written as

$$\begin{aligned} \Psi : \mathbb{C}^* &\longrightarrow Z_L \subset H \\ t = e^\tau &\longmapsto e^{\tau a v}, \end{aligned}$$

where Z_L is the center of the Lie subgroup L corresponding to $\mathfrak{l} \subset \mathfrak{g}'$.

On the other hand, the character χ of the parabolic $\mathfrak{q}$ is dominant, and in particular belongs to the weight lattice $\mathbb{Z}(W)$. Let $X(H)$ be the lattice of characters of H. Sending an element of $X(H)$ to its differential defines a lattice embedding $X(H) \hookrightarrow \mathbb{Z}(W)$ with finite quotient, hence there is an integer b such that $b\chi$ corresponds to a character $\Xi \in X(H)$, i.e. the differential of Ξ is $b\chi$.

Let $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$ be the commutator subalgebra, and $L' = [L, L]$ the commutator subgroup. Recall that a character of $\mathfrak{q}$ factors as $\mathfrak{q} \twoheadrightarrow \mathfrak{l} \twoheadrightarrow \mathfrak{l}/\mathfrak{l}' \twoheadrightarrow \mathbb{C}$; hence χ vanishes on $\mathfrak{l}'$. Thus the character Ξ of H vanishes on $H \cap L'$, so Ξ gives a group homomorphism $L/L' \cong H/(H \cap L') \to \mathbb{C}^*$. Composing with the quotient $Q \twoheadrightarrow L \twoheadrightarrow L/L'$ we obtain a character of Q whose differential is χ. □

LEMMA 5.6. *Let P be a principal G'-bundle over a big open set $U \subset X$ with a reduction P^Q to a parabolic subgroup $Q \subset G'$ on a big open set $U' \subset U$, Ξ be an integer dominant character of Q, and let*

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \dots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E = P(\mathfrak{g}') \tag{5.8}$$

be the balanced algebra filtration associated to it by Lemma 5.4. *Then*

$$\sum_{i=1}^{t+1}(\lambda_{i+1}-\lambda_i)\deg E_{\lambda_i} = \deg P^Q(\Xi) \tag{5.9}$$

where $P^Q(\Xi)$ *is the line bundle associated to* P^Q *by the character* Ξ*.*

Proof. Let $L \subset Q$ be a Levi factor of Q, and Z_L the center of L. For $\mathfrak{z}_\mathfrak{l} = \mathrm{Lie}(Z_L)$ consider the $\mathfrak{z}_\mathfrak{l}$-root decomposition of $\mathfrak{g}'$ (cf. (5.1))

$$\mathfrak{g}' = \bigoplus_{\alpha\in R(\mathfrak{z}_\mathfrak{l})\cup\{0\}} \mathfrak{g}'^{\alpha}.$$

Let $v \in \mathfrak{z}_\mathfrak{l}$ be the element associated to χ by Lemma 5.3. Define an order $<_v$ in the set $R(\mathfrak{z}_\mathfrak{l})\cup\{0\}$ by declaring $\alpha <_v \alpha'$ if $(\alpha-\alpha')(v)<0$. In general, $<_v$ is not a total order, because it may happen that $(\alpha'-\alpha)(v)=0$ even if α and α' are different. Choose a refinement of this to get a total order $\prec$. Number all the roots (including $\alpha=0$) by $\alpha_1 \succ \alpha_2 \succ \cdots \succ \alpha_{l+1}$ in descending order, and define a filtration $\mathfrak{g}'_\bullet$

$$0 \subsetneq \mathfrak{g}'_{\alpha_1} \subsetneq \mathfrak{g}'_{\alpha_2} \subsetneq \ \dots\ \subsetneq \mathfrak{g}'_{\alpha_l} \subsetneq \mathfrak{g}'_{\alpha_{l+1}} = \mathfrak{g}', \quad \text{with } \mathfrak{g}'_{\alpha_i} = \bigoplus_{j=1}^{i} \mathfrak{g}'^{\alpha_j}. \tag{5.10}$$

For the adjoint action of Q on $\mathfrak{g}'$ it is

$$Q\cdot \mathfrak{g}'^{\alpha} \subset \bigoplus_{\beta\geq\alpha}\mathfrak{g}'^{\beta} \subset \bigoplus_{\beta\succeq\alpha}\mathfrak{g}'^{\beta}.$$

This has two consequences: on the one hand, there is an induced action of Q on

$$(\mathrm{gr}\,\mathfrak{g}')^{\alpha_i} := \mathfrak{g}'_{\alpha_i}/\mathfrak{g}'_{\alpha_{i-1}},$$

and, on the other hand, P^Q produces a vector bundle filtration of $E|_{U'}$, and this extends to a saturated filtration on U

$$0 \subsetneq E_{\alpha_1} \subsetneq E_{\alpha_2} \subsetneq \ \dots\ \subsetneq E_{\alpha_l} \subsetneq E_{\alpha_{l+1}} = E. \tag{5.11}$$

Note that, although as vector spaces both $\mathfrak{g}'^{\alpha}$ and $(\mathrm{gr}\,\mathfrak{g}')^{\alpha_i}$ are isomorphic, they are not isomorphic as Q-modules: indeed, while $Q\cdot(\mathrm{gr}\,\mathfrak{g}')^{\alpha_i} \subset (\mathrm{gr}\,\mathfrak{g}')^{\alpha_i}$, in general we only have $Q\cdot\mathfrak{g}'^{\alpha}\subset\bigoplus_{\beta\geq\alpha}\mathfrak{g}'^{\beta}$.

The filtration (5.11) is a refinement of (5.8), with

$$E_{\lambda_i} = E_\alpha, \quad \alpha = \max_{\prec}\left\{\beta\in R(\mathfrak{z}_\mathfrak{l})\cup\{0\} : -(\chi,\alpha) = -\alpha(v) \leq \lambda_i\right\}. \tag{5.12}$$

Furthermore, $E^{\alpha_i} = E_{\alpha_i}/E_{\alpha_{i-1}}$ is isomorphic to the vector bundle associated to P^Q using the action of Q on $(\mathrm{gr}\,\mathfrak{g}')^{\alpha}$. Since this filtration is a refinement of (5.8), it is

$$\deg(E^{\lambda_i}) = \sum_{\alpha(v)=-\lambda_i}\deg(E^{\alpha}), \tag{5.13}$$

where $E^{\lambda_i} = E_{\lambda_i}/E_{\lambda_{i-1}}$.

For each $\mathfrak{z}_{\mathfrak{l}}$-root α the adjoint action of Q on $(\operatorname{gr}\mathfrak{g}')^\alpha$ gives a character

$$\phi_\alpha : Q \xrightarrow{\text{ad}} \mathrm{GL}\big((\operatorname{gr}\mathfrak{g}')^\alpha\big) \xrightarrow{\det} \mathbb{C}^* .$$

Every character of a parabolic subgroup factors through its Levi quotient L, and two characters are equal if they coincide when restricted to its center Z_L. We have a commutative diagram

$$\begin{tikzcd} Q \arrow[r, "\text{ad}"] \arrow[d, two heads] & \mathrm{GL}\big((\operatorname{gr}\mathfrak{g}')^\alpha\big) \arrow[r, "\det"] & \mathbb{C}^* \arrow[d, equal] \\ L \arrow[r, "\text{ad}"] & \mathrm{GL}\big((\operatorname{gr}\mathfrak{g}')^\alpha\big) \arrow[r, "\det"] & \mathbb{C}^* \arrow[d, equal] \\ Z_L \arrow[r, "\text{ad}"] \arrow[u, hook] & \mathrm{GL}\big((\operatorname{gr}\mathfrak{g}')^\alpha\big) \arrow[r, "\det"] & \mathbb{C}^* \end{tikzcd}$$

It follows that

$$\phi_\alpha = \overline{(\dim \mathfrak{g}'^\alpha)\alpha},$$

where we denote by $\overline{(\dim \mathfrak{g}'^\alpha)\alpha}$ the character of Q such that, after restricting to a character $Z_L \to \mathbb{C}^*$, the induced Lie algebra homomorphism $\mathfrak{z}_{\mathfrak{l}} \to \mathbb{C}$ is $(\dim \mathfrak{g}'^\alpha)\alpha$. Hence,

$$\det E^\alpha \cong P^Q\big(\overline{(\dim \mathfrak{g}'^\alpha)\alpha}\big). \tag{5.14}$$

Using equation (5.13), the left-hand side of (5.9) is equal to the degree of the line bundle

$$\bigotimes_{i=1}^{t+1} (\det E^{\lambda_i})^{-\lambda_i} = \bigotimes_{\alpha\in R(\mathfrak{z}_{\mathfrak{l}})\cup\{0\}} (\det E^\alpha)^{\alpha(v)} .$$

Using (5.14), this line bundle is equal to

$$P^Q\Big(\overline{\sum_{\alpha\in R(\mathfrak{z}_{\mathfrak{l}})\cup\{0\}} \alpha(v)(\dim \mathfrak{g}'^\alpha)\alpha}\Big) . \tag{5.15}$$

Claim.

$$\sum_{\alpha\in R(\mathfrak{z}_{\mathfrak{l}})\cup\{0\}} \alpha(v)(\dim \mathfrak{g}'^\alpha)\alpha = \chi .$$

Indeed, if $w \in \mathfrak{z}_{\mathfrak{l}}$, then

$$\chi(w) \; = (v,w) \; = \; \operatorname{tr}([v,.]\circ[w,.]) \; = \sum_{\alpha\in R(\mathfrak{z}_{\mathfrak{l}})\cup\{0\}} (\dim \mathfrak{g}'^\alpha)\alpha(v)\alpha(w) ,$$

and the claim follows.

Since $\overline{\chi} = \Xi$, it follows that the line bundle (5.15) is isomorphic to $P^Q(\Xi)$, and the lemma is proved. $\square$

Corollary 5.7. *A principal G-sheaf $\mathcal{P} = (P, E, \psi)$ is slope-(semi)stable if and only if the associated rational principal G-bundle $P \to U \subset X$ is (semi)stable in the sense of Ramanathan.*

Proof. Without loss of generality, we can assume that G is semisimple. Assume that $\mathcal{P}$ is slope-(semi)stable. Consider a reduction to a parabolic subgroup Q of $P|_{U'} \to U' \subset U$, where U' is a big open set, and consider a dominant character Ξ of Q. This gives a dominant character χ of $\mathfrak{q} = \mathrm{Lie}(Q)$. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{z}$ be a Levi decomposition and $\mathfrak{z}_{\mathfrak{l}}$ the center of $\mathfrak{l}$. A positive integer multiple $\widetilde{\chi} = c\chi$ has the property that $(\widetilde{\chi}, \alpha)$ is integer for all $\mathfrak{z}_{\mathfrak{l}}$-roots α. Consider the balanced algebra filtration $\widetilde{E}^{U'}_{\lambda_\bullet}$ associated to $\widetilde{\chi}$ by Lemma 5.4.

This filtration of $E|_{U'}$ can be extended uniquely to a saturated filtration $\widetilde{E}_{\lambda_\bullet}$ of E on X. By Lemma 5.6, and using the slope-(semi)stability of $\mathcal{P}$,

$$\deg P^Q(\Xi) = \sum_{i=1}^{t+1} \frac{\lambda_{i+1} - \lambda_i}{c} \deg E_{\lambda_i} \ (\leq)\ 0.$$

This means that $P \to U \subset X$ is Ramanathan (semi)stable.

Conversely, assume that $P \to U \subset X$ is Ramanathan (semi)stable. Consider a balanced algebra filtration of E. Let $U' \subset U \subset X$ be the big open set where this is a bundle filtration. Lemma 5.4 produces a reduction P^Q on U' of P to a parabolic subgroup and a dominant character χ of $\mathfrak{q} = \mathrm{Lie}(Q)$. By Lemma 5.5, there is a positive integer b such that $b\chi$ corresponds to a character $\widetilde{\Xi}$ of Q. Then, by Lemma 5.6 and because of the Ramanathan (semi)stability of P, it is

$$\sum_{i=1}^{t+1} (\lambda_{i+1} - \lambda_i) \deg E_{\lambda_i} = \frac{1}{b} \deg P^Q(\widetilde{\Xi}) \ (\leq)\ 0\,;$$

i.e. $\mathcal{P}$ is slope-(semi)stable. □

Corollary 5.8. *If X is a curve, our notion of (semi)stability for principal bundles coincides with that of Ramanathan.*

Let us characterize (semi)stability in terms of the Killing form, as announced in the introduction. An *orthogonal sheaf*, relative to a scheme S, is a pair

$$(E_S, E_S \otimes E_S \longrightarrow \mathcal{O}_{X \times S})$$

such that the bilinear form induced on the fibers of E_S over closed points $(x, s) \in X \times S$ where it is locally free, is nondegenerate. For instance, if (E_S, φ_S) is a $\mathfrak{g}'$-sheaf, the Killing form gives an orthogonal structure to E_S.

Definition 5.9 (Orthogonal filtration). A filtration $E_\bullet \subset E$ of an orthogonal sheaf is said to be orthogonal if $E_i^\perp = E_{-i-1}$ for all i. In terms of $E_{\lambda_\bullet}$: if the integers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_t < \lambda_{t+1}$$

can be denoted

$$\gamma_{-l} < \gamma_{-l+1} < \cdots < \gamma_{l-1} < \gamma_l$$

so that

$$\gamma_{-i} = -\gamma_i, \quad \text{and} \quad E^{\perp}_{\gamma_i} = E_{\gamma_{-i-1}}\,.$$

Observe that an orthogonal filtration is necessarily balanced. These filtrations were introduced in our former article [G-S1] in order to define the (semi)stability of an orthogonal sheaf as the condition of admitting no orthogonal filtration of negative (nonpositive) Hilbert polynomial.

COROLLARY 5.10. *Let $\mathcal{P} = (P, E, \psi)$ be a principal G-sheaf, or just let (E, φ) be a $\mathfrak{g}'$-sheaf. An algebra filtration of E is balanced if and only if it is orthogonal. Therefore, $\mathcal{P}$ is (semi)stable in the sense of Definition* 0.26 *if and only if it is so in the sense of Definition* 0.3.

Proof. We have seen that a balanced algebra filtration of $\mathfrak{g}'$-sheaves is induced from a filtration of Lie algebras as in (5.10). On the other hand, for a semisimple Lie algebra we have

$$(\mathfrak{g}'^{\alpha})^{\perp} = \bigoplus_{\beta \neq -\alpha} \mathfrak{g}'^{\beta}$$

for α, $\beta \in R(\mathfrak{h}) \cup \{0\}$. The first statement follows easily from these two facts. The second follows from the first. □

Finally, we remark that the notion of a semistable sheaf of rank r is not equivalent to the one of semistable principal $\mathrm{GL}(r)$-sheaf. Think, for instance, of vector bundles of rank 2 on $\mathbb{P}^2$ with $c_1 = 1$ and $c_2 = 0$ and their corresponding principal $\mathrm{GL}(2)$-bundles of numerical invariants $(d_1 = 1, c_1 = 0, c_2 = 7)$. It can be seen that the moduli space of semistable sheaves of rank 2, $c_1 = 1$ and $c_2 = 2$, and the one of semistable principal $\mathrm{GL}(2)$-sheaves on $\mathbb{P}^2$ with numerical invariants $(d_1 = 1, c_1 = 0, c_2 = 7)$ are nonisomorphic.

In the next simplest case, the plane $\widetilde{\mathbb{P}}^2$ blown up in a closed point, we can observe that even the notion of (semi)stability itself differs for vector bundles and for their corresponding principal bundles:

Denote D the exceptional divisor, R the fiber of the natural ruling, and p a closed point outside E. For m, $c \in \mathbb{Z}$, let $L = \mathcal{O}_{\widetilde{\mathbb{P}}^2}(mR)$, $M = \mathcal{O}_{\widetilde{\mathbb{P}}^2}(-cD + (c+m)R)$, and F an extension

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \otimes I_p \longrightarrow 0$$

which is locally free, i.e. not in the kernel of α

$$\mathrm{Ext}^1(M \otimes I_p, L) \xrightarrow{\alpha} H^0(\mathcal{E}xt^1(M \otimes I_p, L)) = \mathbb{C} \longrightarrow H^2(M^{\vee} \otimes L) = 0\,.$$

For the vector bundle F and the corresponding principal GL(2)-bundle, the following table of (semi)stability, with respect to the ample line bundle $\mathcal{O}_{\widetilde{\mathbb{P}}^2}(D+2R)$, can be checked (for details, see `math.AG/0206277 v2`):

(c,m)	Vector bundle	Principal bundle
(-1,4)	unstable	unstable
(-1,3)	semistable	unstable
(-1,2)	stable	unstable
(3,-4)	unstable	semistable
(3,-5)	stable	semistable
(4,-5)	unstable	stable
(4,-6)	stable	stable

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