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On the holomorphicity of genus two Lefschetz fibrations

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Abstract

We prove that any genus-2 Lefschetz fibration without reducible fibers and with "transitive monodromy" is holomorphic. The latter condition comprises all cases where the number of singular fibers $\mu \in 10\mathbb{N}$ is not congruent to 0 modulo 40. This proves a conjecture of the authors in [SiTi1]. An auxiliary statement of independent interest is the holomorphicity of symplectic surfaces in S^2-bundles over S^2, of relative degree ≤ 7 over the base, and of symplectic surfaces in $\mathbb{CP}^2$ of degree ≤ 17.

Contents

Introduction
1. Pseudo-holomorphic S^2-bundles
2. Pseudo-holomorphic cycles on pseudo-holomorphic S^2-bundles
3. The $\mathcal{C}^0$-topology on the space of pseudo-holomorphic cycles
4. Unobstructed deformations of pseudo-holomorphic cycle
5. Good almost complex structures
6. Generic paths and smoothings
7. Pseudo-holomorphic spheres with prescribed singularities
8. An isotopy lemma
9. Proofs of Theorems A, B and C
References

Introduction

A differentiable Lefschetz fibration of a closed oriented four-manifold M is a differentiable surjection $p : M \to S^2$ with only finitely many critical points of the form $t \circ p(z, w) = zw$. Here z, w and t are complex coordinates on M and S^2 respectively that are compatible with the orientations. This generalization of classical Lefschetz fibrations in Algebraic Geometry was introduced

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by Moishezon in the late seventies for the study of complex surfaces from the differentiable viewpoint [Mo1]. It is then natural to ask how far differentiable Lefschetz fibrations are from holomorphic ones. This question becomes even more interesting in view of Donaldson's result on the existence of symplectic Lefschetz pencils on arbitrary symplectic manifolds [Do]. Conversely, by an observation of Gompf total spaces of differentiable Lefschetz fibrations have a symplectic structure that is unique up to isotopy. The study of differentiable Lefschetz fibrations is therefore essentially equivalent to the study of symplectic manifolds.

In dimension 4 apparent invariants of a Lefschetz fibration are the genus of the nonsingular fibers and the number and types of irreducible fibers. By the work of Gromov and McDuff [MD] any genus-0 Lefschetz fibration is in fact holomorphic. Likewise, for genus 1 the topological classification of elliptic fibrations by Moishezon and Livné [Mo1] implies holomorphicity in all cases. We conjectured in [SiTi1] that all hyperelliptic Lefschetz fibrations without reducible fibers are holomorphic. Our main theorem proves this conjecture in genus 2. This conjecture is equivalent to a statement for braid factorizations that we recall below for genus 2 (Corollary 0.2).

Note that for genus larger than 1 the mapping class group becomes reasonably general and group-theoretic arguments as in the treatment of the elliptic case by Moishezon and Livné seem hopeless. On the other hand, our methods also give the first geometric proof for the classification in genus 1.

We say that a Lefschetz fibration has *transitive monodromy* if its monodromy generates the mapping class group of a general fiber.

Theorem A. *Let $p : M \to S^2$ be a genus-2 differentiable Lefschetz fibration with transitive monodromy. If all singular fibers are irreducible then p is isomorphic to a holomorphic Lefschetz fibration.*

Note that the conclusion of the theorem becomes false if we allow reducible fibers; see e.g. [OzSt]. The authors expect that a genus-2 Lefschetz fibration with μ singular fibers, t of which are reducible, is holomorphic if $t \leq c \cdot \mu$ for some universal constant c. This problem should also be solvable by the method presented in this paper. One consequence would be that any genus-2 Lefschetz fibration should become holomorphic after fiber sum with sufficiently many copies of the rational genus-2 Lefschetz fibration with 20 irreducible singular fibers. Based on the main result of this paper, this latter statement has been proved recently by Auroux using braid-theoretic techniques [Au].

In [SiTi1] we showed that a genus-2 Lefschetz fibration without reducible fibers is a two-fold branched cover of an S^2-bundle over S^2. The branch locus is a symplectic surface of degree 6 over the base, and it is connected if and only if the Lefschetz fibration has transitive monodromy. The main theorem therefore follows essentially from the next isotopy result for symplectic surfaces in rational ruled symplectic 4-manifolds.

THEOREM B. *Let $p : M \to S^2$ be an S^2-bundle and $\Sigma \subset M$ a connected surface symplectic with respect to a symplectic form that is isotopic to a Kähler form. If $\deg(p|_\sigma) \leq 7$ then Σ is symplectically isotopic to a holomorphic curve in M, for some choice of complex structure on M.*

Remark 0.1. By Gromov-Witten theory there exist surfaces $H, F \subset M$, homologous to a section with self-intersection 0 or 1 and a fiber, respectively, with $\Sigma \cdot H \geq 0$, $\Sigma \cdot F \geq 0$. It follows that $c_1(M) \cdot \Sigma > 0$ unless Σ is homologous to a negative section. In the latter case Proposition 1.7 produces an isotopy to a section with negative self-intersection number. The result follows then by the classification of S^2-bundles with section. We may therefore add the positivity assumption $c_1(M) \cdot \Sigma > 0$ to the hypothesis of the theorem. The complex structure on M may then be taken to be generic, thus leading to $\mathbb{CP}^2$ or the first Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))$.

For the following algebraic reformulation of Theorem A recall that *Hurwitz equivalence* on words with letters in a group G is the equivalence relation generated by

$$g_1 \ldots g_i g_{i+1} \ldots g_k \sim g_1 \ldots [g_i g_{i+1} g_i^{-1}] g_i \ldots g_k.$$

The bracket is to be evaluated in G and takes up the i^{th} position. Hurwitz equivalence in braid groups is useful for the study of algebraic curves in rational surfaces. This point of view dates back to Chisini in the 1930's [Ch]. It has been extensively used and popularized in work of Moishezon and Teicher [Mo2], [MoTe]. In this language Theorem A says the following.

COROLLARY 0.2. *Let $x_1, \ldots, x_{d-1}$ be standard generators for the braid group $B(S^2, d)$ of S^2 on $d \leq 7$ strands. Assume that $g_1 g_2 \ldots g_k$ is a word in positive half-twists $g_i \in B(S^2, d)$ with* (a) $\prod_i g_i = 1$ *or* (b) $\prod_i g_i = (x_1 x_2 \ldots x_{d-1})^d$. *Then $k \equiv 0 \bmod 2(d-1)$ and $g_1 g_2 \ldots g_k$ is Hurwitz equivalent to*

(a) $(x_1 x_2 \ldots x_{d-1} x_{d-1} \ldots x_2 x_1)^{\frac{k}{2d-2}}$

(b) $(x_1 x_2 \ldots x_{d-1} x_{d-1} \ldots x_2 x_1)^{\frac{k}{2d-2} - d(d-1)} (x_1 x_2 \ldots x_{d-1})^d$.

Proof. The given word is the braid monodromy of a symplectic surface Σ in (a) $\mathbb{CP}^1 \times \mathbb{CP}^1$ or (b) $\mathbb{F}_1$ respectively [SiTi1]. The number k is the cardinality of the set $S \subset \mathbb{CP}^1$ of critical values of the projection $\Sigma \to \mathbb{CP}^1$. By the theorem we may assume Σ to be algebraic. A straightfoward explicit computation gives the claimed form of the monodromy for some distinguished choice of generators of the fundamental group of $\mathbb{CP}^1 \setminus S$. The change of generators leads to Hurwitz equivalence between the respective monodromy words. □

In the disconnected case there are exactly two components and one of them is a section with negative, even self-intersection number. Such curves are nongeneric from a pseudo-holomorphic point of view and seem difficult to deal with analytically. One possibility may be to employ braid-theoretic arguments to reduce to the connected case. We hope to treat this case in a future paper.

A similar result holds for surfaces of low degree in $\mathbb{CP}^2$.

THEOREM C. *Any symplectic surface in* $\mathbb{CP}^2$ *of degree* $d \leq 17$ *is symplectically isotopic to an algebraic curve.*

For $d = 1, 2$ this theorem is due to Gromov [Gv], for $d = 3$ to Sikorav [Sk] and for $d \leq 6$ to Shevchishin [Sh]. Note that for other symplectic 4-manifolds homologous symplectic submanifolds need not be isotopic. The hyperelliptic branch loci of the examples in [OzSt] provide an infinite series inside a blown-up S^2-bundle over S^2. Furthermore a quite general construction for homologous, nonisotopic tori in nonrational 4-manifolds has been given by Fintushel and Stern [FiSt].

Together with the classification of symplectic structures on S^2-bundles over S^2 by McDuff, Lalonde, A. K. Liu and T. J. Li (see [LaMD] and references therein) our results imply a stronger classification of symplectic submanifolds up to Hamiltonian symplectomorphism. Here we wish to add only the simple observation that a symplectic isotopy of symplectic submanifolds comes from a family of Hamiltonian symplectomorphisms.

PROPOSITION 0.3. *Let* (M, ω) *be a symplectic* 4*-manifold and assume that* $\Sigma_t \subset M$, $t \in [0, 1]$ *is a family of symplectic submanifolds. Then there exists a family* Ψ_t *of Hamiltonian symplectomorphisms of* M *with* $\Psi_0 = \mathrm{id}$ *and* $\Sigma_t = \Psi_t(\Sigma_0)$ *for every* t.

Proof. At a $P \in \Sigma_{t_0}$ choose complex Darboux coordinates $z = x + iy$, $w = u + iv$ with $w = 0$ describing Σ_{t_0}. In particular, $\omega = dx \wedge dy + du \wedge dv$. For t close to t_0 let f_t, g_t be the functions describing Σ_t as graph $w = f_t(z) + i g_t(z)$. Define

$$H_t = -(\partial_t g_t) \cdot (u - f_t) + (\partial_t f_t) \cdot (v - g_t).$$

Then for every fixed t

$$dH_t = -(u - f_t)\partial_t(dg_t) + (v - g_t)\partial_t(df_t) - (\partial_t g_t)du + (\partial_t f_t)dv.$$

Thus along Σ_t

$$dH_t = -(\partial_t g_t)du + (\partial_t f_t)dv = \omega \neg \big((\partial_t f_t)\partial_u + (\partial_t g_t)\partial_v\big).$$

The Hamiltonian vector field belonging to H_t thus induces the given deformation of Σ_t.

To globalize patch the functions H_t constructed locally over Σ_{t_0} by a partition of unity on Σ_{t_0}. As H_t vanishes along Σ_t, at time t the associated Hamiltonian vector field along Σ_t remains unchanged. Extend H_t to all of M arbitrarily. Finally extend the construction to all $t \in [0,1]$ by a partition of unity argument in t. □

Guide to content. The proofs in Section 9 of the main theorems follow essentially by standard arguments from the Isotopy Lemma in Section 8, which is the main technical result. It is a statement about the uniqueness of isotopy classes of pseudo-holomorphic smoothings of a pseudo-holomorphic cycle $C_\infty = \sum_a m_a C_{\infty,a}$ in an S^2-bundle M over S^2. In analogy with the integrable situation we expect such uniqueness to hold whenever $c_1(M) \cdot C_{\infty,a} > 0$ for every a. In lack of a good parametrization of pseudo-holomorphic cycles in the nonintegrable case we need to impose two more conditions. The first one is inequality $(*)$ in the Isotopy Lemma 8.1

$$\sum_{\{a | m_a > 1\}} \big(c_1(M) \cdot C_{\infty,a} + g(C_{\infty,a}) - 1 \big) < c_1(M) \cdot C_\infty - 1.$$

The sum on the left-hand side counts the expected dimension of the space of *equigeneric* deformations of the multiple components of C_∞. A deformation of a pseudo-holomorphic curve $C \subset M$ is equigeneric if it comes from a deformation of the generically injective pseudo-holomorphic map $\Sigma \to M$ with image C. The term $c_1(M) \cdot C_\infty$ on the right-hand side is the amount of positivity that we have. In other words, on a smooth pseudo-holomorphic curve homologous to C we may impose $c_1(M) \cdot C - 1$ point conditions without loosing unobstructedness of deformations. It is this inequality that brings in the degree bounds in our theorems; see Lemma 9.1.

The Isotopy Lemma would not lead very far if the sum involved also the nonmultiple components. But we may always add spherical ($g = 0$), nonmultiple components to C_∞ on both sides of the inequality. This brings in the second restriction that M is an S^2-bundle over S^2, for then it is a Kähler surface with lots of rational curves. The content of Section 7 is that for S^2-bundles over S^2 we may approximate any pseudo-holomorphic singularity by the singularity of a pseudo-holomorphic sphere with otherwise only nodes. The proof of this result uses a variant of Gromov-Witten theory. As our isotopy between smoothings of C_∞ stays close to the support $|C_\infty|$ it does not show any interesting behaviour near nonmultiple components. Therefore we may replace nonmultiple components by spheres, at the price of introducing nodes. After this reduction we may take the sum on the left-hand side of $(*)$ over all components.

The second crucial simplification is that we may change our limit almost complex structure J_∞ into an almost complex structure $\tilde{J}_\infty$ that is integrable near $|C_\infty|$. This might seem strange, but the point of course is that if $C_n \to C_\infty$

then C_n will generally not be pseudo-holomorphic for $\tilde{J}_\infty$. Hence we cannot simply reduce to the integrable situation. In fact, we will even get a rather weak convergence of almost complex structures $\tilde{J}_n \to \tilde{J}_\infty$ for some almost complex structures $\tilde{J}_n$ making C_n pseudo-holomorphic. The convergence is C^0 everywhere and $C^{0,\alpha}$ away from finitely many points. The construction in Section 5 uses Micallef and White's result on the holomorphicity of pseudo-holomorphic curve singularities [MiWh].

The proof of the Isotopy Lemma then proceeds by descending induction on the multiplicities of the components and the badness of the singularities of the underlying pseudo-holomorphic curve $|C_\infty|$, measured by the virtual number of double points. We sketch here only the case with multiple components. The reduced case requires a modified argument that we give in Step 7 of the proof of the Isotopy Lemma. It would also follow quite generally from Shevchishin's local isotopy theorem [Sh]. By inequality $(*)$ we may impose enough point conditions on $|C_\infty|$ such that any nontrivial deformation of $|C_\infty|$, fulfilling the point conditions and pseudo-holomorphic with respect to a sufficiently general almost complex structure, cannot be equisingular. Hence the induction hypothesis applies to such deformations. Here we use Shevchishin's theory of equisingular deformations of pseudo-holomorphic curves [Sh]. Now for a sequence of smoothings C_n we try to generate such a deformation by imposing one more point condition on C_n that we move away from C_n, uniformly in n. This deformation is always possible since we can use the induction hypothesis to pass by any trouble point. By what we said before the induction hypothesis applies to the limit of the deformed C_n. This shows that C_n is isotopic to a $\tilde{J}_\infty$-holomorphic smoothing of C_∞.

As we changed our almost complex structure we still need to relate this smoothing to smoothings with respect to the original almost complex structure J_∞. But for a J_∞-holomorphic smoothing of C_∞ the same arguments give an isotopy with another $\tilde{J}_\infty$-holomorphic smoothing of C_∞. So we just need to show uniqueness of smoothings in the integrable situation, locally around $|C_\infty|$. We prove this in Section 4 by parametrizing holomorphic deformations of C_∞ in M by solutions of a nonlinear $\bar{\partial}$-operator on sections of a holomorphic vector bundle on $\mathbb{CP}^1$. The linearization of this operator is surjective by a complex-analytic argument involving Serre duality on C, viewed as a nonreduced complex space, together with the assumption $c_1(M) \cdot C_{\infty,a} > 0$.

One final important point, both in applications of the lemma as well as in the deformation of C_n in its proof, is the existence of pseudo-holomorphic deformations of a pseudo-holomorphic cycle under assumptions on genericity of the almost complex structure and positivity. This follows from the work of Shevchishin on the second variation of the pseudo-holomorphicity equation [Sh], together with an essentially standard deformation theory for nodal curves, detailed in [Sk]. The mentioned work of Shevchishin implies that for any sufficiently generic almost complex structure the space of equigeneric deformations is not locally disconnected by nonimmersed curves, and the projection to the base space of a one-parameter family of almost complex structures is open. From this one obtains smoothings by first doing an equigeneric deformation into a nodal curve and then a further small, embedded deformation smoothing out the nodes. Note that these smoothings lie in a unique isotopy class, but we never use this in our proof.

Conventions. We endow complex manifolds such as $\mathbb{CP}^n$ or $\mathbb{F}_1$ with their integrable complex structures, when viewed as almost complex manifolds. A map $F : (M, J_M) \to (N, J_N)$ of almost complex manifolds is *pseudo-holomorphic* if $DF \circ J_M = J_N \circ DF$. A *pseudo-holomorphic curve* C in (M, J) is the image of a pseudo-holomorphic map $\varphi : (\Sigma, j) \to (M, J)$ with Σ a not necessarily connected Riemann surface. If Σ may be chosen connected then C is *irreducible* and its *genus* $g(C)$ is the genus of Σ for the generically injective φ. If $g(C) = 0$ then C is *rational*.

A J*-holomorphic* 2*-cycle* in an almost complex manifold (M, J) is a locally finite formal linear combination $C = \sum_a m_a C_a$ where $m_a \in \mathbb{Z}$ and $C_a \subset M$ is a J-holomorphic curve. The support $\bigcup_a C_a$ of C will be denoted $|C|$. The subset of singular and regular points of $|C|$ are denoted $|C|_{\text{sing}}$ and $|C|_{\text{reg}}$ respectively. If all $m_a = 1$ the cycle is *reduced*. We identify such C with their associated pseudo-holomorphic curve $|C|$. A *smoothing* of a pseudo-holomorphic cycle C is a sequence $\{C_n\}$ of smooth pseudo-holomorphic cycles with $C_n \to C$ in the $\mathcal{C}^0$-topology; see Section 3. By abuse of notation we often just speak of a smoothing $C^\dagger$ of C meaning $C^\dagger = C_n$ with $n \gg 0$ as needed.

For an almost complex manifold $\Lambda^{0,1}$ denotes the bundle of $(0, 1)$-forms. *Complex coordinates* on an even-dimensional, oriented manifold M are the components of an oriented chart $M \supset U \to \mathbb{C}^n$. Throughout the paper we fix some $0 < \alpha < 1$. Almost complex structures will be of class $\mathcal{C}^l$ for some sufficiently large integer l unless otherwise mentioned. The unit disk in $\mathbb{C}$ is denoted Δ. If S is a finite set then $\sharp S$ is its cardinality. We measure distances on M with respect to any Riemannian metric, chosen once and for all. The symbol $\sim$ denotes homological equivalence. An *exceptional sphere* in an oriented manifold is an embedded, oriented 2-sphere with self-intersection number -1.

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1. Pseudo-holomorphic S^2-bundles

In our proof of the isotopy theorems it will be crucial to reduce to a fibered situation. In Sections 1, 2 and 4 we introduce the notation and some of the tools that we have at disposal in this case.

Definition 1.1. Let $p : M \to B$ be a smooth S^2-fiber bundle. If $M = (M, \omega)$ is a symplectic manifold and all fibers $p^{-1}(b)$ are symplectic we speak of a *symplectic S^2-bundle*. If $M = (M, J)$ and $B = (B, j)$ are almost complex manifolds and p is pseudo-holomorphic we speak of a *pseudo-holomorphic* S^2-bundle. If both preceding instances apply and ω tames J then $p : (M, \omega, J) \to (B, j)$ is a *symplectic pseudo-holomorphic S^2-bundle*.

In the sequel we will only consider the case $B = \mathbb{CP}^1$. Then $M \to \mathbb{CP}^1$ is differentiably isomorphic to one of the holomorphic $\mathbb{CP}^1$-bundles $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ or $\mathbb{F}_1 \to \mathbb{CP}^1$.

Any almost complex structure making a symplectic fiber bundle over a symplectic base pseudo-holomorphic is tamed by some symplectic form. To simplify computations we restrict ourselves to dimension 4.

PROPOSITION 1.2. *Let (M, ω) be a closed symplectic 4-manifold and $p : M \to B$ a smooth fiber bundle with all fibers symplectic. Then for any symplectic form ω_B on B and any almost complex structure J on M making the fibers of p pseudo-holomorphic, $\omega_k := \omega + k\, p^*(\omega_B)$ tames J for $k \gg 0$.*

Proof. Since tamedness is an open condition and M is compact it suffices to verify the claim at one point $P \in M$. Write $F = p^{-1}(p(P))$. Choose a frame ∂_u, ∂_v for $T_P F$ with

$$J(\partial_u) = \partial_v, \quad \omega(\partial_u, \partial_v) = 1.$$

Similarly let ∂_x, ∂_y be a frame for the ω-perpendicular plane $(T_P F)^\perp \subset T_P M$ with

$$J(\partial_x) = \partial_y + \lambda\partial_u + \mu\partial_v, \quad \omega(\partial_x, \partial_y) = 1$$

for some $\lambda, \mu \in \mathbb{R}$. By rescaling ω_B we may also assume $(p^*\omega_B)(\partial_x, \partial_y) = 1$. Replacing ∂_x, ∂_y by $\cos(t)\partial_x + \sin(t)\partial_y$, $-\sin(t)\partial_x + \cos(t)\partial_y$, $t \in [0, 2\pi]$, the coefficients $\lambda = \lambda(t)$, $\mu = \mu(t)$ vary in a compact set. It therefore suffices to check that for $k \gg 0$

$$\frac{\omega_{k-1}\big(\partial_x + \alpha\partial_u + \beta\partial_v, J(\partial_x + \alpha\partial_u + \beta\partial_v)\big)}{k + \alpha^2 + \beta^2} = 1 + \frac{\alpha\mu - \beta\lambda}{k + \alpha^2 + \beta^2}$$

is positive for all $\alpha, \beta \in \mathbb{R}$. This term is minimal for

$$\alpha = -\sqrt{\tfrac{k}{1+(\lambda/\mu)^2}}, \quad \beta = \sqrt{\tfrac{k}{1+(\mu/\lambda)^2}},$$

where the value is $1 - \sqrt{\frac{\lambda^2+\mu^2}{4k}}$. This is positive for $k > (\lambda^2 + \mu^2)/4$. □

Denote by $T^{0,1}_{M,J} \subset T^{\mathbb{C}}_M$ the anti-holomorphic tangent bundle of an almost complex manifold (M,J). Consider a submersion $p:(M,J)\to B$ of an almost complex 4-manifold with all fibers pseudo-holomorphic curves. Let $z=p^*(u), w$ be complex coordinates on M with w fiberwise holomorphic. Then

$$T^{0,1}_{M,J} = \langle \partial_{\bar z} + a\partial_z + b\partial_w, \partial_{\bar w}\rangle$$

for some complex-valued functions a,b. Clearly, a vanishes precisely when p is pseudo-holomorphic for some almost complex structure on B. The Nijenhuis tensor $N_J : T_M\otimes T_M \to T_M$, defined by

$$4N_J(X,Y) = [JX,JY]-[X,Y]-J[X,JY]-J[JX,Y],$$

is antisymmetric and J-antilinear in each entry. In dimension 4 it is therefore completely determined by its value on a pair of vectors that do not belong to a proper J-invariant subspace. For the complexified tensor it suffices to compute

$$\begin{aligned} N^{\mathbb{C}}_J&(\partial_{\bar z} + a\partial_z + b\partial_w, \partial_{\bar w})\\ &= -\frac{1}{2}[\partial_{\bar z} + a\partial_z + b\partial_w, \partial_{\bar w}] + \frac{i}{2}J[\partial_{\bar z} + a\partial_z + b\partial_w, \partial_{\bar w}]\\ &= \frac{1}{2}(\partial_{\bar w}a)\big(\partial_z - iJ\partial_z\big) + (\partial_{\bar w}b)\partial_w. \end{aligned}$$

Since $\partial_z - iJ\partial_z$ and ∂_w are linearly independent we conclude:

Lemma 1.3. *An almost complex structure J on an open set $M\subset\mathbb{C}^2$ with $T^{0,1}_{M,J} = \langle \partial_{\bar z} + a\partial_z + b\partial_w, \partial_{\bar w}\rangle$ is integrable if and only if $\partial_{\bar w}a = \partial_{\bar w}b = 0$.*

Example 1.4. Let $T^{0,1}_{M,J} = \langle \partial_{\bar z} + w\partial_w, \partial_{\bar w}\rangle$. Then z and $we^{-\bar z}$ are holomorphic coordinates on M.

The lemma gives a convenient characterization of integrable complex structures in terms of the functions a,b defining $T^{0,1}_{M,J}$. To globalize we need a connection for p. The interesting case will be p pseudo-holomorphic or $a=0$, to which we restrict from now on.

Lemma 1.5. *Let $p:M\to B$ be a submersion endowed with a connection ∇ and let j be an almost complex structure on B. Then the set of almost complex structures J making*

$$p:(M,J)\longrightarrow (B,j)$$

pseudo-holomorphic is in one-to-one correspondence with pairs $(J_{M/B},\beta)$ where

(1) *$J_{M/B}$ is an endomorphism of $T_{M/B}$ with $J^2_{M/B} = -\mathrm{id}$.*

(2) *β is a homomorphism $p^*(T_B)\to T_{M/B}$ that is complex anti-linear with respect to j and $J_{M/B}$:*

$$\beta(j(Z)) = -J_{M/B}(\beta(Z)).$$

Identifying $T_M = T_{M/B} \oplus p^*(T_B)$ *via* ∇ *the correspondence is*

$$J = \begin{pmatrix} J_{M/B} & \beta \\ 0 & j \end{pmatrix}.$$

Proof. The only point that might not be immediately clear is the equivalence of $J^2 = -\,\mathrm{id}$ with complex anti-linearity of β. This follows by computing

$$J^2 = \begin{pmatrix} J^2_{M/B} & J_{M/B} \circ \beta + \beta \circ j \\ 0 & j^2 \end{pmatrix}. \qquad \square$$

LEMMA 1.6. *Let* $p : (M,J) \to (B,j)$ *be a pseudo-holomorphic submersion*, $\dim M = 4$, $\dim B = 2$. *Then locally in* M *there exists a differentiable map*

$$\pi : M \longrightarrow \mathbb{C}$$

inducing a pseudo-holomorphic embedding $p^{-1}(Q) \to \mathbb{C}$ *for every* $Q \in B$.

Moreover, *to any such* π *let*

$$\beta : p^*(T^{0,1}_{B,j}) \longrightarrow T^{1,0}_{M/B,J_{M/B}}$$

be the homomorphism provided by Lemma 1.5 *applied to the connection belonging to* π. *Let* w *be the pull-back by* π *of the linear coordinate on* $\mathbb{C}$ *and* u *a holomorphic coordinate on* B. *Then* $z := p^*(u)$ *and* w *are complex coordinates on* M, *and*

$$\beta(\partial_{\bar u}) = -2bi\partial_w,$$

for the $\mathbb{C}$-*valued function* b *on* M *with*

$$T^{0,1}_{M,J} = \langle \partial_{\bar w}, \partial_{\bar z} + b\partial_w \rangle. \tag{1}$$

Proof. Since p is pseudo-holomorphic, J induces a complex structure on the fibers $p^{-1}(Q)$, varying smoothly with $Q \in B$. Hence locally in M there exists a $\mathbb{C}$-valued function w that fiberwise restricts to a holomorphic coordinate. This defines the trivialization π.

In the coordinates z, w define b via $\beta(\partial_{\bar u}) = -2bi\partial_w$. Then

$$J(\partial_{\bar z}) = -i\partial_{\bar z} - 2bi\partial_w,$$

so the projection of $\partial_{\bar z}$ onto $T^{0,1}_{M,J}$ is

$$(\partial_{\bar z} + iJ(\partial_{\bar z}))/2 = \partial_{\bar z} + b\partial_w. \qquad \square$$

The two lemmas also say how to define an almost complex structure making a given $p : M \to B$ pseudo-holomorphic, when starting from a complex structure on the base, a fiberwise conformal structure, and a connection for p.

For the symplectic isotopy problem we can reduce to a fibered situation by the following device.

PROPOSITION 1.7. *Let $p : (M, \omega) \to S^2$ be a symplectic S^2-bundle. Let $\Sigma \subset M$ be a symplectic submanifold. Then there exists an ω-tamed almost complex structure J on M and a map $p' : (M, J) \to \mathbb{CP}^1$ with the following properties.*

(1) *p' is isotopic to p.*

(2) *p' is pseudo-holomorphic.*

(3) *Σ is J-holomorphic.*

Moreover, if $\{\Sigma_t\}_t$ is a family of symplectic submanifolds there exist families $\{p'_t\}_t$ and $\{J_t\}$ with the analogous properties for every t.

Proof. We explained in [SiTi1, Prop. 4.1] how to obtain a symplectic S^2-bundle $p' : M \to \mathbb{CP}^1$, isotopic to p, so that all critical points of the projection $\Sigma \to \mathbb{CP}^1$ are simple and positive. This means that near any critical point there exist complex coordinates z, w on M with $z = (p')^*(u)$ for some holomorphic coordinate u on $\mathbb{CP}^1$, so that Σ is the zero locus of $z - w^2$. We may take these coordinates in such a way that $w = 0$ defines a symplectic submanifold. This property will enter below when we discuss tamedness.

Since the fibers of p' are symplectic the ω-perpendicular complement to $T_{M/\mathbb{CP}^1}$ in T_M defines a subbundle mapping isomorphically to $(p')^*(T_{\mathbb{CP}^1})$. This defines a connection ∇ for p'. By changing ∇ slightly near the critical points we may assume that it agrees with the connection defined by the projections $(z, w) \to w$.

The coordinate w defines an almost complex structure along the fibers of p' near any critical point. Since at $(z, w) = (0, 0)$ the tangent space of Σ agrees with $T_{M/\mathbb{CP}^1}$, this almost complex structure is tamed at the critical points with respect to the restriction $\omega_{M/\mathbb{CP}^1}$ of ω to the fibers. Choose a complex structure $J_{M/\mathbb{CP}^1}$ on $T_{M/\mathbb{CP}^1}$ that is $\omega_{M/\mathbb{CP}^1}$-tamed and that restricts to this fiberwise almost complex structure near the critical points.

By Lemma 1.5 it remains to define an appropriate endomorphism

$$\beta : (p')^*(T_{\mathbb{CP}^1}) \longrightarrow T_{M/\mathbb{CP}^1}.$$

By construction of ∇ and the local form of Σ we may put $\beta \equiv 0$ near the critical points. Away from the critical points, let $z = (p')^*(u)$ and w be complex coordinates as in Lemma 1.6. Then Σ is locally a graph $w = f(z)$. This graph will be $J = J(\beta)$-holomorphic if and only if

$$\partial_{\bar{z}} f = b(z, f(z)).$$

Here b is related to β via $\beta(\partial_{\bar u}) = -2bi\partial_w$. Hence this defines β along Σ away from the critical points. We want to extend β to all of M keeping an eye on tamedness. For nonzero $X + Y \in (p')^*(T_{\mathbb{CP}^1}) \oplus T_{M/\mathbb{CP}^1}$ the latter requires

$$0 < \omega(X+Y, J(X+Y)) = \omega(X, j(X)) + \omega(Y, J_{M/\mathbb{CP}^1}(Y)) + \omega(Y, \beta(X)).$$

Near the critical points we know that $\omega(X, j(X)) > 0$ because $w = 0$ defines a symplectic submanifold. Away from the critical points, X and $j(X)$ lie in the ω-perpendicular complement of a symplectic submanifold and therefore $\omega(X, j(X)) > 0$ too. Possibly after shrinking the neighbourhoods of the critical points above, we may therefore assume that tamedness holds for $\beta = 0$. By construction it also holds with the already defined β along Σ. Extend this β differentiably to all of M arbitrarily. Let $\chi_\varepsilon : M \to [0,1]$ be a function with $\chi_\varepsilon|_\Sigma \equiv 1$ and with support contained in an ε-neighbourhood of Σ. Then for ε sufficiently small, $\chi_\varepsilon \cdot \beta$ does the job.

If Σ varies in a family, argue analogously with an additional parameter t. □

In the next section we will see some implications of the fibered situation for the space of pseudo-holomorphic cycles.

2. Pseudo-holomorphic cycles on pseudo-holomorphic S^2-bundles

One advantage of having M fibered by pseudo-holomorphic curves is that it allows us to describe J-holomorphic cycles by Weierstrass polynomials, cf. [SiTi2]. Globally we are dealing with sections of a relative symmetric product. This is the topic of the present section. While we have been working with this point of view for a long time it first appeared in print in [DoSm]. Our discussion here is, however, largely complementary.

Throughout $p : (M, J) \to B$ is a pseudo-holomorphic S^2-bundle. To study J-holomorphic curves $C \subset M$ of degree d over B we consider the d-fold relative symmetric product $M^{[d]} \to B$ of M over B. This is the quotient of the d-fold fibered product $M^d_B := M \times_B \cdots \times_B M$ by the permutation action of the symmetric group S_d. Set-theoretically $M^{[d]}$ consists of 0-cycles in the fibers of p of length d.

Proposition 2.1. *There is a well-defined differentiable structure on $M^{[d]}$, depending only on the fiberwise conformal structure on M over B.*

Proof. Let $\Phi : p^{-1}(U) \to \mathbb{CP}^1$ be a local trivialization of p that is compatible with the fiberwise conformal structure; see the proof of Lemma 5.1 for existence. Let u be a complex coordinate on U. To define a chart near a 0-cycle $\sum m_a P_a$ choose $P \in \mathbb{CP}^1 \setminus \{\Phi(P_a)\}$ and a biholomorphism $w : \mathbb{CP}^1 \setminus \{P\} \simeq \mathbb{C}$. The d-tuples with entries disjoint from $\Phi^{-1}(P)$ give an open S_d-invariant sub-

set
$$U \times \mathbb{C}^d \subset M_B^d.$$
Now the ring of symmetric polynomials on $\mathbb{C}^d$ is free. A set of generators $\sigma_1, \ldots, \sigma_d$ together with $z = p^*(u)$ provides complex, fiberwise holomorphic coordinates on $(U \times \mathbb{C}^d)/S_d \simeq U \times \mathbb{C}^d \subset M^{[d]}$.

Different choices lead to fiberwise biholomorphic transformations. The differentiable structure is therefore well-defined. □

We emphasize that different choices of the fiberwise conformal structure on M over B lead to different differentiable structures on $M^{[d]}$. Note also that $M_B^d \longrightarrow M^{[d]}$ is a branched Galois covering with covering group S_d. The branch locus is stratified according to partitions of d, parametrizing cycles with the corresponding multiplicities. The *discriminant locus* is the union of all lower-dimensional strata. The stratum belonging to a partition $d = m_1 + \cdots + m_1 + \cdots + m_s + \cdots + m_s$ with $m_1 < m_2 < \cdots < m_s$ and m_i occurring d_i-times is canonically isomorphic to the complement of the discriminant locus in $M^{[d_1]} \times_B \cdots \times_B M^{[d_s]}$.

PROPOSITION 2.2. *There exists a unique continuous almost complex structure on $M^{[d]}$ making the covering*
$$M_B^d \longrightarrow M^{[d]}$$
pseudo-holomorphic.

Proof. It suffices to check the claim locally in $M^{[d]}$. Let $w : U \times \mathbb{C} \to \mathbb{C}$ be a fiberwise holomorphic coordinate as in the previous lemma. Let $z = p^*(u)$ and b be as in Lemma 1.6, so
$$T^{0,1}_{M,J} = \big\langle \partial_{\bar{w}}, \partial_{\bar{z}} + b(z,w)\partial_w \big\rangle.$$
Let w_i be the pull-back of w by the i^{th} projection $M_B^d \to M$. By the definition of the differentiable structure on $M^{[d]}$, the r^{th} elementary symmetric polynomial $\sigma_r(w_1, \ldots, w_d)$ descends to a locally defined smooth function σ_r on $M^{[d]}$. The pull-back of u to $M^{[d]}$, also denoted by z, and the σ_r provide local complex coordinates on $M^{[d]}$. The almost complex structure on M_B^d is
$$T^{0,1}_{M_B^d} = \big\langle \partial_{\bar{w}_1}, \ldots, \partial_{\bar{w}_d}, \partial_{\bar{z}} + b(z,w_1)\partial_{w_1} + \cdots + b(z,w_d)\partial_{w_d} \big\rangle.$$
The horizontal anti-holomorphic vector field
$$\partial_{\bar{z}} + b(z,w_1)\partial_{w_1} + \cdots + b(z,w_d)\partial_{w_d}$$
is S_d-invariant, hence descends to a continuous vector field Z on $M^{[d]}$. Together with the requirement that fiberwise the map $M_B^d \to M^{[d]}$ be holomorphic, Z determines the almost complex structure on $M^{[d]}$. □

Remark 2.3. The horizontal vector field Z in the lemma, hence the almost complex structure on $M^{[d]}$, will generally only be of Hölder class $\mathcal{C}^{0,\alpha'}$ in the fiber directions, for some $\alpha' > 0$; see [SiTi1]. However, at 0-cycles $\sum m_\mu P_\mu$ with J integrable near $\{P_\mu\}$ it will be smooth and integrable as well. In fact, by Lemma 1.3 integrability is equivalent to holomorphicity of b along the fibers. This observation will be crucial below.

Proposition 2.4. *There is an injective map from the space of J-holomorphic cycles on M of degree d over B and without fiber components to the space of $J_{M^{[d]}}$-holomorphic sections of $q : M^{[d]} \to B$. A cycle $C = \sum m_a C_a$ maps to the section*

$$u \longmapsto \sum_a m_a C_a \cap p^{-1}(u),$$

the intersection taken with multiplicities. The image of the subset of reduced cycles are the sections with image not entirely lying in the discriminant locus.

Proof. We may reduce to the local problem of cycles in $\Delta \times \mathbb{C}$. In this case the statement follows from [SiTi2, Theorem I]. □

Remark 2.5. By using the stratification of $M^{[d]}$ by fibered products $M^{[d_1]} \times_B \cdots \times_B M^{[d_k]}$ with $\sum d_i \le d$ it is also possible to treat cycles with multiple components. In fact, one can show that a pseudo-holomorphic section of $M^{[d]}$ has an image in exactly one stratum except at finitely many points. Now the almost complex structure on a stratum agrees with the almost complex structure induced from the factors. The claim thus follows from the proposition applied to each factor. Because this result is not essential to what follows details are left to the reader.

To study deformations of a J-holomorphic cycle it therefore suffices to look at deformations of the associated section of $M^{[d]}$. Essentially this is what we will do; but as we also have to treat fiber components we describe our cycles by certain polynomials with coefficients taking values in holomorphic line bundles over B. We restrict ourselves to the case $B = \mathbb{CP}^1$.

The description depends on the choice of an integrable complex structure on M fiberwise agreeing with J. Thus we assume now that $p : (M, J_0) \to \mathbb{CP}^1$ is a holomorphic $\mathbb{CP}^1$-bundle. There exist disjoint sections $S, H \subset M$ with $e := H \cdot H \ge 0$. Then $H \sim S + eF$ where F is a fiber, and $S \cdot S = -e$. Denote the holomorphic line bundles corresponding to H, S by L_H and L_S. Let s_0, s_1 be holomorphic sections of L_S, L_H respectively with zero loci S and H. We also choose an isomorphism $L_H \simeq L_S \otimes p^*(L)^e$, where L is the holomorphic line bundle on $\mathbb{CP}^1$ of degree 1.

Note that if $H_d \subset M^{[d]}$ denotes the divisor of cycles $\sum_a m_a P_a$ with $P_a \in H$ for some a then

$$M^{[d]} \setminus H_d = S^d(L^{-e}) = \bigoplus_{\nu=1}^{d} L^{-\nu e}.$$

In fact, $M \setminus H = L^{-e}$.

Proposition 2.6. *Let J be an almost complex structure on M making $p : M \to \mathbb{CP}^1$ pseudo-holomorphic and assume $J = J_0$ near H and along the fibers of p.*

1) *Let $C = \sum_a m_a C_a$ be a J-holomorphic 2-cycle homologous to $dH + kF$, $d > 0$, and assume $H \not\subset |C|$. Let a_0 be a holomorphic section of L^{k+de} with zero locus $p_*(H \cap C)$, with multiplicities.*

Then there are unique continuous sections a_r of $L^{k+(d-r)e}$, $r = 1, \ldots, d$, so that C is the zero locus of

$$p^*(a_0)s_0^d + p^*(a_1)s_0^{d-1}s_1 + \cdots + p^*(a_d)s_1^d,$$

as a cycle.

2) *There exist Hölder continuous maps*

$$\beta_r : \bigoplus_{\nu=1}^{d} L^{-\nu e} \longrightarrow L^{-re} \otimes \Lambda^{0,1}_{\mathbb{CP}^1}, \quad r = 1, \ldots, d,$$

so that a local section $s_0^d + p^*(\alpha_1)s_0^{d-1}s_1 + \cdots + p^*(\alpha_d)s_1^d = 0$ of $M^{[d]} \setminus H_d$ is $J_{M^{[d]}}$-holomorphic if and only if

$$\bar\partial \alpha_r = \beta_r(\alpha_1, \ldots, \alpha_d), \quad r = 1, \ldots, d.$$

3) *Let C be a J-holomorphic 2-cycle homologous to $dH + kF$ and with $H \not\subset |C|$. Decompose $C = \bar C + \sum m_a F_a$ with the second term containing all fiber components. Assume that $J = J_0$ also near*

$$p^{-1}\big(p(|\bar C| \cap H) \cap p(|\bar C| \cap S)\big) \cup \bigcup\nolimits_a F_a.$$

Let a_r^0 be sections of $L^{k+(d-r)e}$ associated to C according to (1). Then there exists a neighbourhood $\mathfrak{D} \subset \bigoplus_{r=0}^d L^{k+(d-r)e}$ of the graph of $(a_0^0, \ldots, a_d^0)$ and Hölder continuous maps

$$b_r : \mathfrak{D} \longrightarrow L^{k+(d-r)e} \otimes \Lambda^{0,1}_{\mathbb{CP}^1}, \quad r = 1, \ldots, d,$$

so that a section $(a_0, \ldots, a_d)$ of $\mathfrak{D} \to \mathbb{CP}^1$ with a_0 holomorphic defines a J-holomorphic cycle if and only if

$$\bar\partial a_r = b_r(a_0, \ldots, a_d), \quad r = 1, \ldots, d. \tag{2}$$

Conversely, any solution of (2) *with* $\delta(a_0,\ldots,a_d)\not\equiv 0$ *corresponds to a* J*-holomorphic cycle without multiple components. Here* δ *is the discriminant.*

Moreover, if J *is integrable near* $|C|$ *then the* b_r *are smooth near the corresponding points of* $\mathfrak{D}$.

Proof. 1) Assume first that $a=1$ and $m_1=1$. Then either C is a fiber and $p^*(a_0)$ is the defining polynomial; or C defines a section of $M^{[d]}$ as in Proposition 2.4. Any 0-cycle of length d on $p^{-1}(Q)\simeq\mathbb{CP}^1$ is the zero locus of a section of $\mathcal{O}_{\mathbb{CP}^1}(d)$ that is unique up to rescaling. The restrictions of $s_0^r s_1^{d-r}$ to any fiber form a basis for the space of global sections of $\mathcal{O}_{\mathbb{CP}^1}(d)$. Hence, after choice of a_0 the a_r are determined uniquely for $r=1,\ldots,d$ away from the zero locus of a_0. If $a_0(Q)=0$ choose a neighbourhood U of Q so that $C|_{p^{-1}(U)}=C'+C''$ with $|C'|\cap S=\emptyset$, $|C''|\cap H=\emptyset$. By the same argument as before we have unique Weierstrass polynomials of the form

$$p^*(a_0')s_0^{d'}+\cdots+p^*(a'_{d'-1})s_0s_1^{d'-1}+s_1^{d'},$$
$$s_0^{d''}+p^*(a_1'')s_0^{d''-1}s_1+\cdots+p^*(a''_{d''})s_1^{d''}$$

defining C' and C'' respectively. Multiplying produces a polynomial defining C. The first coefficient a_0' vanishes to the same order at Q as a_0. In fact, this order equals the intersection index of C' and C with H respectively. This shows $a_0=a_0'\cdot e$ for some holomorphic function e on U with $e(Q)\neq 0$. Therefore $a_1,\ldots,a_d$ extend continuously over Q.

In the general case let F_a be the polynomial just obtained for $C=C_a$. Put

$$F_{(a_0,\ldots,a_d)}=\prod_a F_a^{m_a}.$$

The coefficient of s_0^d has the same zero locus as $p^*(a_0)$; so after rescaling by a constant, $F_{(a_0,\ldots,a_d)}$ has the desired form.

2) Since J and J_0 agree fiberwise and both make p pseudo-holomorphic, the homomorphism $J-J_0$ factors over $p^*T_{\mathbb{CP}^1}$ and takes values in $T_{M/\mathbb{CP}^1}$. Let β be the section of $T^{1,0}_{M/\mathbb{CP}^1,J}\otimes p^*\Lambda^{0,1}_{\mathbb{CP}^1}$ thus defined. Locally β is nothing but the homomorphism obtained by applying Lemma 1.6 to a local J_0-holomorphic trivialization of M. Because $M\setminus H=L^{-e}$ there is a canonical isomorphism

$$T^{1,0}_{M/\mathbb{CP}^1,J_0}\big|_{M\setminus H}\simeq p^*(L^{-e}).$$

This isomorphism understood, we obtain an S_d-invariant map

$$(w_1,\ldots,w_d)\longmapsto(-1)^r\frac{i}{2}\sum_\nu\sigma_{r-1}(w_1,\ldots,\widehat{w}_\nu,\ldots,w_d)\otimes\beta(z,w_\nu)$$

from $(L^{-e})^{\oplus d}$ to $L^{-re}\otimes\Lambda^{0,1}_{\mathbb{CP}^1}$. Define $\beta_r(\alpha_1,\ldots,\alpha_d)$ as the induced map from $S^d(L^{-e})=M^{[d]}\setminus H_d$. The claim on pseudo-holomorphic sections of $M^{[d]}\setminus H_d$

is clear from the definition of $J_{M^{[d]}}$ in Proposition 2.2 and the description of β in Lemma 1.6.

Hölder continuity of the β_r follows from the local consideration in [SiTi1].

3) Let U be a neighbourhood of $p^{-1}\big(p(|\bar{C}| \cap H) \cap p(|\bar{C}| \cap S)\big) \cup \bigcup F_a$ with $J = J_0$ on $p^{-1}(U)$. Over $Q \in \mathbb{CP}^1$ define $\mathfrak{D}$ as those tuples $(a_0, \ldots, a_d)$ with

$$a_0 = 0 \quad \Longrightarrow \quad a_d \neq 0 \quad \text{or} \quad Q \in U.$$

If $a_s = 0$ for $s = 0, \ldots, m-1$ and $a_m \neq 0$ define

$$b_r(a_0, \ldots, a_d) := a_m \cdot \beta_{r-m}^{d-m}(a_{m+1}/a_m, \ldots, a_d/a_m),$$

where $\beta_r^{d'}$ is β_r from (2) for $d = d'$. We also put $b_r(0, \ldots, 0) = 0$. We claim that the b_r are continuous. Over U this is clear as all terms vanish.

Let w be a complex coordinate on M defining a local J_0-holomorphic trivialization of $M \setminus H \to \mathbb{CP}^1$. Let $w_1, \ldots, w_d$ be the induced coordinate functions on $M^d_{\mathbb{CP}^1}$ and b the function encoding J. Pulling back b_r via $M^d_{\mathbb{CP}^1} \to M^{[d]}$ gives

$$a_0 \cdot \sum_{\nu=1}^{d} \sigma_{r-1}(w_1, \ldots, \widehat{w}_\nu, \ldots, w_d) b(z, w_\nu). \tag{3}$$

It remains to show that if $\{\lambda_1^{(n)}, \ldots, \lambda_d^{(n)}\}_n$ and $\{a_0^{(n)}\}_n$ are sequences with $a_r^{(n)} := a_0^{(n)} \sigma_r(\lambda_1^{(n)}, \ldots, \lambda_d^{(n)})$ converging to $(0, \ldots, 0, a_m, \ldots, a_d)$ with $a_m \neq 0$, $a_d \neq 0$, then (3) converges towards $a_m \cdot \beta_{r-m}^{d-m}(a_{m+1}/a_m, \ldots, a_d/a_m)$. After reordering we may assume that $\lambda_1^{(n)}, \ldots, \lambda_m^{(n)}$ correspond to the m points converging to H. By hypothesis $b(z, w) = 0$ for $|w| \gg 0$. Moreover, since $a_d^{(n)}$ converges with nonzero limit and all $a_r^{(n)}$ are bounded, the $\lambda_\nu^{(n)}$ stay uniformly bounded away from 0. Hence for any subset $I \subset \{1, \ldots, d\}$

$$a_0^{(n)} \prod_{\nu \in I} \lambda_\nu^{(n)}$$

converges. The limit is 0 if $\{1, \ldots, m\} \not\subset I$. Evaluating expression (3) at $w_\nu = \lambda_\nu^{(n)}$ and taking the limit gives

$$\begin{aligned}
&\lim_{n\to\infty} \Big(a_0^{(n)} \cdot \sum_{\nu=m+1}^{d} \sigma_{r-1}(\lambda_1^{(n)}, \ldots, \widehat{\lambda}_\nu^{(n)}, \ldots, \lambda_d^{(n)}) \cdot b(z, \lambda_\nu^{(n)}) \Big) \\
&\qquad = \; a_m \cdot \lim_{n\to\infty} \sum_{\nu=m+1}^{d} \sigma_{r-m-1}(\lambda_{m+1}^{(n)}, \ldots, \widehat{\lambda}_\nu^{(n)}, \ldots, \lambda_d^{(n)}) \cdot b(z, \lambda_\nu^{(n)}) \\
&\qquad = \; a_m \cdot \beta_{r-m}^{d-m}(a_{m+1}/a_m, \ldots, a_d/a_m),
\end{aligned}$$

as had to be shown.

The expression for b_r also shows that the local equation for pseudo-holomorphicity of a section $\sigma_r = a_r(z)/a_0(z)$ of $M^{[d]} \setminus H_d$ is

$$\partial_{\bar{z}} a_r(z) = a_0 \beta_r(a_1, \ldots, a_d) = b_r(a_0, \ldots, a_d).$$

This extends over the zeros of a_0. The converse follows from the local situation already discussed at length in [SiTi2].

Finally we discuss regularity of the b_r. The partial derivatives of b_r in the z-direction lead to expressions of the same form as b_r with $b(z, w_\nu)$ replaced by $\nabla_z^k b(z, w_\nu)$. These are continuous by the argument in (2). If J is integrable near $|C|$ then b is holomorphic there along the fibers of p. Hence the b_r and its derivatives in the z-direction are continuous and fiberwise holomorphic. Uniform boundedness thus implies the desired estimates on higher mixed derivatives. □

Remark 2.7. It is instructive to compare the linearizations of the equations characterizing J-holomorphic cycles of the coordinate dependent description in this proposition and the intrinsic one in Proposition 2.4. We have to assume that C has no fiber components. Let σ be the section of $q : M^{[d]} \to \mathbb{CP}^1$ associated to C by Proposition 2.4. There is a PDE acting on sections of $\sigma^*(T_{M^{[d]}/\mathbb{CP}^1})$ governing (pseudo-) holomorphic deformations of σ. For the integrable complex structure this is simply the $\bar{\partial}$-equation. There is a well-known exact sequence

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \bigoplus_{r=0}^{d} L^{k+(d-r)e} \longrightarrow \sigma^*(T_{M^{[d]}/\mathbb{CP}^1}) \longrightarrow 0,$$

describing the pull-back of the relative tangent bundle. The $\bar{\partial}_J$-equation giving J-holomorphic deformations of σ acts on the latter bundle. On the other hand, the middle term exhibits variations of the coefficients $a_0, \ldots, a_d$. The constant bundle on the left deals with rescalings.

The final result of this section characterizes certain smooth cycles.

Proposition 2.8. *In the situation of Proposition* 2.4 *let σ be a differentiable section of $M^{[d]} \to S^2$ intersecting the discriminant divisor transversally. Then the* 2*-cycle C belonging to σ is a submanifold and the projection $C \to S^2$ is a branched cover with only simple branch points. Moreover*, *C varies differentiably under $\mathcal{C}^1$-small variations of σ.*

Proof. Away from points of intersection with the discriminant divisor the symmetrization map $M_B^d \to M^{[d]}$ is locally a diffeomorphism, and the result is clear. Moreover, the discriminant divisor is smooth only at points $\sum m_\mu P_\mu$ with $\sum m_\mu = d - 1$; this is the locus where exactly two points come together. We may hence assume $m_1 = 2$ and $m_a = 1$ for $a > 1$. At $\sum m_\mu P_\mu$ the

$d-2$ coordinates w_μ at P_μ, $\mu > 2$, and $w_1 + w_2$, $w_1 w_2$ descend to complex coordinates on $S^d(\mathbb{CP}^1)$. Similarly, the variation of the P_μ for $\mu > 2$ lead only to multiplication of $\delta(a_0, \ldots, a_d)$ by a smooth function without zeros. It therefore suffices to discuss the case $d = 2$. Then C is the zero locus

$$a_0(z)w^2 + a_1(z)w + a_2(z) = 0.$$

The assumption says that, say, $z = 0$ is a simple zero of

$$\delta(\alpha_1, \alpha_2) = \alpha_1^2 - 4\alpha_2.$$

By assumption there exists a function $h(z)$ with $h(0) \neq 0$ and $\delta(\alpha_1, \alpha_2) = h^2(z) \cdot z$. Replacing w by $u = 2h^{-1}(w + \frac{\alpha_1}{2})$ brings C into standard form $u^2 - z = 0$. Hence C is smooth and the projection to z has a simple branch point over $z = 0$. The same argument is valid for small deformations of σ. $\square$

3. The $\mathcal{C}^0$-topology on the space of pseudo-holomorphic cycles

This section contains a discussion of the topology on the space of pseudo-holomorphic cycles, which we denote $\mathrm{Cyc}_{\mathrm{pshol}}(M)$ throughout. Let $\mathcal{C}(M)$ be the space of pseudo-holomorphic stable maps. An element of $\mathcal{C}(M)$ is an isomorphism class of pseudo-holomorphic maps $\varphi : \Sigma \to M$ where Σ is a nodal Riemann surface, with the property that there are no infinitesimal biholomorphisms of Σ compatible with φ. The $\mathcal{C}^0$-topology on $\mathcal{C}(M)$ is generated by open sets $U_{V,\varepsilon}$ defined for $\varepsilon > 0$ and V a neighbourhood of Σ_{sing} as follows. To compare $\psi : \Sigma' \to M$ with φ consider maps $\kappa : \Sigma' \to \Sigma$ that are a diffeomorphism away from Σ_{sing} and that over a branch of Σ at a node have the form

$$\big\{z \in \Delta \,\big|\, |z| > \tau\big\} \longrightarrow \Delta, \quad re^{i\phi} \longmapsto \frac{r-\tau}{1-\tau} e^{i\phi}$$

for some $0 \le \tau < 1$. Then $\psi : \Sigma' \to M$ belongs to $U_{V,\varepsilon}$ if such a κ exists with maximal dilation over $\Sigma \setminus V$ less than ε and with

$$d_M\big(\psi(z), \varphi(\kappa(z))\big) < \varepsilon \quad \text{for all } z.$$

Recall that the dilation measures the failure of a map between Riemann surfaces to be holomorphic. Note also that an intrinsic measure for the size of the neighbourhood V of the singular set on noncontracted components is the diameter of $\varphi(V)$ in M; on contracted components one may take the smallest ε with V contained in the *ε-thin part.* The latter consists of endpoints of loops around the singular points of length $< \varepsilon$ in the Poincaré metric.

For a fixed almost complex structure of class $\mathcal{C}^{l,\alpha}$, $\mathcal{C}^0$-convergence of pseudo-holomorphic stable maps implies $\mathcal{C}^{l+1,\alpha}$-convergence away from the singular points of the limit. If one wants convergence of derivatives away from the singularities for varying J one needs $\mathcal{C}^{0,\alpha}$-convergence of J for some $\alpha > 0$. We will impose this condition separately each time we need it.

The $\mathcal{C}^0$-topology on $\mathcal{C}(M)$ induces a topology on $\mathrm{Cyc}_{\mathrm{pshol}}(M)$ via the map

$$\mathcal{C}(M) \longrightarrow \mathrm{Cyc}_{\mathrm{pshol}}(M), \quad \big(\varphi : C = \textstyle\bigcup_a C_a \to M\big) \longmapsto \displaystyle\sum_a m_a \varphi(C_a).$$

Here m_a is the covering degree of $C_a \to \varphi(C_a)$. From this point of view the compactness theorem for $\mathrm{Cyc}_{\mathrm{pshol}}(M)$ follows immediately from the version for stable maps. We call this topology on $\mathrm{Cyc}_{\mathrm{pshol}}(M)$ the $\mathcal{C}^0$*-topology*.

Alternatively, one may view $\mathrm{Cyc}_{\mathrm{pshol}}(M)$ as a closed subset of the space of currents on M, or of the space of measures on M. We will not use this point of view here.

Next we turn to a semi-continuity property of pseudo-holomorphic cycles in the $\mathcal{C}^0$-topology. For a pseudo-holomorphic curve singularity (C, P) in a 4-dimensional almost complex manifold M define $\delta(C, P)$ as the virtual number of double points. This is the number of nodes of the image of a small, general, J-holomorphic deformation of the parametrization map from a union of unit disks to M belonging to (C, P). This number occurs in the genus formula. If $C = \bigcup_{a=1}^d C_a$ is the decomposition of a pseudo-holomorphic curve into irreducible components, the genus formula says

$$\sum_{a=1}^{d} g(C_a) = \frac{C \cdot C - c_1(M) \cdot C}{2} + d - \sum_{P \in C_{\mathrm{sing}}} \delta(C, P). \tag{4}$$

We emphasize that in this formula C has no multiple components. For a proof perform a small, general pseudo-holomorphic deformation $\varphi_a : \Sigma_a \to M$ of the pseudo-holomorphic maps with image C_a. This is possible by changing J slightly away from C_{sing}. The result is a J'-holomorphic nodal curve for some small perturbation J' of J. The degree of the complex line bundle $\varphi_a^*(T_M)/T_{\Sigma_a}$ equals $C_a \cdot C_a$ minus the number of double points of C_a. This expresses the genus of Σ_a in terms of $C_a \cdot C_a$, $c_1(M) \cdot C_a$ and $\sum_{P \in (C_a)_{\mathrm{sing}}} \delta(C_a, P)$. Sum over a and adjust by the intersections of C_a with $C_{a'}$ for $a \neq a'$ to deduce (4).

As a measure for how singular the support of a pseudo-holomorphic cycle is, we introduce

$$\delta(C) := \sum_{P \in |C|_{\mathrm{sing}}} \delta(|C|, P).$$

Similarly, as a measure for nonreducedness of a pseudo-holomorphic cycle $C = \sum_a m_a C_a$ put

$$m(C) := \sum_a (m_a - 1).$$

So $\delta(C) = 0$ if and only if $|C|$ is smooth and $m(C) = 0$ if and only if C has no multiple components.

The definition of the $\mathcal{C}^0$-topology on the space of pseudo-holomorphic cycles implies semi-continuity of the pair (m, δ).

LEMMA 3.1. *Let (M, J) be a 4-dimensional almost complex manifold with J tamed by some symplectic form. Let $C_n \subset M$ be J_n-holomorphic cycles with $J_n \to J$ in $\mathcal{C}^0$ and in $\mathcal{C}^{0,\alpha}$ away from a set not containing any closed pseudo-holomorphic curves, also, assume $C_n \to C_\infty$ in the $\mathcal{C}^0$-topology.*

Then $m(C_n) \le m(C_\infty)$ for $n \gg 0$, and if $m(C_n) = m(C_\infty)$ for all n then $\delta(C_n) \le \delta(C_\infty)$. Moreover, if also $\delta(C_n) = \delta(C_\infty)$ for all n then for $n \gg 0$, there is a bijection between the irreducible components of $|C_n|$ and of $|C_\infty|$ respecting the genera.

Proof. By the definition of the cycle topology, for $n \gg 0$ each component of C_∞ deforms to parts of some component of C_n. This sets up a surjective multi-valued map Δ from the set of irreducible components of C_∞ to the set of irreducible components of C_n. The claim on semi-continuity of m follows once we show that the sum of the multiplicities of the components $C_{n,i} \in \Delta(C_{\infty,a})$ does not exceed the multiplicity of $C_{\infty,a}$.

By the compactness theorem we may assume that the C_n lift to a converging sequence of stable maps $\varphi_n : \Sigma_n \to M$. Let $\varphi_\infty : \Sigma_\infty \to M$ be the limit. This is a stable map lifting C_∞. For a component $C_{\infty,a}$ of C_∞ of multiplicity m_a choose a point $P \in C_{\infty,a}$ in the part of $\mathcal{C}^{0,\alpha}$-convergence of the J_n and away from the critical values of φ_∞. Let $H \subset M$ be a local oriented submanifold of real codimension 2 with $\mathrm{cl}(H)$ intersecting $|C|$ transversally and positively precisely in $P \in H$. As $\mathcal{C}^{0,\alpha}$-convergence of almost complex structures implies convergence of tangent spaces away from the critical values, H is transverse to C_n for $n \gg 0$ with all intersections positive. Now any component of C_n with a part degenerating to $C_{\infty,a}$ intersects H, and $H \cdot C_n$ gives the multiplicity of $C_{\infty,a}$ in C_∞. The claimed semi-continuity of multiplicities thus follows from the deformation invariance of intersection numbers.

The argument also shows that equality $m(C_\infty) = m(C_n)$ can only hold if Δ induces a bijection between the nonreduced irreducible components of C_n and C_∞ respecting the multiplicities. This implies convergence $|C_n| \to |C_\infty|$, so we may henceforth assume C_n and C_∞ to be reduced, and φ_n to be injective. Note that φ_∞ may contract some irreducible components of Σ_∞. In any case, $\sum_a g(C_{\infty,a})$ is the sum of the genera of the noncontracted irreducible components of Σ_∞, and it is not larger than the respective sum for Σ_n. The latter equals $\sum_i g(C_{n,i})$ if $C_n = \bigcup_i C_{n,i}$. By the genus formula (4) we conclude

$$\delta(C_n) = \sum_{Q \in (C_n)_{\mathrm{sing}}} \delta(C_n, Q) \le \sum_{P \in (C_\infty)_{\mathrm{sing}}} \delta(C_\infty, P) = \delta(C_\infty).$$

If equality holds, there is a bijection between the singular points of $|C_n|$ and $|C_\infty|$ respecting the virtual number of double points. The genus formula then shows that Δ respects the genera of the irreducible components. $\square$

In the fibered situation of Proposition 2.4 convergence in $\mathrm{Cyc}_{\mathrm{pshol}}(M)$ in the $\mathcal{C}^0$-topology implies convergence of the section of $M^{[d]}$:

PROPOSITION 3.2. *Let $p : M \to B$ be an S^2-bundle. For every n let C_n be a pseudo-holomorphic curve of degree d over B for some almost complex structure making p pseudo-holomorphic. Assume that $C_n \to C$ in the $\mathcal{C}^0$-topology and that C contains no fiber components. Let σ_n and σ be the sections of $M^{[d]} \to B$ corresponding to C_n and C, respectively, according to Proposition* 2.4. *Then*

$$\sigma_n \overset{n\to\infty}{\longrightarrow} \sigma \quad \text{in } \mathcal{C}^0.$$

Proof. We have to show the following. Let $\bar{U} \times S^2 \to M$ be a local trivialization with $\bar{U} \subset B$ a closed ball, and let $V \subset S^2$ be an open set so that $|C| \cap (\bar{U} \times V) \to \bar{U}$ is proper. Let d' be the degree of $C|_{\bar{U}\times V}$ over $\bar{U}$, counted with multiplicities. Then for n sufficiently large $C_n \cap (\bar{U} \times V) \to \bar{U}$ will be a (branched) covering of the same degree d'. In fact, any $P \in |C|$ has neighbourhoods of this form with V arbitrarily small. Away from the critical points of the projection to $\bar{U}$ both C and C_n would then have exactly d' branches on $\bar{U} \times V$, counted with multiplicities. In the coordinates on $M^{[d]}$ exhibited in Proposition 2.1 the components of σ_n are elementary symmetric functions in these branches. As V can be chosen arbitrarily small this implies $\mathcal{C}^0$-convergence of σ_n towards σ.

By the definition of the topology on $\mathcal{C}(M)$ the C_n lie in arbitrarily small neighbourhoods of $|C|$. Properness of $|C| \cap (\bar{U} \times V) \to \bar{U}$ implies

$$\partial(\bar{U} \times V) \cap |C| \subset \partial\bar{U} \times V. \tag{5}$$

By compactness of $(\partial\bar{U} \times V) \cap |C|$ we may replace $|C|$ in this inclusion by a neighbourhood. Therefore (5) holds with C_n replacing $|C|$, for $n \gg 0$. We conclude that $C_n \cap (\bar{U} \times V) \to \bar{U}$ is proper for $n \gg 0$ too, hence a branched covering. Let d_n be its covering degree.

Convergence of the C_n in the $\mathcal{C}^0$-topology implies that for every n there exist stable maps $\varphi_n : \Sigma_n \to M$, $\psi_n : \Sigma_{\infty,n} \to M$ lifting C_n and C respectively and a $\kappa_n : \Sigma_n \to \Sigma_{\infty,n}$ as above with

$$d_M\big(\varphi_n(z), \psi_n(\kappa_n(z))\big) \longrightarrow 0$$

uniformly. Let $Z \subset B$ be the union of the critical values of $p \circ \varphi_n$ and of $p \circ \psi_n$ for all n. This is a countable set, hence has dense complement. Choose $Q \in \bar{U} \setminus Z$ and put $F = p^{-1}(Q)$. By hypothesis F is J_n-holomorphic and transverse to φ_n and ψ_n for every n. Therefore, for each n the cardinality of $A_n := \varphi_n^{-1}(F \cap (\bar{U} \times V))$ and of $\psi_n^{-1}(F \cap (\bar{U} \times V))$ are d_n and d' respectively. Since P is a regular value of $p \circ \psi_n$, for $n \gg 0$ the image $\kappa_n(A_n)$ lies entirely in the regular part of noncontracted components of $\Sigma_{\infty,n}$. On this part the pull-back of the Riemannian metric on M allows uniform measurements of distances. In this metric the distance of $\kappa_n(A_n)$ from $\psi_n^{-1}(F \cap (\bar{U} \times V))$, viewed as 0-cycle, tends to zero for $n \to \infty$. Therefore $d_n = d'$ for $n \gg 0$. □

In a situation where the description of Proposition 2.6(3) applies we obtain convergence of coefficients, even under the presence of fiber components in the limit.

Proposition 3.3. *Given the data* $p : (M, J) \to \mathbb{CP}^1$, J_0, s_0, s_1, C, a_r *of Proposition* 2.6(3) *assume that* J_n *is a sequence of almost complex structures making* p *pseudo-holomorphic and so that* $J_n = J_0$ *on a neighbourhood of* $H \cup p^{-1}\big(p(\bar{C} \cap H) \cap p(\bar{C} \cap S)\big) \cup \bigcup F_a$ *that is independent of* n. *Let* $\{C_n\}_n$ *be a sequence of* J_n*-holomorphic curves converging to* $C = \bar{C} + \sum_a m_a F_a$ *in the* $\mathcal{C}^0$*-topology. Let* $a_{0,n}$ *be holomorphic sections of* L^{k+de} *with zero locus* $p(C_n \cap H)$ *converging uniformly to* a_0.

Then the sections $a_{r,n}$ *of* $L^{k+(d-r)e}$ *corresponding to* C_n *converge uniformly to* a_r *for all* r.

Proof. From Proposition 2.6(3) the $a_{r,n}$ fulfill equations

$$\bar{\partial} a_{r,n} = b_{r,n}(a_{0,n}, \ldots, a_{d,n}),$$

with uniformly bounded right-hand side. Cover $\mathbb{CP}^1$ with 2 disks intersecting in an annulus Ω whose closure does not contain any zeros of a_0. Then $H \cap |C| \cap p^{-1}(\Omega) = \emptyset$. Thus over Ω the branches of C_n stay uniformly bounded away from H; hence the $a_{r,n}$ are uniformly bounded over Ω. The Cauchy integral formula on each of the two disks implies a uniform estimate

$$\|a_{r,n}\|_{1,p} \le c \cdot \big(\|\bar{\partial} a_{r,n}\|_p + \|a_{r,n}|_\Omega\|_\infty\big).$$

Therefore, in view of boundedness of $b_{r,n}$ everywhere and of $a_{r,n}$ on Ω we deduce a uniform estimate on the Hölder norm

$$\|a_{r,n}\|_{0,\alpha} \le c'.$$

Thus it suffices to prove pointwise convergence of the $a_{r,n}$ on a dense set.

Away from the zeros of a_0 union $\bigcup_a p(F_a)$ convergence follows from Proposition 3.2. In fact, the quotients $a_{r,n}/a_{0,n}$ occur as coefficients of the local section

$$s_0^d + p^*(a_{1,n}/a_{0,n}) s_0^{d-1} s_1 + \cdots + p^*(a_{d,n}/a_{0,n}) s_1^d = 0$$

of $M^{[d]} \setminus H_d$; see Proposition 2.6(2). These sections correspond to a sequence of pseudo-holomorphic curves converging to a pseudo-holomorphic cycle without fiber components as considered in Proposition 3.2. □

Note that since $\mathcal{C}^0$-convergence $C_n \to C$ implies convergence of the 0-cycles $H \cap C_n \to H \cap C$, any sequence of holomorphic sections $a_{0,n}$ with zero locus $p(C_n \cap H)$ converges after rescaling.

4. Unobstructed deformations of pseudo-holomorphic cycles

We are interested in finding *unobstructed* deformations of a pseudo-holomorphic cycle C in a pseudo-holomorphic S^2-bundle. In the relevant situations this is possible after changing the almost complex structure. In this section we give sufficient conditions for unobstructedness, while the construction of an appropriate almost complex structure occupies the next section.

The describing PDE follows from Proposition 2.6(3). Recall the assumptions there: J integrable near $|C|$, standard fiberwise and near H union all fiber components of $|C|$ union $p^{-1}\big(p(|\bar{C}| \cap H) \cap p(|\bar{C}| \cap S)\big)$. In the notation of loc.cit., to set up the operator choose $T \subset \mathcal{O}(L^{k+de})$, and an open $\mathfrak{D}' \subset \bigoplus_{r=1}^{d} L^{k+(d-r)e}$ with

$$a_0(\mathbb{CP}^1) \times_{\mathbb{CP}^1} \mathfrak{D}' \subset \mathfrak{D} \quad \text{for all } a_0 \in T.$$

Take $p > 2$ and write $W^{1,p}_{\mathbb{CP}^1}(\mathfrak{D}') \subset \bigoplus_{r=1}^{d} W^{1,p}(\mathbb{CP}^1, L^{k+(d-r)e})$ for the open set of Sobolev sections taking values in $\mathfrak{D}'$. View PDE (2) in Proposition 2.6 as a family of differentiable maps

$$W^{1,p}_{\mathbb{CP}^1}(\mathfrak{D}') \longrightarrow \bigoplus_{r=1}^{d} L^p(\mathbb{CP}^1, L^{k+(d-r)e} \otimes \Lambda^{0,1}_{\mathbb{CP}^1}), \tag{6}$$

$$(a_r)_{r=1,\ldots,d} \longmapsto \big(\bar{\partial} a_r - b_r(a_0, a_1, \ldots, a_d)\big)_{r=1,\ldots,d},$$

parametrized by $a_0 \in T$. Because the b_r depend holomorphically on a_r the linearization of this map takes the form

$$W^{1,p}\big(\textstyle\bigoplus_{r=1}^{d} L^{k+(d-r)e}\big) \longrightarrow L^p\big(\textstyle\bigoplus_{r=1}^{d} L^{k+(d-r)e} \otimes \Lambda^{0,1}_{\mathbb{CP}^1}\big), \quad v \longmapsto \bar{\partial} v - R \cdot v.$$

Here R is a $d \times d$-matrix with entries in $\mathrm{Hom}\,\big(L^{k+(d-r)e}, L^{k+(d-r')e} \otimes \Lambda^{0,1}_{\mathbb{CP}^1}\big)$.

Proposition 4.1. *Assume that there exists a J-holomorphic section $S \subset M$ representing $H - eF$ and that $k \geq 0$. Then $\bar{\partial} - R$ is surjective. Moreover, for any $Q \in \mathbb{CP}^1$ the restriction map*

$$\ker\big(\bar{\partial} - R\big) \longrightarrow \textstyle\bigoplus_{r \geq 1} L_Q^{k+(d-r)e}$$

is surjective.

Proof. Unlike the case of rank 1, the surjectivity of $\bar{\partial} - R$ does not follow from topological considerations. Instead we are going to identify the kernel of this operator with sections of the holomorphic normal sheaf

$$\mathcal{N}_{C|M} = \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$$

of C in M, with zeros along H. Here $\mathcal{I}$ is the ideal sheaf of the possibly nonreduced subspace C of a neighbourhood of $|C|$ in M where J is integrable.

The cokernel of $\bar\partial - R$ is then isomorphic to $H^1(\mathcal{N}_{C|M}(-H))$, which in turn vanishes by an essentially topological argument. The proof proceeds in three lemmas.

Local solutions of $\bar\partial v - R\cdot v = 0$ form a locally free $\mathcal{O}_{\mathbb{CP}^1}$-module $\mathcal{K}$ of rank d, that is, the sheaf of holomorphic sections of a holomorphic vector bundle over $\mathbb{CP}^1$ of rank d. Similarly, because p is holomorphic in a neighbourhood of C the push-forward sheaf

$$p_*(\mathcal{N}_{C|M}(-H)) : U \longmapsto \mathcal{N}_{C|M}(-H)\big(p^{-1}(U)\big)$$

is a locally free $\mathcal{O}_{\mathbb{CP}^1}$-module of the same rank d.

LEMMA 4.2. $\mathcal{K} \simeq p_*\big(\mathcal{N}_{C|M}(-H)\big)$.

Proof. This correspondence holds for any base B, so let us write B instead of $\mathbb{CP}^1$ for brevity. Consider first the case with C smooth and transverse to H and $p : C \to B$ having only simple branch points. Then $\mathcal{N}_{C|M}$ is the sheaf of holomorphic sections of $\big(T_M|_C\big)/T_C$. Let $\mathbf{a}$ be the section of $M^{[d]}$ associated to C according to Proposition 2.4. Away from the critical points of $p|_C$ the inclusion $T_{M/B} \to T_M$ induces an isomorphism $T_{M/B}|_C \simeq \mathcal{N}_{C|M}$. Let Θ be a holomorphic section of $T_{M/B}(-H)$ along $C \cap p^{-1}(U)$, $U \subset B$ open. Since $T_{M^d_B/B} = \bigoplus q_i^*(T_{M/B})$, $q_i : M^d_B \to M$ the i^{th} projection, Θ induces an S_d-invariant holomorphic section $\tilde\Theta$ of $T_{M^d_B/B}$ over $C \times_B \cdots \times_B C \subset M^d_B$. Recall the symmetrization map $\Phi : M^d_B \to M^{[d]}$ from Proposition 2.2. By the definition of the almost complex structure on $M^{[d]}$ this map is holomorphic near $C \times_B \cdots \times_B C$. Thus $\Phi_*(\tilde\Theta)$ is a holomorphic section of $\mathbf{a}^*(T_{M^{[d]}/B})$. The vanishing of Θ on $C \cap H$ translates into the vanishing of the normal component of $\Phi_*(\tilde\Theta)$ along the divisor $H_d \subset M^{[d]}$ introduced before Proposition 2.6. In other words, $\Phi_*(\tilde\Theta)$ is a section of $\mathbf{a}^*\big(T_{M^{[d]}/B}(-\log H_d)\big)$. It is clear that this sets up an isomorphism between $p_*(\mathcal{N}_{C|M}(-H))$ and $\mathbf{a}^*\big(T_{M^{[d]}/B}(-\log H_d)\big)$. We claim that the module of sections over U of the latter sheaf is canonically isomorphic to $\mathcal{K}$. Then define

$$\Psi : p_*\big(\mathcal{N}_{C|M}(-H)\big) \longrightarrow \mathcal{K}$$

away from the critical points of $p|_C$ by sending Θ to $\Phi_*(\tilde\Theta)$.

To prove the claim we have to characterize holomorphic sections of $T_{M^{[d]}/B}$ along the image of $\mathbf{a}$ in the coordinates $(z, w) : M \setminus (H \cup F) \to \mathbb{C}^2$. Let $\sigma_1, \ldots, \sigma_d$ be the fiberwise holomorphic coordinates on $M^{[d]}$ induced by w; see Proposition 2.2. Let

$$Z = \partial_{\bar z} + \beta_1 \partial_{\sigma_1} + \cdots + \beta_d \partial_{\sigma_d}$$

be the antiholomorphic horizontal vector field defining the complex structure. A locally defined function f is holomorphic if and only if it is fiberwise holomorphic and if $Zf = 0$. Thus a fiberwise holomorphic local section $\Theta = \sum_r h_r\partial_{\sigma_r}$ of $T_{M^{[d]}/B}$ is holomorphic if and only if $Z(\Theta f) = 0$ for every fiberwise holomorphic function f with $Zf = 0$. For such f we have $Z(\Theta f) = \sum_{i,j}(Zh_j - h_i\partial_{\sigma_i}\beta_j)\partial_{\sigma_j} f$. Expanding and comparing coefficients gives

$$\partial_{\bar z}h_j + \sum_i(\beta_i\partial_{\sigma_i}h_j - h_i\partial_{\sigma_i}\beta_j) = 0, \tag{7}$$

for all i, j. Now a local section $\sum_r \nu_r(z)\partial_{\sigma_r}$ of $\mathbf{a}^*\big(T_{M^{[d]}/B}\big)$ is holomorphic if and only if there exists a local holomorphic section $\sum_r h_r\partial_{\sigma_r}$ of $T_{M^{[d]}/B}$ with $\nu_r = h_j \circ \mathbf{a}$. Since $\partial_{\bar z}\alpha_r = \beta_r(z, \mathbf{a})$ equation (7) then implies

$$\partial_{\bar z}\nu_r - \textstyle\sum_j \nu_j\partial_{\sigma_j}\beta_r\big|_{(z,\mathbf{a})} = 0. \tag{8}$$

Conversely, if the latter equation is fulfilled, there exist fiberwise holomorphic h_r fulfilling (7) and with $\nu_r = \mathbf{a}^*h_r$. So (8) is the desired characterization of holomorphic sections of $\mathbf{a}^*\big(T_{M^{[d]}/B}\big)$. Rescaling the coordinates σ_r by a_0 leads to $v = (a_0\nu_1, \ldots, a_0\nu_d)$. Noting that $b_r = a_0\beta_r$ we see that v corresponds to a holomorphic section of $\mathbf{a}^*\big(T_{M^{[d]}/B}\big)$ if and only if $\bar\partial v - R \cdot v = 0$.

The map Ψ is an isomorphism wherever defined so far. To finish the case where C is smooth, transverse to H and with only simple branch points it remains to extend Ψ over the critical points of $p|_C$. As we are in a purely holomorphic situation now we are free to work in actual holomorphic coordinates. Note that if C splits into several connected components then $M^{[d]}$ is naturally a fibered product with one factor for each component of C. Since our isomorphism Ψ respects this decomposition the problem is local in C. We may therefore assume C to be defined by $u^2 - z = 0$ with u, z holomorphic coordinates and z descending to B. In this case ∂_z and $-u\partial_z$ are a frame for $p_*(\mathcal{N}_{C|M})$. Away from $z = 0$ take linear combinations with $2u\partial_z + \partial_u \in T_C$ to lift to $T_{M/C}$. Thus

$$\partial_z = -\frac{1}{2u}\partial_u, \quad -u\partial_z = \frac{1}{2}\partial_u$$

in $\mathcal{N}_{C|M}$. Compute

$$\begin{aligned}
\Psi(\partial_z) &= -\Psi\big(\tfrac{1}{2u}\partial_u\big) = -\Big(\tfrac{\sigma_1}{2\sigma_2}\partial_{\sigma_1} + \tfrac{\sigma_1^2-2\sigma_2}{2\sigma_2}\partial_{\sigma_2}\Big)\Big|_{\substack{\sigma_1=0\\ \sigma_2=0}} = \partial_{\sigma_2}\\
\Psi(-u\partial_z) &= \Psi\big(\tfrac12\partial_u\big) = \Big(\partial_{\sigma_1} + \tfrac{\sigma_1}{2}\partial_{\sigma_2}\Big)\Big|_{\substack{\sigma_1=0\\ \sigma_2=0}} = \partial_{\sigma_1}.
\end{aligned}$$

Therefore Ψ is extended to an isomorphism over all of B.

For the general case we know by [SiTi1] that locally in B there exists a holomorphic deformation $\{C_s\}_s$ of C over the 2-polydisk Δ^2, say, with C_s smooth and with $p|_{C_s}$ only simply branched for all $s \neq 0$. The previous reasoning gives an isomorphism of holomorphic vector bundles over $B \times (\Delta^2 \setminus \{0\})$. Since the vector bundles extend over $B \times \{0\}$ this morphism extends uniquely.

Define Ψ as the restriction of this extension to $B \times \{0\}$. Then Ψ is an isomorphism because the determinant of the extension does not vanish in codimension 1; and it is unique because any two deformations of C fit into a joint deformation with the locus of nonsimply branched curves having higher codimension. $\square$

LEMMA 4.3. *Let $C = \sum_{a=0}^{r} m_a C_a$, $m_a > 0$, be a compact holomorphic 1-cycle on a complex manifold X of dimension 2 and let L be a holomorphic line bundle over C. Assume that for all $0 < r' \leq r$, $0 \leq m < m_a$:*

$$c_1(L) \cdot C_0 < 0, \quad \text{and} \quad c_1(L) \cdot C_{r'} < mC_{r'}^2 + \textstyle\sum_{a=0}^{r'-1} m_a C_a \cdot C_{r'}.$$

Then $H^0(C, L) = 0$.

Proof. By induction over $\sum m_a$. We identify effective 1-cycles with compact complex subspaces without further notice. If $\sum m_a = 1$ we are dealing with a holomorphic line bundle of negative degree over a reduced space $C = C_0$, which has no nonzero global sections. In the general case let $\mathcal{L}$ be the sheaf of holomorphic sections of L and $s \in H^0(C, L)$. The effective cycle $C' = C - C_r$ fulfills the induction hypothesis. In view of the exact sequence

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{I}_{C'/C} \longrightarrow \mathcal{L}|_C \longrightarrow \mathcal{L}|_{C'} \longrightarrow 0,$$

the section s lifts to $\mathcal{L} \otimes \mathcal{I}_{C'/C}$. Because $\mathcal{I}_{C_r} \cdot \mathcal{I}_{C'} = \mathcal{I}_C$ the second factor is

$$\mathcal{I}_{C'}/\mathcal{I}_C = \mathcal{I}_{C'} \otimes \mathcal{O}_X/\mathcal{I}_{C_r} = \mathcal{O}_{C_r}(-C').$$

Thus $\mathcal{L} \otimes \mathcal{I}_{C'/C}$ is the sheaf of holomorphic sections of a line bundle over C_r of degree

$$\begin{aligned} \deg_{C_r}\big(\mathcal{L} \otimes \mathcal{I}_{C'}/\mathcal{I}_C\big) &= c_1(L) \cdot C_r - C' \cdot C_r \\ &= c_1(L) \cdot C_r - \textstyle\sum_{a=0}^{r-1} m_a C_a \cdot C_r - (m_r - 1)C_r^2, \end{aligned}$$

which is < 0 by assumption. Hence s vanishes identically. $\square$

LEMMA 4.4. *If $k \geq 0$ then $H^1(C, \mathcal{N}_{C|M}(-H)) = 0$ and $p_*(\mathcal{N}_{C|M}(-H))$ is globally generated.*

Proof. By Serre duality on C

$$H^1(\mathcal{N}_{C|M}(-H))^\vee \simeq H^0(C, \mathcal{I}/\mathcal{I}^2(H) \otimes K_M(C)) = H^0(C, K_M(H)|_C).$$

Now the first part of the statement follows by Lemma 4.3 above with $L = K_M(H)$. To verify the hypotheses write $C = \sum_{a=0}^{r} m_a C_a$ with $C_a = S$ at most for $a = r > 0$. If $C_a \sim d_a H + k_a F$ then for $a < r$

$$0 \leq S \cdot C_a = k_a,$$

and $k_a = 0$ only if $d_a > 0$. Hence $C_a^2 = d_a^2 e + 2d_a k_a \geq 0$ and

$$\deg_{C_a} K_M(H) = -(d_a H + k_a F) \cdot (H + (2-e)F) = -(2d_a + k_a) < 0.$$

If $C_r = S$ then $\deg_{C_r} K_M(H) = e - 2$,

$$\textstyle\sum_{a=0}^{r-1} m_a C_a \sim dH + kF - m_r S = (d - m_r)H + (k + m_r e)F,$$

and, for $0 \leq m < m_r$,

$$mC_r^2 + \textstyle\sum_{a=0}^{r-1} m_a C_a \cdot C_r = -me + (k + m_r e) \geq k + e.$$

Hence in any case the inequalities required in Lemma 4.3 hold.

Global generation follows by a standard dimension argument with the Riemann-Roch formula provided $H^1(\mathcal{N}_{C|M}(-H-F)) = 0$. This is also true here because $\deg_{C_a} K_M(H+F) = -(d_a+k_a) < 0$ and $\deg_S K_M(H+F) = e-1$. □

In summary, Lemma 4.2 identified the sheaf of local solutions of $\bar\partial v - B \cdot v = 0$ with $p_*(\mathcal{N}_{C|M}(-H))$. Because $\bar\partial - B$ is locally surjective, standard arguments of cohomology theory then give an identification

$$\operatorname{coker}\big(\bar\partial - B\big) \simeq H^1\big(\mathbb{CP}^1, p_*(\mathcal{N}_{C|M}(-H))\big).$$

Because $p|_C$ is a finite morphism the latter sheaf equals $H^1(C, \mathcal{N}_{C|M}(-H))$. The latter vanishes by Lemma 4.4. Moreover, since by the same lemma $p_*(\mathcal{N}_{C|M}(-H))$ is globally generated, so is $\mathcal{K}$. This gives the claimed surjectivity of the restriction. □

Remark 4.5. In place of Proposition 2.6 and Proposition 4.1 one can use the fact from complex analytic geometry that the moduli space of compact complex hypersurfaces in a complex manifold X is smooth at points C with $H^1(C, \mathcal{N}_{C|X}) = 0$. Since this result is not trivial we prefer to give the elementary if somewhat cumbersome explicit method described here.

Surjectivity of $\bar\partial - B$ implies a parametrization of pseudo-holomorphic deformations of C by a finite-dimensional manifold. Our unobstructedness result, Proposition 4.7 below, states this in a form appropriate for the isotopy problem. In the proof we need the following version of the Sard-Smale theorem.

Proposition 4.6. *Let S, X, Y be Banach manifolds, $\Phi : S \times X \to Y$ a smooth map with $\Phi|_{\{s\}\times X}$ Fredholm for all $s \in S$. If $Z \subset Y$ is a direct submanifold* (*the differential of the inclusion map has a right-inverse*) *that is transverse to Φ then the set*

$$\{s \in S \mid \Phi|_{\{s\}\times X} \text{ is transverse to } Z\}$$

is of second category in S.

Proof. Apply the Sard-Smale theorem to the projection $\Phi^{-1}(Z) \to S$. □

Proposition 4.7. *Let $p : (M, J_0) \to \mathbb{CP}^1$ be a holomorphic $\mathbb{CP}^1$-bundle with H, S disjoint holomorphic sections, $H \cdot H \geq 0$. For $U, V \subset M$ open sets with $H \subset V$ consider the space $\mathcal{J}_{U,V}$ of almost complex structures J on M with $J = J_0$ fiberwise and on V, integrable on U and making S holomorphic. Write*

$$\mathcal{M}_{U,V} := \coprod_{J \in \mathcal{J}_{U,V}} \mathcal{M}_J$$

with $\mathcal{M}_J$ the space of J-holomorphic cycles in M.

Assume that $C = \bar{C} + \sum m_a F_a$ is a J-holomorphic cycle homologous to $dH + kF$ with $d > 0, k \geq 0$, for $J \in \mathcal{J}_{U,V}$ with $|C| \subset U$, $H \not\subset |C|$, V containing $H \cup p^{-1}\big(p(|\bar{C}| \cap H) \cap p(|\bar{C}| \cap S)\big) \cup \bigcup F_a$. Here $\bar{C}$ contains all nonfiber components of C. Then

(1) *$\mathcal{M}_{U,V}$ and $\mathcal{M}_J$ are Banach manifolds at C.*

(2) *The map $\mathcal{M}_{U,V} \to \mathcal{J}_{U,V}$ is locally around C a projection.*

(3) *The subset of singular cycles in $\mathcal{M}_J$ is nowhere dense and does not locally disconnect $\mathcal{M}_J$ at C. Similarly for $\mathcal{M}_{U,V}$.*

Proof. In Proposition 4.1 we established surjectivity of the linearization of the map (6). An application of the implicit function theorem with $J \in \mathcal{J}_{U,V}$ and a_0 as parameters thus establishes (1) and (2).

We show the density claim in (3) for $\mathcal{M}_J$, the case of $\mathcal{M}_{U,V}$ works analogously. For the time being assume that C has no fiber components. Apply Proposition 4.6 with $S \subset \mathcal{M}_J$ an open neighbourhood of $\{C\}$ and

$$\Phi : S \times \mathbb{CP}^1 \longrightarrow M^{[d]}, \quad (C', Q) \longmapsto C' \cap p^{-1}(Q).$$

For the definition of $M^{[d]}$ see Section 2. This map is well-defined for C' close to C by the absence of fiber components. In local coordinates provided by Proposition 2.6(1) it is evaluation of $(a_0, \ldots, a_d)$ at points of $\mathbb{CP}^1$, hence smooth. For $Z \subset M^{[d]}$ choose the strata $D_i \subset M^{[d]}$ of the discriminant locus parametrizing 0-cycles $\sum m_a C_a$ with fixed partition $d = \sum m_a$ indexed by i. The top-dimensional stratum D_0 parametrizes 0-cycles with exactly one point of multiplicity 2; it is a locally closed submanifold of $M^{[d]}$ of codimension 2. All other strata D_i, $i > 0$, have codimension at least 4. Since X and Y are finite-dimensional here, the Fredholm condition is vacuous. Transversality of Φ to $Z = D_i$ follows for all i from the following lemma.

Lemma 4.8. *For any $Q \in \mathbb{CP}^1$ the map*

$$\mathcal{M}_J \longrightarrow S^d(p^{-1}(Q)) \simeq \mathbb{CP}^d$$

is a submersion at C.

Proof. On solutions of the linearized equation $\bar\partial v - B\cdot v = 0$ the differential of the map in question is evaluation at Q. The claim thus follows from the surjectivity statement in Proposition 4.1. □

Now since $\operatorname{codim}_{\mathbb{R}} D_i > 2$ for $i > 0$ transversality of $\Phi|_{\{C'\}\times\mathbb{CP}^1}$ means that $p^{-1}(Q)\cap C'$ has no point of multiplicity larger than 2, for all $Q \in \mathbb{CP}^1$. On the other hand, by Proposition 2.8 transversal intersections with D_0 translate into smooth points P of C' with the projection $C' \to \mathbb{CP}^1$ being simply branched at P.

For the remaining part of claim (3) we apply Proposition 4.6 with S the space of paths

$$\gamma : [0,1] \to \mathcal{M}_J$$

connecting two smooth curves C', C'' sufficiently close to C. The map is

$$\Phi : S \times \big([0,1]\times\mathbb{CP}^1\big) \longrightarrow M^{[d]}, \quad (\gamma, t, Q) \longmapsto \gamma(t)\cap p^{-1}(Q).$$

and Z runs over the D_i as before. Again, transversality follows by Lemma 4.8. For dimension reasons we still obtain $\gamma(t)\cap D_i = \emptyset$ for $i > 0$. It remains to argue that not only γ is transverse to D_0 but even $\gamma(t)$ is for every $t \in [0,1]$. Let $W \subset [0,1]\times\mathbb{CP}^1$ be the one-dimensional submanifold of (t,Q) with $\gamma(t)$ having a point of multiplicity 2 over Q. Let $v \in T_{\mathbb{CP}^1}$ be in $\ker(Dq)$ where q is the projection

$$q : W \to [0,1].$$

Then since $D_0 \subset M^{[d]}$ is an analytic divisor and the differential of Φ along $t = \text{const}$ is complex linear, it follows that $i\cdot v$ is also in $\ker(Dq)$. But W is one-dimensional, so $v = 0$ as had to be shown. This finishes the proof, provided C does not have fiber components.

In the general case let $Z \subset \mathcal{M}_J$ be the subset of cycles with fiber components. A cycle $C' \in Z$ has a unique decomposition $C' = \bar C' + \sum m_a F_a$ for certain fibers F_a and with $\bar C' \sim dH + (k - \sum m_a)F$. The space of such configurations is of real codimension $2d\sum_a m_a$. Therefore Z is a union of submanifolds of real codimension at least 2, and these may be avoided in any path by small perturbations. Apply the previously established density result to this perturbed path to obtain a path of smooth cycles. □

5. Good almost complex structures

Our objective is now to construct an almost complex structure J as required in Proposition 4.7, making an arbitrary pseudo-holomorphic curve in a pseudo-holomorphic S^2-bundle J-holomorphic. In the next result, we construct an appropriate integrable complex structure J_0.

Lemma 5.1. *Let $p : M \to S^2$ be an S^2-bundle and let $J_{M/B}$ be a complex structure on the fibers of p. Let $H, S \subset M$ be disjoint sections. Then there exists an integrable complex structure J_0 on M with $J_0|_{T_{M/B}} = J_{M/B}$, making p a holomorphic map and H, S holomorphic divisors.*

Moreover, if $U \subset S^2$ is an open subset and $f : p^{-1}(U) \to U \times S^2$ is a trivialization mapping H, S to constant sections then J_0 may be chosen to make this trivialization holomorphic.

Proof. Since $S \cdot S = -H \cdot H$ we may assume $H \cdot H \geq 0$. Put $e := H \cdot H$, and let F be a fiber with $F \subset p^{-1}(U)$ if $U \neq \emptyset$. Denote by P the intersection point of H and F. It suffices to produce a map

$$f : M \setminus \{P\} \longrightarrow \mathbb{CP}^1$$

with the following properties.

(1) $f|_{p^{-1}(Q)}$ is a biholomorphism for every $Q \in S^2 \setminus p(F)$.

(2) $f^{-1}(0) = H \setminus \{P\}$, $f^{-1}(\infty) \subset (S \cup F) \setminus \{p\}$.

(3) There exists a complex coordinate u on an open set $U' \subset S^2$ containing $p(F)$ so that

$$p^*(u)^e \cdot f : p^{-1}(U') \setminus (H \cup S \cup F) \longrightarrow \mathbb{C}$$

extends differentiably to a map $p^{-1}(U') \to \mathbb{CP}^1$ inducing a biholomorphism $F \to \mathbb{CP}^1$.

In fact, away from F this map may be used to define a holomorphic trivialization of p, while near F one may take $p^*(u)^e \cdot f$.

Since $H \cdot H = e$, a tubular neighbourhood of H is diffeomorphic to a neighbourhood of the zero section in the complex line bundle of degree e over $\mathbb{CP}^1$. Let z, w be complex coordinates near P with $z = p^*(u)$, w fiberwise holomorphic and H given by $w = 0$. Consider the zero locus of $w - z^e$. This is a local section of p intersecting H at P of multiplicity $e = H \cdot H$. Hence this zero locus extends to a section H' isotopic to H and with $H \cap H' \subset \{P\}$, $H' \cap S = \emptyset$. Under the presence of a trivialization over $U \subset S^2$ mapping H, S to constant sections, choose H' holomorphic over U.

Away from F we now have an S^2-bundle with three disjoint sections and fiberwise complex structure. The uniformization theorem thus provides a unique map $f : M \setminus F \to \mathbb{CP}^1$ that is fiberwise biholomorphic and maps H, H', S to $0, 1, \infty$ respectively.

It remains to verify (3). Take for u the function with $z = p^*(u)$ as before. Multiplying f by a constant $\lambda \neq 0$ on sections has the effect of keeping H and S fixed, but scaling $H' = f^{-1}(1)$ by λ^{-1}. Thus $p^*(u)^e \cdot f$ corresponds to the family with H' replaced by the graph of $z^e / p^*(u^e) \equiv 1$. This family extends over $u = 0$. □

We are now in position to construct an almost complex structure so that a given pseudo-holomorphic cycle has unobstructed deformations.

LEMMA 5.2. *Let $p : (M, J) \to \mathbb{CP}^1$ be a pseudo-holomorphic S^2-bundle and H, S disjoint J-holomorphic sections. Let $C \subset M$ be a J-holomorphic curve.*

Then for every $\delta > 0$ there exist a $\mathcal{C}^1$-diffeomorphism $\Phi : M \to M$ and an almost complex structure $\tilde{J}$ on M with the following properties.

(1) *Φ is smooth away from a finite subset $A \subset M$, and $D\Phi|_A = \mathrm{id}$, $\Phi(S) = S$, $\Phi(H) = H$.*

(2) $\|D\Phi - \mathrm{id}\|_\infty < \delta$, $\Phi|_{M \setminus B_\delta(A)} = \mathrm{id}$.

(3) *$\Phi(C)$ and H, S are $\tilde{J}$-holomorphic.*

(4) *$p : (M, \tilde{J}) \to \mathbb{CP}^1$ is a pseudo-holomorphic S^2-bundle.*

(5) *$\tilde{J}$ is integrable in a neighbourhood of $|C|$.*

(6) *There exists an integrable complex structure J_0 on M and an open set $V \subset M$ containing H and all fiber components of C, so that $\tilde{J} = J_0$ fiberwise and on V.*

Proof. Define $F = p^{-1}(\infty)$. Without restriction we may assume $F \subset C$ and $S, H \not\subset C$. Decompose $C = \bar{C} \cup \bigcup_a F_a$ with the second term containing the fiber components. To avoid discussions of special cases replace C by the closure of $p^{-1}\big(p(\bar{C} \cap H)\big)$. For the construction of J_0 we would like to apply Lemma 5.1. However, since we want $\tilde{J} = J_0$ near the fiber components of C and $D\Phi|_A = \mathrm{id}$, it is not in general possible to achieve $J_0|_{T_{M/\mathbb{CP}^1}} = J|_{T_{M/\mathbb{CP}^1}}$. For each $Q \in \big(\bar{C} \cap \bigcup_a F_a\big) \setminus (H \cup S)$ take a local J-holomorphic section D_i of p through Q. For each a there exists a local trivialization $p^{-1}(V_a) = V_a \times \mathbb{CP}^1$ restricting to a biholomorphism $(F_a, J|_{T_{F_a}}) \to \mathbb{CP}^1$ and sending D_i and H, S to constant sections. Define the fiberwise complex structure $J_{M/\mathbb{CP}^1}$ near the F_a by pulling back the complex structure on $\mathbb{CP}^1$ via these trivializations. Extend this to the rest of M arbitrarily. Now define the reference complex structure J_0 by applying Lemma 5.1 with the data $M, J_{M/\mathbb{CP}^1}, S, H$ and the chosen trivialization near $\bigcup F_a$.

Next we construct the diffeomorphism Φ. Put

$$A = (C \cup H \cup S)_{\mathrm{sing}} \cup \mathrm{Crit}(p|_{\bar{C}_{\mathrm{reg}}}),$$

where Crit(.) denotes the set of critical points of a map. This is a finite set. For each $P \in A$ let z, w be J_0-holomorphic coordinates near P with $z = p^*(u)$,

$z(P) = w(P) = 0$. If $P \in H \cup S$ we also require $w(H \cup S) = \{0\}$. Let $b(z,w)$ be the function defining J near P via $T^{0,1}_{M,J} = \langle \partial_{\bar w}, \partial_{\bar z} + b_P \partial_w \rangle$. Then

$$\Psi_P = (z, w - b_P(0,0)\bar w) : V_P \longrightarrow \mathbb{C}^2$$

is a chart for M mapping $J|_P$ to the standard complex structure on $T_{\mathbb{C}^2,0}$. Note that in this chart p is the projection onto the first coordinate of $\mathbb{C}^2$. Now $\Psi_P(C \cap V_P)$ is pseudo-holomorphic with respect to an almost complex structure agreeing at $0 \in \mathbb{C}^2$ with the standard complex structure I on $\mathbb{C}^2$. Thus Theorem 6.2 of [MiWh] applies. It gives a diffeomorphism Φ_P of a neighbourhood of the origin in $\mathbb{C}^2$ of class $\mathcal{C}^1$ mapping $\Psi_P(C \cap V_P)$ to a holomorphic curve, and with $D\Phi_P|_0 = \mathrm{id}$. Moreover, by our choices $\Psi_P(H \cup S)$ is already holomorphic and hence, by the construction in [MiWh] remains pointwise fixed under Φ_P. Therefore there exists, for any sufficiently small $\delta > 0$, a diffeomorphism Φ of M with

$$\Phi|_{M \setminus B_\delta(A)} = \mathrm{id}, \quad \|D\Phi\| < \delta, \quad \Phi|_{B_{\delta/2}(P)} = \Psi_P^{-1} \circ \Phi_P \circ \Psi_P \quad \forall P \in A,$$

and with $\Phi|_H = \mathrm{id}$, $\Phi|_S = \mathrm{id}$. This is the desired diffeomorphism of M.

For the definition of $\tilde J$ observe that on $B_{\delta/2}(P)$, for δ sufficiently small and $P \in A$, the transformed curve $\Phi(C)$ is pseudo-holomorphic with respect to $\Psi_P^*(I)$. On this part of M define $\tilde J = \Psi_P^*(I)$. For $P \in A \cap (H \cup S \cup \bigcup F_a)$, moreover, $b_P \equiv 0$; hence $\Psi_P^*(I) = J_0$. This is true at $F_a \cap \bar C$ by the definition of J_0, and for $P \in (H \cup S) \cap \bar C$ because p and H, S are pseudo-holomorphic for both J and J_0. Therefore we may put $\tilde J = J_0$ on $B_{\delta/2}(H \cup S \cup \bigcup F_a)$ for δ sufficiently small. So far we have defined $\tilde J$ on $V := B_{\delta/2}(A \cup H \cup S \cup \bigcup F_a)$. To extend to the rest of M let

$$w : M \setminus (H \cup F) \longrightarrow \mathbb{C}$$

be the restriction of a meromorphic function on (M, J_0) inducing a biholomorphism on each fiber as in the proof of Lemma 5.1. Let $u : \mathbb{CP}^1 \setminus p(F) \simeq \mathbb{C}$ and put $z = p^*(u)$. To define $\tilde J$ agreeing with J^0 fiberwise is equivalent to giving a complex valued function b via

$$T^{0,1}_{M,\tilde J} = \langle \partial_{\bar w}, \partial_{\bar z} + b(z,w)\partial_w \rangle;$$

see Lemmas 1.5, 1.6. The condition that $\Phi(C)$ be pseudo-holomorphic prescribes b along $\Phi(C_{\mathrm{reg}})$. Moreover, $\tilde J$ coincides with J^0 near $H \cup F$ if and only if b has compact support, and the already stated definition of $\tilde J$ on V forces b also to vanish there. This fixes b on $\Phi(C) \cup V$.

LEMMA 5.3. *There exists an extension of b to $\mathbb{C}^2$ with compact support and so that $\partial_{\bar w} b = 0$ in a neighbourhood of $\Phi(C)$.*

Proof. Let $P \in \Phi(C_{\mathrm{reg}})$ be a noncritical point of $\Phi(C_{\mathrm{reg}}) \to \mathbb{CP}^1$. In a neighbourhood $U_P \subset M$ of P write $\Phi(C)$ as graph $w = \lambda(z)$. We define

$$b_P(z,w) = \partial_{\bar z}\lambda(z)$$

on U_P. Cover a neighbourhood of $\Phi(C) \setminus V$ with finitely many such U_P. Let $\{\rho_P\}$ be a partition of unity subordinate to the cover $\{U_P \cap \Phi(C)\}$ of $\Phi(C) \setminus V$. For any P the projection $p|_{U_P \cap \Phi(C)}$ is an open embedding. Hence there exists a function σ_P on $p(U_P \cap \Phi(C))$ with $\rho_P = p^*(\sigma_P)|_{\Phi(C)}$. Then $p^*(\sigma_P)|_{U_P}$ is a partition of unity for $\{U_P\}$ in a neighbourhood U of $\Phi(C) \setminus V$ in M. Put

$$b(z,w) = \begin{cases} \sum_P \sigma_P(z) \cdot b_P(z,w), & (z,w) \in U \\ 0 & (z,w) \in V. \end{cases}$$

For well-definedness it is crucial that w be globally defined on $M \setminus (H \cup F)$. Now $b(z,w)$ is a smooth function on a neighbourhood of $\big(V \cup \Phi(C)\big) \setminus (H \cup F)$ in M with the desired properties. Extend this arbitrarily to $M \setminus (H \cup F)$ with compact support. □

To finish the proof of Lemma 5.2 it remains to remark that $\tilde{J}$ keeps p pseudo-holomorphic by construction. □

The results of this and the last section will be useful for the isotopy problem in combination with the following lemma (cf. also [Sh, Lemma 6.2.5]).

Lemma 5.4. *In the situation of Lemma* 5.2 *let* $\{J_n\}_n$ *be a sequence of almost complex structures making* p *pseudo-holomorphic and converging towards* J *in the* $\mathcal{C}^0$*-topology on all of* M *and in the* $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$*-topology on* $M \setminus A$. *For every* n *let* C_n *be a smooth* J_n*-holomorphic curve with* $C_n \cap A = \emptyset$ *and so that* $C_n \to C$ *in the* $\mathcal{C}^0$*-topology. Let* Φ, $\tilde{J}$ *be a diffeomorphism and almost complex structure from the conclusion of Lemma* 5.2.

Then, possibly after going over to a subsequence, there exists a finite set $\tilde{A} \subset M$ *containing* A *and almost complex structures* $\tilde{J}_n$ *with the following properties.*

(1) p *is* $\tilde{J}_n$*-holomorphic.*

(2) $\Phi(C_n)$ *is* $\tilde{J}_n$*-holomorphic.*

(3) $\tilde{J}_n \to \tilde{J}$ *in* $\mathcal{C}^0$ *on* M *and in* $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$ *on* $M \setminus \tilde{A}$.

An analogous statement holds for sequences of paths $\{C_{n,t}\}_t$, $\{J_{n,t}\}_t$ *uniformly converging to* C *and* J *respectively.*

Proof. By the Gromov compactness theorem a subsequence of the C_n converges as stable maps. Note that $\mathcal{C}^0$-convergence of the almost complex structures is sufficient for this theorem to be applicable [IvSh]. If $\varphi : \Sigma \to M$ is the limit then $C = \varphi_*(\Sigma)$. Define $\tilde{A}$ as the union of A and of $\Phi \circ \varphi$ of the set of critical points of $p \circ \Phi \circ \varphi$. Note that by the definition of convergence of stable maps, away from $\tilde{A}$ the convergence $\Phi(C_n) \to \Phi_*(C)$ is as tuples of sections. In other words, for $P \in \Phi(|C|) \setminus \tilde{A}$, say of multiplicity m in C, there exists a neighbourhood U_P of P and disjoint J_n-holomorphic sections $\lambda_{1,n},\ldots,\lambda_{m,n}$ of p over $p(U_P)$ so that

$$\lambda_{i,n} \overset{n\to\infty}{\longrightarrow} \lambda, \quad i=1,\ldots,m,$$

and λ has image $\Phi(|C|)\cap U_P$. By elliptic regularity and $\mathcal{C}^{0,\alpha}$-convergence of the J_n away from A this convergence is even in $\mathcal{C}^{1,\alpha}$. To save on notation we now write C, C_n, J_n instead of $\Phi_*(C)$, $\Phi(C_n)$, $\Phi_*(J_n)$ respectively. The assumptions remain the same except that J_n may only be continuous at points of A.

A diagonal argument reduces the statement to convergence in $\mathcal{C}^{0,\alpha}(M\setminus B_\varepsilon(\tilde A))$ for any fixed small $\varepsilon>0$ in place of $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$-convergence on $M\setminus\tilde A$. The construction of $\tilde J_n$ proceeds on three types of regions, which are $M\setminus B_{\varepsilon/2}(|C|)$, $B_{3\varepsilon}(\tilde A)$, and $B_\varepsilon(|C|)\setminus B_{2\varepsilon}(A)$.

On $M\setminus B_{\varepsilon/2}(|C|)$ take $\tilde J_n=\tilde J$. For the definition near $\tilde A$ observe that J_n fulfills all requirements except that it is possibly only continuous at A. However, since the distance δ_n from C_n to A is positive, there exist smooth $\tilde J_n$ agreeing with J_n away from $B_{\delta_n/2}(A)$ and still converging to $\tilde J$ in the $\mathcal{C}^0$-topology.

The interesting region is $B_\varepsilon(|C|)\setminus B_{\varepsilon/2}(A)$. Let $z=p^*(u),w$ be local complex coordinates on M near some $P\in|C|$ with u holomorphic and w fiberwise $\tilde J$-holomorphic. Assume that

$$w=\lambda_{i,n}(z),\quad i=1,\ldots,m,$$

describe the branches of C_n near P as above. Since p is pseudo-holomorphic but w need not be fiberwise J_n-holomorphic, the almost complex structure J_n is now equivalent to two functions a_n,b_n via

$$T^{0,1}_{M,J_n}=\langle\partial_{\bar z}+b_n\partial_w,\partial_{\bar w}+a_n\partial_w\rangle.$$

A section $\phi(z)=(z,\lambda(z))$ is J_n-holomorphic if and only if

$$\phi_*\partial_{\bar z}=\partial_{\bar z}+\partial_{\bar z}\lambda\,\partial_w+\partial_{\bar z}\bar\lambda\,\partial_{\bar w}$$

lies in $T^{0,1}_{M,J_n}$. This is the case if and only if

$$\phi_*\partial_{\bar z}=(\partial_{\bar z}+b_n\partial_w)+\partial_{\bar z}\bar\lambda(\partial_{\bar w}+a_n\partial_w).$$

Comparing coefficients gives the equation $\partial_{\bar z}\lambda=b_n+a_n\partial_{\bar z}\bar\lambda$. Thus pseudo-holomorphicity of the i^{th} branch of C_n is equivalent to the equation

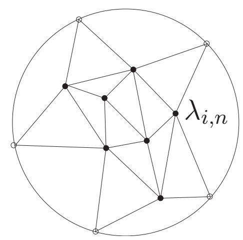
$$\partial_{\bar z}\lambda_{i,n}-a_n(z,\lambda_{i,n})\partial_{\bar z}\bar\lambda_{i,n}=b_n(z,\lambda_{i,n}).$$

To define $\tilde J_n$ fiberwise agreeing with $\tilde J$ requires a function $\tilde b_n$ with

$$\partial_{\bar z}\lambda_{i,n}=\tilde b_n(z,\lambda_{i,n})$$

for all i.

The intersection of the fibers of p with $\overline{B}_\varepsilon(|C|)$ defines a family of closed disks $\overline{\Delta}_z$ near P, $|z| \ll 1$. Take a triangulation of $\overline{\Delta}_0$ with vertices $C_n \cap \overline{\Delta}_0$ in the interior of $\overline{\Delta}_0$ as in the following figure.



Triangulation of $C_n \cap \overline{\Delta}_0$.

The lines in the interior are straight in the w-coordinate. Take δ so small that the triangulation deforms to $\overline{\Delta}_z$ for all $|z| \le \delta$. Let $f_{P,n}$ be the fiberwise piecewise linear function on $B_\varepsilon(|C|) \cap \{|z| < \delta\}$ restricting to $\partial_{\bar z}\lambda_{i,n} - \tilde b(z, \lambda_{i,n})$ along the i^{th} branch of C_n and vanishing at the vertices on the boundary.

Lemma 5.5.
$$\frac{\left|\nabla(\lambda_{i,n} - \lambda_{j,n})\right|}{\left|\lambda_{i,n} - \lambda_{j,n}\right|^\alpha} \longrightarrow 0.$$

Proof. The difference $u = \lambda_{i,n} - \lambda_{j,n}$ of two branches fulfills the elliptic equation

$$\begin{aligned}\partial_{\bar z} u &- a_n(z, \lambda_{i,n})\partial_{\bar z}\bar u \\ &= \big(a_n(z,\lambda_{i,n}) - a_n(z,\lambda_{j,n})\big)\partial_{\bar z}\bar\lambda_{j,n} + b_n(z,\lambda_{i,n}) - b_n(z,\lambda_{j,n}).\end{aligned}$$

Elliptic regularity gives an estimate

$$\|\nabla u\|_{0,\alpha} \le c \cdot \big(\|a_n\|_{0,\alpha}\|\lambda\|_{1,\alpha} + \|b_n\|_{0,\alpha} + 1\big) \cdot \|u\|_\infty^\alpha,$$

with c not depending on n. This implies the desired convergence. □

The lemma implies that the Hölder norm of $f_{P,n}$ tends to 0 for $n \to \infty$. Let $\tilde f_{P,n}$ be a smoothing of $f_{P,n}$ agreeing with $f_{P,n}$ at the vertices of the triangulation and so that

$$\|\tilde f_{P,n}\|_{0,\alpha} \le \|f_{P,n}\|_{0,\alpha} + n^{-1}.$$

Then near P the desired almost complex structure $\tilde J_n$ will be defined by $\tilde b_{P,n} = \tilde f_{P,n} + \tilde b$.

To glue, keeping C_n and p pseudo-holomorphic, we observe that our local candidates for $\tilde J_n$ fiberwise all agree with $\tilde J$. It therefore suffices to glue the corresponding sections $\tilde\beta_n$ of $T^{0,1}_{M/\mathbb{CP}^1} \otimes p^*(\Lambda^{0,1}_{\mathbb{CP}^1})$ (Lemma 1.6) using a partition of unity. Since Lemma 1.6 requires a fiberwise coordinate $\pi = w$ do this in three steps: First on a neighbourhood of a general fiber, minus a general section, then in a neighbourhood of H, and finally on $M \setminus (H \cup F)$.

The statement for paths $\{C_{n,t}\}$, $\{J_{n,t}\}$ follows locally in t by the same reasoning with an additional parameter t; extend this to all t with a partition of unity argument. ☐

6. Generic paths and smoothings

In this section we discuss the existence of certain generic paths of almost complex structures. Let $p : (M, \omega, J) \to \mathbb{CP}^1$ be a symplectic pseudo-holomorphic S^2-bundle. For the purpose of this section $\mathcal{J}$ denotes the space of tamed almost complex structures on M *making p pseudo-holomorphic.* Endowed with the $\mathcal{C}^l$-topology $\mathcal{J}$ is a separable Banach manifold.

We will use the following notion of positivity.

Definition 6.1. An almost complex manifold (M, J) is *monotone* if for every J-holomorphic curve $C \subset M$ it holds

$$c_1(M) \cdot C > 0.$$

LEMMA 6.2. *For any J in a path-connected Baire subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}$ there exist disjoint J-holomorphic sections $S, H \subset M$ with $H^2 = -S^2 \in \{0, 1\}$. Moreover, such J enjoy the following properties*:

(1) *Any irreducible J-holomorphic curve $C \subset M$ not equal to S is homologous to*
$$dH + kF, \quad d, k \geq 0,$$
where F is the class of a fiber.

(2) (M, J) *is monotone* (Definition 6.1).

In particular, there are no J-holomorphic exceptional spheres on M except possibly S.

Proof. For $J = I$ the (generic) integrable complex structure S is a holomorphic section of minimal self-intersection number. Then $c_1(M, J)$ is Poincaré dual to

$$2S + (2 - S \cdot S)F.$$

We first consider the case $M = \mathbb{F}_1$. The expected complex dimension of the space of smooth J-holomorphic spheres representing S is

$$c_1(M, J) \cdot S + \dim_{\mathbb{C}}(M) - \dim_{\mathbb{C}} \operatorname{Aut}(\mathbb{CP}^1) = (2S + 3F) \cdot S - 1 = 0.$$

This is no surprise as S is an exceptional sphere. For $J \in \mathcal{J}$ any reducible J-holomorphic curve representing S is the union of one section representing $S - lF$ and $l > 0$ fibers. The expected complex dimension of such configurations is

$$(2S + 3F)(S - lF) - 1 + l = -l.$$

Standard transversality arguments show that such reducible J-holomorphic representatives of S do not occur for J in a path-connected Baire subset of $\mathcal{J}$. Note that for any curve $C \subset M$ variations of $J \in \mathcal{J}$ span the cokernel of the $\bar{\partial}_J$-operator, so transversality indeed applies. (If $C = F$ is a fiber the cokernel is trivial since deformations of C as fiber span the normal bundle.)

Similar reasoning gives the existence of a J-holomorphic section $H \sim S + F$. Here the expected complex dimension is 2, so we impose two incidence conditions to reduce to dimension 0.

Now if $C \sim dH + kF$ is a J-holomorphic curve different from S then

$$0 \le C \cdot S = (dH + kF) \cdot S = k, \quad 0 \le C \cdot F = (dH + kF) \cdot F = d$$

shows that $d, k \ge 0$, $d + k > 0$ unless $C = S$. In any case

$$c_1(M) \cdot C = (2S + 3F) \cdot (dH + kF) = 2k + 3d > 0.$$

For $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ we have $S \cdot S = 0$, $c_1(M) = 2S + 2F$, the expected complex dimension of spheres representing S is 1, and the expected complex dimension of singular configurations splitting off l fibers is $1 - l$. Impose one incidence condition to reduce the expected complex dimensions by 1. Proceed as before to deduce the existence of disjoint J-holomorphic sections representing $S \sim H$ for generic J.

This time

$$0 \le C \cdot H = (dH + kF) \cdot H = k, \quad 0 \le C \cdot F = (dH + kF) \cdot F = d$$

shows that $d, k \ge 0$ with at least one inequality strict. Thus again

$$c_1(M) \cdot C = (2H + 2F) \cdot (dH + kF) = 2k + 2d > 0$$

for any nontrivial $C \subset M$. $\square$

The other, much deeper genericity result that we will use, is due to Shevchishin. For the readers convenience we state it here adapted to our situation, and give a sketch of the proof.

THEOREM 6.3 ([Sh, Ths. 4.5.1 and 4.5.3]). *Let M be a symplectic 4-manifold and $S \subset M$ a finite subset. There is a Baire subset $\mathcal{J}_{\mathrm{reg}}$ of the space of tamed almost complex structures on M with the following properties.*

(1) *$\mathcal{J}_{\mathrm{reg}}$ is path connected.*

(2) *For a path* $\{J_t\}_{t\in[0,1]}$ *in* $\mathcal{J}_{\mathrm{reg}}$ *let* $\mathcal{M}_{\{J_t\},S}$ *be the disjoint union over* $t\in[0,1]$ *of the moduli spaces of nonmultiple pseudo-holomorphic maps* $\varphi:\Sigma\to(M,J_t)$ *with* $S\subset\varphi(\Sigma)$, *for any closed Riemann surface* Σ. *Then there exists a Baire subset of paths* $\{J_t\}$ *such that* $\mathcal{M}_{\{J_t\},S}$ *is a manifold, at* φ *of dimension*

$$2c_1(M)\cdot\varphi_*[\Sigma]+2g(\Sigma)-1-2\sharp S,$$

and the projection

$$\mathcal{M}_{\{J_t\},S}\longrightarrow[0,1]$$

is open at all φ *except possibly if* $c_1(M)\cdot\varphi_*[\Sigma]-\sharp S\le 0$, $g(\Sigma)>0$ and φ *is an immersion.*

Proof (sketch). We assume $S=\emptyset$; the general case is similar. Let $\mathcal{M}$ be the disjoint union of the moduli spaces of J-holomorphic maps to M for every $J\in\mathcal{J}$. Define $\mathcal{M}_{\mathrm{reg}}$ to be the subset of pairs $(\varphi:\Sigma\to M,J)$ with the cokernel of the linearization of the $\bar\partial_J$-operator at φ having dimension at most 1. Let $\mathcal{J}_{\mathrm{reg}}$ be the complement of the image of $\mathcal{M}\setminus\mathcal{M}_{\mathrm{reg}}$ in $\mathcal{J}$. A standard transversality argument shows that a generic path of almost complex structures lies entirely in $\mathcal{J}_{\mathrm{reg}}$. Then one estimates the codimension of subsets of $\mathcal{M}_{\mathrm{reg}}$ where the cokernel is 1-dimensional. This subset is further stratified according to the so-called *order* and *secondary cusp index* of the critical points of φ. It turns out that a generic path misses all strata except possibly for the case of order 2 and secondary cusp index 1. For this latter case, the crucial point is the existence of explicit second order perturbations of φ showing that the map

$$\nabla D_N:T_{\mathcal{J},J}\longrightarrow\mathrm{Hom}(\ker(D_{N,\varphi}),\mathrm{coker}(D_{N,\varphi}))$$

is surjective; see [Sh, Lemma 4.4.1]. Here $D_{N,\varphi}$ is a $\bar\partial$-operator on the torsion free part of $\varphi^*(T_M)/T_\Sigma$. The implicit function theorem then allows us to compute the dimension of strata with $\mathrm{coker}(D_{N,\varphi})$ of specified dimension. The remaining singularities of order 2 and secondary cusp index 1 have local expressions

$$\varphi(\tau)=\big(\tau^2+O(|\tau|^3),\tau^3+O(|\tau|^{3+\varepsilon})\big),$$

and so are topologically ordinary cusps. Moreover, there must be at least $c_1(M)\cdot\varphi_*[\Sigma]$ cusps present.

A further ingredient of the proof is that the presence of a sufficiently generic cusp contributes complex directions in the second variation of the $\bar\partial_J$-equation. This implies the following: Let φ be a critical point of the projection $\mathcal{M}_{\{J_t\}}\longrightarrow[0,1]$ and assume $c_1(M)\cdot\varphi_*[\Sigma]>0$. Then there is a 2-dimensional submanifold $A\subset\mathcal{M}_{\{J_t\}}$ at φ with real coordinates x,y so that

$$A\longrightarrow\mathcal{M}_{\{J_t\}}\longrightarrow[0,1],\quad(x,y)\longmapsto x^2-y^2.$$

Therefore $\mathcal{M}_{\{J_t\}}\longrightarrow[0,1]$ is open at φ. $\square$

Remarks 6.4. 1. The proof of the theorem shows that we may restrict ourselves to any subspace $\mathcal{J}$ in the space of tamed almost complex structures on M having the following properties: For any $J \in \mathcal{J}$ and any J-holomorphic map $\varphi : (\Sigma, j) \to M$ there exist variations J_t of J in $\mathcal{J}$ and j_t of j so that terms of the form $(\partial_t J_t, \partial_t j_t)$ span the cokernel of the linearization of the $\bar\partial_J$-operator at φ. Moreover, we need enough freedom in varying $\partial_t J_t$ in the normal direction to find solutions of equation (4.4.6) in [Sh].

Variations of J and j enter in the form $(\partial_t J_t) \circ D\varphi \circ j + J \circ D\varphi \circ (\partial_t j_t)$ into this linearization. Therefore, both conditions are fulfilled if, on some open set of smooth points of $\varphi(\Sigma)$ variations inside $\mathcal{J}$ can be prescribed arbitrarily in the normal direction to $\varphi(\Sigma)$ in M.

2. Similarly, if variations of J inside $\mathcal{J}$ fulfill the two conditions only on an open subset $\mathcal{M}' \subset \mathcal{M}$ then the analogous conclusions of the theorem for $\mathcal{M}'$ hold true.

3. The theorem also holds if we replace S by a path $\{S(t)\}_{t\in[0,1]}$.

Proposition 6.5. *If M in Theorem* 6.3 *is the total space of a symplectic S^2-bundle the same conclusions hold for almost complex structures making p pseudo-holomorphic*; *moreover*, *in this case we may additionally assume that for any $J \in \mathcal{J}_{\mathrm{reg}}$ the conclusions of Lemma* 6.2 *hold.*

Proof. By the remark, for $J \in \mathcal{J}$ the only curves that might cause problems are fibers of p. These have always unobstructed deformations. By the same token, in the definition of $\mathcal{J}_{\mathrm{reg}}$ we are free to remove the set of bad almost complex structures from Lemma 6.2. □

The main application of this is the existence of smoothings of J-holomorphic cycles occurring along generic paths of almost complex structures. Our proof uses the unobstructedness of deformations of nodal curves in monotone manifolds, due to Sikorav. It generalizes the well-known unobstructedness lemma for smooth pseudo-holomorphic curves C with $c_1(M) \cdot C > 0$ [Gv, 2.1C1], [HoLiSk]. In the case where all components are rational the result is due to Barraud [Ba]. For the readers convenience we include a sketch of the proof.

Theorem 6.6 ([Sk, Cor. 2]). *Let (M, J) be an almost complex manifold and $C \subset M$ a J-holomorphic curve with at most nodes as singularities and $S \subset C$ a finite set. Assume that for each irreducible component $C_a \subset C$, it holds*

$$c_1(M) \cdot C_a > \sharp(C_a \cap S).$$

Then a neighbourhood of C in the space of J-holomorphic cycles is parametrized by an open set in $\mathbb{C}^d$ with $d = (c_1(M) \cdot C + C \cdot C)/2$. The subset parametrizing *nodal curves is a union of complex coordinate hyperplanes. Each such hyperplane parametrizes deformations of C with one of the nodes unsmoothed.*

In particular, for $d > 0$ a J-holomorphic smoothing of C exists.

Proof (sketch). We indicate the proof for $S = \emptyset$; the general case is similar. Let $\varphi : \Sigma \to M$ be the injective J-holomorphic stable map with image C. Standard gluing techniques for J-holomorphic curves give a parametrization of deformations of φ as J-holomorphic stable maps by the finite-dimensional solution space of a nonlinear equation on $S \times V$; here $S \subset \mathbb{C}^N$ parametrizes a certain universal holomorphic deformation of Σ together with some marked points and V is a linear subspace of finite codimension in a space of sections of $\varphi^*(T_M)$. The differential of the equation in the V-direction is Fredholm and varies continuously with $s \in S$. The precise setup differs from approach to approach; see for example [LiTi], [Si]. The following discussion holds for either of these.

The statement of the theorem follows from this by two observations. First, if Σ has r nodes then S is naturally a product $S_1 \times S_2$, where $S_1 \subset \mathbb{C}^r$ parametrizes deformations of the nodes of Σ, while S_2 takes care of changes of the complex structure of the (normalization of the) irreducible components of Σ together with points marking the position of the singular points of Σ. In particular, the i^{th} coordinate hyperplane in S_1 corresponds to deformations of Σ with the i^{th} node unsmoothed. The observation is that the $\bar\partial_J$-equation is not only differentiable relative to S but even relative to S_1. In fact, variations along the S_2-direction merely change the complex structure of Σ away from the nodes. This variation is manifestly differentiable in all of the gluing constructions.

For the second observation let $\hat\Sigma = \coprod_i \Sigma_i \to \Sigma$ be the normalization of Σ; this is the unique generically injective proper holomorphic map with $\hat\Sigma$ smooth. Write $\hat\varphi$ for the composition with φ. Let D_φ be the linearization of the $\bar\partial_J$-operator acting on sections of $\varphi^*(T_M)$, and $\tilde D_\varphi$ the analogous operator with variations in the S_2-directions included. There is a similar operator $D_{N,\varphi}$ acting on sections of $N := \hat\varphi^*(T_M)/d\hat\varphi(T_\Sigma)$, see e.g. [Sh, §1.5] for details. Both D_φ and $D_{N,\varphi}$ are 0-order perturbations of the $\bar\partial$-operator on $\varphi^*(T_M)$ and on N respectively, for the natural holomorphic structures on these bundles. The adjoint of $D_{N,\varphi}$ thus has an interpretation as 0-order perturbation of the $\bar\partial$-operator acting on sections of $(N \otimes \Lambda^{0,1}_\Sigma)^* \simeq \det \hat\varphi^*(T^*_M)$. Thus $\ker(D^*_{N,\varphi})$ restricted to Σ_i consists of pseudo-analytic sections of a holomorphic line bundle of degree $-c_1(M) \cdot \hat\varphi_*[\Sigma_i]$, which is < 0 by hypothesis. Recall that solutions of linear equations of the form $\partial_{\bar z} f + \alpha(z)\partial_z f + \beta(z) f = 0$ bear much in common with holomorphic functions. They are therefore called *pseudo-analytic functions*; see [Ve]. One standard fact is the existence of a differentiable function g with $e^g f$ holomorphic. It follows that every zero of a pseudo-analytic section contributes positively. Hence a pseudo-analytic section of a complex line bundle of negative degree must be trivial. This shows $\ker(D^*_{N,\varphi})=0$ and in turn $D_{N,\varphi}$ is surjective.

The point is that this well-known surjectivity implies surjectivity of $\widetilde{D}_\varphi$. Partially descending a similar diagram of section spaces on $\hat\Sigma$ to Σ gives the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}^\infty(T_\Sigma) & \longrightarrow & \mathcal{C}^\infty(\varphi^*(T_M)) & \longrightarrow & \mathcal{C}^\infty(N) & \longrightarrow & 0\\
 & & {\scriptstyle\bar\partial}\big\downarrow & & {\scriptstyle D_\varphi}\big\downarrow & & \big\downarrow{\scriptstyle D_{N,\varphi}} & & \\
0 & \longrightarrow & \Omega^{0,1}(T_\Sigma) & \xrightarrow{d\varphi} & \Omega^{0,1}(\varphi^*(T_M)) & \longrightarrow & \Omega^{0,1}(N) & \longrightarrow & 0.
\end{array}$$

Here smoothness of a section at a node means smoothness on each branch plus continuity. Note that the surjectivity of $\mathcal{C}^\infty(\varphi^*(T_M))\to\mathcal{C}^\infty(N)$ holds only in dimension 4 and because C has at most nodes as singularities. The diagram implies

$$\operatorname{coker}(D_{N,\varphi})=\Omega^{0,1}(\varphi^*(T_M))\Big/\Big(d\varphi\big(\Omega^{0,1}(T_\Sigma)\big)+D_\varphi\big(\mathcal{C}^\infty(\varphi^*(T_M))\big)\Big).$$

This is the same as $\operatorname{coker}(\widetilde{D}_\varphi)$. In fact, $\widetilde{D}_\varphi$ applied to variations of the complex structure of Σ spans $d\varphi(\Omega^{0,1}(T_\Sigma))$ modulo $D_\varphi(\mathcal{C}^\infty(T_\Sigma))$, and

$$D_\varphi(\mathcal{C}^\infty(T_\Sigma))\subset D_\varphi(\mathcal{C}^\infty(\varphi^*(T_M))).$$

Summing up, we have shown that $\operatorname{coker}(\widetilde{D}_\varphi)=0$, and $\widetilde{D}_\varphi$ is the differential relative S_1 of the $\bar\partial$-operator acting on sections of $\varphi^*(T_M)$. □

We are now ready for the main result of this section.

LEMMA 6.7. *Let (M,ω) be a symplectic 4-manifold and J a tamed almost complex structure. Assume that $J=J_{t_0}$ for some generic path $\{J_t\}$ as in Theorem 6.3(2). Let $C=\sum_a m_aC_a$ be a J-holomorphic cycle with $c_1(M)\cdot C_a>0$ for all a. Suppose that $C_{a'}$ is an exceptional sphere precisely for $a'>a_0$ and that*

$$C_{a'}\cdot\Big(\sum_{a<a'}m_aC_a\Big)\ge m_{a'}$$

holds for every $a'>a_0$.

Then there exists a J-holomorphic smoothing $C^\dagger$ of C. Moreover, if $U\subset M$ is open and $C|_U=C_1+C_2$ is a decomposition without common irreducible components, then there exists a deformation of C into a nodal curve $C^\dagger$ with all nodes contained in U and $C^\dagger\cap U$ is the union of separate smoothings of C_1 and C_2.

Proof. Assume first that no C_a is an exceptional sphere. Let $\varphi_a:\Sigma_a\to M$ be the generically injective J-holomorphic map with image C_a. Consider the moduli space $\mathcal{M}_{\{J_t\}}$ of tuples of J_t-holomorphic maps

$$(\varphi'_{a,i})_{a,i} : (\Sigma_{1,1}, \dots, \Sigma_{1,m_1}, \dots, \Sigma_{a,1}, \dots, \Sigma_{a,m_a}, \dots) \longrightarrow M^m,$$

together with $t \in [0,1]$. Here $\Sigma_{a,i}$ denotes the closed surface underlying Σ_a with some complex structure depending on i. By Theorem 6.3, $\mathcal{M}_{\{J_t\}}$ is a smooth manifold of expected dimension and the projection

$$\mathcal{M}_{\{J_t\}} \longrightarrow [0,1]$$

is open at any $(\varphi'_{a,i})$. The subset of tuples $(\varphi'_{a,i})$ with one of the maps $\varphi'_{a,i}$ having a critical point is of real codimension at least 2. Similarly, since we have excluded the possibility of exceptional spheres, the condition that $\varphi'_{a,i}$ and $\varphi'_{a',i'}$ for $(a,i) \neq (a',i')$ have a common tangent line is also of real codimension 2. Thus taking $\{J_t\}$ generic we may assume these configurations correspond to a locally finite union of locally closed submanifolds of real codimension at least 2.

We may thus choose a J-holomorphic perturbation $(\varphi'_{a,i})$ of the tuple with entries $\varphi_{a,i} = \varphi_a$ avoiding the subset of singular configurations. The image of $(\varphi'_{a,i})$ is a J-holomorphic curve C with at most nodes as singularities and such that each component evaluates positively on $c_1(M)$. An application of Theorem 6.6 now shows that a smoothing of C exists. This finishes the proof under the absence of exceptional spheres.

In the general case write $C = \bar{C} + \sum_e m_e E_e$ with the second term containing the exceptional spheres. The previous construction applied to $\bar{C}$ yields a smooth J-holomorphic curve $\Sigma = \bar{C}^\dagger \subset M$. Because Σ contains no exceptional sphere and $J = J_{t_0}$, the space of smooth J-holomorphic curves with a common tangent with E_1 and homologous to Σ is nowhere dense in the space of all such J-holomorphic maps. This follows by the same transversality argument and dimension count as in the proof of nodality in [Sh], Theorem 4.5.1. Deforming Σ slightly we may therefore assume the intersection with E_1 to be transverse. Now apply Theorem 6.6 to $\Sigma \cup E_1$. Because $\Sigma \cdot E_1 > 0$ the result is a J-holomorphic smoothing Σ^1 of $\bar{C} + E_1$ not containing E_1. An induction over $\sum_e m_e$ finishes the construction. The assumption on the intersection numbers of exceptional components with the rest of the curve guarantees that in the induction process the intersection of any exceptional component with Σ^i remains nonempty.

The statement on the existence of partial smoothings is clear from the proof. □

7. Pseudo-holomorphic spheres with prescribed singularities

Proposition 7.1. *Let $p : (M,J) \to \mathbb{CP}^1$ be a pseudo-holomorphic S^2-bundle. Let $W \subset M$ be open and*

$$\varphi : \Delta \longrightarrow W$$

a proper, injective, J-holomorphic map. Assume that there exists an integrable complex structure J_0 on M making p a holomorphic map, and with $J|_W = J_0|_W$, $J|_{T_{M/\mathbb{CP}^1}} = J_0|_{T_{M/\mathbb{CP}^1}}$.

Then for any $k > 0$ there exists a J-holomorphic sphere

$$\psi_k : \mathbb{CP}^1 \longrightarrow M$$

approximating φ to k^{th} order at 0:

$$d_M(\varphi(\tau), \psi_k(\tau)) = o(|\tau|^k).$$

Proof. Let S, H be disjoint holomorphic sections of p with $e := H \cdot H > 0$, and F a fiber not containing $\varphi(0)$. There exists a holomorphic map

$$f : M \setminus (F \cap H) \longrightarrow \mathbb{CP}^1$$

with $f^{-1}(\infty) = H \setminus (F \cap H)$, $f^{-1}(0) \subset S \cup F$, inducing an isomorphism on each fiber, cf. Lemma 5.1. Let $u : \mathbb{CP}^1 \setminus p(F) \to \mathbb{C}$ be a holomorphic coordinate. Put $z = p^*(u)$ and $w = f|_{M \setminus (H \cup F)}$. Thus z, w are holomorphic coordinates on $M \setminus (H \cup F)$ with $p : (z, w) \mapsto z$ and so that

$$T^{0,1}_{M,J} = \langle \partial_{\bar{w}}, \partial_{\bar{z}} + b(z, w)\partial_w \rangle;$$

see Lemma 1.6. We may assume that the image of φ is not contained in a fiber, for otherwise the proposition is trivial. By changing the domain of φ slightly we may then assume $(p \circ \varphi)^*(z) = \tau^m$ for the standard coordinate τ on the domain Δ of φ.

The Taylor expansion with respect to the coordinates z, w of φ at $\tau = 0$ up to order k defines a polynomial map $\Delta \to M$. For $k \geq m$ this approximation of φ takes the form

$$\tau \longmapsto (\tau^m, h(\tau))$$

for some polynomial h of degree at most k. Since any holomorphic $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ is projective, (M, J_0) is a projective algebraic variety. Hence for any $k' > k$ the map

$$\tau \longmapsto (\tau^m, h(\tau) + \tau^{k'})$$

extends to a J_0-holomorphic map

$$\psi_0 : \mathbb{CP}^1 \longrightarrow M$$

osculating to φ at $\tau = 0$ to order at least k. By the choice of w it holds $\psi_0(\mathbb{CP}^1) \cap S \cap F = \emptyset$. Hence

$$(\psi_0)_*[\mathbb{CP}^1] \cdot S = k',$$

and $(\psi_0)_*[\mathbb{CP}^1] \sim mH + k'F$. Let $T_{M/\mathbb{CP}^1}$ be the relative tangent bundle. Because p is pseudo-holomorphic this is a complex line bundle. Put $l = \deg \psi_0^*(T_{M/\mathbb{CP}^1})$. Now $c_1(T_{M/\mathbb{CP}^1})$ is Poincaré-dual to $2H - eF$; hence

$$l = (2H - eF)(mH + k'F) = 2k' + me > k.$$

To show the existence of ψ_k in the statement of the proposition we consider a modified Gromov-Witten invariant. Let $\mathcal{J}$ be the space of almost complex structures J' on M of class $\mathcal{C}^l$ with $J'|_W = J_0|_W$, $J'|_{T_{M/\mathbb{CP}^1}} = J_0|_{T_{M/\mathbb{CP}^1}}$ and where p is pseudo-holomorphic. Note that by Lemma 1.2 tamedness on any compact subset in $\mathcal{J}$ is implicit in this definition. Let $\mathcal{M}$ be the disjoint union over $J' \in \mathcal{J}$ of the moduli spaces of J'-holomorphic maps

$$\psi : \mathbb{CP}^1 \longrightarrow M,$$

with

(i) $\psi_*[\mathbb{CP}^1] = (\psi_0)_*[\mathbb{CP}^1]$ in $H_2(M, \mathbb{Z})$.

(ii) in the chosen holomorphic coordinates τ on $\mathbb{CP}^1$ and (z, w) on $M \setminus (H \cup F)$:

$$\psi(\tau) = (\tau^m, h(\tau) + \tau^{l+1} \cdot v(\tau)).$$

In particular, $p \circ \psi = p \circ \varphi$.

Lemma 7.2. *The forgetful map*

$$\mathcal{M} \longrightarrow \mathcal{J}$$

is a Fredholm map of Banach manifolds of index 0.

Proof. Let $\psi(\tau) = (\tau^m, h + \tau^{l+1} v_0)$ be J'-holomorphic. Let $b'(z, w)$ be the function describing J' on $M \setminus (H \cup F) \simeq \mathbb{C}^2$. Then $\partial_{\bar\tau} = m\bar\tau^{m-1}\partial_{\bar z}$, and J'-holomorphicity of ψ is equivalent to

$$\partial_{\bar\tau}(h + \tau^{l+1} v_0) = m\bar\tau^{m-1} b'(\tau^m, h(\tau) + \tau^{l+1} v_0).$$

Therefore a deformation $(\tau^m, h + \tau^{l+1}(v_0 + \eta))$ of ψ inside $\mathcal{M}$ is J'-holomorphic if and only if

$$\partial_{\bar\tau}\, \eta = m\frac{\bar\tau^{m-1}}{\tau^{l+1}} \Big(b'\big(\tau^m, h + \tau^{l+1}(v_0 + \eta)\big) - b'\big(\tau^m, h(\tau) + \tau^{l+1} v_0\big)\Big).$$

Linearizing with fixed b' gives the equation

$$\partial_{\bar\tau}\, \eta = m\bar\tau^{m-1} (\nabla_w b') \cdot \eta + m\bar\tau^{m-1} \left(\frac{\bar\tau}{\tau}\right)^{l+1} (\nabla_{\bar w} b') \cdot \bar\eta.$$

As τ^{l+1} globalizes as a section of $\mathcal{O}_{\mathbb{CP}^1}(l+1)$, the intrinsic meaning of η is as a section of $\psi^*(T_{M/\mathbb{CP}^1})(-l-1)$. So globally our PDE is an equation of CR-type acting on sections of a complex line bundle over $\mathbb{CP}^1$ of degree -1. The index is computed by the Riemann-Roch formula to be 0. Moreover, as ψ is generically injective, the cokernel can be spanned by variations of b'. An application of the implicit function theorem finishes the proof of the lemma. □

Since $\mathcal{J}$ is connected there exists a path $\{J_t\}$ in $\mathcal{J}$ connecting J_0 with $J_1 = J$. By a standard application of the Sard-Smale theorem the restriction of $\mathcal{M}$ to a *generic* path $\{J_t\}$ is a one dimensional manifold $\mathcal{M}_{\{J_t\}}$ over $[0,1]$. We claim that $\mathcal{M}_{\{J_t\}}$ is compact. Let $\psi_i \in \mathcal{M}$ be J_{t_i}-holomorphic, and $t_i \to t_0$. The Gromov compactness theorem gives a J_{t_0}-holomorphic cycle $C_\infty = \sum m_a C_{\infty,a}$ to which a subsequence of $\psi_i(\mathbb{CP}^1)$ converges. By the homological condition and the local form of the elements of $\mathcal{M}$ near $\varphi(0)$ there is exactly one component of C_∞ that projects onto $\mathbb{CP}^1$. All other components are fibers. So the expected dimension drops by 2 for each such bubbling off. Hence bubbling off can be avoided in a generic path as claimed.

We have thus shown that $\mathcal{M}_{\{J_t\}} \to [0,1]$ is a cobordism. It remains to observe the following fact.

LEMMA 7.3. *The fiber of*

$$\mathcal{M}_{\{J_t\}} \longrightarrow [0,1]$$

over 0 *consists of one element.*

Proof. Let $\psi \in \mathcal{M}_{\{J_t\}}$ be J_0-holomorphic. If $\psi \neq \psi_0$ then the intersection index of $\psi(\mathbb{CP}^1)$ with $\psi_0(\mathbb{CP}^1)$ at $\psi(0)$ is at least $m \cdot (l+1)$. But

$$\psi_0(\mathbb{CP}^1) \cdot \psi(\mathbb{CP}^1) = (mH + k'F)^2 = m^2 e + 2mk' = ml,$$

which is absurd. Hence ψ_0 is the only holomorphic map in $\mathcal{M}_{\{J_t\}}$. □

We are now ready to finish the proof of Proposition 7.1. The parity of the cardinality of the fiber stays constant in a one-dimensional cobordism. Hence, by the lemma, the fiber of $\mathcal{M}_{\{J_t\}} \to [0,1]$ over 1 must be nonempty. We may take for ψ_k any element in this fiber (end of proof of proposition). □

8. An isotopy lemma

Our main technical result is the following Isotopy Lemma.

LEMMA 8.1. *Let* $p : (M, J) \to \mathbb{CP}^1$ *be a pseudo-holomorphic* S^2*-bundle with disjoint* J*-holomorphic sections* H, S. *Let* $\{J_n\}$ *be a sequence of almost complex structures making* p *pseudo-holomorphic. Suppose that* $C_n \subset M$, $n \in \mathbb{N}$, *are smooth* J_n*-holomorphic curves and that*

$$C_n \overset{n\to\infty}{\longrightarrow} C_\infty = \sum_a m_a C_{\infty,a}$$

in the $\mathcal{C}^0$*-topology, with* $c_1(M) \cdot C_{\infty,a} > 0$ *for every* a *and* $J_n \to J$ *in* $\mathcal{C}^0$ *and in* $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$ *away from a finite set* $A \subset M$. *Also assume*:

($*$) *If* $C' = \sum_a m'_a C'_a$ *is a nonzero* J'*-holomorphic cycle* $\mathcal{C}^0$*-close to a subcycle of* $\sum_{m_a>1} m_a C_{\infty,a}$, *with* $J' \in \mathcal{J}_{\text{reg}}$ *as in Proposition* 6.5, *then*

$$\sum_{\{a|m'_a>1\}} \big(c_1(M)\cdot C'_a + g(C'_a) - 1\big) < c_1(M)\cdot C' - 1.$$

Then any J*-holomorphic smoothing* $C_\infty^\dagger$ *of* C_∞ *is symplectically isotopic to some* C_n. *The isotopy from* C_n *to* $C_\infty^\dagger$ *can be chosen to stay arbitrarily close to* C_∞ *in the* $\mathcal{C}^0$*-topology, and to be pseudo-holomorphic for a path of almost complex structures that stays arbitrarily close to* J *in* $\mathcal{C}^0$ *everywhere, and in* $\mathcal{C}^{0,\alpha}_{\text{loc}}$ *away from a finite set.*

Remark 8.2. A J-holomorphic smoothing of C_∞ exists by Proposition 6.7 provided $J = J_{t_0}$ for some generic path $\{J_t\}_t$ as in Proposition 6.3. For the condition on exceptional spheres observe that for homological reasons M contains at most one of them, say S. Then either $C_\infty = S$ and there is nothing to prove, or $C_n \cdot S \geq 0$ and

$$S\cdot(C_n - mS) = S\cdot C_n + m \geq m.$$

Proof of the lemma. By Lemma 1.2 there exists a symplectic structure taming J. We may therefore asume J tamed whenever needed.

Recall the functions m, δ defined in Section 3. Order the pairs (m, δ) lexicographically:

$$(\hat m, \hat\delta) < (m,\delta) \quad\Longleftrightarrow\quad \hat m < m \quad\text{or}\quad \big(\hat m = m \quad\text{and}\quad \hat\delta < \delta\big).$$

We do an induction over $(m(C_\infty), \delta(C_\infty))$. The induction process is inspired by the proof of [Sh, Th. 6.2.3], but the logic is different under the presence of multiple components.

The starting point is $(m(C_\infty), \delta(C_\infty)) = (0,0)$. Then C_∞ is a smooth pseudo-holomorphic curve and the statement is trivial.

So let us assume the theorem has been established for all (m,δ) strictly less than $(m(C_\infty), \delta(C_\infty))$. The proof of the induction step proceeds in eight parts, referred to as Steps 1–8.

1) *Enhancing* J. One of the two sections H, S deforms J-holomorphically, say H. We may therefore assume $H \not\subset |C_\infty|$. Apply Lemma 5.2 to $C = |C_\infty| \cup p^{-1}\big(p(|C_\infty| \cap H)\big)$ and J. The result is a $\mathcal{C}^1$-diffeomorphism Φ of M, smooth away from a finite set $A' \subset M$, and an almost complex structure $\tilde J$ integrable in a neighbourhood of $|C_\infty| \cup H$, and making p and H, S pseudo-holomorphic. By perturbing $\tilde J$ slightly away from $|C_\infty|$ we may assume that $(M, \tilde J)$ is monotone. Moreover, $\Phi_*(C_\infty)$ is now a $\tilde J$-holomorphic cycle with unobstructed deformation theory in the sense made precise in Proposition 4.7. The homological condition on C_∞ required there follows because $C_n \cdot S \geq 0$. We

claim that it suffices to prove the theorem under the assumption that the limit almost complex structure J has this special form *for any particular smoothing* $C^\dagger_\infty$. By taking the union with the finite set in the hypothesis we may assume $A' = A$.

Change C_n and J_n slightly to achieve $C_n \cap A = \emptyset$. For example, apply a diffeomorphism that is a small translation near A and the identity away from this set. After going over to a subsequence Lemma 5.4 now provides almost complex structures $\tilde{J}_n$ on M making p pseudo-holomorphic with $\tilde{J}_n \to \tilde{J}$ in $\mathcal{C}^0$ and in $\mathcal{C}^{0,\alpha}$ away from A, and so that $\Phi(C_n)$ is $\tilde{J}_n$-holomorphic. The sequence $\Phi(C_n) \to \Phi_*(C_\infty)$ fulfills the hypothesis of the lemma. Assuming that we can prove the lemma for $J = \tilde{J}$, pick for every $n \gg 0$ an isotopy $\{\tilde{C}_{n,t}\}_{t\in[0,1]}$ between $\Phi(C_n)$ and a smoothing of $\Phi_*(C_\infty)$; here $\tilde{C}_{n,t}$ is pseudo-holomorphic for a path $\{\tilde{J}_{n,t}\}$ with $\tilde{J}_{n,t} \to \tilde{J}$ and $\tilde{C}_{n,t} \to \Phi_*(C_\infty)$ uniformly in the respective $\mathcal{C}^0$-topologies for $n \to \infty$. Changing $\tilde{C}_{n,t}$ and $\tilde{J}_{n,t}$ slightly near A by an appropriate translation we may also assume $\tilde{C}_{n,t} \cap A = \emptyset$ for all t. Then $\{\Phi^{-1}(\tilde{C}_{n,t})\}$ is an isotopy connecting C_n with a smoothing of C_∞. Another application of Lemma 5.4, now to Φ^{-1}, allows us to find paths $\{J_{n,t}\}$ of smooth almost complex structures, converging to J uniformly in $\mathcal{C}^0$ for $n \to \infty$, so that $\Phi^{-1}(\tilde{C}_{n,t})$ is $J_{n,t}$-holomorphic.

To get the statement of the lemma, apply this reasoning both to the original sequence $\{C_n\}$ and to the given sequence of J-holomorphic smoothings of C_∞. This gives an isotopy of C_n and of some smoothing $C^\dagger_\infty$ with the Φ-preimage of some particular $\tilde{J}$-holomorphic smoothing of $\Phi_*(C_\infty)$.

We can henceforth add the following hypotheses to the lemma.

(∗∗) *For some integrable complex structure J_0 on M the pair (C_∞, J) fulfills the hypotheses of Proposition* 4.7.

Recall that Proposition 4.7 in particular required the specification of an open neighbourhood of $|C_\infty|$ where J is integrable. Moreover, by the conclusion of this proposition it now even suffices to produce an isotopy of C_n with some $\hat{J}$-holomorphic smoothing for $\hat{J}$ sufficiently close to J and of the form as required in this proposition. Here we need also Proposition 3.3 to the effect that $\mathcal{C}^0$-convergence of cycles implies convergence of coefficients in the description of Proposition 2.6(3).

In Step 5 we will ask J to have a certain genericity property, which can be achieved by perturbing $\tilde{J}$ in the normal direction near some smooth points of $|C_\infty|$.

2) *Replacement of nonmultiple components of C_∞ by J-holomorphic spheres.* Let $P \in |C_\infty|_{\text{sing}}$ and write

$$\varphi : \Delta \longrightarrow M$$

for any pseudo-holomorphic parametrization of a branch of $|C_\infty|$ that belongs to a nonmultiple component of C_∞ at P. Denote by $\bar{C}_\infty$ the sum of the multiple components of C_∞, as cycles. For the singularities at P of the curves underlying C_∞ and $\bar{C}_\infty$ we have

$$(|C_\infty|, P) = (|\bar{C}_\infty|, P) \cup \bigcup\nolimits_\varphi (\varphi(\Delta), P).$$

Since $J = J_0$ near P, by hypothesis $(**)$ from Step 1, Proposition 7.1 applies. We obtain a sequence

$$\psi_k : \mathbb{CP}^1 \longrightarrow M$$

of J-holomorphic spheres approximating φ at $0 \in \Delta \subset \mathbb{CP}^1$ to order k. Note that the approximation of fiber components is exact, and for k sufficiently large no other approximating sphere is contained in a fiber. We do not indicate the dependence of ψ_k on φ in the notation. For $k \gg 0$ the topological types of the curve singularities $(|C_\infty|, P)$ and of $(|\bar{C}_\infty|, P) \cup \bigcup_\varphi (\psi_k(\mathbb{CP}^1), P)$ agree. Fix such a k. For $\theta > 0$ define the annulus $A_\theta = B_{2\theta}(P) \setminus B_\theta(P)$. For $\theta > 0$ sufficiently small the intersections

$$A_\theta \cap |C_\infty| \quad \text{and} \quad A_\theta \cap \big(|\bar{C}_\infty| \cup \bigcup\nolimits_\varphi \psi_k(\mathbb{CP}^1)\big)$$

are isotopic by an isotopy in A_θ that is the identity near $|\bar{C}_\infty|$. Therefore, for each nonmultiple branch φ there exists a map

$$\psi : \mathbb{CP}^1 \longrightarrow M,$$

which agrees with ψ_k over $M \setminus B_{2\theta}(P)$ and with φ over $B_\theta(P)$, and which is isotopic to φ on A_θ. Let J' be a small perturbation of J agreeing with J near $|\bar{C}_\infty|$ and making p and ψ pseudo-holomorphic. At the expense of further changes of J and ψ away from $|\bar{C}_\infty|$, we obtain an almost complex structure J' and a J'-holomorphic map $\psi' : \mathbb{CP}^1 \to M$ where

- J' agrees with J in a neighbourhood of $|\bar{C}_\infty|$.
- ψ' is isomorphic to φ over a neighbourhood of P.
- Except possibly at P the map ψ' is an immersion intersecting $|\bar{C}_\infty|$ transversely.

Do this inductively for all reduced branches of C_∞ at all singular points of $|C_\infty|$. The result is an almost complex structure J' agreeing with J near $|\bar{C}_\infty|$ and making p pseudo-holomorphic, and a J'-holomorphic cycle C'_∞ all of whose nonmultiple components are rational. We refer to the part of the added components away from the singularities of $|C_\infty|$ as *parasitic part*. Let $U \subset M$ be a neighbourhood of $|\bar{C}_\infty|$ so that C'_∞ agrees with C_∞ on U except for smooth branches of parasitic components *away* from the singularities of $|C_\infty|$. We can take the intersection of U with the parasitic part of $|C'_\infty|$ to be a union of disks.

Similarly adjust the sequence C_n. By $\mathcal{C}^{0,\alpha}$-convergence of the almost complex structures the tangent spaces of C_n converge to the tangent spaces of $|C_\infty|$ away from the multiple components of C_∞. Hence, for all $n \gg 0$ there exist an almost complex structure J'_n making p pseudo-holomorphic, a J'_n-holomorphic immersion $\varphi'_n : \Sigma'_n \to M$ and an open set $V_n \subset \Sigma'_n$ with the following properties.

- $J'_n = J_n$ on U, $J'_n \stackrel{n\to\infty}{\longrightarrow} J'$ in $\mathcal{C}^0$ and in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$ away from $|C_\infty|_{\mathrm{sing}} \cup A$.
- $\varphi'_n(\Sigma'_n) \stackrel{n\to\infty}{\longrightarrow} C'_\infty$.
- $\varphi'_n|_{V_n}$ is isomorphic to the inclusion $C_n \cap U \to M$, which is a part of the original curve, as a pseudo-holomorphic map.

Define $C'_n = \varphi'_n(\Sigma'_n)$. In analogy with C'_∞ we call $\Sigma'_n \setminus V_n$ and $\varphi'_n(\Sigma'_n \setminus V_n)$ the parasitic part of Σ'_n and of C'_n respectively.

A note on notation: Because for most of the proof we work with curves having a parasitic part we henceforth drop the primes on all symbols. To refer to the original curves and almost complex structures we place an upper index 0, so that the original sequences now read $C^0_n = \varphi^0_n(\Sigma^0_n) \to C^0_\infty$ and $J^0_n \to J^0$.

3) *Description of further strategy.* If C_∞ has multiple components the further strategy is to deform φ_n *away* from C_∞, as pseudo-holomorphic map and uniformly in n. In the deformation process we fix enough points so that any occurring degeneration has better singularities, measured by (m, δ). Thus induction applies and we can continue with any smoothing of the degeneration. Therefore, the deformation process is always successful. The result is a sequence of J_n-holomorphic curves $\{C'_n\}$ with C'_n containing the set of chosen points and symplectically isotopic to C_n in the required way, but with a uniform distance from $|C_\infty|$. For the same reason as before, $\lim C'_n$ is a J-holomorphic cycle with better singularities. Next we replace the parasitic part with the removed part of C^0_∞ and apply the induction hypothesis, if necessary. This will eventually lead to a J'-holomorphic smoothing $(C^0_\infty)^\dagger$ of the original cycle C^0_∞ that is isotopic to C^0_n, with J' arbitrarily close to J and fulfilling the other conditions stated in Proposition 4.7. This was left to be shown in Step 1.

If C_∞ is reduced this argument does not work because we cannot prescribe enough points in the deformation to move away. Instead the pseudo-holomorphic deformation $\varphi_{n,t}$ of φ_n will now be with respect to a generic path $\{J_{n,t}\}_t$ of almost complex structures connecting J_n with J. The deformation process is successful as long as $\varphi_{n,t}$ stays close to φ_n. Otherwise a diagonal argument gives a sequence φ_{n,t_n} converging to a J-holomorphic curve with improved singularities as above.

Alternatively, the reduced case follows from the local isotopy theorem due to Shevchishin [Sh, Th. 6.2.3]. In fact, in this case our proof comes down to a global version of the proof given there. For the sake of completeness we nevertheless provide full details here.

4) *Restoration of reduced part.* The rest of the proof will repeatedly require replacements of the parasitic part by the removed part of the original curves C_n^0 or C_∞^0. The purpose of this paragraph is to make this process precise. A point of caution concerns the nodes that have been generated by the introduction of parasitic components. To simplify the proof we use the fibered structure here, although this is not strictly necessary. For the following it is useful to choose a metric d_{cyc} on $\mathrm{Cyc}_{\mathrm{pshol}}(M)$ inducing the $\mathcal{C}^0$-topology. Normalize d_{cyc} in such a way that $d_{\mathrm{cyc}}(C, C_\infty) \le \varepsilon$ implies $|C| \subset B_\varepsilon(|C_\infty|)$.

Denote by C_∞^{hr} the union of the nonmultiple components of C_∞ *that are not fibers of* p ("horizontal, reduced"). Reduced fiber components of C_∞ can only arise from reduced fiber components of C_∞^0, so these do not require any attention. The projection

$$p : C_\infty^{\mathrm{hr}} \longrightarrow \mathbb{CP}^1$$

is a holomorphic (branched) covering. Throughout this step write $Z = p(|C_\infty^0|_{\mathrm{sing}})$. For small θ

$$A := \mathrm{cl}\left(B_{2\theta}(Z)\right) \setminus B_\theta(Z)$$

is a union of annuli containing no branch points of this covering. Take θ also so small that $C_\infty^{\mathrm{hr}} \cap p^{-1}(A) \Subset U$, U the neighbourhood of $\bar{C}_\infty = \bar{C}_\infty^0$ from Step 2. Then

$$\tilde{A} := C_\infty^{\mathrm{hr}} \cap p^{-1}(A) \cap U$$

is also a union of annuli, which are disjoint from $|\bar{C}_\infty|$, one for each branch of C_∞^{hr} near a singular point of $|C_\infty^0|$. Choose

$$\theta' < \min\{d_M(\tilde{A}, |\bar{C}_\infty|), d_M(\tilde{A}, M \setminus U)\}$$

and less than

$$\frac{1}{2} \min\left\{d_M(x,y)\middle| x, y \in p^{-1}(A) \cap |C_\infty^0| \cap U,\, x \neq y,\, p(x) = p(y)\right\}.$$

Then there exists a tubular neighbourhood

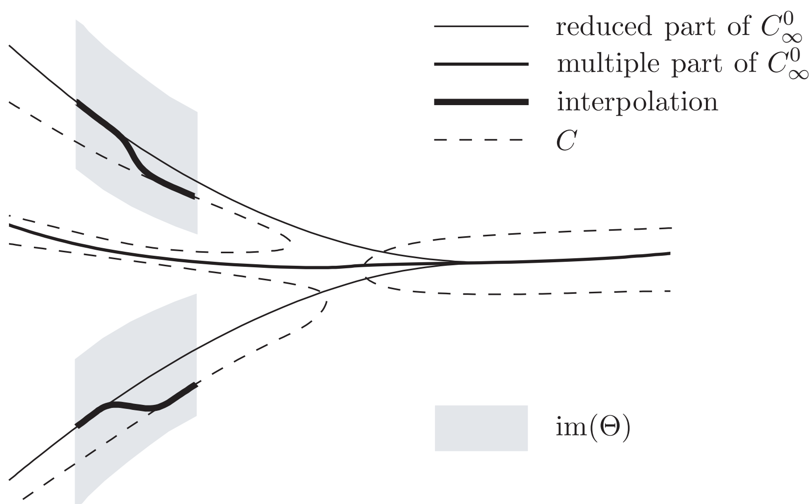
$$\Theta : \tilde{A} \times \Delta \longrightarrow M,$$

of $\tilde{A}$ with image a union of connected components of $B_\theta(C_\infty^{\mathrm{hr}}) \cap p^{-1}(A)$, so that $\Delta_Q := \Theta(\{Q\} \times \Delta)$, $Q \in \tilde{A}$, are J-holomorphic disks contained in the fibers of p. By construction, $\Delta_Q \cdot C_\infty = 1$ for the intersection numbers. Then for $\varepsilon > 0$ sufficiently small it holds also $\Delta_Q \cdot C = 1$ for every J'-holomorphic cycle C with $d_{\mathrm{cyc}}(C, C_\infty) < \varepsilon$ and $\|(J' - J)|_{\mathrm{im}(\Theta)}\|_{0,\alpha} < \varepsilon$. In such cases $\Theta^{-1}(|C|)$ is the graph of a function $\tilde{A} \to \Delta$.

We now choose $\varepsilon > 0$ so small that if $d_{\mathrm{cyc}}(C, C_\infty) < \varepsilon$ then also $(m(C), \delta(C)) \le (m(C_\infty), \delta(C_\infty))$ and equality holds if and only if there is a bijection between the components of C and C_∞ respecting multiplicities (Lemma 3.1). Later we will let ε tend to zero.

Definition 8.3. We say that *restoration applies* to a J'-holomorphic cycle C if (a) $d_{\mathrm{cyc}}(C, C_\infty) < \varepsilon$, $\|(J' - J)|_{\mathrm{im}(\Theta)}\|_{0,\alpha} < \varepsilon$. (b) Let B be an irreducible component of $|C| \setminus p^{-1}(B_\theta(Z))$ intersecting $\mathrm{im}(\Theta)$. Then $B \cap p^{-1}(A) \cap U$ is connected.

Intuitively (b) says that C viewed as deformation of C_∞, does not smooth out self-intersection points involving the parasitic part. The irreducible components of $|C| \setminus (p|_U)^{-1}(B_\theta(Z))$ intersecting $\mathrm{im}(\Theta)$ are then deformations of the nonfiber parasitic part of C_∞. From still another point of view the components of $|C| \setminus (p|_U)^{-1}(B_\theta(Z))$ are pseudo-holomorphic curves with boundary; (b) requires each such curve that intersects $\mathrm{im}(\Theta)$ to have only one boundary component.



Restoration Process.

To define our uniform restoration process choose a smooth function $\rho : A \to [0,1]$ that is identically 0 near the interior boundary components $\partial B_\theta(Z)$ and identically 1 near the exterior boundary components $\partial B_{2\theta}(Z)$. Let C be a cycle to which restoration applies. Let $\lambda, \lambda^0_\infty : \tilde{A} \to \Delta$ be the functions with graphs $\Theta^{-1}(|C|)$ and $\Theta^{-1}(|C^0_\infty|)$ respectively. The *restoration of* C is the following cycle: On $(p|_U)^{-1}(B_\theta(Z))$ take the restriction of C; on $M \setminus (p|_U)^{-1}(B_{2\theta}(Z))$ take the sum of the nonmultiple components of C^0_∞ and all components of $C|_{(p|_U)^{-1}(B_{2\theta}(Z))}$ *not* intersecting $\mathrm{im}(\Theta)$; on $(p|_U)^{-1}(A)$ take Θ of the graph of $p^*(\rho)\lambda^0_\infty + (1 - p^*(\rho))\lambda$. Because the restriction of the nonmultiple components of C^0_∞ to $p^{-1}(A)$ lies in $\mathrm{im}(\Theta)$ the result is indeed a cycle. Let $U' \subset U$ be a neighbourhood of $|\bar{C}_\infty|$ with $U' \cap \mathrm{im}(\Theta) = \emptyset$. This neighbourhood exists by the choice of θ'.

LEMMA 8.4. *Let C be a J'-holomorphic cycle to which restoration applies. Then its restoration C^0 is pseudo-holomorphic for an almost complex structure J^0 with*

$$\|(J^0 - J_\infty)|_{M\setminus U'}\|_{0,\alpha} + \|(J^0 - J')|_U\|_{0,\alpha} < c(\varepsilon),$$

where $c(\varepsilon)$ does not depend on C or J' and $c(\varepsilon) \stackrel{\varepsilon\to 0}{\to} 0$. Moreover, J^0 may be taken to vary continuously under variations of C and J', and to be integrable in U if J' is so.

Proof. Elliptic regularity provides higher estimates on the function $\lambda : \tilde{A} \to \Delta$ defining C on $im(\Theta)$. As in the proof of Lemma 5.4 this allows to interpolate uniformly between J_∞ on $M \setminus U'$ and J' on U by an almost complex structure making C^0 pseudo-holomorphic. □

For the induction argument it will be crucial that restoration is compatible with (m, δ).

LEMMA 8.5. *Let C be a cycle to which restoration applies and assume $(m(C), \delta(C)) < (m(C_\infty), \delta(C_\infty))$. Then for the restoration C^0 of C it holds $(m(C^0), \delta(C^0)) < (m(C^0_\infty), \delta(C^0_\infty))$.*

Proof. Since m depends only on the nonreduced part of a cycle, which restoration does not affect, it suffices to discuss the case $m(C) = m(C_\infty)$. By our choice of ε equality $\delta(C) = \delta(C_\infty)$ then implies the existence of a bijection between the irreducible components of C and C_∞ respecting the multiplicities. This bijection is compatible with restoration. Now the difference between $\delta(C_\infty)$ and $\delta(C^0_\infty)$ has two sources. First, self-intersections of the parasitic part of C_∞. By construction these all lie off U, the neighbourhood of the multiple part $|\bar{C}_\infty|$. Hence this defect equals the virtual number of double points of $|C_\infty|$ on $M \setminus U$. Second, intersections of the parasitic part of C_∞ with the multiple part $|\bar{C}_\infty|$. In view of the mentioned bijection of irreducible components both contributions remain unchanged when comparing $\delta(C)$ and $\delta(C^0)$. Hence $\delta(C_\infty) - \delta(C^0_\infty) = \delta(C) - \delta(C^0)$ and the assertion follows. □

5) *The incidence conditions $|\mathbf{x}| \subset |C_\infty|$.* Write

$$C_\infty = \sum_a m_a C_{\infty,a}.$$

(Note that by our convention from the end of Step 2 the m_a and $C_{\infty,a}$ in the hypothesis of the lemma are now $m^0_a, C^0_{\infty,a}$.) For each a choose points

$x_1^a, \ldots, x_{k_a}^a \in C_{\infty,a}$ in general position with

$$k_a := c_1(M) \cdot C_{\infty,a} + g(C_{\infty,a}) - 1.$$

Since $c_1(M) \cdot C_{\infty,a} > 0$ this number is nonnegative. Hypothesis $(*)$ applied to $C' = \bar{C}_\infty$ together with rationality of the reduced part of C_∞ implies

$$k := \sum_a k_a \leq c_1(M) \cdot C_\infty - 1. \tag{9}$$

The inequality is strict if C_∞ has multiple components. In the reduced case all components of C_∞ are rational and then equality holds. Write $\mathbf{x}^a = (x_1^a, \ldots, x_{k_a}^a)$ and $\mathbf{x} = (x_1, \ldots, x_k)$ for the concatenation of the $\mathbf{x}^a$. Let $\varphi_{\infty,a} : \Sigma_{\infty,a} \to M$ be the generically injective J-holomorphic map with image $C_{\infty,a}$. Consider the moduli space $\mathcal{M}_{a,\mathbf{x}^a}$ of pseudo-holomorphic maps $\varphi : (\Sigma_{\infty,a}, j_a) \to (M, J)$ representing $[C_{\infty,a}]$ in homology and with $|\mathbf{x}^a| \subset \varphi(\Sigma_{\infty,a})$. Here j_a may vary. By Remark 6.4(2) we may assume J to be regular for all small, nontrivial deformations of $\varphi_{\infty,a}$ keeping the incidence conditions $\mathbf{x}^a$; it suffices to change the almost complex structure in the normal direction near a general point of $C_{\infty,a}$. If $m_a > 1$ this leads to a change of $\tilde{J}$ in Step 1; if $m_a = 1$ change J' away from the neighbourhood U of $|\bar{C}_\infty|$ in its construction in Step 2.

Then $\mathcal{M}_{a,\mathbf{x}^a}$ is a manifold of dimension

$$\big(c_1(M) \cdot C_{\infty,a} + g(\Sigma_{\infty,a}) - 1\big) - k_a = 0. \tag{10}$$

Hence $\varphi_{\infty,a}$ is an isolated point of $\mathcal{M}_{a,\mathbf{x}^a}$ for every a. From this we deduce that relevant deformations of C_∞ lead to a drop in (m, δ).

Proposition 8.6. *Let C be a J'-holomorphic cycle with $d_{\mathrm{cyc}}(C, C_\infty) < \varepsilon$ and $|\mathbf{x}| \subset |C|$. Assume that one of the following applies.*

(i) *$J' = J_t$ for a general path $\{J_t\}$ of almost complex structures and $|C|$ is not a nodal curve.*

(ii) *$J' = J$, $C \neq C_\infty$.*

Then

$$(m(C), \delta(C)) < (m(C_\infty), \delta(C_\infty)).$$

Proof. By the definition of ε before Definition 8.3 it holds $(m(C), \delta(C)) \leq (m(C_\infty), \delta(C_\infty))$. Moreover, in the case of equality every component of C is the image of some J'-holomorphic deformation φ_a of $\varphi_{\infty,a}$ with $|\mathbf{x}^a| \subset \mathrm{im}(\varphi_a)$. (Preservation of the incidence condition may require a smaller choice of ε.) By dimension count (10) the parametrizing moduli space of generically injective pseudo-holomorphic maps with incidence conditions has expected relative dimension 0 over the space of almost complex structures. The subspace of nonimmersions is of real codimension at least 2. Hence nonimmersions do not occur over a general path of almost complex structures.

In the second case $\varphi_{\infty,a}$ is an isolated point in $\mathcal{M}_{a,\mathbf{x}^a}$. So equality can only occur if $\varphi_a = \varphi_{\infty,a}$ for all a; that is, $C = C_\infty$. □

6) *The nonreduced case: Deforming C_n uniformly away from C_∞.* Deform C_n and J_n slightly to achieve $|\mathbf{x}| \subset C_n$ for all n. For example, near x_i apply a locally supported diffeomorphism of M moving one branch of C_n to x_i.

We first treat the case when C_∞ has multiple components. The purpose of this step is to establish the following result.

LEMMA 8.7. *For every $n \gg 0$ there exists a pseudo-holomorphic curve $C'_n \subset M$ to which restoration applies and having the following properties.*

(1) *The restoration of C'_n is smooth and isotopic to C_n^0 through a family of pseudo-holomorphic curves within ε-distance to C_∞^0 with respect to d_{cyc}, for a family of almost complex structures with $\mathcal{C}^{0,\alpha}$-distance to J_n^0 getting arbitrarily small with increasing n.*

(2) $|\mathbf{x}| \subset C'_n$.

(3) $d_{\mathrm{cyc}}(C'_n, C_\infty) = \varepsilon/2$.

Proof. The proof is by descending induction over the degree $c_1(M)\cdot C_n$. For a first try take paths $x_n(t)$ on M with $x_n(0) \in C_n \setminus |\mathbf{x}|$ and $\frac{d}{dt} d_M(x_n(t), |C_\infty|) = \varepsilon$. Such paths exist for all n provided that ε is sufficiently small. Write $\tilde{\mathbf{x}}(t)$ for the concatenation $(x_1, \ldots, x_k, x_n(t))$ of $\mathbf{x}$ and $x_n(t)$. Choose a $\mathcal{C}^{0,\alpha}$-small continuous deformation $\{J_{n,t}\}_{t\in[0,1]}$ of J_n so that

- $\{J_{n,t}\}$ fulfills the properties of Theorem 6.3,2 with $S(t) = |\tilde{\mathbf{x}}(t)|$.

Note that the expected (real) dimension of deformations of C_∞ not decreasing (m, δ) and containing $|\tilde{\mathbf{x}}(t)|$ is -2, so these do not occur for generic $\{J_{n,t}\}$; cf. Proposition 8.6.

It now suffices to produce a continuous family $\{C_{n,t}\}_t$ of pseudo-holomorphic cycles, ε-close to C_∞ and with $C_{n,0} = C_n$, such that $C_{n,t}$ is $J_{n,t}$-holomorphic and contains $|\tilde{\mathbf{x}}(t)|$. In fact, by Lemma 6.7 and Remark 8.2 any $C_{n,t}$ has a partial pseudo-holomorphic smoothing $C_{n,t}^\dagger$ to which restoration applies with smooth result $(C_{n,t}^\dagger)^0$. Since $(m(C_{n,t}), \delta(C_{n,t})) < (m(C_\infty), \delta(C_\infty))$ holds for every t, and in view of Lemmas 8.4 and 8.5 the induction hypothesis shows the existence of the requested pseudo-holomorphic isotopy between C_n^0 and $(C_{n,t}^\dagger)^0$ for any t. By application of a continuous family of small diffeomorphisms we can assure $|\mathbf{x}| \subset C_{n,t}^\dagger$ for every t. Because $d_M(x_n(t), |C_\infty|) = \varepsilon t$ there exists $t_0 \leq 1/2$ with $d_{\mathrm{cyc}}(C_{n,t_0}^\dagger, C_\infty) = \varepsilon/2$. Then $C'_n = C_{n,t_0}^\dagger$ is the desired pseudo-holomorphic curve.

Consider the set of times $t_0 \in [0,1]$ such that a continuous family $C_{n,t}$ with the requested properties exists on the intervall $[0, t_0]$. By the Gromov compactness theorem this set is closed, unless $d_{\mathrm{cyc}}(C_{n,t_0}, C_\infty) = \varepsilon$. In the latter case there exists $t < t_0$ with $d_{\mathrm{cyc}}(C_{n,t}, C_\infty) = \varepsilon/2$ and we are done as before. Openness at t_0 follows from Theorem 6.3 as long as C_{n,t_0} is reduced and irreducible.

If C_{n,t_0} is nonreduced or reducible write

$$C_{n,t_0} = \sum_{a=1}^{a_0} C_{n,t_0,a}.$$

Here the $C_{n,t_0,a}$ may not be distinct. Split $\mathbf{x}$ arbitrarily into a_0 tuples $|\mathbf{x}^a|$ with

$$|\mathbf{x}^a| \subset C_{n,t_0,a}.$$

Let k_a be the number of entries of $\mathbf{x}^a$. Since by (9) the number of entries $k = \sum_a k_a$ of $\mathbf{x}$ is less than $c_1(M)\cdot C_{n,t_0} - 1$ there exists an a with

$$k_a < c_1(M)\cdot C_{n,t_0,a} - 1.$$

By Lemma 6.7 there exists a J_{n,t_0}-holomorphic deformation of $C_{n,t_0,a}$ to a nodal curve $\hat{C}_{n,t_0}$ containing $|\mathbf{x}^a|$ in the image. In the further deformation process we want only to deform $\hat{C}_{n,t_0}$ and keep the other components. To this end, change the path $J_{n,t}$ for $t > t_0$ so that it stays constant on a neighbourhood of $\bigcup_{a'\neq a} C_{n,t_0,a'}$. This is possible by Remark 6.4(2). We also choose a new path $x'_n(t)$ so that $x'_n(t) \notin \bigcup_{a'\neq a} C_{n,t_0,a'}$. Now apply the previous reasoning with C_n replaced by $\hat{C}_{n,t_0}$ and $\tilde{\mathbf{x}}(t)$ by the concatenation of $\mathbf{x}^a$ and $x'_n(t)$.

Because the degree $c_1(M)\cdot C_{n,t_\infty,a}$ decreases by an integral amount each time we split the curve, after finitely many changes of $\{J_{n,t}\}$ the deformation will be successful for all t. The result is the desired continuous family of pseudo-holomorphic cycles $C_{n,t}$. □

7) *The reduced case*: *Pseudo-holomorphic deformation of C_n over a path $\{J_{n,t}\}_{t\in[0,1]}$ connecting J_n with J.* If C_∞ is reduced there is no room for imposing one more point constraint without spoiling unobstructedness of the deformation. Instead choose general paths $\{J_{n,t}\}_{t\in[0,1]}$ of tamed almost complex structures connecting J_n with J, such that

$$J_{n,t} \stackrel{n\to\infty}{\longrightarrow} J$$

uniformly in the $\mathcal{C}^0$-topology everywhere and in the $\mathcal{C}^{0,\alpha}$-topology locally on $M \setminus A$. By Remark 6.4(2) we can also arrange $J_{n,t}$ to be integrable on the neighbourhood U of $|\bar{C}_\infty|$ for every $t > 1/2$.

For $n \gg 0$ and $t \in [0,1)$ we seek to find a $J_{n,t}$-holomorphic nodal curve $C_{n,t}$ with

(i) Restoration applies to $C_{n,t}$.

(ii) The restoration of $C_{n,t}$ is smooth and isotopic to C_n^0 through a family of pseudo-holomorphic curves within ε-distance to C_∞^0 with respect to d_{cyc}, for a family of almost complex structure with $\mathcal{C}^{0,\alpha}$-distance to J_n^0 getting arbitrarily small with increasing n.

(iii) $|\mathbf{x}| \subset C'_n$.

LEMMA 8.8. *For every $\varepsilon > 0$ one of the following two cases occurs.*

(1) *There exists an $n > 0$ so that $C_{n,t}$ with properties* (i)–(iii) *exists for all $t < 1$.*

(2) *For every $n \gg 0$ there exists C_{n,t_n} with properties* (i)–(iii) *and so that*

$$d_{\text{cyc}}(C_{n,t_n}, C_\infty) = \varepsilon/2.$$

Proof. For $t = 0$ we may take $C_{n,t} = C_n$. Let $\tau_n \in [0,1]$ be maximal with the property that for each $t \in [0, \tau_n)$ a curve $C_{n,t}$ obeying (i)–(iii) exists. Assume that $\tau_n < 1$. Let $\{C_{n,t_i}\}_i$ be a sequence of pseudo-holomorphic curves obeying (i)–(iii) and with $t_i \nearrow \tau_n$ for $i \to \infty$. By the Gromov compactness theorem, after going over to a subsequence, we may assume cycle-theoretic convergence

$$C_{n,t_i} \stackrel{i\to\infty}{\longrightarrow} C_{\tau_n}.$$

Since $\{J_{n,t}\}$ is generic at $\tau_n < 1$ and $|\mathbf{x}| \subset C_{\tau_n}$, Proposition 8.6 implies that C_{τ_n} is reduced and $\delta(C_{\tau_n}) < \delta(C_\infty)$ unless C_{τ_n} is nodal. If δ drops, we argue as in the nonreduced case: Restoration and application of the induction hypothesis show the existence of $C_{n,t}$ for $t > \tau_n$ with the requested properties, provided $d_{\text{cyc}}(C_{\tau_n}, C_\infty) < \varepsilon$. Otherwise there exists C_{n,t_n} as in (2) (with slightly smaller ε). In the nodal case the existence of $C_{n,t}$ for $t > \tau_n$ follows from the deformation theory of nodal curves Theorem 6.6. □

Note that this line of reasoning fails in the nonreduced case, because in the proof of closedness $|\mathbf{x}|$ may end up unevenly distributed on C_{n,τ_n} if $|C_{n,\tau_n}|$ is reducible near a multiple component of C_∞. A smoothing of C_{n,τ_n} preserving the incidence conditions may then not exist.

8) *Taking the limit.* Assume first that either C_∞ is nonreduced or that Lemma 8.8(2) applies. Let C'_n be the sequence of deformations of C_n constructed in this lemma or in Lemma 8.7 in Step 6 respectively. By the Gromov compactness theorem we may assume convergence

$$C'_n \stackrel{n\to\infty}{\longrightarrow} C'_\infty.$$

By construction $d_{\text{cyc}}(C'_\infty, C_\infty) \geq \varepsilon/2$, hence $C'_\infty \neq C_\infty$. Also because $|\mathbf{x}| \subset C'_\infty$, Proposition 8.6 implies

$$(m(C'_\infty), \delta(C'_\infty)) < (m(C_\infty), \delta(C_\infty)).$$

By construction restoration applies to C'_∞. The result is pseudo-holomorphic for an almost complex structure J'^0 fulfilling the requirements of Proposition 4.7 for the neighbourhoods of $|C_\infty|$ and H chosen in Step 1 once and for all. Hence a J'^0-holomorphic smoothing of this curve exists by this proposition or by Remark 8.2. The induction hypothesis shows that it is symplectically isotopic to the restoration of C'_n, hence to C^0_n. Letting $\varepsilon \to 0$ gives the desired J'^0-holomorphic smoothing of C^0_∞.

In the remaining case Lemma 8.8(1) there is a family of restored curves $C^0_{n,t}$ that is pseudo-holomorphic for a family of almost complex structures that are integrable in the fixed neighbourhood of $|C_\infty| \cup H$ and fiberwise agree with J^0 for $t > 1/2$. In this case the isotopy statement follows for n sufficiently large and t close to 1 from the unobstructedness result, Proposition 4.7.

In any case we thus have produced the desired isotopy between C^0_n and a pseudo-holomorphic smoothing of C^0_∞ under the additional hypothesis $(**)$. This was left to be shown in Step 1. The proof of Lemma 8.1 is finished. $\square$

9. Proofs of Theorems A, B and C

This section provides the proofs of the three main theorems stated in the introduction. We shall use one more lemma.

LEMMA 9.1. *Let (M, J) be an almost complex manifold diffeomorphic to $\mathbb{CP}^2$ or to an S^2-bundle over S^2. Assume that $C \subset M$ is a nontrivial irreducible pseudo-holomorphic curve with $c_1(M) \cdot C > 0$. Then for all $m > 1$*

$$m(c_1(M) \cdot C - 1) > c_1(M) \cdot C + g(C) - 1 \tag{11}$$

in either of the following cases.

(i) *$M = \mathbb{CP}^2$ and $C \sim dH$ with $d \leq 8$, $H \subset \mathbb{CP}^2$ a hyperplane.*

(ii) *$M = \mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{F}_1$ and $C \sim dH + kF$ with $d \leq 3$, $k \geq 0$, where F and H are a fiber and a section of the symplectic S^2-bundle $M \to \mathbb{CP}^1$ respectively with $H \cdot H \in \{0, 1\}$.*

(iii) *$M = \mathbb{F}_1$ and $C \sim dH + kF$ with $d + k \leq 8$, notation as in* (ii).

Proof. Since $c_1(M) \cdot C > 0$ it suffices to establish (11) for $m = 2$. For (i) the genus formula (4) yields

$$g - 1 \leq \frac{d^2 - 3d}{2}.$$

Inequality (11) thus follows from

$$2(3d - 1) > 3d + \frac{d^2 - 3d}{2} \quad \text{or} \quad d^2 - 9d + 4 < 0.$$

This is true for $1 \leq d \leq 8$.

For $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, we note that $H \cdot H = 0$ and the genus formula reads

$$g - 1 \leq \frac{(dH + kF)^2 - (dH + kF)(2H + 2F)}{2} = dk - d - k.$$

So (11) will follow from the fact that

$$2(2d + 2k - 1) > 2d + 2k + (dk - d - k) = dk + d + k,$$

which is equivalent to

$$(k - 3)(3 - d) > -7.$$

This is true for any $k > 0$ as long as $d \leq 3$ and for $k = 0$, $1 \leq d \leq 3$.

Finally the case $M = \mathbb{F}_1$. Then $H \cdot H = 1$, $c_1(M) = 2H + F$,

$$g - 1 \leq \frac{(dH + kF)^2 - (dH + kF)(2H + F)}{2} = \frac{d^2 + 2dk - 3d - 2k}{2},$$

and (11) follows from

$$2(3d + 2k - 1) > 3d + 2k + \frac{d^2 + 2dk - 3d - 2k}{2} = \frac{d^2 + 2dk + 3d + 2k}{2},$$

or

$$k(2d - 6) + d^2 - 9d + 4 < 0. \tag{12}$$

For $k \geq 0$ and $1 \leq d \leq 3$ the first term is nonpositive and

$$d^2 - 9d + 4 < 0,$$

so the result follows. For $d = 0$, $k \geq 1$ we obtain

$$-6k + 4 \leq -2 < 0.$$

If $d \geq 4$ but $k \leq 8 - d$ inequality (12) implies as a sufficient condition

$$(8 - d)(2d - 6) + d^2 - 9d + 4 = -d^2 + 13d - 44 < 0.$$

This inequality holds true for all d. The proof is finished. □

Remark 9.2. We gave (i) in the lemma only to illustrate the origin of the degree restriction $d \leq 17$ for $\mathbb{CP}^2$. For technical reasons we have to do the computation on the $\mathbb{CP}^1$-bundle $\mathbb{F}_1$, with some additional properties covered by case (iii) in the lemma.

Proof of Theorem B. Let ω be the symplectic form on M. Denote by $\Sigma \subset M$ the symplectic submanifold. By Proposition 1.7 there exists a map $p : M \to \mathbb{CP}^1$ and an ω-tamed almost complex structure J on M making p a pseudo-holomorphic map and Σ a J-holomorphic curve. Let H, S, F be two disjoint sections and a fiber of p respectively, with $H \cdot H \in \{0, 1\}$. According to Remark 0.1 we may assume $c_1(M) \cdot \Sigma > 0$. Then deformations of Σ as pseudo-holomorphic curves are unobstructed by the smooth case of Theorem 6.6.

Possibly after deforming Σ slightly we may therefore assume $J \in \mathscr{J}_{\rm reg}$ for the Baire subset $\mathscr{J}_{\rm reg} \subset \mathscr{J}$ introduced in Section 6. Since ω tames also the integrable complex structure I on M, Lemma 6.2, Theorem 6.3 and Proposition 6.5 imply the existence of a path $\{J_t\}_{t\in[0,1]}$ of ω-tamed almost complex structures having the following properties.

(i) $J_0 = J$, $J_1 = I$ and $p : (M, \omega, J_t) \to \mathbb{CP}^1$ is a symplectic pseudo-holomorphic S^2-bundle for every t.

(ii) (M, J_t) is monotone for every t.

(iii) The path $\{J_t\}_{t\in[0,1]}$ is generic in the sense of Theorem 6.3(2).

(iv) Any J_t-holomorphic curve $C \subset M$, $C \not\sim S$ is homologous to $dH + kF$ with $d \geq 0$, $k \geq 0$.

Consider the set Ω of $\tau \in [0,1]$ so that for every $t \leq \tau$ a smooth J_t-holomorphic curve $C_t \subset M$ exists that is isotopic to Σ through an isotopy of ω-symplectic submanifolds. By definition, Ω is an interval. It is open as a subset of $[0,1]$ by the smooth case of Theorem 6.6. It remains to show that it is closed. Let $t_n \nearrow \tau$, $t_n \in \Omega$. For each n choose a J_{t_n}-holomorphic curve C_n symplectically isotopic to Σ. By the Gromov compactness theorem we may assume that the C_n converge to a J_τ-holomorphic cycle $C_\infty = \sum_a m_a C_{\infty,a}$.

Now $J_{t_n} \to J_\tau$, $C_n \to C_\infty$ fulfill the hypotheses of Lemma 8.1. Condition $(*)$ follows from Lemma 9.1(ii). The conclusion of Lemma 8.1 in conjunction with Remark 8.2 now shows that $\tau \in \Omega$. Hence $\Omega = [0,1]$ (end of proof of Theorem B). $\square$

Proof of Theorem C. The proof is essentially the same as that of Theorem B. As we are not in a fibered situation we proceed as follows. Let $\sigma : \mathbb{F}_1 \to \mathbb{CP}^2$ be the blowing up in a point $P \in \mathbb{CP}^2 \setminus \Sigma$. Choose a Kähler form $\tilde\omega$ on $\mathbb{F}_1$ that agrees with $\sigma^*(\omega)$ away from $\sigma^{-1}(B_\varepsilon(P))$ for $\varepsilon < d_M(P,\Sigma)/2$. Now run the program in the proof of Theorem B to the $\tilde\omega$-symplectic surface $\sigma^{-1}(\Sigma)$. By Remark 6.4 we are free to take the J_t and all intermediately occurring almost complex structures to agree with the standard integrable complex structure on $\sigma^{-1}(B_\varepsilon(P))$. Then J_t descends to an ω-tamed almost complex structure $\bar J_t$ on $\mathbb{CP}^2$. If C_t is a smooth J_t-holomorphic curve homologous to $\sigma^{-1}(\Sigma)$ then C_t is disjoint from $S = \sigma^{-1}(P)$ for homological reasons. Thus $\sigma(C_t)$ is a smooth $\bar J_t$-holomorphic curve, hence symplectic with respect to ω.

It remains to verify assumption $(*)$ in Lemma 8.1. Let $C' = \sum_{a\geq 0} m'_a C'_a$ be a J_t-holomorphic cycle homologous to dH, where H is a section with $H \cdot H = 1$. By Lemma 6.2(1) there are no J_t-holomorphic curves homologous to $aH + bF$ with $b < 0$ except $S \sim H - F$. Therefore, if $C'_a \sim d_a H + k_a F$

with $k_a \geq 0$ for $a > 0$, and $m'_0 C'_0 = mS$ then

$$d = m + \sum_{a>0} m'_a d_a \quad \text{and} \quad m = \sum_{a>0} m'_a k_a.$$

Hence $d = \sum_{a>0} m'_a(d_a + k_a)$. Now if $d \leq 17$ and C' has a multiple component then $d_a + k_a \leq 8$ for all a. Assumption $(*)$ therefore follows from Lemma 9.1(iii) (end of proof of Theorem C). $\square$

Proof of Theorem A. The main theorem of [SiTi1] implies that $M \to S^2$ factors over a degree 2 cover of an S^2-bundle $p : P \to S^2$, branched along a symplectic surface $B \subset P$ with $p|_B$ a simply branched cover of S^2. The assumption on the monodromy means that B is connected. By Theorem B there exists a symplectic isotopy B_t connecting $B = B_0$ to a holomorphic curve B_1 with respect to the integrable complex structure. Apply Proposition 1.7 with $\Sigma_t = B_t$. The result is a family J_t of almost complex structures and a family of S^2-bundles $p_t : P \to S^2$ such that

(1) B_t is J_t-holomorphic.

(2) $p_t : (P, J_t) \to \mathbb{CP}^1$ is pseudo-holomorphic.

Perturbing B_t slightly using the smooth case of Theorem 6.6 we can achieve that $B_t \to \mathbb{CP}^1$ has only simple branch points for every t. Let M_t be the two-fold cover of P branched along B_t. Then $M_t \to S^2$ is a genus-2 symplectic Lefschetz fibration for every t. This shows that $M \to S^2$ is even isotopic to a holomorphic Lefschetz fibration (end of proof of Theorem A). $\square$

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