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The uniqueness of the helicoid

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In this paper we will discuss the geometry of finite topology properly embedded minimal surfaces M in $\mathbb{R}^3$. M of finite topology means M is homeomorphic to a compact surface $\widehat{M}$ (of genus k and empty boundary) minus a finite number of points $p_1, ..., p_j \in \widehat{M}$, called the punctures. A closed neighborhood E of a puncture in M is called an end of M. We will choose the ends sufficiently small so they are topologically $S^1 \times [0, 1)$ and hence, annular. We remark that $\widehat{M}$ is orientable since M is properly embedded in $\mathbb{R}^3$.

The simplest examples (discovered by Meusnier in 1776) are the helicoid and catenoid (and a plane of course). It was only in 1982 that another example was discovered. In his thesis at Impa, Celso Costa wrote down the Weierstrass representation of a complete minimal surface modelled on a 3-punctured torus. He observed the three ends of this surface were embedded: one top catenoid-type end[1], one bottom catenoid-type end, and a middle planar-type end[2] [8]. Subsequently, Hoffman and Meeks [15] proved this example is embedded and they constructed for every finite positive genus k embedded examples of genus k and three ends.

In 1993, Hoffman, Karcher and Wei [14] discovered the Weierstrass data of a complete minimal surface of genus one and one annular end. Computer generated pictures suggested this surface is embedded and the end is asymptotic to an end of a helicoid. Hoffman, Weber and Wolf [17] have now given a proof that there is such an embedded surface. Moreover, computer evidence suggests that one can add an arbitrary finite number k of handles to a helicoid to obtain a properly embedded genus k minimal surface asymptotic to a helicoid.

For many years, the search went on for simply connected examples other than the plane and helicoid. We shall prove that there are no such examples.

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[1] Asymptotic to the end of some catenoid.

[2] Asymptotic to the end of some plane.

Theorem 0.1. *A properly embedded simply-connected minimal surface in* $\mathbb{R}^3$ *is either a plane or a helicoid.*

In the last decade, it was established that the unique 1-connected example is the catenoid. First we proved such an example is transverse to a foliation of $\mathbb{R}^3$ by planes [23], and then Pascal Collin [6] proved this property implies it is a catenoid.

There is an important difference between M with one end and those with more than one end. The latter surfaces have the property that one can find planar or catenoid type ends in their complement. This limits the surface to a region of space where it is more accessible to analysis. Clearly the helicoid admits no such end in its compliment. To find planar and catenoidal type ends in the compliment of an M with at least two ends, one solves Plateau problems in appropriate regions of space and passes to limits to obtain complete stable minimal surfaces. Then the stable surface has finite total curvature by [10], and hence has a finite number of standard ends.

In addition to proving the uniqueness of the helicoid, we also describe the asymptotic behavior of any properly embedded minimal annulus A in $\mathbb{R}^3$, A diffeomorphic to $S^1 \times [0,1)$. We prove that either A has finite total Gaussian curvature and is asymptotic to the end of a plane or catenoid or A has infinite total Gaussian curvature and is asymptotic to the end of a helicoid. In fact, if A has infinite total curvature, we prove that A has a special conformal analytic representation on the punctured disk D^* which makes it into a minimal surface of "finite type" (see [12], [26], [27]). In this case the stereographic projection of the Gauss map $g\colon D^* \to \mathbb{C} \cup \{\infty\}$ has finite growth at the puncture in the sense of Nevanlinna. Since a nonplanar properly embedded minimal surface in $\mathbb{R}^3$ with finite topology and one end always has infinite total curvature and one annular end, such a surface always has finite type.

Theorem 0.2. *Suppose M is a properly embedded nonplanar minimal surface with finite genus k and one end. Then, M is a minimal surface of finite type, which means, after a possible rotation of M in $\mathbb{R}^3$, that:*

1. *M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at a single point p_∞;*

2. *If $g\colon M \to \mathbb{C} \cup \{\infty\}$ is the stereographic projection of the Gauss map, then dg/g is a meromorphic 1-form on $\overline{M}$ with a double pole at p_∞;*

3. *The holomorphic 1-form $dx_3 + i dx_3^*$ extends to a meromorphic 1-form on $\overline{M}$ with a double pole at p_∞ and with zeroes at each pole and zero of g of the same order as the zero or pole of g. The meromorphic function g has k zeroes and k poles counted with multiplicity.*

In fact, this analytic representation of M implies M is asymptotic to a helicoid.

A consequence of the above theorem is that the moduli space of properly embedded one-ended minimal surfaces of genus k is an analytic variety; we conjecture that this variety always consists of a single point, or equivalently, there exists a unique properly embedded minimal surface with one end for each integer k.

Theorem 0.2 and the main theorem in [6] have the following corollary:

COROLLARY 1. *If M is a properly embedded minimal surface in $\mathbb{R}^3$ of finite topology, then each annular end of M is asymptotic to the end of a plane, a catenoid or a helicoid.*

The above corollary demonstrates the strong geometric consequences that finite topology has for a properly embedded minimal surface. In particular, the Gaussian curvature of M is uniformly bounded.

The validity of the following "bounded curvature conjecture" would show that the hypotheses of Theorems 0.1 and 0.2 can be weakened by changing "proper" to "complete", since by Theorem 1.6, a complete embedded minimal surface of bounded curvature is proper.

CONJECTURE 1. *Any complete embedded minimal surface in $\mathbb{R}^3$ with finite genus has bounded Gaussian curvature.*

This paper is organized as follows. In Section 1 we establish the following properties for minimal laminations of $\mathbb{R}^3$. A minimal lamination consists of either one leaf, which is a properly embedded minimal surface, or if there is more than one leaf in the lamination, then there are planar leaves. The set of planar leaves P is closed and each limit leaf is planar. In each open slab or halfspace in the complement of P there is at most one leaf of the lamination, which (if it exists) has unbounded curvature and is proper in the slab or halfspace. Each plane in the slab or halfspace separates such a leaf into exactly two components. Furthermore, if the lamination has more than one leaf, then each leaf of finite topology is a plane.

In Section 2 we begin the study of a properly embedded simply-connected minimal surface M, which we will always assume is not a plane. The starting point is the theorem of Colding and Minicozzi concerning homothetic blow-downs of M. They prove that any sequence of homothetic scalings of M, with the scalings converging to zero, has a subsequence $\lambda(i)M$ that converges to a minimal foliation $\mathcal{L}$ in $\mathbb{R}^3$ consisting of parallel planes and such that the convergence is smooth except along a connected Lipschitz curve $S(\mathcal{L})$ that meets each leaf in a single point. They also prove $S(\mathcal{L})$ is contained in a double cone C around the line passing through the origin and orthogonal to the planes in $\mathcal{L}$. Notice that if N is a properly embedded triply-periodic minimal surface, then no sequence of homothetic blow-downs of N can converge to a lamination.

Also notice that if N is a vertical helicoid, then any homothetic blow-down of N is the foliation by horizontal planes and the singular set of convergence is the x_3-axis. In this section we prove that for a given M, a homothetic blow-down $\mathcal{L}$ is independent of the choice of scalings converging to zero and that M is transverse to the planes in $\mathcal{L}$. In particular, the Gauss map of M omits the two unit vectors orthogonal to the planes in $\mathcal{L}$.

We denote the unique homothetic blow-down foliation of M by $\mathcal{L}(M)$, which we may assume consists of horizontal planes. From the uniqueness of $\mathcal{L}(M)$, we get the following useful picture of M in Section 2. Let C be the vertical double cone mentioned above which contains the singular set of convergence $S(\mathcal{L}(M))$. There exists a solid hyperboloid $\mathcal{H}$ of revolution with boundary asymptotic to the boundary of the cone C such that for $\mathcal{W}$ defined to be the closure of $\mathbb{R}^3 - \mathcal{H}$, $\mathcal{W} \cap M$ consists of two multisheeted graphs of asymptotically zero gradient over their projection on the x_1x_2-plane.

In Section 3 we prove that there is a positive integer n_0 such that if G is a minimal graph over a proper subdomain D in $\mathbb{R}^2 \times \{0\}$ with zero boundary values and bounded gradient, then G can have at most n_0 components that are not contained in the x_1x_2-plane. Motivated by this result, Li and Wang [19] have shown that one can drop our bounded gradient hypothesis and still obtain the finite connectivity property for G. In Section 4 we use our finite connectedness result, on minimal graphs of bounded gradient and our description of $\mathcal{W} \cap M$, to prove that each plane in $\mathcal{L}(M)$ intersects M transversely in one proper arc. Furthermore, we prove in Theorem 4.4, using results in [7], that M can be conformally parametrized by $\mathbb{C}$ and in this parametrization the third coordinate function can be expressed as $x_3 = \mathbb{R}e(z)$. In Section 5 we use Theorem 4.4 and the uniqueness of $\mathcal{L}(M)$ to prove that the stereographically projected Gauss map is $g(z) = e^{az+b}$ from which it follows that M is a vertical helicoid. In Section 6 we prove that if M has finite genus and one end, then M is a surface of finite type.

1. Minimal laminations of R^3

A closed set $\mathcal{L}$ in $\mathbb{R}^3$ is called a *minimal lamination* if $\mathcal{L}$ is the union of pairwise disjoint connected complete injectively immersed minimal surfaces. Locally we require that there are $C^{1,\alpha}$ coordinate charts $f\colon D\times(0,1) \to \mathbb{R}^3, 0 < \alpha < 1$, with $\mathcal{L}$ in $f(D\times(0,1))$ the image of the $D\times\{t\}$, t varying over a closed subset of $(0,1)$. The minimal surfaces in $\mathcal{L}$ are called the *leaves* of $\mathcal{L}$.

A leaf L of a minimal lamitation $\mathcal{L}$ is smooth (even analytic), and if K is a compact set of an L which is a limit leaf of $\mathcal{L}$, then the leaves of $\mathcal{L}$ converge smoothly to L over K; the convergence is uniform in the C^k-topology for any k.

Our work will depend upon the following (very important) curvature estimates of Colding and Minicozzi [4], which we will refer to as the curvature estimates $\mathcal{C}$. There exists an $\varepsilon > 0$ such that the following holds. Let $y \in \mathbb{R}^3$,

$r > 0$ and $\Sigma \subset B_{2r}(y) \cap \{x_3 > x_3(y)\} \subset \mathbb{R}^3$ be a compact embedded minimal disk with $\partial\Sigma \subset \partial B_{2r}(y)$. For any connected component Σ' of $B_r(y) \cap \Sigma$ with $B_{\varepsilon r}(y) \cap \Sigma' \neq \emptyset$, one has $\sup_{\Sigma'}|A_{\Sigma'}|^2 \leq r^{-2}$.

A consequence of these curvature estimates is the following. Let Σ be any compact smooth surface passing through the origin with boundary contained in the boundary of the ball $B(1)$ of radius one centered at the origin. There is an ε and a constant c such that if D is an embedded minimal disk in $B(1)$, disjoint from Σ, and with boundary contained in the boundary of $B(1)$, then in $B(\varepsilon)$, the curvature of D is bounded by c. This can be seen by homothetically expanding Σ; the ε depends on the norm of the second fundamental form of Σ in the ball $B(\frac{1}{2})$. In our applications Σ will be a stable minimal disk for which one always has a bound on the norm of the second fundamental form in $B(\frac{1}{2})$, by curvature estimates for stable surfaces.

In this section we will prove a general structure theorem that explains some of the geometric properties that hold for a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$. A properly embedded minimal surface is the simplest example of a minimal lamination.

The only known examples of minimal laminations of $\mathbb{R}^3$ with more than one leaf are closed sets of parallel planes in $\mathbb{R}^3$ and the second author conjectures that these are the only ones. In fact, we will prove that in the case $\mathcal{L}$ has more than one leaf, then every leaf of $\mathcal{L}$ with finite topology is a plane.

We say that a minimal surface M in $\mathbb{R}^3$ has *locally bounded curvature* if the intersection of M with any closed ball has Gaussian curvature bounded from below by a constant that only depends on the ball. Every leaf L of a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ has locally bounded Gaussian curvature. The reason that the curvature is locally bounded is that the intersection of $\mathcal{L}$ with a closed ball is compact and the Gaussian curvature function is continuous.

LEMMA 1.1. *Suppose M is a complete connected embedded minimal surface in $\mathbb{R}^3$ with locally bounded Gaussian curvature. Then one of the following holds*:

(1) *M is properly embedded in $\mathbb{R}^3$*;

(2) *M is properly embedded in an open halfspace of $\mathbb{R}^3$ with limit set the boundary plane of this halfspace*;

(3) *M is properly embedded in an open slab of $\mathbb{R}^3$ with limit set consisting of the boundary planes.*

Proof. Let x_n be any sequence of points in M, converging to some x in $\mathbb{R}^3$. Since M has locally bounded curvature, there is a $\delta = \delta(x)$ such that for n sufficiently large, M is a graph F_n over the disk $D_\delta(x_n)$ in the tangent plane to M at x_n, of radius δ and centered at x_n. Moreover each such local graph F_n has bounded geometry.

Choose a subsequence of the x_n so that the tangent planes to M at the subsequence converge to some plane P at x. Then the F_n of this subsequence will be graphs (for n large) over the disk D of radius $\delta/2$ in P centered at x. By compactness of minimal graphs, a subsequence of the F_n will converge to a minimal graph F_∞ over $D, x \in F_\infty$.

Notice that F_∞ at x does not depend on the subsequence of the x_n. If $y_n \in M$ is a sequence converging to x with the tangent planes of M at y_n converging to a plane Q at x. Then $P = Q$ and the local graphs G_n of M at y_n converge to F_∞ as well. If this were not the case then F_∞ and G_∞ would cross each other near x (i.e, $x \in F_\infty \cap G_\infty$ and the maximum principle implies there are points of $F_\infty \cap G_\infty$ near x where they meet transversely). Now F_∞ is the uniform limit of the graphs F_n and G_∞ is the uniform limit of the graphs G_n so near a point of transverse intersection of F_∞ and G_∞ we would have F_i intersecting G_j transversely for i, j large. This contradicts that M is embedded. Notice also that each F_n is disjoint from F_∞; this follows by the same reasoning as above. Thus we have a local lamination contained in the closure $\overline{M}$ of M.

Each point $y \in \partial F_\infty$ is also an accumulation point of M so there is a limit graph $F_\infty(y)$ over a disk of radius $\delta(y)$ centered at y. By uniqueness of limits, $F_\infty(y) = F_\infty$ where they intersect. Thus F_∞ may be continued analytically to obtain a complete minimal surface in $\bar{M}$. The lamination $\mathcal{L}$ is obtained by taking the closure of all the limit surfaces so obtained.

Next we will prove that any limit leaf of $\mathcal{L}$ is a plane.

Let L be a limit leaf and $\widehat{L}$ the universal covering space of L. The exponential map of L is a local diffeormorphism and there is a normal bundle ν over $\widehat{L}$, of varying radius, that submerses in $\mathbb{R}^3$. Give ν the flat metric induced by the submersion; $\widehat{L}$ is the zero section of ν.

Let $\widehat{D}$ be a compact simply-connected domain of $\widehat{L}$, D its projection into L. Each point of D has a neighborhood that is a uniform limit of (pairwise disjoint) local graphs of M. The usual holomony construction allows one to lift these local graphs along the lifting of paths in D to obtain $\widehat{D}$ as a uniform limit of pairwise-disjoint embedded minimal surfaces E_n in ν.

It is known that any compact domain F (here $F = \widehat{D}$) that is a limit of disjoint minimal domains E_n is stable. Here is a proof. If F were unstable, the first eigenvalue λ_1 of the stability operator L of the minimal surface F (here $L = \Delta - 2K$) is negative. Let $\vec{n}$ denote the unit normal vector field along F in ν and f the eigenfunction of $\lambda_1, L(f) + \lambda_1 f = 0, f > 0$ in F and $f = 0$ on ∂F.

Consider the variation of F: $F(t) = \{x + tf(x)\vec{n}(x) \mid x \in F\}$. The first variation $\dot{H}(0)$ of the mean curvature of $F(t)$ at $t = 0$ is given by $L(f)$. Since $\lambda_1 < 0$, and $f(x) > 0$ for $x \in \mathrm{Int}(F)$, it follows that the mean curvature vector of $F(t)$, for t small, points away from F, i.e, $\langle \vec{H}_t(x), \vec{n}(x) \rangle > 0$.

Now for t_0 small, choose n large so that $E(n)$ is close enough to F so there is a nonempty intersection of $F(t_0)$ and $E(n)$. As t decreases from t_0 to 0, the

$F(t)$ go from $F(t_0)$ to F. So there will be a smallest positive t so that $D(t)$ has a nonempty intersection with $E(n)$. Let $y \in F(t) \cap E(n)$. Near y, $E(n)$ is on the mean convex side of $F(t)$. Since $E(n)$ is a minimal surface, this is impossible.

Hence, by the stability theorem of Fischer-Colbrie and Schoen [11] or do Carmo and Peng [9], $\widehat{L}$ is a plane, hence L as well, and so each limit leaf of $\mathcal{L}$ is a plane. Lemma 1.1 follows immediately from the fact that the limit leaves of $\mathcal{L}$ are planes. $\square$

Remark 1.2. F. Xavier [29] proved that a complete nonflat immersed minimal surface of bounded curvature in $\mathbb{R}^3$ is not contained in a halfspace. Hence, if one replaces "locally bounded curvature" by "bounded curvature," the possibilities 2 and 3 cannot occur in Lemma 1.1.

LEMMA 1.3. *Suppose M is a complete connected embedded minimal surface in $\mathbb{R}^3$ with locally bounded curvature. If M is not proper and P is a limit plane of M, then, for any $\varepsilon > 0$, the closed ε-neighborhood of P intersects M in a connected set and the curvature of this set is unbounded.*

Proof. Suppose P is a limit plane of M and, to be concrete, suppose P is the x_1x_2-plane and that M lies above P. Let $P(\varepsilon)$ be the plane at height ε and suppose that M intersects the closed slab S between P and $P(\varepsilon)$ in at least two components $M(1)$, $M(2)$. By Sard's theorem, we may assume that $P(\varepsilon)$ intersects M transversely. We know that M is proper in the open slab between P and $P(\varepsilon)$ since through any accumulation point of M in the open slab there would pass a limit plane of M.

Let R be the region of $S - P$ bounded by $M(1) \cup M(2)$. Since P is a limit plane of both M_1 and M_2, then R is a complete flat 3-manifold whose boundary is a good barrier for solving Plateau problems (see [24]). Consider a smooth compact exhaustion $\Sigma(1), \Sigma(2), \ldots, \Sigma(n), \ldots$ of $M(1)$. Let $\tilde{\Sigma}(i) \subset R$ with $\partial\tilde{\Sigma}(i) = \partial\Sigma(i)$ be least-area surfaces $\mathbb{Z}_2$-homologous to $\Sigma(i)$ in R. Standard curvature estimates and local area bounds imply that a subsequence of the $\tilde{\Sigma}(i)$ converges to a complete properly embedded stable minimal surface Σ in R with boundary $\partial M(1)$. Since R is proper in $S - P$ and since $S - P$ is simply-connected, Σ separates S. Therefore, Σ is orientable and the curvature estimates of Schoen [28] then imply curvature estimates at any uniform distance from $P(\varepsilon)$.

By the Halfspace Theorem in [16], or rather its proof, Σ cannot be proper in S. As in the previous lemma, the limit set of Σ is a plane $P' \subset S$, and clearly $P' = P$. Since Σ has curvature estimates near P, there exists a δ, $0 < \delta < \varepsilon/2$, such that the normal lines to $\Sigma(\delta) = \Sigma \cap \{(x_1, x_2, x_3) \mid 0 < x_3 < \delta\}$ are close to vertical lines. Hence, the orthogonal projection $\pi\colon \Sigma(\delta) \to P$ is a submersion onto its image. Furthermore, given any compact disk $D \subset P$, every component of $\pi^{-1}(D)$ is compact. Using this compactness property, and a slight variation of the following lemma, it follows that π is injective on each component Δ of $\Sigma(\delta)$. Therefore, Δ is proper graph in S, which we observed before cannot occur. This argument gives a contradiction and the end of the argument proves that M has unbounded curvature in S. $\square$

LEMMA 1.4. *Suppose M and N are smooth connected manifolds of the same dimension such that N is simply-connected and M may have boundary. If $\pi\colon M \to N$ is a proper submersion onto its image and $\pi|\partial M$ is injective on each boundary component of M, then π is injective. In particular, if M is a smooth immersed surface with boundary in $\mathbb{R}^3$, the projection $\pi\colon M \to \mathbb{R}^2$ to the x_1x_2-plane is a proper submersion onto its image and $\pi|\partial M$ is injective, then M is a graph over $\pi(M) \subset \mathbb{R}^2 \times \{0\}$.*

Proof. If M has no boundary, then $\pi\colon M \to N$ is a connected covering space and the lemma follows since N is simply-connected.

If ∂ is a boundary component of M, then $\pi(\partial)$ is a properly embedded codimension-one submanifold of N. Since N is simply-connected, $\pi(\partial)$ separates N into two open components. We label these components of $N - \pi(\partial)$ by $C(M)$ and $C(\partial)$, where $C(M)$ is the component such that the closure of $\pi^{-1}(C(M))$ contains ∂ as boundary component. Now consider the quotient space $\hat{M}$ obtained from the disjoint union of M with all the closures of $\overline{C}(\partial_\alpha)$, ∂_α a boundary component of M, with identification map π, $\pi\colon \partial M \to \cup\overline{C}(\partial_\alpha)$. Let $\hat{\pi}\colon \hat{M} \to N$ be the natural projection that extends π on $M \subset \hat{M}$. It is straighforward to check that $\hat{\pi}$ is a connected covering space of N. Since N is simply-connected, $\hat{\pi}$ is injective which proves the lemma. $\square$

LEMMA 1.5. *If M is a complete embedded minimal surface in $\mathbb{R}^3$ with finite topology and locally bounded curvature, then M is properly embedded in $\mathbb{R}^3$.*

Proof. Suppose now that M has finite topology and lies in the upper halfspace of $\mathbb{R}^3$ with limit set the x_1x_2-plane P. If M has bounded curvature in some ε-neighborhood of P, then it was proved above that M is proper in this neighborhood and has a plane in its closure. This is impossible by the Halfspace Theorem. It remains to prove that M has bounded curvature in an ε-neighborhood of P.

Arguing by contradiction, assume that M does not have bounded curvature. In this case, there is an annular end $E \subset M$ whose Gaussian curvature is not bounded in the slab $S = \{(x_1, x_2, x_3) \mid 0 \le x_3 \le 1\}$. After a homothety of M, we may assume that ∂E is contained in the ball B_0 of radius one centered at the origin. Since M has locally bounded curvature, the part of E inside $B(0)$ has bounded curvature.

Since $E \cap S$ does not have bounded curvature, there exists a sequence $p(1), \dots, p(i), \dots$ in $E \cap S$ with $\|p(i)\| \ge i$ and $|K(p(i))| \ge i$. After possibly rotating M around the x_3-axis and choosing a subsequence, we may assume that the sequence $(5/\|p(i)\|) \cdot p(i)$ converges to the point $(5, 0)$ in the x_1x_2-plane. Let B be the ball of radius one in the x_1x_2-plane and centered at $(5, 0)$. Notice that there is no compact connected minimal surface with one boundary curve in B_0 and the other boundary curve in B (pass a catenoid between B_0 and B). Using the convex hull property of a compact minimal surface, it is easy to check that $[(5/\|p(i)\|)E] \cap B$ consists only of simply-connected components which are disjoint from the boundary of $(5/\|p(i)\|)E$. The curvature estimates $\mathcal{C}$ defined at the beginning of this section imply that, as $i \to \infty$, the curvature of $(5/\|p(i)\|)E$ at $(5/\|p(i)\|) \cdot p(i)$ converges to 0. But the Gaussian curvature at such points approaches $-\infty$ as $i \to \infty$. This contradiction proves the lemma. □

The next theorem follows immediately from the previous lemmas.

THEOREM 1.6. *Suppose $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3$. If $\mathcal{L}$ has one leaf, then this leaf is a properly embedded surface in $\mathbb{R}^3$. If $\mathcal{L}$ has more than one leaf, then $\mathcal{L}$ consists of the disjoint union of a nonempty closed set of parallel planes $\mathcal{P} \subset \mathcal{L}$ together with a collection of complete minimal surfaces of unbounded Gaussian curvature that are properly embedded in the open slabs and halfspaces of $\mathbb{R}^3 - \mathcal{P}$ and each of these open slabs and halfspaces contains at most one leaf of $\mathcal{L}$. In this case every plane, parallel to but different from the planes in $\mathcal{P}$, intersects at most one of the leaves of $\mathcal{L}$ and separates such a leaf into two components. Furthermore, in the case $\mathcal{L}$ contains more than one leaf, the leaves of $\mathcal{L}$ of finite topology are planes.*

Remark 1.7. Meeks, Perez and Ros [20] have shown that the properness conclusion of Lemma 1.5 holds if M has finite genus and locally bounded curvature. In particular, by Theorem 1.6, if $\mathcal{L}$ is a minimal laminiation of $\mathbb{R}^3$ with more than one leaf, then the leaves of $\mathcal{L}$ of finite genus are planes. Theorem 1.6 and its aforementioned generalization by Meeks, Perez and Ros plays an important role in recent advances in classical minimal surface theory ([20], [21], [22]).

2. The transversality of the homothetic blow-down $\mathcal{L}$ of M with M

The main goal of this section is to prove that M is transverse to any homothetic blow-down of M. To accomplish this we will need the following theorem which is due to Colding and Minicozzi [4].

THEOREM 2.1. *Let $\Sigma_i \subset B_{R_i} \subset \mathbb{R}^3$ be a sequence of embedded minimal disks with $\partial\Sigma_i \subset \partial B_{R_i}$ where $R_i \to \infty$. If $\sup_{B_1 \cap \Sigma_i} |A|^2 \to \infty$, then there exists a subsequence, Σ_j, and (after a rotation of $\mathbb{R}^3$) a Lipschitz curve $\mathcal{S}\colon \mathbb{R} \to \mathbb{R}^3$*

with $x_3(\mathcal{S}(t)) = t$ *so for any* $\alpha < 1$ *the sequence* $\Sigma_j \backslash \mathcal{S}$ *are multi-valued graphs and converges in the* C^α*-topology to the foliation*, $\mathcal{F} = \{x_3 = t \mid t \in \mathbb{R}\}$, *of* $\mathbb{R}^3$. *Moreover, for all* $r > 0, t$, *then* $\sup_{B_r(\mathcal{S}(t))\cap\Sigma_j} |A|^2 \to \infty$.

THEOREM 2.2. *Suppose* M *is a properly embedded nonflat simply-connected minimal surface in* $\mathbb{R}^3$ *and* $\lambda(i) \in \mathbb{R}^+$ *with* $\lambda(i) \to 0$. *Then*:

(1) *A subsequence* $M(i_j)$ *of the surfaces* $\lambda(i)M$ *converges to a foliation* $\mathcal{L}$ *of* $\mathbb{R}^3$ *by planes, which is independent of the sequence* $\lambda(i) \to 0$. *The sense of convergence will be made clear in the proof.*

(2) *The planes in* $\mathcal{L}$ *are transverse to* M. *In particular, the Gauss map of* M *misses the pair of unit vectors orthogonal to the planes in* $\mathcal{L}$.

Proof. Given a sequence $\lambda(i) \in \mathbb{R}^+$, $\lambda(i) \to 0$, define the related sequence $M(i) = \lambda(i)M$. Given any ball $B \subset \mathbb{R}^3$, every boundary component of every component of $M(i) \cap B$ bounds a disk in $M(i)$, which is contained in B by the convex ball property; hence, every component of $M(i) \cap B$ is simply-connected. Also note that in arbitrarily small neighborhoods of the origin every subsequence of the surfaces $\{M(i)\}_{i\in\mathbb{N}}$ fails to have bounded curvature. Although it is not stated in the hypothesis of Theorem 2.1 that the simply-connected surfaces Σ_i need not be connected, the conclusion of Theorem 2.1 still holds under this weaker hypothesis.

It follows that for any sequence $\lambda(i) \in \mathbb{R}^+$, $\lambda(i) \to 0$, a subsequence $M(i_j) = \lambda(i_j)M$ converges to a foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes. Furthermore, the convergence $M(i_j) \to \mathcal{L}$ is smooth except along a Lipschitz curve $S(\mathcal{L})$. The curve $S(\mathcal{L})$ passes through the origin and is contained in a double cone $C = C(\mathcal{L})$ with cone point at the origin and axis orthogonal to $\mathcal{L}$. Furthermore, the aperature of the cone C only depends on the Lipschitz constant of $S(\mathcal{L})$ which, in turn, depends only on curvature estimates $\mathcal{C}$ defined at the beginning of Section 1. Colding and Minicozzi also prove a unique extension result, Theorem 0.2 of [2] for the multigraphs $M(i_j)$ that begin around $S(\mathcal{L})$ near the origin. Specifically, they prove that these beginning flat multigraphs extend all the way to infinity as flat multigraphs with well-defined limiting normal vector and consequently: $\mathcal{L}$ *is independent of the sequence* $\lambda(i) \to 0$. This proves statement 1 in Theorem 2.2.

Assume now that $\mathcal{L}$ is the foliation of $\mathbb{R}^3$ by horizontal planes. Let E be the compact vertical cylinder centered along the x_3-axis and such that $E \cap C = E \cap \partial C = \partial E$, which consists of the circles $S_+ = \partial C \cap \{(x_1, x_2, 1)\}$ and $S_- = \partial C \cap \{(x_1, x_2, -1)\}$.

Assume that the sequence $M(i)$ converges to $\mathcal{L}$ with singular set $S(\mathcal{L})$. For each i large, $M(i) \cap E$ consists of a positive finite number of compact arcs $\alpha(i, 1), \dots, \alpha(i, n(i))$ with one end point on each boundary curve of E, a finite number of boundary arcs $\beta(i, 1), \dots, \beta(i, k(i))$ with end points in the same component of ∂E and another subset which consists of a finite set of points. For i large, the tangent lines of the associated α and β curves are almost horizontal and each β arc is a graph over its projection to the boundary component of E containing its end points. As $i \to \infty$, the associated β curves converge to the graphs of a constant function with a value of ± 1. For i large, each associated α curve is a high order multisheeted graph over its projection on S_- with winding number going to infinity as $i \to \infty$. Colding and Minicozzi [4] have proven that the number of α curves is two, which simplifies some of the arguments that follow. However, we will not use this fact and instead prove this fact at the end of the proof of Theorem 2.2.

Let $\tilde{E}(i)$ be the closure of the component of $E - \bigcup_{j=1}^{k(i)} \beta(i,j)$ that contains the α curves. After a small deformation of the top and bottom curves of $\tilde{E}(i)$ into $\tilde{E}(i)$, these new curves $\gamma(i,+)$, $\gamma(i,-)$ bound a cylinder $E(i) \subset E$ such that $E(i)$ intersects $M(i)$ only along the old α curves and $E(i)$ intersects each α curve in a connected arc. We will assume that in this construction, $\gamma(i,+)$ and $\gamma(i,-)$ converge C^1 to the original boundary curves of E as $i \to \infty$.

After a small C^1-perturbation of the horizontal circle foliation of $E(i)$, we may assume that $\gamma(i,+)$ and $\gamma(i,-)$ are leaves of the perturbed foliation and that each leaf of the induced foliation by closed curves on $E(i)$ intersects each α-type curve transversely in a single point. Fix a parametrization $\gamma(i,t)$ of the leaves of this foliation of $E(i)$, $-1 \le t \le 1$, such that $\gamma(i,t)$ is approximately a horizontal circle at height t and such that for any fixed t, $\gamma(i,t)$ converges C^1 to the circle of height t on E as $i \to \infty$.

Since each $\gamma(i,t)$ is a graph over the convex circle S_- in the plane $\{(x_1,x_2,-1)\}$, Rado's theorem [18] implies that each $\gamma(i,t)$ is the boundary of a unique minimal disk $D(i,t)$ which is a graph over the disk $\{(x_1,x_2,x_3) \mid x_1^2 + x_2^2 \le 1,\ x_3 = -1\}$. These disks $D(i,t)$ vary in a C^1-manner with the parameter t and give rise to a C^1-foliation of the solid "cylinder" $W(i) = \bigcup_{-1 \le t \le 1} D(i,t)$. Furthermore, as $\partial D(i,t)$ converges C^1 to the circle on E of height t, these foliations of $W(i)$ converge in the C^1-norm to the foliation by horizontal disks of the solid cylinder with boundary E.

By Sard's theorem, we may assume that the disks $D(i,\pm 1)$, $D(i,\pm\frac{1}{2})$ are each transverse to $M(i) \cap W(i)$. We now prove that each disk $D(i,t)$, $-\frac{1}{2} \le t \le \frac{1}{2}$, is transverse to $M(i)$ for i large. Suppose to the contrary that there exists a sequence $t(i_j)$, $-\frac{1}{2} < t(i_j) < \frac{1}{2}$, such that $D(i_j, t(i_j))$ intersects a disk component $K(i_j)$ of $M(i_j) \cap W(i_j)$ nontransversely. After replacing by a subsequence, we will assume that $i_j = i$.

Since $D(i,t(i)) \cap K(i)$ is an analytic subset of $K(i)$ with a singularity at an interior point of $K(i)$, this subset separates the disk $K(i)$ into at least four components, with at least two components on each side of $D(i,t(i))$. To see this, first note that each component of $K(i)-D(i,t(i))$ contains points on the associated α curves. The reason for this is that if $\Delta \subset [K(i)-D(i,t(i))]$ is a connected component with boundary disjoint from $E(i)$, then there exists a closed curve γ in $[K(i) \cap D(i,t(i))] - \partial D(i,t(i))$. The curve γ bounds a disk $\tilde{\Delta}$ in $K(i)$ that is contained in $W(i)$ by the convex hull property for i large. Since $D(i,1)$ and $D(i,-1)$ are disjoint from $\tilde{\Delta}$, there is a highest or lowest graph $D(i,t_0)$ such that $D(i,t_0) \cap \tilde{\Delta} \neq \emptyset$. Since $t_0 \neq t(i)$, the disk $D(i,t_0)$ intersects $\tilde{\Delta}$ with $\tilde{\Delta}$ on one side of $D(i,t_0)$, which contradicts the maximum principle for minimal surfaces. This proves that for $W(i,\frac{1}{2}) = \bigcup_{\frac{1}{2}\leq t} D(i,t)$, the set $K(i) \cap W(i,\frac{1}{2})$ consists of at least two components, $K(i,1)$, $K(i,2)$, each with associated α-type curves in their boundary.

Without loss of generality, we may assume that the disk $D(i,\frac{2}{3}) \subset W(i,\frac{1}{2})$ intersects $K(i,1)$ and $K(i,2)$ transversely. Let $\gamma(i)$ be one of the arcs in $K(i,1) \cap D(i,\frac{2}{3})$ or in $K(i,2) \cap D(i,\frac{2}{3})$. Since $\partial D(i,\frac{2}{3})$ intersects each α-type curve in $E(i)$ in a single point, the end points of $\gamma(i)$ lie on different α-type curves on $\partial W(i,\frac{1}{2})$. Since $M(i)$ separates $\mathbb{R}^3$, for i large the unit normals to $M(i)$ at the points of $\partial D(i,t) \cap M(i)$ alternate up and down as one traverses $\partial D(i,t)$. In particular, for every such $\gamma(i)$ there is a point $p(\gamma(i))$ on $\gamma(i)$ where the Gauss map is horizontal.

Since, away from the singular set $S(\mathcal{L})$, all the points of $M(i)$ have normal vectors converging to the vertical, the points $p(\gamma(i))$ on $M(i)$ converge to the point $S(\frac{2}{3}) = S(\mathcal{L}) \cap \{q \in \mathbb{R}^3 \mid x_3(q) = \frac{2}{3}\}$ as $i \to \infty$. Hence, for any $\varepsilon > 0$ the maximum absolute curvature in the ε-neighborhood of the $p(\gamma(i))$ in either $K(i,1)$ or $K(i,2)$ goes to infinity as $i \to \infty$. Since $D(i,\frac{1}{2})$ converges to the horizontal disk of height $\frac{1}{2}$, there exists an $\varepsilon > 0$ such that for i large, $d(S(\frac{2}{3}), \partial W(i,\frac{1}{2})) > \varepsilon$. A theorem of Meeks-Yau [24] implies that $\partial K(i,1)$ bounds a stable embedded minimal disk $F(i)$ in the closure of the component of $W(i,\frac{1}{2}) - [K(i,1) \cup K(i,2)]$ containing $K(i,1) \cup K(i,2)$ in its boundary. Since $K(i,1)$ and $K(i,2)$ each have $S(\frac{2}{3})$ as a limit coming from the points $p(\gamma(i))$ and $F(i)$ separates $K(i,1)$ and $K(i,2)$ in the topological ball $W(i,\frac{1}{2})$, there are points $q(i) \in F(i)$ converging to $S(\frac{2}{3})$ as $i \to \infty$. But then the curvature estimates for stable minimal surfaces in [28] imply that the Gaussian curvature of the collection of $F(i)$ is uniformly bounded near $S(\frac{2}{3})$ and so the curvature estimates $\mathcal{C}$ defined at the beginning of Section 1 imply that the curvatures of $K(i,1)$ and of $K(i,2)$ are uniformly bounded in some fixed neighborhood of $S(\frac{2}{3})$ for i large, which we observed at the beginning of this paragraph leads to a contradiction. This contradiction proves that every disk $D(i,t)$, $-\frac{1}{2} \leq t \leq \frac{1}{2}$, is transverse to $M(i)$ for i large.

Let $\tilde{W}(i) = \bigcup_{-\frac{1}{2}\leq t\leq \frac{1}{2}} D(i,t)$. Now consider the homothetic expansions $(1/\lambda(i))\tilde{W}(i)$ with the related foliations $\mathcal{F}(i)$ obtained from homothetically expanding the disks $D(i,t)$, $-\frac{1}{2} \leq t \leq \frac{1}{2}$, by the factor $1/\lambda(i)$. The foliations $\mathcal{F}(i)$ converge in the C^1-norm to the foliation of $\mathbb{R}^3$ by horizontal planes, which is $\mathcal{L}$. Since the leaves in $\mathcal{F}(i)$ are transverse to M and the Gauss map of M is an open mapping, the planes in $\mathcal{L}$ must be transverse to M. This proves statement 2 that M is transverse to $\mathcal{L}$.

For the sake of completeness, we now prove that for i large, the number $n(i)$ of α-type arcs equals two. A slight modification of the arguments used to prove the transversality of the planes in $\mathcal{L}$ with M shows that for i large $M \cap \tilde{W}(i)$ is a connected disk $K(i)$. The disk $K(i)$, for i large, has boundary consisting of $n(i)$ α-type curves. The parametrized disk foliation of $\tilde{W}(i)$ induces a natural parameter function $F\colon \tilde{W}(i) \to [-\frac{1}{2}, \frac{1}{2}]$ restricted to $K(i)$. Since the leaves $D(i,t)$ are transverse to $K(i)$, $F|K(i)$ has no interior critical points. By elementary Morse theory, the lack of interior critical points for $F|K(i)$ implies $n(i)$ is two.

This completes the proof of Theorem 2.2. $\square$

Uniqueness of the homothetic blow-down $\mathcal{L}(M)$ implies strong asymptotic convergence properties for M outside the cone C associated to $\mathcal{L}(M)$. For the remainder of this section we will assume that $\mathcal{L}(M)$ is the foliation of $\mathbb{R}^3$ by horizontal planes.

Definition 2.1. Let $\mathcal{H}$ be a solid hyperboloid of revolution with boundary asymptotic to the boundary of the cone C and let $K = \mathcal{H} \cap M$. We define $\mathcal{W}$ to be the closure of $\mathbb{R}^3 - \mathcal{H}$.

THEOREM 2.3. *After a possible homothety of M, $M \cap \mathcal{W}$ is a multisheeted graph over its projection onto the x_1x_2-plane with two simply-connected components, $M(1), M(2)$, each with one boundary component which is a proper arc. After choosing an orientation of M, for any divergent sequence of points in $M(1)$, the sequence of unit normal vectors at these points converges to* $(0,0,1)$ *and for any divergent sequence of points in $M(2)$ the unit normals converge to* $(0,0,-1)$. *In particular, the multisheeted graphs $M(1), M(2)$ have asymptotically zero gradient and sublinear growth. Note that $K = M - [\mathrm{Int(M(1))} \cup \mathrm{Int(M(2))}]$ is a strip in M.*

Proof. Fix a divergent sequence of points $\{p(i)\} \in M \cap \mathcal{W}$ and consider the associated sequence $(1/\|p(i)\|)M = M(i)$ of homothetic scalings of M. If the normal line of M at $p(i)$ does not converge to a vertical line, then the points $q(i) = p(i)/\|p(i)\|$ must have a subsequence $q(i_j)$ that converges to a point $q \in S^2 - \mathrm{Int}(C)$. Since $\mathcal{L}(M)$ is a foliation by horizontal planes, any convergent subsequence of $M(i_j)$ must have q as a singular point. But $q \notin \mathrm{Int}(C)$ which contradicts the property that the singular set of convergence is contained in $\mathrm{Int}(C) \cup \{(0,0,0)\}$. This contradiction proves that outside some compact subset of $M \cap \mathcal{W}$, the Gauss map of M is bounded away from the horizontal. Hence, after a homothety of M, we may assume that the Gauss map on $M \cap \mathcal{W}$ is bounded away from the horizontal and is asymptotic to the vertical. By the last paragraph in the proof of Theorem 2.2, $\partial(M \cap \mathcal{W})$ has two boundary curves and hence $M \cap \mathcal{W}$ consists of two components which are multisheeted graphs over their projections to the x_1x_2-plane. Since M separates $\mathbb{R}^3$, these two multisheeted graphs have opposite orientations. The theorem now follows from these observations. □

3. Finiteness of minimal graphs of bounded gradient

The main theorem of this section concerns nontrivial minimal graphs with bounded gradient and zero boundary values.

THEOREM 3.1. *Suppose that G is a minimal graph of a function f defined on a proper domain $\mathcal{D}$ of $\mathbb{R}^n$. Assume f has zero boundary values and the gradient of f is bounded. Then there are at most a finite number of components of $\mathcal{D}$ where f is nonconstant.*

It is a conjecture of Meeks that in dimension two a graph G with non-planar components described above has at most two components and only one component in the sublinear growth case.

Note that the theorem follows by proving the special case where the function f is nonnegative. We will consider positive minimal graphs u over a possibly disconnected domain $\mathcal{D}$ of $\mathbb{R}^n$, with zero boundary values. Let H^d denote graphs over $\mathcal{D}$ whose growth is at most polynomial of degree d; i.e., a graph u is in H^d if there is some constant $C > 0$ such that

$$|u(x)| \leq C(1 + r^d),$$

where $r = \|x\|$ is the Euclidean norm. The proof of Theorem 3.1 is inspired by the techniques in the paper [5].

The paper of Colding and Minicozzi [5] solves affirmatively a conjecture of Yau: On an open complete manifold with nonnegative Ricci curvature, the space of harmonic functions with polynomial growth of a fixed rate, is finite dimensional. More generally, they prove this result for open complete manifolds which have the doubling property (a volume growth condition) and satisfy a uniform Neumann-Poincaré inequality. Both of these conditions hold for $\mathbb{R}^n$ with the canonical metric.

Harmonicity is used in their proof only to obtain a reverse Poincaré inequality.

Reverse Poincaré inequality. If $\Omega > 1$, there is a constant $C = C(\Omega)$ such that if u is harmonic on M, $p \in M$ and $r > 0$, then

$$r^2 \int_{B_p(r)} |\nabla u|^2 \leq C \int_{B_p(\Omega r)} u^2.$$

A reverse Poincaré inequality also holds for solutions of uniformly quasi-elliptic operators L in divergence form on $\mathbb{R}^n$. (The minimal surface operator is not of this type.) Colding-Minicozzi use this to show the space of L-harmonic functions on $\mathbb{R}^n$, with polynomial growth of a fixed rate, is finite dimensional.

Now suppose u is a minimal graph over a domain D of $\mathbb{R}^n$ with zero boundary values. Extend u to $\mathbb{R}^n$ to be zero on the complement of D and denote this extension by u as well. Clearly the extended u is not an entire minimal graph over $\mathbb{R}^n$ but it is reasonable to believe the space of such u is finite dimensional. We remark that the minimal graph operator is

$$L(u) = \operatorname{div}(\tfrac{\nabla u}{w}), w = \sqrt{1 + |\nabla u|^2},$$

and this operator is not quite in the form considered in [5]. This leads to the gradient bound we have as hypothesis.

Definition 3.1. Let $B(r)$ denote the ball of radius r centered at the origin of $\mathbb{R}^n$, and $W^{2,1}(B(r))$ be the $(2,1)$ Sobolev space on $B(r)$. Following [5], we define $W_{k^2}(B(r)) = \{u \in \; W^{2,1}(B(r)) \mid \int_{B(r)} u^2 + r^2 \int_{B(r)} |\nabla u|^2 \leq k^2\}$.

A set $\{f_j\} \subset W^{2,1}(B(r))$ is orthonormal on $B(r)$ if

$$\int_{B(r)} f_i f_j = \delta_i^j.$$

We state a special case of Proposition 2.5 in [5].

Proposition 3.2. *Given $k > 0$, there exists an $N = N(k^2) > 0$ such that there exist at most $N - 1$ functions in $W_{k^2}(B(2R))$, that are orthonormal on $B(R)$.*

We refer to [5] for the proof; they prove the more general result on open complete manifolds satisfying the doubling property and a uniform Neumann-Poincaré inequality.

Let $P(d)$ denote the space of functions u on $\mathbb{R}^n$ such that there exists $K > 0$ so that

$$\int_{B(r)} u^2 \leq K(r^{2d+n} + 1).$$

Clearly if u is the entire extension of a minimal graph with zero boundary values, and $u \in H^d$, then $u \in P(d)$.

We quote the result:

Proposition 3.3 ([5, 4.16]). *Suppose $u_1, \ldots, u_{2l} \in P(d)$ are linearly independent. Given $\Omega > 0$ and $m_0 > 0$, there exists $m \geq m_0$, $\hat{l} \geq \frac{l}{2}\Omega^{-4d-2n}$, and functions $v_1, \ldots, v_{\hat{l}}$ in the linear span of the u_i such that*

$$\int_{B(\Omega^{m+1})} {v_j}^2 \leq 2\Omega^{4d+2n},$$

and

$$\int_{B(\Omega^m)} v_i v_j = \delta_i^j.$$

Finally, we need a version of the

Reverse Poincaré inequality. Let u be the entire extension of a minimal graph with zero boundary values. Let $C = C(u)$ be a positive upper bound for $|\nabla u|$. For $\lambda > 1$, one has

$$r^2 \int_{B(r)} |\nabla u|^2 \leq \frac{4c}{(\lambda-1)^2} \int_{B(\lambda r)} u^2,$$

where $c = \sqrt{1 + C(u)^2}$.

Proof. We know that u satisfies the equation $L(u) = \operatorname{div}\,(\frac{\nabla u}{w}) = 0$, $w = \sqrt{1 + |\nabla u|^2}$. Let ϕ be a cut-off function on $B(\lambda r)$, $\phi = 1$ on $B(r)$, $\phi = 0$ on $\partial B(\lambda r)$ and $|\nabla \phi| \leq \frac{1}{r(\lambda-1)}$. Then

$$\begin{aligned}
\operatorname{div}\left(\phi^2 u \frac{\nabla u}{w}\right) &= \phi^2 u \operatorname{div}\left(\frac{\nabla u}{w}\right) + \left\langle \nabla(\phi^2 u), \frac{\nabla u}{w} \right\rangle \\
&= 2\phi u \left\langle \nabla \phi, \frac{\nabla u}{w} \right\rangle + \phi^2 \left\langle \nabla u, \frac{\nabla u}{w} \right\rangle.
\end{aligned}$$

Since

$$\int_{B(\lambda r)} \operatorname{div}\left(\phi^2 u \frac{\nabla u}{w}\right) = \int_{\partial B(\lambda r)} \phi^2 u \left\langle \frac{\nabla u}{w}, u \right\rangle = 0,$$

we have

$$0 = -\int_{B(\lambda r)} 2\phi u, \left\langle \nabla \phi, \frac{\nabla u}{w} \right\rangle - \int_{B(\lambda r)} \phi^2 \left\langle \nabla u, \frac{\nabla u}{w} \right\rangle.$$

The Cauchy-Schwarz inequality yields

$$-\int_{B(\lambda r)} 2\phi u \left\langle \nabla \phi, \frac{\nabla u}{w} \right\rangle \leq 2 \left(\int_{B(\lambda r)} \frac{\phi^2}{w} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{B(\lambda r)} \frac{u^2}{w} |\nabla \phi|^2 \right)^{\frac{1}{2}}.$$

Thus,

$$\int_{B(\lambda r)} \phi^2 \frac{|\nabla u|^2}{w} \leq 4 \int_{B(\lambda r)} \frac{u^2}{w} |\nabla \phi|^2,$$

so

$$\int_{B(r)} \frac{|\nabla u|^2}{w} \leq 4 \int_{B(\lambda r)} u^2 |\nabla \phi|^2,$$

since $\phi = 1$ on $B(r)$ and $w \geq 1$. Then the estimate for $|\nabla \phi|$ and sup w, yields the Reverse Poincaré inequality. $\square$

Now we can prove the main result of this section, which has Theorem 3.1 as a corollary.

THEOREM 3.4. *Let C be a positive constant. The number of positive minimal graphs over disjoint domains of $\mathbb{R}^n$ with zero boundary values and gradient at most C is bounded.*

Proof. Suppose that $u_1, \ldots, u_{2l}$ are positive minimal graphs with zero boundary values, over disjoint domains of $\mathbb{R}^n$, with $|\nabla u_j| \leq C$. The entire extensions of $\{u_j\}$ to $\mathbb{R}^n$ are linearly independent. Notice that bounded gradient implies linear growth, hence each $u_i \in H^d$, with $d = 1$.

Let $\Omega > 2$. By Proposition 3.3, there exists m arbitrarily large, $\hat{l} \geq \frac{l}{2}\Omega^{-4d-2n}$ and functions $v_1, \ldots, v_{\hat{l}}$ in the linear span of the $\{u_j\}$, such that

$$\int_{B(\Omega^{m+1})} v_i^2 \leq 2\Omega^{4d+2n},$$

and

$$\int_{B(\Omega^m)} v_i v_j = \delta_i^j.$$

Apply the Reverse Poincaré-inequality to v_i, with $r = 2\Omega^m$, $\lambda = \Omega/2$, $\lambda r = \Omega^{m+1}$, to obtain

$$r^2 \int_{B(r)} |\nabla v_i|^2 \leq \frac{4c}{(\Omega/2-1)^2} \int_{B(\lambda r)} v_i^2.$$

Notice that the Reverse-Poincaré-inequality is linear in functions with disjoint supports; so it applies to the v_i. Then for each v_i one obtains

$$r^2 \int_{B(r)} |\nabla v_i|^2 + \int_{B(r)} v_i^2 \leq \left(\frac{8c}{(\Omega/2-1)^2} + 2 \right) \Omega^{4d+2n}.$$

Let k^2 denote the constant on the right side of the last inequality. This inequality implies each v_i is in $W_{k^2}(B(r))$. Apply Proposition 3.3 (with $2R = r$), to conclude there exists an $N = N(k^2)$ (independent of m), such that $\hat{l} < N$. Then $l < 2N\Omega^{4d+2n}$ and the theorem is proved. □

4. Nonexistence of asymptotic curves and consequences

In Theorem 2.3 we described a decomposition of M into three components – a smooth proper strip K and two disks $M(1), M(2)$, each having one noncompact boundary component and such that the projection π of $M(1) \cup M(2)$ to the x_1x_2-plane is a proper submersion onto its image. In this section, we will use this special decomposition of M to prove that the holomorphic function $h = x_3 + ix_3^*\colon M \to \mathbb{C}$ is a conformal diffeomorphism. This result will follow from the proof of the nonexistence of certain asymptotic curves for h.

Definition 4.1. An integral curve $\gamma\colon [0,\infty) \to M$ of ∇x_3 or $-\nabla x_3$ is an *asymptotic curve* with *asymptotic value* $\gamma(\infty) \in \mathbb{R}$ if $\lim_{t\to\infty} x_3(\gamma(t)) = \gamma(\infty)$.

PROPOSITION 4.1. *If none of the integral curves of ∇x_3 or $-\nabla x_3$ are asymptotic curves, then $h = x_3 + ix_3^*\colon M \to \mathbb{C}$ is a conformal diffeomorphism.*

Proof. Recall from Theorem 2.2 that ∇x_3 is never zero and so all integral curves of M are proper on M. Suppose that there are no asymptotic curves for ∇x_3 or $-\nabla x_3$. Let $\Gamma \subset M$ be one of the proper arcs in $x_3^{-1}(0)$. Let $F\colon M \times \mathbb{R} \to M$ be the flow of ∇x_3 and $\mathcal{D} \subset M$ be the open subdomain that is the image $F(\Gamma \times \mathbb{R})$. We first prove that $\mathcal{D} = M$. If not, let $p \in \partial(M - \mathcal{D})$ and let α be the integral curve of ∇x_3 with $\alpha(0) = p$. By hypothesis, $\alpha|[0,\infty)$ is not an asymptotic curve of ∇x_3. Without loss of generality, we may assume that $x_3(p) = -1$. Choose an embedded arc $\delta\colon [0,1] \to M$ with $\delta(0) = p$, $\delta((0,1]) \subset \mathcal{D}$ and $x_3(\delta([0,1]) = -1$. Let $F(p)\colon \delta([0,1]) \times [0,\infty) \to M$ denote the flow $F(p)$ for ∇x_3 across $\delta([0,1])$. Since α is not an asymptotic curve for ∇x_3, then for some $t_0 \in [0,\infty)$, $x_3(\alpha(t_0)) = x_3((F(p)(\delta(0),t_0))) = 0$. Since $p \notin \mathcal{D}$, $\alpha(t_0)$ is contained in a component $\widetilde{\Gamma}$ of $x_3^{-1}(0)$ different from Γ.

By continuity of F, points near p on δ flow to points near $\alpha(t_0)$ on $\widetilde{\Gamma}$ for t near t_0. But $x_3^{-1}(F(\delta(\frac{1}{n},t_n)))$ is on Γ for some t_n. Since x_3 is strictly increasing on the integral curves of ∇x_3, this contradicts that $\Gamma \cap \widetilde{\Gamma} = \emptyset$. This contradiction proves that $\mathcal{D} = M$.

Since the gradient of the function $h = x_3 + ix_3^*\colon M \to \mathbb{C}$ is never zero, h is a local diffeomorphism that is injective on each integral curve of ∇x_3. Since ∇x_3^* is also never zero, h is injective on Γ. Since $\mathcal{D} = M$, h maps Γ diffeomorphically to an open interval I on the imaginary axis of $\mathbb{C}$ and maps each integral curve of ∇x_3 diffeomorphically to a complete horizontal line in $\mathbb{C}$ passing through some point of I. Hence, M is conformally diffeomorphic to $I \times \mathbb{R}$. If I were not the entire imaginary axis, then $M(+) = x_3^{-1}([0,\infty))$ would be conformally diffeomorphic to the unit disk with a closed interval removed from its boundary. This contradicts Theorem 3.1 in [7] that states that for any properly immersed minimal surface Σ in $\mathbb{R}^3$, each component of the proper subdomain $\Sigma(+) = \{p \in \Sigma \mid x_3(p) \geq 0\}$ has full harmonic measure, which implies $M(+)$ is a closed disk with a closed subset of Lebesgue measure zero removed from its boundary. Hence, I is the entire imaginary axis and $h(M) = \mathbb{C}$, which proves the proposition. $\square$

Proposition 4.1 reduces the proof of showing $h\colon M \to \mathbb{C}$ is a conformal diffeomorphism to demonstrating that ∇x_3 and $-\nabla x_3$ have no asymptotic integral curves. We will prove that the existence of an asymptotic integral curve yields an infinitely disconnected graph G contained in $M(1)\cup M(2)$ with boundary values in a fixed horizontal plane, which is impossible by Theorem 3.1.

Lemma 4.2. *If W is a component of $x_3^{-1}((-\infty, 0])$ with an infinite number of boundary curves, then W is conformally diffeomorphic to the closed unit disk D with a closed countable set E removed from ∂D. The set E has a single limit point $*$ and $*$ is a limit of points in E on both sides of $*$ on ∂D. Also, $K \cap W$, considered to be a subset of D, is noncompact with $*$ as its unique limit point in E, where K is the strip in M defined in Definition* 2.1.

Proof. A simple application of the maximum principle for harmonic functions shows that each component of $x_3^{-1}((-\infty, 0])$ is simply-connected and every boundary component is noncompact. By Theorem 3.1 in [7], W is conformally diffeomorphic to D with a closed subset $E \subset \partial D$ of Lebesgue measure zero in ∂D removed. We will identify W with $D - E$. Since W has an infinite number of boundary curves, E is infinite and has at least one limit point. We first prove that such a limit point is also a limit point of $K \cap W$ in D.

Suppose that there was a limit point $*$ of E that was not a limit point of $K \cap W$. Then for some ε sufficiently small, the half ball $V = B(*, \varepsilon) = \{p \in D - E \mid d(p, *) \leq \varepsilon\}$ is disjoint from K and $\delta = \{p \in D - E \mid d(p, *) = \varepsilon\}$ is a compact arc.

Now consider the family of solid cylinders

$$C(t) = \left\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq t^2\right\}$$

of radius t in $\mathbb{R}^3$. Let $K(t) = K \cup [C(t) \cap M]$. Let $P = x_3^{-1}(0)$ and note that $K(t) \cap \partial W \subset K(t) \cap C(t) \cap P$ is compact. Also, note that $K(t) \cap D$ and $K \cap D$ have the same limit points, if any, in E because $K(t) - \mathrm{Int}(K)$ is compact. Choose t_0 large enough so that the compact arc δ is contained in $K(t_0)$.

Let $\mathcal{D}$ be the closure in $D - E$ of the component of $V - K(t_0)$ that contains the limit point $*$. Elementary separation arguments, applied in the simply-connected surface M, show that $\mathcal{D}$ has an infinite number of boundary arcs and exactly one of these boundary arcs contains a finite positive number of arcs and points in $\partial K(t_0)$ with the remainder of the boundary components of $\mathcal{D}$ being disjoint from $K(t_0)$. To see this consider the open domain $M - \mathcal{D}$, each component of which has one boundary arc. Since the interior of $K(t_0)$ is connected, it is contained in one of the components of $M - \mathcal{D}$ and so its closure in M intersects only one of the arcs of $\partial(M - \mathcal{D}) = \partial\mathcal{D}$, which proves our claim concerning $\partial\mathcal{D}$.

Let π denote the projection of $\partial\mathcal{D}$ to the x_1x_2-plane P. The map $\pi|\partial\mathcal{D}$ is injective on the portion of $\partial\mathcal{D}$ outside $C(t_0)$, since π is the identity function on this part of the boundary of $\mathcal{D}$. But $\pi|[\partial\mathcal{D} \cap \partial C(t_0)]$ is also injective. To see this first note that $\partial\mathcal{D} \cap \partial C(t_0)$ consists of a finite collection of pairwise disjoint compact arcs which lie below the plane P and have end points on the circle $\partial C(t_0) \cap P$. Since the projection π on each of these arcs is a submersion, each of these arcs is a graph over an arc in the image circle. Note that $\mathcal{D}$ separates the region R defined to be the intersection of the following three regions: the halfspace below the plane P, the set $\mathbb{R}^3 - \text{Int}(C(t_0))$ and $\mathcal{W}$. The reason $\mathcal{D}$ separates the region R is that the generator of $H_1(R, \mathbb{Z}) = \mathbb{Z}$ can be chosen to be a circle γ on $C(t_0) \cap R$ disjoint from $\mathcal{D} \cap C(t_0)$ which is compact; such a γ must have odd intersection number with any properly embedded surface Σ in R, $\partial\Sigma \subset \partial R$, that fails to separate R. Since $\mathcal{D}$ separates R, observe that for an arc $\alpha \subset \partial\mathcal{D} \cap \partial C(t_0)$, there cannot be another such arc β of $\partial\mathcal{D} \cap \partial C(t_0)$ which lies immediately above or below α. Otherwise, since $\mathcal{D}$ separates R, the normal vectors of $\mathcal{D}$ along α and the normal vectors to $\mathcal{D}$ along β would lie in different hemispheres but the projection π orients $\mathcal{D}$, so this is impossible. Hence, $\pi|[\partial\mathcal{D} \cap \partial C(t_0)]$ is injective from which it follows that $\pi|\partial\mathcal{D}$ is injective.

Let W' denote the closed subdomain of M with boundary being the proper arc σ in $\partial\mathcal{D}$ that contains arcs of $\partial K(t_0)$ and such that W' is disjoint from K. Since $\pi|W'$ is a proper submersion and $\pi|\partial W'$ is injective, W' is a graph over its projection onto the x_1x_2-plane. However, $\mathcal{D} \subset W'$ and $W' - [\text{Int}(\mathcal{D}) \cup \sigma]$ consists of an infinite number of subgraphs of W' whose union is a graph over a proper infinitely disconnected subdomain of P and the boundary of this graph lies in P. The existence of this graph contradicts Theorem 3.1. This contradiction proves that every limit point of $E \subset \partial D$ is a limit point of $K \cap W$ in E.

Since $K \cap \partial W$ is compact and $K \cap W$ has at most one end, $K \cap W \subset D$ has exactly one limit point in E, which must be a limit point of E. Hence, E has exactly one limit point $*$, which implies E is countable. It remains to prove that $*$ is a limit of points in E on both sides of $*$ on the circle ∂D.

Suppose $*$ is only a single sided limit in $E \subset \partial D$ and we will derive a contradiction. Recall the enlargements $K(t)$ of K and note that $*$ is their unique limit point in D as well. Let $M(1,t)$ and $M(2,t)$ denote the components of $[W - \text{Int}(K(t))] \subset D$ that have the limit point $*$. Elementary separation properties (as applied in the fourth paragraph of this proof) imply that $M(1,t)$ and $M(2,t)$ each have exactly one boundary component that contains points of $\partial K(t)$. The assumption that $*$ is a single sided limit implies that $M(1,t)$ or $M(2,t)$ has a finite number of boundary components but not both. Suppose $M(1,t)$ has a finite number of boundary components and $M(2,t)$ has an infinite number of boundary components. By choosing t_0 sufficiently large (large enough so that the component of $K(t_0) \cap W$ containing the end of $K(t_0)$ intersects the boundary arc in $M(1,t)$ with end point $*$), $\partial M(1,t_0)$ is connected. Furthermore, one end of $\partial M(1,t_0)$ is on $\partial K(t_0)$ and $x_3 \to -\infty$ on this end; the other end is in the plane P. Both ends of $\partial M(1,t_0)$, when considered to lie in D, have $*$ as their limit point on ∂D.

Let $\mathcal{H}(t_0) = \mathcal{H} \cup C(t_0)$ be the simply-connected solid region of $\mathbb{R}^3$ from which we obtained $K(t_0)$ as $\mathcal{H}(t_0) \cap M$. Let R be the closure of the complement of $\mathcal{H}(t_0)$ in the lower halfspace $\{(x_1, x_2, x_3) \mid x_3 \leq 0\}$. Although $M(2,t_0)$ does not separate R, it does separate $R - M(1,t_0)$. We check this by showing that $M(1,t_0) \cup M(2,t_0)$ separates R into two components. If not, there exists a closed curve α in R that intersects $M(2,t_0)$ transversely in a single point and is disjoint from $M(1,t_0)$.

Note that ∂R is an annulus that contains the generator of the fundamental group of R. $M(1,t_0)$ and $M(2,t_0)$ each have exactly one boundary component that fails to separate ∂R, namely, the boundary components that contain a proper noncompact arc of $\partial K(t_0)$. The curve α is homologous in R to some nonzero integer multiple of the generating circle C in ∂R. But the intersection number of C with $\partial M(1,t_0)$ and with $\partial M(2,t_0)$ is ± 1. Since the intersection number is well-defined on homology, α must intersect $M(1,t_0)$, a contradiction. Hence, $M(2,t_0)$ separates $R - M(1,t_0)$.

Recall that $K(t_0)$ lies in one component of the complement of the interior of $M(2,t_0)$ and each boundary component of $M(2,t_0)$ separates M into two components. Let $\{\beta(i) \mid i \in \mathbb{N}\}$ be an enumeration of the boundary components of $\partial M(2,t_0)$ that are different from the boundary arc β containing portions of $\partial K(t_0)$ and note that $\beta(i) \subset (P - C(t_0))$. Given a $\beta(i)$, let $F(i)$ be the closure of the component of $P - \beta(i)$ that is disjoint from $C(t_0)$. Let $\hat{F}(i)$ be the region in M, disjoint from $K(t_0)$, with $\partial \hat{F}(i) = \beta(i)$. Since $\pi|\hat{F}(i)$ is a proper submersion onto its image and $\pi|\partial\hat{F}(i)$ is injective, $\hat{F}(i)$ is a graph over a domain in P disjoint from $C(t_0)$, and hence, $\hat{F}(i)$ is a graph over $F(i)$.

We now check that $F(i) \cap F(j) = \emptyset$ for $i \neq j$. If not, we may assume that for some $i \neq j$, $F(i) \subsetneqq F(j)$ and $F(i)$ is a maximal such domain contained in $F(j)$. Since $\hat{F}(n)$ is a graph over $F(n)$ for all n, points of $M(2,t_0)$ close to $\beta(i)$ lie below the strip $T = F(j) - \mathrm{Int}(F(i))$ and points of $M(2,t_0)$ close to $\beta(j)$ lie below $P - \mathrm{Int}(F(j))$. Since $C(t_0)$ contains the end points of every component in $M(1,t_0) \cap P$ and the disk $F(j)$ is disjoint from $C(t_0)$, the strip T is disjoint from $M(1,t_0)$ (since $T \subset F(j)$). Since $M(2,t_0)$ is a multisheeted graph, the closure of the component $X \subset R - [M(1,t_0) \cup M(2,t_0)]$ that lies above $M(2,t_0)$ near $\beta(i)$ is the same component that lies above $M(2,t_0)$ near $\beta(j)$. But then a small arc in P intersecting $\beta(j)$ transversely in a single point has its end points in ∂X but intersects $M(2,t_0)$ transversely in a single point, which means that $M(2,t_0)$ does not separate $R - M(1,t_0)$ as previously shown. This contradiction proves $F(i) \cap F(j) = \emptyset$ for $i \neq j$. Hence, $\bigcup_{i=1}^{\infty} \hat{F}(i)$ is a graph over $\bigcup_{i=1}^{\infty} F(i)$. The existence of this graph contradicts Theorem 3.1 and proves the lemma. $\square$

Definition 4.2. Suppose that $\gamma\colon [0,\infty) \to M$ is an asymptotic integral curve for ∇x_3. Let $W(\gamma)$ be the component of $x_3^{-1}((-\infty, \gamma(\infty)])$ that contains γ. If γ is an asymptotic curve for $-\nabla x_3$, then let $W(\gamma)$ denote the component of $x_3^{-1}([\gamma(\infty), \infty))$ containing γ.

Recall, by Lemma 4.2, that $W(\gamma)$ is conformally $D - E(\gamma)$, where $E(\gamma)$ is a finite set or $E(\gamma)$ is countable with exactly one (two-sided) limit point. As before, we will identify $W(\gamma)$ with $D-E(\gamma)$; note that the set $E(\gamma)$ corresponds to the ends of $W(\gamma)$ under this identification.

Definition 4.3. Since an asymptotic curve γ of ∇x_3 or $-\nabla x_3$ is proper in $D - E(\gamma)$ and $E(\gamma)$ corresponds to the ends of $W(\gamma)$, γ has a unique limit point $L(\gamma) \in E(\gamma)$. We say that γ is a *simple* asymptotic curve if $L(\gamma)$ is an isolated point of $E(\gamma)$.

LEMMA 4.3. *None of the asymptotic integral curves of* ∇x_3 *or* $-\nabla x_3$ *are simple.*

Proof. Suppose to the contrary that there exists a simple asymptotic integral curve γ for ∇x_3 and we will derive a contradiction. In order to be a simple asymptotic curve, there must exist a compact embedded arc δ in $W(\gamma)$ with one end point in each of the local pair of arcs $\alpha(1), \alpha(2)$ in $\partial D - E(\gamma)$, in a small neighborhood of $L(\gamma)$ in D, which have an end point equal to $L(\gamma)$. The curve δ separates $W(\gamma)$ into two closed components; let $D(\delta)$ denote the component that has the limit point $L(\gamma)$.

The disk $D(\delta)$ is conformally the closed unit disk with a single point removed from its boundary. A straightforward application of Schwartz reflection and Riemann's removable singularities theorem shows that $h = x_3 + ix_3^*\colon D(\delta) \to \mathbb{C}$ extends smoothly to the puncture $L(\gamma) \in \partial D(\delta)$ with value $\lim_{t\to\infty} h \circ \gamma(t) = * \in \{z \in \mathbb{C} \mid \mathbb{R}e(z) = \gamma(\infty)\}$. Note that along an integral curve of ∇x_3 or $-\nabla x_3$ the conjugate function x_3^* remains constant. It follows that the image of the two arcs $\alpha(1), \alpha(2)$, by h, are contained in $\{z \in \mathbb{C} \mid \mathbb{R}e(z) = \gamma(\infty)\}$ and these arcs have limiting value $*$. In particular, $W(\gamma)$ has two distinct boundary components, $\beta_-(0), \beta_+(0)$, containing $\alpha(1), \alpha(2)$, respectively. Here, we may assume that $h(\beta_-(0)) \subset \{\mathbb{R}e(z) = \gamma(\infty), \operatorname{Im}(z) \le \operatorname{Im}(*)\}$ and $h(\beta_+(0)) \subset \{\mathbb{R}e(z) = \gamma(\infty), \operatorname{Im}(z) \ge \operatorname{Im}(*)\}$. Assume that $\beta_\pm(0)\colon \mathbb{R} \to W(\gamma) \subset D$ are parametrizations so that $\lim_{t\to\infty} \beta_\pm(0)(t) = L(\gamma)$.

Let $W(\gamma, 1)$ be the component of $x_3^{-1}([\gamma(\infty), \infty))$ that contains $\beta_+(0)$ in its boundary and let $W(\gamma, -1)$ be the component with $\beta_-(0)$ in its boundary. By Lemma 4.2, $\lim_{t\to\infty} \beta_\pm(0)(t)$ corresponds to an isolated end of $W(\gamma, \pm 1)$. Let $\beta_+(1)$ be the "other" boundary curve in $W(\gamma, 1)$ next to $\beta_+(0)$ in the sense that $\lim_{t\to\infty} \beta_+(0)(t) = \lim_{t\to\infty} \beta_+(1)(t)$.

The curve $\beta_+(1)$ is distinct from $\beta_+(0)$ and, after correctly parametrizing $\beta_+(1)$, $\lim_{t\to\infty} h \circ \beta_+(1)(t) = *$. In a similar manner we define the curve $\beta_-(1) \subset W(\gamma, -1)$. We next consider the two components $W(\gamma, \pm 2)$ in $x_3^{-1}((-\infty, \gamma(\infty)])$ such that $\beta_\pm(1) \subset \partial W(\gamma, \pm 2)$. Another application of Lemma 4.2 gives rise to an associated pair of boundary components, $\beta_+(2) \subset W(\gamma, 2)$ and $\beta_-(2) \subset W(\gamma, -2)$. Continuing in this manner yields an infinite sequence of domains $\{W(\gamma, i) \mid i \in \mathbb{Z}\}$ in M, each of which has a distinguished pair of oriented boundary components, $\beta_+(i), \beta_+(i-1)$, for $i > 0$, and, for $i < 0$, $\beta_-(-i), \beta_-(-i-1)$ and where $W(\gamma, 0) = W(\gamma)$. A straightforward separation argument proves that all of these domains and arcs are distinct.

Since K intersects only a finite number of the curves $\{\beta_\pm(i) \mid i \in \mathbb{N}\}$ and these curves are distinct, there is an $n \in \mathbb{N}$ such that $\beta_+(n)$ is disjoint from K. After possibly reversing the signed index, we may assume that K lies in the same component of $M - \beta_+(n)$ that contains $\beta_+(n-1)$. By elementary separation properties, all of the domains $W(\gamma, i)$, $i \geq n$, lie in the closure of the component of $M - \beta_+(n)$ disjoint from K.

Let G be the subdomain of M with boundary $\beta_+(n)$ that is disjoint from K. By Lemma 1.4, G is a graph with boundary values in the plane $P(\gamma) = \{(x_1, x_2, x_3) \mid x_3 = \gamma(\infty)\}$. The collection $\{W(\gamma, i) \mid i \geq n\}$ represents an infinite number of subgraphs whose boundary is in $P(\gamma)$. The existence of these graphs contradicts Theorem 3.1 and proves the lemma. $\square$

THEOREM 4.4. *There are no asymptotic curves for ∇x_3 or $-\nabla x_3$. In particular, by Propostion 4.1, $h = x_3 + ix_3^*\colon M \to \mathbb{C}$ is a conformal diffeomorphism.*

Proof. Suppose γ is an asymptotic integral curve of ∇x_3. By Lemma 4.3, γ is not simple. In particular, $L(\gamma) \in E(\gamma) \subset \partial D$ is a limit point of $E(\gamma)$ and so $W(\gamma)$ contains an infinite number of boundary components. Lemma 4.2 implies that $L(\gamma)$ is the unique limit point of $E(\gamma)$, $L(\gamma)$ is a two-sided limit point and $L(\gamma)$ is the unique limit point of $W(\gamma) \cap K$ in D. Since K intersects only a finite number of the boundary curves in $W(\gamma)$, there exists an embedding of the compact interval $\sigma\colon I = [-1, 1] \to \partial D$ with end points $\sigma(\pm 1) \in \partial D - E(\gamma)$, $\sigma(0) = L(\gamma)$ and $\sigma(I) \cap K = \emptyset$. Furthermore, we may assume that $\sigma^{-1}(E(\gamma)) = \{0, \pm\frac{1}{n} \mid n \in \mathbb{N} \text{ and } n \geq 2\}$.

Consider a point $p \in \sigma(I) \cap (E(\gamma) - L(\gamma))$. There exists an $\varepsilon(p) > 0$ such that, for the closed metric ball $\widetilde{B}(p, \varepsilon(p)) \subset D$, $B(p, \varepsilon(p)) = \widetilde{B}(p, \varepsilon(p)) - \{p\}$ is disjoint from K and $B(p, \varepsilon(p))$ is conformally equivalent to the unit disk with one point removed from the boundary. Here the boundary of $B(p, \varepsilon(p))$ consists of two arcs on $\sigma(I) - E(\gamma)$ that limit to p and a compact arc in D corresponding to the points in D of distance $\varepsilon(p)$ from p. A slight modification of the proof of Lemma 4.3 shows that $h = x_3 + ix_3^*\colon B(p, \varepsilon(p)) \to \mathbb{C} \cup \{\infty\}$ has a well-defined limiting value $p(*)$ at p contained in the circle $\{z \in \mathbb{C} \cup \{\infty\} \mid z = \infty \text{ or } \mathbb{R}e(z) = \gamma(\infty)\}$. If $p(*) \neq \infty$, then, there is a simple asymptotic curve in M which is contained in the subset $W(\gamma)$ with limit point $p \in E(\gamma)$, which contradicts Lemma 4.3. Hence, we may assume that $p(*) = \infty$.

Since $h\colon B(p,\varepsilon(p)) \to \mathbb{C}\cup\{\infty\}$ has a well-defined limit at p which is ∞, an application of the Schwartz reflection principle, as in the proof of Lemma 4.3, proves the existence of some closed neighborhood $N(p)$ of p in $B(p,\varepsilon(p))$ that goes diffeomorphically under h to $\{z \in \mathbb{C} \mid \mathbb{R}e(z) \leq \gamma(\infty) \text{ and } |z| \geq R_0\}$ for some large positive R_0.

Consider the inverse image $\beta(p)$ of the proper arc $(-\infty, -R_0] \subset \mathbb{R} \subset \mathbb{C}$ under $(h|N(p))^{-1}$. Then $\beta(p)$ has limit point p and the x_3-coordinate of $\beta(p)$ is less than or equal to $-R_0$. Suppose we choose p to be the point $\sigma(\frac{1}{2}) \in E(\gamma)$ or the point $\sigma(-\frac{1}{2}) \in E(\gamma)$. These choices yield proper embedded arcs $\beta(\frac{1}{2})$ and $\beta(-\frac{1}{2})$, which we may assume are disjoint. Now join the end points of $\beta(-\frac{1}{2})$ and $\beta(\frac{1}{2})$ in $W(\gamma)$ by an embedded compact arc so that the union of the three arcs is a properly embedded curve β in $\mathrm{Int}(W(\gamma))$ with limit points $\sigma(\pm\frac{1}{2})$ in $E(\gamma)$.

The arc β bounds a closed subdomain $W(\gamma,\beta)$ of $W(\gamma)$ that contains the end of γ limiting to $L(\gamma)$. Each boundary component δ in $\partial W(\gamma,\beta) - \beta$ is contained in $\sigma(I)$ and so is disjoint from K. For each such δ let $\Lambda(\delta)$ denote the closed subdomain of M bounded by δ that is disjoint from K. Define $W(\beta)$ to be the union of $W(\gamma,\beta)$ with all of the components $\Lambda(\delta)$ where δ is a boundary component of $\partial W(\gamma,\beta) - \beta$. Since $K \cap \Lambda(\delta) = \emptyset$ for each such δ, $W(\beta) \cap K = W(\gamma,\beta) \cap K$ and so it follows that $\sup(x_3|(W(\beta)\cap K)) = \sup(x_3|(W(\gamma,\beta)\cap K)) \leq \gamma(\infty)$.

Since β is the union of three arcs, where on each such arc x_3 is bounded from above by $\gamma(\infty)$ by some fixed amount, there exists an $\varepsilon > 0$ such that $-\infty < x_3(\beta) < \gamma(\infty) - 2\varepsilon$. Now consider the component $F(\gamma)$ of $x_3^{-1}([\gamma(\infty)-\varepsilon,\infty))$ that contains the end of γ with limiting x_3-value $\gamma(\infty)$. If $F(\gamma)$ had a finite number of boundary components, then $F(\gamma)$ is conformally the disk D with a finite set $E \subset \partial D$ removed. In this case our previous arguments show that $h = x_3 + ix_3^*\colon F(\gamma) \to \{z \in \mathbb{C} \mid \mathbb{R}e(z) \geq \gamma(\infty)-\varepsilon\} \subset \mathbb{C}\cup\{\infty\}$ extends smoothly across the finite set $E \subset \partial D$ to the constant function $\gamma(\infty) - \varepsilon$. But since the end of γ lies in $F(\gamma)$ and γ takes on an asymptotic value that is not the value of the extended map on ∂D, the extended map could not be continuous. Thus, $F(\gamma)$ must have an infinite number of boundary components.

Since $F(\gamma)$ intersects $W(\beta)$ (they both contain the end of γ with limiting x_3-value $\gamma(\infty)$) and β is disjoint from $F(\gamma)$, then $F(\gamma) \subset W(\beta)$. In particular, $x_3|(F(\gamma)\cap K)$ is bounded from above by $\gamma(\infty)$ but, by definition of $F(\gamma)$, $x_3|(F(\gamma)\cap K)$ is bounded from below by $\gamma(\infty)-\varepsilon$. Therefore, $F(\gamma)\cap K$ is compact. Since $F(\gamma)$ contains an infinite number of boundary components and $F(\gamma)\cap K$ is compact, Lemma 4.2 now gives the desired contradiction. This completes the proof that ∇x_3 and $-\nabla x_3$ have no asymptotic integral curves. The second statement in the theorem now follows from Proposition 4.1. $\square$

5. The Gauss map and the uniqueness of the helicoid

Throughout this section, unless otherwise stated, M will denote a properly embedded simply-connected minimal surface in $\mathbb{R}^3$ with $\mathcal{L}(M)$ being the foliation of $\mathbb{R}^3$ by horizontal planes.

Every conformal minimal immersion $f\colon \Sigma \to \mathbb{R}^3$ has an analytic representation in terms of a holomorphic 1-form dh, where $h = x_3 + ix_3^*$, and the stereographic projection $g\colon \Sigma \to \mathbb{C} \cup \{\infty\}$ of its Gauss map $G\colon \Sigma \to S^2$. Namely, assuming $f(p_0) = (0,0,0)$, then f can be recovered from dh and g by integration:

$$f(p) = \mathbb{R}e \int_{p_0}^{p} \left(\frac{1}{2}\left(\frac{1}{g} - g\right) dh, \frac{1}{2}\left(\frac{1}{g} + g\right) i\, dh,\, dh \right).$$

The above representation is called the Weierstrass representation of $f\colon \Sigma \to \mathbb{R}^3$. If the surface in question is a vertical helicoid, then we can take $\Sigma = \mathbb{C}$, $g(z) = e^{\alpha z}$ for some $\alpha \in i\mathbb{R}$, and $dh = dz$.

By Theorem 4.4 and elementary covering space theory, we know that, after a possible rotation, a properly embedded simply-connected minimal surface M in $\mathbb{R}^3$ can be parametrized by $\mathbb{C}$ with $x_3 = \mathbb{R}e(z)$ and $g(z) = e^{H(z)}$ for some entire function $H(z)$ on $\mathbb{C}$. It remains to prove that $H(z) = \alpha z$ for some $\alpha \in i\mathbb{R} - \{0\}$.

If $H(z) = a_n z^n + \cdots + a_0$ is a nonconstant polynomial, then $g(z) = e^{H(z)}$ has an essential singularity at infinity. In this case there exists a sequence of points $p(i)$ in $\mathbb{C}$ with $|p(i)| \to \infty$ and $|g(z)| \to 1$. Since the Gaussian curvature can be expressed in $\mathbb{C}$ coordinates by

$$K = -16 \left(\frac{|g||g'|}{(1+|g|^2)^2} \right)^2 = -16|H'|^2 \left(\frac{|g|^2}{(1+|g|^2)^2} \right)^2,$$

then $K(p(i)) \to -\infty$ if $H(z)$ is not a polynomial of degree one. If $H(z) = az+b$, then after a conformal affine change of coordinates w, $H(w) = iw$ and $dw = cdz$ for some $c \in \mathbb{C}$, which means that M is an associate surface to a vertical helicoid. But such a surface is known to be embedded only if it is actually a vertical helicoid (see, for example, [25]). This gives a proof of the following lemma.

LEMMA 5.1. *A complete minimal surface M defined as $f\colon \mathbb{C} \to \mathbb{R}^3$ with $x_3 = \mathbb{R}e(z)$ and $g(z) = e^{H(z)}$, where $H(z)$ is a nonconstant polynomial, has bounded nonzero Gaussian curvature if and only if $H(z)$ is linear. In particular, by the above discussion, if M is embedded, has bounded curvature and $H(z)$ is a polynomial, then M is a vertical helicoid.*

PROPOSITION 5.2. *$H(z)$ is a linear function.*

We will prove $H(z)$ is linear through a series of lemmas.

Lemma 5.3. *If, for some latitude γ in S^2, the inverse image $g^{-1}(\gamma)$ is a connected embedded arc, then $H(z)$ is linear.*

Proof. Recall that $e^z\colon \mathbb{C} \to S^2 - \{0, \infty\}$ is a covering map and $g(z) = e^{H(z)}$. Suppose $g^{-1}(\gamma)$ is a connected embedded arc $\alpha\colon \mathbb{R} \to M$. It follows that g has no branch points on $g^{-1}(\gamma)$, so H has no branch points on $g^{-1}(\gamma)$. Then $H \circ \alpha\colon \mathbb{R} \to \mathbb{C}$ parametrizes an arc $\tilde{\alpha}$ that is a vertical line in $\mathbb{C}$. For any point $p \in \tilde{\alpha}$, $H^{-1}(p)$ consists of one point in M. Hence, H is a holomorphic function defined on $M = \mathbb{C}$ of degree one, which implies H is linear. $\square$

Lemma 5.4. *Suppose $H(z)$ is not a polynomial. Then, for some latiude $\gamma \subset S^2$, $G^{-1}(\gamma)$ consists of an infinite number of proper arcs $\Gamma = \{\alpha(k) \mid k \in \mathbb{N}\}$ contained in the solid hyperboloid $\mathcal{H}$ defined in Section 2. For every $n \in \mathbb{N}$, there exists a large $T(n) \in \mathbb{R}^+$ such that for all $t \geq T(n)$ or for all $t \leq -T(n)$, the horizontal plane $P(t)$ at height t intersects at least n arcs in Γ.*

Proof. By Sard's theorem, almost every value of $x_3 \circ G\colon M \to [-1, 1]$ is a regular value. By our choice of $\mathcal{H}$, there exists an $\varepsilon > 0$ such that $(x_3 \circ G)^{-1}((-\varepsilon, \varepsilon)) \subset \mathcal{H}$. Let $r \in (-\varepsilon, \varepsilon)$ be a regular value of $x_3 \circ G$ and γ be a latitude at height r. Since $H(z)$ is not a polynomial, $G^{-1}(\gamma)$ consists of an infinite number of arcs $\Gamma = \{\alpha(k) \mid k \in \mathbb{N}\}$. Since $\mathcal{H}$ has only two ends, one of the ends of $\mathcal{H}$ must contain the ends of an infinite number of ends of arcs in Γ. Suppose that the top end of $\mathcal{H}$ contains this infinite set of ends of Γ. Since every horizontal slab intersects $\mathcal{H}$ in a compact set, for every integer n, there exists a large positive $T(n)$ such that $G^{-1}(\gamma) \cap x_3^{-1}([T(n), \infty))$ contains noncompact arcs with boundary at height $T(n)$ representing ends of at least n components in Γ. Since x_3 restricted to each of these components is proper, every horizontal plane $P(t)$, $t \geq T(n)$, must intersect each of these components, which completes the proof of the lemma. $\square$

Now consider a sequence of homothetic scalings $\lambda(i)M$, $\lambda(i) \to 0$, such that $\lambda(i)M$ converges to a foliation of $\mathbb{R}^3$ by horizontal planes with singular curve S that intersects each of these planes in one point. Let $p \in S$ be the singular point of height 4 and let B be the ball of radius 1 centered at p. Denote by $d\colon B \to [0, 1]$ the distance function to ∂B. Let K denote the Gaussian curvature function on $\lambda(i)M$. For each i, choose a point $p(i) \in \lambda(i)M$ in B where the function

$$k_i\colon \lambda(i)M \cap B \to [0, \infty), \quad k_i(x) = \sqrt{|K(x)|} \cdot d(x),$$

has its maximum value. Suppose $K(i)$ is the absolute value of the Gaussian curvature at $p(i)$. Let $M(i) = \sqrt{K(i)}[\lambda(i)M - p(i)]$, where $-p(i)$ refers to translation by $-p(i)$. It is straightforward to prove that the Gaussian curvature of the collection $\{M(i)\}$ is uniformly bounded on compact subsets of $\mathbb{R}^3$. Also, a subsequence $M(i_j)$ of the $M(i)$ converges smoothly to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$.

LEMMA 5.5. *$\mathcal{L}$ is a properly embedded simply-connected minimal surface $\hat{M}$ with bounded curvature whose Gauss map misses the values $\{0, \infty\}$ and whose Gaussian curvature is -1 at the origin. Furthermore, the convergence of $M(i_j)$ to $\hat{M}$ is of multiplicity one and $\mathcal{L}(\hat{M}) = \mathcal{L}(M)$. (Recall that $\mathcal{L}(\Sigma)$ is the unique homothetic blow-down of a properly embedded simply-connected minimal surface Σ.)*

Proof. Note that the leaf of $\mathcal{L}$ passing through the origin has Gaussian curvature -1 at the origin. Since the curvature of an $M(i)$ defined above is in fact actually bounded from below by -2 in any fixed compact subset for i sufficiently large, the leaves of $\mathcal{L}$ have bounded Gaussian curvature. Theorem 1.6 implies $\mathcal{L}$ is a properly embedded connected minimal surface $\hat{M}$.

Since the convergence of the $M(i_j)$ to $\hat{M}$ is smooth and $\hat{M}$ is proper, the $M(i_j)$ converge to $\hat{M}$ with multiplicity one, otherwise $\hat{M}$ would be stable and flat. Since this multiplicity one convergence property will be important in the proof that M is a helicoid, we give the argument here. Let Σ be a smooth compact subdomain of $\hat{M}$ (which we can suppose is simply-connected by passing to a covering space) and let $N(\Sigma)$ be a neighborhood of Σ obtained by exponentiating an ε-neighborhood of the zero section of the normal bundle to Σ. Since $\hat{M}$ is proper and the surfaces $M(i_j)$ converge smoothly to $\hat{M}$, for i_j large, $N(\Sigma)$ intersects $M(i_j)$ in a finite number of normal graphs over Σ. By taking the difference of two distinct such graphs, and normalizing so that the difference has norm one at some fixed interior point of Σ, one can pass to a convergent subsequence that converges to a positive solution of the linearized equation, which yields a positive Jacobi function on Σ. Therefore Σ, and hence $\hat{M}$ is stable. By [9] and [11], $\hat{M}$ is a flat plane, which contradicts the fact that $\hat{M}$ has curvature -1 at the origin.

We now observe that $\hat{M}$ is simply-connected. Let γ be a simple closed curve in $\hat{M}$. Since $M(i_j)$ converges smoothly on compact subsets of $\mathbb{R}^3$ with multiplicity one to $\hat{M}$, for i_j large, we can lift γ to a unique closest closed curve $\gamma(j) \subset M(i_j)$. The curve $\gamma(j)$ bounds a disk $D(j)$ on $M(i_j)$ and the disks $D(j)$ converge (they have uniformly bounded curvature and by the isoperimetric inequality have uniformly bounded area) to a disk in $\hat{M}$ bounding γ. This proves $\hat{M}$ is simply-connected.

The fact that the Gauss map of $\hat{M}$ misses $\{0, \infty\}$ follows from the observation that the minimal surfaces $M(i_j)$ converging to $\hat{M}$ do not have vertical normals and the Gauss map of $\hat{M}$ is an open map. Consider $\mathcal{L}(\hat{M})$; since $\hat{M}$ is conformally $\mathbb{C}$ by Theorem 4.4, Picard's theorem implies its Gauss map cannot miss four points in S^2. Hence, Theorem 2.2 implies $\mathcal{L}(\hat{M})$ is the foliation of $\mathbb{R}^3$ by horizontal planes. $\square$

We now return to the proof of Proposition 5.2.

Proof. By Lemma 5.3, it suffices to prove that there exists a latitude $\gamma \subset S^2$ such that $G^{-1}(\gamma) \subset M$ is a connected embedded arc. We first will prove that $H(z)$ is a polynomial.

Reasoning by contradiction, suppose that $H(z)$ is not a polynomial. By Lemma 5.4, we can assume that for any integer n there is a $T(n)$ such that for every t greater than or equal to $T(n)$, the horizontal plane $P(t)$ at height t intersects at least n components of $G^{-1}(\gamma)$.

Consider a sequence of homothetic scalings $\lambda(i)M$, $\lambda(i) \to 0$, such that $\lambda(i)M$ converges to the foliation of $\mathbb{R}^3$ by horizontal planes. Let $p \in S(\mathcal{L})$ be the point at height 4 and let B denote the closed ball of radius one centered at p. For appropriately chosen points $p(i) \in \lambda(i)M$ and after passing to a subsequence, the previous arguments imply that the homothetic expansions $M(i) = \sqrt{K(i)}[\lambda(i)M - p(i)]$ converge smoothly (with multiplicity one) to a simply-connected minimal $\hat{M}$ of nonzero bounded curvature.

Since the point $p(i) \in \lambda(i)M$ lies in B, whose center is at height 4 and whose radius is one, it follows that $x_3(p(i)) \geq 3$. From this fact and using that $\lambda(i) \to 0$, we deduce that for all $n \in \mathbb{N}$, $p(i)$ lies above height $\lambda(i)T(n)$ for i large, and therefore at least n components of the preimage of γ through the Gauss map of $\lambda(i)M$ will intersect the horizontal plane containing $p(i)$. After a translation and homothetic expansion, this intersection property implies that the horizontal plane P at height zero must intersect at least n components of $G_i^{-1}(\gamma)$, where G_i stands for the Gauss map of $M(i)$ (for i large).

Since almost every $r \in [-1, 1]$ is a regular value, we may assume r is chosen sufficiently small, a regular value of both $x_3 \circ G$ and $x_3 \circ \hat{G}$, so that the latitude γ at height r satisfies $G^{-1}(\gamma) \subset \hat{\mathcal{H}}$ where $\hat{\mathcal{H}}$ is the hyperboloid associated to $\hat{M}$. Since $\hat{\Gamma} = \hat{G}^{-1}(\gamma)$ contains a finite number k of arcs which enter the slab ${x_3}^{-1}([-2, 2])$, for i large, there are at most k arcs in $\Gamma_i = {G_i}^{-1}(\gamma)$ that enter the region $\hat{\mathcal{H}} \cap {x_3}^{-1}([-1, 1])$. But by the previous paragraph, for i large, there is at least one other arc $\alpha(i) \in \Gamma_i$ that intersects the plane P at a point $q(i)$ outside $\hat{\mathcal{H}}$; $q(i)$ will be a divergent sequence in $P \cap M(i)$.

Now consider a new sequence of surfaces: $N(i) = (1/\|q(i)\|)M(i)$. By the theorems in Section 2 and Lemma 5.5, a subsequence of the $N(i)$ converges to the foliation $\mathcal{L}(\hat{M}) = \mathcal{L}(M)$ of $\mathbb{R}^3$ by horizontal planes with associated singular curve $S(\mathcal{L}(\hat{M}))$ passing through the origin and contained in a solid vertical double cone C with vertex at the origin. Since $\mathcal{L}(\hat{M})$ is the foliation by horizontal planes and the tangent plane of $N(i)$ at $q(i)/\|q(i)\|$ is bounded away from the horizontal, then any limit point of the sequence of points $q(i)/\|q(i)\|$ must be contained in C. Since this sequence of points of $N(i)$ lies on the unit circle in P, we obtain a contradiction which proves $H(z)$ is a polynomial.

We now know that since $H(z)$ is not linear, then it is a polynomial of degree greater than one. Our previous arguments imply that there is a lattitude

γ such that $G^{-1}(\gamma)$ consists of at least two proper arcs in $\mathcal{H}$. By a slight modification of the proof of Lemma 5.4, we can assume that there is a T such that for every t greater than or equal to T, the horizontal plane $P(t)$ at height t intersects at least two components of $G^{-1}(\gamma)$.

As in the proof that $H(z)$ is a polynomial, we can construct a sequence $M(i)$ of minimal surfaces by translating and homothetically altering the surface M, which converges to a properly embedded minimal surface $\hat{M}$. As before $\hat{M}$ is simply-connected and has bounded curvature. By our previous arguments, the related function $\hat{H}(z)$ is a polynomial and $\mathcal{L}(\hat{M}) = \mathcal{L}(M)$ is a foliation of $\mathbb{R}^3$ by horizontal planes. By Theorem 4.4, $h = x_3 + ix_3^*\colon \hat{M} \to \mathbb{C}$ is a conformal diffeomorphism. By Lemma 5.1, $\hat{M}$ is a vertical helicoid and so for i large, $G_i^{-1}(\gamma) \cap \hat{\mathcal{H}}$ contains exactly one component which intersects the horizontal plane P. Since $G_i^{-1}(\gamma) \cap P$ contains another point $q(i)$ outside of $\hat{\mathcal{H}}$, our previous argument used to show $H(z)$ is a polynomial, gives a contradiction, which completes the proof of Proposition 5.2. $\square$

Finally, using Proposition 5.2 and Lemma 5.1, we deduce that M is a vertical helicoid. This completes the proof of Theorem 0.1 stated in the introduction.

6. Minimal surfaces of finite type

Recently Hoffman, Weber and Wolf [17] gave a rigorous analytic proof of the existence of a properly embedded minimal surface M in $\mathbb{R}^3$ that is conformally diffeomorphic to a rhombus torus punctured in a single point. This surface was constructed earlier by Hoffman, Karcher and Wei [13], [14]. The surface M is an example of a minimal surface of finite type, which means, among other things, that its Gauss map has finite growth in the sense of Nevanlinna.

In the case the minimal surface is properly embedded, Theorem 0.2 in the introduction gives an explicit description of what it means to be a properly embedded minimal surface of finite type. In the general case we refer the interested reader to the following papers for a complete discussion of the general theory of immersed minimal surfaces of finite type [12], [25], [27].

For our purposes we would like to consider the more general problem of describing the asymptotic geometry of a properly embedded minimal annulus $A = [0, \infty) \times S^1$ in $\mathbb{R}^3$. The main result in this section is that such an annulus is either asymptotic to a plane, an end of a catenoid or to a helicoid. If A has finite total curvature, then it is well-known that A is asymptotic to the end of a catenoid or to a plane. This result follows easily from the Weierstrass representation and from the facts that A is conformally a punctured disk and that the Gauss map on A extends meromorphically across the puncture.

We now analyze the case where A has infinite total curvature. What we will prove is that even though A is not simply-connected and has boundary, our analysis of the simply-connected empty-boundary case given in the previous sections can be adapted to deal with A. Most of this proof will consist of an analysis of the modifications in the proof of Theorem 0.1 that need to be taken into account because A is not simply-connected or because A has boundary. We now give this proof.

Consider $\lambda(i) \in \mathbb{R}^+$, $\lambda(i) \to 0$, and let $A(i) = \lambda(i)A$. Since ∂A is compact, it follows that for any compact ball $B \subset \mathbb{R}^3$ disjoint from the origin and for i sufficiently large, every component of $A(i) \cap B$ is simply-connected. Hence, by the compactness and regularity results of Colding and Minicozzi that are used to prove Theorem 2.1, a subsequence of the $A(i)$ (denoted again by $A(i)$) converges to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by planes and the singular set $S(\mathcal{L})$ of convergence of the $A(i)$ to $\mathcal{L}$ is a connected transverse Lipschitz curve. Thus, with minor modifications, Theorem 2.1 holds in our context with A replacing M.

Statement 1 of Theorem 2.2 holds in our new situation for the same reason as before. However, statement 2 in Theorem 2.2 that the limit foliation $\mathcal{L}$ is transverse to A fails because of the introduction of a finite number of critical points of the coordinate function orthogonal to the planes in $\mathcal{L}$ that may arise from ∂A. However, after removing a compact set from A, which does not affect the limit blow-down, the Gauss map on A misses the unit normal vectors to $\mathcal{L}$.

The uniqueness of $\mathcal{L}$ implies that Theorem 2.3 also essentially holds for A.

In Section 4 we proved that $h = x_3 + ix_3^*\colon M \to \mathbb{C}$ is a conformal diffeomorphism. In the present case, what we would like to prove is that, after removing a compact subset of A, $h\colon A \to \mathbb{C}$ is a proper embedding of A. If A is the end of a properly embedded minimal surface, then, by Cauchy's theorem, the conjugate function x_3^* is well-defined. In the general case the function ix_3^* may have a nonzero imaginary period $\lambda \in i\mathbb{R}$. In this case $h\colon A \to \mathbb{C}/\Lambda = \mathbb{R} \times (i\mathbb{R}/\lambda) = \mathbb{R} \times S^1$ is well-defined. The analysis that $\pm\nabla x_3$ has no asymptotic curves does not now change in any substantial way from the simply-connected case, which shows that in these coordinates $x_3(z) = \mathbb{R}e(z)$

Finally, the proof that $g = e^{H(z)}$, $H(z)$ linear, given in the simply-connected case, goes through without modification in the case the period λ is 0 and, in the case $\lambda \neq 0$, there is little modification in the original proof to be made as e^{x+iy} is periodic in the imaginary direction like h. It now follows from the analysis in [26] and [12] that A is of finite type and is asymptotic to a helicoid. This completes the proof of Theorem 0.2 stated in the introduction.

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