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Quasi-isometry invariance of group splittings

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Abstract

We show that a finitely presented one-ended group which is not commensurable to a surface group splits over a two-ended group if and only if its Cayley graph is separated by a quasi-line. This shows in particular that splittings over two-ended groups are preserved by quasi-isometries.

0. Introduction

Stallings in [St1], [St2] shows that a finitely generated group splits over a finite group if and only if its Cayley graph has more than one end. This result shows that the property of having a decomposition over a finite group for a finitely generated group G admits a geometric characterization. In particular it is a property invariant by quasi-isometries.

In this paper we show that one can characterize geometrically the property of admitting a splitting over a virtually infinite cyclic group for finitely presented groups. So this property is also invariant by quasi-isometries.

The structure of group splittings over infinite cyclic groups was understood only recently by Rips and Sela ([R-S]). They developed a 'JSJ-decomposition theory' analog to the JSJ-theory for three manifolds that applies to all finitely presented groups. This structure theory underlies and inspires many of the geometric arguments in this paper. A different approach to the JSJ-theory for finitely presented groups has been given by Dunwoody and Sageev in [D-Sa]. Their approach has the advantage of applying also to splittings over $\mathbb{Z}^n$ or even, more generally, over 'slender groups'.

Bowditch in a series of papers [Bo 1], [Bo 2], [Bo 3] showed that a one-ended hyperbolic group that is not a 'triangle group' splits over a two-ended group if and only if its Gromov boundary has local cut points. This characterization implies that the property of admitting such a splitting is invariant under quasi-isometries for hyperbolic groups. Swarup ([Sw]) and Levitt ([L]) contributed to the completion of Bowditch's program which led also to the solution of the cut point conjecture for hyperbolic groups.

To state the main theorem of this paper we need some definitions: If Y is a path-connected subset of a geodesic metric space (X,d) then one can define a metric on Y, d_Y, by defining the distance of two points to be the infimum of the lengths of the paths joining them that lie in Y. A *quasi-line* $L \subset X$ is a path-connected set such that (L,d_L) is quasi-isometric to $\mathbb{R}$ and such that for any two sequences $(x_n),(y_n) \in L$ if $d_L(x_n,y_n) \to \infty$ then $d(x_n,y_n) \to \infty$.

We say that a quasi-line L *separates* X if $X - L$ has at least two components that are not contained in any finite neighborhood of L.

With this notation we show the following:

Theorem 1. *Let G be a one-ended, finitely presented group that is not commensurable to a surface group. Then G splits over a two-ended group if and only if the Cayley graph of G is separated by a quasi-line.*

This easily implies that admitting a splitting over a two-ended group is a property invariant by quasi-isometries. More precisely we have the following:

Corollary. *Let G_1 be a one-ended, finitely presented group that is not commensurable to a surface group. If G_1 splits over a two-ended group and G_2 is quasi-isometric to G_1 then G_2 splits also over a two-ended group.*

We note that a different generalization of Stalling's theorem was obtained by Dunwoody and Swenson in [D-Sw]. They show that if G is a one-ended group, which is not virtually a surface group, then it splits over a two-ended group if and only if it contains an infinite cyclic subgroup of 'codimension 1'. We recall that a subgroup J of G is of codimension 1 if the quotient of the Cayley graph of G by the action of J has more than one end. The disadvantage of this characterization is that it is not 'geometric'; in particular our corollary does not follow from it. On the other hand [D-Sw] contains a more general result that applies to splittings over $\mathbb{Z}^n$. Our results build on [D-Sw] (in fact we only need Proposition 3.1 of this paper dealing with the 'noncrossing' case).

The idea of the proof of Theorem 1 can be grasped more easily if we consider the special case of $G = \mathbb{Z}^3 \star_{\mathbb{Z}} \mathbb{Z}^3$. One can visualize the Cayley graph of G as a tree in which the vertices are blown to copies of $\mathbb{Z}^3$ and two adjacent vertices (i.e. $\mathbb{Z}^3$'s) are identified along a copy of $\mathbb{Z}$. Now the copies of $\mathbb{Z}^3$ are 'fat' in the sense that they cannot be separated by a 'quasi-line'. The Cayley graph of G on the other hand is not fat as it is separated by the cyclic groups corresponding to the edge of the splitting. This is a pattern that stays invariant under quasi-isometry: A geodesic metric space quasi-isometric to the Cayley graph of G is also like a tree; the vertices of the tree are 'fat' chunks of space that cannot be separated by 'quasi-lines' and two adjacent such 'fat' pieces are glued along a 'quasi-line'.

The proof of the general case is along the same lines but one has to take account of the 'hanging-orbifold' vertices of the JSJ decomposition of G.

The main technical problem is to show that when the Cayley graph of a group is separated by a quasi-line then 'fat' pieces do indeed exist. To be more precise one has to show that if any two points that are sufficiently far apart are separated by a quasi-line then the group is commensurable to a surface group. For this it suffices to show that the Cayley graph of G is quasi-isometric to a plane. So what we are after is an up to quasi-isometry characterization of planes.

The first such characterization was given by Mess in his work on the Seifert conjecture ([Me]). There have been some more such characterizations obtained recently by Bowditch ([Bo 4]), Kleiner ([Kl]) and Maillot ([Ma]).

The characterization that we need for this work is quite different from the previous ones. 'Large scale' geometric problems are often similar to topological problems. Our problem is similar to the following topological characterization of the plane:

Let X be a one-ended, simply connected geodesic metric space such that any two points on X are separated by a line. Then X is homeomorphic to a plane.

We outline a proof of this in the appendix. It is based on the classic characterization of the sphere given by Bing ([Bi]).

The proof of the large scale analog to this runs along the same line but is more fuzzy as a quasi-prefix has to be added to the definitions and arguments. Although we could carry out the analogy throughout the proof, we simplify the argument in the end using the homogeneity of the Cayley graph. We use in particular Varopoulos' inequality to conclude in the nonhyperbolic case and the Tukia, Gabai, Casson-Jungreis theorem on convergence groups ([T], [Ga], [C-J]) to deal with the hyperbolic case.

The topological characterization of the plane presented in the appendix is quite crucial for understanding the quasi-isometric characterization of planar groups used here. We advise the reader to understand the topological argument of the appendix before reading its 'large scale' generalization (Sections 1–3 of this paper). A principle underlying this work is that many topological results have, when reformulated appropriately, large scale analogs. Both the proofs and the statements of these analogs can be involved but this is more due to the difficulty of 'translation' to large scale than genuine mathematical difficulty. We hope that the statement and proof of Proposition 2.1 offers a good introduction to 'translating' from topology to large scale.

We explain now how this paper is organized: In Section 2 we show (Prop. 2.1) that if a quasi-line L separates a Cayley graph in three pieces then points on L cannot be separated by quasi-lines. We state below Proposition 2.1 (we state it in fact in a slightly different, but equivalent, way in Section 2):

Proposition 2.1. *Let X be a locally finite simply connected complex and let L be a quasi-line separating X, such that $X - L$ has at least three distinct essential connected components X_1, X_2, X_3. If L_1 is another quasi-line in X then L is contained in a finite neighborhood of a single component of $X - L_1$.*

We call a component X_i essential if $X_i \cup L$ is one-ended. We remark that the proposition above is similar to the following topological fact: Let X be the space obtained by gluing three half-planes along their boundary line. Then points on the common boundary line of the three half-planes cannot be separated by any line in X. We will actually need a stronger and somewhat less obvious form of this that is proved in Lemma A.1 of the appendix. The proof of Proposition 2.1 is a 'large scale' version of the proof of Lemma A.1.

Proposition 2.1 is used in Section 3 to give a new 'quasi-isometric' characterization of planar groups:

Theorem. *Let G be a one-ended finitely presented group and let $X = X_G$ be a Cayley complex of G. Suppose that there is a quasi-line L such that for any $K > 0$ there is an $M > 0$ such that any two points x, y of X with $d(x, y) > M$ are K-separated by some translate of L, gL $(g \in G)$. Then G is commensurable to a fundamental group of a surface.*

The theorem above is in fact slightly weaker than Theorem 3.1 that we prove in Section 3. The proof of this is a 'large scale' version of the proof of the main theorem of the appendix:

Theorem A. *Let X be a locally compact, geodesic metric space and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\lim_{x\to 0} f(x) = 0$. If X satisfies the following three conditions then it is homeomorphic to the plane.*

1) *X is one-ended.*

2) *X is simply connected.*

3) *For any two points $a, b \in X$ there is an f-line separating them.*

We refer to the appendix for the definition of f-lines which is somewhat technical. To make sense of the theorem above think of f-lines as proper lines, i.e. homeomorphic images of $\mathbb{R}$ in X.

It turns out that to carry out our proof we need a stronger version of Theorem 3.1 proved in Section 4. It says roughly that if G is not virtually planar then its Cayley graph has an unbounded connected subset S such that no two points on S can be separated by a quasi-line (Theorem 4.1). We call such subsets solid. In the example $G = \mathbb{Z}^3 \star_{\mathbb{Z}} \mathbb{Z}^3$ this subset corresponds to a $\mathbb{Z}^3$-subgroup.

The proof of Theorem 4.1 is based on the homogeneity of the Cayley graph of G. The characterization theorem of virtual surface groups given in Section 4 allows us to pass from large scale geometry to splittings. The idea is that maximal unbounded solid sets are at finite Hausdorff distance from vertex groups of the JSJ-decomposition of G. This is easier to show when these sets are 'big', i.e. they are not themselves quasi-lines. This is the case for example if $G = \mathbb{Z}^3 \star_{\mathbb{Z}} \mathbb{Z}^3$. If on the other hand G is, say, a Baumslag-Solitar group then all solid sets in its Cayley graph are quasi-lines.

In Section 5 we show (Proposition 5.3) that solid subsets correspond to subgroups when they are not quasi-isometric to quasi-lines. In fact they are vertex groups for the Bass-Serre tree corresponding to a splitting of G over a two-ended group. We prove then Theorem 1, in case there are solid subsets of X which are not quasi-lines, by applying [D-Sw].

In Section 6 we deal with the 'exceptional' case in which all solid subsets are quasi-lines. This is split in several cases. We show depending on the case either directly that G splits over a two ended subgroup by applying again [D-Sw], or that G admits a free action on an $\mathbb{R}$-tree, in which case we conclude by Rips' theory ([B-F]). This completes the proof of Theorem 1.

We note that Section 6 is essentially self-contained. It does not require the technical results of the appendix and their large scale analogs. It could be read directly after the preliminaries and the definition of solid sets in Section 4 as it offers a good illustration of how one can derive splitting results from a mild geometric assumption which is valid in many cases (for example this assumption holds for Baumslag-Solitar groups).

In Section 7 we show that JSJ decompositions are invariant under quasi-isometries. More precisely we have the following:

THEOREM 7.1. *Let G_1, G_2 be one-ended finitely presented groups, let Γ_1, Γ_2 be their respective* JSJ-*decompositions and let X_1, X_2 be the Cayley graphs of G_1, G_2.*

Suppose that there is a quasi-isometry $f : G_1 \to G_2$. Then there is a constant $C > 0$ such that if A is a subgroup of G_1 conjugate to a vertex group, an orbifold hanging vertex group or an edge group of the graph of groups Γ_1, then $f(A)$ contains in its C-neighborhood (*and it is contained in the C-neighborhood of*) *respectively a subgroup of G_2 conjugate to a vertex group, an orbifold hanging vertex group or an edge group of the graph of groups Γ_2.*

It is an interesting question whether Theorem 1 is true for finitely generated groups in general. The existence of a characterization like the one in Theorem 1 was posed as a question by Gromov in the 1996 Group Theory Conference in Canberra.

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1. Preliminaries

A metric space X is called a geodesic metric space if for any pair of points x, y in X there is a path p joining x, y such that length$(p) = d(x, y)$. We call such a path a geodesic. A geodesic triangle in a geodesic metric space X consists of three geodesics a, b, c whose endpoints match. A geodesic metric space X is called (δ)-hyperbolic if there is a $\delta \geq 0$ such that for all triangles a, b, c in X any point on one side is in the δ-neighborhood of the two other sides. If G is a finitely generated group then its Cayley graph can be made a geodesic metric space by giving to each edge length 1. A finitely generated group is called (Gromov) hyperbolic if its Cayley graph is a (δ)-hyperbolic geodesic metric space. A path $\alpha : [0, l] \to X$ is called a (K, L)-*quasigeodesic* if there are $K \geq 1, L \geq 0$ such that length$(\alpha|_{[t,s]}) \leq Kd(\alpha(t), \alpha(s)) + L$ for all t, s in $[0, l]$. In what follows we will always assume paths to be parametrized with respect to arc length. A (not necessarily continuous) map $f : X \to Y$ is called a (K, L) *quasi-isometry* if every point of Y is in the L-neighborhood of the image of f and for all $x, y \in X$

$$\frac{1}{K}d(x, y) - L \leq d(f(x), f(y)) \leq Kd(x, y) + L.$$

Definition 1.1. Let X, Y be metric spaces. A map $f : X \to Y$ is called *uniformly proper* if for every $M > 0$ there is an $N > 0$ such that for all $A \subset Y$,

$$\operatorname{diam}(A) < M \Rightarrow \operatorname{diam}(f^{-1}(A)) < N.$$

We remark that this notion is due to Gromov. In [G2] embeddings that are uniformly proper maps are called uniform embeddings. It is easy to see that the inclusion map of a finitely generated group H in a finitely generated group G is a uniformly proper map (where G and H are given the word *metric* corresponding to some choice of system of generators for each).

In what follows we consider $\mathbb{R}$ as a metric space.

Definition 1.2. Let X be a metric space. Let $L : \mathbb{R} \to X$ be a one-to-one, continuous map. We suppose that L is parametrized with respect to arc length (i.e. length$(L[x, y]) = d(x, y)$ for all x, y). We then call L a *line* if it is *uniformly proper.*

There is a distortion function associated to L, $D_L : \mathbb{R}^+ \to \mathbb{R}^+$ defined as follows:

$$D_L(t) = \sup\{\operatorname{diam}(L^{-1}(A)), \text{ where diam } (A) \leq t\}.$$

We often identify L with its image $L(\mathbb{R})$ and write $L \subset X$. If $a = L(a'), b = L(b')$ are points in L, we denote by $[a, b]$ the interval between a, b in L (so $[a, b] = L([a', b']))$, and by $|b - a|$ the length of this interval. We write $a < b$ if $a' < b'$. If $t \in \mathbb{R}$ we denote by $a - t$ the point $L(a' - t)$.

Definition 1.3. Let X be a metric space. We call $L \subset X$ a *quasi-line* if L is path connected and if there is a line $L' \subset L$ and $N > 0$ such that every point in L can be joined to L' by a path lying in L of length at most N.

One can also define quasi-lines as follows: Let $L \subset X$ be a path connected subset of X. We consider L as a metric space by defining the distance of two points in L to be the length of the shortest path in L joining them (or the infimum of the lengths if there is no shortest path). Then L is a quasi-line if:

i) L is quasi-isometric to $\mathbb{R}$.

ii) L is uniformly properly embedded in X.

We say that $L \subset X$ is an (f, N)-quasi-line, where f is a proper increasing function, $f : \mathbb{R}^+ \to \mathbb{R}^+$, if L lies in the N-neighborhood of a line L' and $D_{L'}(t) \leq f(t)$ for all $t > 0$.

Suppose that the quasi-line L lies in the N-neighborhood of a line L'. We define then a map $a \in L \to a' \in L'$ where $d(a, a') \leq N$. Clearly there are many possible choices for this map; we choose one such map arbitrarily. If $a, b \in L$ we define the interval between a, b in L as follows:

$$[a, b]_L = \{x \in L : d(x, [a', b']) \leq N\}.$$

Clearly this depends on the map $a \to a'$. It is convenient to talk about the 'length' of the intervals of L. We define $\text{length}([a, b]_L) = \text{length}([a', b'])$.

We similarly define a *partial order* on L by $a < b$ if and only if $a' < b'$. If $t \in \mathbb{R}$ and $a \in L$ then $a + t$ is by definition the point $a' + t \in L' \subset L$. In what follows when we write that a quasi-line L is in the N-neighborhood of a line L' we will tacitly imply that a map $a \to a'$ is also given.

We will use throughout the notation for lines corresponding to quasi-lines, so if L is an (f, N)- quasi-line we will denote by L' the line corresponding to L (see Def. 1.3).

The following definition is abusive but useful:

Definition 1.4. Let X be a metric space and let L be a quasi-line in X. We call a connected component of $X - L$, Y, *essential* if $Y \cup L$ is one-ended. We say that a quasi-line L *separates* X, if $X - L$ has at least two essential connected components and there is an $M > 0$ such that every nonessential component of $X - L$ is contained in the M-neighborhood of L.

The following proposition shows that our definition is equivalent to a weaker and more natural notion of separation.

PROPOSITION 1.4.1. *Let X be a Cayley graph of a finitely presented one-ended group G and let L be an (f, N)-quasi-line such that for every $n > 0$ there are $x, y \in X$ such that $d(x, L) > n, d(y, L) > n$ and x, y lie in different components of $X - L$. Then there is an (f, N)-quasi-line L_1 that separates X.*

Proof. We show first that there is an (f,N)-quasi-line L_0 such that $X-L_0$ has at least two essential components. For any $r>0$ sufficiently big and for any $t\in L$ there is a path in $X-L$ joining the two infinite components of $L-B_t(r)$. Without loss of generality we can assume that this path (except its endpoints) is contained in a single component of $X-L$. We call this path $p(t,r)$.

Since X is locally finite and G is finitely presented we can assume that there are a $t\in L$ and an $r_0>0$ such that $p(t,r)$ lies for every $r>r_0$ in the same component of $X-L$, say C. Since G is one-ended C is clearly essential. By our hypothesis we have that there is a sequence y_n such that $d(y_n,L)>n$ and $y_n\notin C$. Let q_n be a geodesic joining y_n to L with endpoint $t_n\in L$ and such that $\text{length}(q_n)=d(y_n,L)$. Let us denote by T_n the union $L\cup p_n$. We then pick $g_n\in G$ such that $g_nt_n=t$ and next consider the sequence g_nT_n. It is clear that there is a subsequence of g_n, denoted for convenience also by g_n, so that g_nT_n converges on compact sets to a union $L_0\cup p$ where L_0 is a quasi-line and p is an infinite half geodesic lying in the same component of $X-L_0$. By passing if necessary to a subsequence we can ensure that $X-L_0$ has at least one essential component disjoint from p.

Indeed, note that there is a sequence $r_n\in\mathbb{N}$, $r_n\to\infty$, such that for any $x\in L$ there are simple paths $p(x,n)$ with the following properties (see Fig. 1):

1. $p(x,n)$ is contained in $\bar{C}$ and $p(x,n)$ joins the two unbounded components of $L-B_x(r_n)$.
2. $p(x,n)\cap B_x(r_n)=\emptyset$ and $p(x,n)\subset B_x(r_{n+1})$.
3. There is a path $q(x,n)$ contained in $B_x(r_{n+2})\cap C$ joining $p(x,n)$ to $p(x,n+1)$.

By passing to a subsequence we can ensure that for every $k>0$ the following holds: For every $n>k$,

$$g_n(p(t_n,n)\cup q(t_n,n))=g_k(p(t_k,k)\cup q(t_k,k)).$$

This clearly implies that $X-L_0$ has at least one essential component disjoint from p.

Let C_1 be the component of $X-L_0$ containing p. Suppose that $C_1\cup L_0$ is not one-ended. Then there is a compact K such that $(C_1\cup L_0)-K$ is two-ended and there is an infinite component of L_0-K, say L_0^+, such that $C_1\cup L_0^+$ is one-ended. We can then pick $x_n\in L_0^+$, $x_n\to\infty$ and $h_n\in G$ such that $h_nx_n=t$. By passing, if necessary, to a subsequence we can assume that h_nL_0 converges on compact sets to a quasi-line, denoted, to simplify notation, still by L_0. As before we can ensure that $X-L_0$ has at least two essential components.

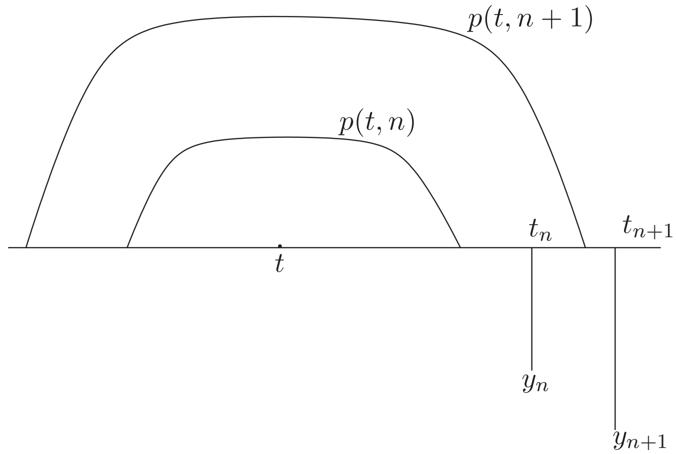


Figure 1

We have shown therefore that there is a quasi-line L_0 such that $X - L_0$ has at least two essential components. Note also that if L is an (f, N)-quasi-line L_0 is also an (f, N)-quasi-line.

Showing that there is a quasi-line satisfying the conclusion of the proposition is proved in the same way: Suppose that there is a sequence $z_n \in X$ such that $d(z_n, L_0) > n$ for all $n \in \mathbb{N}$ and such that the z_n do not belong to any essential component of $X - L_0$. We then pick geodesics q_n joining z_n to L with length$(q_n) = d(z_n, L_0)$ and we pick $k_n \in G$ such that $k_n z_n = e$ (where e is a fixed vertex). We show as above that there is a subsequence of $k_n L_0$ converging on compact sets to a quasi-line L_1 such that $X - L_1$ has at least three essential components.

We continue in the same way to produce new quasi-lines. It is clear that this procedure terminates and produces a quasi-line, which we call, as in the conclusion of the lemma, L_1, such that if $z_n \in X$ satisfies that $d(z_n, L_0) \to \infty$ then almost all z_n lie in essential components of $X - L_1$.

We remark that the procedure terminates because given f, N there is an $M > 0$ such that for any (f, N)-quasi-line L, $X - L$ has less than M essential components. $\square$

Remark 1.4.2. We can show in the same way the following slightly stronger result: Let X be a Cayley graph of a finitely presented one-ended group G and let L_n be a sequence of (f, N)-quasi-lines such that for every $n > 0$ there are $x, y \in X$ such that $d(x, L_n) > n, d(y, L_n) > n$ and x, y lie in different components of $X - L_n$. Then there is a quasi-line L that separates X.

It is clear that a finite neighborhood of a quasi-line is itself a quasi-line. The next proposition strengthens Proposition 1.4.1 to neighborhoods of quasi-lines.

Proposition 1.4.3. *Let X be a Cayley graph of a finitely presented one-ended group G. If an (f,N)-quasi-line L separates X then there is an (f,N)-quasi-line L_0 such that for every $r>0$, $N_r(L_0)$ separates X.*

Proof. We define a sequence of (f,N)-quasi-lines L_n $(n>0)$ such that $N_k(L_n)$ separates X for all $k\le n$. If $N_1(L)$ separates X we define $L_1=L$. Otherwise we show as in Proposition 1.4.1 that there is an (f,N)-quasi-line L_1 such that $N_1(L_1)$ separates X. We continue inductively: if $N_{k+1}(L_k)$ separates X we define $L_{k+1}=L_k$ otherwise we modify L_k as in Proposition 1.4.1 to obtain L_{k+1}. We can assume that all L_k contain the identity vertex e.

We note that by their construction the L_k satisfy the following:

For every $r>0$ there is an $M>0$ such that for all $k\ge r$ every nonessential component of $X-N_r(L_k)$ is contained in $N_M(L_k)$.

By passing to a subsequence we can assume that $B_e(k)\cap L_n$ does not depend on n for $n\ge k$. We define L_0 by $x\in L_0$ if $x\in B_e(k)\cap L_k$. Clearly L_0 has the property required. □

The following proposition shows that the essential components of $X-L$ have a property that one can consider as a 'large scale' version of local connectedness.

Proposition 1.4.4. *Let X be a Cayley graph of a finitely presented one-ended group G. If an (f,N)-quasi-line L separates X then there is an (f,N)-quasi-line L_0 which separates X and has the following property*:

There is an $r_0>0$ such that for each $r>r_0$ there is an $R>r$ such that if $d(x,L_0)=r=d(y,L_0)$, $d(x,y)<f(3r)$ and x,y lie in the same essential component of $X-L_0$, then x,y can be joined by a path of length less than R which does not meet L_0.

Proof. We will show this by contradiction. Let L_0 be a separating (f,N)-quasi-line which satisfies the following 2 properties:

1. The number of essential components of $X-L_0$ is the maximum possible.

2. If L_1 is a separating (f,N)-quasi-line satisfying property 1 then $\sup\{d(x,L_1)\}\le\sup\{d(x,L_0)\}$ where the supremum is taken over all x that lie in a nonessential component of $X-L_1$ on the left side and respectively of $X-L_0$ on the right side. Loosely speaking 2 just says that the nonessential components of $X-L_0$ are as 'big' as possible.

Let r_0 be such that if $d(x,L_0)\ge r_0$ then x lies in an essential component of $X-L_0$. Suppose that L_0 does not satisfy the conclusion of the proposition for r_0. There are then some $r>r_0$ and sequences $(x_n),(y_n)$ such that $d(x_n,y_n)=r$, x_n,y_n lie in the same component of $X-L_0$ and x_n,y_n cannot be joined in $X-L_0$ by any path of length less than n. We pick $g_n\in G$ such that $g_nx_n=e$ (where e is a fixed vertex). We have then as in Proposition 1.4.1 that a subsequence of $g_n L_0$ converges on compact sets to a quasi-line L_1 such that $X - L_1$ has the same number of essential components as $X - L_0$. By passing if necessary to a subsequence we have that $g_n x_n$ and $g_n y_n$ converge respectively to x_0, y_0. Clearly x_0, y_0 do not lie in the same essential component of $X - L_1$. It follows that at least one of them lies in a nonessential component of $X - L_1$. This however contradicts our assumption that L_0 satisfies property 2. □

It is easy to see that Proposition 1.4.4 can be strengthened so that is applies to finite neighborhoods of quasi-lines as well:

PROPOSITION 1.4.5. *Let X be a Cayley graph of a finitely presented one-ended group G. If an (f, N)-quasi-line L separates X then there is an (f, N)-quasi-line L_0 which satisfies the conclusion of Proposition* 1.4.3 *and has the following property*:

For any $M > 0$ there is an $r_M > 0$ such that for each $r > r_M$ there is an $R > r$ such that if $d(x, L_0) = r = d(y, L_0)$, $d(x, y) < f(3r)$ and x, y lie in the same essential component of $X - N_M(L_0)$, then x, y can be joined by a path of length less than R which does not meet $N_M(L_0)$.

Proof. Left to the reader.

Definition 1.5. We say that $a, b \in X$ are *K-separated by a quasi-line* L if $d(a, L) > K$, $d(b, L) > K$ and $a \in X_1, b \in X_2$ where X_1, X_2 are two distinct essential connected components of $X - L$

It is easy to see that these notions are invariant under quasi-isometries:

LEMMA 1.6. *Let $f : X \to Y$ be a quasi-isometry of the geodesic metric spaces X, Y. Let $L \subset X$ be a quasi-line of X. Then there is an $M > 0$ such that the M-neighborhood of $f(L)$, $N_M f(L)$, is a quasi-line of Y.*

Proof. Left to the reader.

LEMMA 1.7. *Let $f : X \to Y$ be a quasi-isometry of the geodesic metric spaces X, Y. Let $L \subset X$ be a quasi-line separating X. Then there is an $M > 0$ such that $N_M(f(L))$ is a quasi-line separating Y.*

Proof. Left to the reader.

Our interest in quasi-lines comes from the following:

LEMMA 1.8. *Let G be a finitely presented group that splits over a* 2-*ended subgroup J. Let X be a Cayley graph of G. Then there is a neighborhood of J in X that is a quasi-line separating X.*

Proof. Since separation is invariant by quasi-isometries we show this using a complex naturally associated to the splitting of G (see [Sc-W]): If $G = A *_J B$ let K_J, K_A, K_B be finite complexes with $\pi_1(K_J) = J, \pi_1(K_A) = A$, $\pi_1(K_B) = B$. We consider $K_J \times [-1,1]$. Let $f : K_J \to K_A$, $g : K_J \to K_B$ be cellular maps inducing on π_1 the monomorphisms from J to A, B in $G = A*_J B$. We glue $K_J \times \{-1\}$, $K_J \times \{1\}$, respectively to K_A, K_B by f, g and we obtain a complex C with $\pi_1(C) = G$. A similar construction applies if the splitting is an HNN-extension. We make metric the 1-skeleton of the universal cover of C, each $\tilde{C}$ being given edge length 1. With this metric $\tilde{C}^{(1)}$ is quasi-isometric to X.

If T is the Bass-Serre tree of the splitting $G = A*_J B$ there is a natural map $p : \tilde{C} \to T$ sending copies of $\tilde{K}_J \times [-1,1]$ to edges of T and collapsing copies of $\tilde{K}_A, \tilde{K}_B$ to vertices of T. We note that p implies distance nonincreasing. It follows that if Z is a copy of $\tilde{K}_J \times \{0\}$ in $\tilde{C}$, $\tilde{C} - Z$ has two components, C_1, C_2 neither of which is contained in a neighborhood of Z.

It remains to show that C_1 and C_2 are one-ended. We note that since $\tilde{C}$ is one-ended, if C_1 is not one-ended, and K is a compact set such that $C_1 - K$ has more than one unbounded component, then the closure in $\tilde{C}$ of each unbounded component of $C_1 - K$ has unbounded intersection with Z. We note further that if U is such an unbounded component of $C_1 - K$ and a, b are two vertices of Z lying in the closure of U and $\bar{U}$, then there is a path in Z joining a, b which lies in $\bar{U}$ as well. Indeed consider a path u joining a, b in U and a path w joining them in Z. Take a Van-Kampen diagram, D, for the closed path $u \cup w$ (see [L-S, Ch. 6] for a definition of Van-Kampen diagrams). Take the maximal connected subdiagram of D containing u which maps to $\bar{U}$. Clearly the boundary of this subdiagram contains a path joining a, b that maps to a path in Z. We conclude that an unbounded component of $Z - K$ is contained in a finite neighborhood of U.

From the discussion above it follows that in order to show that C_1 is one-ended it suffices to prove the following:

If x is a fixed vertex in Z and if $B_x(n)$ is the ball of radius n centered at x then there is a path p_n in C_1 joining the distinct unbounded components of $Z - B_x(n)$. Note that for small n, $Z - B_x(n)$ might have only one unbounded component (and the condition becomes void) while for sufficiently big n, $Z - B_x(n)$ has exactly two unbounded components. We show below how to construct the paths p_n.

We fix now a vertex of Z, x, and we consider an infinite path, q, in C_1 such that $q(0) = x$ and such that $d(q(n), Z) \to \infty$ as $n \to \infty$. We note that a conjugate of J acts co-compactly on Z. By passing, if necessary, to an index 2 subgroup, say J_0, we obtain a group acting co-compactly on Z which preserves C_1. So there is a $k > 0$ such that for any vertex $y \in Z$ there is $g \in J_0$ such that $d(gy, x) < k$.

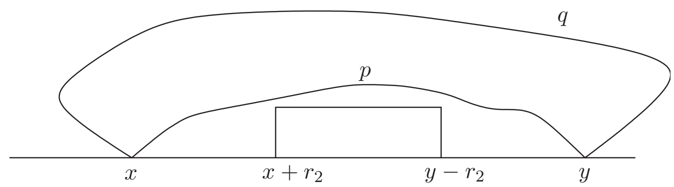


Figure 2

Given $n > 0$ there is a vertex $y \in Z$ and $R > 0$ such that $d(y, q) > n + k$ and $B_y(n + k) \subset B_x(R)$. Since G is one-ended there is a path v joining some vertex $q(t)$ of q to Z without intersecting $B_x(R)$. Consider now the path $q_n = v \cup q([0, t])$. Clearly $d(q_n, y) > n + k$. Let $g \in J_0$ is such that $d(gy, x) < k$. It is easy to see now that we can take p_n to be the path gq_n. □

Lemma 1.9. *Let G be a finitely presented group and let X be a Cayley graph of G. Let L be a quasi-line separating X and let Y be an essential component of $X - L$. Then given $r_1 > 0$ there is $r_2 > 0$ such that any $x < y \in L$ with* $\mathrm{length}([x, y]_L) > 2r_2$ *can be joined by a path p lying in $Y \cup L$ such that*

a. $p \cap N_{r_1}([x + r_2, y - r_2]_L) = \emptyset$,

b. $p \subset N_{r_2}([x, y]_L)$.

Proof. By choosing r_2 sufficiently big we can ensure that

$$N_{r_1}([x + r_2, y - r_2]_L \cap (-\infty, x]_L = \emptyset$$

and

$$N_{r_1}([x + r_2, y - r_2]_L \cap [y, \infty)_L = \emptyset.$$

Since $Y \cup L$ is one-ended there is a path, q, joining x, y in X–$N_{r_1}([x+r_2, y-r_2]_L)$. Let w be a path joining x, y which is contained in $[x, y]_L$. We consider a Van-Kampen diagram, D, for the closed path $q \cup w$ (see Figure 2).

Let $f : D^{(1)} \to X$ be the natural map from the 1-skeleton of D to X. We remark that $f^{-1}((X - N_{r_2}([x, y]_L)) \cup N_{r_1}([x + r_2, y - r_2]_L))$ does not separate $f^{-1}(x)$ from $f^{-1}(y)$ in D. That is, there is a vertex in $f^{-1}(x)$ which can be joined in D to a vertex in $f^{-1}(y)$ by a path which does not meet $f^{-1}((X - N_{r_2}([x, y]_L)) \cup N_{r_1}([x + r_2, y - r_2]_L))$. Let us call this path p'. Using the fact that L separates and that each point in L is at distance less than N from L' we can easily modify (if necessary) p' to a path p satisfying the conclusion of the lemma. □

Convention. To translate topological arguments (like the ones in the appendix) to 'quasi-isometric' arguments one has to look at a space with larger and larger scales. These scales are determined by constants that one can explicitly compute. This is not very rewarding and so we use the following convention: We write that the statement $P(r_1, r_2)$ that depends on two numbers r_1, r_2 holds for $r_2 \gg r_1 \gg 0$ if there is an $R_1 > 0$ such that for each $r_1 > R_1$ there is an $R_2 > r_1$ such that for all $r_2 > R_2$ the statement $P(r_1, r_2)$ is true. Similarly we write for $r_3 \gg r_2 \gg r_1 \gg 0$, $P(r_1, r_2, r_3)$ holds etc.

2. Separation properties of quasi-lines

The main result of this section is Proposition 2.1. It is a technical result that will allow us to assume in the next section that quasi-lines separate X in at most two essential components. We note that Proposition 2.1 is a 'large scale' analog of a topological result (Lemma A.1 of the appendix). Its proof is a good illustration of the techniques used in this paper, namely the 'translation' of topological arguments into 'large scale geometry' arguments.

Although the results in this section can be stated for (large scale simply connected) metric spaces in general we will state and prove them only for locally finite, simply connected complexes. The reason is that we are interested in applying them to Cayley complexes of finitely presented groups.

As usual we make metric the 1-skeleton of such complexes by giving each edge length 1, and defining the distance of two vertices to be the length of the shortest path joining them. In what follows we will also assume that quasi-lines are *simply connected* ; this is done to simplify notation. The results that follow are valid in general for 'large scale' simply connected complexes as by definition quasi-lines are 'large scale', simply connected.

We can always 'fill the holes' of a given (f, N)-quasi-line L and replace it by a simply connected one, as long as the quasi-line is contained in the Cayley complex of a finitely presented group G. Indeed a quasi-line is contained in the N-neighborhood of a line L'. We join each vertex of L to a vertex of L' by a path of length less than or equal to N. We add now to the presentation of the group all words corresponding to simple closed curves of length less than $2N+1+f(2N+1)$ in the Cayley graph of G. By this construction any closed curve c in a quasi-line L is homotopic to a curve in L' and therefore can be contracted to a point. Moreover there is an $M > 0$ such that for any closed curve in L the filling disc for c is contained in the M-neighborhood of c. In other words the filling radius of closed curves in L is bounded by M. We will assume in what follows that quasi-lines also have this property. We will also assume that all separating quasi-lines considered satisfy the conclusion of Propositions 1.4.3 and 1.4.5.

The proposition and the proof that follow give a 'large scale analog' of Lemma A.1 of the appendix.

PROPOSITION 2.1. *Let X be a locally finite simply connected complex and let L be a quasi-line separating X, such that $X - L$ has at least three distinct essential connected components X_1, X_2, X_3. Then for any proper, increasing*

$f:\mathbb{R}^+\to\mathbb{R}^+$ and $N>0$ there is a $K>0$ such that any two vertices $a,b\in L$ that are sufficiently far apart cannot be K-separated by any (f,N) quasi-line of X.

Proof. Suppose that L is contained in the M-neighborhood of a line $L'\subset L$ (see Def. 1.3). It is clear that it suffices to show that there is $K>0$ such that any two points $a,b\in L'$ that are sufficiently far apart cannot be K-separated by any (f,N)-quasi-line. Indeed this implies that any two points on L that are sufficiently far apart cannot be $K+2M$-separated by any (f,N)-quasi-line.

In the argument that follows we use four constants K_1,K,R such that $K_1\gg K\gg R\gg 0$. It will be clear that the argument is valid if $R\gg 0$, $K\gg R$ and $K_1\gg K$. One can of course give explicit estimates (in terms of f and N) for K_1,K,R but we leave this to the reader.

Let $a<b\in L'$ so that a,b are K_1-separated by an (f,N) quasi-line L_1. By Lemma 1.9 a,b can be joined in X_i by a simple path p_i that does not intersect the $2K$-neighborhood of the interval $[a+\frac{K_1}{4},b-\frac{K_1}{4}]'_L$ of L'. We choose p_i so that an initial and a terminal subpaths of p_i are contained in L while in between these subpaths p_i does not intersect L. We call these initial and terminal subpaths, respectively, p_{i0},p_{i1} and call the subpath between them p'_i.

We consider the simple closed paths $q_i=p_i\cup[a,b]_{L'}$ (see Figure 3) and note that by Van-Kampen's theorem $X_i\cup L$ is simply connected for all i. Let D_i be Van-Kampen diagrams representing a contraction of q_i to a point inside $X_i\cup L$.

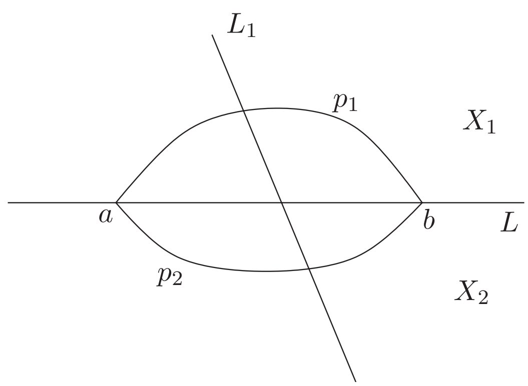


Figure 3

To simplify notation we denote by p_i the subpath of ∂D_i mapped onto p_i. Likewise we denote by $[a,b]$ the subpath of ∂D_i mapped onto $[a,b]_{L'}$.

We consider now the Van Kampen diagram $D=D_1\sqcup D_2/\sim$ where $\sim$ is given by the identification of the subpaths of D_1,D_2 that map onto $[a,b]$. We call g the natural map $g:D\to X$ sending ∂D to $p_1\cup p_2$. Let $a_1,b_1\in[a,b]_{L'}$ be such that the following hold:

a_1 lies in the same essential component of $X - L_1$ as a, $d(a_1, L_1) \geq 2R$ and a_1 is the maximal vertex in $[a,b]_{L'}$ with these properties for the order of L'.

b_1 is $2R$- separated from a by L_1 and is the first vertex after a_1 on $[a,b]_{L'}$ with this property.

We assume that K is chosen so that the following holds: There is at most one interval, $[x,y]_{L_1'}$ of L_1' with the following properties:

a. $[x,y]_{L_1}$ intersects $[a_1,b_1]_L$.

b. $B_N(x)$ and $B_N(y)$ intersect $g(\partial D)$.

c. If $z \in (x,y)_{L_1'}$ then $B_N(z)$ does not intersect $g(\partial D)$.

We will show now that there is an interval $[x,y]_{L_1'}$ with the above properties such that $[x,y]_{L_1}$ contains a path joining $B_N(x)$ to $B_N(y)$ which is contained in $X_1 \cup X_2$. We assume that no such interval exists and we argue by contradiction. Consider all maximal subdiagrams of D, say U, with the property that ∂U is in $g^{-1}(L_1)$. Since L_1 is simply connected we can modify all such U so that $U \subset g^{-1}(L_1)$ (we cut away all such diagrams U and glue back diagrams contracting ∂U to a point inside L_1).

We consider now the connected components of $D - g^{-1}(L)$. Let V_1 be the component containing p_1' and V_2 be the component containing p_2'. Let w be a simple path in ∂V_1 separating p_1' from p_2'. Then $g(w)$ is contained in L. If I is a minimal interval of L containing $g(w)$ then one sees easily that I contains $[a + \frac{K_1}{4}, b - \frac{K_1}{4}]_L$. We conclude that $g(w)$ intersects both $B_R(a_1), B_R(b_1)$.

Let a_1', b_1' be such that $g(a_1') \in B_R(a_1)$, $g(b_1') \in B_R(b_1)$. Then a_1', b_1' are separated in D by a path, say q, such that $g(q) \subset L_1$ and $\partial q \subset \partial D$. Now, ∂q separates ∂D in two paths, say c_1, c_2. Clearly neither c_1 nor c_2 is contained in L_1. We consider the shortest path in L_1, with the same endpoints as q, which is contained in $X_1 \cup X_2$. For convenience we still call this path q. We consider Van-Kampen diagrams for $c_1 \cup q$ and $c_2 \cup q$. Since q does not intersect $[a_1,b_1]_L$ one of either $c_1 \cup q$ or $c_2 \cup q$ has the property that any Van-Kampen diagram corresponding to it contains two points, say a_1', b_1', such that $g(a_1') \in B_R(a_1)$, $g(b_1') \in B_R(b_1)$. Let us say this is the case for $c_1 \cup q$. We pick a Van-Kampen diagram for $c_1 \cup q$, and repeat the procedure. This is bound to stop after finitely many steps, producing a diagram in which the preimage of $B_R(a_1)$ is not separated from the preimage of $B_R(b_1)$ by the preimage of L_1, a contradiction.

We showed therefore that there exists an interval $[x,y]_{L_1'}$ of L_1' with the properties a,b,c described above such that $[x,y]_{L_1}$ contains a path joining $B_N(x)$ to $B_N(y)$ which is contained in $X_1 \cup X_2$.

By considering D_1, D_3 we see that $[x,y]_{L_1}$ contains another path joining $B_N(x)$ to $B_N(y)$ which is contained in $X_1 \cup X_3$. This implies that $B_N(x)$, $B_N(y)$ are both contained in X_1. On the other hand by considering D_2, D_3 we conclude that $B_N(x)$, $B_N(y)$ are both contained in X_2, a contradiction. This proves Proposition 2.1. □

3. A geometric characterization of virtually planar groups

In this section we give a quasi-isometric characterization of virtual surface groups. It is modeled after Theorem A of the appendix and its proof follows closely the proof of this theorem. Roughly what we show is that if G is a one-ended finitely presented group such that any two points in its Cayley graph which are sufficiently far away are separated by a quasi-line, then G is virtually a surface group. In the proof of Theorem A one shows that every simple closed curve separates and uses a classical theorem of Bing [Bi] to conclude the proof. For Theorem 3.1 below we distinguish two cases: in the nonhyperbolic case we show also that appropriately chosen (with big filling radius) 'thickened' simple closed curves separate while in the hyperbolic case we show that 'thickened' geodesics separate. In the first case we conclude using Varopoulos' isoperimetric inequality and in the second using the Tukia-Gabai theorem on convergence groups. To show that thickened simple closed curves separate we argue as for Theorem A: We use separating quasi-lines to define what it means for a point to be 'inside' a simple closed curve. (see the definition after Lemma 3.2.1). The main technical result is Lemma 3.2 which parallels Lemma A.3.3 of the appendix. This is used to show later in Lemma 3.4 that the definition of 'inside' does not depend essentially on the quasi-line picked.

We state now the main result of this section:

THEOREM 3.1. *Let G be a one-ended group and let $X = X_G$ be a Cayley complex of G. Suppose that there is a proper increasing $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $N > 0$ such that for any $K > 0$ there is an $M > 0$ such that any two points x, y of X with $d(x,y) > M$ are K-separated by some (f,N) quasi-line. Then G is commensurable to a fundamental group of a surface.*

Proof. In the proof that follows we suppose that $K \gg N$. It will be clear that the argument is valid for K sufficiently bigger than N, and it is easy to obtain an explicit estimate for K. We will need some technical lemmas:

Let L be a quasi-line separating X and let L' be its corresponding line. By Proposition 2.1 $X - L$ contains exactly two essential connected components. We denote them respectively L^+, L^-. We denote $\bar{L}^+$ the union $L^+ \cup L$. Similarly $\bar{L}^- = L^- \cup L$.

The next lemma is the 'large scale' analog of Lemma A.3.3 of the appendix.

LEMMA 3.2. *Let L_1, L_2 be separating (f, N) quasi-lines and let L_1', L_2' be the corresponding lines. For $r_2 \gg r_1 \gg 0$ the following hold*:

Let $a, b \in L_1'$ be such that $a, b \in L_2^-$, $d(a, L_2) \geq r_1, d(b, L_2) \geq r_1$ and for all $t \in (a, b)_{L_1'}$, $d(t, \bar{L}_2^+) < r_1$. Let I be a minimal interval of L_2 containing $[a, b]_{L_1} \cap L_2$.

Let $x \in I \cap L_2'$, $y \in (L_2 - I) \cap L_2'$ be such that $d(x, L_1) > r_2$, $d(y, L_1) > r_2$. Then any path p joining x, y and lying in $\bar{L}_2^+$ intersects $[a, b]_{L_1}$.

Proof. The proof is similar to that of Lemma A.3.3 of the appendix. We need a lemma similar to Lemma A.3.3.1 of the appendix:

LEMMA 3.2.0. *With the notation of Lemma 3.2 the following holds*:

Let $S = ([a, b]_{L_1} \cap L_2) - ([a, b]_{L_1} \cap [x, y]_{L_2})$. There is a path p' in $\bar{L}_2^-$ joining x, y and a Van-Kampen diagram $g : D \to X$ for $p' \cup [x, y]_{L_2'}$ such that:

a) $d(g(D), S) > r_1$.

b) $d(p', [a, b]_{L_1} \cap [x, y]_{L_2}) > r_1$.

Proof. Let p_1 be a path in $\bar{L}_2^-$ joining x, y such that $d(p_1, [a, b]_{L_1}) > 3r_1$. We note that such a path exists by our assumption that $r_2 \gg r_1$ and Lemma 1.9. Let $g_1 : D_1 \to X$ be a Van-Kampen diagram for $p_1 \cup [x, y]_{L_2'}$. Let $s \in g_1^{-1}(S)$ and let O_s be the component of s in $D_1 - g_1^{-1}(N_{r_1}([x, y]_{L_2'}) \cap L_2)$. If $\partial O_s \subset g_1^{-1}(N_{r_1}([x, y]_{L_2'}) \cap L_2)$ we modify D_1 as follows: Since $\partial O_s \subset L_2$ we fill ∂O_s in L_2, so after the change $g_1(O_s) \subset N_{2r_1}([x, y]_{L_2'}) \cap L_2$.

By performing this 'cut and paste' operation for all $s \in g_1^{-1}(S)$ as above we get a Van-Kampen diagram, that we still call D_1, such that all $s \in g_1^{-1}(S)$ belong to a single component of $D_1 - g_1^{-1}(N_{r_1}([x, y]_{L_2'}) \cap L_2)$.

We consider now the subdiagram D_{r_1} of D_1 consisting of all closed 2-cells σ of D_1 such that $d(g_1(\sigma), [x, y]_{L_2'}) \leq 2r_1$. Let D_1' be the connected component of D_{r_1} containing $[x, y]_{L_2'}$. Clearly there is a path $p_2 \in \partial D_1'$ joining x, y such that $d(g(p_2), [a, b]_{L_1} \cap [x, y]_{L_2}) > r_1$. We then take $p' = g_1(p_2)$. Clearly if we take D to be the subdiagram of D_1 bounded by $[x, y]_{L_2'} \cup p_2$ and $g = g_1|_D$ both conditions a) and b) are satisfied. □

We return now to the proof of Lemma 3.2 arguing by contradiction. Let p be a path joining x, y in $\bar{L}_2^+$ such that p does not intersect $[a, b]_{L_1}$. Let $h_1 : E_1 \to X$ be a Van-Kampen diagram for $p \cup [x, y]_{L_2'}$ such that $h_1(E_1) \subset \bar{L}_2^+$. Let $h : E \to X$ be the Van-Kampen diagram obtained by identifying D, E_1 along $[x, y]_{L_2'}$.

Let r_0 be such that $r_1 \gg r_0 \gg 0$. By Remark 3.3, there are $z_1, z_2 \in [x, y]_{L_2'}$ such that z_1, z_2 are separated by L_1 and $d(z_1, [a, b]_{L_1}) = d(z_2, [a, b]_{L_1}) = 2r_0$. Let $t_1 \in h^{-1}(N_{r_0}(z_1)), t_2 \in h^{-1}(N_{r_0}(z_2))$. Now, t_1, t_2 are separated by $h^{-1}(L_1)$. Therefore there is a minimal simple path c in $h^{-1}(L_1)$ separating t_1, t_2. If c is a closed path then either t_1 or t_2 is contained in the region it bounds. We can eliminate then one of t_1, t_2 by cutting the region bounded by c and gluing back a Van-Kampen diagram for c whose image is contained in L_1. If c is not closed its endpoints are contained in ∂E. If c_1, c_2 are the endpoints of c then both $c_1, c_2 \in p'$ and (by Lemma 3.2.0) $[h(c_1), h(c_2)]_{L_1}$ does not intersect $[z_1, z_2]_{L_2}$.

We cut E open along $c - \{c_1, c_2\}$ and obtain thus a diagram with a region, say F, in its interior that is bounded by two copies of c. We join c_1, c_2 in F by a path $\bar{c}$. We extend h to $\bar{c}$ by mapping it to a path lying in $[h(c_1), h(c_2)]_{L_1}$ that joins $h(c_1), h(c_2)$. F is subdivided by $\bar{c}$ in two regions. Each region is bounded by $\bar{c} \cup c$. We fill this regions by Van-Kampen diagrams that contract $h(\bar{c} \cup c)$ to a point inside $[h(c_1), h(c_2)]_{L_1}$. We obtain thus a diagram, that we still call E for convenience, in which t_1, t_2 are separated by $\bar{c}$. We remark that $h(\bar{c})$ does not intersect $[z_1, z_2]_{L_2}$.

It is easy to see that repeating this 'cut and paste' operation finitely many times we obtain a Van-Kampen diagram $h : E \to X$, for $p \cup p'$ such that the following holds: If $t_1 \in h^{-1}(N_{r_0}(z_1), t_2 \in h^{-1}(N_{r_0}(z_2)$ then t_1, t_2 are separated in E by a simple path $c \in h^{-1}(L_1)$ such that $h(c)$ does not intersect $[z_1, z_2]_{L_2}$. This is clearly impossible. $\square$

Lemma 3.2 holds also for infinite intervals. More precisely we have the following:

LEMMA 3.2.1. *Let L_1, L_2 be separating (f, N) quasi-lines and let L_1', L_2' be the corresponding lines. For $r_2 \gg r_1 \gg 0$ the following holds*:

Let $a, c \in L_1'$ be such that $a \in L_2^-$, $c \in L_2^+$ and $d(a, L_2) \geq r_1, d(c, L_2) \geq r_1$ and for all $t \in (a, \infty]_{L_1'}$ $d(t, \bar{L}_2^+) < r_1$. Let $b \in [a, c]_{L_1} \cap L_2$ and let I_1, I_2 be the infinite connected components of $L_2 - B_{r_1}(b)$.

Let $x \in I_1$, $y \in I_2$ be such that $d(x, L_1) > r_2$, $d(y, L_1) > r_2$. Then any path p joining x, y and lying in $\bar{L}_2^+$ intersects $[a, \infty)_{L_1}$.

Proof. Left to the reader. $\square$

The following definition is similar to Definition A.4 of the appendix:

Definition. Let C be a closed curve in X and let L be an (f, N) quasi-line. Let $x \in L$ be such that $d(x, C) > R$. We say that a subpath of C lying in L^+ (or in L^-) is R-above x if the following are satisfied:

1) $\partial I \subset L$.

2) x lies in the interval of L determined by ∂I.

3) I is a maximal subpath satisfying 1), 2).

We say that x is an (R, L)-*interior point* of C if there is an odd number of subpaths of C in L^+ that are R-above x.

Note that for sufficiently big R this does not depend on our choice of line L' for L. When such an R is given we say simply that x is an *L-interior point* of C an (rather than an (R, L)-interior point).

To state the next lemma we need a definition:

Definition. Let C be a simple closed curve. We say that C is *locally* (c_1, c_2)*-quasigeodesic* if every subpath of C of length less than length$(C)/2$ is a (finite) (c_1, c_2)-quasigeodesic.

The next lemma is an analogue of Lemma A.4.2 of the appendix:

LEMMA 3.3. *Let $c_1, c_2, R > 0$ be given. For any sufficiently big $R_1 > R$ the following holds: Let C be a simple closed curve that is a locally (c_1, c_2) quasi-geodesic and let L be a separating quasi-line of X, R_1-separating $a, b \in C$. Then there is an $x \in L$ such that x is an (R, L)-interior point of C.*

The proof, left to the reader, is the same as the proof of Lemma A.4.2 of the appendix. □

We need a definition:

Definition. We say that a quasi-line L_1, *r-crosses* a quasi-line L_2 at $[x, y]_{L_1}$ if x, y are r-separated by L_2. We also say that $[x, y]_{L_1}$ *r-crosses* L_2.

Remark 3.3. As in Lemma A.2 of the appendix we remark that there is a proper function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that if L_1 r-crossses L_2 at $[x, y]_{L_1}$ then L_2 $g(r)$-crosses L_1 at an interval $[a, b]_{L_2}$ contained in the r-neighborhood of $[x, y]_{L_1}$.

The following lemma is an analogue to Lemma A.4.3 of the appendix.

LEMMA 3.4. *Let C be a simple closed curve in X and let L_1, L_2 be separating quasi-lines. For any sufficiently big $r > 0$ and $R \gg r$ the following holds: Let $x < y$ be points on L_1' such that x is r-separated from y by L_2. Suppose that $d([x, y]_{L_1}, C) > R$. If $t \in [x, y]_{L_1} \cap L_2$ then t is an (R, L_1)-interior point of C if and only if it is an (R, L_2)-interior point of C.*

Proof. The proof is similar to the proof of Lemma A.4.3 of the appendix. Some modifications however have to be made since L_1, L_2 are quasi-lines and not lines, and so it is not possible to have a 'planar' picture of L_1, L_2 such that points of L_1 (or L_2) separated by L_2 are mapped on the plane to points that are separated by the image of L_2. What one can show roughly is that points on L_1 (or L_2) 'far away' from L_2 (or L_1) are correctly mapped to the plane. We explain this here in some detail.

We define first a map from $L_1' \sqcup L_2'$ to the plane. Mapping L_2' to the x-axis in a length-preserving way, for convenience, we identify L_2' to its image. We assume $r \gg r_2 \gg r_1$ where r_1, r_2 are such that lemma 3.2 holds. To each minimal interval $[x_1, x_2]_{L_1'}$ of L_1 such that $[x_1, x_2]_{L_1}$ r_1-crosses L_2 we associate a point lying in $[x_1, x_2]_{L_1} \cap L_2'$.

Let $J = [a, b]_{L_1'}$ be an interval of L_1' satisfying the conditions of Lemma 3.2. Using J we will define a map from a smaller interval contained in J to the plane. Let $[a, a_1]_{L_1'}$ and $[b_1, b]_{L_1'}$ be minimal intervals such that $[a, a_1]_{L_1}$ and $[b_1, b]_{L_1}$ r_1-cross L_2. Let a_2, b_2 be the points on L_2' corresponding to these intervals. Let $a_2', b_2' \in L_1'$ be points on L_1' such that $d(a_2, a_2') \leq N$, $d(b_2, b_2') \leq N$ (where a_2', b_2' are obtained by the usual map from the quasi-line L_1 to the line L_1').

We map then $[a_2', b_2']_{L_1'}$ to a polygonal path joining a_2, b_2. This polygonal path intersects the x-axis only at its endpoints and lies in the half-plane $y > 0$ if $a, b \in L_2^-$ and in the half-plane $y < 0$ if $a, b \in L_2^+$.

In a similar way, we map infinite intervals of L_1' using intervals of the form $[a, \infty)_{L_1'}$ or of the form $(-\infty, a]_{L_1'}$ satisfying the conditions of Lemma 3.2.1.

We note now that we can write L_1' as a union of intervals satisfying the conditions of either Lemma 3.2 or 3.2.1. Two such intervals J_1, J_2 can intersect 'near' their endpoints. There is a natural order on this set of intervals of L_1' as each interval intersects exactly one other interval near each of its endpoints. Call two intervals that intersect *adjacent*. If two intervals satisfying 3.2 or 3.2.1 are adjacent then using these intervals we define a map from two intervals contained in them. These two new intervals intersect at exactly one point and are mapped to polygonal lines which also intersect at exactly one point. We can, therefore, using these maps define a map from L_1' to the plane; each point of L_1' belongs to an interval $[a_2', b_2']_{L_1'}$ as above and is mapped to the plane by the corresponding map. We call $g : L_1' \to \mathbb{R}^2$ the map obtained in this way and note that g might not be one-to-one. It is possible that two intervals, say $[a, b]_{L_1'}$, $[c, d]_{L_1'}$, and are mapped to the plane so that $g(a), g(b), g(c), g(d)$ lie on the x-axis, $(a, b)_{L_1'}$, $(c, d)_{L_1'}$ are mapped both, either in the upper or in the lower half-plane and $g(c) \in [g(a), g(b)]$ while $g(d) \notin [g(a), g(b)]$. In this case the image of $[a, b]_{L_1'}$ intersects the image of $[c, d]_{L_1'}$. By changing the map g if necessary we can assume that $g([a, b]_{L_1'})$ intersects $g([c, d]_{L_1'})$ at exactly one point. We can further assume that for any pair of intervals as above for which $g(c), g(d)$ are either both inside or both outside $[g(a), g(b)]$ the images of $[a, b]_{L_1'}$, $[c, d]_{L_1'}$ do not intersect.

We explain now how to modify g so that $g(L_1')$ is a line. We fix an interval $[a, b]_{L_1'}$ and consider all intervals such that their images intersect $g([a, b]_{L_1'})$. We say $g([c, d]_{L_1'})$ intersects $g([a, b]_{L_1'})$, $g(a), g(b), g(c), g(d)$ lie on the x-axis, and $(a, b)_{L_1'}$, $(c, d)_{L_1'}$ are both mapped in the upper half-plane. Let I be the bounded interval of $L_1' - ((a, b)_{L_1'} \cup (c, d)_{L_1'})$. I then joins an endpoint of $[c, d]_{L_1'}$ to an endpoint of $[a, b]_{L_1'}$. To fix ideas let us say that the endpoints of I are a, c. We change g so that $g(I) = g(a)$ and so that $g([c, d]_{L'_1})$ becomes a polygonal path in the upper half-plane joining $g(d)$ to $g(a)$. We pick this polygonal path so that it does not intersect $g((a, b]_{L'_1})$ and any other paths of $g(L')$ in the upper half-plane that have both their endpoints either inside or outside $[g(a), g(d)]$.

If $r_3 \gg r_2$ we have that length$(I) < r_3$ by Lemmas 3.2, 3.2.1. This modification is made for every interval $[c, d]_{L'_1}$ whose image intersects $g((a, b)_{L'_1})$. In this way eventually $g((a, b)_{L'_1})$ does not intersect any other polygonal path. We continue by picking another interval and changing the map in the same away to eliminate intersections with the image of this interval. As there are countably many intervals, it is clear that we can eliminate all self -intersections of $g(L'_1)$. Note that after these modifications some intervals I of L'_1 are mapped to a point but all such intervals have length smaller than r_3.

For r_3 big enough one can verify easily that if $a, b \in L'_1$ are such that $d(a, L_2) > r_3$, $d(b, L_2) > r_3$, then a, b are r_3-separated by L_2 if and only if $g(a), g(b)$ are separated by $g(L'_2)$.

The next lemma shows that this holds also for $a, b \in L'_2$:

Lemma 3.4.1. *If $a, b \in L'_2$ are such that $d(a, L_1) > r_3$, $d(b, L_1) > r_3$ then a, b are r_3-separated by L_1 if and only if $g(a), g(b)$ are separated by $g(L'_1)$.*

Proof. We show first that if $a, b \in L'_2$ are not separated by L_1 then $g(a), g(b)$ are not separated by $g(L'_1)$. We note that there is a path p joining a, b in X that does not intersect the r_2-neighborhood of L_1. Indeed, if this were not so, $X - N_{r_2}(L_1)$ would have more that two essential components (see Def. 1.4). This contradicts the hypothesis of Theorem 3.1 (see Prop. 2.1). We decompose p as a union $p = p_1 \cup \cdots \cup p_n$ where the p_i's are successive subpaths lying in $\bar{L}_2^+$ or in $\bar{L}_2^-$. Let a_i, b_i be the endpoints of p_i. We fix some p_i and suppose that, say, $p_i \subset \bar{L}_2^+$. Note that if there is no path joining the points $g(a_i), g(b_i)$ in $\mathbb{R}^2 - g(L'_1)$ then a_i, b_i are separated in $\bar{L}_2^+$ by an interval of L_1 as in Lemma 3.2 (or in Lemma 3.2.1). This is impossible. We conclude that $g(a), g(b)$ are not separated by $g(L'_1)$.

Similarly, if for a, b as in the lemma, $g(a), g(b)$ are not separated by $g(L'_1)$ then it is easy to see that $g(a), g(b)$ can be joined by a path that does not meet $g(N_{r_2}(L_1))$. This in turn implies that a, b can be joined by a path that does not meet L_1. □

Note that $g(L'_1)$ and $g(L'_2)$ separate the plane in two pieces. We define $g(L_1^+)$ to be the component of $\mathbb{R}^2 - g(L_1)$ which contains $g(a)$ for some $a \in L'_2$ such that $a \in L_1^+$ and $d(a, L_1) > r_3$. We define similarly $g(L_1^-)$,$g(L_2^+)$,$g(L_2^-)$, and now extend g to C, so that g will be defined on $L_1 \sqcup L_2 \sqcup C$.

Assuming that $r \gg r_3$, we show how to map C to the plane so that $t \in [x, y]_{L'_1} \cap L_2$ is an (R, L_1)-interior point ((R, L_2)-interior point) of C if and only if $g(t)$ is a $g(L'_1)$ ($g(L'_2)$) interior point of the image of C.

We extend g to $L_1 \cup L_2$ in the obvious way by defining $g(a) = g(a')$, where $a \to a'$ is the usual map from a line to the corresponding quasi-line.

We decompose C as a union of successive paths $C = C_1 \cup \cdots \cup C_n$ where the endpoints of C_i are on L_2 and C_i is contained in $\bar{L}_2^+$ or in $\bar{L}_2^-$. We explain now how to map each C_i to the plane. Let us say that the endpoints of C_i are a_i, b_i and, to fix ideas, suppose that C_i is contained in $\bar{L}_2^+$. We will map C_i to the plane so that the following conditions hold:

1) If $a \in C_i \cap L_1$ then $g(a) \in g(C_i)$.

2) If for some $a \in L_1$, $g(a) \in g(C_i)$, then $d(a, C_i) < r_3$.

We orient C_i from a_i to b_i. Let $c_1, \ldots, c_r$ be the vertices of intersection of C_i with L_1 in the order they appear. If $c_1 \in L_2^+$ we map the subpath $[a_i, c_1]$ of C_i to be a polygonal line in $g(L_2^+)$ joining $g(a_i)$ to $g(c_1)$ and having the minimum possible number of intersection points with $g(L_1)$. Note that any intersection point of $g([a_i, c_1])$ with $g(L_1)$ lies in an interval of $g(L_1)$ that lies in $g(L_2^+)$ and separates a_i, c_1. By the definition of g and Lemma 3.2 one sees easily that such an interval contains a point $g(t)$ such that either $d(t, a_i) < r_3$ or $d(t, c_1) < r_3$. We choose $g([a_i, c_1])$ so that it intersects this interval exactly at a point $g(t)$ with the above property.

If $c_1 \in L_1^-$ then we pick a point s on L_2 such that $d(s, c_1) < r_3$ and we define $g([a_i, c_1])$ to be the union of three polygonal paths: the first joins $g(a_i)$ to $g(s)$ and lies in $g(L_2^+)$; the second joins $g(s)$ to $g(c_1)$ and lies in $g(L_2^-)$; and the third is the inverse of the second. This path might intersect $g(L_1)$ at points other than c_1. We can however arrange, as before, that if $g(t)$ is such an intersection point then either $d(t, a_i) < r_3$ or $d(t, c_1) < r_3$.

We continue in the same way defining $g([c_1, c_2])$ to be a polygonal path joining $g(c_1), g(c_2)$ and lying in $g(L_2^+)$ if $g(c_1), g(c_2)$ lie in $g(L_2^+)$. If one of them or both lie in $g(L_2^-)$ then we define this path as before using an auxiliary point on $g(L_2)$ close to the point in $g(L_2^-)$. We define g on all subpaths C_i in the same way.

By the remarks made above we can arrange so that g satisfies the following: If $t \in C$ is such that $g(t) \in g(C) \cap g(L_1)$ or $g(t) \in g(C) \cap g(L_2)$ then, respectively, $d(t, L_1) < r_3$ or $d(t, L_2) < r_3$.

Note that x is an L_2 interior point of C if and only if $g(x)$ is a $g(L_2)$ interior point of $g(C)$. Indeed this follows easily from the definition of g, the extra intersection points with $g(L_2)$ that we might create defining $g(C)$ do not change the parity of intervals above $g(x)$.

We will show now that x is an L_1 interior point of C if and only if $g(x)$ is a $g(L_1)$ interior point of $g(C)$. Let $p = [b_1, b_2]$ be a subpath of C lying in $\bar{L}_1^+$ with $b_1, b_2 \in L_1$. Then p can be written as a union of successive subpaths $p = p_1 \cup \cdots \cup p_n$ such that for each i one of the following two holds:

1) $g(p_i) \subset \overline{\mathbb{R}^2 - g(L_1)}$ and the endpoints of $g(p_i)$ lie on $g(L_1)$ and are separated on $g(L_1)$ by $g(x)$, or

2) $g(p_i) \subset \overline{\mathbb{R}^2 - g(L_1)}$ and the endpoints of $g(p_i)$ lie on $g(L_1)$ and are not separated on $g(L_1)$ by $g(x)$.

We will show that in case 1) $g(p_i)$ lies in $g(\bar{L}_1^+)$. Let q be a subpath of p_i satisfying the following:

Each endpoint of $g(q)$ lies either on $g(L_1)$ or on $g(L_2)$, $g(q)$ is contained either in $g(\bar{L}_2^+)$ or in $g(\bar{L}_2^-)$ and there is a point t on q at distance bigger than r_3 from L_1. To fix ideas we assume that $g(q)$ lies in $g(\bar{L}_2^+)$. We then join t to a point s on L_2 by a path that meets L_2 only at s and does not intersect the r_2 neighborhood of L_1. Clearly $s \in L_1^+$. From Lemmas 3.2, 3.2.1 it follows that $g(s)$ and the endpoints of $g(q)$ lie in the same component of $g(\bar{L}_2^+) - g(L_1)$. This implies that $g(q)$ is at the same component of $\mathbb{R}^2 - g(L_1)$ as $g(s)$. Therefore $g(p_i) \subset g(\bar{L}_1^+)$.

From these observations it follows that the number of subpaths of $g(p)$ lying above $g(x)$ in $g(L_1^+)$ is odd. This in turn implies that $g(x)$ is a $g(L_1)$ interior point for $g(C)$ if and only if it is an L_1-interior point for C. Since $g(x)$ is a $g(L_1)$ interior point for $g(C)$ if and only if it is an $g(L_2)$ interior point for $g(C)$ the lemma follows. □

We show now how to approximate any path in X by "polygonal paths" i.e. paths made by intervals of quasi-lines. This is similar to Lemma A.4.4 of the appendix.

Remark 3.5. Note that for any $r > 0$ there is an $R > r$ such that for any separating (f, N)-quasi-line L the following holds: If $O \in X - N_r(L)$ and $d(O, L) > R$ then O lies in an essential component of $X - N_r(L)$.

This is because (see §2) all separating quasi-lines considered satisfy the conclusion of Proposition 1.4.3.

Lemma 3.6. *For any $r > 0$ there is an $R > r$ such that for any separating (f, N)-quasi-line L the following holds: Let $x, y \in X$ be such that $d(x, L) = d(y, L) = r + 1$ and let $x', y' \in L$ be such that $d(x, x') = d(y, y') = r + 1$. If x, y lie in the same essential component of $X - N_r(L)$ then x, y can be joined by a path lying in $(X - N_r(L)) \cap N_R([x', y']_L$.*

Proof. The lemma follows easily from Proposition 1.4.5. □

To construct the "polygonal paths" needed we use some constants M, R, r. In what follows we suppose that $M \gg R \gg r \gg 0$. We describe below some of the properties of these constants.

Suppose that $r > N$ is big enough so that Lemma 3.4 holds. We take R big enough so that Remark 3.5 and Lemma 3.6 hold for $R/4$ (where r is as given above). Let $M > R$ be such that any two points $x, y \in X$ with $d(x, y) \geq M$ can be R-separated by a quasi-line.

We fix now $O \in X$ and let $B = B_M(O)$ be the ball of radius M around O. Clearly each vertex t in $S_M(O)$ is R-separated from O by a quasi-line L_t. We call S the set of all these quasi-lines. For a quasi-line $L \in S$ we denote by L^+ the essential component of $X - L$ containing O and by L^- the other essential component of $X - L$. For each quasi-line $L_i \in S$ we denote by I_i a minimal interval of L_i containing $B_{2M}(O) \cap L_i$.

Definition. Two quasi-lines L_1, $L_2 \in S$ *cross* if either there are $x_1, y_1 \in I_1$ such that $[x_1, y_1]_{L_1}$ r-crosses L_2 or if there are $x_2, y_2 \in I_2$ such that $[x_2, y_2]_{L_2}$ r-crosses L_1.

Definition. We call a subset T of S *cross-connected* if for any $L_t, L_s \in T$ there is a sequence $L_1 = L_t, L_2, \ldots, L_n = L_s$ such that all members of the sequence lie in T and any two successive quasi-lines in this sequence cross.

We note that $S_M(O)$ is not necessarily connected. Since X is one ended there is exactly one connected component of $S_M(O)$ that meets the frontier of an unbounded connected component of $X - B$. We denote this connected component by F.

We define now a finite sequence of subsets of S. We pick a point $t \in F$ and pick a quasi-line $L_t \in S$ such that L_t R-separates O from t. Let T_1 be a maximal cross-connected subset of S containing L_t.

Given $T_1, \ldots, T_i$ we explain how to define T_{i+1}: Suppose that there is a vertex $s \in F$ such that s is not $R/2$-separated from O by any quasi-line lying in $T_1 \cup \cdots \cup T_i$. We then pick $L_s \in S$ such that L_s R-separates s from O. We define T_{i+1} to be a *maximal cross-connected subset* of S containing L_s. If there is no such vertex s then T_{i+1} is not defined.

Let $T_1, \ldots, T_n$ be the sequence defined in this way. We have the following:

LEMMA 3.7. *For any $t \in F$ there is a quasi-line in T_n that $R/4$-separates t from O.*

Proof. We will prove this by contradiction. Let x be a vertex on F such that there is a quasi-line $L \in T_n$ such that L_1 R-separates x from O and such that no quasi-line in $T_1 \cup \cdots \cup T_{n-1}$ $R/2$-separates x from O. We consider the set of points of F that are $R/4$-separated from O by some quasi-line in T_n. Let F_1 be the connected component of x in this set. Let $y \in \partial F_1 \cap F$. Let $L_2 \in T_1 \cup \cdots \cup T_{n-1}$ such that L_2 $R/2$-separates y from O. We note that L_2 does not $R/2$-separate x from O. Let p be a path in F_1 joining x to y. We distinguish two cases:

Case 1: L_2 intersects p.

Case 2: L_2 does not intersect p.

We will arrive at a contradiction by showing in both cases that $L_2 \in T_n$. We deal first with case 1. Suppose that L_2 intersects p at a point u. There is a quasi-line $L_3 \in T_n$ that $\frac{R}{4}$-separates u from O. Let u_1 be the closest point to u such that $u_1 \in L_2^+$ and $d(u_1, L_2) = r + 1$. Let q be a path joining O to u_1 in $B_{2M}(O)$ such that q does not intersect $N_r(L_2)$. Such a path exists by Lemma 3.6. Clearly q intersects L_3. Therefore there is a point on L_3 lying in L_2^+ at a distance bigger than r from L_2.

We distinguish again two cases: Either $d(y, L_3) \leq R/4$ or $d(y, L_3) > R/4$. In the first case there is a point $y_1 \in L_3$ with $d(y, y_1) \leq R/4$. Therefore $y_1 \in L_2^-$ and $d(y_1, L_2) > r$. Therefore in this case L_3 cross L_2 and $L_2 \in T_n$.

In the second case Lemma 3.6 implies that there is a path q_1 in B_{2M} joining O to y_1 such that q_1 does not intersect $N_r(L_3)$. Clearly L_2 intersects q_1 at a point lying in L_3^+. Therefore L_2 crosses L_3 and again $L_3 \in T_n$. This settles case 1. Case 2 is treated similarly by considering L_1, L_2: Since L_2 separates x from O there is a point $x_1 \in L_2$ with $d(x, x_1) \leq R/2$. Therefore $x_1 \in L_1^-$ and $d(x_1, L_1) > r$. If $d(y, L_1) > R/4$ by Remark 3.5 and Lemma 3.6 there is a path joining O to y in $B_{2M}(O)$ in $X - N_r(L_1)$. This path intersects L_2 so that L_2 crosses L_1 and $L_2 \in T_n$.

If $d(y, L_1) \leq R/4$ we pick a closest point to y, say y_1, with the properties that $y_1 \in L_1^+$ and $d(y_1, L_1) = r+1$. By Lemma 3.6 there is a path in $X - N_r(L_1)$ lying in $B_{2M}(O)$ joining O to y_1. This path intersects L_2 at a point x_2. Therefore L_2 crosses L_1 and $L_2 \in T_n$. □

LEMMA 3.7.1. *There is a cross-connected subset of S, S', such that any point of $S_M(O)$ is $R/4$-separated from O by some element of S'.*

Proof. Indeed for each $t \in S_M(O)$ that is not $R/4$-separated from O by some element of T_n we pick a quasi-line L_t in S that R-separates t from O. Clearly L_t intersects F so it crosses some element of T_n. We therefore obtain S' by adding to T_n all quasi-lines L_t as above. □

The proof of Theorem 3.1 now proceeds by distinguishing two cases: Either G is not a hyperbolic group or G is a hyperbolic group.

First case. G is not hyperbolic. We show in this case that G is commensurable with $\mathbb{Z}^2$. We recall the definition of the *filling radius of a simple closed curve in X* given in [G2]:

Definition. The *filling radius of a simple closed curve c in X* is the smallest r such that c can be contracted to a point in $N_r(c)$. If D is a Van-Kampen diagram for c and if $f : D \to X$ is such that $f(\partial D) = c$ then the radius of D is the maximum of $d(f(x), c)$ where x ranges over the vertices of D.

It is easy to construct long simple closed curves in X that are locally (c_1, c_2) quasi-geodesics and have 'big' filling radii. This is a standard fact for

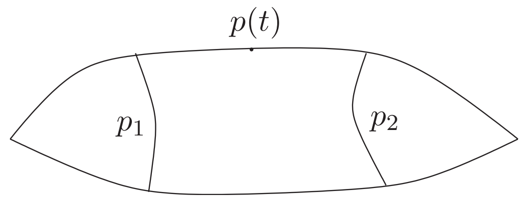


Figure 4

nonhyperbolic spaces, a consequence of the fact that the isoperimetric inequality satisfied by X is at least quadratic. We sketch below one such construction. Note that the constants (c_1, c_2) obtained are by no means optimal.

Since G is not hyperbolic for any $R > 0$ there is an R-thick bigon in X (see [P]). That is, there are two finite geodesic paths p, q with common endpoints such that p is not contained in the R-neighborhood of q (see Fig. 4). We parametrize p, q with respect to arclength as usual. Let $\max\{d(p(t), q)\} = m > R$ and let t_0 be such that $d(p(t_0), q) = m$. Let $t_1 \in [t_0 - 3m, t_0 - \frac{2m}{3}]$, $t_2 \in [t_0 + \frac{2m}{3}, t_0 + 3m]$ be such that:

$$m/4 < d(p(t_1), q) < d(p(t_1 + s), q) + 2s/3 \text{ for all } s \in [0, t_0 - t_1]$$

and

$$m/4 < d(p(t_2), q) < d(p(t_2 - s), q) + 2s/3 \text{ for all } s \in [0, t_2 - t_0].$$

It is easy to see that such t_1, t_2 exist.

Let $q(s_1)$ be a point on q such that $d(q(s), p(t_1))$ attains a minimum and similarly let $q(s_2)$ be a point on q such that $d(q(s), p(t_2))$ attains a minimum. We join $p(t_1)$ to $q(s_1)$ by a geodesic p_1 and $p(t_2)$ to $q(s_2)$ by a geodesic p_2. We consider now the simple closed curve $c = p[t_1, t_2] \cup p_1 \cup p_2 \cup q[s_1, s_2]$. We have:

$$R < 2m < \text{length}(c) < 16m.$$

It is easy to verify that c is a local $(100, 1)$-quasi-geodesic. We state this as a lemma:

LEMMA 3.8. *Suppose that G is not hyperbolic. Then for any $R > 0$ there is a simple closed curve c in X (the Cayley complex of G) that is a local* $(100, 1)$-*quasi-geodesic, has* $\text{length}(c) > R$ *and, such that the filling radius of c is bigger than* $\text{length}(c)/100$.

Proof. Indeed it is easy to verify that the simple closed curves c constructed above have filling radius bigger than $\text{length}(c)/100$. □

Remark 3.9. It is easy to see that the filling radius of c^n is also bigger than $\text{length}(c)/100$, where c^n is the curve obtained by traversing c n times in the same direction.

To finish the proof of Theorem 3.1 in this case it suffices to show that neighborhoods of 'long' simple closed curves that are $(100,1)$ local-quasi-geodesics separate X. We explain this below.

Let M, R, R_1, r be as in Remark 3.5 and Lemma 3.6. The arguments below hold for $K \gg m \gg M \gg 0$. Let c be a simple closed curve in X that is a $(100,1)$ local-quasigeodesic and has $\text{length}(c) \gg K$. By Lemma 3.3 there are a separating quasi-line L and $x \in L$ such that x is a (L,K)-interior point of c. Now, there is a path p, which does not intersect the $2m$-neighborhood of c, joining x to a point y such that $d(y,c) > \text{length}(c)/100$.

Indeed consider a Van-Kampen diagram, D, for c. Let $f : D \to X$ be the simplicial map induced by the labelling of D for which $f(\partial D) = c$. Consider now $f^{-1}(N_{2m}(c))$. $D - f^{-1}(N_{2m}(c))$ has several connected components say $D_1, \ldots, D_n$ with ∂D_i $(i = 1, \ldots, n)$ mapping to $N_{2m}(c)$. We define for each i a map $f_i : \partial D_i \to c$ as follows: If v is a vertex of ∂D_i we define $f_i(v)$ to be the closest vertex of $f(v)$ lying on c. We extend this to the edges by mapping the edge $[v_1, v_2]$ to the smallest subpath of c joining $f_i(v_1), f_i(v_2)$. We note now that for some i, say i_0, the degree of $f_{i_0} : \partial(D_{i_0}) \to c$ is bigger than 1. Indeed it is clear that $\deg(f_1) + \cdots + \deg(f_n) = \deg(f) = 1$. By Remark 3.9 above we have that the radius of D_{i_0} is bigger than $\text{length}(c)/100$. On the other hand by the separation properties of L we have that there is a vertex v of D such that $d(f(v), x) \leq N$.

Let v_1 be a vertex of D such that $d(f(v_1), c) > \text{length}(c)/100$. It is clear then that we can join x to $f(v)$ by a path not intersecting $N_{2m}(c)$ and then join $f(v)$ to $f(v_1)$ by a path lying in $f(D_{i_0})$ and hence not intersecting $N_{2m}(c)$. The union of these two paths is a path p joining x to $f(v_1) = y$. With the above notation we have the following:

Lemma 3.10. *y lies in a bounded component of $X - N_m(c)$.*

Proof. We will show that there is a point $y_1 \in p$ with $d(y, y_1) < M$ that lies in a bounded component of $X - N_m(c)$. This clearly implies the lemma since y, y_1 lie in the same component of $X - N_m(c)$. Indeed, by Lemma 3.7.1, for each point O on p we can pick a finite cross-connected set of separating quasi-lines that separate O from $S_M(O)$. Using these separating quasi-lines we can find a finite sequence of separating quasi-lines $L_1, \ldots, L_k = L$ and a sequence of points, $x = a_0 \in L_0 = L, a_1 \in L_1, \ldots, y_1 = a_n \in L_n$ such that for all i, $d(a_i, p) \leq M$ and one of the following two conditions holds for L_i, L_{i+1}:

1) An interval $[t,s]_{L_i}$ of L_i with $t, s \in B_m(a_i)$ r-crosses L_{i+1}. Moreover $a_i \in [t,s]_{L_i}$.

2) An interval $[t,s]_{L_{i+1}}$ of L_{i+1} with $t, s \in B_m(a_{i+1})$ r-crosses L_i. Moreover $a_i \in [t,s]_{L_{i+1}}$.

We explain here briefly how this sequence is constructed, starting with $O = x$ and using the construction of Lemma 3.7.1: If we call S' the set of quasi-lines separating x from $S_M(x)$ we note that there is a quasi-line $L_1 \in S'$ such that an interval of L_1 with endpoints in $B_{2M}(x)$, r-crosses L. We pick now a point $a \in p$ that lies on some quasi-line L_a in S'. Clearly a is at distance bigger than R from x. Since S' is cross-connected and L_1 r-crosses L we can find quasi-lines $L_0 = L, L_1, \dots, L_i = L_a$ in S' and intervals that satisfy at least one of the above two conditions. We pick points $a_1 = x, a_2, \dots, a_i = a$ lying on the intersection of these intervals with $S_M(x)$. We repeat the same procedure replacing x by $a = a_i$, picking in the same way a point on p that comes after a as we traverse p from x to y. We continue until we arrive at a point $y_1 = a_n \in p$ with $d(y_1, y) < M$.

Note now that by Lemmas 3.7.1 and 3.4, since x is an L-interior point of c, y_1 is an L_n-interior point of c.

Since X is one ended $X - N_m(c)$ has exactly one unbounded component, say Y. Since L_n is infinite it intersects Y. Let z be a point in $Y \cap L_n$ with $d(z, c) > \text{length}(c)$. Clearly z is an L_n exterior point of c.

On the other hand if y_1 and z both lie in Y we can join them by a path q that does not intersect $N_m(c)$. But then we can find a sequence of quasi-lines, as we did above, 'joining' y_1 to z. This however would imply that z is an L_n-interior point of c, a contradiction. We conclude therefore that y_1 lies in a bounded component of $X - N_m(c)$. Since y, y_1 lie on the same component of $X - N_m(c)$ the same is true for y. □

Lemma 3.10 easily implies that in this case G is virtually $\mathbb{Z}^2$ using Varopoulos' isoperimetric inequality ([V]) and Gromov's theorem on groups of polynomial growth ([G1]). We recall here Varopoulos' inequality as stated in [C-S]: Let H be a finitely generated group and let Y be the Cayley graph of H for a finite set of generators of cardinality, say, k. For $\Omega \subset Y$ we denote by $|\Omega|$ the number of vertices in Ω and by $\partial\Omega$ the number of vertices of Y at distance 1 from Ω (i.e., $\partial\Omega = \{t \in Y : d(t, \Omega) = 1\}$ where $d(t, \Omega) = \min\{d(t, s) : s \in \Omega\}$).

Let e be the vertex of Y corresponding to the identity element of G. Let $B_n(e) = \{x \in Y : d(x, e) \leq n\}$ and let $V(n) = |B_n(e)|$. Let $\phi : \mathbb{R}^+ \to \mathbb{N}$ be the function:

$$\phi(\lambda) = \inf\{n \in \mathbb{N} : V(n) > \lambda\}.$$

Varopoulos' inequality. H satisfies the isoperimetric inequality:

$$\frac{|\Omega|}{\phi(2|\Omega|)} \leq 8k|\partial\Omega|$$

for all $\Omega \subset H$.

As noted in [C-S] the inequality $V(n) \geq Cn^D$ implies the inequality

$$(*) \qquad |\Omega|^{\frac{D-1}{D}} \leq C|\partial\Omega|.$$

We now apply this inequality to G where we choose Ω to be the component of $X - N_m(c)$ containing y. To simplify notation we denote length$(c) = n$. In this case we have the inequalities:

$$|\Omega| \geq V(n/100),$$

$$|\partial\Omega| \leq C(m)n$$

where $C(m)$ is a constant depending only on m and G.

By the inequality $(*)$ above we have that there is a constant $C > 0$ such that $V(n) \leq Cn^2$ holds for G. Since G is one-ended we conclude that G is commensurable with $\mathbb{Z}^2$. This clearly implies Theorem 3.1 in case 1.

Case 2. We assume that G is hyperbolic. As in case 1 we can show that there is an $m > 0$ such that the m neighborhood of any bi-infinite geodesic l in X separates X. This however implies that any pair of points of the Gromov boundary of X, ∂X, separates ∂X. This is turn implies that ∂X is homeomorphic to S^1 (see [N, Ch. IV, Thm. 12.1]). Hence by the Tukia-Gabai theorem ([T], [Ga]) on convergence groups of S^1, G is commensurable to the fundamental group of the surface of genus 2. $\square$

Remark 3.11. We did not use the full force of the hypothesis of Theorem 3.1. Indeed instead of using the fact that for all $K > 0$ there is an M such that any x, y with $d(x, y) > M$ are K-separated by a quasi-line we used the weaker hypothesis that there is a sufficiently big K_1 for which there is M_1 such that any x, y with $d(x, y) > M_1$ are K_1-separated by a quasi-line, and there is a sufficiently big $K_2 > M_1$ for which there is an M_2 such that any x, y with $d(x, y) > M_2$ are K_2-separated by a quasi-line. The value of K_1 for which the above argument holds depends on f, N, G and the value of K_2 for which it holds depends on M_1, f, N, G. Roughly speaking it is enough to be able to K-separate points that are sufficiently far apart in X for some K 'big enough' rather than for all K. This will be important in the next section.

4. A more refined characterization of virtually planar groups

We keep in this section the notation of Theorem 3.1. To prove the quasi-isometry invariance of splittings over 2-ended groups we need a stronger (and more technical) version of Theorem 3.1. Roughly speaking we need to show that if a group is not virtually planar then there are unbounded connected subsets of its Cayley graph that cannot be 'cut' by quasi-lines. We make this precise below.

Definition. A connected subset of X, Y is an *r-solid subset* of X if for any separating quasi-line L, Y is contained in the r-neighborhood of a component of $X - L$.

We can now state the main result of this section:

THEOREM 4.1. *Let G be a one-ended, finitely presented group that is not commensurable to a planar group. Let X be a Cayley complex of G. Then there is an $r > 0$ such that X contains an unbounded r-solid subset.*

Proof. We will prove this by contradiction. We start with a lemma that follows easily from Theorem 3.1:

LEMMA 4.2. *Let G be a one-ended group that is not commensurable to a planar group. Let X be a Cayley complex of G. Then for any $x \in X$ there is an $r > 0$ such that there is an unbounded, connected $Y \subset X$ such that $x \in Y$ and if $y \in Y$ then any quasi-line L separating x, y intersects either $B_r(x)$ or $B_r(y)$.*

Proof. Suppose not. Note that since G acts transitively on X the conclusion of the lemma does not hold for any $x \in X$. We pick $r_1 \gg 0$ and define $F(r_1)$ to be the set of all $y \in X$ such that a quasi-line separating x from y intersects either $B_r(x)$ or $B_r(y)$. Let $Y(r_1)$ be the connected component of x in $F(r_1)$. If $Y(r_1)$ is bounded then any vertex in $\partial Y(r_1)$ is r_1-separated from x by some quasi-line. So we can apply the construction of Lemmas 3.7.1–3.7.3 with $B_M(x)$ replaced by $Y(r_1)$. If $M_1 = \mathrm{diam}(Y(r_1))$. We can define similarly $F(r_2)$ and $Y(r_2)$ for $r_2 \gg r_1$. If $Y(r_2)$ is bounded we can use the same construction and prove Lemma 3.10 under our weaker hypothesis. Therefore the proof of Theorem 3.1 applies in our setting and G is commensurable to a planar group, a contradiction. □

In the next three lemmas we show, respectively, that a 'bigon', a 'triangle' and a 'rectangle', made by crossing intervals of separating quasi-lines, separate X.

With the notation of Theorem 4.1 we have the following:

LEMMA 4.3. *Let L_1, L_2 be (f, N)-quasi-lines separating X. Let $r \gg M \gg 0$. Let I_1, I_2 be two disjoint intervals of L_1 such that I_1, I_2 r-cross L_2. Let I be an interval of L_1 containing $I_1 \cup I_2$ and let J be an interval of L_2 containing $N_r(I \cap L_2) \cap L_2$. Then there is a point x such that $d(x, I \cup J) \geq M$ and x lies in a bounded component of $X - (I \cup J)$.*

Proof. Let $r \gg r_2 \gg r_3 \gg M$. Let $I_1 = [a_1, b_1]_{L_1}, I_2 = [a_2, b_2]_{L_1}$ and suppose that $b_1 < a_2$. We assume without loss of generality that $d(b_1, L_2) \geq r$ and $b_1 \in L_2^+$. Let I_0 be a maximal subinterval of I containing b_1 such that

$I_0 \subset N_{r_3}(L_2^-)$. If J_0 is a minimal subinterval of J containing $J \cap I_0$ one sees easily that there is an $x_1 \in J_0$ such that $d(x_1, I) \geq r_2$. Clearly there is an $x \in L_1^+$ such that $x \in B_{r_2}(x_1)$ and $d(x, J \cup I_0) \geq M$. Lemma 3.2 implies then that $d(x, I \cup J) \geq M$ and that x satisfies the conclusion of Lemma 4.3. $\square$

The next two lemmas treat the cases of a 'triangle' and a 'rectangle'. To make sure that these are not degenerate we have to assume that their sides are not close to each other. We make this precise in the following definition:

Definition. Let L_1, L_2 be separating quasi-lines and let I_1, I_2 be intervals of L_1, L_2 respectively. We say that I_1 (r_1, r_2)-*fellowtravels* I_2 if there is a subinterval of I_2 of length greater than r_2 that is contained in the r_1-neighborhood of I_1.

LEMMA 4.4. *Let L_1, L_2, L_3 be (f, N)-quasi-lines separating X. Let $R \gg r_1 \gg r_2 \gg r_3 \gg r \gg M \gg 0$. Suppose that no L_i contains a pair of disjoint intervals such that either both intervals r-cross some L_j or one of them r-crosses L_j and the other (r_3, r_2)-fellowtravels L_j, $(i, j \in \mathbb{Z}/3\mathbb{Z})$. Suppose that the following hold*:

1. *There are intervals $J_i \subset L_i$ such that J_i r_1-crosses L_{i-1} and L_{i+1} $(i \in \mathbb{Z}/3\mathbb{Z})$.*
2. *If I_1, I_2 are minimal subintervals of J_1, J_2 with the property that I_1, I_2 r_1-cross L_3 then $d(I_1, I_2) \geq R$.*

Then there is a point x such that $d(x, J_1 \cup J_2 \cup J_3) \geq M$ and x lies in a bounded component of $X - (J_1 \cup J_2 \cup J_3)$.

Proof. Let J_1', J_2' be minimal subintervals of J_1, J_2 respectively such that J_1' r_1-crosses L_2 and J_2' r_1-crossses L_1. Condition 2 in the statement of the lemma implies that either $J_1' \cap I_1 = \emptyset$ or $J_2' \cap I_2 = \emptyset$ (or both these hold). We may assume without loss of generality that $J_1' \cap I_1 = \emptyset$. Then J_1' by our hypothesis does not r-cross L_3. We may suppose without loss of generality that $J_1' \subset N_r(L_3^+)$. Let I_3 be a minimal interval of L_3 containing $(I_1 \cap L_3) \cup (I_2 \cap L_3)$. Let $x_1 \in I_3$ be such that $d(x_1, J_2 \cup J_3) > r_3$. Note that such a point exists by our hypothesis that there is no interval of L_1 disjoint from I_1 that (r_3, r_2)-fellowtravels L_3. We pick $x \in L_3^+$ such that $d(x, x_1) \leq r_3/2$ and $d(x, J_1 \cup J_2 \cup J_3) \geq M$. By lemmas 3.2 and 3.2.1 it follows that x lies in a bounded component of $X - (J_1 \cup J_2 \cup J_3)$. $\square$

LEMMA 4.5. *Let L_1, L_2, L_3, L_4 be (f, N)-quasi-lines separating X. Let $R \gg r_1 \gg r_2 \gg r_3 \gg r \gg M \gg 0$. Suppose that no L_i contains a pair of disjoint intervals such that either both intervals r-cross some L_j or one of them*

r-crosses L_j and the other (r_3, r_2)-fellowtravels L_j. $(i, j \in \mathbb{Z}/4\mathbb{Z})$. Suppose that the following hold: There are intervals $J_i \subset L_i$ such that J_i r_1-crosses L_{i-1} and L_{i+1} $(i \in \mathbb{Z}/4\mathbb{Z})$. Assume further that if I_1, I_3 are minimal subintervals of J_1, J_3 with the property that I_1, I_3 r_1-cross L_4 then $d(I_1, I_3) \geq R$ and that $J_2 \subset N_r(L_4^+)$. Then there is a point x such that $d(x, J_1 \cup J_2 \cup J_3 \cup J_4) \geq M$ and x lies in a bounded component of $X - (J_1 \cup J_2 \cup J_3 \cup J_4)$.

Proof. We note that under our assumptions on L_1, L_2, L_3, L_4 the following holds (see Lemma 3.2.1): Let $J = [c, d]_{L_i}$ be a minimal interval of some L_i such that J r-crosses some L_j. Let us say that $c \in L_j^-$ and let $c_1 \in J \cap L_j$. Then $[c_1, \infty)_{L_i}$ is contained in $N_r(L_j^+)$. Moreover if x, y lie in distinct connected components of $L_j - B_r(c)$ and if p is a path joining them in $\bar{L}_j^+$ then either p intersects L_i or at least one of x, y is at distance smaller than r_2 from $[c_1, \infty)_{L_i}$. The lemma follows easily by repeated applications of this observation. We explain this in detail below and note that by our hypothesis it follows that $J_i \cap J_{i+1} \neq \emptyset$ $(i \in \mathbb{Z}/4\mathbb{Z})$.

Let I_4 be a minimal subinterval of J_4 containing $(I_1 \cap J_4) \cup (I_3 \cap J_4)$. It is easy to see that there is an $x_1 \in I_4$ such that $d(x_1, L_1 \cup L_2 \cup L_3) > r_3$. We pick $x \in B_{x_1}(r) \cap L_4^+$ such that $d(x, L_1 \cup L_2 \cup L_3 \cup L_4) > M$. To fix ideas we assume that $x \in L_i^+$ for all i. We claim that x satisfies the conclusion of the lemma. Indeed if this is not the case there is a path p lying in $\bar{L}_4^+$ joining x to a point on $L_4 - J_4$ such that p does not intersect $J_1 \cup J_2 \cup J_3 \cup J_4$. Let y be the first point of intersection of p with $L_1 \cup L_2 \cup L_3 \cup L_4$. Suppose that $y \in L_1$. The observation we made above implies that $d(y, L_2) < r_2$. This in turn implies that $d(y, L_3) < 2r_2$. This however is impossible. We argue similarly if y lies in one of L_2, L_3, L_4. □

Now, let $x \in X$ and let $r \gg 0$ be such that $F(r), Y(r)$ are unbounded where

$$F(r) = \{y \in X\colon \text{ if a quasi-line } L \text{ separates } x, y \text{ then } L \text{ intersects } B_r(x) \cup B_r(y)\}$$

and $Y(r)$ is the connected component of x in $F(r)$.

Let $(y_1, y_2, \dots)$ be an infinite path of distinct vertices in $Y(r)$ such that $d(y_i, y_{i+1}) = 1$ for all i. Then by our hypothesis that Theorem 4.1 does not hold we have that for any $n > 0$ there are $i > j$ such that y_i, y_j are n-separated by a quasi-line L_n. By the definition of $Y(r)$ L_n intersects $B_r(x)$. Let us say that L_n intersects $B_r(x)$ at a_n and the path $(y_j, \dots, y_i)$ at a vertex b_n. Clearly length$([a_n, b_n]_{L_n})$ tends to infinity as $n \to \infty$. We can choose n and $r_1 > r$ big enough in this construction to ensure that no vertex of $[a_n, b_n]_{L_n}$ is r_1-separated from x by some quasi-line. We explain this in detail: Suppose that some $t \in [a_n, b_n]_{L_n}$ is r_1-separated from x by some quasi-line L. For sufficiently big r_1 there are $t_1 < t < t_2$ on L_n such that t_1, t_2 are r-separated from t by L (see Remark 3.3). Since L_n n-separates some vertices $y_i, y_j \in Y(r)$

and since L does not intersect $B_x(r)$ we have that either x, y_i, y_j lie in the same component of $X - L$ or L separates y_i or y_j or both y_i, y_j from x and in these cases L intersects respectively $B_r(y_i)$, $B_r(y_j)$ or both $B_r(y_i)$,$B_r(y_j)$. In all cases we see easily that L contains two disjoint intervals which r_1-cross L_n. Lemma 4.3 implies that for some M such that $r_1 \gg M \gg 0$ there are intervals $I \subset L$ $J \subset L_n$ such that the ball $B_x(M)$ is contained in a bounded component of $X - (I \cup J)$.

We can now repeat the same argument replacing r_1 by $r_2 \gg r_1$. As we noted in Remark 3.11 these constructions imply that G is virtually a planar group. We assume therefore that r_1 is chosen big enough so that no vertex of $[a_n, b_n]_{L_n}$ is r_1-separated from x by some quasi-line.

Since X is locally finite, we can find a subsequence of L_n that we still denote L_n for convenience, so that the following holds:

(∗) For all $i > j$, $[a_i, b_i]_{L_i} \cap [a_j, b_j]_{L_j}$ contains an interval $[a_j, c_j]_{L_j}$ and $\lim(\text{length}([a_j, c_j]_{L_j})) = \infty$, as $j \to \infty$.

Let us denote by L the set of all $t \in X$ such that $t \in [a_j, c_j]_{L_j}$ for all except possibly finitely many j. We remark that L is connected unbounded and contained in $Y(r_1)$ (where $Y(r_1)$ is defined in a similar way as $Y(r)$. We note also that if F is a finite set of points in L then there is an interval $[a_j, c_j]_{L_j}$ containing F.

Now, let $(z_1, z_2, \dots)$ be an infinite path of distinct vertices in L such that $d(z_1, x) \leq r$ and $d(z_i, z_{i+1}) = 1$ for all i. Let $R \gg r_2 \gg r_1$. By our hypothesis that Theorem 4.1 does not hold we have that for any $n > 0$ there are $i > j > n$ such that z_i, z_j are r_2-separated by a quasi-line l_n. Clearly we can pick k big enough so that l_n r_2-crosses L_k. We can also suppose that l_n does not contain two disjoint intervals that r_1-cross L_k. Indeed it suffices to choose r_1 big enough, as Lemma 4.3 implies that if this fails for all r_1 then G is a virtually surface group.

Let r_3, r_4 be such that $r_2 \gg r_3 \gg r_4 \gg r_1$. Let I_n be a minimal interval of l_n that r_2-crosses L_k. We can assume that l_n does not contain an interval disjoint from I_n which (r_4, r_3)-fellowtravels L_k.

Indeed suppose that this is not the case for any $r_2 \gg r_3 \gg 0$. Let J be an interval of L_k of length bigger than r_3 that is contained in the r_4-neighborhood of an interval of l_n. By our hypothesis that Theorem 4.1 does not hold and since $r_3 \gg 0$, there are $x, y \in J$ and a quasi-line L_0 such that L_0 is r_2 separating x, y. Let J_1 be a minimal interval of l_n containing I_n such that J is contained in the r_4 neighborhood of J_1. Let J_2 be a minimal interval of L_k containing $J_1 \cap L_k$ and J. Let J_3 be a minimal interval of L_0 that intersects $[x, y]_{L_k}$ and r_3-crosses both L_k and l_n. We see then as in Lemma 4.4 that there is a point $z \in X$ such that z lies in a bounded component of $X - (J_1 \cup J_2 \cup J_3)$ and $d(x, J_1 \cup J_2 \cup J_3) \gg M$ where $M \gg r_1$. Clearly we can repeat this construction choosing bigger values for r_2, r_3, M. By Remark 3.11 this implies that G is virtually a planar group.

Since all l_n intersect the r_1 neighborhood of x, given $R \gg r_2$ we can find quasi-lines l_t, l_s such that $l_t \cap l_s$ contains an interval I of length bigger than R intersecting the r_1-neighborhood of x. We observe now that we can pick k big enough such that l_t, l_s r_2-cross L_k. We suppose, to fix ideas, that $I_t \subset l_t, I_s \subset l_s, I_k \subset L_k \cap L$ are bounded intervals such that I_t, I_s r_2-cross L_k, $I \subset I_t \cap I_s$, I_k contains $I_t \cap L_k$ and $I_s \cap L_k$ and I_k r_3-crosses both l_s, l_t. We can also choose l_t, l_s so that if $J_t \subset I_t, J_s \subset I_s$ are minimal subintervals of I_t, I_s that r_1-cross L_k then $d(J_t, J_s) > R$.

We can suppose,by picking r_1 big enough, that I_t does not r_1-cross l_s. Indeed if this is not the case, by Lemma 4.5 and Remark 3.11 we have that G is virtually a planar group.

Since $R \gg 0$ there are points $x, y \in I$ and a quasi-line L_0 such that L_0 r_2-separates x, y. Let J be a minimal interval of L_0 that contains $I \cap L_0$ which r_3-crosses l_t, l_s. By applying Lemma 4.5 to the intervals I_t, I_s, J, I_k and by Remark 3.11 we see that G is virtually a planar group. This is a contradiction. $\square$

5. Large scale geometry of groups

In this section we assume that G is an one-ended group that is not virtually a surface group. We denote as usual by X the Cayley complex of G. We suppose that an (f, N) quasi-line separates X.

Our purpose here is to prove Theorem 1 when the Cayley graph of G contains 'big' solid subsets. We showed in Section 4 that there are always unbounded solid subsets. Here we will require something stronger, namely that there are solid subsets that are not contained in finite neighborhoods of quasi-lines. Note that this is in fact always the case when the JSJ-decomposition of G has some nonhanging orbifold vertex group which is not 2-ended. An example to keep in mind is $G = \mathbb{Z}^3 *_{\mathbb{Z}} \mathbb{Z}^3$. Maximal solid subsets in this group are contained in finite neighborhoods of $\mathbb{Z}^3$ subgroups. An important step is Lemma 5.2 below stating that if X contains an unbounded r-solid subset, F, which is not a quasi-line then there is a finitely generated subgroup of G acting co-compactly on this subset. This lemma provides a first link between geometry and algebra. The idea then is to show that a finite neighborhood of F separates X. The intersection of the closure of a component of $X - F$ with F is a quasi-line. By an argument similar to that of Lemma 5.2 we show that this quasi-line is contained in a finite neighborhood of a 2-ended group. Using [D-Sw] we conclude that Theorem 1 holds in this case.

We continue this section by characterizing groups which do not contain 'big' solid subsets (Lemma 5.1). We think of this case as 'exceptional' and will treat it in the next section.

By Theorem 4.1, X contains an unbounded r-solid subset. Let F be a maximal, r-solid subset of X (where we order r-solid subsets by inclusion). Note that a maximal r-solid subset exists by Zorn's lemma. We distinguish now two cases:

Case 1. For any $n > 0$ there is an $x \in F$ such that $B_n(x) \cap F$ is contained in the r-neighborhood of some (f, N) separating quasi-line of X.

Case 2. The hypothesis of case 1 does not hold.

Case 1 will be treated in detail in the next section. It is in a sense 'exceptional'. In this case there are maximal r-solid subsets that are contained in quasi-lines. More precisely we have the following lemma:

LEMMA 5.1. *Let F be a maximal r-solid subset of X. Suppose that for any $n > 0$ there is an $x \in F$ such that $B_n(x) \cap F$ is contained in the r-neighborhood of some (f, N) quasi-line of X. Then there are a $K > 0$ and a maximal r-solid subset of X, F_1, such that F_1 is contained in an $(f, N + r)$ quasi-line L and L is contained in the K-neighborhood of F_1.*

Proof. Clearly the r-neighborhood of an (f, N) quasi-line is an (f, N')-quasi-line where $N' = N + r$. We pick K so that the following holds: For any (f, N') quasi-line L if a path p with endpoints a, b is contained in L then $[a, b]_L$ is contained in the K-neighborhood of p.

For each $n \in \mathbb{N}$ we pick x_n such that $B_n(x) \cap F$ is contained in some (f, N') quasi-line L_n. Let I_n be an interval of L_n contained in the K-neighborhood of $B_n(x) \cap F$ with $\mathrm{diam}(I_n) > n/2$. We translate I_n to the origin to an interval $g_n I_n = [a_n, b_n]$ so that $d(e, a_n) \geq n/4, d(e, b_n) \geq n/4$.

Since X is locally finite we can find a subsequence $[a_{n_i}, b_{n_i}]$ such that the following two conditions are satisfied:

(1) For all $i < j$ the intersection of $[a_{n_i}, b_{n_i}] \cap [a_{n_j}, b_{n_j}]$ contains an interval of an (f, N') quasi-line, $[c_i, d_i]$, such that $d(c_i, e) > i, d(d_i, e) > i$.

(2) $[c_i, d_i] \cap g_{n_i} F = [c_j, d_j] \cap g_{n_j} F$.

We re-index this subsequence to avoid double indices, so we write $[a_i, b_i]$ for $[a_{n_i}, b_{n_i}]$. Let now L be the set of points lying in all but finitely many of the $[c_i, d_i]$. Clearly L is an (f, N') quasi-line. Let F_1 be the set of points that lie in $[c_i, d_i] \cap g_i F$ for all but finitely many i. It is easy to see that F_1 is a maximal r-solid subset verifying the conditions of the lemma. $\square$

We now turn our attention to case 2.

LEMMA 5.2. *Let F be a maximal r-solid subset of X. Suppose that there is an $n > 0$ such that for all $x \in F$, $B_n(x) \cap F$ is not contained in the*

r-neighborhood of any (f,N) quasi-line of X. Then there is a finitely generated subgroup of G, H, such that $hF = F$ for all $h \in H$ and H acts co-compactly on F.

Proof. We consider the set of subsets of $B_e(n)$ and call it S. Let t be a vertex of F and let $g \in G$ be such that $gt = e$. We say that the type of a vertex t of F is s, where $s \in S$, if $gB_n(t) \cap F = s$. Clearly there are only finitely many types. Let $x \in F$ and $R > 0$ be such that for every vertex v of F there is a vertex in $B_R(x) \cap F$ with the same type as v. For each vertex $v_i \in B_{2R}(x) \cap F$ we pick $h_i \in G$ such that $h_i(v_i)$ is a vertex in $B_R(x) \cap F$ with the same type as v_i. We claim that the subgroup, H, of G generated by the h_i's satisfies the requirements of the lemma. We note that for all i,

$$F \cap B_n(h_i v_i) \subset h_i F \cap F.$$

If a vertex $v \in h_i F$ does not lie in F then there is a quasi-line L that r-separates v from a vertex of F, say u. Since $F \cap B_n(h_i v_i)$ is not contained in the r-neighborhood of any quasi-line, there is a point in $F \cap B_n(h_i v_i)$ that is r-separated by L either from v or from u. But this contradicts the fact that F and $h_i F$ are r-solid subsets. Hence $h_i F = F$.

We show now that H acts co-compactly on F: Let $v \in F$, $v \notin B_n(x)$, such that the shortest path in F that joins v to x is of length n. We call this path p and let v_j be the first vertex in p lying in $B_{2R}(x)$. Then $h_j v$ is joined to x by a path lying in F of length less than $n - R$. This shows that for any $v \in F$ there is an $h \in H$ such that $hv \in B_R(x)$. $\square$

If we consider F as a metric space, where the distance between any two points in F is defined to be the infimum of the lengths of the paths in F joining them, then it follows from Lemma 5.2 that the obvious map $i : F \to X$ is uniformly proper.

For $K \geq 0$ we consider the components of $X - N_K(F)$. Since H acts co-compactly on F there are finitely many such components modulo the action of H. Let C be a component of $X - N_K(F)$. We denote by ∂C the intersection $\overline{C} \cap N_K(F)$. Since F is connected and X simply connected we see that ∂C is connected. Arguing as in Lemma 5.2 we can see that there is a finitely generated subgroup of H, say H_C, such that, for all $h \in H_C$, $hC = C$ and H_C acts co-compactly on ∂C. Indeed, by fixing a compact $D \subset N_K(F)$ such that $HD = N_K(F)$ we have that only finitely many components of $X - N_K(F)$ meet D. Let us call the set of these components S, and let C be any component of $X - N_K(F)$. For any $v \in \partial C$ we pick an $h_v \in H$ such that $h_v v \in D$. We say then that the type of v is $h_v(C) \in S$. Clearly there are finitely many types and by considering $D_1 \subset \partial C$, D_1 compact, connected, containing at least one vertex of each type, we see, as in Lemma 5.2, that there is a finitely generated subgroup of H, say H_C, such that $H_C D_1 = \partial C$.

If C is a connected component of $X - N_K(F)$ we can consider ∂C as a metric space where the distance of two points is defined to be the length of the shortest path in ∂C joining them. Then the obvious inclusion map $i : \partial C \to X$ is uniformly proper.

We show now that if a quasi-line separates X and we are in case 2 then G splits over a two-ended group.

PROPOSITION 5.3. *Let G be a one-ended finitely presented group that is not virtually planar, and let X be a Cayley complex of G. Suppose that an (f, N)-quasi-line separates X and let F be an unbounded maximal r-solid subset of X. With the assumption that there is an $n > 0$ such that for all $x \in F$, $B_n(x) \cap F$ is not contained in the r-neighborhood of any (f, N)-quasi-line, G splits over a two-ended group.*

Proof. We assume in this proof that $r \gg 0$. We can do that since if there is an r_1-solid subset of X, F_1, with the property required in Proposition 5.3, then for any $r > r_1$ there is a maximal r-solid subset of X, F, containing F_1. By Lemma 5.2 there is a subgroup of G, H, acting co-compactly on F. If F is contained in an r-neighborhood of a separating quasi-line then H is 2-ended and X/H has more than one end. In this case the result follows by the 'annulus theorem' of Dunwoody-Swenson ([D-Sw]).

We assume also that for every separating quasi-line L, $X - L$ has exactly two essential components. Indeed, if not, there is an $r > 0$ so that L is an r-solid subset of X. If the maximal r-solid subset of X containing L is as in case 1 then the result follows by our treatment of case 1 in the next section. If the maximal r-solid subset of X containing L is as in case 2 then clearly there is a 2-ended subgroup of G acting co-compactly on L and the result follows by the 'annulus theorem' of Dunwoody-Swenson ([D-Sw]).

We will distinguish two cases:

Case 1: F with the path metric is quasi-isometric to $\mathbb{R}$.

Case 2: F with the path metric is not quasi-isometric to $\mathbb{R}$.

For case 1, it suffices to show that some neighborhood of F separates X in at least two components, none of which is contained in a neighborhood of F. Indeed there is a two-ended subgroup of G, say H, acting co-compactly on F. If a neighborhood of F separates X then X/F has at least two ends and by the annulus theorem of Dunwoody-Swenson ([D-Sw]) G splits over a two-ended group.

If there is a $c > 0$, separating quasi-lines L_n and intervals $[x_n, y_n]_{L_n}$ such that $d([x_n, y_n]_{L_n}, F) < c$ and $\text{length}([x_n, y_n]_{L_n}) > n$ then clearly $N_c(F)$ separates X as required above. Indeed using H we can translate all L_n so that $d([x_n, y_n]_{L_n}, x) < c$ and $d(x_n, x), d(y_n, x) \to \infty$ as $n \to \infty$ (where x is a fixed vertex of F). Then there is a subsequence of L_n converging on compact sets to a separating quasi-line L_∞ contained in $N_c(F)$, so that $N_c(F)$ clearly separates. We suppose therefore that no such sequence L_n exists.

Since the proposition is clearly true if $X-F$ has more than one component that is not contained in a neighborhood of F, we assume that $X-F$ has exactly one such component, say A. We denote by ∂F the intersection $\bar{A}\cap F$. Since X is simply connected ∂F is connected and is in fact a quasi-line. Let l be a line contained in ∂F. Using the fact that H acts co-compactly on F and the remark made earlier it is easy to see that the following holds:

For every $v\in\partial F$ there is a separating quasi-line L such that L $(r-1)$-separates v from some vertex of F and v is contained in a bounded component of $F-L$. Moreover for any $c>0$ every connected component of the set $N_c(F)\cap L$ is bounded.

If we denote by F' a maximal unbounded $(r-1)$-solid subset of F we have that F' is contained in the $(r-1)$-neighborhood of an essential component of $X-L$ that does not contain v. If a quasi-line L has this property we say that L $(r-1)$-separates v from F'.

In fact it is sufficient to pick finitely many quasi-lines having this property for a finite set of vertices of ∂F and then consider the images of these quasi-lines by H. In this way we obtain a set S of quasi-lines such that for each $v\in\partial F$ there is an $L\in S$ such that $d(v,L)=r-1$ and L $(r-1)$-separates v from F'.

Let L^+ be the essential component of $X-L$ which has the property that its $(r-1)$-neighborhood contains F' and L^- be the essential component of $X-L$ containing a vertex $v\in F$ such that $d(v,L)=r-1$. We will say that L $(r-k)$-separates x from F' if $x\in L^-$ and $d(x,L)\geq r-K$.

Some auxiliary lemmas about quasi-lines in S are needed. We state them below using the above notation.

Lemma 5.3.1. *Let $K>0$ be a given constant. Then for r sufficiently big there is a constant $C>0$ so that the following holds. Let $L_1,\ L_2\in S$ and let $v_1,v_2\in l$ such that L_i $(i=1,2)$, $(r-1)$-separates v_i from F'. Let $(a_1,b_1)_l,(a_2,b_2)_l$ be, respectively, the connected components of $l-L_1$, $l-L_1$ containing v_1, v_2. Suppose that L_2 $(r-1)$-separates b_1 from F'. Then there is a point x in $[a_2-C,b_2+C]_{L_2}$ such that $d(x,L_1)>K$ and $x\in L_1^+$.*

Proof. As usual we do not give explicit estimates for C,r. The argument below holds for $C\gg r\gg K\gg 0$. Let C_1,C_2 be such that $C\gg C_2\gg C_1\gg r$. Let b_1' be the closest point to b_1 in L_1^+ such that $d(b_1,L_1)>K$. Let $c\in l$ be a point in $B_{C_1}(a_2)$ such that $c\in L_1^+\cap L_2^+$ and $d(c,L_1)>K$. Let p be a path joining c to b_1' in $X-N_K(L_1)$. Such a path exists by Lemma 3.6 and p then intersects $[a_2-C,b_2+C]_{L_2}$ at a point x as required by the lemma since c,b_1' are separated by L_2. $\square$

LEMMA 5.3.2. *Let $K > 0$ be a given constant. Then for r sufficiently big there is a constant $C > 0$ so that the following holds. Let L_1, $L_2 \in S$ be such that L_i $(r-1)$-separates v_i from F' $(i = 1, 2)$. Let $(a_1, b_1)_l, (a_2, b_2)_l$ be, respectively, the connected components of of $l - L_1, l - L_2$ containing v_1, v_2. Suppose that L_1 $(r-1)$-separates some point $a \in [a_1, b_1]_l$ from F. Suppose that L_2 $(r-k)$-separates b_1 from F' but that it does not $(r-k)$-separate a from F'. Then there is a point y in $[a_2 - C, b_2 + C]_{L_2}$ such that $d(y, L_1) > K$ and $y \in L_1^-$.*

Proof. The argument below holds for $C \gg r \gg K$. Clearly if $a \in L_2^-$ the conclusion holds. We assume therefore that $a \in L_2^+$. Let C_1 be such that $C \gg C_1 \gg r$. Let p be a path joining a to b_1, lying in $L_1^- \cap B_{C_1}(a_2)$ that satisfies the following: if a point of p is at distance less than K from L_1 then it lies at distance less than $(r-k)$ from b_1. It is easy to see that such a path exists for $r \gg K$. Clearly L_2 intersects p at a point y satisfying the conclusion of the lemma. $\square$

In order to show that a neighborhood of F separates we will show first how to 'approximate' F by a sequence of quasi-lines such that any two successive quasi-lines cross near F. We make this precise and explain it below. We order as usual the points of l by identifying them with $\mathbb{R}$.

We define a sequence of quasi-lines L_n as follows: Fix $a \in l$ and pick $L_1 \in S$ so that L_1 $(r-1)$-separates a from F'. If $[a_1, b_1]_l$ (where $b_1 > a_1$) is the connected component of $l \cap \bar{L}_1^-$ containing a we pick $L_2 \in S$ so that it $(r-1)$-separates b_1 from F'. Continuing inductively, i.e. given L_k so that $[a_k, b_k]_l$ $(b_k > a_k)$ is the connected component of $l \cap \bar{L}_k^-$ containing b_{k-1}, we pick L_{k+1} so that it $(r-1)$-separates b_k from F'.

Given $K, C > 0$, we define two relations on the sequence $\{L_n : n \in \mathbb{N}\}$:'$>$' and 'cross'. We write $L_m > L_n$ if L_m $(r-k)$-separates b_n from F'. We say that L_m crosses L_n if $[a_m - C, b_m - C]_{L_m}$ K-crosses L_n.

Lemmas 5.3.1, 5.3.2 imply that we can choose r, K, C so that the following hold: $L_n > L_{n-1}$ for all $n > 1$ and if $L_n > L_{n-1}, L_n > L_{n-2}, \ldots, L_n > L_{n-m}$ hold but $L_n > L_{n-m-1}$ does not hold then L_n cross L_{n-m}. We note that there is an m_0 such that for any n and for $m > m_0$, $L_n > L_{n-m}$ does not hold. Now, we have the following easy lemma:

LEMMA 5.3.3. *Let $(L_1, L_2, \ldots)$ be an infinite sequence for which two relations '$>$' and 'cross' are defined that satisfy the following properties:*

a) *$L_n > L_{n-1}$ for all $n > 1$.*

b) *There is an m_0 such that for any n and for $m > m_0$, $L_n > L_{n-m}$ does not hold.*

c) *If* $L_n > L_{n-1}, L_n > L_{n-2}, \dots, L_n > L_{n-m}$ *and* $L_n > L_{n-m-1}$ *does not hold then* L_n *crosses* L_{n-m}.

Then there is an infinite subsequence $L_{i_1}, L_{i_2}, \dots$ *such that for all* n, L_{i_n} *crosses* $L_{i_{n-1}}$ *and either* $L_{i_1} > L_1$ *or* $L_{i_1} = L_1$.

Proof. Clearly a), b), c) imply that if $n > m_0$ then there is a $k < n$ so that L_n crosses L_k. In fact by a) and c) we see also that if $L_n > L_1$ does not hold then L_n crosses some L_k where $n > k \geq 1$. So we can construct inductively arbitrarily big finite subsequences satisfying the conclusion of the lemma. But then by a standard argument we see that such an infinite subsequence also exists. $\square$

We now pick $K > 0$ so that Lemma 3.4 holds for quasi-lines that K-cross (i.e. K plays the role of r in the notation of Lemma 3.4). We pick r, C so that Lemmas 5.3.1, 5.3.2 hold and as in Lemma 5.3.3 we construct an infinite subsequence of $(L_1, L_2, \dots)$ satisfying the conclusion of 5.3.3. We re-index this subsequence, to simplify notation, and we call it still $(L_1, L_2, \dots)$.

Let us say that L_i $(r-1)$-separates from F' a vertex v_i. Since H acts cocompactly on F there is an $M > 0$ so that for each $i \in \mathbb{N}$ there is $h_i \in H$ so that $d(h_i v_i, v_1) < M$. We consider then the sequence $(h_i L_1, h_i L_2, \dots)$. Recall that by the construction of S, $h_i L_n \in S$ for all n. By a standard argument similar to the one of Lemma 5.3.3 we see that there is a bi-infinite sequence of quasi-lines in S, such that for all $v \in l$ there is some $v' \in L$ with $d(v, v') < M$ such that v' is $(r-1)$-separated from F' by some element of this bi-infinite sequence. To keep notation simple we denote this bi-infinite sequence by $(\dots, L_{-1}, L_0, L_1, L_2, \dots)$.

To prove Proposition 5.3 it is enough, by the torus theorem ([D-Sw]) to show now that there is a $C > 0$ so that the C-neighborhood of F separates X, i.e. we will show that $X - N_C(F)$ contains at least two components that are not contained in any neighborhood of F. For this it suffices to show the following: For each $n > 0$ there are u_n, v_n such that $d(u_n, F) > n$, $d(v_n, F) > n$ and u_n, v_n are separated by $N_C(F)$. We will explain below how to pick u_n, v_n and show that the C-neighborhood of F separates X where C is a constant so that Lemmas 5.3.1, 5.3.2 (and the previous construction) hold.

Let $v_0 \in l$ be a vertex $(r-1)$-separated from F' by L_0. We consider vertices x_n, y_n lying in distinct infinite connected components of $L_0 - B_C(v_0)$. We suppose that $d(x_n, F) \gg n, d(y_n, F) \gg n$ and $n \gg C$. We join x_n, y_n in L_0^- by a simple path p such that $d(p, F) \gg n$. We pick $u_n \in p$ so that $d(u_n, L_0) > n$ and join x_n, y_n in L_0^+ by a simple path q such that $d(v_0, q) \gg n$. We arrange so that p, q intersect only at x_n, y_n.

Considering the simple closed curve $c = p \cup q$, we see clearly that v_0 is a (C, L_0)-interior point of this curve. We consider now the connected components of the intersection $q \cap N_C(F)$. Note that $F - B_C(v_0)$ has two infinite connected components, say F_1, F_2. Let $t_1, \dots, t_m$ be vertices of q so that $q \cap N_C(F)$ is contained in $[t_1, t_m]_q$, $[t_i, t_{i+1})_q$ intersects exactly one of $N_C(F_1), N_C(F_2)$ for all i and $t_i \in q \cap N_C(F)$ for all i. If n is sufficiently big we can choose $s_i \in (t_i, t_{i+1})_q$ for all i so that $d(s_i, N_C(F)) > n$. We claim now that at least one of the s_i is separated from u_n by $N_C(F)$. Suppose not, then we can join each s_i to u_n by a path, say p_i that does not meet $N_C(F)$. Without loss of generality we can assume that this path meets exactly one of the infinite components of $L_0 - B_{r+2N}(v_0)$ and that it meets q only at s_i. Moreover we can assume that $p_i \cap L_0$ is connected.

If there is some i so that p_i, p_{i+1} intersect distinct infinite components of $L_0 - B_{r+2N}(v_0)$, we consider the simple closed curve $c_i = p_i \cup [s_i, s_{i+1}]_q \cup p_{i+1}$. Clearly v_0 is a (C, L_0)-interior point of this curve. On the other hand c_i intersects exactly one of $N_C(F_1), N_C(F_2)$, say $N_C(F_1)$. If we pick a point w sufficiently far from v_0 on $N_C(F_2)$ lying on some quasi-line L_i then w is a (C, L_i)-exterior point of this curve. We see then as in the proof of Lemma 3.10 that $N_C(F_2)$ has to intersect c_i, a contradiction.

If for all i, p_i, p_{i+1} intersect the same component of $L_0 - B_{r+2N}(v_0)$, say the one containing y_n, then we can argue similarly by considering the curve obtained by the union of q_1 with the subpath of $p \cup q$ with endpoints s_1, u_n containing x_n. Therefore in either case we see that there is a point v_n separated from u_n by $N_C(F)$ and neither u_n nor v_n is contained in the n-neighborhood of F. This proves Proposition 5.3 in case 1.

Case 2 can be treated similarly to case 1. We recall as remarked before Proposition 5.3 that the following holds: If we consider for $C > 0$ a connected component, say A, of $X - N_C(F)$ and if we denote by ∂A the intersection $\bar{A} \cap N_C(F)$ we have that a finitely generated subgroup of H acts co-compactly on ∂A. Here we denote as usual by H the finitely generated subgroup of G acting co-compactly on F, provided by Lemma 5.2.

By the torus theorem ([D-Sw]), to conclude in case 2, it is enough to show that for some $C > 0$ there is a connected component, say A, of $X - N_C(F)$, that does not lie in a neighborhood of F and such that ∂A is a quasi-line. Now, let A be a connected component of $X - F$, that does not lie in a neighborhood of F, and let $\partial A = \bar{A} \cap F$. Let l be a line in ∂A. As in case 1, for every $v \in l$ there is a separating quasi-line L $(r-1)$-separating v from F. We pick now C, r, K, $C \gg r \gg K$ so that Lemmas 5.3.1, 5.3.2 hold and we 'approximate' l by crossing separating quasi-lines as we did in case 1. Note that in this case we cannot suppose that there is a subgroup of G preserving l and acting on it with finite quotient. We can however construct, as in Lemmas 5.3.1, 5.3.2, 5.3.3, a sequence of separating quasi-lines $(L_1, L_2, \dots)$ so that L_i $(r-1)$-separates from F a point v_i in l and L_i crosses L_{i-1} for all $i > 1$ (where 'cross' is defined exactly as in case 1). Since H acts co-compactly on F there is an $M > 0$ so that for each $i > 1$ there is an $h_i \in H$ so that

$d(h_i v_i, v_1) < M$. As in case 1 we remark that we can extract a subsequence of the sequence of sequences $(h_i L_1, h_i L_2, \dots)$ $(i = 1, 2, \dots)$ 'converging' to a bi-infinite sequence of separating quasi-lines. To keep notation simple we denote this bi-infinite sequence by $(\dots, L_{-1}, L_0, L_1, \dots)$. We can arrange when extracting this subsequence so that $h_i l$ also converges to a line $l_1 \subset F$.

We note that each L_i crosses L_{i-1} $(i \in \mathbb{Z})$ and each L_i $(r-1)$-separates a point, $u_i \in F$ from F. As in case 1 we can construct a sequence $w_n \in X$ such that the C-neighborhood of l_1 separates w_n from F and $d(w_n, F) > n$. Clearly there is a connected component B of $X - N_C(F)$ that contains infinitely many elements of the sequence (w_n) and ∂B is a quasi-line. This implies Proposition 5.3 in case 2. □

6. The exceptional case

In this section we treat case 1 of Section 5 assuming that there is a maximal r'-solid subset of X, F, such that F is contained in the K-neighborhood of some separating (f, N)-quasi-line L and L is contained in the K-neighborhood of F for some $K > 0$. We distinguish two cases:

Case 1: $X - L$ has more than two essential components.

Case 2: $X - L$ has exactly two essential components.

We give an informal outline before going into the technical details: This section is about the geometry of groups which have the property that all vertex groups of their JSJ decomposition which are not hanging orbifold groups are two-ended (for example, Baumslag-Solitar groups have this property). In such groups maximal solid subsets are at finite Hausdorff distance from edge groups, so they are quasi-lines. So solid subsets in this case are 'small' and we cannot directly conclude that they are at finite distance from subgroups as in Section 5. We show however that this is true under the additional assumption that a maximal solid subset L separates X into at least 3 components (case 1). Note that this is the case for example in Baumslag-Solitar groups. In such groups bigger and bigger finite neighborhoods of L separate X into more and more components. The idea of the argument now is the following: if $x, y \in L$ and $g \in G$ such that $gx = y$ then gL is contained in a bounded Hausdorff neighborhood of L (where the bound does not depend on x, y). Indeed gL cannot 'escape' far away from L, since gL would not separate X then. We show finally that by carefully picking g we can find an infinite cyclic group $\langle g \rangle$ at finite Hausdorff distance from L. We then apply [D-Sw] to conclude the outline.

In fact all groups dealt with in this section (except virtual surface groups) fall in case 1. Case 2 is there for the following technical reason: Let $G = \pi_1(S)$ be the fundamental group of a surface and assume that our L corresponds to a leaf of a lamination of S. Of course if we enlarge the set of quasi-lines, L will not be solid. The problem is that we have no natural way to increase the set of quasi-lines and see directly that G is a surface group. So in this case we pass from the set of translates of L to an $\mathbb{R}$-tree and use Rips' theory to conclude.

We will deal first with case 1. Assume first that the following holds:

$(*)$: There are an $x \in L$ and an $R > 0$ such that if $x \in gL \cap L$ then gL is contained in the R-neighborhood of L. We denote by $C_1, \dots, C_n$ the essential components of $X - L$ and by $D_1, \dots, D_m$ the essential components of $X - N_R(L)$ (where $N_R(L)$ is the R-neighborhood of L).

Suppose now that for some $g \in G$, $x \in gL \cap L$. Then for each $i = 1, \dots, n$ there are $k_1, \dots, k_i$ such that gC_i is contained in a finite neighborhood of $D_{k_1} \cup \dots \cup D_{k_i}$ and $D_{k_1} \cup \dots \cup D_{k_i}$ is contained in a finite neighborhood of gC_i. We pick $x_i \in L$ and $g_i \in G$ such that $d(x, x_i) \to \infty$ and $g_i x_i = x$. By the remark above we see that there are infinitely many pairs x_i, x_j such that $g_i^{-1} g_j C_k$ is contained in a finite neighborhood of C_k and C_k is contained in a finite neighborhood of $g_i^{-1} g_j C_k$, for all $k = 1, \dots, n$. We pick x_i, x_j satisfying this with $d(x_i, x_j) \gg 0$ and write $g = g_i^{-1} g_j$. Clearly C_t is contained in a finite neighborhood of $g^s C_t$ for all $s \in \mathbb{Z}$ and for all $t = 1, \dots, n$.

If $g^k L \cap L = \emptyset$ for some k then the essential component of $X - L$ containing $g^k L$ contains more than one of $g^k C_i$ $(i = 1, \dots, n)$, a contradiction. This implies that $g^k L$ is contained in the R-neighborhood of L for all $k \in \mathbb{Z}$. By our assumption that $d(x_i, x_j) \gg 0$, the order of g is infinite. Clearly then $X/\langle g \rangle$ has more than one end and by [D-Sw] it follows that G splits over a virtually cyclic group.

Assume now that $(*)$ does not hold. We fix $x \in L$ and we pick $x_i \in L$ and $g_i \in G$ such that $d(x, x_i) \to \infty$ and $g_i x_i = x$. By passing if necessary to a subsequence we can arrange so that the following holds: If $i > j$ then $g_j L \cap B_j(x) = g_i L \cap B_j(x)$. Since L is r-solid, for some $r > 0$, for each i, j we have that $g_i L$ is contained in the $r + 2N$-neighborhood of some essential component of $X - g_j L$.

It is clear then, by the argument given, assuming $(*)$, that G splits over a virtually cyclic group if for some i there are infinitely many j so that $g_i L$ is contained in the $r+2N$-neighborhood of $g_j L$. Therefore, by passing if necessary to a subsequence, we can assume that for each i, j neither $g_i L$ is contained in the $r + 2N$-neighborhood of $g_j L$ nor is $g_j L$ contained in the $r + 2N$-neighborhood of $g_i L$.

We fix now $i, j > 0$ and we consider $g_i L, g_j L$. Let $C_1, \dots, C_n$ be the essential components of $X - g_i L$ and let $D_1, \dots, D_n$ be the essential components of $X - g_j L$. We may assume without loss of generality that $g_i L$ is contained in the $r+2N$-neighborhood of D_1 and $g_j L$ is contained in the $r+2N$-neighborhood of C_1.

Let $y \in g_j L$ such that $d(y, g_i L) > r + 2N$. If $y' \in D_k$, where $k \neq 1$, and $d(y', g_i L) > r + 2N$ then y', y both lie in C_1. This shows that for all $k \neq 1$ D_k is contained in the $r + 2N$-neighborhood of C_1. Similarly for $k \neq 1$, C_k is contained in the $r + 2N$-neighborhood of D_1.

We pick now $M > 0$ such that for each i, $B_M(x)$ intersects all essential components of $X - g_i L$. For $a, b \in B_M(x) - g_M L$ we write $a \sim b$ if there is an i_0 such that for all $i > i_0$ a, b lie in the same essential component of $X - g_i L$. Clearly $\sim$ is an equivalence relation. By the observations above we have that there are at least $2n - 2$ equivalence classes for $\sim$.

We consider now the (f, N) quasi-line L_1 that is the limit of $g_i L$. More precisely $x \in L_1$ if there is i_0 such that for all $i > i_0$, $x \in g_i L$. Clearly L_1 is a separating quasi-line and if $a, b \in B_M(x) - g_M L$ are not equivalent then a, b lie in different essential components of $X - L_1$. We conclude that $X - L_1$ has at least $2n - 2 > n$ essential components. We can now repeat the argument replacing L by L_1. Note that L_1 is also contained in a neighborhood of a maximal r-solid subset of X. If G does not split over $\mathbb{Z}$ and is not a virtually planar group, we arrive at a contradiction as the number of essential components of $X - L$ is bounded by a uniform bound for all separating (f, N)-quasi-lines L. This proves our claim in case 1.

Case 2: $X - L$ has exactly two essential components. Below, we denote by $d_{\mathcal{H}}$ the Hausdorff distance between two sets. Thus, if A, B are subsets of X, $d_{\mathcal{H}}(A, B)$ is the infimum of all r such that A, B are contained in the r-neighborhood of each other. We distinguish again two cases:

Case 2a: The set $S = \{g \in G : d_{\mathcal{H}}(gL, L) < \infty\}$ is infinite and

Case 2b: The set $S = \{g \in G : d_{\mathcal{H}}(gL, L) < \infty\}$ is finite.

We treat first case 2a, noting that if there is an $M > 0$ such that $d_{\mathcal{H}}(gL, L) < M$ for all $g \in S$ then we can find a $g \in G$ of infinite order such that $X/\langle g \rangle$ has at least two ends. Noting that S is a group and is infinite and since if $x \in L$ the orbit of x, Sx, is contained in the M-neighborhood of L, we have that S is quasi-isometric to $\mathbb{R}$, hence virtually cyclic. So by [D-Sw] we conclude that either G is virtually planar or it splits over a virtually cyclic group.

We denote by C_1, C_2 the essential components of $X - L$, letting (g_i) be an infinite sequence of elements of S such that $d_{\mathcal{H}}(g_i L, L) < \infty$ and $d_{\mathcal{H}}(g_i L, L) \to \infty$. We can assume that $g_i C_1$ is contained in a finite neighborhood C_1 for each i. Indeed if this does not hold we can replace our sequence by $g_1 g_i$. Now L is an r-solid set for some $r > 0$. This implies that for each i either $g_i C_1$ is contained in $N_r(C_1 \cup L)$ or $g_i C_2$ is contained in $N_r(C_2 \cup L)$. By passing to a subsequence we can assume, without loss of generality, that $g_i C_1$ is contained in $N_r(C_1 \cup L)$ for all i.

Let now $R \gg r$ and let j be such that for some $t \in L$, $d(g_j t, L) > R$. We claim that g_j is then an element of infinite order. We will show in fact by induction that $d(g_j^n t, L) > R/2$ for all n. This is true by hypothesis for $n = 1$. Assume it is true for $n = k$. Since L is r-solid for all $x \in L$ if $g_j^k x \notin C_1$ then $d(g_j^k x, L) < r$. So $g_j^k C_1$ is contained in the r neighborhood of $C_1 \cup L$. Now

$$d(g_j^{k+1} t, g_j^k L) = d(g_j t, L) > R.$$

Let y be the point on L closest to $g_j^{k+1} t$. If y does not lie in $g_j^k(C_1)$ then since $g_j^{k+1} t \in g_j^k(C_1)$, $d(g_j^{k+1} t, y) > R$. Otherwise since $g_j^k C_1$ is contained in the r-neighborhood of $C_1 \cup L$ and $R \gg r$ we have that $d(y, g_j^k L) < R/2$ and $d(g_j^{k+1} t, L) > R/2$.

We consider now the connected components of $X - (L \cup g_j L)$. If there are more than two such components that are not contained in a finite neighborhood of L then there is a neighborhood of L, denoted by L_1, that is a quasi-line and has the property that $X - L_1$ has more than two essential components. But then by replacing L by L_1 we conclude by case 1 that either G is virtually planar or it splits over a virtually cyclic group. We can therefore assume that all components of $X - (L \cup g_j L)$ except two are contained in $N_M(L)$ for some $M > 0$ and moreover, that $g_j L$ is also contained in $N_M(L)$.

We note now that for any $t \in C_1$ there is an n_0 such that for all $n > n_0$, $t \notin g_j^n C_1$. Indeed it is clear that if for some n_0, $t \notin g_j^{n_0} C_1$ and $d(t, g_j^{n_0} L) > M$, then for all $n > n_0$, $t \notin g_j^n C_1$.

Let p be a finite path joining t to L. If $t \in N_M(g_j^n C_1)$ for all n then $g_j^n(L)$ intersects $N_M(p)$ for all $n > 0$. This implies that $d(g^{-n}(t), L) < M$ for all $n > 0$. It easily follows that $X/\langle g_j \rangle$ has more than one end and we can conclude using the torus theorem ([D-Sw]).

We have, therefore, that the following holds: For every $t \in X$ there is an $n \in \mathbb{Z}$ such that $d(t, g_j^n L) < M$, or, put it differently $\langle g_j \rangle N_M(L) = X$. We have also that for all n, $d_{\mathcal{H}}(g_j^n L, g_j^{n+1} L) < M$. So X is 'foliated' by translates of L. It is easy to see now, as in Section 3, that neighborhoods of simple closed curves in X separate and neighborhoods of (bi)-infinite geodesics separate. So by Varopoulos' inequality and the Tukia-Gabai theorem G is commensurable to a surface group. This concludes our argument in case 2a.

Passing now to case 2b, we suppose that the set $S = \{g \in G : d_{\mathcal{H}}(gL, L) < \infty\}$ is finite and S is a group. We define an equivalence relation on the set $V = \{gL : g \in G\}$. For $A, B \in V$ we say that $A \sim B$ if $d_{\mathcal{H}}(A, B) < \infty$. Note that since S is finite there is an M which does not depend on A, B such that if $d_{\mathcal{H}}(A, B) < \infty$ then $d_{\mathcal{H}}(A, B) < M$. We denote the equivalence class of A by $[A]$. Let $P = \{[A] : A \in V\}$. We note that G acts on P. Moreover P is equipped with a 'separation' relation, namely, $[A]$ separates $[B]$ from $[C]$ if B, C lie in finite neighborhoods of distinct essential components of $X - A$.

Using this separation relation we will construct an $\mathbb{R}$-tree T such that $P \subset T$ and the separation relation on P is induced by separation on T.

To simplify notation below we drop the brackets and write simply A for $[A] \in P$. If $A, B \in P$ we define the interval (A, B) to be the set of all $C \in P$ that separate A from B. We define $[A, B]$ to be $(A, B) \cup \{A, B\}$. It is obvious that $[A, B] = [B, A]$. Separation induces a linear order on intervals: If $C, D \in [A, B]$ we say that $C < D$ if C separates A from D.

We note now that it is possible to show, and by applying Swenson's construction ([Swe]) using our countable set P, that G acts by homeomorphisms on an $\mathbb{R}$-tree.

We sketch below a more direct way to construct such an action using the geometry of X. Let $[A, B]$ be an interval in P and let $C_n \in [A, B]$ be a sequence such that $C_i < C_{i+1}$ for all $i \in \mathbb{N}$. One sees easily then that C_n has a subsequence C_{n_k} that converges on compact sets. To be precise, there is point $x \in X$ such that for any $r \in \mathbb{N}$ there is a k_0 such that for all $s, t > k_0$, $B_x(r) \cap C_{n_s} = B_x(r) \cap C_{n_t}$. Denote by C the limit of this subsequence which is clearly a quasi-line separating X and $X - C$ has at least two essential components. Noting here that C_n is actually an equivalence class of quasi-lines, to make sense of convergence we pick arbitrarily a quasi-line from each equivalence class. Change of choice leads to a limit quasi-line C' that lies in a finite neighborhood of C.

If $X - C$ has more than two essential components we can conclude as in case 2a. We assume therefore that this does not happen for any sequence C_n as above. We note that we can extend our separation relation to C in the obvious way and that we have $C_n < C \leq B$. We remark that if there is some quasi-line $D \in [A, B]$ such that $d_{\mathcal{H}}(C, D) < \infty$ then in fact $d_{\mathcal{H}}(C, D) \leq M_1$ for some $M_1 \gg M$ which does not depend on C, D. Indeed note that since C, D are r-solid C is contained in the r-neighborhood of an essential component of $X - D$, say X_1, and D is contained in the r-neighborhood of an essential component of $X - C$, say X_2. If $d_{\mathcal{H}}(C, D) > M_1$ there is a point t in $X_1 \cap X_2$ at distance bigger than $r + M$ from both C, D. Consider then a translate of L, gL, containing t. Since gL is not contained in a finite neighborhood of D, it either r-crosses C or D (a contradiction) or $X - (C \cup D)$ has more than two essential components. If however $X - (C \cup D)$ has more than two essential components we can conclude the argument as in case 1 and assume that this does not happen. We consider now the set of all quasi-lines C obtained as limits of convergent subsequences as described above. We have a natural equivalence relation on this set: $C \sim C'$ if $d_{\mathcal{H}}(C, C') < \infty$. We denote by $[C]$ the equivalence class of C and by $\bar{P}$ the set obtained by adding all these new equivalence classes of quasi-lines to P. If there is some quasi-line $D \in P$ such that $d_{\mathcal{H}}(C, D) < \infty$ we simply identify $[C]$ and $[D]$. Note however that there is an $M_2 \gg M_1$ (which does not depend on C) such that if $d_{\mathcal{H}}(C, D) < \infty$ then $d_{\mathcal{H}}(C, D) < M_2$ for any two quasi-lines in $\bar{P}$ (here we abuse notation as $\bar{P}$ consists of equivalence classes of quasi-lines; we simply consider all quasi-lines contained in these equivalence classes). Indeed this is shown by the same argument used above: if there are two such quasi-lines sufficiently far away there is a translate of L 'between' them and either $X - (C \cup D)$ has more than two essential components (in which case we conclude as in case 1) or the translate of L r-crosses at least one of C, D a contradiction.

From the observation above, the cardinality of the set of elements g of G such that $d_{\mathcal{H}}(gC, C) < \infty$ is bounded by a constant that does not depend on C (where C is always a quasi-line obtained as a limit as described above).

We define intervals and order for $\bar{P}$ as we did for P noting now that any interval of $\bar{P}$ has the supremum property. We say that $A, B \in \bar{P}$ are adjacent if $(A, B) = \emptyset$ and define an equivalence relation on the set of unordered pairs of adjacent elements of $\bar{P}$ as follows: $\{A, B\} \sim \{A, C\}$ if $A \notin (B, C)$. We take the transitive closure of $\sim$. Denote by $C(A, B)$ the equivalence class of the pair of adjacent vertices $\{A, B\}$. For each equivalence class $C(A, B)$, pick a number $l_{A,B}$ so that the sum $\sum l_{A,B}$ over all equivalence classes is finite. Note that there are only countably many pairs $\{A, B\}$ of adjacent elements of $\bar{P}$ and thus countably many equivalence classes $C(A, B)$.

We will now turn $\bar{P}$ into an $\mathbb{R}$-tree by joining any pair of adjacent elements of $\bar{P}$ by an interval. We perform this operation first on intervals of $\bar{P}$ and then glue these intervals together. Let $[A, B]$ be an interval of $\bar{P}$. Clearly there are at most countably many pairs of adjacent elements of $\bar{P}$. Let $\{\{A_i, B_i\}\}$ be the set of adjacent pairs in $[A, B]$. We consider now the union

$$[A, B] \cup I_{A_i, B_i}$$

where I_{A_i, B_i} is an interval of $\mathbb{R}$ of length l_{A_i, B_i}. For convenience we assume that $A_i < B_i$ for all i. We extend the order $<$ of $[A, B]$ in the obvious way on $[A, B] \cup I_{A_i, B_i}$ (i.e. if $t \in I_{A_i, B_i}$ we define $A_i < t < B_i$ etc.). Now $[A, B] \cup I_{A_i, B_i}$ is a linearly ordered set that has the supremum property. There are no adjacent points for '$<$' on this set and it has a countable dense set with respect to the order topology so it is homeomorphic to $[0, 1]$ with respect to the order topology.

It is easy to supply a metric for these new intervals: We note that an interval $[A, B]$ of $\bar{P}$ that contains no adjacent points is homeomorphic to $[0, 1]$ with respect to the order topology. Let us denote by P' the set of points of P that are contained in some nondegenerate interval $[A, B]$ that does not contain any adjacent points. Clearly there is a countable set Σ of intervals $[A_i, B_i]$ such that $P' = \cup [A_i, B_i]$ and such that $[A_i, B_i] \cap [A_j, B_j]$ is either empty or a point if $i \neq j$. For each $[A_i, B_i]$ we pick an $l_i > 0$ such that $\sum l_i < \infty$ and we choose an order-preserving map $h_i : [A_i, B_i] \to [0, l_i]$. If $[X, Y] \subset [A_i, B_i]$ we define the length of $[X, Y]$ to be the length of $h_i([X, Y])$. If $C, D \in \bar{P}$ we define $d_{\mathcal{H}}(C, D)$ to be the sum of the lengths of the intervals of $[C, D]$ corresponding to adjacent points in $[C, D]$ plus the sum of the lengths of the intervals $[C, D] \cap [A_i, B_i]$ where $[A_i, B_i] \in \Sigma$.

We explain now how to glue together all intervals constructed above to obtain an $\mathbb{R}$-tree. Denote by $I[A, B]$ the interval $[A, B] \cup I_{A_i,B_i}$. If $I[A, B]$, $I[C, D]$ are two such intervals we identify $I[A, B]$, $I[C, D]$ along $[A, B] \cap [C, D]$. We claim that this is an interval $I[E, F]$ (if it is nonempty). Indeed let $X \in [A, B] \cap [C, D]$. We have then $A < X < B$ and $C < X < D$ (assuming that A, B, C, D are distinct). The set of all Y such that $A < Y < X$ and $C < Y < X$ has a maximum, say E. To see this note that if p_1, p_2 are paths joining X to A, B then Y intersects p_1, p_2. Therefore if Y_n is a sequence such that $A < Y_{n+1} < Y_n < X$ and $C < Y_{n+1} < Y_n < X$, a subsequence of Y_n converges to a quasi-line $\bar{Y}$. From this it follows that the set of Y's has a maximum. Similarly the set of Y such that $X < Y < B$ and $X < Y < D$ has a minimum, say F and $[A, B] \cap [C, D] = I[E, F]$.

If E is adjacent to a point $E_1 \in [A, B]$ and to a point $E_2 \in [C, D]$ and if $\{E, E_1\}$ and $\{E, E_2\}$ are equivalent then we identify also the midpoint of $I_{E,E_1} \subset I[A, B]$ with the midpoint of $I_{E,E_2} \subset [C, D]$. We make a similar identification for F. Finally we identify $I[A, B]$, $I[C, D]$ along the convex closure of the points identified.

By making these identifications on the set $\cup I[A, B]$ we clearly obtain an $\mathbb{R}$-tree T. Since G acts on $\bar{P}$ preserving intervals one sees easily that G acts on T by homeomorphisms.

We show now that we can assume that the action is nonnesting. Indeed suppose that for some element $g \in G$ we have $g([A, B]) \subset [A, B]$ and $g([A, B]) \neq [A, B]$. Note that by $[A, B]$ here we denote a segment of T. Clearly we can assume that $A, B \in P$. We have then that a subsequence of $g^n A$ converges on compact sets on X to a quasi-line $C \in \bar{P}$. Clearly $[C]$ is invariant by g so that $X/\langle g \rangle$ has more than two ends and we can conclude the argument [D-Sw].

Since the cardinality of the stabilizer of any vertex of T lying in $\bar{P}$ is bounded by a fixed $M > 0$ we see that the action is stable. One sees easily that the action of G on T has no global fixed point. By a result of Levitt ([L]) G admits a nontrivial, stable action by isometries on an $\mathbb{R}$-tree with finite segment stabilizers. So by Rips' theory ([B-F]) we have that G splits over a virtually cyclic group. By Proposition 5.3 and the preceding treatment of 'exceptional cases' we arrive at the following:

THEOREM 1. *Let G be a one-ended, finitely presented group that is not commensurable to a surface group. Then G splits over a two-ended group if and only if the Cayley graph of G is separated by a quasi-line.*

Corollary. *Let G_1 be a one-ended, finitely presented group that is not commensurable to a surface group. If G_1 splits over a two-ended group and G_2 is quasi-isometric to G_1 then G_2 splits also over a two-ended group.*

7. Quasi-isometry invariance of JSJ-decompositions

One can see, using the results of the previous sections, that JSJ-decompositions are preserved by quasi-isometries. We will consider the JSJ-decomposition of one-ended finitely presented groups corresponding to splittings over 2-ended groups (see [D-Sa]).

Theorem 7.1. *Let G_1, G_2 be one-ended finitely presented groups, let Γ_1, Γ_2 be their respective* JSJ*-decompositions and let X_1, X_2 be the Cayley graphs of G_1, G_2. Suppose that there is a quasi-isometry $f : G_1 \to G_2$. Then there is a constant $C > 0$ such that if A is a subgroup of G_1 conjugate to a vertex group, an orbifold hanging vertex group or an edge group of the graph of groups Γ_1, then $f(A)$ contains in its C-neighborhood (and it is contained in the C-neighborhood of) respectively a subgroup of G_2 conjugate to a vertex group, an orbifold hanging vertex group or an edge group of the graph of groups Γ_2.*

Proof. It suffices to show that images of edge groups of Γ_1 by f are contained in finite neighborhoods of edge groups of Γ_2. We will argue by contradiction. The idea of the proof that follows is that the images of the edge groups of Γ_1 furnish a new set of separating quasi-lines for G_2; so we can use this set and apply the machinery of the previous sections to show that some vertex group of Γ_2, which is not a hanging orbifold vertex group, admits a splitting over a 2-ended group in which all its adjacent edge groups are elliptic. This contradicts the properties of JSJ-decompositions (see [D-Sa]).

We will consider a set S of separating quasi-lines on X_2. We define S inductively: S_0 is the set of all edge groups of Γ_2 (to be more precise we 'thicken' these groups so that they verify the properties of quasi-lines stated in Sections 1, 2). If E is an edge group of Γ_1, such that $f(E)$ is not contained in a finite neighborhood of an edge group of Γ_2 we thicken $f(E)$ so that it satisfies the conditions of Sections 1, 2 and we add it to S_0. We do this for all edge groups of Γ_1. We then add all translates of these quasi-lines by the action of G_2. We call S_1 the set obtained.

We thicken all quasi-lines in $f^{-1}(S_1)$ so that they satisfy the requirements of Sections 1, 2 and we consider all their translates by G_1. Let us call this set S_2'. We take S_2 to be the quasi-lines obtained by all translates of the quasi-lines in $f(S_2')$ (where we thicken quasi-lines as before). Let L be a quasi-line in S_2 and let $E \in S_0$.

By the results of Section 3 it follows that there is an $N \gg 0$ such that if $x, y \in L$ are separated by E and $d(x,E), d(y,E) > N$ then $L - E$ has two unbounded components which are contained in distinct components of $X_2 - E$.

We say then that L crosses E. If no edge group of Γ_2 is crossed by an $L \in S_2$ we define $S = S_1$. Otherwise we collapse all edges of Γ_2 which correspond to quasi-lines which are crossed and we obtain a new graph Γ_2'.

We thicken all quasi-lines in $f^{-1}(S_2)$ so that they satisfy the requirements of Sections 1, 2 and we consider all their translates by G_1. Let us call this set S_3'. We take S_3 to be the quasi-lines obtained by all translates of the quasi-lines in $f(S_3')$ (where we thicken quasi-lines as before). If no edge group of Γ_2' is crossed by a quasi-line in S_3 we define $S = S_2$. Otherwise we collapse the edges of Γ_2' which correpond to quasi-lines which were crossed and we repeat. It is clear that this procedure terminates and produces a set S. Let us call Γ the graph of groups obtained from Γ_2 by successively collapsing edges as in the previous procedure. Thus, no edge groups of Γ are crossed by quasi-lines in S (of course Γ may be a single vertex in which case this is trivially true).

We note that the set S has the following property: There are $A, B > 0$ such that for any $L \in S$ and $x, y \in L$ there is an (A, B)-quasi-isometry $g : X_2 \to X_2$ such that $d(g(x), y) < A$ and $g(L)$ is contained in the A-neighborhood of L. Indeed edge groups of Γ_2 clearly have this property and at each inductive step we just added translates either by the action of G_2 or of G_1 and so one sees easily that this property is preserved.

Assume first that $S = S_1$. In this case by our assumption there are a vertex group V of Γ_2 and an edge group E of Γ_1, such that V minus a finite neighborhood of $f(E)$ has at least two components none of which is contained in a finite neighborhood of $f(E)$. We distinguish two cases depending on whether V is a hanging orbifold group or not. If V is a hanging orbifold group we pick a vertex group, V_1, adjacent to E in Γ_1 which is not a hanging orbifold group. If some edge adjacent to V_1 is not mapped by f in a finite neighborhood of V then there is an edge group of Γ_2 which is mapped by f^{-1} in a finite neighborhood of V_1 and separates V_1 so that we can interchange the role of G_1, G_2 and argue as in the next case. We can assume therefore that all edge groups adjacent to V_1 are mapped by f in a finite neighborhood of V. This implies that V_1 is quasi-isometric to a subset of V hence is virtually free. Moreover the set of ends of V_1 maps to the set of ends of V. Every quasi-line in V_1 is mapped by f to a quasi-line in V which can be thickened to a separating quasi-line. This induces a natural cyclic order (see [Bo 1]) on the set of ends of V_1 preserved by the action of V_1. By Theorem 4.8 and Proposition 4.9 of [Bo 1] it follows that V_1 is a hanging orbifold group, a contradiction.

Assume now that V is not a hanging orbifold group. Let F be a maximal r-solid subset of V, where $r \gg 0$. It is clear that V is not contained in a finite neighborhood of F. There are again two cases: Either every such F is contained in a finite neighborhood of some edge group of V or not. We claim that if the first holds then V is virtually free. Indeed suppose not and let $\bar{V}$ be a one ended subgroup of V. If $\bar{V}$ does not have any unbounded r-solid subsets then by the results of Sections 3, 4 we have that G_2 is virtually a surface group. So A contains some unbounded r-solid subset F which is at finite distance from an edge group of V. In the same way as in Section 5 we see that A is not contained in any finite neighborhood of $X_2 - F$. By Proposition 3.1 of [D-Sw] it follows that V splits over a 2-ended group and edge groups of Γ_2 are elliptic with respect to this splitting. So Γ_2 can be refined, a contradiction.

Since all r-solid subsets of V are contained in finite neighborhoods of edge groups of V one sees easily that every quasi-line in V can be thickened so that it separates X_2 in at least two essential components. It follows by Theorem 4.8 and Proposition 4.9 of [Bo 1] that V is a hanging orbifold group, a contradiction.

We assume therefore that there is a maximal r-solid subset of V, F, which is not contained in a finite neighborhood of an edge group of V. If F is as in case 2 of Section 5 then by Proposition 3.1 of [D-Sw] it follows that V splits over a 2-ended group and the edge groups of Γ_2 are elliptic with respect to this splitting. So Γ_2 can be refined using this splitting. This however contradicts the fact that JSJ-decompositions 'encode' all splittings over 2-ended groups (see [D-Sa]).

Therefore we can assume that F is as in case 1 of Section 5. In this case, using Lemma 5.1 one produces a maximal r-solid subset of V, F', which is a quasi-line. We claim that if F' is contained in a finite neighborhood of an edge group, E', of V then there is some $L \in S$ which is an r-solid separating quasi-line of V and which is not contained in a finite neighborhood of an edge group of V. Note first that E' is not contained in a finite neighborhood of F. Indeed in this case translates of F along E' would be contained in F and so (by the construction of Lemma 5.1) F' would not be contained in a neighborhood of E'. Therefore a neighborhood of F intersects E' along connected sets of bigger and bigger diameter but there is an unbounded connected subset of E' which is not contained in this neighborhood of F. Therefore there are quasi-lines in S separating this unbounded connected subset of E' from F which 'contain' bigger and bigger connected subsets of E'. To be more precise there is $N > 0$ and $L_n \in S$ such that the N-neighborhood of L_n intersects E' along a connected subset of diameter bigger than n. We claim that for n big enough L_n is r-solid. Indeed, if some $L \in S$ crosses L_n then we can translate L along L_n by (A, B)-quasi-isometries. So a translate of L crosses E', which is impossible. We can assume therefore that F' is not contained in a neighborhood of an edge group of V; if this is not the case we just replace it by some L_n.

Now we can apply the arguments of Section 6 to show again that the JSJ-decomposition of G_2 can be refined. We recall that in Section 6 there are several cases to consider. Proposition 3.1 of [D-Sw] can be applied in all cases except in the case where one obtains an action on an $\mathbb{R}$-tree. In the case of a nonnesting action we can refine Γ_2 by Theorem 9.6 of [B-F] (Theorem 12.72 of

[Ka]). We explain now how to obtain a refinement of Γ_2 in the nesting $\mathbb{R}$-trees action case.

We recall that in the case of nesting actions translates of F' converge on compact sets to a quasi-line, $\bar{L}$, which is contained in a finite neighborhood of a 2-ended group. One uses Proposition 3.1 of [D-Sw] to obtain a splitting. It is possible however that L_1 is contained in a finite neighborhood of an edge group, E', of V. So in this case the new splitting does not give a refinement of Γ_2. In this case too as before we have that there are $N > 0$ and a sequence $L_n \in S$ such that the N-neighborhood of L_n interects E' along a connected subset of diameter bigger than n. So we can again replace F' by some $L \in S$. As before, using L we can either refine Γ_2 using the arguments of Section 6 or obtain a nesting action on an R-tree. Since L is not contained in a finite neighborhood of an edge group of V then one of the following holds:

- Either there is an $N > 0$ such that the N-neighborhood of L intersects distinct edge groups of V along connected sets of arbitrarily big diameter or
- translates of L converge on compact sets on a solid quasi-line $\bar{L}$ such that for any $N > 0$ there is an $M > 0$ such that each connected component of the intersection of the N-neighborhood of $\bar{L}$ with an edge group of V is of diameter less than M.

In the second case if we replace L by $\bar{L}$ and apply the machinery of Section 6 we obtain a splitting of V which can be used to refine Γ_2 thereby arriving at a contradiction.

In the first case we can translate distinct edge groups of V along L by (A, B)-quasi-isometries and obtain translates of edge groups which cross, again a contradiction. This finishes the proof in the case that $S = S_1$.

The general case is treated in a similar way: Let V be a vertex group of Γ which is obtained by collapsing some vertex group of Γ_2 to a single vertex. We consider maximal r-solid subsets of V. If each such subset of V is contained in a finite neighborhood of an edge group of Γ then, as before, using Theorem 4.8 and Proposition 4.9 of [Bo 1] we conclude that V is a hanging orbifold group, contradicting the properties of JSJ-decompositions (see [D-Sa]).

We assume therefore that V contains some r-solid subset F and that F is contained in a nonhanging orbifold vertex group of Γ_2. Indeed if no such F exists in V we enlarge S by including sufficiently many separating quasi-lines, from the orbifold vertex groups of Γ_2 included in V, so that there is no r-solid set in V with respect to this enlarged set of quasi-lines. Then by the previous argument it follows that V is a hanging orbifold vertex group, a contradiction.

If F is as in case 2 of Section 5, using Proposition 3.1 of [D-Sw] one obtains a splitting over some 2-ended group which is not contained (up to finite index) in an edge group or a hanging orbifold vertex group of Γ_2, a contradiction. If

F is as in case 1 of Section 5 then (by Lemma 5.1) one obtains an r-solid set F' which is a separating quasi-line. Again there are several cases to consider. As before one can show that F' can be chosen so that it is not contained in a finite neighborhood of an edge group of Γ. One then applies the arguments of Section 6. In the cases where Proposition 3.1 of [D-Sw] applies one obtains a splitting over a 2-ended subgroup which is not contained (up to finite index) in an edge or hanging orbifold subgroup of Γ_2, a contradiction. In the case of a nonnesting $\mathbb{R}$-tree action one notes that edge groups of Γ_2 act elliptically on this $\mathbb{R}$-tree so by Proposition 9.6 of [B-F] one obtains a refinement of Γ_2, a contradiction.

Finally in the case of a nesting action on an $\mathbb{R}$-tree, either one obtains a refinement of Γ_2 applying Proposition 3.1 of [D-Sw] or one shows as earlier that there is a quasi-line, L_1, in S which 'fellow-travels' for bigger and bigger intervals, distinct edge groups of Γ. In this case we can translate one edge group by (A, B)-quasi-isometries along L_1 so that it crosses another. So some edges group of Γ is crossed by a quasi-line in S, a contradiction. $\square$

We note that one can show by a similar argument that a half quasi-line never separates the Cayley graph of a one-ended finitely presented group G.

Appendix

The problem of finding topological characterizations for the 2-sphere or more generally for 2-dimensional manifolds was quite popular in the first half of the twentieth century. Moore, Kuratowski, Janiszewski, Zippin, Van-Kampen, Bing, Young (see [Mo], [Ku], [Z], [V], [Bi]) and others contributed to the solution of this problem. In this paper we used a 'large scale' characterization of 'quasi-planes' that was modeled on a new topological characterization of the plane. We prove here the theorem that served as 'model' for the large scale characterization. The statement of the theorem will be somehow unnatural from the point of view of point set topology. The reason for that is that we choose a statement that is similar to the 'large scale' characterization. A similar characterization theorem more natural from the standpoint of point set topology will appear elsewhere.

We will use the following theorem due to Bing (see [Bi]).

Theorem 1. *Let X be a compact, connected and locally connected nondegenerate metric space. If X is separated by each of its simple closed curves but by no pair of its points then it is homeomorphic to a 2-sphere.*

The characterization of the plane (='open' 2-cell) that we prove here differs from this theorem in two ways: First we use separation by lines rather than simple closed curves and second we assume that any two points in X are separated by a line (rather than assuming that *every* line separates). This second difference (unlike the first) is quite crucial. For example if one relaxes condition 2 of Theorem 1 to: Any two points in X are separated by some simple closed curve, then the conclusion is no longer true. Indeed any compact surface X satisfies this. One can find wilder spaces satisfying this condition, for example add smaller and smaller handles to a sphere so that the handles converge to a point.

To make up for this we will assume that X is simply connected. This is quite natural for us as the 'large scale' analog of the characterization given here concerns Cayley graphs of finitely presented groups which are 'large scale'-simply connected.

Definitions. A metric space X is called a *geodesic metric space* if any two points in X can be joined by an arc whose length is equal to the distance between the two points. Also, X will be called *one-ended* if for any compact K, $X - K$ has exactly one unbounded component. A *line* in X is a *proper*, one-to-one, *continuous*, *rectifiable map* $L : \mathbb{R} \to X$. We will often identify a line L with its image $L(\mathbb{R})$.

It is convenient to parametrize L with respect to arclength. In the rest of this appendix we will assume all lines to be parametrized with respect to arclength (so that length$(L([a, b])) = |b - a|$).

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\lim_{x\to 0} f(x) = 0$. We say that a line L is an f-line if $|a - b| \le f(d(L(a), L(b)))$ for all $a, b \in \mathbb{R}$.

We define an *order* on a line as follows: If $a = L(a'), b = L(b')$ where L is a line we write $a < b$ if $a' < b'$. We denote by $[a, b]_L$ the set of all $t \in L$ such that $a \le t \le b$. Similarly we define $(a, b)_L$. When there is no ambiguity we write $[a, b]$ instead of $[a, b]_L$. If $t \in \mathbb{R}$ we denote by $a + t$ the point $L(a' + t)$.

We say that a line L *separates* X if $X - L$ has at least two connected components and, for each connected component C of $X - L$, L is contained in the closure $\bar{C}$, of C. If $a, b \in X$ we say that a line L *separates* a *from* b if L separates X and a, b belong to distinct components of $X - L$.

Remarks. Note that our definition of separation is quite nonstandard. If we take X in the above definitions to be a locally compact, connected and locally connected space then X is *arcwise connected*. Moreover if L is a line separating X and C is a component of $X - L$ then C is arcwise connected. Theorem A stated below holds also if we replace f-lines by lines. This requirement of 'uniformity' for lines is made so that the proof can be translated easier to the 'large scale' argument used in the proof of the quasi-isometry invariance of group splittings. We note that it is used only at the end of the proof (after Lemma A.4.3).

One can weaken the hypothesis of the theorem further by assuming that X has no cut points rather than assuming that it is one-ended. A more general result without superfluous hypotheses will appear elsewhere.

Now we can state the main result of this appendix:

THEOREM A. *Let X be a locally compact, geodesic metric space and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\lim_{x\to 0} f(x) = 0$. If X satisfies the following three conditions then it is homeomorphic to the plane.*

1) *X is one-ended.*

2) *X is simply connected.*

3) *For any two points $a, b \in X$ there is an f-line separating them.*

Proof. In the proof that follows we will call f-lines simply lines most of the time. We will revert to the 'f-line' notation when it becomes relevant.

As the proof is rather long and technical we give now a brief outline of our argument: In the classical characterization theorems (see [Z], [V]) one uses separating curves to create finer and finer grids which approximate the space and 'in the limit' converge to a plane. We cannot, however, do this in our setting. Think for example of the (simply connected) space obtained by gluing three closed half-planes along their boundary line. Take then two lines L_1, L_2 in two distinct half-planes which intersect at exactly one point. A pair of lines on the plane intersecting at one point cuts up the plane into four pieces, while the lines L_1, L_2 cut up our space into only three pieces. The way to get around this problem is to use cross-points (see Figure A.2 and the definition before Lemma A.2) rather than intersection points. To carry out the proof we appeal to the known characterization theorems (see e.g. [Bi]). This reduces the problem to showing that simple closed curves in X separate. In other words we have to show that the Jordan curve theorem holds in X. In Definition A.4 we explain how to pick a point 'inside' a simple closed curve. This definition however depends on a separating line on which the point lies. We show then in Lemma A.4.3 that being 'inside' is well defined for cross-points. This relies on Lemma A.3 which says that lines that cross cut up X as they would cut up the plane. Completing the proof of the Jordan curve theorem after Lemma A.4.3 is quite easy. We start the proof by showing that lines separate X in exactly two pieces (Lemma A.1). This is quite intuitive, if one considers the space defined above by gluing three half-planes one notes that points on the common boundary line cannot be separated by lines. So Lemma A.1 is a generalization of this fact. The simple connectedness assumption is quite crucial for this, as it is for Lemma A.3 where similar arguments are used.

We will need the following lemma from plane topology:

LEMMA A.0. *Let O be a connected open subset of the open unit disk D. Let K be a connected component of ∂O and let $x, y \in O$ be such that $d(x, K) < \varepsilon$ and $d(y, K) < \varepsilon$. Then there is a path p in O connecting x to y such that p is contained in the ε neighborhood of ∂O.*

Proof. We consider D to be lying on the plane. Let U be the union of the open balls $B_x(\varepsilon)$ with center $x \in \partial O$ and radius ε. Let $W = U \cap D$. Let V be the connected component of W containing K. Clearly $x, y \in V$ so there is a path in D joining them that does not intersect ∂V.

On the other hand, $x, y \in O$ so there is a path in D joining them that does not intersect ∂O.

Noting that $\partial O \cap \partial V = \emptyset$, we recall now Alexander's lemma: Let x, y be points and let F_1, F_2 be disjoint closed sets on the plane. If there are paths joining x, y in $\mathbb{R}^2 - F_1$ and in $\mathbb{R}^2 - F_2$ then there is a path joining them in $\mathbb{R}^2 - (F_1 \cup F_2)$ (see [N, Th. 9.2, p. 112]).

Since ∂O, ∂V are closed there is a path p lying in D joining x to y that intersects neither ∂O nor ∂V. Clearly p is contained in O and lies in the ε neighborhood of ∂O. $\square$

LEMMA A.1. *Let L be a line separating X. Then $X - L$ has exactly two components.*

Proof. We will prove this by contradiction: Assume that $X - L$ has at least three components C_1, C_2, C_3. Let $a, b \in L$. We will show that a, b cannot be separated by any line in X. Without loss of generality we assume that $a < b$. Suppose that a line L_1 separates a from b. Say $a \in A$, $b \in B$ where A, B are connected components of $X - L_1$. Let $c = \min(d(L_1, \{a, b\}), d(a, b))$. Let $\varepsilon_1 = c/4$. We take $a_i, b_i \in L$ $(i = 1, 2, 3)$ such that:

i) a_i is joined to b_i by a path $p_i \in \bar{C}_i$ such that $p_i \cap L = \{a_i, b_i\}$ and

ii) $\text{length}([a_i, a]) < \varepsilon_1$, $\text{length}([b_i, b]) < \varepsilon_1$.

Without loss of generality we can assume that $p_i \cap L_1$ has finitely many connected components. Indeed if this is not the case for some p_i we modify it as follows: If J is a maximal interval of L_1 with the properties that J is entirely contained in C_i and its endpoints lie on p_i then we modify p_i by replacing the subpath of p_i with the same endpoints as J by J.

We can further assume that the p_i are simple paths and we consider the simple closed paths $q_i = p_i \cup [a_i, b_i]$.

The $\bar{C}_i$ are simply connected. Indeed let w be a closed curve in $\bar{C}_i$ and let $f : D \to X$ be such that $f(\partial D) = w$. Also, $f^{-1}(X - \bar{C}_i)$ is an open set in D and the frontier of this open set is sent to L by f. Using Tietze's theorem we can modify f so that the whole open set is mapped to L. This shows that $\bar{C}_i$ is simply connected.

Let the D_i be discs and let $f_i : D_i \to \bar{C}_i$ be maps such that $f_i(\partial D_i) = q_i$. We assume further that f_i restricted to ∂D_i is injective. To simplify notation we denote by p_i the subpath of ∂D_i mapped onto p_i. Likewise we denote by a_i, b_i the points of ∂D_i mapped onto a_i, b_i. Let $\bar{x} = \sup\{t \in L \cap A,\ t < b\}$ and let $[\bar{x}, \bar{y}]_L$ be the connected component of $L \cap L_1$ containing $\bar{x}$ (see Figure A.1).

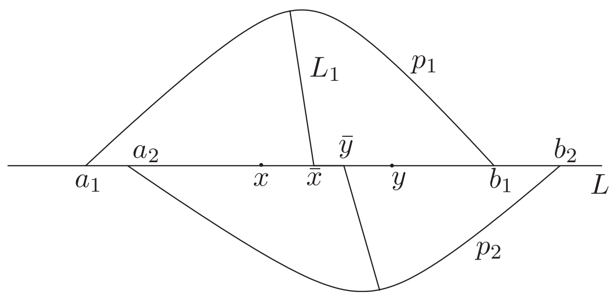


Figure A.1

For each D_i there is a segment on ∂D_i mapped onto $[a_i, b_i]$. We consider now the disc $D = D_1 \sqcup D_2/\sim$ where $t \sim s$ if $t \in \partial D_1$, $s \in \partial D_2$ and $f_1(t) = f_2(s)$. We define $f : D \to X$ by $f(t) = f_1(t)$ if $t \in D_1$ and $f(t) = f_2(t)$ if $t \in D_2$.

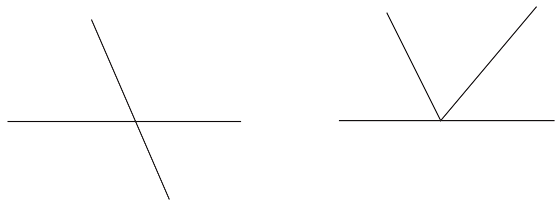
For convenience we will consider D to be the closed unit disc in the plane. We will denote by U the open unit disc. So $U = D - \partial D$. Let $\varepsilon > 0$ be such that the following conditions are satisfied:

1) There are $x < \bar{x}$, $y > \bar{y}$ on L such that $B_x(2\varepsilon) \subset A$ and $B_y(2\varepsilon) \cap A = \emptyset$.
2) If a segment I of L_1 intersects $[x, y]_L$ and $I \cap (p_1 \cup p_2 \cup p_3) = \partial I$ then I contains $[\bar{x}, \bar{y}]_L$.

We consider now the components of $D - f^{-1}(L_1)$ intersecting $f^{-1}(B_x(\varepsilon))$ or $f^{-1}(B_y(\varepsilon))$. Clearly there are finitely many such components as $d(x, L_1) > 2\varepsilon$, $d(y, L_1) > 2\varepsilon$. Let O be such a component. We denote by ∂O the closure of O, $\bar{O}$ in the plane, minus the interior of O in the plane. Suppose that ∂O intersects ∂D at a subset of $f^{-1}(L_1)$. Using Tietze's theorem we can modify f so as to map $\bar{O}$ to L_1. We do this for all components of $D - f^{-1}(L_1)$ intersecting $f^{-1}(B_x(\varepsilon))$ or $f^{-1}(B_y(\varepsilon))$. We obtain thus a new map from D to X which by abuse of notation we still call f.

Consider now the connected components of $U - f^{-1}(L)$. Let us call U_1 the connected component of $U - f^{-1}(L)$ such that the boundary of this component in the plane contains $\partial D_1 \cap \partial D$ and U_2 the connected component of $U - f^{-1}(L)$ such that the boundary of this component in the plane contains $\partial D_2 \cap \partial D$. There is a connected component, say K_1, of the boundary of $\bar{U}_1$ in U such that the boundary of K_1 in the plane contains a_1, b_1. Indeed, if not, then by Alexander's lemma (see [N, Th. 9.1.2, p. 110]) p_1, p_2 are joined by a path in $D - f^{-1}(L)$, a contradiction. We have then $f(K_1) \subset L$ and $f(K_1) \supset [a_1, b_1]$.

We consider again the components of $D - f^{-1}(L_1)$ intersecting $f^{-1}(B_x(\varepsilon))$ or $f^{-1}(B_y(\varepsilon))$. If O is such a component then $\partial O \cap \partial D$ is not contained in $f^{-1}(L_1)$. Then $f(\partial O)$ contains a segment of L_1 intersecting ∂D at least two points. We consider all segments J of L_1 that are minimal segments with the



Crossing lines

Lines which intersect but do not cross

Figure A.2

properties that J lies in $\bar{C}_1 \cup C_2$ and the endpoints of J lie on ∂D. Clearly there are finitely many such segments. We will show that some segment J contains $[\bar{x}, \bar{y}]_L$.

Indeed suppose that this is not the case and pick a J as above. Say $J \subset (\bar{y}, \infty)_{L_1}$. The endpoints of J separate ∂D in two arcs, say c_1, c_2. We consider the closed curves $c_1 \cup J$, $c_2 \cup J$. Either $c_1 \cup J$ or $c_2 \cup J$ has the property that every disc filling it contains two points that map to x, y. Say this is the case for $c_1 \cup J$. To simplify notation we still call D a disc filling $c_1 \cup J$ and let $g : \bar{D} \to X$ sending ∂D to $c_1 \cup J$. Since L_1 separates x from y, $g^{-1}(L_1)$ separates $g^{-1}(x)$ from $g^{-1}(y)$. We consider all the connected components of $D - g^{-1}(L_1)$ that intersect either $g^{-1}(x)$ or $g^{-1}(y)$. Clearly there are finitely many such components. Let O be such a component. If $g(\partial O) \subset L_1$, using Tietze's theorem we can modify g so that $\bar{O}$ maps to L_1.We do this to all such components obtaining a new filling disc for $c_1 \cup J$ that we still call D. As before we conclude that there is a component O of $D - g^{-1}(L_1)$ such that O intersects either $g^{-1}(x)$ or $g^{-1}(y)$ and $g(\partial O)$ is an interval of L_1 with both its endpoints on c_1. As before we consider the set of segments that lie on $\bar{C}_1 \cup C_2$ which are minimal with respect to this property. We pick one of them and repeat the operation. It is clear that this procedure stops after finitely many steps, producing a disc D that contains two points mapping to x, y which are not separated by the preimage of L_1 in D. This is a contradiction.

We showed therefore that there is a segment $J \subset L_1$ with its endpoints on $c_1 \cup c_2$, contained in $\bar{C}_1 \cup C_2$ which contains $[\bar{x}, \bar{y}]_L$. By applying the same argument to $D_1 \sqcup D_3$ we conclude that there is an interval of L_1 containing $[\bar{x}, \bar{y}]_L$ with its endpoints on $\partial(D_1 \sqcup D_3)$ that is contained in $\bar{C}_1 \cup C_3$. This is clearly a contradiction. □

For the rest of the proof of Theorem A we need more definitions:

Definitions. Let L be a separating line of X. We denote the two components of $X - L$ by L^+, L^-.

Let L_1, L_2 be separating lines of X. We say that L_1 *crosses* L_2 at $x \in L_1 \cap L_2$ if for any neighborhood of x in L_2, $(x - \varepsilon, x + \varepsilon)_{L_2}$, there are $a, b \in$

$(x-\varepsilon, x+\varepsilon)_{L_2}$ separated by L_1. More generally if $[x_1, x_2]$ is a connected component of $L_1 \cap L_2$ we say that L_1 *crosses* L_2 at $[x_1, x_2]$ if for any neighborhood of $[x_1, x_2]$ in L_2, $(x_1-\varepsilon, x_2+\varepsilon)_{L_2}$, there are $a, b \in (x_1-\varepsilon, x_2+\varepsilon)_{L_2}$ separated by L_1.

If $I_1 \subset L_1$, $I_2 \subset L_2$ are subintervals of L_1, L_2 containing x in their interior we say that I_1, I_2 *cross* at x. Similarly we define what it means for two intervals to cross at a common subinterval. We call x (resp. $[x_1, x_2]$) a *cross-point* (resp. *cross-interval*) of L_1, L_2. We say that I_1, I_2 cross if they cross at some point x or at some interval $[x_1, x_2]$.

If $\{L_i\}$ is a set of separating lines, a set of intervals $S = \{I_i \subset L_i\}$ is called *cross-connected* if for all $I, J \in S$ there is a sequence $I_k, k = 1, \ldots, n$ where $I_k \in S$, $I_1 = I$, $I_n = J$ and for $k = 1, \ldots, n-1$, I_k crosses I_{k+1}. We say then that $(I_1, \ldots, I_n)$ is a *cross-path* between I, J.

We define in a similar way what it means for a path p to cross a separating line L.

Now, there is an easy lemma:

Lemma A.2. *If L_1 crosses L_2 at x (or at $[x_1, x_2]$) then L_2 crosses L_1 at x (or at $[x_1, x_2]$).*

Proof. Indeed suppose that an interval $I \subset L_1$ containing x at its interior lies in (say) $\bar{L}_2^+$. Then using the simple-connectedness of $\bar{L}_2^+$ we see that any two points in $L_2 - L_1$ sufficiently close to x can be joined by a path that avoids L_1, which contradicts the hypothesis of the lemma. We argue similarly if L_1 crosses L_2 at an interval $[x_1, x_2]$. $\square$

The set of points that are either cross-points or lie on cross-intervals of two lines is closed. So $L_1 - \{$cross-points and cross intervals of L_1, $L_2\}$ is a union of intervals. If (a, b) is an interval in this set we say that a, b are successive cross-points of L_1, L_2.

To simplify notation in the arguments that follow we suppose that two lines cross always at points and not at intervals. It is quite clear how to modify the proofs in order to take care of cross intervals. We leave this to the reader.

Lemma A.3. *Let L_1, L_2 be separating lines and let $a < b$ be two successive cross-points of L_1, L_2 on L_1. Let $I_1 = [a, b]_{L_1}$, $I_2 = [a, b]_{L_2}$. Then $I_1 \cup I_2$ separates X.*

Proof. Let $x \in I_2 - L_1$ (it is clear that such a point exists). Say $L_2 - I_2 = A \cup B$. We orient A, B towards I_2. Let O_x be a connected open neighborhood of x that does not meet L_1. Let $x_1, x_2 \in O_x$ lying respectively in L_2^+, L_2^-. We will show that any path p joining x_1, x_2 intersects $I_1 \cup I_2$. This clearly implies Lemma A.3.

We argue by contradiction: Let p be a path joining x_1, x_2 which does not intersect $I_1 \cup I_2$. We join x_1 to x by a path p_1 in $O_x \cap L_2^+$ and x_2 to x by a path p_2 in $O_x \cap L_2^-$. Now consider the closed path $p \cup p_1 \cup p_2$. To simplify notation we again call this path p.

Without loss of generality we assume that p is a simple path which crosses L_2 necessarily (possibly several times). We have the following:

LEMMA A.3.1. *There are* $[c_1, d_1] \subset p$, $[c_2, d_2] \subset I_1$ *such that* $c_1, d_1, c_2, d_2 \in L_2$, c_1, d_1 *separate* c_2, d_2 *on* L_2 *and* $[c_1, d_1], [c_2, d_2]$ *are contained both in the closure of the same component of* $X - L_2$.

Proof. We will prove this by contradiction defining a map f from $L_2 \sqcup p \sqcup I_1$ to the plane as follows:

We map L_2 to the x-axis and then map the segments of $p \sqcup I_1$ in L_2^+ to the upper half-plane ($y > 0$) and the the segments of $p \sqcup I_1$ in L_2^- to the lower ($y < 0$) half-plane. The map sends subsegments of $p \sqcup I_1$ lying in L_2^+ (or in L_2^-) injectively on the plane but it might create intersections among different such subsegments of $p \cup I_1$.

We arrange f also so that the following two conditions are satisfied:

a) If $[a_1, b_1], [a_2, b_2] \subset (p \cup I_1) \cap \bar{L}_2^+$ (or $[a_1, b_1], [a_2, b_2] \subset (p \cup I_1) \cap \bar{L}_2^-$ and $a_1 < a_2 < b_1 < b_2$ we define f so that $f([a_1, b_1]), f([a_2, b_2])$intersect at exactly one point.

b) If $[a_1, b_1], [a_2, b_2] \subset (p \cup I_1) \cap \bar{L}_2^+$ (or $[a_1, b_1], [a_2, b_2] \subset (p \cup I_1) \cap \bar{L}_2^-$) and a_1, b_1 do not separate a_2, b_2 on L_2 then $f([a_1, b_1]) \cap f([a_2, b_2]) = \emptyset$.

Now $f(p)$ is a closed curve so that it separates the plane. On the other hand $f(I_1) \cap f(p) = \emptyset$ and $f(I_1)$ is connected so that $f(I_1)$ lies in the same component of $\mathbb{R}^2 - f(p)$. On the other hand the endpoints of $f(I_1)$ are equal to the endpoints of $f(I_2)$ so they lie in distinct components of $\mathbb{R}^2 - f(p)$. $\square$

LEMMA A.3.2. *There are* $[c_1, d_1] \subset p$, $[c_2, d_2] \subset I_1$ *as in Lemma* A.3.1 *so that* c_2, d_2 *are cross-points of* I_1 *with* L_2.

Proof. We argue as in the proof of Lemma A.3.1 and we remark that we can modify $f(I_1)$ so that all its points of intersection with $f(L_2)$ are cross-points. If a pair of intervals with the properties required at Lemma A.3.2 does not exist we arrive at a contradiction as in Lemma A.3.1. $\square$

LEMMA A.3.3. *Let* $[c_1, d_1] \subset p$, $[c_2, d_2] \subset I_1$ *be as in Lemma* A.3.2. *Then* $[c_1, d_1]$ *intersects* $[c_2, d_2]$.

Proof. We assume without loss of generality that $c_2 \in [c_1, d_1]_{L_2}$. We will need a lemma:

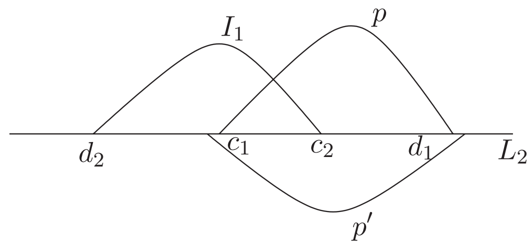


Figure A.3

Lemma A.3.3.1. *There is a path p' in $\bar{L}_2^-$ joining c_1 to d_1, and not intersecting $(c_1, d_1)_{L_2}$, that satisfies the following*:

If q is the closed curve $p' \cup [c_1, d_1]_{L_2}$ then there is a homotopy $g_1 : D_1 \to \bar{L}_2^-$ such that $d_2 \notin g_1(D_1)$ where D_1 is a disk and $g_1(\partial D_1) = q$.

Proof. Let p' be a simple path in $\bar{L}_2^-$ joining c_1 to d_1 and not intersecting $(c_1, d_1)_{L_2}$. We suppose also that p' is a union of three successive subpaths, the first lying in L_2 the second intersecting L_2 only at its endpoints and the third lying in L_2 (see Figure A.3). This first and third subpaths maybe reduced to a point. We denote the second subpath p''.

Let q_1 be the closed curve $p' \cup [c_1, d_1]_{L_2}$. Let D_1 be a disc and let $g_1 : D_1 \to \bar{L}_2^-$ be a map such that $g_1(\partial D_1) = q_1$. We assume that g_1 restricted to ∂D_1 is injective. As usual we call p'' the subpath of ∂D_1 mapped to p''.

Let U be the union of all connected components O of $D_1 - g_1^{-1}(L_2)$ such that $\partial O \subset g_1^{-1}(L_2)$. Then $\partial U \subset g_1^{-1}(L_2)$ and by Tietze's theorem we can modify g_1 so that $g_1(U) \subset L_2$. After this modification, $D_1 - g_1^{-1}(L_2)$ is connected and contains p'' (except its endpoints). Of course it is possible that $d_2 \in g_1(D_1)$. We will show below how to modify p' and D_1 so that $d_2 \notin g_1(D_1)$.

For convenience we will consider D_1 to be a square on the plane with sides parallel to the axes. We assume moreover that $[c_1, d_1]_{L_2}$ corresponds to the upper side of this square. For simplicity we will still denote this side by $[c_1, d_1]_{L_2}$.

Let $x \in g_1^{-1}(d_2)$. We consider the connected component of x, say O_x, in $D_1 - g_1^{-1}([c_1, d_1]_{L_2})$. If $\partial O_x \subset g_1^{-1}([c_1, d_1]_{L_2})$ then using Tietze's theorem we change g_1 so that $g_1(O_x) \subset [c_1, d_1]_{L_2}$. Since there are finitely many components O_x as above after finitely many steps either D_1 satisfies the conditions of the lemma or there is a single connected component of $D_1 - g_1^{-1}([c_1, d_1]_{L_2})$ containing $g_1^{-1}(d_2)$.

Let $\varepsilon > 0$ be such that if $x_1 \in g_1^{-1}([c_1, d_1]_{L_2})$ and $x_2 \in g_1^{-1}(d_2)$ then $d(x_1, x_2) > 4\varepsilon$. We consider an ε-grid on the plane which partitions D_1 and join each square on the grid that meets $g_1^{-1}(d_2)$ to p'' by a path that avoids $g_1^{-1}([c_1, d_1]_{L_2})$. Let $\delta > 0$ be smaller than the distance of any such path from $g_1^{-1}([c_1, d_1]_{L_2})$. We consider now a δ-grid on the plane which partitions D_1 and call B_δ the set of all squares on the grid intersecting $g_1^{-1}([c_1, d_1]_{L_2})$. Let C_δ be the connected component of $[c_1, d_1]_{L_2}$ in B_δ. Clearly $g_1^{-1}(d_2)$ does not meet C_δ. Therefore there is a simple path p_δ lying on ∂C_δ such that p_δ separates D_1 in two components one of which contains $[c_1, d_1]_{L_2}$ (the upper side of D_1) and the other $g_1^{-1}(d_2)$. Let C'_δ be the component of $D_1 - p_\delta$ containing $[c_1, d_1]_{L_2}$. Now, $g_1(p_\delta)$ is the p' required by the lemma and we can take C'_δ for D_1. □

Let D_2 be a disc and let $g_2 : D_2 \to \bar{L}_2^+$ be a map such that $g_2(\partial D_2) = [c_1, d_1]_{L_2} \cup [c_1, d_1]$ (here $[c_1, d_1] \subset p$ as in Lemma A.3.3). We identify D_1, D_2 along $[c_1, d_1]_{L_2}$ and obtain a disk D and a map $g : D \to X$ such that $g(\partial D) = p' \cup p$ (where p', D_1 are as in Lemma A.3.3.1).

Let $a, b \in [c_1, d_1]_{L_2}$ be such that the following two conditions hold:

i) a, b are separated by L_1;

ii) if $[x_1, x_2]_{L_1}$ has $x_1, x_2 \in \partial D$ and does not contain $[c_2, d_2]_{L_1}$ then $[x_1, x_2]_{L_1}$ does not intersect $[a, b]_{L_2}$.

Such points exist since $c_2 \in [c_1, d_1]_{L_2}$ is a cross-point of L_1, L_2. So we can find a, b as above sufficiently close to c_2.

We consider now $g^{-1}(a), g^{-1}(b)$. If x lies in $g^{-1}(a)$ or in $g^{-1}(b)$ we consider the connected component of x, say U_x, to be in $D - g^{-1}(L_1)$. If $g(\partial U_x) \subset L_1$ we can, using Tietze's theorem, change g so that $g(U_x) \subset L_1$. We assume therefore that for each x in $g^{-1}(a)$ or in $g^{-1}(b)$ the connected component of x in $D - g^{-1}(L_1)$ contains an interval of ∂D.

Let x, U_x be as above. Then $\partial D - \partial U_x$ is a union of open intervals. Let y be a point in $g^{-1}(a)$ or in $g^{-1}(b)$ such that $y \notin U_x$. Then it is clear that U_y intersects at most one component of $\partial D - \partial U_x$. Since there is a finite number of components of $D - g^{-1}(L_1)$ that intersect $g^{-1}(\{a, b\})$ it is easy to see that there is an $x \in g^{-1}(\{a, b\})$ such that there is exactly one component, say α, of $\partial D - \partial U_x$ with the following property: for every $y \in g^{-1}(\{a, b\})$ such that $y \notin U_x$, U_y intersects α.

We consider the connected component of α in $D - \partial U_x$ and call it O. Now, O intersects ∂D at α and $K = \partial O$ is connected and separates x from α.

We show now how to modify D without altering $g^{-1}(a), g^{-1}(b)$ so that for any $y \in g^{-1}(\{a, b\})$, $y \notin U_x$, x, y are separated by a simple path that maps to an interval of L_1 that does not contain c_2. Note that $\partial D - \alpha$ is a simple path β and both endpoints of β lie in K.

We define now two discs E_1, E_2 and maps: $h_1 : E_1 \to X$, $h_2 : E_2 \to X$. Both E_1, E_2 are copies of D. To define h_1, h_2 we modify g as follows: $h_1 = g$ on $K \cap \beta$ and h_1 is thus defined on a closed subset of β. The complement of this set is a union of open intervals. If (y_1, y_2) is such an interval, $h_1(y_1) = g(y_1) \in L_1$ and $h_1(y_2) = g(y_2) \in L_1$. We extend h_1 on (y_1, y_2) by mapping this interval injectively to the open interval of L_1 with endpoints $h_1(y_1), h_1(y_2)$. In this way we extend h_1 on β and note that h_1 maps β to an interval of L_1 that does not contain c_2. We consider the complement of O in D. This is a closed subset of the plane and h_1 maps the frontier of this set to an interval of L_1 that does not contain d_2. We extend h_1 to the whole set using Tietze's theorem. Finally we define $h_1 = g$ on $\bar{O}$.

We define h_2 similarly: We set $h_2 = g$ on $D - O$ and we define h_2 on α as we defined h_1 on β; i.e., if $i : \partial D \to \partial D$ is a one-to-one map interchanging α, β we define $h_2(x) = h_1(i(x))$. Also, O is an open set whose boundary is mapped to an interval of L_1 that does not contain d_2 by h_2. We extend h_2 on this open set using Tietze's theorem.

As E_1, E_2 are simply copies of D we will use the same labellings for them as for D. We can identify the arc β of E_1 to the arc α of E_2 and create a new disc $\tilde{D}$. We define $h : \tilde{D} \to X$ in the obvious way and note that $h(\partial\tilde{D}) = p' \cup p$, so that we can replace D by $\tilde{D}$. We note that passing from D to $\tilde{D}$ we did not change the inverse images of a, b. What we gained is that U_x is separated from any $y \in h^{-1}(\{a, b\})$ by the arc β and $h(\beta)$ is an interval of L_1 that does not contain c_2.

We repeat this operation as follows: Let $\tilde{D}$ be the disc obtained at some step and $h : \tilde{D} \to X$ be such that $h(\partial\tilde{D}) = p' \cup p$. There is a finite set of points $x_1 = x, x_2, \ldots, x_n$ lying in $h^{-1}(\{a, b\})$ such that the corresponding open sets U_{x_i} are disjoint and each x_i is separated from any $y \in h^{-1}(\{a, b\})$ by some arc γ such that $h(\gamma)$ is an interval of L_1 that does not contain c_2. If there is still some $z \in h^{-1}(a)$ that is not separated from some $y \in h^{-1}(b)$ by an arc γ as above we choose a point $z \in h^{-1}(\{a, b\})$ in $\tilde{D}$ with the property that for every $y \in h^{-1}(\{a, b\})$, such that U_y is not equal to some U_{x_i}, U_y intersects the same component of $\partial\tilde{D} - \partial U_z$. We repeat then the previous operation where z plays now the role of x.

By repeating this operation a finite number of times we obtain a disc that we still denote by $\tilde{D}$ and a map $h : \tilde{D} \to X$, with $h(\partial\tilde{D}) = p' \cup p$ which has the following property: If $x \in h^{-1}(a)$ and $y \in h^{-1}(b)$ then there is a simple arc γ separating x from y and $h(\gamma)$ is an interval of L_1 (with its endpoints on p') that does not contain c_2. This however is impossible: there is a connected set $S \subset \tilde{D}$ such that $h(S) \supset [c_1, d_1]_{L_2}$. Since no arc γ as above intersects $[a, b]_{L_2}$ one sees easily that there are points $x \in h^{-1}(a)$ and $y \in h^{-1}(b)$ that are not separated by any such arc. This contradiction proves Lemma A.3.3. □

Clearly Lemmas A.3.1, A.3.2, A.3.3 imply Lemma A.3. □

Remark A.3. There is a stronger form of Lemma A.3 that can be proved in the same way. Although not needed here the 'large scale' version of it is important and we state it here: I is an interval of intersection of L_1, L_2 if it is a connected component of $L_1 \cap L_2$ that is not a single point. Now the stronger version of Lemma A.3 is:

Let L_1, L_2 be separating lines. Let $a < b$ be two points of L_1 lying in $L_1 \cap L_2$ such that each one of them is either a cross-point of L_1, L_2 or an endpoint of an interval of intersection of L_1, L_2. Assume moreover that $(a,b)_{L_1}$ does not contain any cross-points of L_1, L_2. Let $I_1 = [a,b]_{L_1}$, $I_2 = [a,b]_{L_2}$. Then $I_1 \cup I_2$ separates X.

The following lemma is the main step in the proof of Theorem A.

LEMMA A.4. *Let C be a simple closed curve in X. Then C separates X.*

Proof. We will need a technical definition:

Definition A.4. Let C be a simple closed curve in X and let L be a separating line of X. Let $x \in L - C$. We say that a *subpath* of C, I, lying in $\bar{L}^+$ (or in L^-) is above x if the following are satisfied:

1) $\partial I \subset L$.

2) x lies in the interval of L determined by ∂I.

3) I is a maximal subpath satisfying 1), 2).

We say that x is an *L-interior point* of C if there is an odd number of subpaths of C in L^+ that are above x.

Remark A.4. Note that if C is a simple closed curve on the plane and L is a straight line then an L-interior point of C lies inside C (i.e. lies in the bounded component of $(\mathbb{R}^2 - C)$).

We have now some lemmas related to the above definition.

LEMMA A.4.1. *Let C be a simple closed curve and L a separating line of X. Let $x \in L - C$. If there is an odd number of subpaths of C above x in L^+ then there is an odd number of subpaths of C above x in L^-.*

The proof is left to the reader. □

LEMMA A.4.2. *Let C be a simple closed curve and let L be a separating line of X separating $a, b \in C$. Then there is an $x \in L - C$ such that x is an L-interior point of C.*

Proof. There is a $y \in L$ such that there is at least one subpath I of C above it in L^+ and a subpath J above it in L^-. Indeed, suppose this is not the case. Let $[c,d]$ be a maximal interval of L such that for each $x \in (c,d)_L - C$ there is at least one subpath of C above x in L^+. Then c, d are joined by a subpath of C in $\bar{L}^+$. Indeed if not there is a subpath I of C in L^+ such that at least one endpoint of I lies in $(c,d)_L$ and every neighborhood of this endpoint in C does not lie in $\bar{L}^+$. Then there are points in L close to this endpoint such that there are subpaths of C above them in L^+ and in L^-, a contradiction. Now let C_1 be the subpath joining c to d in L^+. We may assume without loss of generality that $c < d$ in L. Now, C is oriented so that C_1 is traversed from c to d in the positive direction. If we now traverse C_1, the subpaths of C after C_1 lying in L^+ intersect L in $[d, \infty)$ (otherwise $[c, d]$ would not be maximal) and by our hypothesis the same is true for the subintervals of C lying in L^-. This is clearly impossible; therefore there is a $y \in L$ such that there are subpaths of C above it both in L^+ and in L^-.

Assume now that y is not an L-interior point of C. Consider now the subpaths of C above y in L^+, L^-. We traverse them successively and note that there are two such paths I, J traversed successively such that $I \in L^+$, $J \in L^-$ (or the inverse). Say $I = [a_1, a_2] \subset L^+$, $J = [b_1, b_2] \subset L^-$ traversed in the order: $a_1 \to a_2 \to b_1 \to b_2$. There is an $\varepsilon > 0$ such that $C - [a_1, b_2]_C$ is at distance $> \varepsilon$ from a_2, b_1. We consider then the points $a_2 - \varepsilon, a_2 + \varepsilon, b_1 - \varepsilon, b_1 + \varepsilon$ on L. We remark that the parity of the number of subintervals of $C - [a_2, b_1]_C$ lying above $a_2 - \varepsilon$ and $a_2 + \varepsilon$ in L^+ is different. The same is true for the number of subintervals of $C - [a_2, b_1]_C$ lying above $b_1 - \varepsilon$ and $b_1 + \varepsilon$ in L^-.

We note now that if $a_2 = b_1$ or $[a_2, b_1]_C = [a_2, b_1]_L$ the above observations imply that for at least one of $a_2 - \varepsilon, a_2 + \varepsilon, b_1 - \varepsilon, b_1 + \varepsilon$ there is an odd number of subintervals of C lying above it in L^+ (in fact if $a_2 < b_1$ in L the parity of such intervals is different for $a_2 - \varepsilon$ and $b_1 + \varepsilon$ while if $a_2 > b_1$ the parity of such intervals is different for $b_1 - \varepsilon$ and $a_2 + \varepsilon$).

Clearly the parity of the number of subintervals of $C - [a_2, b_1]_C$ lying above points of $[a_2, b_1]_L - C$ in L^+ or in L^- remains constant. Indeed, otherwise the parity of the number of subintervals of $[a_2, b_1]_C$ above points of $[a_2, b_1]_L - C$ would have to change too. This would imply that C intersects itself on $[a_2, b_1]_L$, a contradiction.

Without loss of generality we suppose that for each point in $[a_2, b_1]_L - C$ there is an odd number of subintervals of $C - [a_2, b_1]_C$ above it in L^+ and an even number of such subintervals in L^-. We consider now the biggest subinterval of L, $[a, b]_L$,(say $a < b$ on L) with the following properties:

1) $[a_2, b_1]_L \subset [a, b]_L$.
2) For all $x \in [a, b]_L - C$ the number of subintervals of $[a_2, b_1]_C$ above x in L^+ is odd.

If $([a, b]_L - C) - ([a_2, b_1]_L - C) = \emptyset$ then the parity of the number of intervals of C above $a - \varepsilon, b + \varepsilon$ is different and we are done.

If $([a, b]_L - C) - ([a_2, b_1]_L - C) \neq \emptyset$ then there is a subinterval $[x, y]_L$ of $[a, b]_L$ equal to a connected component of $C \cap [a, b]_L$ (possibly reduced to a or b) so that the parity of the number of subintervals of C above $x - \delta, y + \delta$ is different for some $\delta < \varepsilon$ chosen sufficiently small to ensure that $x - \delta, y + \delta \notin C$.

□

LEMMA A.4.3. *Let C be a simple closed curve in X and let L_1, L_2 be separating lines. If x is a cross-point of L_1, L_2 then x is an L_1-interior point of C if and only if it is an L_2-interior point of C.*

Proof. Let $p : S^1 \to C$ be a parametrization of C. We define a map f from $L_1 \sqcup L_2 \sqcup S^1$ to the plane as follows:

We map L_1 to the x-axis in a length-preserving way. Then we map the segments of L_2 -((cross-points of L_1, L_2)$\cup$(cross-intervals of L_1, L_2)) in $\bar{L}_1^+$ to the upper half-plane ($y > 0$) and the segments of L_2 in $\bar{L}_1^-$ to the lower ($y < 0$) half-plane. If (a, b) is such a segment lying in $\bar{L}_1^+$ we map it to the polygonal line made out of three segments and having as vertices $(f(a), 0)$, $(f(a), b-a))$, $(f(b), b-a))$, $(f(b), 0)$. Note that the image is the 'open' polygonal line, so that $(f(a), 0)$ and $(f(b), 0)$ do not lie in the image of (a, b). The map sends subsegments of L_2 lying in $\bar{L}_1^+$ (or in $\bar{L}_1^-$) injectively on the plane. If t is a cross-point of L_1, L_2 then we define $f(t)$ by continuity so that $f(t)$ is a cross-point of $f(L_1), f(L_2)$. We extend this map to the cross intervals of L_1, L_2 in the obvious way and note that if $t \in L_2$ is in $L_1 \cap L_2$ but is not a cross-point and does not lie on a cross-interval $f(t)$ does not lie on $f(L_1)$. Lemma A.3 ensures that we can find a map f from $L_1 \sqcup L_2$ (satisfying the above conditions) which maps L_2 injectively on the plane.

We remark that $f(L_1), f(L_2)$ are lines and they divide the plane in two pieces. Also, $a, b \in L_1$ are separated by L_2 if and only if $f(a), f(b)$ are separated by $f(L_2)$. Indeed suppose that $a, b \in L_1$ are not separated by L_2. Then they can be joined by a path p such that $p \cap L_1$ has finitely many connected components and $p \cap L_2 = \emptyset$. We can write p as a union $p = p_1 \cup \cdots \cup p_k$ where each p_i lies in $\bar{L}_1^+$ or in $\bar{L}_1^-$. By Lemma A.3.3 the endpoints of p_i are not separated in $\bar{L}_1^+$ (or in $\bar{L}_1^-$) by any interval of L_2. This implies that their images are not separated by an interval of $f(L_2)$. Therefore we can join $f(a), f(b)$ by a path which does not meet $f(L_2)$.

If there are $a, b \in L_1$ which are separated by L_2 such that $f(a), f(b)$ are not separated by $f(L_2)$ then by the argument above we have that L_1 lies in the closure of a single component of $\mathbb{R}^2 - f(L_2)$ which is impossible.

We will call $f(L_1^+)$ the component of $\mathbb{R}^2 - f(L_1)$ containing the images of points of L_2 that lie in L_1^+. We define similarly $f(L_1^-)$,$f(L_2^+)$,$f(L_2^-)$.

We explain now how to map S^1 to the plane: We modify first $p : S^1 \to X$ as follows: Let $q = [a, b]$ be an arc of S^1 such that $p(q) \subset \bar{L}_1^+$, $p(q) \cap L_1 = p(\partial q)$ and $x \in [p(a), p(b)]_{L_1}$. Let a, b be the endpoints of ∂q. We modify p so that it becomes constant in an ε neighborhood of the a, b (we 'thicken' a, b to small intervals).

Still calling the new map p, we then have $p([a-\varepsilon, a+\varepsilon]) = p(a)$, $p([b-\varepsilon, b+\varepsilon]) = p(b)$ and p maps injectively the rest of S^1 to C. We perform this modification for all arcs q as above and also for arcs with $p(q) \subset \bar{L}_1^-$

or $p(q) \subset \bar{L}_2^+$ or $p(q) \subset \bar{L}_2^-$ and still call this map p. Clearly the image of p is still C. Let $q = [a+\varepsilon, b-\varepsilon]$ be an arc of S^1 such that $p(q) \subset \bar{L}_1^+$, $p(q) \cap L_1 = p(\partial q)$, $x \in [p(a+\varepsilon), p(b-\varepsilon)]_{L_1}$ and $p(\partial q) \cap L_2 = \emptyset$. We can then define $\bar{p} : q \to \mathbb{R}^2$ so that the following holds:

1) For any $t \in q$, $p(t) \in L_2^+$ if and only if $\bar{p}(t) \in f(L_2^+)$ and $p(t) \in L_2^-$ if and only if $\bar{p}(t) \in f(L_2^-)$.

2) $\bar{p}(q) \subset \overline{f(L_1^+)}$.

The existence of $\bar{p}$ is an easy consequence of Lemma A.3.3. We note that $\bar{p}$ is not necessarily injective and define in this case $\bar{p}([a-\varepsilon, a+\varepsilon]) = \bar{p}(a+\varepsilon)$ and $\bar{p}([b-\varepsilon, b+\varepsilon]) = \bar{p}(b-\varepsilon)$.

If q is an arc as above for which $p(\partial q) \cap L_2 \neq \emptyset$ then if $p(\partial q)$ intersects L_2 only at cross-points of L_1, L_2 it is still possible to define $\bar{p}$ on q so that 1), 2) are satisfied. We explain what to do if an endpoint of q is an intersection point of L_1, L_2 that is not a cross-point: Say $p(a-\varepsilon) \in p(\partial q)$ lies on an interval of L_2:$[a_1, a_2]_{L_2}$ where a_1, a_2 are cross-points of L_1, L_2 and $[a_1, a_2]_{L_2}$ lies in $\bar{L}_1^+$ or in $\bar{L}_1^-$.

If $[a_1, a_2]_{L_2}$ lies in $\bar{L}_1^-$ then we define $\bar{p}$ on $[a-\varepsilon, a+\varepsilon]$ so that $\bar{p}([a-\varepsilon, a+\varepsilon])$ is an arc in $\overline{f(L_1^-)}$ joining the image of a by f on $f(L_1)$ to the image of a by f on $f(L_2)$ and which intersects $f(L_1 \cup L_2)$ only at its endpoints.

If $[a_1, a_2]_{L_2}$ lies in $\bar{L}_1^+$ then we define $\bar{p}$ on $[a-\varepsilon, a+\varepsilon]$ so that $\bar{p}([a-\varepsilon, a+\varepsilon])$ is an arc in $\overline{f(L_1^+)}$ joining the image of a by f on $f(L_1)$ to the image of a by f on $f(L_2)$ and which intersects $f(L_1 \cup L_2)$ only at its endpoints. We define $\bar{p}$ similarly on $[b-\varepsilon, b+\varepsilon]$ and finally on $(a+\varepsilon, b-\varepsilon)$ so that conditions 1), 2) above hold.

We define $\bar{p}$ in the same way for all arcs which are as above but where L_1^+ is replaced by one of L_1^-, L_2^+, L_2^-. It is possible that two such arcs overlap but it is easy to see that we can define $\bar{p}$ on the overlap in the same way for both arcs.

We explain now how to extend $\bar{p}$ to the rest of S^1: Let (c, d) be a connected component of S^1 minus the union of arcs on which $\bar{p}$ is already defined. If both $p(c), p(d)$ lie on L_1 then $\bar{p}$ is defined on (c, d) so that for any $t \in (c, d)$, $\bar{p}(t) \in f(L_1)$ if and only if $p(t) \in L_1$ and $\bar{p}(t) = f(p(t))$ and so that if $\bar{p}(t) \in L_2$ then $p(t) \in L_2$ and $\bar{p}(t) = f(p(t))$. It is clear by Lemma A.3.3 that this can be done. We define $\bar{p}$ similarly if both $p(c), p(d)$ lie on L_2. Otherwise if, say $p(c) \in L_1$ and $p(d) \in L_2$ then if $p((c, d))$ does not intersect $L_1 \cup L_2$ we define $\bar{p}$ on (c, d) so that $\bar{p}([c, d])$ is an arc joining $\bar{p}(c), \bar{p}(d)$. This arc intersects $f(L_1), f(L_2)$ only at its endpoints. Otherwise we write $[c, d] = [c, c_1] \cup [c_1, d]$ where one of $[c, c_1), (c_1, d]$ intersects only one of the L_1, L_2 and the other has both its endpoints on L_1, L_2. Finally we map each one of $[c, c_1], [c_1, d]$ by $\bar{p}$ as described above. In all cases we have that for any $t \in (c, d)$ if $\bar{p}(t) \in L_1$ then

$p(t) \in f(L_1)$ and if $\bar{p}(t) \in L_2$ then $p(t) \in f(L_2)$. Also if either of these occurs then $\bar{p}(t) = f(p(t))$.

In this way we define $\bar{p}$ on the whole of S_1. We note now that by definition x is an $L_1(L_2)$-interior point of C if and only if $f(x)$ is an $f(L_1)(f(L_2))$-interior point of $\bar{p}(S^1)$. Since $f(x)$ is an $f(L_1)$-interior point of $\bar{p}(S^1)$ if and only if it is an $f(L_2)$-interior point of $\bar{p}(S^1)$ we have that x is an L_1-interior point of C if and only if x is an L_2-interior point of C. □

To finish the proof of Lemma A.4 we need to show that every path in X can be approximated by 'polygonal paths', i.e. paths made out of intervals of separating lines. We explain now how this can be done.

Let $O \in X$. Let $t \in S_1(O)$ (i.e. $d(O,t) = 1$). There is a line L_t separating O from t. For each $t \in S_1(O)$ choose a line L_t separating t from O. Since X is locally compact there is a finite set of lines $S' = \{L_1, \ldots, L_n\}$ such that each point $t \in S_1(O)$ is separated from O by some L_i. Suppose that this set of lines is minimal with respect to this property. Clearly there is an $\varepsilon > 0$ such that:

a) $d(L_i, O) > \varepsilon$ for $i = 1, \ldots, n$.

b) For every $t \in S_1(O)$ there is an L_i separating t, O such that $d(L_i, t) > \varepsilon$.

Denote by $\bar{L}_i^+$ the closure of the component of $X - L_i$ that contains O and denote by I_i the smallest interval of L_i containing $L_i \cap B_1(O)$ (where $i = 1, \ldots, n$).

We call S' the set of all the I_i and let S be a maximal cross-connected subset of S'. Note that the following holds: If $x \in I_s \cap S_1(O)$ is separated from O by L_t then I_s crosses I_t. Indeed let $y \in S_1(O)$ be a point that is separated from O by L_s and is not separated from O by L_t. Let q be a geodesic joining O to y and let y_1, y_2 be, respectively, the first and the last point of intersection of q with L_t. Clearly if q intersects L_s before y_1 then I_s crosses I_t. We now modify q by replacing $[y_1, y_2]_q$ with $[y_1, y_2]_{L_t}$. The new path still joins O to y, so that it has to cross L_s. This clearly implies that there is a point of I_s lying in L_t^+ which in turn implies that I_s crosses I_t.

Lemma A.4.4. *$X - S$ is disconnected and O lies in a bounded component of $X - S$.*

Proof. Without loss of generality we suppose that $L_1, \ldots, L_k$ are the lines in S. Let C be the component of O in $X - (L_1 \cup \cdots \cup L_k)$. Note that ∂C is connected. Indeed let $a, b \in L_i \cap \partial C$. There is path q joining a, b in $\bar{C}$. Consider $q \cup [a,b]_{L_i}$ and a filling disc for this path, say $f : D \to X$. Then, as in Lemma A.1 we see that a, b are contained in the same component of the boundary of the connected component of $f^{-1}(C)$ that contains q (or to be precise $f^{-1}(q) \cap \partial D$). It follows that a, b can be joined to each other by a path lying in ∂C. Since S is cross-connected, ∂C is connected.

Clearly ∂C cannot contain an infinite interval of some L_i $(i = 1, \dots, k)$. In fact if an interval of some L_i is contained in ∂C then it is either contained in $B_{1-\varepsilon}(O)$ or it lies outside $B_1(O)$. Since ∂C is connected this implies that ∂C is contained in $B_{1-\varepsilon}(O)$; therefore it is contained in S. $\square$

Of course we could have made the same construction using $S_\delta(O)$ for any $\delta > 0$ instead of 1.

We return now to the proof of Lemma A.4. Let L be a line separating two points $a, b \in C$. Then by Lemma A.4.2 there is an $x \in L - C$ that is an L-interior point of C. Let $y \in L$ be a point such that $L \cap C \subset (-\infty, y - 1)_L$. Then y is not an L-interior point of C. We will show that any path joining x to y intersects C. Indeed suppose not. Then there is a path p joining x to y and a $1 > \varepsilon > 0$ such that $d(p, C) > \varepsilon$, where $d(p, C) = \min\{d(t, s) | t \in p,\, s \in C\}$.

Let $\delta > 0$ be such that if $t, s \in L$ and $d(t, s) \leq \delta$ then $[t, s]_L$ is contained in $B_\varepsilon(t)$. Note here that by our hypothesis that lines are f-lines, δ depends only on ε and not on L. This is the only place where we use the hypothesis that lines in this proof are f-lines.

As in Lemma A.4.4 for each point $O \in p$ we pick a finite number of lines separating O from $S_\delta(O)$. Using the compactness of p and Lemma A.4.4 we see that there is a finite sequence of lines $L_1, \dots, L_n$ such that L crosses L_1 at x_0, L_k crosses L_{k-1} at x_{k-1} $(k = 1 \dots n)$, L_n crosses L at x_n, $x_0, x_1, \dots, x_n$ are contained in the δ neighborhood of p and $d(x_0, x) < \delta$, $d(x_n, y) < \delta$.

By Lemma A.4.3 we have that x_0 is an L-interior point of C if and only if x_n is an L-interior point of C. As $d(x_0, x) < \delta$, $d(x_n, y) < \delta$ we conclude that x, y are either both L-interior points of C or neither is an L-interior point of C. This is clearly a contradiction and finishes the proof of Lemma A.4. $\square$

To prove Theorem A we will use Theorem 1. Since X is one-ended no pair of points separates X. We can compactify X by adding one point at infinity. Clearly the compact space obtained thus is not separated by any pair of its points. We note that the argument of Lemma A.4 applies also to lines, so that every line in X separates X. We conclude that each simple closed curve in the space obtained by the one point compactification of X separates. So by Theorem 1 this space is a sphere. Clearly then X is homeomorphic to the plane. $\square$

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