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Statistical properties of unimodal maps: the quadratic family

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Abstract

We prove that almost every nonregular real quadratic map is Collet-Eckmann and has polynomial recurrence of the critical orbit (proving a conjecture by Sinai). It follows that typical quadratic maps have excellent ergodic properties, as exponential decay of correlations (Keller and Nowicki, Young) and stochastic stability in the strong sense (Baladi and Viana). This is an important step in achieving the same results for more general families of unimodal maps.

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Introduction

Here we consider the quadratic family, $f_a = a - x^2$, where $-1/4 \leq a \leq 2$ is the parameter, and we analyze its dynamics in the invariant interval.

The quadratic family has been one of the most studied dynamical systems in the last decades. It is one of the most basic examples and exhibits very

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rich behavior. It was also studied through many different techniques. Here we are interested in describing the dynamics of a typical quadratic map from the statistical point of view.

0.1. *The probabilistic point of view in dynamics.* In the last decade Palis [Pa] described a general program for (dissipative) dynamical systems in any dimension. In short, he shows that 'typical' dynamical systems can be modeled stochastically in a robust way. More precisely, one should show that such typical systems can be described by finitely many attractors, each of them supporting an (ergodic) physical measure: time averages of Lebesgue-almost-every orbit should converge to spatial averages according to one of the physical measures. The description should be robust under (sufficiently) random perturbations of the system; one asks for stochastic stability.

Moreover, a typical dynamical system was to be understood, in the Kolmogorov sense, as a set of full measure in generic parametrized families.

Besides the questions posed by this conjecture, much more can be asked about the statistical description of the long term behavior of a typical system. For instance, the definition of physical measure is related to the validity of the Law of Large Numbers. Are other theorems still valid, like the Central Limit or Large Deviation theorems? Those questions are usually related to the rates of mixing of the physical measure.

0.2. *The richness of the quadratic family.* While we seem still very far away from any description of dynamics of typical dynamical systems (even in one-dimension), the quadratic family has been a remarkable exception. Let us describe briefly some results which show the richness of the quadratic family from the probabilistic point of view.

The initial step in this direction was the work of Jakobson [J], where it was shown that for a positive measure set of parameters the behavior is stochastic; more precisely, there is an absolutely continuous invariant measure (the physical measure) with positive Lyapunov exponent: for Lebesgue almost every x, $|Df^n(x)|$ grows exponentially fast. On the other hand, it was later shown by Lyubich [L2] and Graczyk-Swiatek [GS1] that regular parameters (with a periodic hyperbolic attractor) are (open and) dense. While stochastic parameters are predominantly expanding (in particular have sensitive dependence to initial conditions), regular parameters are deterministic (given by the periodic attractor). So at least two kinds of very distinct observable behavior are present in the quadratic family, and they alternate in a complicated way.

It was later shown that stochastic behavior could be concluded from enough expansion along the orbit of the critical value: the Collet-Eckmann condition, exponential growth of $|Df^n(f(0))|$, was enough to conclude a positive Lyapunov exponent of the system. A different approach to Jakobson's Theorem in [BC1] and [BC2] focused specifically on this property: the set of

Collet-Eckmann maps has positive measure. After these initial works, many others studied such parameters (sometimes with extra assumptions), obtaining refined information of the dynamics of CE maps, particularly information about exponential decay of correlations[1] (Keller and Nowicki in [KN] and Young in [Y]), and stochastic stability (Baladi and Viana in [BV]). The dynamical systems considered in those papers have generally been shown to have excellent statistical descriptions[2].

Many of those results also generalized to more general families and sometimes to higher dimensions, as in the case of Hénon maps [BC2].

The main motivation behind this strong effort to understand the class of CE maps was certainly the fact that such a class was known to have positive measure. It was known however that very different (sometimes wild) behavior coexisted. For instance, it was shown the existence of quadratic maps without a physical measure or quadratic maps with a physical measure concentrated on a *repelling* hyperbolic fixed point ([Jo], [HK]). It remained to see if wild behavior was observable.

In a big project in the last decade, Lyubich [L3] together with Martens and Nowicki [MN] showed that almost all parameters have physical measures: more precisely, besides regular and stochastic behavior, only one more behavior could (possibly) happen with positive measure, namely infinitely renormalizable maps (which always have a uniquely ergodic physical measure). Later Lyubich in [L5] showed that infinitely renormalizable parameters have measure zero, thus establishing the celebrated *regular or stochastic* dichotomy. This further advancement in the comprehension of the nature of the statistical behavior of typical quadratic maps is remarkably linked to the progress obtained by Lyubich on the answer of the Feigenbaum conjectures [L4].

0.3. *Statements of the results.* In this work we describe the asymptotic behavior of the critical orbit. Our first result is an estimate of hyperbolicity:

THEOREM A. *Almost every nonregular real quadratic map satisfies the Collet-Eckmann condition*:

$$\liminf_{n\to\infty} \frac{\ln(|Df^n(f(0))|)}{n} > 0.$$

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[1] CE quadratic maps are not always mixing and finite periodicity can appear in a robust way. This phenomena is related to the map being renormalizable, and this is the only obstruction: the system is exponentially mixing after renormalization.

[2] It is now known that weaker expansion than Collet-Eckmann is enough to obtain stochastic behavior for quadratic maps, on the other hand, exponential decay of correlations is actually equivalent to the CE condition [NS], and all current results on stochastic stability use the Collet-Eckmann condition.

The second is an estimate on the recurrence of the critical point. For regular maps, the critical point is nonrecurrent (it actually converges to the periodic attractor). Among nonregular maps, however, the recurrence occurs at a precise rate which we estimate:

Theorem B. *Almost every nonregular real quadratic map has polynomial recurrence of the critical orbit with exponent* 1:

$$\limsup_{n\to\infty} \frac{-\ln(|f^n(0)|)}{\ln(n)} = 1.$$

In other words, the set of n such that $|f^n(0)| < n^{-\gamma}$ is finite if $\gamma > 1$ and infinite if $\gamma < 1$.

As far as we know, this is the first proof of polynomial estimates for the recurrence of the critical orbit valid for a positive measure set of nonhyperbolic parameters (although subexponential estimates were known before). This also answers a long standing conjecture of Sinai.

Theorems A and B show that typical nonregular quadratic maps have enough good properties to conclude the results on exponential decay of correlations (which can be used to prove Central Limit and Large Deviation theorems) and stochastic stability in the sense of L^1 convergence of the densities (of stationary measures of perturbed systems). Many other properties also follow, like existence of a spectral gap in [KN] and the recent results on almost sure (stretched exponential) rates of convergence to equilibrium in [BBM]. In particular, this answers positively Palis's conjecture for the quadratic family.

0.4. *Unimodal maps.* Another reason to deal with the quadratic family is that it seems to open the doors to the understanding of unimodal maps. Its universal behavior was first realized in the topological sense, with Milnor-Thurston theory. The Feigenbaum-Coullet-Tresser observations indicated a geometric universality [L4].

A first result in the understanding of measure-theoretical universality was the work of Avila, Lyubich and de Melo [ALM], where it was shown how to relate metrically the parameter spaces of nontrivial analytic families of unimodal maps to the parameter space of the quadratic family. This was proposed as a method to relate observable dynamics in the quadratic family to observable dynamics of general analytic families of unimodal maps. In that work the method is used successfully to extend the regular or stochastic dichotomy to this broader context.

We are also able to adapt those methods to our setting. The techniques developed here and the methods of [ALM] are the main tools used in [AM1] to obtain the main results of this paper (except the exact value of the polynomial recurrence) for nontrivial real analytic families of unimodal maps (with negative Schwarzian derivative and quadratic critical point). This is a rather general set of families, as trivial families form a set of infinite codimension. For a different approach (still based on [ALM]) which does not use negative Schwarzian derivative and obtains the exponent 1 for the polynomial recurrence, see [A], [AM3].

In [AM1] we also prove a version of Palis conjecture in the smooth setting. There is a residual set of k-parameter C^3 (for the equivalent C^2 result, see [A]) families of unimodal maps with negative Schwarzian derivative such that almost every parameter is either regular or Collet-Eckmann with subexponential bounds for the recurrence of the critical point.

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1. General definitions

1.1. *Maps of the interval.* Let $f : I \to I$ be a C^1 map defined on some interval $I \subset \mathbb{R}$. The *orbit* of a point $p \in I$ is the sequence $\{f^k(p)\}_{k=0}^{\infty}$. We say that p is *recurrent* if there exists a subsequence $n_k \to \infty$ such that $\lim f^{n_k}(p) = p$.

We say that p is a *periodic point of period* n of f if $f^n(p) = p$, and $n \geq 1$ is minimal with this property. In this case we say that p is *hyperbolic* if $|Df^n(p)|$ is not 0 or 1. Hyperbolic periodic orbits are *attracting* or *repelling* according to $|Df^n(p)| < 1$ or $|Df^n(p)| > 1$.

We will often consider the restriction of iterates f^n to intervals $T \subset I$, such that $f^n|_T$ is a diffeomorphism. In this case we will be interested on the *distortion* of $f^n|_T$,

$$\operatorname{dist}(f^n|_T) = \frac{\sup_T |Df^n|}{\inf_T |Df^n|}.$$

This is always a number bigger than or equal to 1; we will say that it is small if it is close to 1.

1.2. *Trees.* We let Ω denote the set of finite sequences of nonzero integers (including the empty sequence). Let Ω_0 denote Ω without the empty sequence. For $\underline{d} \in \Omega$, $\underline{d} = (j_1, \ldots, j_m)$, we let $|\underline{d}| = m$ denote its length.

We denote $\sigma^+ : \Omega_0 \to \Omega$ by $\sigma^+(j_1, \ldots, j_m) = (j_1, \ldots, j_{m-1})$ and $\sigma^- : \Omega_0 \to \Omega$ by $\sigma^-(j_1, \ldots, j_m) = (j_2, \ldots, j_m)$.

For the purposes of this paper, one should view Ω as a (directed) tree with root $\underline{d} = \emptyset$ and edges connecting $\sigma^+(\underline{d})$ to $\underline{d}$ for each $\underline{d} \in \Omega_0$. We will use Ω to label objects which are organized in a similar tree structure (for instance, certain families of intervals ordered by inclusion).

1.3. *Growth of functions.* Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. We say that f grows *at least exponentially* if there exists $\alpha > 0$ such that $f(n) > e^{\alpha n}$ for all n sufficiently big. We say that f grows *at least polynomially* if there exists $\alpha > 0$ such that $f(n) > n^\alpha$ for all n sufficiently big.

The standard *torrential* function T is defined recursively by $T(1) = 1$, $T(n+1) = 2^{T(n)}$. We say that f grows *at least torrentially* if there exists $k > 0$ such that $f(n) > T(n-k)$ for every n sufficiently big. We will say that f grows *torrentially* if there exists $k > 0$ such that $T(n-k) < f(n) < T(n+k)$ for every n sufficiently big.

Torrential growth can be detected from recurrent estimates easily. A sufficient condition for an unbounded function f to grow at least torrentially is an estimate,

$$f(n+1) > e^{f(n)^\alpha}$$

for some $\alpha > 0$. Torrential growth is implied by an estimate,

$$e^{f(n)^\alpha} < f(n+1) < e^{f(n)^\beta}$$

with $0 < \alpha < \beta$.

We will also say that f decreases at least exponentially (respectively torrentially) if $1/f$ grows at least exponentially (respectively torrentially).

1.4. *Quasisymmetric maps.* Let $k \geq 1$ be given. We say that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is *quasisymmetric* with constant k if for all $h > 0$

$$\frac{1}{k} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq k.$$

The space of quasisymmetric maps is a group under composition, and the set of quasisymmetric maps with constant k preserving a given interval is compact in the uniform topology of compact subsets of $\mathbb{R}$. It also follows that quasisymmetric maps are Hölder.

To describe further the properties of quasisymmetric maps, we need the concept of quasiconformal maps and dilatation so we just mention a result of Ahlfors-Beurling which connects both concepts: any quasisymmetric map extends to a quasiconformal real-symmetric map of $\mathbb{C}$ and, conversely, the restriction of a quasiconformal real-symmetric map of $\mathbb{C}$ to $\mathbb{R}$ is quasisymmetric. Furthermore, it is possible to work out upper bounds on the dilatation (of an optimal extension) depending only on k and conversely.

The constant k is awkward to work with: the inverse of a quasisymmetric map with constant k may have a larger constant. We will therefore work with a less standard constant: we will say that h is γ-quasisymmetric (γ-qs) if h admits a quasiconformal symmetric extension to $\mathbb{C}$ with dilatation bounded by γ. This definition behaves much better: if h_1 is γ_1-qs and h_2 is γ_2-qs then $h_2 \circ h_1$ is $\gamma_2\gamma_1$-qs.

If $X \subset \mathbb{R}$ and $h : X \to \mathbb{R}$ has a γ-quasisymmetric extension to $\mathbb{R}$ we will also say that h is γ-qs.

Let $QS(\gamma)$ be the set of γ-qs maps of $\mathbb{R}$.

2. Real quadratic maps

If $a \in \mathbb{C}$ we let $f_a : \mathbb{C} \to \mathbb{C}$ denote the (complex) quadratic map $a - z^2$. For real parameters in the range $-1/4 \leq a \leq 2$, there exists an interval $I_a = [\beta, -\beta]$ with

$$\beta = \frac{-1 - \sqrt{1 + 4a}}{2}$$

such that $f_a(I_a) \subset I_a$ and $f_a(\partial I_a) \subset \partial I_a$. For such values of the parameter a, the map $f = f_a|_{I_a}$ is unimodal; that is, it is a self map of I_a with a unique turning point. To simplify the notation, we will usually drop the dependence on the parameter and let $I = I_a$.

2.1. *The combinatorics of unimodal maps.* In this subsection we fix a real quadratic map f and define some objects related to it.

2.1.1. *Return maps.* Given an interval $T \subset I$ we define the *first return map* $R_T : X \to T$ where $X \subset T$ is the set of points x such that there exists $n > 0$ with $f^n(x) \in T$, and $R_T(x) = f^n(x)$ for the minimal n with this property.

2.1.2. *Nice intervals.* An interval T is *nice* if it is symmetric around 0 and the iterates of ∂T never intersect $\operatorname{int} T$. Given a nice interval T we notice that the domain of the first return map R_T decomposes in a union of intervals T^j, indexed by integer numbers (if there are only finitely many intervals, some indexes will correspond to the empty set). If 0 belongs to the domain of R_T, we say that T is *proper*. In this case we reserve the index 0 to denote the component of the critical point: $0 \in T^0$.

If T is nice, it follows that for all $j \in \mathbb{Z}$, $R_T(\partial T^j) \subset \partial T$. In particular, $R_T|_{T^j}$ is a diffeomorphism onto T unless $0 \in T^j$ (and in particular $j = 0$ and T is proper). If T is proper, $R_T|_{T^0}$ is symmetric (even) with a unique critical point 0. As a consequence, T^0 is also a nice interval.

If $R_T(0) \in T^0$, we say that R_T is *central*.

If T is a proper interval then both R_T and R_{T^0} are defined, and we say that R_{T^0} is the generalized renormalization of R_T.

2.1.3. *Landing maps.* Given a proper interval T we define the *landing map* $L_T : X \to T^0$ where $X \subset T$ is the set of points x such that there exists $n \geq 0$ with $f^n(x) \in T^0$, and $L_T(x) = f^n(x)$ for the minimal n with this property. We notice that $L_T|_{T^0} = \mathrm{id}$.

2.1.4. *Trees.* We will use Ω to label iterations of noncentral branches of R_T, as well as their domains. If $\underline{d} \in \Omega$, we define $T^{\underline{d}}$ inductively in the following way. We let $T^{\underline{d}} = T$ if $\underline{d}$ is empty and if $\underline{d} = (j_1, \dots, j_m)$ we let $T^{\underline{d}} = (R_T|_{T^{j_1}})^{-1}(T^{\sigma^-(\underline{d})})$.

We denote $R_T^{\underline{d}} = R_T^{|\underline{d}|}|_{T^{\underline{d}}}$ which is always a diffeomorphism onto T.

Notice that the family of intervals $T^{\underline{d}}$ is organized by inclusion in the same way as Ω is organized by (right side) truncation (the previously introduced tree structure).

If T is a proper interval, the first return map to T naturally relates to the first landing to T^0. Indeed, denoting $C^{\underline{d}} = (R_T^{\underline{d}})^{-1}(T^0)$, the domain of the first landing map L_T is easily seen to coincide with the union of the $C^{\underline{d}}$, and furthermore $L_T|_{C^{\underline{d}}} = R_T^{\underline{d}}$.

Notice that this allows us to relate R_T and R_{T^0} since $R_{T^0} = L_T \circ R_T$.

2.1.5. *Renormalization.* We say that f is *renormalizable* if there is an interval $0 \in T$ and $m > 1$ such that $f^m(T) \subset T$ and $f^j(\operatorname{int} T) \cap \operatorname{int} T = \emptyset$ for $1 \leq j < m$. The maximal such interval is called the *renormalization interval of period* m, with the property that $f^m(\partial T) \subset \partial T$.

The set of renormalization periods of f gives an increasing (possibly empty) sequence of numbers m_i, $i = 1, 2, \dots$, each related to a unique renormalization interval $T^{(i)}$ which forms a nested sequence of intervals. We include $m_0 = 1$, $T^{(0)} = I$ in the sequence to simplify the notation.

We say that f is *finitely renormalizable* if there is a smallest renormalization interval $T^{(k)}$. We say that $f \in \mathcal{F}$ if f is finitely renormalizable and 0 is recurrent but not periodic. We let $\mathcal{F}_k$ denote the set of maps f in $\mathcal{F}$ which are exactly k times renormalizable.

2.1.6. *Principal nest.* Let Δ_k denote the set of all maps f which have (at least) k renormalizations and which have an orientation reversing nonattracting periodic point of period m_k which we denote p_k (that is, p_k is the fixed point of $f^{m_k}|_{T^{(k)}}$ with $Df^{m_k}(p_k) \leq -1$). For $f \in \Delta_k$, we denote $T_0^{(k)} = [-p_k, p_k]$. We define by induction a (possibly finite) sequence $T_i^{(k)}$, such that $T_{i+1}^{(k)}$ is the component of the domain of $R_{T_i^{(k)}}$ containing 0. If this sequence is infinite, then either it converges to a point or to an interval.

If $\cap_i T_i^{(k)}$ is a point, then f has a recurrent critical point which is not periodic, and it is possible to show that f is not $k+1$ times renormalizable. Obviously in this case we have $f \in \mathcal{F}_k$, and all maps in $\mathcal{F}_k$ are obtained in this way: if $\cap_i T_i^{(k)}$ is an interval, it is possible to show that f is $k+1$ times renormalizable.

We can of course write $\mathcal{F}$ as a disjoint union $\cup_{i=0}^{\infty} \mathcal{F}_i$. For a map $f \in \mathcal{F}_k$ we refer to the sequence $\{T_i^{(k)}\}_{i=1}^{\infty}$ as the *principal nest*.

It is important to notice that the domain of the first return map to $T_i^{(k)}$ is always dense in $T_i^{(k)}$. Moreover, the next result shows that, outside a very special case, the return map has a hyperbolic structure.

LEMMA 2.1. *Assume $T_i^{(k)}$ does not have a nonhyperbolic periodic orbit in its boundary. For all $T_i^{(k)}$ there exists $C > 0$, $\lambda > 1$ such that if $x, f(x), \dots, f^{n-1}(x)$ do not belong to $T_i^{(k)}$ then $|Df^n(x)| > C\lambda^n$.*

This lemma is a simple consequence of a general theorem of Guckenheimer on hyperbolicity of maps of the interval without critical points and nonhyperbolic periodic orbits (Guckenheimer considers unimodal maps with negative Schwarzian derivative, and so this applies directly to the case of quadratic maps, the general case is also true by Mañé's Theorem, see [MvS]). Notice that the existence of a nonhyperbolic periodic orbit in the boundary of $T_i^{(k)}$ depends on a very special combinatorial setting; in particular, all $T_j^{(k)}$ must coincide (with $[-p_k, p_k]$), and the k-th renormalization of f is in fact renormalizable of period 2.

By Lemma 2.1, the maximal invariant of $f|_{I \setminus T_i^{(k)}}$ is an expanding set, which admits a Markov partition (since $\partial T_i^{(k)}$ is preperiodic, see also the proof of Lemma 6.1); it is easy to see that it is indeed a Cantor set[3] (except if $i = 0$ or in the special period 2 renormalization case just described). It follows that the geometry of this Cantor set is well behaved; for instance, its image by any quasisymmetric map has zero Lebesgue measure.

In particular, one sees that the domain of the first return map to $T_i^{(k)}$ has infinitely many components (except in the special case above or if $i = 0$) and that its complement has well behaved geometry.

2.1.7. *Lyubich's regular or stochastic dichotomy.* A map $f \in \mathcal{F}_k$ is called *simple* if the principal nest has only finitely many central returns; that is, there are only finitely many i such that $R|_{T_i^{(k)}}$ is central. Such maps have many good features; in particular, they are stochastic (this is a consequence of [MN] and [L1]).

In [L3], it was proved that almost every quadratic map is either regular or simple or infinitely renormalizable. It was then shown in [L5] that infinitely renormalizable maps have zero Lebesgue measure, which establishes the regular or stochastic dichotomy.

Due to Lyubich's results, we can completely forget about infinitely renormalizable maps; we just have to prove the claimed estimates for almost every simple map.

[3]Dynamically defined Cantor sets with such properties are usually called *regular Cantor sets*.

During our discussion, for notational reasons, we will fix a renormalization level κ; that is, we will only analyze maps in Δ_κ. This allows us to fix some convenient notation: given $g \in \Delta_\kappa$ we define $I_i[g] = T_i^{(\kappa)}[g]$, so that $\{I_i[g]\}$ is a sequence of intervals (possibly finite). We use the notation $R_i[g] = R_{I_i[g]}$, $L_i[g] = L_{I_i[g]}$ and so on (so that the domain of $R_i[g]$ is $\cup I_i^j[g]$ and the domain of $L_i[g]$ is $\cup\, C_i^{\underline{d}}[g]$). When doing phase analysis (working with fixed f) we usually drop the dependence on the map and write R_i for $R_i[f]$.

(Notice that, once we fix the renormalization level κ, for $g \in \Delta_\kappa$, the notation $I_i[g]$ stands for $T_i^{(\kappa)}[g]$, even if g is more than κ times renormalizable.)

2.1.8. *Strategy.* To motivate our next steps, let us describe the general strategy behind the proofs of Theorems A and B.

(1) We consider a certain set of nonregular parameters of full measure and describe (in a probabilistic way) the dynamics of the principal nest. This is our phase analysis.

(2) From time to time, we transfer the information from the phase space to the parameter, following the description of the parapuzzle nest which we will make in the next subsection. The rules for this correspondence are referred to as *phase-parameter relation* (which is based on the work of Lyubich on complex dynamics of the quadratic family).

(3) This correspondence will allow us to exclude parameters whose critical orbit behaves badly (from the probabilistic point of view) at infinitely many levels of the principal nest. The phase analysis coupled with the phase-parameter relation will assure us that the remaining parameters still have full measure.

(4) We restart the phase analysis for the remaining parameters with extra information.

After many iterations of this procedure we will have enough information to tackle the problems of hyperbolicity and recurrence.

We first describe the phase-parameter relation, and we will delay all statistical arguments until Section 3.

A larger outline of this strategy, including the motivation and organization of the statistical analysis, appeared in [AM2].

2.2. *Parameter partition.* Part of our work is to transfer information from the phase space of some map $f \in \mathcal{F}$ to a neighborhood of f in the parameter space. This is done in the following way. We consider the first landing map L_i: the complement of the domain of L_i is a hyperbolic Cantor set $K_i = I_i \setminus \cup C_i^{\underline{d}}$. This Cantor set persists in a small parameter neighborhood J_i of f, changing in a continuous way. Thus, loosely speaking, the domain of L_i induces a persistent partition of the interval I_i.

Along J_i, the first landing map is topologically the same (in a way that will be clear soon). However the critical value $R_i[g](0)$ moves relative to the partition (when g moves in J_i). This allows us to partition the parameter piece J_i in smaller pieces, each corresponding to a region where $R_i(0)$ belongs to some fixed component of the domain of the first landing map.

THEOREM 2.2 (topological phase-parameter relation). *Let $f \in \mathcal{F}_\kappa$. There is a sequence $\{J_i\}_{i \in \mathbb{N}}$ of nested parameter intervals* (*the* principal parapuzzle nest *of f*) *with the following properties.*

(1) *J_i is the maximal interval containing f such that for all $g \in J_i$ the interval $I_{i+1}[g] = T_{i+1}^{(\kappa)}[g]$ is defined and changes in a continuous way.* (*Since the first return map $R_i[g]$ has a central domain, the landing map $L_i[g] : \cup C_i^{\underline{d}}[g] \to I_i[g]$ is defined.*)

(2) *$L_i[g]$ is topologically the same along J_i; there exist homeomorphisms $H_i[g] : I_i \to I_i[g]$, such that $H_i[g](C_i^{\underline{d}}) = C_i^{\underline{d}}[g]$. The maps $H_i[g]$ may be chosen to change continuously.*

(3) *There exists a homeomorphism $\Xi_i : I_i \to J_i$ such that $\Xi_i(C_i^{\underline{d}})$ is the set of g such that $R_i[g](0)$ belongs to $C_i^{\underline{d}}[g]$.*

The homeomorphisms H_i and Ξ_i are not uniquely defined, since it is easy to see that we can modify them inside each $C_i^{\underline{d}}$ window keeping the above properties. However, H_i and Ξ_i are well defined maps if restricted to K_i.

This fairly standard phase-parameter result can be proved in many different ways. The most elementary proof is probably to use the monotonicity of the quadratic family to deduce the topological phase-parameter relation from Milnor-Thurston's kneading theory by purely combinatorial arguments. Another approach is to use Douady-Hubbard's description of the combinatorics of the Mandelbrot set (restricted to the real line) as does Lyubich in [L3] (see also [AM3] for a more general case).

With this result we can define, for any $f \in \mathcal{F}_\kappa$, intervals $J_i^j = \Xi_i(I_i^j)$ and $J_i^{\underline{d}} = \Xi_i(I_i^{\underline{d}})$. From the description given it immediately follows that two intervals $J_{i_1}[f]$ and $J_{i_2}[g]$ associated to maps f and g are either disjoint or nested, and the same happens for intervals J_i^j or $J_i^{\underline{d}}$. Notice that if $g \in \Xi_i(C_i^{\underline{d}}) \cap \mathcal{F}_\kappa$ then $\Xi_i(C_i^{\underline{d}}) = J_{i+1}[g]$.

We will concentrate on the analysis of the regularity of Ξ_i for the special class of simple maps f: one of the good properties of the class of simple maps is better control of the phase-parameter relation. Even for simple maps, however, the regularity of Ξ_i is not great; there is too much dynamical information contained in it. A solution to this problem is to forget some dynamical information.

2.2.1. *Gape interval.* If $i > 1$, we define the *gape interval* $\tilde{I}_{i+1}$ as follows.

We have that $R_i|_{I_{i+1}} = L_{i-1} \circ R_{i-1} = R_{i-1}^{\underline{d}} \circ R_{i-1}$ for some $\underline{d}$, so that $I_{i+1} = (R_{i-1}|_{I_i})^{-1}(C_{i-1}^{\underline{d}})$. We define the gape interval $\tilde{I}_{i+1} = (R_{i-1}|_{I_i})^{-1}(I_{i-1}^{\underline{d}})$.

Notice that $I_{i+1} \subset \tilde{I}_{i+1} \subset I_i$. Furthermore, for each I_i^j, the gape interval $\tilde{I}_{i+1}$ either contains or is disjoint from I_i^j.

2.2.2. *The phase-parameter relation.* As discussed before, the dynamical information contained in Ξ_i is entirely given by $\Xi_i|_{K_i}$; a map obtained by Ξ_i by modification inside a $C_i^{\underline{d}}$ window still has the same properties. Therefore it makes sense to ask about the regularity of $\Xi_i|_{K_i}$. As anticipated before we must erase some information to obtain good results.

Let $f \in \mathcal{F}_\kappa$ and let τ_i be such that $R_i(0) \in I_i^{\tau_i}$. We define two Cantor sets, $K_i^\tau = K_i \cap I_i^{\tau_i}$ which contains refined information restricted to the $I_i^{\tau_i}$ window and $\tilde{K}_i = I_i \setminus (\cup I_i^j \cup \tilde{I}_{i+1})$, which contains global information, at the cost of erasing information inside each I_i^j window and in $\tilde{I}_{i+1}$.

THEOREM 2.3 (phase-parameter relation). *Let f be a simple map. For all $\gamma > 1$ there exists i_0 such that for all $i > i_0$,*

PhPa1: $\Xi_i|_{K_i^\tau}$ *is γ-qs,*

PhPa2: $\Xi_i|_{\tilde{K}_i}$ *is γ-qs,*

PhPh1: $H_i[g]|_{K_i}$ *is γ-qs if $g \in J_i^{\tau_i}$,*

PhPh2: *the map $H_i[g]|_{\tilde{K}_i}$ is γ-qs if $g \in J_i$.*

The phase-parameter relation follows from the work of Lyubich [L3], where a general method based on the theory of holomorphic motions was introduced to deal with this kind of problem. A sketch of the derivation of the specific statement of the phase-parameter relation from the general method of Lyubich is given in the appendix. The reader can find full details (in a more general context than quadratic maps) in [AM3].

Remark 2.1. One of the main reasons why the present work is restricted to the quadratic family is related to the topological phase-parameter relation and the phase-parameter relation. The work of Lyubich uses specifics of the quadratic family, specially the fact that it is a full family of quadratic-like maps, and several arguments involved have indeed a global nature (using for instance the combinatorial theory of the Mandelbrot set). Thus we are only able to conclude the phase-parameter relation in this restricted setting.

However, the statistical analysis involved in the proofs of Theorem A and B in this work is valid in much more generality. Our arguments suffice (without any changes) for any one-parameter analytic family of unimodal maps f_λ with the following properties:

(1) For every λ, f_λ has a quadratic critical point and negative Schwarzian derivative,[4]

(2) For almost every nonregular parameter λ, f_λ has all periodic orbits repelling (so that Lemma 2.1 holds), is conjugate to a quadratic simple map, and the topological phase-parameter relation[5] and the phase-parameter relation[6] are valid at λ.

The assumption of a quadratic critical point is probably the hardest to remove at this point, so our analysis does not apply, say, for the families $a - x^{2n}$, $n > 1$. It is worthwhile to point out that most of the arguments developed in this paper go through for higher criticality. The key missing links are in the starting points of this paper: zero Lebesgue measure of infinitely renormalizable parameters and of finitely renormalizable parameters without exponential decay of geometry (in the sense of [L1]), and growth of moduli of parapuzzle annuli (in the sense of [L3]) for almost every parameter.

3. Measure and capacities

3.1. *Quasisymmetric maps.* If $X \subset \mathbb{R}$ is measurable, let us denote $|X|$ its Lebesgue measure. Let us make explicit the metric properties of γ-qs maps to be used.

For each γ, there exists a constant $k \geq 1$ such that for all $f \in QS(\gamma)$, for all $J \subset I$ intervals,

$$\frac{1}{k}\left(\frac{|J|}{|I|}\right)^k \leq \frac{|f(J)|}{|f(I)|} \leq \left(\frac{k|J|}{|I|}\right)^{1/k}.$$

Furthermore $\lim_{\gamma\to 1} k(\gamma) = 1$. So for each $\varepsilon > 0$ there exists $\gamma > 1$ such that $k(2\gamma - 1) < 1 + \varepsilon/5$. From now on, once a given γ close to 1 is chosen, ε will always denote a small number with this property.

3.2. *Capacities and trees.* The γ-capacity of a set X in an interval I is defined as follows:

$$p_\gamma(X|I) = \sup_{h\in QS(\gamma)} \frac{|h(X\cap I)|}{|h(I)|}.$$

[4]More generally it is enough to ask that the first return map to a sufficiently small nice interval have negative Schwarzian derivative.

[5]Actually one only needs the topological phase-parameter relation to be valid for all deep enough levels of the principal nest.

[6]In [AM1] it is shown how to work around this condition for most families satisfying condition (1). The results obtained are weaker though, and the statistical analysis is slightly harder.

This geometric quantity is well adapted to our context, since it is well behaved under tree decompositions of sets. In other words, if I^j are disjoint subintervals of I and $X \subset \cup I^j$ then

$$p_\gamma(X|I) \leq p_\gamma(\cup_j I_j|I) \sup_j p_\gamma(X|I^j).$$

3.3. *A measure-theoretic lemma.* Our procedure consists in obtaining successively smaller (but still full-measure) classes of maps for which we can give a progressively refined statistical description of the dynamics. This is done inductively as follows: we pick a class X of maps (which we have previously shown to have full measure among nonregular maps) and for each map in X we proceed to describe the dynamics (focusing on the statistical behavior of return and landing maps for deep levels of the principal nest); then we use this information to show that a subset Y of X (corresponding to parameters for which the statistical behavior of the *critical orbit* is not anomalous) still has full measure. An example of this parameter exclusion process is given by Lyubich in [L3] where he shows using a probabilistic argument that the class of simple maps has full measure in $\mathcal{F}$.

Let us now describe our usual argument (based on the argument of Lyubich which in turn is a variation of the Borel-Cantelli Lemma). Assume at some point we know how to prove that almost every simple map belongs to a certain set X. Let Q_n be a (bad) property that a map may have (usually some anomalous statistical parameter related to the n-th stage of the principle nest). Suppose we prove that if $f \in X$ then the probability that a map in $J_n(f)$ has the property Q_n is bounded by $q_n(f)$ which is shown to be summable for all $f \in X$. We then conclude that almost every map does not have property Q_n for n big enough.

Sometimes we also apply the same argument, proving instead that $q_n(f)$ is summable where $q_n(f)$ is the probability that a map in $J_n^{\tau_n}(f)$ has property Q_n, (recall that τ_n is such that $f \in J_n^{\tau_n}(f)$).

In other words, we apply the following general result.

Lemma 3.1. *Let $X \subset \mathbb{R}$ be a measurable set such that for each $x \in X$ a sequence $D_n(x)$ of nested intervals converging to x is defined such that for all $x_1, x_2 \in X$ and any n, $D_n(x_1)$ is either equal or disjoint to $D_n(x_2)$. Let Q_n be measurable subsets of $\mathbb{R}$ and $q_n(x) = |Q_n \cap D_n(x)|/|D_n(x)|$. Let Y be the set of all $x \in X$ which belong to at most finitely many Q_n. If $\sum q_n(x)$ is finite for almost any $x \in X$ then $|Y| = |X|$.*

Proof. Let $Y_n = \{x \in X | \sum_{k=n}^{\infty} q_k(x) < 1/2\}$. It is clear that $Y_n \subset Y_{n+1}$ and $| \cup Y_n| = |X|$.

Let $Z_n = \{x \in Y_n | |Y_n \cap D_m(x)|/|D_m(x)| > 1/2, m \geq n\}$. It is clear that $Z_n \subset Z_{n+1}$ and $| \cup Z_n| = |X|$.

For $m \geq n$, let $T_n^m = \cup_{x \in Z_n} D_m(x)$. Let $K_n^m = T_n^m \cap Q_m$. Of course,

$$|K_n^m| = \int_{T_n^m} q_m \leq 2 \int_{Y_n} q_m.$$

And, also,

$$\sum_{m \geq n} \int_{Y_n} q_m \leq \frac{1}{2}|Y_n|.$$

This shows that $\sum_{m \geq n} |K_n^m| \leq |Y_n|$, so that almost every point in Z_n belongs to at most finitely many K_n^m. We conclude then that almost every point in X belongs to at most finitely many Q_m. $\square$

The following obvious reformulation will often be convenient:

Lemma 3.2. *In the same context as above, assume that there exist sequences $Q_{n,m}$, $m \geq n$ of measurable sets and let Y_n be the set of x belonging to at most finitely many $Q_{n,m}$. Let $q_{n,m}(x) = |Q_{n,m} \cap D_m(x)|/|D_m(x)|$. Let $n_0(x) \in \mathbb{N} \cup \{\infty\}$ be such that $\sum_{m=n}^{\infty} q_{n,m}(x) < \infty$ for $n \geq n_0(x)$. Then for almost every $x \in X$, $x \in Y_n$ for $n \geq n_0(x)$.*

In practice, we will estimate the capacity of sets in the phase space: that is, given a map f we will obtain subsets $\tilde{Q}_n[f]$ in the phase space, corresponding to bad branches of return or landing maps. We will then show that for some $\gamma > 1$ we have $\sum p_\gamma(\tilde{Q}_n[f]|I_n[f]) < \infty$ or $\sum p_\gamma(\tilde{Q}_n[f]|I_n^{\tau_n}[f]) < \infty$. We will then use PhPa2 or PhPa1, and the measure-theoretical lemma above to conclude that with total probability among nonregular maps, for all n sufficiently big, $R_n(0)$ does not belong to a bad set.

From now on when we prove that almost every nonregular map has some property, we will just say that with total probability (without specifying) such a property holds.

(To be strictly formal, we have fixed the renormalization level κ (in particular to define the sequence J_n without ambiguity), so that applications of the measure theoretical argument will actually be used to conclude that for almost every parameter in $\mathcal{F}_\kappa$ a given property holds. Since almost every nonregular map belongs to some $\mathcal{F}_k$, this is equivalent to the statement regarding almost every nonregular parameter.)

4. Statistics of the principal nest

4.1. *Decay of geometry.* As before, let $\tau_n \in \mathbb{Z}$ be such that $R_n(0) \in I_n^{\tau_n}$. An important parameter in our construction will be the scaling factor

$$c_n = \frac{|I_{n+1}|}{|I_n|}.$$

This variable of course changes inside each $J_n^{\tau_n}$ window, however, not by much. From PhPh1, for instance, we get that with total probability

$$\lim_{n\to\infty} \sup_{g_1,g_2\in J_n^{\tau_n}} \frac{\ln(c_n[g_1])}{\ln(c_n[g_2])} = 1.$$

This variable is by far the most important in our analysis of the statistics of return maps. Often considering other variables (say, return times), we will show that the distribution of those variables is concentrated near some average value. Our estimates will usually give a range of values near the average, and c_n will play an important role. Due (among other issues) to the variability of c_n inside the parameter windows, the ranges we select will depend on c_n up to an exponent (say, between $1-\varepsilon$ and $1+\varepsilon$), where ε is a small, but fixed, number. From the estimate we just obtained, for big n the variability (margin of error) of c_n will fall comfortably in such range, and we need not elaborate more.

A general estimate on the rates of decay of c_n was obtained by Lyubich: he shows that (for a finitely renormalizable unimodal map with a recurrent critical point), c_{n_k} decays exponentially (on k), where n_k-1 is the subsequence of noncentral levels of f. For simple maps, the same is true with $n_k=k$, as there are only finitely many central returns. Thus we can state:

Theorem 4.1 (see [L1]). *If f is a simple map then there exists $C>0$, $\lambda<1$ such that $c_n < C\lambda^n$.*

Let us use the following notation for the combinatorics of a point $x \in I_n$. If $x\in I_n^j$ we let $j^{(n)}(x)=j$ and if $x \in C_n^{\underline{d}}$ we let $\underline{d}^{(n)}(x)=\underline{d}$.

Lemma 4.2. *With total probability, for all n sufficiently big,*

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k | I_n) < kc_n^{1-\varepsilon/2}, \tag{4.1}$$

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k | I_n) < e^{-kc_n^{1+\varepsilon/2}}. \tag{4.2}$$

Also,

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k | I_n^{\tau_n}) < kc_n^{1-\varepsilon/2}, \tag{4.3}$$

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k | I_n^{\tau_n}) < e^{-kc_n^{1+\varepsilon/2}}. \tag{4.4}$$

Proof. Let us compute the first two estimates.

Since I_n^0 is in the middle of I_n, we have as a simple consequence of the Real Schwarz Lemma (see [L1] and (4.8) in Lemma 4.5 below) that

$$\frac{c_n}{4} < \frac{|C_n^{\underline{d}}|}{|I_n^{\underline{d}}|} < 4c_n.$$

As a consequence

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| = m|I_n) < (4c_n)^{1-\varepsilon/3}$$

and we get the estimate (4.1) summing on $0 \le m \le k$.

For the same reason, we get that

$$\begin{aligned} p_{2\gamma-1}(|\underline{d}^{(n)}(x)| &\ge m+1|I_n) \\ &< \left(1 - \left(\frac{c_n}{4}\right)^{1+\varepsilon/3}\right) p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \ge m|I_n). \end{aligned}$$

This implies

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \ge m|I_n) \le \left(1 - \left(\frac{c_n}{4}\right)^{1+\varepsilon/3}\right)^m.$$

Estimate (4.2) follows from

$$\begin{aligned} \left(1 - \left(\frac{c_n}{4}\right)^{1+\varepsilon/3}\right)^k &< (1 - c_n^{1+\varepsilon/2})^k \\ &< ((1 - c_n^{1+\varepsilon/2})^{c_n^{-1-\varepsilon/2}})^{kc_n^{1+\varepsilon/2}} \\ &< e^{-kc_n^{1+\varepsilon/2}}. \end{aligned}$$

The two remaining estimates are analogous. $\square$

Let us now transfer this result (more precisely the second pair of estimates) to the parameter in each $J_n^{\tau_n}$ window using PhPa1. To do this notice that the measure of the complement of the set of parameters in $J_n^{\tau_n}$ such that $c_n^{-1+2\varepsilon} < s_n < c_n^{-1-2\varepsilon}$ can be estimated by $2c_n^{\varepsilon}$ for n big which is summable for all ε by Theorem 4.1. So we have:

Lemma 4.3. *With total probability,*

$$\lim_{n\to\infty} \frac{\ln(s_n)}{\ln(c_n^{-1})} = 1.$$

The parameter s_n influences the size of c_{n+1} in a determinant way.

Corollary 4.4. *With total probability,*

$$\liminf_{n\to\infty} \frac{\ln(\ln(c_{n+1}^{-1}))}{\ln(c_n^{-1})} \ge 1. \tag{4.5}$$

In particular, c_n *decreases at least torrentially fast.*

Proof. It is easy to see (by, for instance, the Real Schwarz Lemma; see [L1]; see also item (4.9) in Lemma 4.5 below) that there exists a constant $K > 0$ (independent of n) such that for each $\underline{d} \in \Omega$, both components of $I_n^{\sigma^+(\underline{d})} \setminus I_n^{\underline{d}}$

have size at least $(e^K - 1)|I_n^d|$. In particular, by induction, if $R_n(0) \in C_n^d$ we have that both gaps of $I_n \setminus C_n^d$ have size at least $(e^{Ks_n} - 1)|C_n^d|$. Taking the preimage by R_n, and using the Real Schwarz Lemma again, we see that $c_{n+1} < Ce^{Ks_n/2}$ for some constant $C > 0$ independent of n. We conclude that

$$\liminf \frac{\ln(c_{n+1}^{-1})}{s_n} \geq \frac{K}{2},$$

and since $c_n \to 0$ as $n \to \infty$ we have

$$\liminf \frac{\ln(\ln(c_{n+1}^{-1}))}{\ln(s_n)} \geq 1$$

which together with Lemma 4.3 implies (4.5). $\square$

Remark 4.1. In the proof of Corollary 4.4, the constant $K > 0$ is related to the real bounds. In our situation, since we have decay of geometry, we can actually take $K \to \infty$ as $n \to \infty$, so we actually have

$$\frac{\ln(c_{n+1}^{-1})}{s_n} \to \infty$$

torrentially fast.

4.2. *Fine partitions.* We use Cantor sets K_n and $\tilde{K}_n$ to partition the phase space. In many circumstances we are directly concerned with intervals of this partition. However, sometimes we just want to exclude an interval of given size (usually a neighborhood of 0). This size does not usually correspond to (the closure of) a union of gaps, so we instead should consider in applications an interval which is a union of gaps, with approximately the given size [7]. The degree of relative approximation will always be torrentially good (in n), so we usually won't elaborate on this. In this section we just give some results which will imply that the partition induced by the Cantor sets are fine enough to allow torrentially good approximations.

The following lemma summarizes the situation. The proof is based on estimates of distortion, the Real Schwarz Lemma and the Koebe Principle (see [L1]), and is very simple, so we just sketch the proof.

[7] We need to consider intervals which are unions of gaps due to our phrasing of the phase-parameter relation, which only gives information about such gaps. However, this is not absolutely necessary, and we could have proceeded in a different way: the proof of the phase-parameter relation actually shows that there is a *holonomy map* between phase and parameter *intervals* (and not only Cantor sets) corresponding to a *holomorphic motion* for which we can obtain good qs estimates. While this map is not canonical, the fact that it is a holonomy map for a holomorphic motion with good qs estimates would allow our proofs to work.

LEMMA 4.5. *The following estimates hold*:

$$\frac{|I_n^j|}{|I_n|} = O(\sqrt{c_{n-1}}), \tag{4.6}$$

$$\frac{|I_n^{\underline{d}}|}{|I_n^{\sigma^+(\underline{d})}|} = O(\sqrt{c_{n-1}}), \tag{4.7}$$

$$\frac{c_n}{4} < \frac{|C_n^{\underline{d}}|}{|I_n^{\underline{d}}|} < 4c_n, \tag{4.8}$$

$$\frac{|\tilde{I}_{n+1}|}{|I_n|} = O(e^{-s_{n-1}}). \tag{4.9}$$

Proof (Sketch). Since $R_n^{\underline{d}}$ has negative Schwarzian derivative, it immediately follows that the Koebe space[8] of $C_n^{\underline{d}}$ inside $I_n^{\underline{d}}$ has at least order c_n^{-1}.

It is easy to see that $R_{n-1}|_{I_n}$ can be written as $\phi \circ f$ where ϕ extends to a diffeomorphism onto I_{n-2} with negative Schwarzian derivative and thus with very small distortion. Since $R_{n-1}(I_n^j)$ is contained on some $C_{n-1}^{\underline{d}}$, we see that the Koebe space of I_n^j in I_n is at least of order $c_{n-1}^{-1/2}$ which implies (4.6).

Let us now consider an interval $I_n^{\underline{d}}$. Let I_n^j be such that $R_n^{\sigma^+(\underline{d})}(I_n^{\underline{d}}) = I_n^j$. We can pullback the Koebe space of I_n^j inside I_n by $R_n^{\sigma^+(\underline{d})}$, so that (4.6) implies (4.7). Moreover, this shows by induction that the Koebe space of $I_n^{\underline{d}}$ inside I_n is at least of order $c_{n-1}^{-|\underline{d}|/2}$. Since $R_{n-1}(\tilde{I}_{n+1}) \subset I_{n-1}^{\underline{d}}$ with $|\underline{d}| = s_{n-1}$, the Koebe space of $\tilde{I}_{n+1}$ in I_n is at least $c_{n-2}^{-|\underline{d}|/4}$, which implies (4.9).

It is easy to see that $R_n^{\underline{d}}|_{I_n^{\underline{d}}}$ can be written as $\phi \circ f \circ R_n^{\sigma^+(\underline{d})}$, where ϕ has small distortion. Due to (4.6), $R_n^{\sigma^+(\underline{d})}|_{I_n^{\underline{d}}}$ also has small distortion, so that a direct computation with f (which is purely quadratic) gives (4.8). $\square$

In other words, distances in I_n can be measured with precision $\sqrt{c_{n-1}}|I_n|$ in the partition induced by $\tilde{K}_n$, due to (4.6) and (4.9) (since $e^{-s_{n-1}} \ll c_{n-1}$).

Distances can be measured much more precisely with respect to the partition induced by K_n; in fact we have good precision in each $I_n^{\underline{d}}$ scale. In other words, inside $I_n^{\underline{d}}$, the central gap $C_n^{\underline{d}}$ is of size $O(c_n|I_n^{\underline{d}}|)$ (by (4.8)) and the other gaps have size $O(\sqrt{c_{n-1}}|C_n^{\underline{d}}|)$ (by (4.7) and (4.8)).

[8]The Koebe space of an interval T' inside an interval $T \supset T'$ is the minimum of $|L|/|T'|$ and $|R|/|T'|$ where L and R are the components of $T \setminus T'$. If the Koebe space of T' inside T is big, then the Koebe Principle states that a diffeomorphism onto T' which has an extension with negative Schwarzian derivative onto T has small distortion. In this case, it follows that the Koebe space of the preimage of T' inside the preimage of T is also big.

4.3. *Initial estimates on distortion.* To deal with the distortion control we need some preliminary known results. Those estimates are based on the Koebe Principle and the estimates of Lemma 4.5. All needed arguments are already contained in the proof of Lemma 4.5, so we won't get into details.

Proposition 4.6. *The following estimates hold*:

(1) *For any* j, *if* $R_n|_{I_n^j} = f^k$, $\operatorname{dist}(f^{k-1}|_{f(I_n^j)}) = 1 + O(c_{n-1})$.

(2) *For any* $\underline{d}$, $\operatorname{dist}(R_n^{\sigma^+(\underline{d})}|_{I_n^{\underline{d}}}) = 1 + O(\sqrt{c_{n-1}})$.

We will use the following immediate consequence for the decomposition of certain branches.

Lemma 4.7. *With total probability*,

(1) $R_n|_{I_n^0} = \phi \circ f$ *where* ϕ *has torrentially small distortion.*

(2) $R_n^{\underline{d}} = \phi_2 \circ f \circ \phi_1$ *where* ϕ_2 *and* ϕ_1 *have torrentially small distortion and* $\phi_1 = R_n^{\sigma^+(\underline{d})}$.

4.4. *Estimating derivatives.*

Lemma 4.8. *Let* w_n *denote the relative distance in* I_n *of* $R_n(0)$ *to* $\partial I_n \cup \{0\}$:

$$w_n = \frac{d(R_n(0), \partial I_n \cup \{0\})}{|I_n|}, \quad \text{where } d(x, X) = \inf_{y \in X} |y - x|.$$

With total probability,

$$\limsup_{n \to \infty} \frac{-\ln(w_n)}{\ln(n)} \leq 1.$$

In particular $R_n(0) \notin \tilde{I}_{n+1}$ *for all* n *large enough.*

Proof. This is a simple consequence of PhPa2, by the fact that $n^{-1-\delta}$ is summable, for all $\delta > 0$ (by (4.9) to obtain the last conclusion). $\square$

From now on we suppose that f satisfies the conclusions of the above lemma.

Lemma 4.9. *With total probability*,

$$\limsup_{n \to \infty} \frac{\sup_{j \neq 0} \ln(\operatorname{dist}(f|_{I_n^j}))}{\ln(n)} \leq 1/2.$$

Proof. Denote by $P_n^{\underline{d}}$ a $|C_n^{\underline{d}}|/n^{1+\delta}$ neighborhood of $C_n^{\underline{d}}$. Notice that the gaps of the Cantor sets K_n inside $I_n^{\underline{d}}$ which are different from $C_n^{\underline{d}}$ are torrentially (in n) smaller than $C_n^{\underline{d}}$, so that we can take $P_n^{\underline{d}}$ as a union of gaps of K_n up to torrentially small error.

It is clear that if h is a γ-qs homeomorphism (γ close to 1) then

$$|h(P_n^{\underline{d}} \setminus C_n^{\underline{d}})| \leq n^{-1-\delta/2}|h(C_n^{\underline{d}})|.$$

Notice that if $C_n^{\underline{d}}$ is contained in I_n^j with $j \neq \tau_n$, then $P_n^{\underline{d}}$ does not intersect $I_n^{\tau_n}$. Since the $C_n^{\underline{d}}$ are disjoint,

$$p_\gamma(\cup(P_n^{\underline{d}} \setminus C_n^{\underline{d}})|I_n^{\tau_n}) \leq n^{-1-\delta/2}$$

which is summable.

Transferring this estimate to the parameter using PhPa1 we see that with total probability, if n is sufficiently big, if $R_n(0)$ does not belong to $C_n^{\underline{d}}$ then $R_n(0)$ does not belong to $P_n^{\underline{d}}$ as well. In particular, if n is sufficiently big, the critical point 0 will never be in a $n^{-1/2-\delta/5}|I_{n+1}^j|$ neighborhood of any I_{n+1}^j with $j \neq 0$ (the change from $n^{-1-\delta}$ to $n^{-1/2-\delta/5}$ is due to taking the inverse image by $R_n|_{I_{n+1}}$, which corresponds, up to torrentially small distortion, to taking a square root, and causes the division of the exponent by two). This implies the required estimate on distortion since f is quadratic. $\square$

LEMMA 4.10. *With total probability*,

$$\limsup_{n\to\infty} \frac{\sup_{\underline{d}\in\Omega} \ln(\operatorname{dist}(R_n^{\underline{d}}))}{\ln(n)} \leq \frac{1}{2}. \tag{4.10}$$

In particular, for n big enough, $\sup_{\underline{d}\in\Omega} \operatorname{dist}(R_n^{\underline{d}}) \leq 2^n$ *and* $|DR_n(x)| > 2$, $x \in \cup_{j\neq 0} I_n^j$.

Proof. By Lemma 4.7, Lemma 4.9 implies (4.10). If $j \neq 0$, by (4.6) of Lemma 4.5 we get that $|R_n(I_n^j)|/|I_n^j| = |I_n|/|I_n^j| > c_{n-1}^{-1/3}$, so that $\operatorname{dist}(R_n|_{I_n^j})$ $\leq 2^n$ implies that for all $x \in I_n^j$, $|DR_n(x)| > c_{n-1}^{-1/3}2^{-n} > 2$. $\square$

Remark 4.2. Lemma 4.9 has also an application for approximation of intervals. This result implies that if $I_n^j = (a,b)$ and $j \neq 0$, we have $1/2^n < b/a$ $< 2^n$. As a consequence, for any symmetric (about 0) interval $I_{n+1} \subset X \subset I_n$, there exists a symmetric (about 0) interval $X \subset \tilde{X}$, which is union of I_n^j and is such that $|\tilde{X}|/|X| < 2^n$ (approximation by union of $C_n^{\underline{d}}$, with $|\tilde{X}|/|X|$ torrentially close to 1, follows more easily from the discussion on fine partitions).

We will also need to estimate derivatives of iterates of f, and not only of return branches.

LEMMA 4.11. *With total probability, if n is sufficiently big and if $x \in I_n^j$, $j \neq 0$, and $R_n|_{I_n^j} = f^K$, then for $1 \leq k \leq K$, $|(Df^k(x))| > |x| c_{n-1}^3$.*

Proof. First notice that by Lemmas 4.8 and 4.7, $R_n|_{I_n^0} = \phi \circ f$ with $|D\phi| > 1$, provided n is big enough (since ϕ has small distortion and there is a big macroscopic expansion from $f(I_n^0)$ to $R_n(I_n^0)$). Also, by Lemma 4.4, $|I_n|$ decays so fast that $\prod_{r=1}^n |I_n| > c_{n-1}^{3/2}$ for n big enough. Finally, by Lemma 4.10, for n big enough, $|DR_n(x)| > 1$ for $x \in I_n^j$, $j \neq 0$. Let n_0 be so big that if $n \geq n_0$, all the above properties hold.

From hyperbolicity of f restricted to the complement of I_{n_0} (from Lemma 2.1), there exists a constant $C > 0$ such that if s_0 is such that $f^s(x) \notin I_{n_0}^0$ for every $s_0 \leq s < k$ then $|Df^{k-s_0}(f^{s_0}(x))| > C$.

Let us now consider some $n \geq n_0$. If $k = K$, we have a full return and the result follows from Lemma 4.10.

Assume now $k < K$. Let us define $d(s)$, $0 \leq s \leq k$ such that $f^s(x) \in I_{d(s)} \setminus I_{d(s)}^0$ (if $f^s(x) \notin I_0$ we set $d(s) = -1$). Let $m(s) = \max_{s \leq t \leq k} d(t)$. Let us define a finite sequence $\{k_r\}_{r=0}^l$ as follows. With $k_0 = 0$ and when $k_r < k$ we let $k_{r+1} = \max\{k_r < s \leq k | d(s) = m(s)\}$. Notice that $d(k_i) < n$ if $i \geq 1$, since otherwise $f^{k_i}(x) \in I_n$ so that $k = k_i = K$ which contradicts our assumption.

The sequence $0 = k_0 < k_1 < \cdots < k_l = k$ satisfies $n = d(k_0) > d(k_1) > \cdots > d(k_l)$. Let θ be maximal with $d(k_\theta) \geq n_0$. Now

$$|Df^{k-k_\theta}(f^{k_\theta}(x))| > C|Df(f^{k_\theta}(x))|,$$

and so if $\theta = 0$ then $Df^k(x) > |2Cx|$ and we are done.

Assume now $\theta > 0$. Then

$$|Df^{k-k_\theta}(f^{k_\theta}(x))| > C|Df(f^{k_\theta}(x))| > C|I_{d(k_\theta)+1}|.$$

For $1 \leq r \leq \theta$, the action of $f^{k_r - k_{r-1}}$ near $f^{k_{r-1}}(x)$ is obtained by applying the central component of $R_{d(k_r)}$ followed by several noncentral components of $R_{d(k_r)}$. Since $d(k_r) \geq n_0$, we can estimate

$$|Df^{k_r - k_{r-1}}(f^{k_{r-1}}(x))| > |DR_{d(k_r)}(f^{k_{r-1}}(x))| > |Df(f^{k_{r-1}}(x))|.$$

For $r = 1$, this argument gives $|Df^{k_1}(x)| \geq |Df(x)|$, while for $r > 1$ we can estimate

$$|Df^{k_r - k_{r-1}}(f^{k_{r-1}}(x))| > |Df(f^{k_{r-1}}(x))| > |I_{d(k_{r-1})+1}|.$$

Combining it all we get

$$\begin{aligned}|Df^k(x)| &= |Df^{k_1}(x)| \cdot |Df^{k-k_\theta}(f^{k_\theta}(x))| \prod_{r=2}^{\theta} |Df^{k_r-k_{r-1}}(f^{k_{r-1}}(x))| \\ &> |2x| \cdot C \cdot |I_{d(k_\theta)+1}| \prod_{r=2}^{\theta} |I_{d(k_{r-1})+1}| = |2Cx| \prod_{r=1}^{\theta} |I_{d(k_r)+1}| \\ &\geq |2Cx| \prod_{r=0}^{n} |I_r| > |x| c_{n-1}^3. \qquad \square\end{aligned}$$

5. Sequences of quasisymmetric constants and trees

5.1. *Preliminary estimates.* From now on, we will need to transfer estimates on the capacity of certain sets from level to level of the principal nest. In order to do so we will need to consider not only γ-capacities with some γ fixed, but different constants for different levels of the principal nest. Next, we will make use of sequences of constants converging (decreasing) to a given value γ. We recall that γ is some constant very close to 1 such that $k(2\gamma-1) < 1+\varepsilon/5$, with ε very small.

We define the sequences $\rho_n = (n+1)/n$ and $\tilde{\rho}_n = (2n+3)/(2n+1)$, so that $\rho_n > \tilde{\rho}_n > \rho_{n+1}$ and $\lim \rho_n = 1$. We define the sequence $\gamma_n = \gamma\rho_n$ and an intermediate sequence $\tilde{\gamma}_n = \gamma\tilde{\rho}_n$.

As we know, the generalized renormalization process relating R_n to R_{n+1} has two phases, first R_n to L_n and then L_n to R_{n+1}. The following remarks shows why it is useful to consider the sequence of quasisymmetric constants due to losses related to distortion.

Remark 5.1. Let S be an interval contained in I_n^d. Using Lemma 4.7 we have $R_n^d|_S = \psi_2 \circ f \circ \psi_1$, where the distortion of ψ_2 and ψ_1 are torrentially small and $\psi_1(S)$ is contained in some I_n^j, $j \neq 0$. If S is contained in I_n^0 we may as well write $R_n|_S = \phi \circ f$, and the distortion of ϕ is also torrentially small.

In either case, if we decompose S in $2km$ intervals S_i of equal length, where k is the distortion of either $R_n^d|_S$ or $R_n|_S$ and m is subtorrentially big (say, $m < 2^n$), the distortion obtained restricting to any interval S_i will be bounded by $1+m^{-1}$. Indeed, in the case $S \subset I_n^0$, we have $\operatorname{dist}(R_n|_{S_i}) \leq \operatorname{dist}(\phi)\operatorname{dist}(f|_{S_i})$. Now $k = \operatorname{dist}(R_n|_S) \geq \operatorname{dist}(\phi)^{-1}\operatorname{dist}(f|_S)$. Since f is quadratic,

$$\begin{aligned}\operatorname{dist}(f|_{S_i}) - 1 &\leq \frac{|S_i|}{|S|}(\operatorname{dist}(f|_S) - 1) \leq \frac{1}{2km}(k\operatorname{dist}(\phi) - 1) \\ &\leq \frac{\operatorname{dist}(\phi)}{2m}.\end{aligned}$$

Since $\operatorname{dist}(\phi)-1$ is torrentially small, $\operatorname{dist}(f|_{S_i}) \leq 1+(2/3)m^{-1}$ and $\operatorname{dist}(R_n|_{S_i}) \leq 1+m^{-1}$. The case $S \subset I_n^d$ is entirely analogous, when we consider $\operatorname{dist}(R_n^d|_{S_i})$

$\leq \operatorname{dist}(\psi_2) \operatorname{dist}(f|_{\psi_1(S_i)}) \operatorname{dist}(\psi_1)$, and use torrentially small distortion of ψ_1 and ψ_2. The estimate now becomes

$$\begin{aligned}\operatorname{dist}(f|_{\psi_1(S_i)}) - 1 &\leq \frac{|\psi_1(S_i)|}{|\psi_1(S)|}(\operatorname{dist}(f|_{\psi_1(S)}) - 1)\\ &\leq \frac{\operatorname{dist}(\psi_1)}{2km}(k \operatorname{dist}(\psi_1) \operatorname{dist}(\psi_2) - 1)\\ &\leq \frac{\operatorname{dist}(\psi_1)^2 \operatorname{dist}(\psi_2))}{2m}\end{aligned}$$

and we conclude again that $\operatorname{dist}(R_n^d|_{S_i}) \leq 1 + m^{-1}$.

Remark 5.2. Now, let us fix γ such that the corresponding ε is small enough. We have the following estimate for the effect of the pullback of a subset of I_n by the central branch $R_n|_{I_n^0}$. With total probability, for all n sufficiently big, if $X \subset I_n$ satisfies

$$p_{\tilde{\gamma}_n}(X|I_n) < \delta \leq n^{-1000}$$

then

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X)|I_{n+1}) < \delta^{1/5}.$$

Indeed, let V be a $\delta^{1/4}|I_{n+1}|$ neighborhood of 0. Then $R_n|_{I_{n+1}\setminus V}$ has distortion bounded by $2\delta^{1/4}$.

Let $W \subset I_n$ be an interval of size $\lambda|I_n|$. Of course

$$p_{\tilde{\gamma}_n}(X \cap W|W) < \delta\lambda^{-1-\varepsilon}.$$

We decompose each side of $I_{n+1} \setminus V$ as a union of $n^3\delta^{-1/4}$ intervals of equal length. Let W be such an interval. From Lemma 4.8, it is clear that the image of W covers at least $\delta^{1/2}n^{-4}|I_n|$ and then that

$$p_{\tilde{\gamma}_n}(X \cap R_n(W)|R_n(W)) < \delta^{(1-\varepsilon)/2}n^{4+4\varepsilon}.$$

So we conclude that (since the distortion of $R_n|_W$ is of order $1 + n^{-3}$ by Remark 5.1)

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X) \cap W|W) < \delta^{(1-\varepsilon)/2}n^5$$

(we use the fact that the composition of a γ_{n+1}-qs map with a map with small distortion is $\tilde{\gamma}_n$-qs). Since

$$p_{\gamma_{n+1}}(V|I_{n+1}) < (2\delta^{1/4})^{1-\varepsilon},$$

we get the required estimate.

5.2. *More on trees.* We will need the following application of the above remarks:

LEMMA 5.1. *With total probability, for all n sufficiently big*

$$p_{\tilde{\gamma}_n}((R_n^d)^{-1}(X)|I_n^d) < 2^n p_{\gamma_n}(X|I_n).$$

Proof. Decompose $I_n^{\underline{d}}$ in $n^{\ln(n)}$ intervals of equal length, say, $\{W_i\}_{i=1}^{n^{\ln(n)}}$. Then by Lemma 4.10, $|R_n^{\underline{d}}(W_i)| > n^{-2\ln n}|I_n|$, and so we get

$$p_{\gamma_n}(R_n^{\underline{d}}(W_i) \cap X | R_n^{\underline{d}}(W_i)) < n^{4\ln(n)} p_{\gamma_n}(X|I_n).$$

Applying Remark 5.1, we see that

$$p_{\tilde{\gamma}_n}((R_n^{\underline{d}})^{-1}(X) \cap W_i | W_i) < n^{4\ln(n)} p_{\gamma_n}(X|I_n),$$

(we use the fact that the composition of a $\tilde{\gamma}_n$-qs map with a map with small distortion is γ_n-qs) which implies the desired estimate. $\square$

By induction we get:

Lemma 5.2. *With total probability, for n big enough, if $X_1, \ldots, X_m \subset \mathbb{Z} \setminus \{0\}$*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, \ldots, j_m, \ldots, j_{|\underline{d}^{(n)}(x)|}), j_i \in X_i, 1 \le i \le m | I_n) \le 2^{mn} \prod_{i=1}^{m} p_{\gamma_n}(j^{(n)}(x) \in X_i | I_n).$$

The following is an obvious variation of the previous lemma fixing the start of the sequence.

Lemma 5.3. *With total probability, for n big enough, if $X_1, \ldots, X_m \subset \mathbb{Z} \setminus \{0\}$, and if $\underline{d} = (j_1, \ldots, j_k)$,*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, \ldots, j_k, j_{k+1}, \ldots, j_{k+m}, \ldots, j_{|\underline{d}^{(n)}(x)|}), j_{i+k} \in X_i, 1 \le i \le m | I_n^{\underline{d}}) \le 2^{mn} \prod_{i=1}^{m} p_{\gamma_n}(j^{(n)}(x) \in X_i | I_n).$$

In particular, with $\underline{d} = (\tau_n)$,

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (\tau_n, j_1, \ldots, j_m, j_{m+1}, \ldots, j_{|\underline{d}^{(n)}(x)|}), j_i \in X_i, 1 \le i \le m | I_n^{\tau_n}) \le 2^{mn} \prod_{i=1}^{m} p_{\gamma_n}(j^{(n)}(x) \in X_i | I_n).$$

The last part of the above lemma will often be necessary in order to apply PhPa1.

Sometimes we are more interested in the case where the X_i are all equal.

Let $Q \subset \mathbb{Z} \setminus \{0\}$. Let $Q(m,k)$ denote the set of $\underline{d} = (j_1, \ldots, j_m)$ such that $\#\{1 \le i \le m, \, j_i \in Q\} \ge k$.

Define $q_n(m,k) = p_{\tilde{\gamma}_n}(\cup_{\underline{d} \in Q(m,k)} I_n^{\underline{d}} | I_n)$.

Let $q_n = p_{\gamma_n}(\cup_{j \in Q} I_n^j | I_n)$.

Lemma 5.4. *With total probability, for n large enough,*

$$q_n(m,k) \leq \binom{m}{k}(2^n q_n)^k. \tag{5.1}$$

Proof. We have the following recursive estimates for $q_n(m,k)$:

(1) $q_n(1,0)=1$, $q_n(1,1) \leq q_n \leq 2^n q_n$, and $q_n(m+1,0) \leq 1$ for $m \geq 1$,

(2) $q_n(m+1,k+1) \leq q_n(m,k+1) + 2^n q_n q_n(m,k)$.

Indeed, (1) is completely obvious and if $(j_1,\dots,j_{m+1}) \in Q(m+1,k+1)$ then either $(j_1,\dots,j_m) \in Q(m,k+1)$ or $(j_1,\dots,j_m) \in Q(m,k)$ and $j_{m+1} \in Q$, so that (2) follows from Lemma 5.1. It is clear that (1) and (2) imply, by induction, (5.1). □

We recall that by Stirling's formula,

$$\binom{m}{qm} < \frac{m^{qm}}{(qm)!} < \left(\frac{3}{q}\right)^{qm}.$$

So we can get the following estimate. For $q \geq q_n$,

$$q_n(m,(6\cdot 2^n)qm) < \left(\frac{1}{2}\right)^{(6\cdot 2^n)qm}. \tag{5.2}$$

This is also used in the following form. If $q^{-1} > 6 \cdot 2^n$ (it is usually the case, since q will be torrentially small)

$$\sum_{k>q^{-2}} q_n(k,(6\cdot 2^n)qk) < 2^{-n}q^{-1}\left(\frac{1}{2}\right)^{(6\cdot 2^n)q^{-1}}. \tag{5.3}$$

This can be interpreted as a large deviation estimate in this context.

6. Estimates on time

Our aim in this section is to estimate the distribution of return times to I_n: they are concentrated around c_{n-1}^{-1} up to an exponent close to 1.

The basic estimate is a large deviation estimate which is proved in the next subsection (Corollary 6.5) and states that for $k \geq 1$ the set of branches with time larger than kc_n^{-4} has capacity less than e^{-k}.

6.1. *A large deviation lemma for times.* Let $r_n(j)$ be such that $R_n|_{I_n^j} = f^{r_n(j)}$. We will also use the notation $r_n(x) = r_n(j^{(n)}(x))$, the n-th return time of x (there should be no confusion for the reader, since we consistently use j for an integer index and x for a point in the phase space).

Let

$$A_n(k) = p_{\gamma_n}(r_n(x) \geq k | I_n).$$

Since f restricted to the complement of I_{n+1} is hyperbolic, from Lemma 2.1, it is clear that $A_n(k)$ decays exponentially with k:

LEMMA 6.1. *With total probability, for all* $n > 0$, *there exists* $C > 0$, $\lambda > 1$ (*which depend on* n) *such that* $A_n(k) < C\lambda^k$.

Proof. Consider a Markov partition for $f|_{I \setminus I_{n+1}}$, that is, a finite union of intervals $M_1, \dots, M_m$ such that

(1) $\cup_{i=1}^m M_i = I \setminus I_{n+1}$.

(2) For every $1 \leq i \leq m$, $f|_{M_i}$ is a diffeomorphism.

(3) $f(\cup_{i=1}^m \partial M_i) \subset \cup_{i=1}^m \partial M_i$.

It is easy to see that such a Markov partition also satisfies

(4) For every $1 \leq i \leq m$, either

$$f(M_i) = \bigcup_{M_j \subset f(M_i)} M_j \quad \text{or} \quad f(M_i) = I_{n+1} \cup \bigcup_{M_j \subset f(M_i)} M_j.$$

(To construct such a Markov partition, notice first that the boundary of I_{n+1} is preperiodic to a periodic orbit q (of period p). In particular, $f^s(\partial I_{n+1}) = q$ for some integer $s > p$. Let K be the (finite) set of all x which never enter $\operatorname{int} I_{n+1}$ and such that $f^j(x) = q$ for some $j \leq s$. Since I_{n+1} is nice, $\partial I_{n+1} \subset K$, and since $s > p$, $q \in K$. In particular K is forward invariant. It is easy to see that the connected components of $I \setminus (K \cup I_{n+1})$ form a Markov partition of $I \setminus I_{n+1}$.)

It follows that if $f^j(x) \in \cup_{i=1}^m \operatorname{int} M_i$, $0 \leq j \leq k$, then there exists a unique interval $x \in M^k(x)$ such that $f^k|_{M^k(x)}$ is a diffeomorphism onto some M_j. Notice that if $k \geq 1$, $f(M^k(x)) = M^{k-1}(f(x))$.

By Lemma 2.1, if $y \in M^k(x)$, $|Df^k(y)|$ is exponentially big in k. In particular, $\sum_{j=0}^{k-1} |f^j(M^k(x))| < C'$ for some constant $C' > 0$ independent of $M^k(x)$. Since f is C^2, $\operatorname{dist}(f|_{M^k(x)})$ is uniformly bounded in k. Notice that the bounds on distortion depend on n. (An alternative to this classical argument is to obtain the bounded distortion from the negative Schwarzian derivative.)

By Lemma 2.1 again, the set of points $x \in I$ which never enter I_{n+1} has empty interior: for every $T \subset I$ there is an iterate $f^r(T)$ which intersects I_{n+1} (otherwise the exponentially growing intervals $f^r(T) \subset I$ would eventually become bigger than I). So there exists $r > 0$ such that, for every M_j, there exists $x \in M_j$ and $t_j < r$ with $f^{t_j}(x) \in \operatorname{int} I_{n+1}$. It follows that there exists an interval $E_j \subset M_j$ such that $f^{t_j}(E_j) \subset \operatorname{int} I_{n+1}$.

Fixing some $M^k(x)$ with $f^k(M^k(x)) = M_j$, let $E^k(x) = (f^k|_{M^k(x)})^{-1}(E_j)$. By bounded distortion, it follows that $|E^k(x)|/|M^k(x)|$ is uniformly bounded from below independently of $M^k(x)$. In particular, $p_{2\gamma}(M^k(x) \setminus E^k(x)|M^k(x)) < \lambda$ for some constant $\lambda < 1$.

Let M^k be the union of the $M^k(x)$ and E^k be the union of the $E^k(x)$. Then $M^{k+r} \cap E^k = \emptyset$. In particular, $p_{2\gamma}(M^{(k+1)r}|I) < \lambda p_{2\gamma}(M^{kr}|I)$.

We conclude that $p_{2\gamma}(M^k|I_n) < C\lambda^{k/r}$ for some constant $C > 0$. If $k > r_n(0)$, then $M^k \cap I_n$ contains the set of points $x \in I_n$ such that $f^j(x) \notin I_n$, $1 \leq j \leq k$, that is, all points $x \in I_n$ with $r_n(x) > k$. Adjusting C and λ if necessary, we have $A_n(k) < C\lambda^k$. □

Remark 6.1. It turns out that λ depends strongly on n. Indeed, it is possible to show that λ is torrentially close to 1. The argument above does not give any estimate on the behavior of λ as n grows, but it will be used below as the basis of an inductive argument which will give explicit estimates on λ for n big.

Let ζ_n be the maximum $\zeta \leq c_{n-1}$ such that for all $k \geq \zeta^{-1}$ we have

$$A_n(k) \leq e^{-\zeta k} \tag{6.1}$$

and finally let $\alpha_n = \min_{1 \leq m \leq n} \zeta_m$.

Our main result in this section is to estimate α_n. We will show that with total probability, for n big we have $\alpha_{n+1} \geq c_n^4$. For this we will have to do a simultaneous estimate for landing times, which we define now.

Let $l_n(\underline{d})$ be such that $L_n|_{I_n^{\underline{d}}} = f^{l_n(\underline{d})}$. We will also use the notation $l_n(x) = l_n(\underline{d}^{(n)}(x))$.

Let us define

$$B_n(k) = p_{\tilde{\gamma}_n}(l_n(x) > k|I_n). \tag{6.2}$$

$$B_n^{\tau_n}(k) = p_{\tilde{\gamma}_n}(l_n(x) > k + r_n(\tau_n)|I_n^{\tau_n}). \tag{6.3}$$

Lemma 6.2. *If* $k > c_n^{-3/2}\alpha_n^{-3/2}$ *then*

$$B_n(k) < e^{-c_n^{3/2}\alpha_n^{3/2}k}, \tag{6.4}$$

and

$$B_n^{\tau_n} < e^{-c_n^{-3/2}\alpha_n^{3/2}k}. \tag{6.5}$$

Proof. We first show (6.4). Let $k > c_n^{-3/2}\alpha_n^{-3/2}$ be fixed. Let $m_0 = \alpha_n^{3/2}k$.

Notice that by Lemma 4.2

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq m_0|I_n) \leq e^{-c_n^{5/4}\alpha_n^{3/2}k}. \tag{6.6}$$

Fix now $m < m_0$. Let us estimate

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) > k | I_n). \tag{6.7}$$

For each $\underline{d} = (j_1, \dots, j_m)$ we can associate a sequence of m positive integers r_i such that $r_i \leq r_n(j_i)$ and $\sum r_i = k$. The average value of r_i is at least k/m so we conclude that

$$\sum_{r_i \geq k/2m} r_i > k/2. \tag{6.8}$$

Recall also that

$$\frac{k}{2m} > \frac{1}{(2\alpha_n^{3/2})} > \alpha_n^{-1}. \tag{6.9}$$

Given a sequence of m positive integers r_i as above we can make the following estimate using Lemma 5.2

$$\begin{aligned} p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, \dots, j_m), r_n(j_i) \geq r_i | I_n) \\ \leq 2^{mn} \prod_{j=1}^{m} p_{\gamma_n}(r_n(x) \geq r_j | I_n) \\ \leq 2^{mn} \prod_{r_j \geq \alpha_n^{-1}} p_{\gamma_n}(r_n(x) \geq r_j | I_n) \\ \leq 2^{mn} \prod_{r_j \geq k/2m} e^{-\alpha_n r_j} \\ \leq 2^{mn} e^{-\alpha_n k/2}. \end{aligned} \tag{6.10}$$

The number of sequences of m positive integers r_i with sum k is

$$\begin{aligned} \binom{k+m-1}{m-1} &\leq \frac{1}{(m-1)!}(k+m-1)^{m-1} \\ &\leq \frac{1}{m!}(k+m)^m \leq \left(\frac{2ek}{m}\right)^m. \end{aligned} \tag{6.11}$$

Notice that

(6.12)
$$\begin{aligned} 2^{mn}\left(\frac{2ek}{m}\right)^m &\leq \left(\frac{2^{n+3}k}{m}\right)^{\frac{m}{k2^{n+3}}k2^{n+3}} \\ &\leq \left(\frac{2^{n+3}k}{m_0}\right)^{\frac{m_0}{k2^{n+3}}k2^{n+3}} \qquad \text{(since } x^{1/x} \text{ decreases for } x > e\text{)} \\ &\leq \left(\frac{2^{n+3}}{\alpha_n^{3/2}}\right)^{m_0} \leq e^{\alpha_n^{5/4}k}. \end{aligned}$$

So we can finally estimate

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) \geq k | I_n) \leq 2^{mn} \left(\frac{2ek}{m}\right)^m e^{-\alpha_n k/2} \tag{6.13}$$
$$< e^{(\alpha_n^{1/4} - 1/2)\alpha_n k}.$$

Summing up on m we get

$$\begin{aligned} p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| < m_0, l_n(x) \geq k | I_n) & \\ \leq m_0 e^{(\alpha_n^{1/4} - 1/2)\alpha_n k} & \\ < e^{(2\alpha_n^{1/4} - 1/2)\alpha_n k} & \quad (\text{since} \frac{\ln(m_0)}{k} \leq \frac{\ln(k)}{k} \leq \alpha_n^{5/4}) \\ \leq e^{-\alpha_n k/3}. & \end{aligned} \tag{6.14}$$

As a direct consequence we get

$$B_n(k) < e^{-\alpha_n k/3} + e^{-c_n^{5/4}\alpha_n^{3/2} k} < e^{-c_n^{3/2}\alpha_n^{3/2} k}, \tag{6.15}$$

concluding the proof of (6.4).

For the proof of (6.5) one proceeds analogously. Take k and m_0 as before. By Lemma 4.7 one gets

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq m_0 | I_n^{\tau_n}) \leq e^{-c_n^{5/4}\alpha_n^{3/2} k}. \tag{6.16}$$

For any $m < m_0$, if $\underline{d} = (\tau_n, j_1, \ldots, j_m)$ and $l_n(\underline{d}) > k + r_n(\tau_n)$ then there exists $r_i \leq r_n(j_i)$ with $\sum_{i=1}^m r_i = k$. Repeating the argument of (6.10) (and using Lemma 5.3 instead of Lemma 5.2) one gets, for any such sequence $r_1, \ldots, r_m$,

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (\tau_n, j_1, \ldots, j_m), r_n(j_i) \geq r_i | I_n^{\tau_n}) \leq 2^{mn} e^{-\alpha_n k/2}. \tag{6.17}$$

The previous combinatorial estimate can be applied again to obtain

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m+1, l_n(x) > k + r_n(\tau_n) | I_n^{\tau_n}) < e^{(\alpha_n^{1/4} - 1/2)\alpha_n k}. \tag{6.18}$$

Summing up (6.18) on $m < m_0$ and using (6.16) we obtain estimate (6.5). $\square$

Let $v_n = r_n(0)$ be the return time of the critical point.

Lemma 6.3. *With total probability, for n large enough,*

$$v_{n+1} < c_n^{-2}\alpha_n^{-2}/2.$$

Proof. By the definition of α_n and PhPa2, it follows that with total probability, for n large enough,

$$r_n(\tau_n) < c_{n-1}^{-1}\alpha_n^{-1}.$$

Recall that $\underline{d}^{(n)}(0)$ is such that $R_n(0) \in C_n^{\underline{d}^{(n)}(0)}$. Using Lemma 6.2, more precisely estimate (6.5), together with PhPa1, we get with total probability, for n large enough,

$$l_n(\underline{d}^{(n)}(0)) - r_n(\tau_n) < n\alpha_n^{-3/2}c_n^{-3/2},$$

and thus

$$v_{n+1} < v_n + c_{n-1}^{-1}\alpha_n^{-1} + n\alpha_n^{-3/2}c_n^{-3/2} < v_n + \alpha_n^{-2}c_n^{-2}/4.$$

Notice that α_n decreases monotonically; thus for n_0 big enough and for $n > n_0$,

$$v_{n+1} < v_{n_0} + \sum_{k=n_0}^{n} \alpha_k^{-2}c_k^{-2}/4 < v_{n_0} + \alpha_n^{-2}c_n^{-2}/3.$$

which for n big enough implies $v_{n+1} < c_n^{-2}\alpha_n^{-2}/2$. $\square$

LEMMA 6.4. *With total probability, for n large enough,*

$$\alpha_{n+1} \geq \min\{\alpha_n^4, c_n^4\}.$$

Proof. Let $k \geq \max\{\alpha_n^{-4}, c_n^{-4}\}$. From Lemma 6.3 one immediately sees that if $r_{n+1}(j) \geq k$ then $R_n(I_{n+1}^j))$ is contained on some $C_n^{\underline{d}}$ with $l_n(\underline{d}) = r_{n+1}(j) - v_n \geq k/2 \geq n\alpha_n^{-3/2}c_n^{-3/2}$.

Applying Lemma 6.2 we have $B_n(k/2) < e^{-\alpha_n^{3/2}c_n^{3/2}k/2}$.

Applying Remark 5.2 we get

$$A_{n+1}(k) < e^{-k\alpha_n^{3/2}c_n^{3/2}/200} < e^{-k\min\{\alpha_n^4, c_n^4\}}. \qquad \square$$

Since c_n decreases torrentially, we get

COROLLARY 6.5. *With total probability, for n large enough $\alpha_{n+1} \geq c_n^4$.*

Remark 6.2. In particular, by Lemma 6.3, for n big, $v_n < c_{n-1}^{-2}\alpha_{n-1}^{-2}/2 < c_{n-1}^{-4}$.

6.2. *Consequences.*

LEMMA 6.6. *With total probability, for all n sufficiently large,*

$$p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-1+\varepsilon}|I_n) < c_n^{\varepsilon/2}, \tag{6.19}$$

$$p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-1-5\varepsilon/3}|I_n) \leq e^{-c_n^{-\varepsilon/4}}, \tag{6.20}$$

$$p_{\tilde{\gamma}_n}(l_n(x) - r_n(x) < c_n^{-1+\varepsilon}|I_n^{\tau_n}) < c_n^{\varepsilon/2}, \tag{6.21}$$

$$p_{\tilde{\gamma}_n}(l_n(x) - r_n(x) > c_n^{-1-5\varepsilon/3}|I_n^{\tau_n}) \leq e^{-c_n^{-\varepsilon/4}}. \tag{6.22}$$

Proof. We will concentrate on estimates (6.19) and (6.20), since (6.21) and (6.22) are analogous.

We have $l_n(\underline{d}) \geq |\underline{d}|$, and from Lemma 4.2

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \leq c_n^{-1+\varepsilon}|I_n) \leq c_n^{\varepsilon/2},$$

which implies (6.19).

On the other hand, by the same lemma,

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq c_n^{-1-\varepsilon}|I_n) \leq e^{-c_n^{-\varepsilon/2}}.$$

Defining

$$X_m = \bigcup_{\substack{\underline{d}=(j_1,\ldots,j_m),\\ r_n(j_m)>c_n^{-\varepsilon/2}c_{n-1}^{-4}}} I_n^{\underline{d}}$$

we have

$$p_{\tilde{\gamma}_n}(X_m|I_n) \leq 2^n e^{-c_n^{-\varepsilon/2}} < e^{-c_n^{-\varepsilon/3}}.$$

Since

$$c_n^{-1-\varepsilon}c_n^{-\varepsilon/2}c_{n-1}^{-4} < c_n^{-1-5\varepsilon/3},$$

we conclude that if x satisfies $l_n(x) > c_n^{-1-5\varepsilon/3}$ and $|\underline{d}_n(x)| < c_n^{-1-\varepsilon}$ then x belongs to some X_m with $1 \leq m \leq c_n^{-1-\varepsilon}$. So we get

$$p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-1-5\varepsilon/3}|I_n) \leq e^{-c_n^{-\varepsilon/2}} + c_n^{-1-\varepsilon}e^{-c_n^{-\varepsilon/3}} < e^{-c_n^{-\varepsilon/4}}$$

which implies (6.20). □

Corollary 6.7. *With total probability, for all n sufficiently large,*

$$p_{\gamma_{n+1}}(r_{n+1}(x) < c_n^{-1+\varepsilon}|I_{n+1}) < c_n^{\varepsilon/10}, \tag{6.23}$$

$$p_{\gamma_{n+1}}(r_{n+1}(x) > c_n^{-1-2\varepsilon}|I_{n+1}) \leq e^{-c_n^{-\varepsilon/5}} \leq c_n^n. \tag{6.24}$$

Proof. Notice that $r_{n+1}(j) = v_n + l_n(\underline{d})$, where $R_n(I_{n+1}^j) \subset C_n^{\underline{d}}$. By Remark 6.2, we can estimate $v_n < c_{n-1}^{-10}$. The distribution of $r_{n+1}(j) - v_n$ can then be estimated by the distribution of $l_n(\underline{d})$ from Lemma 6.6, with a slight loss given by Remark 5.2. □

Using PhPa2 we get

Lemma 6.8. *With total probability, for all n sufficiently big,*

$$\lim_{n\to\infty} \frac{\ln(r_n(\tau_n))}{\ln(c_{n-1}^{-1})} = 1. \tag{6.25}$$

Corollary 6.9. *With total probability, for all n sufficiently large,*

$$p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-1+\varepsilon} | I_n^{\tau_n}) \leq c_n^{\varepsilon/10}, \tag{6.26}$$

$$p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-1-11\varepsilon/6} | I_n^{\tau_n}) \leq e^{-c_n^{-\varepsilon/5}}. \tag{6.27}$$

Proof. Just use Lemma 6.8 together with estimates (6.21) and (6.22) of Lemma 6.6. □

Corollary 6.10. *With total probability,*

$$\lim_{n\to\infty} \frac{\ln(v_{n+1})}{\ln(c_n^{-1})} = 1.$$

Proof. Notice that $v_{n+1} = v_n + l_n(\underline{d})$ where $R_n(0) \in C_n^{\underline{d}}$. Using Corollary 6.9 and PhPa1 we get $c_n^{-1+\varepsilon} < l_n(\underline{d}) < c_n^{-1-11\varepsilon/6}$. By Remark 6.2, $v_n < c_{n-1}^{-10}$, so $c_n^{-1+\varepsilon} < v_{n+1} < c_n^{-1-2\varepsilon}$. Letting ε go to 0 we get the result. □

Remark 6.3. Using Lemma 4.8, we see that $|R_n(I_{n+1})| > 2^{-n}|I_n|$. Since $|Df(x)| < 4$, $x \in I$, it follows that $|DR_n(x)| < 4^{v_n}$, $x \in I_{n+1}$. In particular, Corollary 6.10 implies that with total probability, for all $\varepsilon > 0$, for all n big enough,

$$2^{-n} c_n^{-1} < \frac{|R_n(I_{n+1})|}{|I_{n+1}|} < 4^{v_n} < 4^{c_{n-1}^{-1-\varepsilon}},$$

so that $\ln(c_n^{-1}) < c_{n-1}^{-1-2\varepsilon}$. Together with Corollary 4.4, This implies that

$$\lim_{n\to\infty} \frac{\ln(\ln(c_n^{-1}))}{\ln(c_{n-1}^{-1})} = 1,$$

and so c_n^{-1} grows torrentially (and not faster).

7. Dealing with hyperbolicity

In this section we show by an inductive process that the great majority of branches are reasonably hyperbolic. In order to do that, in the following subsection, we define some classes of branches, with “very good” distribution of times, which are not too close to the critical point. The definition of very good distributions of times has an inductive component: they are compositions of many very good branches of the previous level. The fact that most branches are very good is related to the validity of some type of Law of Large Numbers estimate.

7.1. *Some kinds of branches and landings.*

7.1.1. *Standard landings.* Let us define the set of standard landings of level n, $\mathrm{LS}(n) \subset \Omega$, as the set of all $\underline{d} = (j_1, \ldots, j_m)$ satisfying the following.

LS1: (m not too small or large). $c_n^{-1/2} < m < c_n^{-1-2\varepsilon}$.

LS2: (No very large times). $r_n(j_i) < c_{n-1}^{-14}$ for all i.

LS3: (Short times are sparse in large enough initial segments). For $c_{n-1}^{-2} \leq k \leq m$

$$\#\{1 \leq i \leq k,\, r_n(j_i) < c_{n-1}^{-1+2\varepsilon}\} < (6 \cdot 2^n) c_{n-1}^{\varepsilon/10} k.$$

LS4: (Large times are sparse in large enough initial segments) For $c_n^{-1/n} \leq k \leq m$

$$\#\{1 \leq i \leq k,\, r_n(j_i) > c_{n-1}^{-1-2\varepsilon}\} < (6 \cdot 2^n) e^{-c_{n-1}^{\varepsilon/5}} k.$$

Lemma 7.1. *With total probability, for all n sufficiently big,*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin \mathrm{LS}(n) | I_n) < c_n^{1/3}/2, \tag{7.1}$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin \mathrm{LS}(n) | I_n^{\tau_n}) < c_n^{1/3}/2. \tag{7.2}$$

Proof. Let us start with estimate (7.1) (on I_n). Let us estimate the complement of the set of landings which violate each item of the definition.

(LS1) This was estimated before (see Lemma 4.2); an upper bound is $c_n^{1/3}/3$ (with ε small).

(LS2) By Corollary 6.5 the γ_n-capacity of $\{r_n(x) > c_{n-1}^{-14}\}$ is at most $e^{-c_{n-1}^{-10}} \ll c_n^3$. Using Lemma 5.1, we see that the $\tilde{\gamma}_n$-capacity of the set of $\underline{d} = (j_1, \ldots, j_m)$ with $r_n(j_i) > c_n^{-14}$ for some $i \leq c_n^{-1-2\varepsilon}$ (in particular for some $i \leq m$ if m is as in LS1) is bounded by $2^n c_n^{-1-2\varepsilon} c_n^3 \ll c_n$.

(LS3) This is a large deviation estimate, and so we follow the ideas of §5.2, particularly estimate (5.2). Put $q = c_{n-1}^{\varepsilon/10}$. By the inequality (6.23) of Corollary 6.7, we can estimate the $\tilde{\gamma}_n$-capacity corresponding to the violation of LS3 for some fixed $c_{n-1}^{-2} \leq k \leq c_n^{-1-2\varepsilon}$ by

$$\left(\frac{1}{2}\right)^{(6 \cdot 2^n) qk} \leq \left(\frac{1}{2}\right)^{c_{n-1}^{-3/2}} \ll c_n^3.$$

Summing up over $k \leq c_n^{-1-2\varepsilon}$ (and in particular for $k \leq m$ as in LS1) we get the upper bound c_n.

(LS4) We use the method of the previous item. Put $q = e^{-c_{n-1}^{-\varepsilon/5}}$. By estimate (6.24) of Corollary 6.7, we can bound the $\tilde{\gamma}_n$-capacity corresponding to the violation of LS4 for some fixed $c_n^{-1/n} \leq k \leq c_n^{-1-2\varepsilon}$ by

$$\left(\frac{1}{2}\right)^{(6 \cdot 2^n) qk} \ll c_n^3.$$

Summing up over $k \leq c_n^{-1-2\varepsilon}$ (and in particular for $k \leq m$ as in LS1) we get the upper bound c_n.

Adding the losses of the four items, we get the estimate (7.1). To establish estimate (7.2) (on $I_n^{\tau_n}$), the only item which changes is LS2; we have to be careful since if $r_n(\tau_n)$ is very large then automatically LS2 is violated for every $\underline{d}$ which starts by τ_n. But this was taken care of in Lemma 6.8, and with this observation the estimates are the same. □

7.1.2. *Very good returns and excellent landings.* Define the set of very good returns, $\mathrm{VG}(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ by induction as follows. We let $\mathrm{VG}(n_0, n_0) = \mathbb{Z} \setminus \{0\}$ and supposing $\mathrm{VG}(n_0, n)$ already defined, we define $\mathrm{LE}(n_0, n) \subset \mathrm{LS}(n)$ (excellent landings) as the set of standard landings satisfying the following extra condition:

LE: (Not very good moments are sparse in large enough initial segments). For all $c_{n-1}^{-2} < k \leq m$,

$$\#\{1 \leq i \leq k,\, j_i \notin \mathrm{VG}(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{1/20} k.$$

And we define $\mathrm{VG}(n_0, n+1)$ as the set of j such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \in \mathrm{LE}(n_0, n)$ and the satisfying the extra condition:

VG: (distant from 0). The distance of I_{n+1}^j to 0 is bigger than $c_n^{1/3}|I_{n+1}|$.

LEMMA 7.2. *With total probability, for all n_0 sufficiently big and all $n \geq n_0$, if*

$$p_{\gamma_n}(j^{(n)}(x) \notin \mathrm{VG}(n_0, n) | I_n) < c_{n-1}^{1/20} \tag{7.3}$$

then

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin \mathrm{LE}(n_0, n) | I_n) < c_n^{1/3}, \tag{7.4}$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin \mathrm{LE}(n_0, n) | I_n^{\tau_n}) < c_n^{1/3}. \tag{7.5}$$

Proof. We first use Lemma 7.1 to estimate the $\tilde{\gamma}_n$-capacity of branches not in $\mathrm{LS}(n)$ by $c_n^{1/3}/2$.

Let $q = c_{n-1}^{1/20}$. Using the hypothesis and estimate (5.2) of §5.2 (see also the estimate of the complement of LS3 in Lemma 7.1) we first estimate the $\tilde{\gamma}_n$-capacity of the set of landings which violate LE for a specific value of k with $k \geq c_{n-1}^{-2}$ by $(1/2)^{(6 \cdot 2^n)qk}$ and then summing on k we get

$$\sum_{k \geq c_{n-1}^{-2}} \left(\frac{1}{2}\right)^{(6 \cdot 2^n)qk} \ll c_n.$$

This argument works both for (7.4) (in I_n) and (7.5) (in $I_n^{\tau_n}$). □

LEMMA 7.3. *With total probability, for all n_0 sufficiently big and for all $n \geq n_0$,*

$$p_{\gamma_n}(j^{(n)}(x) \notin \mathrm{VG}(n_0, n)|I_n) < c_{n-1}^{1/20}. \tag{7.6}$$

Proof. It is clear that with total probability, for n_0 sufficiently big and $n \geq n_0$, the set of branches I_n^j at distance at least $c_{n-1}^{1/3}|I_n|$ from 0 has γ_n-capacity bounded by $c_{n-1}^{1/8}$.

For $n = n_0$, (7.6) holds (since all branches are very good except the central one). Using Lemma 7.2, if (7.6) holds for n then (7.4) also holds for n. Pulling back estimate (7.4) by $R_n|_{I_{n+1}}$ (using Remark 5.2), we get (7.6) for $n+1$. By induction on n, (7.6) holds for all $n \geq n_0$. □

Using PhPa2 we get (using the measure-theoretical argument of Lemma 3.2)

LEMMA 7.4. *With total probability, for all n_0 big enough, for all n big enough, $\tau_n \in \mathrm{VG}(n_0, n)$.*

LEMMA 7.5. *With total probability, for all n_0 big enough and for all $n \geq n_0$, if $j \in \mathrm{VG}(n_0, n+1)$ then*

$$\frac{1}{2} m c_{n-1}^{-1+2\varepsilon} < r_{n+1}(j) < 2 m c_{n-1}^{-1-2\varepsilon},$$

where as usual, m is such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ and $\underline{d} = (j_1, \ldots, j_m)$.

Proof. Notice that $r_{n+1}(j) = v_n + \sum r_n(j_i)$. To estimate the total time $r_{n+1}(j)$ from below we use LS3 and get

$$\frac{1}{2} m c_{n-1}^{-1+2\varepsilon} < (1 - 6 \cdot 2^n c_n^{\varepsilon/10}) m c_{n-1}^{-1+2\varepsilon} < r_{n+1}(j).$$

To estimate from above, we notice that $v_n < c_{n-1}^{-4}$ and by LS2 and LS4

$$\sum_{r_n(j_i) > c_{n-1}^{-1-2\varepsilon}} r_n(j_i) < 6 \cdot 2^n c_{n-1}^{-14} e^{-c_{n-1}^{-\varepsilon/5}} m < m,$$

so that

$$r_{n+1}(j) < m c_{n-1}^{-1-2\varepsilon} + m + c_{n-1}^{-4} < 2 m c_{n-1}^{-1-2\varepsilon}. \qquad \square$$

Remark 7.1. Using LS1 we get the estimate $c_n^{-1/2} < r_{n+1}(j) < c_n^{-1-3\varepsilon}$ for $j \in \mathrm{VG}(n_0, n+1)$.

Let $j \in \mathrm{VG}(n_0, n+1)$. We can write $R_{n+1}|_{I_{n+1}^j} = f^{r_{n+1}(j)}$, that is, a big iterate of f. One may consider which proportion of those iterates belongs to very good branches of the previous level. More generally, we can truncate the return R_{n+1}, that is, we may consider $k < r_{n+1}(j)$ and ask which proportion of iterates up to k belongs to very good branches.

LEMMA 7.6. *With total probability, for all n_0 big enough and for all $n \geq n_0$, the following holds.*

Let $j \in \mathrm{VG}(n_0, n+1)$, and let $\underline{d} = (j_1, \ldots, j_m)$ be such that $R_n(I^j_{n+1}) = C^{\underline{d}}_n$. Let m_k be biggest possible with

$$v_n + \sum_{j=1}^{m_k} r_n(j_i) \leq k$$

(*the amount of full returns to level n before time k*) *and let*

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in \mathrm{VG}(n_0, n)}} r_n(j_i)$$

(*the total time spent in full returns to level n which are very good before time k*). *Then $1 - \beta_k/k < c_{n-1}^{1/100}$ if $k > c_n^{-2/n}$.*

Proof. Let us estimate first the time i_k which is not spent on noncritical full returns:

$$i_k = k - \sum_{j=1}^{m_k} r_n(j_i).$$

This corresponds exactly to v_n plus some incomplete part of the return $j_{m_{k+1}}$. This part can be bounded by $c_{n-1}^{-4} + c_{n-1}^{-14}$ (use Corollary 6.10 to estimate v_n and LS2 to estimate the incomplete part).

Using LS2 we conclude now that

$$m_k > (k - c_{n-1}^{-4} - c_{n-1}^{-14}) c_{n-1}^{14} > c_n^{-1/n}$$

and so m_k is not too small.

Let us now estimate the contribution h_k from full returns j_i with time higher than $c_{n-1}^{-1-2\varepsilon}$. Since m_k is big, we can use LS4 to conclude that the number of such high time returns must be less than $c_{n-1}^n m_k$, so that their total time is at most $c_{n-1}^{n-14} m_k$.

The not-very-good full returns on the other hand can be estimated by LE (given the estimate on m_k); they are at most $c_{n-1}^{1/21} m_k$. So we can estimate the total time l_k of not-very-good full returns with time less than $c_{n-1}^{-1-2\varepsilon}$ by

$$c_{n-1}^{1/25} c_{n-1}^{-1-2\varepsilon} m_k.$$

Since m_k is big, we can use LS3 to estimate the proportion of branches with not-too-small time, and so we conclude that at most $c_{n-1}^{\varepsilon/11} m_k$ branches are not very good or have time less than $c_{n-1}^{-1+2\varepsilon}$. Thus, β_k can be estimated from below as

$$(1 - c_{n-1}^{\varepsilon/11}) c_{n-1}^{-1+2\varepsilon} m_k.$$

It is easy to see then that $i_k/\beta_k \ll c_{n-1}^{1/100}$, $h_k/\beta_k \ll c_{n-1}^{1/100}$. If ε is small enough, we also have

$$l_k/\beta_k < 2c_{n-1}^{1/25-4\varepsilon} < c_{n-1}^{1/90}.$$

So $(i_k + h_k + l_k)/\beta_k$ is less than $c_{n-1}^{1/100}$. Since $i_k + h_k + l_k + \beta_k = k$ we have $1 - \beta_k/k < (i_k + h_k + l_k)/\beta_k$. □

7.1.3. *Cool landings.* Let us define the set of cool landings $\mathrm{LC}(n_0, n) \subset \Omega$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all $\underline{d} = (j_1, \ldots, j_m)$ in $\mathrm{LE}(n_0, n)$ satisfying

LC1: (Starts very good). $j_i \in \mathrm{VG}(n_0, n)$, $1 \leq i \leq c_{n-1}^{-1/30}$.

LC2: (Short times are sparse in large enough initial segments). For $c_{n-1}^{-\varepsilon/5} \leq k \leq m$

$$\#\{1 \leq i \leq k,\ r_n(j_i) < c_{n-1}^{-1+2\varepsilon}\} < (6 \cdot 2^n) c_{n-1}^{\varepsilon/10} k.$$

LC3: (Not very good moments are sparse in large enough initial segments). For all $c_{n-1}^{-1/30} \leq k \leq m$

$$\#\{1 \leq i \leq k,\ j_i \notin \mathrm{VG}(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{1/60} k.$$

LC4: (Large times are sparse in large enough initial segments). For $c_{n-1}^{-200} \leq k \leq m$

$$\#\{1 \leq i \leq k,\ r_n(j_i) > c_{n-1}^{-1-2\varepsilon}\} < (6 \cdot 2^n) c_{n-1}^{100} k.$$

LC5: (Starts with no large times). $r_n(j_i) < c_{n-1}^{-1-2\varepsilon}$, $1 \leq i \leq e^{c_{n-1}^{-\varepsilon/5}}/2$.

Notice that LC4 and LC5 overlap, since $c_{n-1}^{-200} < e^{c_{n-1}^{-\varepsilon/5}}/2$ as do LC1 and LC3. From this we can conclude that we can control the proportion of large times or not-very-good times in all moments (and not only for large enough initial segments).

Lemma 7.7. *With total probability, for all n_0 sufficiently big and all $n \geq n_0$,*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin \mathrm{LC}(n_0, n) | I_n) < c_{n-1}^{1/100} \tag{7.7}$$

and for all n big enough

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin \mathrm{LC}(n_0, n) | I_n^{\tau_n}) < c_{n-1}^{1/100}. \tag{7.8}$$

Proof. We follow the ideas of the proof of Lemma 7.1. Let us start with estimate (7.7). Notice that by Lemmas 7.3 and 7.2 we can estimate the $\tilde{\gamma}_n$-capacity of the complement of excellent landings by $c_n^{1/3}$. The computations below indicate what is lost going from excellent to cool due to each item of the definition:

(LC1) This is a direct estimate analogous to LS2. By Lemma 7.3, the γ_n-capacity of the complement of very good branches is bounded by $c_{n-1}^{1/20}$, so an upper bound for the $\tilde{\gamma}_n$-capacity of the set of landings which do not start with $c_{n-1}^{-1/30}$ very good branches is given by

$$2^n c_{n-1}^{1/20} c_{n-1}^{-1/30} \ll c_{n-1}^{1/100}.$$

(LC2) This is essentially the same large deviation estimate of LS3. We put $q = c_{n-1}^{\varepsilon/10}$. By estimate (6.23) of Corollary 6.7, the $\tilde{\gamma}_n$-capacity of the set of landings violating LC2 for a specific value of k is bounded by $(1/2)^{(6\cdot 2^n)qk}$, and summing up on k (see also estimate (5.3)) we get the upper bound

$$\sum_{k \geq c_{n-1}^{-\varepsilon/5}} \left(\frac{1}{2}\right)^{(6\cdot 2^n) c_{n-1}^{\varepsilon/10} k} \leq (2^{-n} c_{n-1}^{-\varepsilon/10}) \left(\frac{1}{2}\right)^{(6\cdot 2^n) c_{n-1}^{-\varepsilon/10}} \ll c_{n-1}^{1/100}.$$

(LC3) In analogy to the previous item, we set $q = c_{n-1}^{1/60}$ and using Lemma 7.3 we get an upper bound

$$\sum_{k \geq c_{n-1}^{-1/30}} \left(\frac{1}{2}\right)^{(6\cdot 2^n) c_{n-1}^{1/60} k} \leq (2^{-n} c_{n-1}^{-1/60}) \left(\frac{1}{2}\right)^{(6\cdot 2^n) c_{n-1}^{-1/60}} \ll c_{n-1}^{1/100}.$$

(LC4) As before, we set $q = c_{n-1}^{100}$ and using estimate (6.24) of Corollary 6.7 we get

$$\sum_{k \geq c_{n-1}^{-200}} \left(\frac{1}{2}\right)^{(6\cdot 2^n) c_{n-1}^{100} k} \leq (2^{-n} c_{n-1}^{-100}) \left(\frac{1}{2}\right)^{(6\cdot 2^n) c_{n-1}^{-100}} \ll c_{n-1}^{1/100}.$$

(LC5) This is a direct estimate as in LC1; using estimate (6.24) of Corollary 6.7 we get

$$2^n e^{-c_{n-1}^{\varepsilon/5}} e^{c_{n-1}^{-\varepsilon/5}/2} \ll c_{n-1}^{1/100}.$$

Putting those together, we obtain (7.7). For (7.8), we must be careful to have $\tau_n \in \mathrm{VG}(n_0, n)$ and $r_n(\tau_n) < c_{n-1}^{-1-2\varepsilon}$; otherwise we would have immediate problems due to LC1 and LC5. But we took care of those properties in Lemmas 7.4 and 6.8, and with this observation the estimates are the same as before. $\square$

Transferring the result to the parameter, using PhPa1, we get (using the measure-theoretical argument of Lemma 3.2).

Lemma 7.8. *With total probability, for all n_0 big enough, for all n big enough, $R_n(0) \in C_n^{\underline{d}}$ with $\underline{d} \in \mathrm{LC}(n_0, n)$.*

7.2. *Hyperbolicity.*

7.2.1. *Preliminaries.* For $j \neq 0$, we define

$$\lambda_n(j) = \inf_{x \in I_n^j} \frac{\ln |DR_n(x)|}{r_n(j)}.$$

And $\lambda_n = \inf_{j \neq 0} \lambda_n(j)$. As a consequence of the exponential estimate on distortion for returns (which competes with torrential expansion from the decay of geometry), together with hyperbolicity of f in the complement of I_n^0 we immediately have the following

Lemma 7.9. *With total probability, for all n sufficiently big, $\lambda_n > 0$.*

Proof. By Lemma 2.1, there exists a constant $\tilde{\lambda}_n > 0$ such that each periodic orbit p of f whose orbit is entirely contained in the complement of I_{n+1} must satisfy $\ln |Df^m(p)| > \tilde{\lambda}_n m$, where m is the period of p. On the other hand, each noncentral branch $R_n|_{I_n^j}$ has a fixed point. By Lemma 4.10, $\sup \operatorname{dist}(R_n|_{I_n^j}) \leq 2^n$ and of course $\lim_{j \to \pm\infty} r_n(j) = \infty$, and so we have

$$\liminf_{j \to \pm\infty} \lambda_n(j) \geq \tilde{\lambda}_n.$$

On the other hand, for any $j \neq 0$, $\lambda_n(j) > 0$ by Lemma 4.10, and so $\lambda_n > 0$. ☐

7.2.2. *Good branches.* The "minimum hyperbolicity" $\liminf \lambda_n$ of the parameters we will obtain will in fact be positive, as it follows from one of the properties of Collet-Eckmann parameters (uniform hyperbolicity on periodic orbits, see [NS]), together with our estimates on distortion.

However our strategy is not to show that the minimum hyperbolicity is positive, but that the typical value of $\lambda_n(j)$ stays bounded away from 0 as n grows (and is in fact bigger than $\lambda_{n_0}/2$ for $n > n_0$ big). Since we also have to estimate the hyperbolicity of truncated branches it will be convenient to introduce a new class of branches with good hyperbolic properties.

We define the set of good returns $G(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all j such that

G1: (hyperbolic return).

$$\lambda_n(j) \geq \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2}.$$

G2: (hyperbolicity in truncated return). For $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$

$$\inf_{x \in I_n^j} \frac{\ln |Df^k(x)|}{k} \geq \lambda_{n_0} \frac{1 + 2^{n_0 - n + 1/2}}{2} - c_{n-1}^{2/(n-1)}.$$

Notice that since c_n decreases torrentially, for n sufficiently big G2 implies that if j is good then for $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$,

$$\inf_{x \in I_n^j} \frac{\ln |Df^k(x)|}{k} \geq \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2}.$$

LEMMA 7.10. *With total probability, for all n_0 big enough and for all $n > n_0$, $\mathrm{VG}(n_0, n) \subset G(n_0, n)$.*

Proof. Let us prove that if G1 is satisfied for all $j \in \mathrm{VG}(n_0, n)$, then $\mathrm{VG}(n_0, n+1) \subset G(n_0, n+1)$ (notice that by definition of λ_{n_0} the hypothesis is satisfied for n_0). Fix $j \in \mathrm{VG}(n_0, n+1)$ and define

$$a_k = \inf_{x \in I_{n+1}^j} \frac{\ln |Df^k(x)|}{k}.$$

Consider values of k in the range $c_n^{-3/n} \leq k \leq r_{n+1}(j)$ (notice that if $k = r_{n+1}(j)$ belongs to this range by Remark 7.1).

We let (as usual) $R_n(I_{n+1}^j) \subset C_n^{\underline{d}}$, $\underline{d} = (j_1, \ldots, j_m)$. Notice that by Corollary 6.10, $v_n < c_{n-1}^{-4} < k$. Let us say that j_i was completed before k if $v_n + r_n(j_1) + \cdots + r_n(j_i) \leq k$. We let the queue be defined as

$$q_k = \inf_{x \in C_n^{\underline{d}}} \ln |Df^{k-r} \circ f^r(x)|$$

where $r = v_n + r_n(j_1) + \cdots + r_n(j_{m_k})$ with j_{m_k} the last complete return.

We show first that $|DR_n(x)| > 1$ if $x \in I_{n+1}^j$. Indeed, by Lemma 4.7, $DR_n|_{I_{n+1}} = \phi \circ f$, where ϕ has small distortion, so that by Lemma 4.8,

$$|D\phi(x)| > \frac{|R_n(I_{n+1})|}{2|f(I_{n+1})|} > \frac{2^{-n}|I_n|}{|I_{n+1}|^2},$$

while by VG, $|Df(x)| = |2x| \geq c_n^{1/3}|I_{n+1}|$, so that $|DR_n(x)| > c_n^{-1/2}$.

By Lemma 4.10, any complete return before k produces some expansion; that is, the absolute value of the derivative of such a return is at least 1. On the other hand, $-q_k$ can be bounded from above by $-\ln(c_n c_{n-1}^5)$ by Lemma 4.11. We have

$$-\frac{q_k}{k} \leq \frac{-\ln(c_n c_{n-1}^5)}{c_n^{-3/n}} \ll c_n^{2/n}.$$

Now we use Lemma 7.6 and get

$$\begin{aligned} a_k &> \frac{\beta_k}{k} \frac{\lambda_{n_0}(1 + 2^{n_0 - n})}{2} - \frac{-q_k}{k} \\ &\geq \frac{\lambda_{n_0}(1 + 2^{n_0 - n - 1/2})}{2} - \frac{-q_k}{k} \end{aligned}$$

which gives G2. If $k = r_{n+1}(j)$ then $q_k = 0$, which gives G1. $\square$

7.2.3. *Hyperbolicity in cool landings.*

LEMMA 7.11. *With total probability, if n_0 is sufficiently big, for all n sufficiently big, if $\underline{d} \in \mathrm{LC}(n_0, n)$ then for all $c_{n-1}^{-4/(n-1)} < k \le l_n(\underline{d})$,*

$$\inf_{x \in C_n^{\underline{d}}} \frac{\ln |Df^k(x)|}{k} \ge \frac{\lambda_{n_0}}{2}.$$

Proof. Fix such $\underline{d} \in \mathrm{LC}(n_0, n)$, and let as usual $\underline{d} = (j_1, \dots, j_m)$. Let

$$a_k = \inf_{x \in C_n^{\underline{d}}} \frac{\ln |Df^k(x)|}{k}.$$

Analogously to Lemma 7.6, we define m_k as the number of full returns before k, so that m_k is the biggest integer such that

$$\sum_{i=1}^{m_k} r_n(j_i) \le k.$$

We define

$$\beta_k = \sum_{\substack{1 \le i \le m_k, \\ j_i \in \mathrm{VG}(n_0, n)}} r_n(j_i)$$

(counting the time up to k spent in complete very good returns) and

$$i_k = k - \sum_{i=1}^{m_k} r_n(j_i).$$

(counting the time in the incomplete return at k).

Let us now consider two cases: either all iterates are part of very good returns (that is, all j_i, $1 \le i \le m_k$ are very good and if $i_k > 0$ then j_{m_k+1} is also very good), or some iterates are not part of very good returns.

Case 1 (*All iterates are part of very good returns*). Since full good returns are very hyperbolic by G1 and very good returns are good, we just have to worry about possibly losing hyperbolicity in the incomplete time. To control this, we introduce the queue

$$q_k = \inf_{x \in C_n^{\underline{d}}} \ln |Df^{i_k} \circ f^{k-i_k}(x)|.$$

We have $-q_k \le -\ln(c_{n-1}^{1/3} c_{n-1}^5)$ by Lemma 4.11 and VG, using that the incomplete time is in the middle of a very good branch. Let us split again in two cases: i_k big or otherwise.

Subcase 1a ($i_k \ge c_{n-1}^{-4/(n-1)}$). If the incomplete time is big, we can use G2 to estimate the hyperbolicity of the incomplete time (which is part of a very good return): $q_k/i_k > \lambda_{n_0}/2$. We have

$$a_k > \lambda_{n_0}\frac{(1+2^{n_0-n})}{2}\cdot\frac{k-i_k}{k}+\frac{q_k}{i_k}\cdot\frac{i_k}{k}>\frac{\lambda_{n_0}}{2}.$$

Subcase 1b ($i_k < c_{n-1}^{-4/(n-1)}$). If the incomplete time is not big, we cannot use G2 to estimate q_k, but in this case i_k is much less than k: since $k > c_{n-1}^{-4/(n-1)}$, at least one return was completed ($m_k \geq 1$), and since it must be very good we conclude that $k > c_{n-1}^{-1/2}$ by Remark 7.1, so that for n big

$$a_k > \lambda_{n_0}\frac{(1+2^{n_0-n})}{2}\cdot\frac{k-i_k}{k}-\frac{-q_k}{k}>\frac{\lambda_{n_0}}{2}.$$

Case 2 (*Some iterates are not part of a very good return*). By LC1, $m_k > c_{n-1}^{-1/30}$. Notice that by LC2, if $m_k > c_{n-1}^{-\varepsilon/5}$ then

$$k-i_k > c_{n-1}^{-1+2\varepsilon}m_k/2.$$

So it follows that $m_k > c_{n-1}^{-1/30}$ implies that $k > c_{n-1}^{-35/34}$ (with small ε).

For the incomplete time we have $-q_k \leq -\ln(c_n c_{n-1}^5) < c_{n-1}^{-1-\varepsilon}$, and so $-q_k/k < c_{n-1}^{1/100}$.

Arguing as in Lemma 7.6, we split $k-\beta_k-i_k$ (time of full returns which are not very good) in a part relative to returns with high time (more than $c_{n-1}^{-1-2\varepsilon}$) which we denote h_k and in a part relative to returns with low time (less than $c_{n-1}^{-1-2\varepsilon}$) which we denote l_k. Using LC4 and LC5 to bound the number of returns with high time, and using LS2 to bound their time, we get

$$h_k < c_{n-1}^{-14}(6\cdot 2^n)c_{n-1}^{100}m_k,$$

and using LC1 and LC3 we have

$$l_k < c_{n-1}^{-1-2\varepsilon}(6\cdot 2^n)c_{n-1}^{1/60}m_k < c_{n-1}^{-79/80}m_k,$$

provided ε is small enough.

Since $k > c_{n-1}^{-1+2\varepsilon}m_k/2$ we have

$$\frac{h_k+l_k}{k} < 4c_{n-1}^{1/85},$$

provided ε is small enough.

Now if $i_k < c_{n-1}^{-1-2\varepsilon}$ then $i_k/k < c_{n-1}^{1/80}$ (with ε small), and if $i_k > c_{n-1}^{-1-2\varepsilon}$ then by LC5, $m_k \geq e^{c_{n-1}^{-\varepsilon/5}} > c_{n-1}^{-n}$, so that by LS2, $i_k/k < i_k/m_k < c_{n-1}^{-14}/c_{n-1}^{-n}$. Thus, in both cases $i_k/k < c_{n-1}^{1/80}$.

From our estimates on i_k and on h_k and l_k we have $1-(\beta_k/k) < c_{n-1}^{1/90}$. Now very good returns are very hyperbolic, and full returns (even not very good ones) always give derivative at least 1 from Lemma 4.10. Now, we have the estimate

$$a_k > \lambda_{n_0} \frac{(1+2^{n-n_0})}{2} \cdot \frac{\beta_k}{k} - \frac{-q_k}{k} > \frac{\lambda_{n_0}}{2}$$

for n big. $\square$

8. Main Theorems

8.1. *Proof of Theorem* A. We must show that with total probability, f is Collet-Eckmann. We will use the estimates on hyperbolicity of cool landings to show that if the critical point always falls in a cool landing then there is uniform control of the hyperbolicity along the critical orbit.

Let

$$a_k = \frac{\ln |Df^k(f(0))|}{k}$$

and $e_n = a_{v_n - 1}$.

It is easy to see that if n_0 is big enough such that the conclusions of both Lemmas 7.8 and 7.11 are valid, we obtain for n large enough that

$$e_{n+1} \geq e_n \frac{v_n - 1}{v_{n+1} - 1} + \frac{\lambda_{n_0}}{2} \cdot \frac{v_{n+1} - v_n}{v_{n+1} - 1}$$

and so

$$\liminf_{n \to \infty} e_n \geq \frac{\lambda_{n_0}}{2}. \tag{8.1}$$

Let now $v_n - 1 < k < v_{n+1} - 1$. Define $q_k = \ln |Df^{k-v_n}(f^{v_n}(0))|$.

Assume first that $k \leq v_n + c_{n-1}^{-4/(n-1)}$. From LC1 we know that τ_n is very good; so by LS1, $r_n(\tau_n) > c_{n-1}^{-1/2}$, and so k is in the middle of this branch (that is, $v_n \leq k \leq v_n + r_n(\tau_n) - 1$). Using that $|R_n(0)| > |I_n|/2^n$ (see Lemma 4.8), we get by Lemma 4.11 that

$$-q_k < -\ln(2^{-n} c_{n-1} c_{n-1}^5) < c_{n-2}^{-1-\varepsilon}.$$

Since $v_n > c_{n-1}^{-1+\varepsilon}$ (by Lemma 6.10) we have

$$a_k \geq e_n \frac{v_n - 1}{k} - \frac{-q_k}{k} > \left(1 - \frac{1}{2^n}\right) e_n - \frac{1}{2^n}. \tag{8.2}$$

If $k > v_n + c_{n-1}^{-4/(n-1)}$, using Lemma 7.11 we get

$$a_k \geq e_n \frac{v_n - 1}{k} + \frac{\lambda_{n_0}}{2} \cdot \frac{k - v_n + 1}{k}. \tag{8.3}$$

It is clear that estimates (8.1), (8.2) and (8.3) imply that $\liminf_{k \to \infty} a_k \geq \lambda_{n_0}/2$ and so f is Collet-Eckmann. $\square$

8.2. *Proof of Theorem* B. We must obtain, with total probability, upper and lower (polynomial) bounds for the recurrence of the critical orbit. It will be easier to study first the recurrence with respect to iterates of return branches, and then estimate the total time of those iterates.

8.2.1. *Recurrence in terms of return branches.* The principle of the phase analysis is very simple: for the essentially Markov process generated by iteration of the noncentral branches of R_n, most orbits (in the qs sense) approach 0 at a polynomial rate before falling in I_{n+1}. From this we conclude, using the phase-parameter relation, that with total probability the same holds for the critical orbit.

LEMMA 8.1. *With total probability, for n big enough and for* $1 \leq i \leq c_{n-1}^{-2}$,

$$\frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} < (1+4\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Proof. Notice that due to torrential (and monotonic) decay of c_n, we can estimate $|I_n| = c_{n-1}^{1+\delta_n}$, with δ_n decaying torrentially fast.

From Lemma 4.8, we have

$$\frac{\ln |R_n(0)|}{\ln c_{n-1}} < \frac{\ln(2^{-n}|I_n|)}{\ln c_{n-1}} < 1+4\varepsilon$$

and the result follows for $i = 1$.

For $1 \leq j \leq 2\varepsilon^{-1}$, let $X_j \subset I_n$ be a $c_{n-1}^{(1+2\varepsilon)(1+j\varepsilon)}$ neighborhood of 0. For n big, we can estimate (due to the relation between $|I_n|$ and c_{n-1})

$$\frac{|X_j|}{|I_n|} < \frac{c_{n-1}^{(1+2\varepsilon)(1+j\varepsilon)}}{c_{n-1}^{1+2\varepsilon}} = c_{n-1}^{j\varepsilon(1+2\varepsilon)}$$

(we of course consider X_j as a union of $C^{\underline{d}}_n$, so that its size is near the required size; the precision is high enough for our purposes due to Remark 4.2).

We have to make sure that the critical point does not land in some X_j for $c_{n-1}^{(1-j)\varepsilon} < i \leq c_{n-1}^{-j\varepsilon}$. This requirement can be translated on $R_n(0)$ not belonging to a certain set $Y_j \subset I_n$ such that

$$Y_j = \bigcup_{c_{n-1}^{(1-j)\varepsilon} \leq |\underline{d}| < c_{n-1}^{-j\varepsilon}} (R^{\underline{d}}_n)^{-1}(X_j).$$

By Lemma 4.8, it is clear that no X_j intersects $I_n^{\tau_n}$, so we easily get

$$p_\gamma(Y_j | I_n^{\tau_n}) \leq c_{n-1}^{-j\varepsilon} c_{n-1}^{(1+\varepsilon)j\varepsilon} \leq c_{n-1}^{\varepsilon^2}$$

and

$$p_\gamma\Big(\bigcup_{j=1}^{2\varepsilon^{-1}} Y_j | I_n^{\tau_n}\Big) < 2\varepsilon^{-1} c_{n-1}^{\varepsilon^2}.$$

Applying PhPa1, the probability that for some $1 \leq j \leq 2\varepsilon^{-1}$ and $c_{n-1}^{(1-j)\varepsilon} < i \leq c_{n-1}^{-j\varepsilon}$ we have $|R_n^i(0)| < c_{n-1}^{(1+2\varepsilon)(1+j\varepsilon)}$ is at most $2\varepsilon^{-1}c_{n-1}^{\varepsilon^2}$, which is summable. In particular, with total probability, for j and i as above, we have for n big enough

$$\begin{aligned}\frac{\ln|R_n^i(0)|}{\ln(c_{n-1})} &\leq (1+2\varepsilon)(1+j\varepsilon)\\ &< (1+4\varepsilon)(1+(j-1)\varepsilon) < (1+4\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right). \qquad \square\end{aligned}$$

LEMMA 8.2. *With total probability, for n big enough and for $c_{n-1}^{-1-\varepsilon} < i \leq s_n$,*

$$\frac{\ln|R_n^i(0)|}{\ln(c_{n-1})} < (1+4\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Proof. The argument is the same as for the previous lemma, but the decomposition has a slightly different geometry. Let

$$x_j = c_{n-1}^{(1+2\varepsilon)(1+(1+\varepsilon)^{j+1})},$$

so that

$$\frac{x_j}{|I_n|} < \frac{c_{n-1}^{(1+2\varepsilon)(1+(1+\varepsilon)^{j+1})}}{c_{n-1}^{1+\varepsilon}} < c_{n-1}^{(1+2\varepsilon)(1+\varepsilon)^{j+1}}.$$

Let K be biggest with $x_K > c_n^{1-\varepsilon}$. For $0 \leq j \leq K$, let $X_j \subset I_n$ be an x_j neighborhood of 0 (approximated as union of $C_n^{\underline{d}}$, notice that $x_j > c_n^{1-\varepsilon} \gg |I_{n+1}|$ for $0 \leq j \leq K$, so that the approximation is good enough for our purposes due to Remark 4.2). Let $Y_j \subset I_n$ be such that

$$Y_j = \bigcup_{c_{n-1}^{-(1+\varepsilon)^j} \leq |\underline{d}| < c_{n-1}^{-(1+\varepsilon)^{j+1}}} (R_n^{\underline{d}})^{-1}(X_j).$$

By Lemma 4.8, it is clear that no X_j intersects $I_n^{\tau_n}$, so we easily get

$$p_\gamma(Y_j|I_n^{\tau_n}) \leq c_{n-1}^{-(1+\varepsilon)^{j+1}} c_{n-1}^{(1+\varepsilon)^{j+2}} \leq c_{n-1}^{\varepsilon(1+j\varepsilon)}$$

and

$$p_\gamma(\bigcup_{j=0}^{K} Y_j|I_n^{\tau_n}) < \sum_{j=0}^{\infty} c_{n-1}^{\varepsilon(1+j\varepsilon)} = \frac{c_{n-1}^{\varepsilon}}{1-c_{n-1}^{\varepsilon^2}} < c_{n-1}^{\varepsilon/2}.$$

Applying PhPa1, the probability that for some $0 \leq j \leq K$ and

$$c_{n-1}^{-(1+\varepsilon)^j} < i \leq c^{-(1+\varepsilon)^{j+1}}$$

we have

$$|R_n^i(0)| < c_{n-1}^{(1+2\varepsilon)(1+(1+\varepsilon)^{j+1})}$$

is at most $c_{n-1}^{\varepsilon/2}$, which is summable. In particular, with total probability, for j and i as above, we have

$$\begin{aligned}\frac{\ln|R_n^i(0)|}{\ln(c_{n-1})} &< (1+2\varepsilon)(1+(1+\varepsilon)^{j+1}) \\ &< (1+4\varepsilon)(1+(1+\varepsilon)^j) < (1+4\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).\end{aligned}$$

This covers the range $c_{n-1}^{-1} < i \le c_{n-1}^{-(1+\varepsilon)^{K+1}}$. For $c_{n-1}^{-(1+\varepsilon)^{K+1}} < i \le s_n$, notice that $R_n^i(0) \notin I_{n+1}$, so that

$$\begin{aligned}\frac{\ln|R_n^i(0)|}{\ln c_{n-1}} &< \frac{\ln(|I_{n+1}|/2)}{\ln c_{n-1}} \\ &< \frac{1+4\varepsilon}{1+2\varepsilon}\cdot\frac{\ln c_n^{1-\varepsilon}}{\ln c_{n-1}} \\ &\le \frac{1+4\varepsilon}{1+2\varepsilon}\cdot\frac{\ln x_{K+1}}{\ln c_{n-1}} \qquad \text{(by definition of } K) \\ &\le (1+4\varepsilon)(1+(1+\varepsilon)^{K+1}) \\ &\le (1+4\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).\end{aligned}$$

□

Both cases are summarized below:

Corollary 8.3. *With total probability, for n big enough and for $1 \le i \le s_n$,*

$$\frac{\ln|R_n^i(0)|}{\ln(c_{n-1})} < (1+4\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

8.2.2. *Total time of full returns.* We must now relate the return times in terms of R_n to the return times in terms of f.

For $1 \le i \le s_n$, let k_i be such that $R_n^i(0) = f^{k_i}(0)$.

Lemma 8.4. *With total probability, for n big enough and for $c_{n-1}^{-\varepsilon} < i \le s_n$,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > (1-3\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Proof. By Lemma 7.8, $R_n(0)$ belongs to a cool landing, so that using LC2 (which allows us to estimate the average of return times over a large initial segment of cool landings) we get

$$\frac{k_i}{i-1} > c_{n-1}^{-1+3\varepsilon}.$$

This immediately gives

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > (1-3\varepsilon) + \frac{\ln(i-1)}{\ln(c_{n-1}^{-1})} > (1-3\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right). \qquad \square$$

Using that $v_n > c_{n-1}^{-1+\varepsilon}$ (from Corollary 6.10) and that $k_i \geq v_n$ we get for $1 \leq i \leq c_{n-1}^{-\varepsilon}$,

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} \geq \frac{\ln(v_n)}{\ln(c_{n-1}^{-1})} > \frac{\ln(c_{n-1}^{-1+\varepsilon})}{\ln(c_{n-1}^{-1})} > (1-3\varepsilon)(1+\varepsilon) \geq (1-3\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Together with Lemma 8.4, this gives

Corollary 8.5. *With total probability, for n big enough and for $1 \leq i \leq s_n$,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > (1-3\varepsilon)\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

8.2.3. *Upper and lower bounds.* Notice that $|R_n(0)| = |f^{v_n}(0)| \leq c_{n-1}$, so using Lemma 6.10 we get

$$\limsup_{n\to\infty} \frac{-\ln|f^n(0)|}{\ln(n)} \geq \limsup_{n\to\infty} \frac{-\ln|f^{v_n}(0)|}{\ln(v_n)} \geq \limsup_{n\to\infty} \frac{-\ln(c_{n-1})}{\ln(v_n)} \geq 1.$$

Let now $v_n \leq k < v_{n+1}$. If $|f^k(0)| < k^{-1-10\varepsilon}$ then Lemma 6.10 implies that $f^k(0) \in I_n$ and so $k = k_i$ for some i. It follows from Corollaries 8.3 and 8.5 that

$$|f^{k_i}(0)| > k_i^{-1-10\varepsilon}.$$

Varying ε we get

$$\limsup_{n\to\infty} \frac{-\ln|f^n(0)|}{\ln(n)} \leq 1.$$

Appendix: Sketch of the proof of the phase-parameter relation

The proof of the phase-parameter relation uses ideas from complex analysis. We will provide a sketch of the proof assuming familiarity with the work [L3]. For a more general result (with all details fully worked out), see [AM3].

Given a simple map f, one can define (as in §3 of [L3]) a sequence of *holomorphic families of generalized quadratic-like maps* R_i, $i \geq 1$, related by *generalized renormalization*. To fix notation, the parameter space of those families will be denoted $\Lambda_i[f]$, so that for each $g \in \Lambda_i[f]$ the family defines a *generalized quadratic-like map* $R_i[g] : U_i^j[g] \to U_i[g]$. Moreover, the family R_i

is *equipped* (with a holomorphic motion h_i of the U_i^j and U_i) and *proper*. The following properties of this sequence of families will be important for us:

(1) $\Lambda_i[f] \cap \mathbb{R} = J_i[f]$, and for $g \in J_i[f]$, $U_i[g] \cap \mathbb{R} = I_i[g]$ and $U_i^j[g] \cap \mathbb{R} = I_i^j[g]$.

(2) For $g \in J_i[f]$, the map $R_i[g] : \cup U_i^j[g] \to U_i[g]$ is an extension of the *real first return map* $R_i[g] : I_i^j[g] \to I_i[g]$ defined before. For $j \neq 0$ (respectively for $j = 0$), $R_i[g] : U_i^j[g] \to U_i[g]$ is a homeomorphism (respectively a double covering).

(3) The modulus of $U_i[f] \setminus \overline{U_{i+1}[f]}$ grows at least linearly in i ([GS2], [L2]).

(4) The modulus of $\Lambda_i[f] \setminus \overline{\Lambda_{i+1}[f]}$ grows at least linearly in i ([L3]).

Define τ_i as before by $R_i[f](0) \in I_i^{\tau_i}[f]$. Let $\Lambda_i^j[f]$ denote the set of $g \in \Lambda_i[f]$ such that $R_i[g](0) \in U_i^j[g]$. We have:

(5) $\Lambda_i^j[f] \cap \mathbb{R} = J_i^j[f]$ and in particular $\Lambda_i^{\tau_i}[f] \cap \mathbb{R} = J_i^{\tau_i}[f]$.

By Lemma 4.8 of [L3], the holomorphic motion h_i of U_i, U_i^j (corresponding to R_i) has uniformly bounded dilation (independently of i) *when restricted to* $U_i \setminus \overline{U_i^0}$. Item (3) above implies that for i big, there is an annulus of big modulus (linear growth in i) contained in $U_i[f] \setminus \overline{(U_i^0[f] \cup U_i^{\tau_i}[f])}$ and going around $U_i^{\tau_i}[f]$. By transverse quasiconformality of holomorphic motions (Corollary 2.1 of [L3]), and the λ-Lemma of [MSS], this estimate can be transferred to the parameter space, and so we get:

(6) The modulus of $\Lambda_i[f] \setminus \overline{\Lambda_i^{\tau_i}[f]}$ grows at least linearly in i.

For each $g \in \Lambda_i[f]$, denote by $L_i[g]$ the first landing map to $U_i^0[g]$ obtained by iteration of noncentral branches of $R_i[g]$. By item (2) above, we have:

(7) For $g \in J_i[f]$, the domain of $L_i[g]$ is a union $\cup_{\underline{d} \in \Omega} W_i^{\underline{d}}[g]$ of disks such that $W_i^{\underline{d}}[g] \cap \mathbb{R} = C_i^{\underline{d}}[g]$ and $L_i[g]$ extends the *real first landing map* $L_i[g] : \cup_{\underline{d} \in \Omega} C_i^{\underline{d}}[g] \to I_i^0[g]$ defined before.

The family L_i is also equipped with a holomorphic motion $\hat{h}_i$ of U_i and the $W_i^{\underline{d}}$ (see §3.5 of [L3]). Define $\Gamma_i^{\underline{d}}[f]$ as the set of $g \in \Lambda_i[f]$ such that $R_i[g](0) \in W_i^{\underline{d}}[g]$. The λ-Lemma and (6) imply:

(8) For $g \in J_i^{\tau_i}[f]$, there exists a real-symmetric qc map of $\mathbb{C}$, whose dilation goes to 1 as i grows, taking $U_i[f]$ to $U_i[g]$, and taking any $W_i^{\underline{d}}[f]$ to $W_i^{\underline{d}}[g]$.

Items (7) and (8) prove PhPh1 in the phase-parameter relation.

Item (6) and transverse quasiconformality of holomorphic motions imply:

(9) There is a real-symmetric qc map of $\mathbb{C}$, whose dilation goes to 1 as i grows, taking $U_i^{\tau_i}[f]$ to $\Lambda_i^{\tau_i}[f]$, and taking any $W_i^{\underline{d}}[f]$ *contained in* $U_i^{\tau_i}[f]$ to $\Gamma_i^{\underline{d}}[f]$.

Items (5), (7) and (9) prove PhPa1 in the phase-parameter relation.

Notice that for any $g \in \Lambda_i[f]$, and for every $\underline{d}$ the map $L_i[g] : W_i^{\underline{d}}[g] \to U_i^0[g]$ extends to a holomorphic diffeomorphism $R_i^{\underline{d}}[g] : U_i^{\underline{d}}[g] \to U_i[g]$. It is easy to see that $\hat{h}_i$ (as defined in §3.5 of [L3]) is also a holomorphic motion of the $U_i^{\underline{d}}$.

Define $\tilde{\Lambda}_{i+1}[f]$ as the set of g such that $R_i[g](0) \in U_i^{\underline{d}_i}[g]$, where $\underline{d}_i$ is chosen such that $R_i[f](0) \in C_i^{\underline{d}_i}[f]$. It follows that $\Lambda_{i+1}[f] = \Gamma_i^{\underline{d}_i}[f] \subset \tilde{\Lambda}_{i+1}[f] \subset \Lambda_i^{\tau_i}[f]$. By (3), the modulus of $U_i^{\underline{d}_i}[f] \setminus \overline{W_i^{\underline{d}_i}[f]}$ grows at least linearly in i. By (6) and transverse quasiconformality of holomorphic motions, this implies:

(10) The modulus of $\tilde{\Lambda}_i[f] \setminus \overline{\Lambda_i[f]}$ grows at least linearly in i.

For $g \in \Lambda_i[f]$, the map $R_i[g] : U_i^0[g] \to U_i[g]$ extends to a bigger domain $\tilde{U}_{i+1}[g] = (R_{i-1}[g]|_{U_i[g]})^{-1}(U_{i-1}^{\underline{d}_{i-1}}[g])$, as a double covering map onto $U_{i-1}[g]$ (notice that $U_i^0[g] \subset \tilde{U}_{i+1}[g] \subset U_i[g]$). It follows:

(11) If $g \in J_i[f]$ then $\tilde{U}_{i+1}[g] \cap \mathbb{R} = \tilde{I}_{i+1}[g]$.

The holomorphic motion $\hat{h}_{i-1}$ (corresponding to L_{i-1}) naturally lifts to a holomorphic motion $\tilde{h}_i$ of U_i, $\tilde{U}_{i+1}$ and all U_i^j *not contained in* $\tilde{U}_{i+1}$, which is defined (in principle) over $\Lambda_i[f]$, but extends to a holomorphic motion defined over $\tilde{\Lambda}_i[f]$.

Item (10) and yet another application of the λ-Lemma imply:

(12) For $g \in J_i[f]$, there exists a real-symmetric qc map of $\mathbb{C}$, whose dilation goes to 1 as i grows, taking $U_i[f]$ to $U_i[g]$, and taking any $U_i^j[f]$ *not contained in* $\tilde{U}_{i+1}[f]$ to $U_i^j[g]$.

Items (2), (11) and (12) prove PhPh2 in the phase-parameter relation.

Item (10) and transverse quasiconformality of holomorphic motions imply:

(13) There is a real-symmetric qc map of $\mathbb{C}$, whose dilation goes to 1 as i grows, taking $U_i[f]$ to $\Lambda_i[f]$, and taking any $U_i^j[f]$ *not contained in* $\tilde{U}_{i+1}[f]$ to $\Lambda_i^j[f]$.

Items (2), (11) and (13) prove PhPa2 in the phase-parameter relation. All items of the phase-parameter relation are proved.

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