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Quadratic forms of signature $(2,2)$ and eigenvalue spacings on rectangular 2-tori

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1. Introduction

The Oppenheim conjecture, proved by Margulis [Mar1] (see also [Mar2]), asserts that for a nondegenerate indefinite irrational quadratic form Q in $n \geq 3$ variables, the set $Q(\mathbb{Z}^n)$ is dense. In [EMM] (where we used the lower bounds established in [DM]) a quantitative version of the conjecture was established. Namely:

Let ρ be a continuous positive function on the sphere $\{v \in \mathbb{R}^n \mid \|v\| = 1\}$, and let $\Omega = \{v \in \mathbb{R}^n \mid \|v\| < \rho(v/\|v\|)\}$. We denote by $T\Omega$ the dilate of Ω by T. For an indefinite quadratic form Q in n variables, let $N_{Q,\Omega}(a,b,T)$ denote the cardinality of the set

$$\{x \in \mathbb{Z}^n \ : \ x \in T\Omega \text{ and } a < Q(x) < b\}.$$

We recall from [EMM] that for any such Q there exists a constant $\lambda_{Q,\Omega}$ such that for any interval (a,b), as $T \to \infty$,

$$\mathrm{Vol}(\{x \in \mathbb{R}^n \ : \ x \in T\Omega \text{ and } a \leq Q(x) \leq b\}) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}. \tag{1}$$

Theorem 1.1 ([EMM, Th. 2.1]). *Let Q be an indefinite quadratic form of signature (p,q), with $p \geq 3$ and $q \geq 1$. Suppose Q is not proportional to a rational form. Then for any interval (a,b), as $T \to \infty$,*

$$N_{Q,\Omega}(a,b,T) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}, \tag{2}$$

where $n = p+q$, and $\lambda_{Q,\Omega}$ is as in (1).

If the signature of Q is $(2,1)$ or $(2,2)$ then Theorem 1.1 fails; in fact there are irrational forms for which along a subsequence T_j, $N_{Q,\Omega}(a,b,T_j) > T_j^{n-2}(\log T_j)^{1-\varepsilon}$. Such forms may be obtained by consideration of irrational

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forms which are very well approximated by split rational forms. It should be noted that the asymptotically exact lower bounds established by Dani and Margulis (see [DM]) hold for any irrational indefinite quadratic form in $n \geq 3$ variables.

Observe also that whenever a form of signature $(2, 2)$ has a rational isotropic subspace L then $L \cap T\Omega$ contains on the order of T^2 integral points x for which $Q(x) = 0$; hence $N_{Q,\Omega}(-\varepsilon, \varepsilon, T) \geq cT^2$, independently of the choice of ε. Thus to obtain an asymptotic formula similar to (2) in the signature $(2, 2)$ case, we must exclude the contribution of the rational isotropic subspaces. We remark that an irrational quadratic form of signature $(2, 2)$ may have at most four rational isotropic subspaces (see Lemma 10.3).

The space of quadratic forms in four variables is a linear space of dimension 10. Fix a norm $\| \cdot \|$ on this space.

Definition 1.2. (EWAS) A quadratic form Q is called *extremely well approximable by split forms* (EWAS) if for any $N > 0$ there exists a split integral form Q' and $2 \leq k \in \mathbb{R}$ such that

$$\left\| Q - \frac{1}{k} Q' \right\| \leq \frac{1}{k^N}.$$

Our main result is:

THEOREM 1.3. *Suppose Ω is as above. Let Q be an indefinite quadratic form of signature $(2, 2)$ which is not* EWAS. *Then for any interval (a, b), as $T \to \infty$,*

$$\tilde{N}_{Q,\Omega}(a, b, T) \sim \lambda_{Q,\Omega}(b - a)T^2, \tag{3}$$

where the constant $\lambda_{Q,\Omega}$ is as in (1), *and $\tilde{N}_{Q,\Omega}$ counts the points not contained in isotropic subspaces.*

As observed above, lattice points belonging to isotropic rational 2-dimensional subspaces have to be excluded. It turns out also that points belonging to a wider class of subspaces (which we shall call "quasinull") have to be treated separately. Given the form Q consider the orthogonal group $\mathrm{SO}(Q) \subset \mathrm{SL}(4, \mathbb{R})$ of all the orientation preserving linear transformations preserving Q. It acts on the 6-dimensional space $\wedge^2\mathbb{R}^4$. This representation is reducible and $\wedge^2\mathbb{R}^4$ decomposes into a direct sum of two irreducible 3-dimensional spaces, $\wedge^2\mathbb{R}^4 = V_1 \oplus V_2$ (see Lemma 2.1). We observe that a 2-dimensional subspace $L \subset \mathbb{R}^4$ is isotropic if and only if the corresponding 1-dimensional subspace $\wedge^2 L \subset \wedge^2\mathbb{R}^4$ lies in one of the subspaces V_i. Equivalently, if we let $\{w_1, w_2\}$ be a basis of L then L is isotropic if and only if $\|\pi_1(w_1 \wedge w_2)\| \cdot \|\pi_2(w_1 \wedge w_2)\| = 0$, where $\pi_i : \wedge^2\mathbb{R}^4 \to V_i$, $i = 1, 2$, are the projections to V_i so that $v = \pi_1(v) + \pi_2(v)$. Given a rational 2-dimensional subspace L let $\{w_1, w_2\}$ be an integral basis of $L \cap \mathbb{Z}^4$. The subspace will be called μ_1-*quasinull* if

$$\|\pi_1(w_1 \wedge w_2)\| \cdot \|\pi_2(w_1 \wedge w_2)\| < \mu_1,$$

where $\mu_1 > 0$ is a fixed constant, and $\| \cdot \|$ is a Euclidean norm on $\wedge^2\mathbb{R}^4$.

Since most results do not depend on the choice of the parameter μ_1, we will often use the term quasinull subspace to refer to a μ_1-quasinull subspace. We also define the norm of a 2-dimensional rational subspace L to be the norm of $w_1 \wedge w_2$ where $\{w_1, w_2\}$ is any integral basis of $L \cap \mathbb{Z}^4$.

The following theorem is valid without any diophantine conditions:

THEOREM 1.4. *Suppose Ω is as above. Let Q be any indefinite quadratic form of signature $(2, 2)$ not proportional to a rational form. Then for any interval (a, b), as $T \to \infty$,*

$$\limsup \frac{1}{T^2} N'_{Q,\Omega}(a, b, T) \leq \lambda_{Q,\Omega}(b - a), \tag{4}$$

where the constant $\lambda_{Q,\Omega}$ is as in (1), *and $N'_{Q,\Omega}$ counts the points not contained in quasinull subspaces of norm at most T.*

With a diophantine condition we have:

THEOREM 1.5. *Suppose Q is a quadratic form of signature $(2, 2)$ which is not* EWAS. *Let $X_{Q,\Omega}(a, b, T)$ denote the number of integral points $v \in T\Omega$ such that $a < Q(v) < b$ and v lies in some nonisotropic quasinull subspace of norm at most T. Then, as $T \to \infty$,*

$$X_{Q,\Omega}(a, b, T) = o(T^2). \tag{5}$$

We also recall:

THEOREM 1.6 (Dani-Margulis). *Suppose Ω is as above. Let Q be any indefinite*
quadratic form in $n \geq 3$ variables, not proportional to a rational form. Then for any interval (a, b), as $T \to \infty$,

$$\liminf \frac{1}{T^{n-2}} N_{Q,\Omega}(a, b, T) \geq \lambda_{Q,\Omega}(b - a), \tag{6}$$

where the constant $\lambda_{Q,\Omega}$ is as in (1).

To deduce Theorem 1.3 from Theorem 1.4, Theorem 1.6 and Theorem 1.5 one can argue as follows: Suppose Q of signature $(2, 2)$ is not EWAS (see Definition 1.2); then by Theorem 1.5, $X_{Q,\Omega}(a, b, T)$ is $o(T^2)$. Hence, if $0 \notin (a, b)$ we get

$$\limsup_{T\to\infty} \frac{N_{Q,\Omega}(a,b,T)}{T^2} = \limsup_{T\to\infty} \frac{\tilde{N}_{Q,\Omega}(a,b,T)}{T^2} = \limsup_{T\to\infty} \frac{N'_{Q,\Omega}(a,b,T)}{T^2} \leq \lambda_{Q,\Omega}(b - a).$$

However, the universal lower bound of Dani-Margulis (Theorem 1.6) implies that for any irrational Q, $\liminf_{T\to\infty} \frac{1}{T^2} N_{Q,\Omega}(a,b,T) = \lambda_{Q,\Omega}(b-a)$. Hence for Q not EWAS and $0 \notin (a,b)$ we get $N_{Q,\Omega}(a,b,T) \sim \lambda_{Q,\Omega}(b-a)T^2$ which is equivalent to Theorem 1.3.

Eigenvalue spacings on flat 2*-tori.* It has been suggested by Berry and Tabor that the eigenvalues of the quantization of a completely integrable Hamiltonian follow the statistics of a Poisson point-process, which means their consecutive spacings should be independent and identically distributed exponentially distributed. For the Hamiltonian which is the geodesic flow on the flat 2-torus, it was noted by P. Sarnak [Sar] that this problem translates to one of the spacing between the values at integers of a binary quadratic form, and is related to the quantitative Oppenheim problem in the signature $(2,2)$ case. We briefly recall the connection following [Sar].

Let $\Delta \subset \mathbb{R}^2$ be a lattice and let $M = \mathbb{R}^2/\Delta$ denote the associated flat torus. The eigenfunctions of the Laplacian on M are of the form $f_v(\cdot) = e^{2\pi i\langle v,\cdot\rangle}$, where v belongs to the dual lattice Δ^*. The corresponding eigenvalues are $4\pi^2\|v\|^2$, $v \in \Delta^*$. These are the values at integral points of the binary quadratic $B(m,n) = 4\pi^2\|mv_1 + nv_2\|^2$, where $\{v_1, v_2\}$ is a $\mathbb{Z}$-basis for Δ^*. We will identify Δ^* with $\mathbb{Z}^2$ using this basis.

We label the eigenvalues (with multiplicity) by

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \dots .$$

It is easy to see that Weyl's law holds, i.e.

$$|\{j \;:\; \lambda_j(M) \leq T\}| \sim c_M T,$$

where $c_M = (\operatorname{area} M)/(4\pi)$. We are interested in the distribution of the local spacings $\lambda_j(M) - \lambda_k(M)$. In particular, for $0 \notin (a,b)$, set

$$R_M(a,b,T) = \frac{|\{(j,k) \;:\; \lambda_j(M) \leq T, \lambda_k(M) \leq T, a \leq \lambda_j(M) - \lambda_k(M) \leq b\}|}{T}.$$

The statistic R_M is called the pair correlation. The Poisson-random model predicts, in particular, that

$$\lim_{T\to\infty} R_M(a,b,T) = c_M^2(b-a). \tag{7}$$

Note that the differences $\lambda_j(M) - \lambda_k(M)$ are precisely the integral values of the quadratic form $Q_M(x_1,x_2,x_3,x_4) = B(x_1,x_2) - B(x_3,x_4)$.

P. Sarnak showed in [Sar] that (7) holds on a set of full measure in the space of tori. Some remarkable related results for forms of higher degree and higher dimensional tori were proved in [V1], [V2] and [V3]. These methods, however, cannot be used to explicitly construct a specific torus for which (7) holds. A corollary of Theorem 1.3 is the following:

THEOREM 1.7. *Let M be a 2-dimensional flat torus rescaled so that one of the coefficients in the associated binary quadratic form B is 1. Let A_1, A_2 denote the two other coefficients of B. Suppose that there exists $N > 0$ such that for all triples of integers (p_1, p_2, q) with $q \geq 2$,*

$$\max_{i=1,2} \left| A_i - \frac{p_i}{q} \right| > \frac{1}{q^N}.$$

Then, for any interval (a, b) not containing 0, (7) holds, i.e.

$$\lim_{T\to\infty} R_M(a, b, T) = c_M^2(b - a).$$

In particular, the set of $(A_1, A_2) \subset \mathbb{R}^2$ for which (7) does not hold has zero Hausdorff dimension.

Thus, if one of the A_i is diophantine (e.g. algebraic), then M has a spectrum whose pair correlation satisfies the Berry-Tabor conjecture.

This establishes the pair correlation for the flat torus or "boxed oscillator" considered numerically by Berry and Tabor. We note that without some diophantine condition, (7) may fail.

Let $\Omega_M \subset \mathbb{R}^4$ denote the set $\{x : \max(B(x_1, x_2)^{1/2}, B(x_3, x_4)^{1/2}) \leq 1\}$. By construction, if $(m_1, n_1, m_2, n_2) \in \mathbb{Z}^4 \cap T\Omega$, then $B(m_1, n_1)$ and $B(m_2, n_2)$ are two eigenvalues of size at most T^2, and thus $Q_M(m_1, n_1, m_2, n_2)$ is a difference between two points in the spectrum of size at most T^2. An elementary calculation (cf. [Sar]) shows that $\lambda_{Q_M,\Omega_M} = (\operatorname{area} M/4\pi)^2 = c_M^2$. Thus, one arrives at the following:

PROPOSITION 1.8 (see [Sar]). *The equation (7) holds for M if and only if (3) holds with $Q = Q_M$, $\Omega = \Omega_M$.*

Thus in view of Proposition 1.8, Theorem 1.7 is indeed a corollary of Theorem 1.3.

Remark. The case where M is rectangular is simpler since in that case the quasinull subspaces can be easily described, allowing for an elementary proof of Theorem 1.5. This is written out in Section 9.

Outline of the proofs. Both [EMM] and this paper are based on the following approach. In order to estimate $N_{Q,\Omega}(a, b, T)$ we make a transition to considering certain integrals on the space of unimodular lattices in $\mathbb{R}^n$, i.e. $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$. This transition is based on the transitivity of the action of the orthogonal group $\mathrm{SO}(Q)$ on the level sets of the quadratic form Q. We fix a suitably chosen compact set $U \subset \mathbb{R}^n$-$\{0\}$ such that for $v \in U$, $a \leq Q(v) \leq b$. Let $\hat{f}$ denote the function on the space of lattices $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ which associates to each lattice Δ the number of points in $\Delta \cap U$. Because of the transitivity of the action of $\mathrm{SO}(Q)$, any vector $v \in \mathbb{Z}^n$ with $T/2 \leq \|v\| \leq T$ and

$a \le Q(v) \le b$ can be brought into U by an appropriate element $g \in \mathrm{SO}(Q)$. Note that the number of points in the lattice $g\mathbb{Z}^4$ lying in U is exactly equal to the number of points in the lattice $\mathbb{Z}^n$ lying in $g^{-1}U$. Hence the number of points in $\mathbb{Z}^n$ lying in $g^{-1}U$ is equal to $\hat{f}(g\mathbb{Z}^n)$. By varying $g \in \mathrm{SO}(Q)$ in an appropriate way, the sets $g^{-1}U$ can be made to cover the set $R = \{v \in \mathbb{R}^4 \ : \ a \le Q(v) \le b \text{ and } T/2 \le \|v\| \le T\}$. Thus, the number of integer points in R, i.e. $N_Q(a,b,T) - N_Q(a,b,T/2)$ may be approximated by an integral of the form $T^2 \int_H \hat{f}(g\mathbb{Z}^4)\,dg$, where H is a suitably chosen subset of $\mathrm{SO}(Q)$. Note that one needs on the order of T^2 translates of U to cover R, since g is volume-preserving and the volume of R is asymptotic to T^2.

Observe that for any choice of U, $\hat{f}$ is unbounded. In [DM] S.G. Dani and G.A. Margulis have used integrals of the form $\int_H \phi(g\mathbb{Z}^n)\,dg$ where ϕ is a bounded function with $\phi \le \hat{f}$ to give asymptotically exact lower bounds on the number of lattice points in sets of the form $\{v \in \mathbb{R}^n \ : \ a < Q(v) < b\} \cap T\Omega$ for indefinite irrational quadratic forms, $n \ge 3$. The proof of these estimates uses M. Ratner's measure classification theorem (see [Ratner]) as well as the methods developed by Dani and Margulis for studying unipotent flows via "linearization"; see [DM]. One also needs to have an estimate of the contribution of elements of lattices lying at the "cusps" of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, i.e., outside of large compact subsets.

At this point there is a significant difference between the cases considered in [EMM] and the present case. (Note that the main result of [EMM] does not hold for all forms of signature $(2,2)$.) However, the contributions of lattices having either a single short vector or three independent short vectors is still estimated as in [EMM]. In estimating the contribution of those lattices having a two-dimensional sublattice spanned by short vectors one has to exclude the lattice points belonging to quasinull subspaces. One of the main technical tools of the current paper is the estimate provided by Theorem 2.6 for the contribution of these lattices excluding the points coming from quasinull subspaces. The proof of Theorem 2.6 (which is outlined in Section 5) occupies Sections 5–8. The contribution of quasinull spaces is estimated in Section 9 for the case of rectangular tori, and in Section 10 in the general case.

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2. Passage to the space of lattices

Fix a quadratic form Q of signature $(2,2)$ and discriminant 1. Then there exists an element $g_Q \in \mathrm{SL}(4,\mathbb{R})$ such that for any $v \in \mathbb{R}^n$, $Q(v) = B(g_Q v)$, where B is the "standard" quadratic form $B(z_1,z_2,z_3,z_4) = z_1 z_4 - z_2 z_3$. Let Λ denote $g_Q\mathbb{Z}^4$ so that $Q(\mathbb{Z}^4) = B(\Lambda)$.

We identify $\mathbb{R}^4$ with $M_2(\mathbb{R})$ by sending (z_1, z_2, z_3, z_4) to $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$. Then the determinant function on $M_2(\mathbb{R})$ becomes the quadratic form B of signature $(2, 2)$. This shows that the orthogonal group $H = \mathrm{SO}(B)$ is locally isomorphic to $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ with the action of $g = (g_1, g_2) \in \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ on $v \in M_2(\mathbb{R})$ given by $v \to g_1 v g_2^{-1}$. Write $H = H_1 \times H_2$ (almost direct product).

Let $e_{ij} \in M_2(\mathbb{R})$ denote the elementary matrix with 1 in row i column j and 0's elsewhere. Then e_{11}, e_{12}, e_{21} and e_{22} form a basis for $M_2(\mathbb{R}) \approx \mathbb{R}^4$. The following lemma is standard:

Lemma 2.1 (Reducibility of the representation of SO(2,2) on $\wedge^2\mathbb{R}^4$).

(a) *The space* $\wedge^2\mathbb{R}^4 \approx \wedge^2 M_2(\mathbb{R})$ *splits as a direct sum of two invariant subspaces* $V_1 \oplus V_2$ *each of dimension* 3. *A basis for* V_1 *is given by* $f_1^{(1)} = e_{11} \wedge e_{12}$, $f_2^{(1)} = e_{21} \wedge e_{22}$, $f_3^{(1)} = e_{11} \wedge e_{22} - e_{12} \wedge e_{21}$. *A basis for* V_2 *is given by* $f_1^{(2)} = e_{11} \wedge e_{21}$, $f_2^{(2)} = e_{12} \wedge e_{22}$, $f_3^{(2)} = e_{11} \wedge e_{22} + e_{12} \wedge e_{21}$.

(b) H_1 *fixes each vector in* V_2 *and* H_2 *fixes each vector in* V_1. *On* V_i, H_i *fixes the quadratic form* $Q(x f_1^{(i)} + y f_2^{(i)} + z f_3^{(i)}) = z^2 - xy$.

(c) *Let* π_i *denote the projection* $\wedge^2\mathbb{R}^4 \to V_i$. *Let* L *be a two-dimensional subspace of* $\mathbb{R}^4$, *and let* v_1, v_2 *be a basis for* L. *Then*

$$\begin{aligned} Q(\pi_1(v_1 \wedge v_2)) &= Q(\pi_2(v_1 \wedge v_2)) \\ &= \text{the discriminant of the restriction of } B \text{ to } L\ . \end{aligned}$$

The restriction of B *to* L *is identically* 0 *if and only if* $\pi_1(v_1 \wedge v_2) = 0$ *or* $\pi_2(v_1 \wedge v_2) = 0$.

Let Λ be a lattice in $\mathbb{R}^4$. A subspace L of $\mathbb{R}^4$ is called Λ-rational if $L \cap \Lambda$ is a lattice in L. Let L be a 2-dimensional Λ-rational subspace L of $\mathbb{R}^4$, and let v_1, v_2 be an integral basis for $L \cap \Lambda$. Write $v^L = v_1 \wedge v_2$. Note that up to sign, v^L depends only on L and Λ and does not depend on the choice of integral basis. In view of Lemma 2.1 (c), the restriction of B to L is identically 0 if and only if $\|\pi_1(v^L)\|\|\pi_2(v^L)\| = 0$.

Definition 2.2 (Quasinull subspace). Let Λ be a lattice in $\mathbb{R}^4$. Fix μ_1, $0 < \mu_1 < 1$. A Λ-rational subspace L of $\mathbb{R}^4$ is called μ_1*-quasinull with respect to* Λ if $\dim(L) = 2$ and

$$\|\pi_1(v^L)\|\|\pi_2(v^L)\| < \mu_1. \tag{8}$$

Instead of the standard quadratic form B and the lattice $\Lambda = g_Q\mathbb{Z}^4$, we often consider the given form $Q(v) = B(g_Q v)$ and the standard lattice $\mathbb{Z}^4$. We will then say that a $\mathbb{Z}^4$-rational subspace L is μ_1-quasinull if and only if $g_Q L$ is μ_1-quasinull with respect to Λ. We will also occasionally omit μ_1.

Instead of working directly with $\mathrm{SO}(2,2)$ we work with $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, which is locally isomorphic. Let $b_t \in \mathrm{SL}(2,\mathbb{R})$ denote the matrix $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{pmatrix}$, let $a_t = (b_t, b_t)$ and let $K = \mathrm{SO}(2)\times\mathrm{SO}(2)$ denote the standard maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. Let dk denote the normalized Haar measure on K.

As in [EMM], for a lattice $\Delta \subset \mathbb{R}^4$ and a function $f \in C_0^\infty(\mathbb{R}^4 - \{0\})$, let

$$\hat{f}(\Delta) = \sum_{v\in\Delta} f(v).$$

Recall that in [EMM, Th. 2.3] we proved that for Q of signature (p,q) with $p \geq 3$ and $q \geq 1$, and as long as Q is not proportional to a rational form (so that $\mathrm{SO}(Q)\Lambda$ is not closed), and for any continuous function ν on K, we have

$$\lim_{n\to\infty}\int_K \hat{f}(a_t k\Lambda)\nu(k)\,dk = \int_K \nu(k)\,dk \int_{\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SL}(p+q,\mathbb{Z})} \hat{f}(\Delta)\,d\mu(\Delta). \tag{9}$$

This was used to obtain the main result [EMM, Th. 2.1] (which is restated as Theorem 1.1 in the present paper).

In the case of signature $(2,2)$, (9) may fail in general because of the contribution of quasinull subspaces. We will need the following modification: Let $X_T(\Lambda)$ denote the set of $v \in \Lambda$ which do not belong to any quasinull subspace of Λ of norm at most T, and let

$$\tilde{f}(g;\Lambda) = \sum_{v\in X_T(\Lambda)} f(gv). \tag{10}$$

In order to prove Theorem 1.4 we prove the following theorem:

THEOREM 2.3. *Let Q be any quadratic form of signature $(2,2)$ and discriminant 1 which is not proportional to a rational form. Let $g_Q \in \mathrm{SL}(4,\mathbb{R})$ be such that for all $v \in \mathbb{R}^4$, $Q(v) = B(g_Q v)$, and let $\Lambda = g_Q\mathbb{Z}^4$. Then, for any function $f \in C_0^\infty(\mathbb{R}^4 - \{0\})$, and any continuous function ν on K,*

$$\limsup_{t\to\infty}\int_K \tilde{f}(a_t k;\Lambda)\nu(k)\,dk \leq \int_K \nu(k)\,dk \int_{\mathrm{SL}(4,\mathbb{R})/\mathrm{SL}(4,\mathbb{Z})} \hat{f}(\Delta)\,d\mu(\Delta), \tag{11}$$

where μ denotes the normalized Haar measure on the space of lattices $\mathrm{SL}(4,\mathbb{R})/\mathrm{SL}(4,\mathbb{Z})$.

Theorem 1.4 follows from Theorem 2.3 by an argument identical to that used in [EMM, §3.4, §3.5] to deduce [EMM, Th. 2.1] (i.e. Theorem 1.1) from [EMM, Th. 2.3] (i.e. (9)), with the natural modifications due to the fact that we assert and assume only inequalities. Essentially, the proof is based on the identity of the form

$$\int_K \tilde{f}(a_t k;\Lambda)\nu(k)\,dk = \sum_{v\in X_T(\Lambda)} \int_K f(a_t k v)\nu(k)\,dk, \tag{12}$$

obtained by integrating (10). The right-hand side of (12) is then related to the number of lattice points $v \in [e^t/2, e^t]\partial\Omega$ with $a < Q(v) < b$ which are not contained in quasinull subspaces of norm at most T (where $T = e^t$).

Let Δ be a lattice in $\mathbb{R}^4$. As in [EMM], for any Δ-rational subspace L, we denote by $d_\Delta(L)$ or simply by $d(L)$ the volume of $L/(L \cap \Delta)$. Let us note that $d(L)$ is equal to the norm of $e_1 \wedge \cdots \wedge e_\ell$ in the exterior power $\wedge^\ell \mathbb{R}^4$ where $\ell = \dim L$ and $\{e_1, \cdots, e_\ell\}$ is a basis over $\mathbb{Z}$ of $L \cap \Delta$. If $L = \{0\}$ we write $d(L) = 1$.

We recall the following:

Lemma 2.4 ([EMM, Lemma 5.6]). *For any two Δ-rational subspaces L and M*

$$d(L)d(M) \geq d(L \cap M)d(L + M). \tag{13}$$

As in [EMM] we use the notation:

(14)

$$\alpha_i(\Delta) = \sup\left\{ \frac{1}{d(L)} \,\middle|\, L \text{ is a } \Delta\text{-rational subspace of dimension } i \right\}, \quad 0 \leq i \leq 4,$$

$$\alpha(\Delta) = \max_{0 \leq i \leq 4} \alpha_i(\Delta).$$

We recall the following theorem:

Theorem 2.5. *Suppose $i = 1$ or 3. Then for any $\xi > 0$,*

$$\sup_{t \geq 0} \int_K \alpha_i(a_t k \Lambda)^{2-\xi}\, dk < \infty.$$

Hence there exists a constant c depending only on ξ and Λ such that for all $t > 0$ and all $\delta > 0$,

$$|\{k \in K \;:\; \alpha_i(a_t k \Lambda) > 1/\delta\}| < c\delta^{2-\xi}.$$

Proof. The first assertion is proved in [EMM, §5] (see the proof of equation (5.75)). The second assertion follows from the first and Chebechev's inequality. □

The analogous assertion for α_2 is false even when $\xi = 1$; this accounts for the failure of Theorem 1.1 in the signature $(2,2)$ case. However, in this situation, we prove a substitute: see Theorem 2.6 below.

Suppose $g \in \mathrm{SL}(4,\mathbb{R})$, and Λ is as above. Let

$$\hat{\alpha}_2(g;\Lambda) = \sup\left\{ \frac{1}{d(gL)} \;:\; \dim(L) = 2,\, L \text{ is rational and not } \mu_1\text{-quasinull} \right\}.$$

Theorem 2.6. *There exists a constant c depending only on μ_1 and Λ such that for all $0 < \delta \ll 1$ and all $t > 0$,*

$$|\{k \in K \ : \ \hat{\alpha}_2(a_t k; \Lambda) > 1/\delta\}| < c\delta^{1.05}. \tag{15}$$

Hence, for any $\theta < 1.05$,

$$\sup_{t \geq 0} \int_K \hat{\alpha}_2(a_t k; \Lambda)^\theta < \infty.$$

In Section 3 we derive Theorem 2.3 from Theorem 2.5 and Theorem 2.6. After making some preliminary calculations in Section 4, we will begin the proof of Theorem 2.6 (which is the main technical point of this paper) in Section 5.

3. Proof of Theorem 2.3

In this section, we deduce Theorem 2.3 from Theorem 2.5, and Theorem 2.6. This argument is very similar to the proof of [EMM, Th. 3.5] in [EMM, §5].

Lemma 3.1. *For every function $f \in C_0^\infty(\mathbb{R}^4\text{-}\{0\})$ there exists a constant $c = c(f)$ such that for all $g \in \mathrm{SL}(4, \mathbb{R})$,*

$$\tilde{f}(g; \Lambda) \leq c\hat{\alpha}(g; \Lambda)$$

where $\hat{\alpha}(g; \Lambda) = \max(1, \alpha_1(g\Lambda), \hat{\alpha_2}(g; \Lambda), \alpha_3(g\Lambda))$.

Proof of Lemma 3.1. This is an easy modification of a well known lemma of Schmidt; see [Sch, Lemma 2]. □

Proof of Theorem 2.3. We may assume that $\tilde{f}$ is nonnegative. Let

$$A(r) = \{\Delta \in \mathrm{SL}(4, \mathbb{R})/\mathrm{SL}(4, \mathbb{Z}) \ : \ \alpha_1(\Delta) > r\}.$$

Choose a continuous nonnegative function g_r on $\mathrm{SL}(4, \mathbb{R})/\mathrm{SL}(4, \mathbb{Z})$ such that $g_r(\Delta) = 1$ if $\Delta \in A(r+1)$, $g_r(\Delta) = 0$ if $\Delta \notin A(r)$ and $0 \leq g_r(\Delta) \leq 1$ if $\Delta \in A(r) - A(r+1)$. Then

$$\int_K \tilde{f}(a_t k; \Lambda)\, dk = \int_K \tilde{f}(a_t k; \Lambda) g_r(a_t k\Lambda)\, dk + \int_K \tilde{f}(a_t k; \Lambda)(1 - g_r(a_t k\Lambda))\, dk. \tag{16}$$

But,

$$\begin{aligned}\tilde{f}(a_t k; \Lambda) g_r(a_t k\Lambda) &\leq c\hat{\alpha}(a_t k; \Lambda) g_r(a_t k\Lambda) \\ &= c\hat{\alpha}(a_t k; \Lambda)^{-0.04} g_r(a_t k\Lambda)\hat{\alpha}(a_t k; \Lambda)^{1.04} \\ &\leq B_1 r^{-0.04}\hat{\alpha}(a_t k; \Lambda)^{1.04}\end{aligned}$$

(the last inequality is true because $\hat{\alpha}(a_t k; \Lambda) \geq \alpha_1(a_t k\Lambda)$ and $g_r(a_t k\Lambda) = 0$ if $\alpha_1(a_t k\Lambda) \leq r$). Therefore

(17)
$$\int_K \tilde{f}(a_t k; \Lambda) g_r(a_t k\Lambda)\nu(k)\, dk \leq B_1 r^{-0.04} \left(\sup_{k\in K} |\nu(k)|\right) \int_K \hat{\alpha}(a_t k; \Lambda)^{1.04}\, dk.$$

According to Theorem 2.5 and Theorem 2.6 there exists B such that

$$\int_K \hat{\alpha}(a_t k; \Lambda)^{1.04}\, dk < B,$$

for any $t \geq 0$. Then (17) implies that

$$\int_K \tilde{f}(a_t k; \Lambda) g_r(a_t k\Lambda)\nu(k)\, dk \leq BB_1 r^{-1.04} \left(\sup_{k\in K} |\nu(k)|\right). \tag{18}$$

Let $h(\Delta) = \hat{f}(\Delta)(1 - g_r(\Delta))$. Then for all $t > 0$ and all $k \in K$,

$$\tilde{f}(a_t k; \Lambda)(1 - g_r(a_t k\Lambda)) \leq h(a_t k\Lambda). \tag{19}$$

Since the function h is continuous and has compact support, [EMM, Th. 4.5] implies that for every $\varepsilon > 0$ and every quadratic form Q not proportional to a rational form, there exists $t_0 > 0$ such that for $\Lambda = g_Q\mathbb{Z}^4$ and every $t > t_0$,

$$\left| \int_K h(a_t k\Lambda)\nu(k)\, dk - \int_K \nu(k)\, dk \int_{\mathrm{SL}(4.\mathbb{R})/\mathrm{SL}(4,\mathbb{Z})} h(\Delta)\, d\mu(\Delta) \right| < \frac{\varepsilon}{2}. \tag{20}$$

It is easy to see that (16), (18), (19) and (20) imply (11) if r is sufficiently large. ☐

4. The rectangles

We will eventually establish (15), but first we need to study carefully the sets $\{k \in K \ :\ d(a_t kL) < \eta\}$, where L is a two-dimensional subspace. This will be done in this section.

Let $b_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, and $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Then $a_t = (b_t, b_t)$ and $k = (k_\theta, k_{\theta'})$. If we use the norm

$$\|v\|' = \max(\|\pi_1(v)\|, \|\pi_2(v)\|)$$

for defining α_2 and d then the set $\{k \in K \ :\ d(a_t kL) < \eta\}$ is the direct product of the sets $\{\theta \ :\ \|b_t k_\theta \pi_1(v^L)\| < \eta\}$ and $\{\theta' \ :\ \|b_t k_{\theta'} \pi_2(v^L)\| < \eta\}$. As we shall see below, each set $\{\theta \ :\ \|b_t k_\theta \pi_1(v^L)\| < \eta\}$ is either an interval or a union of two intervals; thus our set $\{k \in K \ :\ d(a_t kL) < \eta\}$ is a union of up to four rectangles. Note that these rectangles (and all rectangles appearing in this paper) have sides parallel to the coordinate axes.

More generally, we may also consider sets of the form

$$\{(\theta,\theta') \ : \ \|b_t k_\theta \pi_1(v^L)\| < \eta_1, \ \|b_t k_{\theta'} \pi_2(v^L)\| < \eta_2\}.$$

Let $M_t(L) = \min_{k\in k} \|a_t k v^L\|'$.

Implied constants. In this section, all implied constants are either absolute, or depend only on μ_1 and Λ.

LEMMA 4.1. *There is a constant $c \ll 1$ such that the following holds: Suppose L is not μ_1-quasinull with respect to Λ, $t \gg 1$ and $M_t(L) \ll 1$. Then, both of the sets $\{\theta \ : \ \|b_t k_\theta \pi_1(v^L)\| < c\}$ and $\{\theta' \ : \ \|b_t k_{\theta'} \pi_2(v^L)\| < c\}$ are not all of $\mathrm{SO}(2)$. Also, if $\|v^L\| \gg 1$ then for some $j \in \{1,2\}$ the set $\{\phi \ : \ \|b_t k_\phi \pi_j(v^L)\| < M_t(L)^{-0.1}\}$ is not all of $\mathrm{SO}(2)$.*

Proof. It is easy to check that for any $v \in \mathbb{R}^3$ and $t \gg 1$,

$$\|b_t\|^{-1}\|v\| \le \min_\theta \|b_t k_\theta v\| \le \max_\theta \|b_t k_\theta v\| \approx \|b_t\|\|v\|. \tag{21}$$

Write $v^L = \pi_1(v^L) + \pi_2(v^L)$. Without loss of generality we may assume $\|\pi_2(v^L)\| \ge \|\pi_1(v^L)\|$. Since $M_t(L) \ll 1$, from (21) we see that $\|\pi_2(v^L)\| \ll \|b_t\|$. Hence since L is not μ_1-quasinull with respect to Λ, $\|\pi_1(v^L)\| \ge \|b_t\|^{-1}$. Then, by (21), $\max_\theta \|b_t k_\theta \pi_1(v^L)\| \approx \|b_t\|\|\pi_1(v^L)\| \ge c$ as required. Since L is Λ-rational, $\|v^L\| \ge c_0$ where $c_0 = c_0(\Lambda)$; hence $\|\pi_2(v^L)\| \ge c_0/2$. This implies (for any $\theta < 1$),

$$\begin{aligned}\max_\theta \|b_t k_\theta \pi_2(v^L)\| \approx \|b_t\|\|\pi_2(v^L)\| &\ge (c_0/2)\|b_t\| \ge (c_0/2)\|b_t\|^{-\theta} \\ &\ge (c_0/4)(\|b_t\|\|\pi_2(v^L)\|)^{-\theta} \ge cM_t(L)^{-\theta},\end{aligned}$$

as required. (Here c depends only on μ_1 and Λ.) □

For an interval $I \subset S^1$ and $c > 0$, let cI denote the interval with the same center as I and length $c|I|$. Similarly for a rectangle $R = I_1 \times I_2$ and $c > 0$, let cR denote the rectangle $cI_1 \times cI_2$

We approximate the sets

$$\{(\theta,\theta') \ : \ \|b_t k_\theta \pi_1(v^L)\| < \eta_1, \ \|b_t k_{\theta'} \pi_2(v^L)\| < \eta_2\}$$

by sets $R_t^{L,\pm,\pm}(\eta_1,\eta_2)$ which are rectangles (rather than unions of rectangles), and whose center does not depend on t. A major difficulty is that the aspect ratio of the rectangles is not bounded (even if $\eta_1 = \eta_2$). However, the modified family has the following properties:

PROPOSITION 4.2. *Suppose L is not μ_1-quasinull with respect to Λ, $t \gg 1$, and $M_t(L) \ll 1$. There exist rectangles $R_t^{L,\pm\pm}(\delta_1,\delta_2) = I_t^{\pi_1(L),\pm}(\delta_1) \times I_t^{\pi_2(L),\pm}(\delta_2)$, such that the following properties hold:*

(a) *The rectangles* $R_t^{L,\pm\pm}(\eta_1,\eta_2)$ *approximate the sets*

$$\{(\theta,\theta') \,:\, \|b_t k_\theta \pi_1(v^L)\| < \eta_1,\ \|b_t k_{\theta'} \pi_2(v^L)\| < \eta_2\};$$

i.e. there exists $0 < c < 1$ *such that if* $M_t(L) \ll \delta_i \ll \max_k \|\pi_i(a_t k v^L)\|$ *then*

$$\begin{aligned}(22)\quad & R_t^{L,\pm\pm}(c\delta_1, c\delta_2)\\ & \subset \{(\theta,\theta') \,:\, \|b_t k_\theta \pi_1(v^L)\| < \delta_1,\ \|b_t k_{\theta'} \pi_2(v^L)\| < \delta_2\}\\ & \subset R_t^{L,++}(c^{-1}\delta_1, c^{-1}\delta_2)\\ & \cup R_t^{L,+-}(c^{-1}\delta_1, c^{-1}\delta_2) \cup R_t^{L,-+}(c^{-1}\delta_1, c^{-1}\delta_2) \cup R_t^{L,--}(c^{-1}\delta_1, c^{-1}\delta_2).\end{aligned}$$

(b) *The rectangles* R_t^L *shrink as* t *increases*; *i.e. there exists an absolute constant* $C < \infty$ *such that if* $\max_k \|\pi_i(a_t k v^L)\| \gg \delta_i \gg M_t(L)$, $\tau \gg 0$ *and* $\delta_i \gg M_{t+\tau}(L)$, *then*

$$R_{t+\tau}^{L,\pm\pm}(\delta_1,\delta_2) \subset Ce^{-\tau/2} R_t^{v,\pm\pm}(\delta_1,\delta_2). \tag{23}$$

(c) *Measure ratio estimate. If* $\max_k \|\pi_i(a_t k v^L)\| \gg \eta_i \gg \delta_i \gg M_t(L)$, *then*

$$\frac{|R_t^{L,\pm\pm}(\delta_1,\delta_2)|}{|R_t^{L,\pm\pm}(\eta_1,\eta_2)|} \ll \left(\frac{\delta_1\delta_2}{\eta_1\eta_2}\right)^{1/2}. \tag{24}$$

(d) *The alternative. Either one side is determined*; *i.e. for* $\max_k \|\pi_i(a_t k v^L)\| \gg \delta_i \gg M_t(L)$,

$$R_t^{L,\pm\pm}(\delta_1,\delta_2) \text{ has at least one side of length } \approx e^{-t} M_t(L)^{-1/2}\delta_i^{1/2}, \tag{25}$$

or the minimum does not decrease; *i.e. for* $\tau > 0$,

$$M_{t+\tau}(L) \geq M_t(L). \tag{26}$$

Proposition 4.2 is proved in Appendix A.

Notation. We use the notation $R_t^{L,\pm\pm}(\eta)$ to denote $R_t^{L,\pm\pm}(\eta,\eta)$.

5. The scheme of the proof of Theorem 2.6

Let $\mathcal{L}_t(\delta)$ denote the set of non-μ_1-quasinull Λ-rational 2-dimensional subspaces L such that for some $k \in K$, $d(a_t k L) < \delta$. We have:

$$\begin{aligned}(27)\quad & \{k \in K \,:\, \hat\alpha_2(a_t k;\Lambda) > 1/\delta\}\\ & = \bigcup_{L\in\mathcal{L}_t(\delta)} \{k \in K \,:\, d(a_t k L) < \delta\} \subset \bigcup_{\substack{\sigma\in\pm\\ \rho\in\pm}} \left(\bigcup_{L\in\mathcal{L}_t(\delta)} R_t^{L,\sigma\rho}(c^{-1}\delta)\right),\end{aligned}$$

where c is as in Proposition 4.2 (a). Let $\tau > 0$ be a parameter to be chosen later. Let

$$\mathcal{L}_{t,\text{new}}(\delta) = \mathcal{L}_t(\delta) - \mathcal{L}_{t-\tau}(\delta), \qquad \text{and} \qquad \mathcal{L}_{t,\text{old}}(\delta) = \mathcal{L}_t(\delta) \cap \mathcal{L}_{t-\tau}(\delta).$$

We will consider old and new subspaces separately. The reason for this will become apparent in Section 7. The proof of Theorem 2.6 consists of the following steps:

Step 1 (Contribution of old subspaces). *There exists a constant $\tau > 0$ (depending only on μ_1 and Λ) such that for any $\omega > 1$ and any $\xi > 0$ there exist constants $\kappa = \kappa(\omega, \xi, \mu_1, \Lambda) < 1$, $t_0 = t_0(\mu_1, \Lambda)$, $\delta_0 = \delta_0(\omega, \xi, \tau, \mu_1, \Lambda) > 0$, and $B = B(\omega, \xi, \tau, \mu_1, \Lambda) > 0$ such that for any $\delta < \delta_0$, any $t > t_0 + \tau$, and any $\sigma, \rho \in \{+, -\}$,*

$$\left| \bigcup_{L \in \mathcal{L}_{t,old}(\delta)} R_t^{L,\sigma\rho}(\omega\delta) \right| \leq \kappa \left| \bigcup_{L \in \mathcal{L}_{t-\tau}(\delta)} R_{t-\tau}^{L,\sigma\rho}(\omega\delta) \right| + B\delta^{2-\xi}.$$

Step 2 (Contribution of new subspaces). *For any $\omega > 1$ there exist constants $t_0 = t_0(\mu_1, \Lambda)$, $\delta_0 = \delta_0(\mu_1, \Lambda, \omega) > 0$, and $B_1 = B_1(\mu_1, \Lambda, \omega)$ such that for any $\tau > 0$, any $\delta < \delta_0$, any $t > t_0 + \tau$, and any $\sigma, \rho \in \{+, -\}$,*

$$\left| \bigcup_{L \in \mathcal{L}_{t,\text{new}}(\delta)} R_t^{L,\sigma\rho}(\omega\delta) \right| \leq B_1 \delta^{1.05}.$$

Proof of Theorem 2.6 *assuming Step* 1 *and Step* 2. To prove (15) it is clearly enough to estimate the measure of the right-hand side of (27) with fixed σ and ρ. Henceforth we fix σ and ρ and drop the $\pm\pm$ from the notation.

Choose $\xi > 0$ such that $1.05 < 2 - \xi$, and choose τ such that Step 1 holds. Choose $\omega_1 = \omega_1(\tau) > 1$ such that for any $v \in \wedge^2\mathbb{R}^4$ and any s, $0 < s < \tau$, $\omega_1^{-1}\|v\| \leq \|a_s v\| \leq \omega_1\|v\|$. Choose $\omega = c^{-1}\omega_1$, where c is as in Proposition 4.2 (a). Suppose δ is sufficiently small and t is sufficiently large so that both Step 1 and Step 2 hold. In the argument below, all implied constants depend on μ_1, Λ, τ and ω. Let

$$h(t) = \left| \bigcup_{L \in \mathcal{L}_t(\delta/\omega_1)} R_t^L(\omega\delta) \right|,$$

$$h_{\text{old}}(t) = \left| \bigcup_{L \in \mathcal{L}_{t,old}(\delta/\omega_1)} R_t^L(\omega\delta) \right|,$$

and

$$h_{\rm new}(t) = \left| \bigcup_{L\in\mathcal{L}_{t,\rm new}(\delta/\omega_1)} R_t^L(\omega\delta) \right|,$$

so that $h(t) \le h_{\rm old}(t) + h_{\rm new}(t)$. By Step 2, $h_{\rm new}(t) = O(\delta^{1.05})$. By Step 1,

$$h_{\rm old}(t+\tau) \le \kappa h(t) + O(\delta^{2-\xi}) \le \kappa h_{\rm old}(t) + O(\delta^{1.05}). \tag{28}$$

We may assume that δ is small enough so that $h(0) = 0$. Hence, since $\kappa < 1$, (28) implies that for $n \in \mathbb{N}$, $h_{\rm old}(n\tau) = O(\delta^{1.05})$. Then, for $n \in \mathbb{N}$, $h(n\tau) = O(\delta^{1.05})$.

Now let $t > 0$ be arbitrary. We may write $t = n\tau + s$, where $0 < s < \tau$. Then, by the definition of ω_1, for each $L \in \mathcal{L}_t(\delta/\omega_1)$, $L \in \mathcal{L}_{n\tau}(\delta)$, and $R_t^L(c^{-1}\delta) \subset R_{n\tau}^L(\omega\delta)$. Hence, the measure of the right-hand side of (27) is bounded by $h(n\tau) = O(\delta^{1.05})$. This proves (15). □

6. Proof of Step 1

The proof is based on Proposition 4.2 (b); indeed, if the rectangles $R^L(\omega\delta)$ were disjoint, then Step 1 would follow directly from Proposition 4.2 (b). In general, these rectangles are not disjoint, but the overlap is contained in the region where either α_1 or α_3 is large. In fact, we have the following lemma (which follows immediately from Lemma 2.4):

LEMMA 6.1. *Suppose L and M are two distinct 2-dimensional Δ-rational lattices with $d(L) < \delta$, $d(M) < \delta$. Then $\alpha_1(\Delta) > 1/\delta$ or $\alpha_3(\Delta) > 1/\delta$. Hence, by Proposition 4.2 (a),*

$$\begin{aligned} R_t^L(\omega\delta) \cap R_t^M(\omega\delta) \subset \{k \in K \ : \ \alpha_1(a_t k\Lambda) > c/(\omega\delta)\} \\ \cup \{k \in K \ : \ \alpha_3(a_t k\Lambda) > c/(\omega\delta)\}, \end{aligned} \tag{29}$$

where c is as in Proposition 4.2 (a).

Now the measure of the right-hand side of (29) can be bounded by using Theorem 2.5. However, to complete the proof, we will need a certain covering lemma (Lemma 6.4 below).

6.1. *Two covering lemmas.*

Definition 6.2. A rectangle $R \in \mathbb{R}^2$ will be called a quasi-square if the ratio of its sides is between 1/2 and 2.

LEMMA 6.3 (Covering by quasi-squares). *Let $\{S_j\}$ be a countable collection of quasi-squares in $\mathbb{R}^2$, with sides parallel to the coordinate axes, and let $E = \bigcup_j S_j$. Then there is a subcollection $\mathcal{D}$ of pairwise disjoint quasi-squares* *such that* $E \subset \bigcup_{\mathcal{D}} 6S_j$ *and so that the total area of the remaining quasi-squares satisfies*

$$\left| \bigcup_{S \notin \mathcal{D}} S \right| \leq \left| \bigcup_{i \neq j} (6S_i) \cap (6S_j) \right| . \tag{30}$$

Proof. This is well known. The proof consists of ordering the quasi-squares by size, and repeatedly adding to $\mathcal{P}$ the largest quasi-square which is disjoint from the union of the quasi-squares already in $\mathcal{D}$. Note that each remaining quasi-square S is contained in $(6S) \cap (6T)$ where T is a quasi-square already in $\mathcal{D}$. □

LEMMA 6.4 (Covering by rectangles). *Let* $\{R_j\}$ *be a countable collection of rectangles in* $\mathbb{R}^2$ *with sides parallel to the coordinate axes. Suppose*

$$\left| \bigcup_{j \neq k} (6R_j) \cap (6R_k) \right| \leq \varepsilon. \tag{31}$$

Then for $\nu \ll 1$,

$$\left| \bigcup_{j} \nu R_j \right| \leq 42\nu \left| \bigcup_{j} R_j \right| + 21\varepsilon. \tag{32}$$

Proof. Using cuts perpendicular to its longer side, each of the rectangles R_j may be divided into a finite set of quasi-squares. Partitioning these into 21 classes $\mathcal{P}_j^{(k)}$, $k = 0 \dots 20$, so that a quasi-square from class $\mathcal{P}_j^{(k+1)}$ follows a quasi-square from class $\mathcal{P}_j^{(k)}$ (mod 21), we obtain $R_j = \sqcup_{k=0}^{20} \sqcup_{S \in \mathcal{P}_j^{(k)}} S$. For each $1 \leq k \leq 21$ let $\mathcal{P}^{(k)} = \bigcup \mathcal{P}_j^{(k)}$.

If S is a quasi-square $[-a, a] \times [-b, b]$, let $S^\dagger$ denote the "cross" $[-\nu a, \nu a] \times [-b, b] \cup [-a, a] \times [-\nu b, \nu b]$. Clearly $|S^\dagger| \leq 2\nu |S|$, and

$$\bigcup_j (\nu R_j) \subset \bigcup_{j,k} \bigcup_{S \in \mathcal{P}_j^{(k)}} S^\dagger = \bigcup_k \bigcup_{S \in \mathcal{P}^{(k)}} S^\dagger.$$

Hence to estimate $\left| \bigcup_j (\nu R_j) \right|$, we need to estimate $\left| \bigcup_{S \in \mathcal{P}^{(k)}} S^\dagger \right|$ for each k. Note that if $S, T \in \mathcal{P}^{(k)}$ and $S \neq T$, then $(6S) \cap (6T) \subset \bigcup_{i \neq j} (6R_i) \cap (6R_j)$.

Thus, by Lemma 6.3, we obtain a subcollection $\mathcal{D}^{(k)} \subset \mathcal{P}^{(k)}$ of disjoint quasi-squares such that

$$\left| \bigcup_{S \in \mathcal{P}^{(k)} - \mathcal{D}^{(k)}} S \right| \leq \left| \bigcup_{\substack{S, T \in \mathcal{P}^{(k)} \\ S \neq T}} (6S) \cap (6T) \right| \leq \left| \bigcup_{i \neq j} (6R_i) \cap (6R_j) \right| \leq \varepsilon. \tag{33}$$

Observe that $\left|\bigcup_{S\in\mathcal{D}^{(k)}} S^\dagger\right| \le 2\nu \left|\bigcup_{S\in\mathcal{D}^{(k)}} S\right|$. Thus for each k,

$$\left|\bigcup_{S\in\mathcal{P}^{(k)}} S^\dagger\right| \le \left|\bigcup_{S\in\mathcal{D}^{(k)}} S^\dagger\right| + \left|\bigcup_{S\in\mathcal{P}^{(k)}-\mathcal{D}^{(k)}} S^\dagger\right|$$
$$\le 2\nu \left|\bigcup_{S\in\mathcal{D}^{(k)}} S\right| + \left|\bigcup_{S\in\mathcal{P}^{(k)}-\mathcal{D}^{(k)}} S\right|.$$

By (33) the lemma follows. □

6.2. *The proof of Step* 1. We may assume that ω is sufficiently large and δ sufficiently small so that by Proposition 4.2 (b), for $L \in \mathcal{L}_{t,old}(\delta)$, $R_t^L(\omega\delta) \subset Ce^{-\tau}R^L_{t-\tau}(\omega\delta)$. Also by Lemma 6.1 and Theorem 2.5, for any $\xi > 0$,

$$\left|\bigcup_{\substack{L,M\in\mathcal{L}_{t-\tau}(\delta)\\ L\neq M}} R^L_{t-\tau}(\omega\delta)\cap R^M_{t-\tau}(\omega\delta)\right| \le |\{k \;:\; \alpha_1(a_t k\Lambda) > c/(\omega\delta)\}|$$
$$+ |\{k \;:\; \alpha_3(a_t k\Lambda) > c/(\omega\delta)\}| = O(\delta^{2-\xi}),$$

where the implied constant depends only on μ_1, Λ, ξ and ω. Choose $\tau > 0$ such that $Ce^{-\tau/2} < 1/2$, and let $\kappa = 2Ce^{-\tau/2}$. Then, by Lemma 6.4,

$$\left|\bigcup_{L\in\mathcal{L}_{t,old}(\delta)} R_t^L(\omega\delta)\right| \le 2Ce^{-\tau/2}\left|\bigcup_{L\in\mathcal{L}_{t,old}(\delta)} R_{t-\tau}(\omega\delta)\right| + O(\delta^{2-\xi})$$
$$\le \kappa \left|\bigcup_{L\in\mathcal{L}_{t-\tau}(\delta)} R_{t-\tau}(\omega\delta)\right| + O(\delta^{2-\xi}). \qquad \square$$

7. The nesting property

In this section we begin the proof of Step 2. In this section (and the rest of the proof of Step 2) τ and t are fixed; we also assume that t is sufficiently large and δ sufficiently small so that Proposition 4.2 holds. In this section all constants are independent of t and δ, but may depend on μ_1, Λ, ω, τ and the constants in Section 4.

Let $c < 1$ be as in Lemma 4.1. Let $N > 1$ be a parameter to be chosen later (see Proposition 7.4 below), and let $c_0 = c/N^3$. If L is not μ_1-quasinull, then by Lemma 4.1 at least one of the rectangles $R_1^L = R_t^L(c_0\delta^{-0.1}, c_0)$ or $R_2^L = R_t^L(c_0, c_0\delta^{-0.1})$ does not wrap around the torus. We call this rectangle R^L_{big}. We will also use the "intermediate" rectangles $R^L_{\text{int}} = R^L(\delta^{0.1})$.

The strategy of the proof of Step 2 is as follows: For every $L \in \mathcal{L}_{t,\text{new}}(\delta)$, $R^L(\omega\delta) \subset R^L_{\text{int}} \subset R^L_{\text{big}}$. We have the next estimates (which follow immediately from Proposition 4.2 (c)):

Lemma 7.1. *Suppose* $L \in \mathcal{L}_t(\delta)$. *Then,*

$$|R_t^L(\omega\delta)| = O_N\left(\delta^{0.9}|R_{\text{int}}^L|\right), \tag{34}$$

and

$$|R_t^L(\omega\delta)| = O_N\left(\delta^{1.05}|R_{\text{big}}^L|\right). \tag{35}$$

If the rectangles $\{R_{\text{big}}^L \;:\; L \in \mathcal{L}_t(\delta)\}$, were disjoint, Step 2 would follow immediately from Lemma 7.1. However, in general, these rectangles may intersect. Their main combinatorial property is the following:

Lemma 7.2. *Suppose* L *and* M *are two distinct elements in* $\mathcal{L}_t(\delta)$, *and* $R_t^L(\omega\delta)$ *intersects* R_{big}^M. *Then* L *intersects* M *nontrivially* (*i.e.* $\dim(L \cap M) > 0$). *More generally, for any* $N > 1$, *there exists* $\delta_0 = \delta_0(N, \mu_1, \Lambda, \omega)$ *such that if* $\delta < \delta_0$ *and* $R_t^L(\omega\delta)$ *intersects* NR_{big}^M, *then* L *and* M *intersect nontrivially.*

Proof of Lemma 7.2. Pick $k \in R_t^L(\omega\delta) \cap NR_{\text{big}}^M$. Then $d(a_tkL) = O_N(\delta)$, and $d(a_tkM) = O_N(\delta^{-0.1})$. In particular, $d(a_tkL)d(a_tkM) = O_N(\delta^{0.9}) \ll 1$. Hence, by Lemma 2.4 a_tkL and a_tkM intersect nontrivially. Thus, L and M intersect nontrivially. □

Note that Lemma 7.2 can be applied only when a "small" rectangle $R_t^L(\omega\delta)$ intersects a "big" rectangle R_{big}^M; there is no information gained when two "big" rectangles overlap. The following property is useful for overcoming this difficulty:

Definition 7.3 (the nesting property). Let R_j be a collection of rectangles in $\mathbb{R}^2$. This collection has the nesting property if there is a constant $N > 1$ such that for any i, j, $i \neq j$, one of the following holds:

(i) R_i and R_j are disjoint.

(ii) $R_j \subset NR_i$.

(iii) $R_i \subset NR_j$.

For example, collections of squares have the nesting property. However, in general, collections of rectangles do not (unless for example the aspect ratio is bounded).

The main result of this section is the following proposition:

Proposition 7.4 (New subspaces have the nesting property). *There are a constant* $N > 1$, *independent of* t *and* δ, *and a decomposition of* $\mathcal{L}_{t,\text{new}}(\delta)$ *into four disjoint subsets* $\mathcal{L}_t^q(\delta)$ *such that for each* q, *the collections* $\{R_{\text{int}}^L \;:\; L \in \mathcal{L}_t^q(\delta)\}$ *and* $\{R_{\text{big}}^L \;:\; L \in \mathcal{L}_t^q(\delta)\}$ *both have the nesting property.*

Proposition 7.4 is the main advantage of working with $\mathcal{L}_{t,\text{new}}(\delta)$ instead of $\mathcal{L}_t(\delta)$. Before beginning the proof of Proposition 7.4 we state a corollary:

Corollary 7.5. *Suppose* $L, M \in \mathcal{L}_t^q(\delta)$ *and* R^L_{big} *intersects* R^M_{big}. *Then* L *intersects* M *nontrivially.*

Proof of Corollary 7.5. Suppose R^L_{big} intersects R^M_{big}. Using Proposition 7.4 we may assume without loss of generality that $R^L_{\mathrm{big}} \subset NR^M_{\mathrm{big}}$; hence $R^L_t(\omega\delta) \subset NR^M_{\mathrm{big}}$. Now the corollary follows from Lemma 7.2. □

We now start the proof of Proposition 7.4, first noting the following:

Lemma 7.6. *Suppose* $L \in \mathcal{L}_{t,\mathrm{new}}(\delta)$. *Then* $M_t(L) \approx \delta$ *(where* $M_t(L)$ *is as defined in* §4).

Proof. Since $L \in \mathcal{L}_t(\delta)$, $M_t(L) \leq \delta$. Since $L \neq \mathcal{L}_{t-\tau}(\delta)$, $M_{t-\tau}(L) \geq \delta$; then $M_t(L) \geq \delta/\|a_\tau\|$. □

The advantage of $\mathcal{L}_{t,\mathrm{new}}(\delta)$ compared to $\mathcal{L}_t(\delta)$ is the following:

Lemma 7.7 (Fixed side). *Suppose* $L \in \mathcal{L}_{t,\mathrm{new}}(\delta)$. *Then for any* $\eta \gg \delta$, *the rectangle* $R^L_t(\eta)$ *has at least one side of length* $\approx e^{-t}\delta^{-1/2}\eta^{1/2}$. *For any* $\eta_1, \eta_2 \gg \delta$, $R^L_t(\eta_1, \eta_2)$ *has at least one side of length either* $\approx e^{-t}\delta^{-1/2}\eta_1^{1/2}$ *or* $\approx e^{-t}\delta^{-1/2}\eta_2^{1/2}$.

Proof. Since $L \in \mathcal{L}_t(\delta)$, $M_t(L) \leq \delta$. Since $L \notin \mathcal{L}_{t-\tau}(\delta)$, $M_{t-\tau}(L) > \delta$. Hence the second alternative of part (d) of Proposition 4.2 cannot hold. Hence the first alternative holds, and thus $R^L_{t-\tau}(\eta)$ has a side of length $\approx e^{-t}M_{t-\tau}(L)^{-1/2}\eta^{1/2} \approx e^{-t}\delta^{-1/2}\eta^{1/2}$. Thus the same holds for $R^L_t(\eta)$. The proof of the second assertion is identical. □

Proof of Proposition 7.4. We only prove the assertion for the "big" rectangles; the proof for the "intermediate" rectangles is identical. Suppose $L \in \mathcal{L}_{t,\mathrm{new}}(\delta)$. By Lemma 7.7, there exist constants c_3, c_4 depending only on μ_1, Λ, ω and τ such that R^L_{big} either has a side of length between $c_3e^{-t}\delta^{-0.55}$ and $c_4e^{-t}\delta^{-0.55}$ or has a side of length between $c_3e^{-t}\delta^{-1/2}$ and $c_4e^{-t}\delta^{-1/2}$; this side can be either "vertical" or "horizontal". Thus there are four possibilities for each L. For $1 \leq q \leq 4$ let $\mathcal{L}_t^q(\delta)$ denote the set of $L \in \mathcal{L}_{t,\mathrm{new}}(\delta)$ for which the q'th possibility occurs. Now let $N = 2c_4/c_3$; the proposition follows. □

We will also use the following version of the nesting property:

Lemma 7.8 (the nesting lemma). *Suppose* $L_1, \dots L_m \in \mathcal{L}_t^q(\delta)$. *Then one of the following holds*:

(a) $\bigcap_{j=1}^m R^{L_j}_{\mathrm{big}} = \emptyset$.

(b) *The sets can be reordered so that for* $1 \leq i < j \leq m$,

$$R^{L_i}_{\mathrm{big}} \subset NR^{L_j}_{\mathrm{big}}. \tag{36}$$

Proof. In view of Lemma 7.7, one can just order the rectangles R^L_{big} by increasing size. $\square$

8. Proof of Step 2

In this section all implied constants depend on μ_1, Λ, τ, ω and N (and hence ultimately only on μ_1 and Λ).

8.1. *Preliminaries.* In view of Corollary 7.5, if some point belongs to many sets of the form $R^{L_i}_{\text{big}}$ where all $L_i \in \mathcal{L}^q_t(\delta)$, then for $i \neq j$, $\dim(L_i \cap L_j) = 1$. The following lemma is immediately obvious:

LEMMA 8.1. *Suppose $L_1, L_2, \dots, L_m$ are distinct two-dimensional subspaces of $\mathbb{R}^4$ and for $1 \le i < j \le m$, $\dim(L_i \cap L_j) = 1$. Then at least one of the following holds*:

(a) $\dim(L_1 \cap L_2 \cap \cdots \cap L_m) = 1$.

(b) $\dim(L_1 + L_2 + \cdots + L_m) = 3$. $\square$

Thus, if many rectangles R^{L_j} have a common point, then either all the L_j have a common vector, or all the L_j lie in a common 3-dimensional subspace, or both. In view of this, we will need some information about the sets $F(\eta) = \{k : d(a_t kH) < \eta\}$, where H is either a vector or a three-dimensional subspace. Thus, we will use the following lemma:

LEMMA 8.2. *Let H be a Λ-rational subspace of dimension 1 or 3. For $\eta > 0$ let $F(\eta)$ denote the set $\{k \in K : d(a_t kH) < \eta\}$. Suppose R_1 and R_2 are disjoint rectangles such that for $i = 1, 2$, $|R_i \cap F(\eta)| \ge 0.9|R_i|$, and let $\lambda > 1$ be such that $3\lambda R_1$ and $3\lambda R_2$ are still disjoint and of diameter at most $\pi/8$. Then, for some $i \in \{1, 2\}$,*

$$|R_i| \le c_1 \lambda^{-1} |(\lambda R_i) \cap F(c_2 \eta)|,$$

where $c_1 > 0$ and $c_2 > 1$ depend only on Λ.

Proof. See Appendix B. $\square$

8.2. *Proof of Step* 2. *Notational convention.* In this subsection, we use the notation $\alpha_{13}(\Delta) = \max(\alpha_1(\Delta), \alpha_3(\Delta))$. Let $\mathcal{L}^q_t(\delta)$ be as in Section 7. We fix q.

Definition 8.3 (The class Ω_0). Let $\Omega_0 \subset \mathcal{L}^q_t(\delta)$ denote the subspaces $L \in \mathcal{L}^q_t(\delta)$ such that

$$R^L_t(\omega\delta) \subset \{k \in K : \alpha_{13}(a_t k \Lambda) > b_0 \delta^{-0.55}\}, \tag{37}$$

where b_0 is a constant to be chosen later, depending only on ω, τ, μ_1, Λ and N (and independent of t and δ).

LEMMA 8.4 (The contribution of Ω_0).

$$\left|\bigcup_{L\in\Omega_0} R_t^L(\omega\delta)\right| \leq \left|\{k \in K \ : \ \alpha_{13}(a_t k\Lambda) \geq b_0\delta^{-0.55}\}\right| = O(\delta^{1.05}).$$

Proof. This follows immediately from (37) and Theorem 2.5. □

LEMMA 8.5 (Intermediate rectangles are disjoint). *Suppose $L, M \in \mathcal{L}_t^q(\delta)$ - Ω_0, and $L \neq M$. Then R_{int}^L and R_{int}^M are disjoint.*

Proof. Suppose not. Then by Proposition 7.4, we may assume that $R_{\text{int}}^L \subset NR_{\text{int}}^M$. Then $R_t^L(\omega\delta) \subset NR_{\text{int}}^M$. Let $k \in R_t^L(\omega\delta) \subset NR_{\text{int}}^M$ be arbitrary. Then $d_{a_tk\Lambda}(a_tkL) = O(\delta)$ and $d_{a_tk\Lambda}(a_tkM) = O(\delta^{0.1})$. Hence

$$d_{a_tk\Lambda}(a_tkL)\, d_{a_tk\Lambda}(a_tkM) = O(\delta^{1.1}).$$

Then by Lemma 2.4,

$$d_{a_tk\Lambda}(a_tk(L\cap M))\, d_{a_tk\Lambda}(a_tk(L+M)) = O(\delta^{1.1}).$$

Thus, either $d_{a_tk\Lambda}(a_tk(L\cap M)) = O(\delta^{0.55})$ or $d_{a_tk\Lambda}(a_tk(L+M)) = O(\delta^{0.55})$. In either case, $\alpha_{13}(a_tk\Lambda) \geq b_0\delta^{-0.55}$ as long as b_0 is sufficiently small. Since $k \in R_t^L(\omega\delta)$ is arbitrary, this implies that L belongs to Ω_0 which is a contradiction. □

We now choose b_0 in Definition 8.3 so that Lemma 8.5 holds.

Definition 8.6 (The class Ω_1). Let $\Omega_1 \subset \mathcal{L}_t^q(\delta)$ - Ω_0 denote the set of subspaces $L \in \mathcal{L}_t^q(\delta)$ - Ω_0 such that

$$|R_t^L(\omega\delta) \cap \{k \in K \ : \ \alpha_{13}(a_tk\Lambda) > b_1\delta^{-0.55}\}| \geq 0.1|R_t^L(\omega\delta)|, \tag{38}$$

where b_1 is a constant to be chosen later, depending only on ω, τ, μ_1, Λ and N (and independent of t and δ).

LEMMA 8.7 (The contribution of Ω_1).

$$\left|\bigcup_{L\in\Omega_1} R_t^L(\omega\delta)\right| = O(\delta^{1.05}).$$

Proof. In view of Lemma 8.5 and the fact that $R_t^L(\omega\delta) \subset R_{\text{int}}^L$, the rectangles $\{R_t^L(\omega\delta) \ : \ L \in \Omega_1\}$ are pairwise disjoint. Now the estimate follows immediately from (38) and Theorem 2.5. □

The following lemma is a refinement of Lemma 7.2:

LEMMA 8.8. *Suppose* $L, M \in \mathcal{L}_t^q(\delta)\text{-}\Omega_0\text{-}\Omega_1$, $L \neq M$ *and* $R_t^L(\omega\delta) \subset NR_{\text{big}}^M$. *Then* $\dim(L \cap M) = 1$, *and for all*

$$k \in R_t^L(\omega\delta) \cap \{k \in K \ : \ \alpha_{13}(a_t k\Lambda) \leq b_1\delta^{-0.55}\},$$

$$d_{a_t k\Lambda}(a_t k(L \cap M)) = O(\delta^{0.35}) \qquad \text{and} \qquad d_{a_t k\Lambda}(a_t k(L + M)) = O(\delta^{0.35}).$$

Proof. Suppose $k \in R_t^L(\omega\delta)$. Then $d_{a_t k\Lambda}(a_t kL) = O(\delta)$ and $d_{a_t k\Lambda}(a_t kM) = O(\delta^{-0.1})$. Hence, by Lemma 2.4,

$$d_{a_t k\Lambda}(a_t k(L \cap M))\, d_{a_t k\Lambda}(a_t k(L + M)) = O(\delta^{0.9}). \tag{39}$$

Hence, at least one of the factors in (39) is $O(\delta^{0.45}) \ll 1$. This implies that $a_t kL$ and $a_t kM$ intersect nontrivially; thus L and M intersect nontrivially, and so $\dim(L \cap M) = 1$.

Now suppose $k \in R_t^L(\omega\delta) \cap \{k \ : \ \alpha_{13}(a_t k\Lambda) \leq b_1\delta^{-0.55}\}$. Then,

$$d_{a_t k\Lambda}(a_t k(L \cap M)) \geq \alpha_3^{-1}(a_t k\Lambda) \geq b_1^{-1}\delta^{0.55}, \tag{40}$$

and

$$d_{a_t k\Lambda}(a_t k(L + M)) \geq \alpha_1^{-1}(a_t k\Lambda) \geq b_1^{-1}\delta^{0.55}. \tag{41}$$

From (39), (40) and (41) it follows that for $k \in R_t^L(\omega\delta) \cap \{k \ : \ \alpha_{13}(a_t k\Lambda) \leq b_1\delta^{-0.55}\}$,

$$d_{a_t k\Lambda}(a_t k(L \cap M)) = O(\delta^{0.35}) \qquad \text{and} \qquad d_{a_t k\Lambda}(a_t k(L + M)) = O(\delta^{0.35}). \tag{42}$$

But, since $L \notin \Omega_1$, $|R_t^L(\omega\delta) \cap \{k \ : \ \alpha_{13}(a_t k\Lambda) \leq b_1\delta^{-0.55}\}| \geq 0.9|R_t^L(\omega\delta)|$. $\square$

We now choose b_1 in Definition 8.6 so that Lemma 8.8 holds.

Definition 8.9 (The class Ω_2). Let $\Omega_2 \subset \mathcal{L}_t^q(\delta)\text{-}\Omega_0\text{-}\Omega_1$ denote the set of subspaces $L \in \mathcal{L}_t^q(\delta)\text{-}\Omega_0\text{-}\Omega_1$ such that

$$|R_t^L(\omega\delta)| \leq b_3\delta^{0.45}|R_{\text{int}}^L \cap \{k \in K \ : \ \alpha_{13}(a_t k\Lambda) \geq b_4\delta^{-0.35}\}|,$$

where the constants $b_3 < \infty$ and $b_4 > 0$ are to be chosen later, depending only on ω, τ, μ_1, Λ and N (and independently of t and δ).

LEMMA 8.10 (The contribution of Ω_2).

$$\left| \bigcup_{L \in \Omega_2} R_t^L(\omega\delta) \right| = O(\delta^{1.05}).$$

Proof. This is a formal consequence of Lemma 8.5 and Theorem 2.5. $\square$

LEMMA 8.11. *Suppose* $L_1, L_2, L_3 \in \mathcal{L}_t^q(\delta) - \Omega_0 - \Omega_1 - \Omega_2$. *Then* $R_{\rm big}^{L_1} \cap R_{\rm big}^{L_2} \cap R_{\rm big}^{L_3} = \emptyset$.

Proof. Suppose not. Then by Lemma 7.8, after possibly renumbering the L_j, we may assume that for $1 \leq j < k \leq 3$, $R_{\rm big}^{L_j} \subset NR_{\rm big}^{L_k}$. Then by the first assertion of Lemma 8.8 (or by Lemma 7.2), for $1 \leq j < k \leq 3$, $\dim(L_j \cap L_k) = 1$. Now by Lemma 8.1, either $\dim(L_1 \cap L_2 \cap L_3) = 1$ or $\dim(L_1 + L_2 + L_3) = 3$ (or both). Without loss of generality, we may assume that the former holds; then let $H = L_1 \cap L_2 \cap L_3$.

Let $R_j = R_t^{L_j}(\omega\delta)$. Since for $j = 1, 2$, $R_j \subset NR_{\rm big}^{L_3}$, by Lemma 8.8 for every $k \in R_j$ either $\alpha_{13}(a_t k\Lambda) > b_1\delta^{-0.55}$ or $d_{a_t k\Lambda}(a_t kH) \leq b_5\delta^{0.35}$ where b_5 is the implied constant in Lemma 8.8. Since $L_j \notin \Omega_1$, the set $R_j \cap \{k : \alpha_{13}(a_t k\Lambda) \leq b_1\delta^{-0.55}\}$ is of area at least $0.9|R_j|$, and for each of its points k, $d_{a_t k\Lambda}(a_t kH) \leq b_5\delta^{0.35}$. Hence $|R_j \cap \{k : d_{a_t k\Lambda}(a_t kH) \leq b_5\delta^{0.35}\}| \geq 0.9|R_j|$.

We now choose λ to be the largest number such that $3\lambda R_j \subset R_{\rm int}^{L_j}$. Then by Lemma 8.5, $3\lambda R_1$ and $3\lambda R_2$ are disjoint. In view of the construction of the rectangles $R^L(\eta)$ we have $\lambda^{-1} = O(\delta^{0.45})$. Observe that the diameters of $3\lambda R_j$ are smaller than $\pi/8$. We may now apply Lemma 8.2 with $\eta = b_5\delta^{0.35}$ and conclude (as long as b_3 in Definition 8.9 is sufficiently big and b_4 in Definition 8.9 is sufficiently small), that for some $j \in \{1, 2\}$, $L_j \in \Omega_2$. This is a contradiction. $\square$

We now choose b_3, b_4 in Definition 8.9 so that Lemma 8.11 holds.

LEMMA 8.12 (The contribution of the rest of the subspaces). *Let* $\mathcal{L}'$ *denote* $\mathcal{L}_t^q(\delta) - \Omega_0 - \Omega_1 - \Omega_2$. *Then*

$$\left| \bigcup_{L \in \mathcal{L}'} R_t^L(\omega\delta) \right| = O(\delta^{1.05}).$$

Proof. In view of Lemma 8.11, the rectangles $\{R_{\rm big}^L : L \in \mathcal{L}'\}$ cover each point at most twice. Now the estimate follows from Lemma 7.1. $\square$

Now Step 2 follows immediately from Lemmas 8.4, 8.7, 8.10 and 8.12. This completes the proof of Theorem 2.6. $\square$

9. Quasinull subspaces and rectangular tori

For the case of the rectangular torus of sides π and π/β, the eigenvalues are values at integers of the binary quadratic form $q_\beta(m, n) = m^2 + \beta^2 n^2$. Let $Q_\beta(x_1, x_2, x_3, x_4) = q_\beta(x_1, x_2) - q_\beta(x_3, x_4)$. In this case we have four isotropic subspaces, i.e. $\{x_1 = \pm x_3, x_2 = \pm x_4\}$.

Let $B(\cdot)$ be the standard form as defined in Section 2. Then $Q_\beta(v) = B(g_{Q_\beta} v)$, where g_{Q_β} is given by the change of variable $z_1 = x_1 - x_3$, $z_2 = \beta(x_2 - x_4)$, $z_3 = \beta(x_2 + x_4)$, $z_4 = x_1 + x_3$. Let $\Lambda_\beta = g_{Q_\beta}\mathbb{Z}^4$; then Λ_β consists of the vectors $(w_1, \beta w_2, \beta w_3, w_4)$, with $w_i \in \mathbb{Z}$, and also $w_1 \pm w_4 \in 2\mathbb{Z}$, $w_2 \pm w_3 \in 2\mathbb{Z}$. By construction, $Q_\beta(\mathbb{Z}^4) = B(\Lambda_\beta)$.

In this section, we give an elementary proof of Theorem 1.5 for the special case of rectangular tori (i.e. $Q = Q_\beta$). We first make the following:

Definition 9.1 (Exceptional subspaces). Fix a constant $0 < \mu < 1/2$. A subspace L of $\mathbb{R}^4$ is called *exceptional* if $\dim L = 2$ and L spanned by the vectors $u_1 = (A, B, -A, B)$ and $u_2 = (C, D, C, -D)$ or by the vectors $u_1 = (A, B, -A, -B)$ and $u_2 = (C, D, C, D)$ where $A, B, C, D \in \mathbb{Z} - \{0\}$ satisfy the condition

$$\left| \frac{AD}{BC} - \beta^2 \right| < \frac{\mu}{(BC)^2}. \tag{43}$$

Observe that (43) implies that AD/BC is a convergent p_n/q_n in the continued fraction expansion of b^2. Note also that n is determined by L and let us denote $\Phi(L) = n$.

Note that $Q_\beta(u_1) = Q_\beta(u_2) = 0$, but the restriction of Q_β to L is nonzero. In fact,

$$Q_\beta(\lambda_1 u_1 + \lambda_2 u_2) = (AD - \beta^2 BC)\lambda_1\lambda_2. \tag{44}$$

LEMMA 9.2 (Quasinull implies exceptional). *Let Q_β, g_{Q_β}, Λ_β be as in the beginning of this section. There exist constants $\mu_1 > 0$, $c > 1$ (depending only on μ in Definition 9.1 and β) such that the following holds: Let L be any Λ_β-rational subspace with $\|v^L\| > c$ which is μ_1-quasinull with respect to Λ_β. Then the $\mathbb{Z}^4$-rational subspace $g_{Q_\beta}^{-1} L$ is an exceptional subspace in the sense of Definition 9.1.*

Proof. Let $v_1 = (w_1, \beta w_2, \beta w_3, w_4)$ and $v_2 = (w_1', \beta w_2', \beta w_3', w_4')$ be a reduced integral basis for $L \cap \Lambda_\beta$. From the definition of Λ_β, $w_1, w_2, w_3, w_4 \in \mathbb{Z}$ and $w_1', w_2', w_3', w_4' \in \mathbb{Z}$. Since the basis is reduced, $\|v^L\| = \|v_1 \wedge v_2\| \geq c_0\|v_1\|\|v_2\|$ where c_0 is an absolute constant. Let $\mu_1 = c_0\mu/4$ and let $c = 4\mu_1/\beta$.

Since $\|v^L\| = \|\pi_1(v^L)\| + \|\pi_2(v^L)\| > c$, (8) implies that either

$$\|\pi_1(v^L)\| < \frac{2\mu_1}{\|v^L\|} < \frac{\beta}{2}, \tag{45}$$

or

$$\|\pi_2(v^L)\| < \frac{2\mu_1}{\|v^L\|} < \frac{\beta}{2}. \tag{46}$$

Without loss of generality, we may assume that (46) holds. Using Lemma 2.1, we get

$$\begin{aligned}(47)\quad \pi_2(v^L) = {}&\beta(w_1w_3' - w_3w_1')e_{11}\wedge e_{21} + \beta(w_2w_4' - w_4w_2')e_{12}\wedge e_{22}\\ &+\frac{1}{2}((w_1w_4' - w_4w_1') + \beta^2(w_2w_3' - w_3w_2'))(e_{11}\wedge e_{22} + e_{12}\wedge e_{21}).\end{aligned}$$

Considering the coefficient of $e_{11}\wedge e_{21}$ we get $\beta|w_1w_3' - w_3w_1'| < \beta$. Hence $|w_1w_3' - w_3w_1'| < 1$. But $w_1w_3' - w_3w_1' \in \mathbb{Z}$, thus $w_1w_3' - w_3w_1' = 0$. Similarly, by considering the coefficient of $e_{12}\wedge e_{22}$ we get $w_2w_4' - w_4w_2' = 0$. Thus, v_1 and v_2 may be written as

$$(48)\qquad \begin{aligned} v_1 &= \lambda_1(y_1, 0, \beta y_3, 0) + \lambda_2(0, y_2, 0, \beta y_4) \text{ and}\\ v_2 &= \lambda_1'(y_1, 0, \beta y_3, 0) + \lambda_2'(0, y_2, 0, \beta y_4),\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_1', \lambda_2', y_1, y_2, y_3, y_4 \in \mathbb{Z}$. Substituting into the coefficient of $e_{11}\wedge e_{22} + e_{12}\wedge e_{21}$ in (47) and using (46) we get

$$|(\lambda_1\lambda_2' - \lambda_2\lambda_1')(y_1y_4 + \beta^2 y_2y_3)| < \frac{4\mu_1}{\|v^L\|}.$$

Since $\lambda_1\lambda_2' - \lambda_2\lambda_1' \in \mathbb{Z}$ and $\lambda_1\lambda_2' - \lambda_2\lambda_1' \neq 0$ (otherwise v_1 and v_2 would be linearly dependent), $|\lambda_1\lambda_2' - \lambda_2\lambda_1'| \geq 1$. Also $\|v^L\| \geq c_0\|v_1\|\|v_2\| \geq c_0|y_2y_3|$. Hence

$$(49)\qquad |y_1y_4 + \beta^2 y_2y_3| < \frac{4\mu_1}{c_0|y_2y_3|} < \frac{\mu}{|y_2y_3|}.$$

Now the lemma follows from (48), (49) and the definition of g_{Q_β}. $\square$

LEMMA 9.3 (Contribution of exceptional lattices). *Suppose* β^2 *is* diophantine, *i.e. there exists* $N > 0$ *such that for all relatively prime pairs of integers* (p, q), $|\beta^2 - p/q| > q^{-N}$. *Then, for any interval* (a, b), $\limsup_{T\to\infty} \frac{1}{T^2} X_\beta(a, b, T) = 0$, *where* X_β *counts the points in the exceptional subspaces.*

Proof. Let p_n/q_n denote the continued fraction approximations to β^2. Then, since β^2 is diophantine, $\varepsilon_n = |\beta^2 - p_n/q_n| > q_n^{-N}$ for all n.

Suppose L is an exceptional subspace. Without loss of generality, we may assume that L is spanned by the vectors $u_1 = (A, B, -A, B)$ and $u_2 = (C, D, C, -D)$ where $A, B, C, D \in \mathbb{Z}$ and because of (43), $(AD)/(BC) = p_n/q_n$ for $n = \Phi(L)$. Hence, there exists $\nu > 0$ such that $AD = \nu p_n$ and $BC = \nu q_n$.

Let $N_L(a, b, T)$ denote the number of nondiagonal solutions in L, i.e the number of vectors $v = \lambda_1u_1 + \lambda_2u_2$ with $\lambda_1, \lambda_2 \in \mathbb{Z} - \{0\}$ such that $\|v\| < T$ and $a < Q_\beta(v) < b$.

Note that u_1 and u_2 are orthogonal. Since we are considering nondiagonal solutions both λ_1 and λ_2 are nonzero. Then the condition that $\|v\| < T$ implies (by Schmidt's lemma) that $N_L < T^2/(\|u_1\|\|u_2\|) \ll T^2/(\nu q_n)$.

By (44), the quadratic form restricted to L is $Q(\lambda_1 u_1 + \lambda_2 u_2) = (AD - \beta^2 BC)\lambda_1\lambda_2 = \nu q_n \varepsilon_n \lambda_1 \lambda_2$, where $\varepsilon_n = \frac{AD}{BC} - \beta^2$ satisfies $\|\varepsilon_n\| > q_n^{-N}$. Thus, if there exists a nondiagonal solution, (i.e. with $|\lambda_1\lambda_2| \geq 1$), we must have $\nu q_n \varepsilon_n < \rho$ where $\rho = \max(|a|, |b|)$. Hence we need only consider exceptional subspaces with $\nu < \rho/(q_n\varepsilon_n)$. The total contribution $N_n = \sum_{\Phi(L)=n} N_L$ is bounded by

$$
\begin{aligned}
(50) \qquad N_n(a,b,T) &= \sum_{\Phi(L)=n} N_L(a,b,T) \\
&\leq \sum_{\nu=1}^{\rho/(q_n\varepsilon_n)} \sum_{\substack{AD=\nu p_n \\ BC=\nu q_n}} \frac{T^2}{\nu q_n} \leq \sum_{\nu=1}^{\rho/(q_n\varepsilon_n)} \tau(\nu p_n)\tau(\nu q_n)\frac{T^2}{\nu q_n},
\end{aligned}
$$

where τ is the divisor function. Using the estimate $\tau(q) < c_\varepsilon q^\varepsilon$ which holds for any $\varepsilon > 0$ as well as the diophantine condition on β by which $\|\varepsilon_n\| > q_n^{-N}$, one deduces that

$$
N_n(a,b,T) \leq C_\varepsilon \frac{T^2}{q_n^{1-(2N+4)\varepsilon}}, \tag{51}
$$

where C_ε is independent of T and n. Fix $\varepsilon < 1/(2N+4)$ so that $\theta = 1 - (2N+4)\varepsilon > 0$.

Also note that because of the form of Q_β restricted to L, any N_n is bounded by a constant R_n independent of T. Pick any $M > 0$. Then

$$
\begin{aligned}
\limsup_{T\to\infty} \frac{1}{T^2} X_n(a,b,T) &= \limsup_{T\to\infty} \left(\frac{1}{T^2} \sum_{n=1}^{M-1} N_n(a,b,T) + \frac{1}{T^2} \sum_{n=M}^{\infty} N_n(a,b,T) \right) \\
&\leq \limsup_{T\to\infty} \frac{1}{T^2} \sum_{n=1}^{M-1} R_n + C_\varepsilon \sum_{n=M}^{\infty} q_n^{-\theta} \leq C_\varepsilon \sum_{n=M}^{\infty} q_n^{-\theta}.
\end{aligned}
$$

Letting $M \to \infty$ we obtain the desired estimate, since the sum $\sum_{n=1}^{\infty}(1/q_n^\theta)$ converges by the properties of continued fractions. □

Now Theorem 1.5, in the special case $Q = Q_\beta$, follows from Lemma 9.2 and Lemma 9.3. This completes the proof of Theorem 1.3 in the special case of rectangular tori.

10. The contribution of quasinull subspaces

10.1. *Structural preliminaries.*

LEMMA 10.1. *Given three transversal* 2*-dimensional subspaces* V_1, V_2 *and* V_3 *of* $\mathbb{R}^4$, *there is a unique* (*up to proportionality*) *quadratic form* Q *such that the restriction of* Q *to each* V_i *is zero. If the subspaces are defined over* $\mathbb{Q}$ *then* Q *is rational and split.*

Proof. We may choose a basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$ such that $\{e_1, e_2\}$ is a basis for V_1 and $\{e_3, e_4\}$ is a basis for V_2. Now the (symmetric) matrix of Q in this basis is of the form $\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}$. The remaining entries of this matrix are determined up to proportionality by the three linear equations implied by the condition that the restriction of Q to V_3 is zero. □

PROPOSITION 10.2. *Let Q be a quadratic form of signature $(2, 2)$. Then $\wedge^2\mathbb{R}^4$ decomposes into two invariant subspaces V_1 and V_2.*

(a) *The subspaces V_1 and V_2 are orthogonal with respect to the 6-variable bilinear form $Q^{(6)}(v, w) = v \wedge w$. The restriction of $Q^{(6)}$ to V_i has signature $(2, 1)$.*

(b) *The pair of subspaces (V_1, V_2) determines Q up to proportionality.*

(c) *The map f taking (V_1, V_2) to Q/proportionality is a rational map defined over $\mathbb{Q}$.*

(d) *If V_1 is rational and the restriction of $Q^{(6)}$ to V_1 is split over $\mathbb{Q}$ then $Q = f(V_1, V_1^{\perp})$ is a split form defined over $\mathbb{Q}$.*

Proof of (a). Follows from Lemma 2.1.

Proof of (b). If V_1 is given, then V_2 is determined by (a). This implies that all the isotropic 2-dimensional subspaces of Q are given. Since any null vector of Q is the intersection of two isotropic 2-subspaces, we have that the null cone of Q is determined. Hence Q is determined up to proportionality.

Proof of (c). The rationality of f is clear from the construction. The fact that f is defined over $\mathbb{Q}$ follows from the fact that f is bijective. Note that the pair $(V_1, V_1^{\perp})$ may be defined over $\mathbb{Q}$ even if V_1 is not rational.

Proof of (d). The restriction of $Q^{(6)}$ to V_1 is split over $\mathbb{Q}$ if and only if there exists a basis for V_1 consisting of null-vectors for $Q^{(6)}$. Now the statement follows from Lemma 10.1. □

LEMMA 10.3. *Let Q be an irrational quadratic form of signature $(2, 2)$. Then Q has at most four rational isotropic subspaces.*

Proof. Suppose Q has at least five rational isotropic subspaces. Let N denote the light cone of $Q^{(6)}$. Then for some i, $N \cap V_i$ has at least three linearly independent rational points, and thus the lemma follows from Lemma 10.1. □

10.2. *The number of quasinull subspaces.* For a rational subspace $L \subset \mathbb{R}^4$, let the norm of L denote the length of $v^L = v_1 \wedge v_2 \in \wedge^2\mathbb{R}^4$, where $\{v_1, v_2\}$ is any integral basis for $L \cap \mathbb{Z}^4$. The norm is equal to the volume of $L/(L \cap \mathbb{Z}^4)$.

THEOREM 10.4. *Suppose Q is not* EWAS (*see Definition* 1.2). *Then there exists $\delta > 0$ such that the number of quasinull subspaces of norm between $T/2$ and T is $O(T^{1-\delta})$.*

To prove Theorem 10.4 we work in $\wedge^2\mathbb{R}^4 \cong \mathbb{R}^6$. Instead of counting quasinull subspaces $L \subset \mathbb{R}^4$, we count the associated vectors $v^L \in \wedge^2\mathbb{R}^4$. The following obvious lemma summarizes the properties of this transition.

LEMMA 10.5. *Given $\mu_1 > 0$ there exists $c > 0$ such that for any μ_1-quasinull subspace L of norm between $T/2$ and T, the corresponding primitive vector $w = v^L \in \wedge^2\mathbb{R}^4 \cong \mathbb{R}^6$ satisfies the following conditions*:

(a) $Q^{(6)}(w) = 0$ (*recall $Q^{(6)}(w)$ denotes $w \wedge w$*).

(b) *For some $i \in \{1, 2\}$, $\|\pi_i(w)\| < c/T$ (where π_i is the orthogonal projection on V_i).*

(c) $T/2 \leq \|w\| \leq T$.

Conversely, the above conditions, with an appropriate choice of $c = c(\mu_1)$, imply that w corresponds to a 2-dimensional quasinull subspace of $\mathbb{R}^4$ of norm between $T/2$ and T. □

Hence to prove Theorem 10.4, it suffices to count the number of vectors in $\mathbb{R}^6$ satisfying (a)–(c) of Lemma 10.5. Without loss of generality we may assume $i = 2$.

Define

$$Q^{(3)}(v) = Q^{(6)}(\pi_1(v)).$$

Note that for any vector w satisfying (a)–(c) of Lemma 10.5,

$$|Q^{(3)}(w)| < \text{const.}$$

Let $g_T \in \mathrm{GL}(6, \mathbb{R})$ denote the linear transformation which is the identity on V_1 and stretches by a factor of T on V_2. Note that g_T preserves $Q^{(3)}$. Let Δ_T denote the image of $\mathbb{Z}^6$ under g_T.

Thus it suffices to estimate the number of primitive vectors $w \in \Delta_T$ satisfying the following conditions:

(a′) $|Q^{(3)}(w)| < \text{const.}$

(b′) $\|\pi_2(w)\| < \text{const}'$.

(c′) $T/2 \leq \|w\| \leq T$.

Let $H_1 \subset \mathrm{SL}(6,\mathbb{R})$ denote the group which acts via the identity on V_2 and preserves $Q^{(3)}$. Then $H_1 \cong \mathrm{SO}(2,1)$. Let $\{w_1, w_2, w_3\}$ be a basis for V_1 in which $Q^{(3)}(xw_1 + yw_2 + zw_3) = 2xz - y^2$. Let $a_t \in H_1$ be defined by

$$\begin{aligned} a_t w_1 &= e^t w_1, \\ a_t w_2 &= w_2, \\ a_t w_3 &= e^{-t} w_3. \end{aligned}$$

Let $K \cong \mathrm{SO}(2)$ be the intersection of H_1 with the group preserving the norm $\|xw_1+yw_2+zw_3\| = (x^2+y^2+z^2)^{1/2}$. Then K is a maximal compact subgroup of H_1.

Proposition 10.6. *There exists a bounded function f supported on a compact set in $\mathbb{R}^6$ such that the number of vectors in Δ_T satisfying* (a′)–(c′) *is bounded above by*

$$\int_K \hat{f}(a_t k \Delta_T)\, dm(k),$$

where $dm(k)$ is the normalized Haar measure on K, $t = \log T$, and

$$\hat{f}(\Delta) = \sum_{v\in\Delta} f(v).$$

Proof. See [EMM, §3], and also the paragraph following the statement of Theorem 2.3 in the present paper. □

Recall the definitions of the functions α_i and α from Section 2. We note the following:

Lemma 10.7.

(i) $\alpha_6(\Delta_T) = T^{-3}$.

(ii) *For any j, $1 \le j \le 6$, $\alpha_j(\Delta_T) \le 1$.*

(iii) *There is an absolute constant c such that for any j, $2 \le j \le 6$, $\alpha_j(\Delta_T) \le c\alpha_{j-1}(\Delta_T)$.*

(iv) *There is a constant $C < \infty$ such that $\alpha_4(\Delta_T) \le CT^{-1/3}$.*

Proof. (i) is immediate from the fact that $\alpha_6(\Lambda) = 1$ and the definition of Δ_T. (ii) and (iii) are also clear from the definitions.

To prove (iv) let $\tilde{W}$ denote the Δ_T-rational 4-dimensional subspace such that $1/d_4(\tilde{W}) = \alpha_4(\tilde{W})$, and let $W = g_T^{-1}(\tilde{W})$ denote the corresponding rational subspace. Suppose $d_4(\tilde{W}) \le C^{-1}T^{1/3}$; then $\tilde{W}$ has a reduced basis $\tilde{w}_1, \dots \tilde{w}_4$, where all vectors have length at most $C^{-1}T^{1/3}$. Let $w_i = g_T^{-1}\tilde{w}_i$

and $S_T = \{w \in \mathbb{R}^4 \ : \ \|w\| \leq T \text{ and } \|\pi_2(w)\| \leq 1/T\}$. Let $B \subset W$ denote the intersection of W with the ball of radius $C^{-1}T^{1/3}$. By construction, $w_i \in B \cap S_T$, and the w_i are four linearly independent vectors in $\mathbb{Z}^4$. Then the 4-dimensional volume of $B \cap S_T$ must be bounded below by an absolute constant. But, from the form of S_T, the 4-dimensional volume of $B \cap S_T$ is at most an absolute constant times C^{-3}. This is a contradiction if C is sufficiently large. $\square$

LEMMA 10.8. *There exist absolute constants $\sigma > 0$, and $\rho > 0$ such that for every sufficiently small $\delta > 0$ and any $T > 2$, the following holds: let Q, T, Δ_T, g_T be as in Lemma 10.7. Suppose $\alpha_3(\Delta_T) \geq T^{-\delta}$, and let U_3 denote the Δ_T-rational 3-dimensional subspace such that $1/d_3(U_3) = \alpha_3(\Delta_T)$. Then one of the following holds:*

(a) *The restriction of $Q^{(6)}$ to $g_T^{-1}U_3$ is anisotropic over $\mathbb{Q}$.*

(b) *There exists a split integral form Q', with $\|Q'\| \leq T^{\sigma\delta}$, such that for some $1 \leq \lambda \in \mathbb{R}$, $\|Q - \frac{1}{\lambda}Q'\| \leq T^{-\rho}$. If $w \in g_T^{-1}U_3$ is any rational point satisfying $Q^{(6)}(w) = 0$, then w corresponds to a rational isotropic subspace of Q'.*

Proof. Suppose that $Q^{(6)}$ restricted to $g_T^{-1}U_3$ is isotropic, hence split over $\mathbb{Q}$. Let $Q'' = f(g_T^{-1}U_3, g_T^{-1}U_3^{\perp})$ denote a quadratic form of determinant ± 1 associated with $g_T^{-1}U_3$ as in Proposition 10.2. Let Q' be the unique primitive integral form which is a multiple of Q''. Since $g_T^{-1}U_3$ has a reduced integral basis of vectors of norm at most T^{δ}, $g_T^{-1}U_3^{\perp}$ also has an integral basis bounded by a fixed power of T^{δ}. Hence, the coefficients of Q' are also bounded by a fixed power of T^{δ}, see Lemma 10.1 and Proposition 10.2 (d). Choose $\lambda \in \mathbb{R}$ so that Q and $\frac{1}{\lambda}Q'$ have the same determinant. By Proposition 10.2 (c), $\|Q - \frac{1}{\lambda}Q'\|$ is bounded by a fixed power of $T^{\delta-1}$. This, for sufficiently small $\delta < 1$, implies that (b) holds. $\square$

For $1 \leq j \leq 6$, consider the representation of H_1 on $\wedge^j(\mathbb{R}^6)$. We have $\wedge^j(\mathbb{R}^6) = W_0 \oplus \left(\oplus_{i=1}^{m_j} W_i\right)$ where H_1 acts trivially on W_0, and acts via the 3-dimensional representation on each W_i, $1 \leq i \leq m_j$. Let p_i denote the associated projections.

Now, $\|\cdot\|^*$ is the norm on $\mathbb{R}^3$ defined in [EMM, (5.23)]. Let $\|\cdot\|^{\#}$ denote the norm on $\wedge^j(\mathbb{R}^6)$ defined by

$$\|w\|^{\#} = \max(\|p_0(w)\|, \max_{1 \leq i \leq m_j} \|p_i(w)\|^*).$$

For a lattice Δ in $\mathbb{R}^4$ and a Δ-rational subspace M, we define $d^{\#}_{\Delta}(M) = \|e_1 \wedge \cdots \wedge e_m\|^{\#}$, where $\{e_1, \ldots, e_m\} \subset \Delta$ is a basis for M.

We now continue the proof of Theorem 10.4. Let δ be as in Lemma 10.8. We may assume $\delta < 1/3$. Let $U_1 \subset U_2 \subset U_3 \subset U_4 \subset U_5 \subset U_6 \cong \mathbb{R}^6$ be defined as follows: U_1 is spanned by the shortest vector in Δ_T, U_2 is spanned by U_0 and the shortest vector in Δ_T outside U_1, etc. It is a standard fact from reduction theory that there exists a constant C_6 depending only on the dimension such that

$$\frac{1}{d_i(U_i)} \leq \alpha_i(\Delta_T) \leq C_6 \frac{1}{d_i(U_i)}.$$

By Lemma 10.7 (iv), $d_4(U_4) > C_6^{-1}C^{-1}T^{1/3}$. Let $k \geq 0$ be the largest integer such that $d_k(U_k) < C_6^{-1}C^{-1}T^{\delta}$. Then, $k \leq 3$. For $1 \leq i \leq k$, and $g \in \mathrm{SL}(6, \mathbb{R})$ let

$$\alpha_i^{(-)}(g\Delta_T) = \sup\left\{ \frac{1}{d^{\#}_{g\Delta_T}(gL)} \;\middle|\; L \text{ is a } \Delta_T\text{-rational subspace} \right.$$
$$\left. \text{of dimension } i \text{ not contained in } U_k \right\}$$

(this is an abuse of notation because $\alpha_i^{(-)}$ depends on g as well as on $g\Delta_T$). For $k+1 \leq i \leq 6$, set $\alpha_i^{(-)} = \alpha_i$. We now claim that for $1 \leq i \leq 3$,

$$\alpha_i^{(-)}(\Delta_T) \leq \text{const} \cdot T^{-\delta/3}. \tag{52}$$

Indeed, if $k = 0$, (52) is clear. let v denote the shortest vector in U_i' which is not contained in U_k. Note that $d(U_i') \geq \text{const} \cdot \|v\|$ (since the length of every vector in Δ_T is bounded from below). Also note that $\|v\| \geq \text{const} \cdot d(U_k)^{1/k}$ (otherwise v would belong to U_k). Thus,

$$C_6^{-1}C^{-1}T^{\delta} \leq d(\mathbb{R}v + U_k) \leq \|v\| d(U_k) \leq \text{const} \cdot \|v\|^{k+1},$$

hence $\|v\| \geq \text{const} \cdot T^{\frac{\delta}{k+1}}$ which implies the result for $k = 1, 2$. For $k = 3$, we have by Lemma 10.7 (iv),

$$C^{-1}T^{1/3} \leq d(\mathbb{R}v + U_k) \leq \|v\| d(U_k);$$

thus $\|v\| \geq T^{\frac{1}{3}-\delta}$. Now, (52) follows in this case as well.

As in [EMM, §5] let A_τ be the averaging operator defined by

$$(A_\tau f)(x) = \int_K f(a_\tau k x)\, dm(k).$$

Then, arguing as in the proof of [EMM, Lemma 5.7] we see that the following inequalities hold with $\omega = \omega(\tau)$:

$$\begin{aligned} A_\tau \alpha_1^{(-)} &\leq \alpha_1^{(-)} + \omega^2 \sqrt{\alpha_2^{(-)}}, \\ A_\tau \alpha_2^{(-)} &\leq \alpha_2^{(-)} + \omega^2 \sqrt{\alpha_1 \alpha_3^{(-)}} + \omega^2 \sqrt{\alpha_4}, \\ A_\tau \alpha_3^{(-)} &\leq \alpha_3^{(-)} + \omega^2 \sqrt{\alpha_2 \alpha_4} + \omega^2 \sqrt{\alpha_1 \alpha_5} + \omega^2 \sqrt{\alpha_6}, \\ A_\tau \alpha_4 &\leq \alpha_4 + \omega^2 \sqrt{\alpha_2 \alpha_6} + \omega^2 \sqrt{\alpha_3 \alpha_5}, \\ A_\tau \alpha_5 &\leq \alpha_5 + \omega^2 \sqrt{\alpha_4 \alpha_6}. \end{aligned} \tag{53}$$

In the above, we used the fact that if L is a subspace which is not contained in U_k, and M is any other subspace, then $L+M$ is also not contained in U_k. (However, $L \cap M$ may belong to U_k.)

We also have

$$\begin{aligned} A_\tau \alpha_1 &\le \alpha_1 + \omega^2 \sqrt{\alpha_2}, \\ A_\tau \alpha_2 &\le \alpha_2 + \omega^2 \sqrt{\alpha_1 \alpha_3} + \omega^2 \sqrt{\alpha_4}, \\ A_\tau \alpha_3 &\le \alpha_3 + \omega^2 \sqrt{\alpha_2 \alpha_4} + \omega^2 \sqrt{\alpha_1 \alpha_5} + \omega^2 \sqrt{\alpha_6}. \end{aligned} \tag{54}$$

Let

$$\begin{aligned} \beta_1 &= T^{\delta/16} \alpha_1^{(-)} + \alpha_1, \\ \beta_2 &= T^{\delta/8} \alpha_2^{(-)} + \alpha_2, \\ \beta_3 &= T^{\delta/4} \alpha_3^{(-)} + \alpha_3, \\ \beta_4 &= T^{\delta/2} \alpha_4, \\ \beta_5 &= T^{\delta} \alpha_5, \\ \beta_6 &= \alpha_6 = T^{-3}. \end{aligned} \tag{55}$$

Now from (53) and (54) we get

$$\begin{aligned} A_\tau \beta_1 &\le \beta_1 + \omega^2 \sqrt{\beta_2}, \\ A_\tau \beta_2 &\le \beta_2 + \omega^2 \sqrt{\beta_1 \beta_3} + \omega^2 \sqrt{\beta_4}, \\ A_\tau \beta_3 &\le \beta_3 + \omega^2 \sqrt{\beta_2 \beta_4} + \omega^2 \sqrt{\beta_1 \beta_5} + \omega^2 \sqrt{\beta_6}, \\ A_\tau \beta_4 &\le \beta_4 + \omega^2 \sqrt{\beta_2 \beta_6} + \omega^2 \sqrt{\beta_3 \beta_5}, \\ A_\tau \beta_5 &\le \beta_5 + \omega^2 \sqrt{\beta_4 \beta_6}. \end{aligned} \tag{56}$$

From these inequalities we derive the following:

CLAIM 10.9. *For any $\rho > 0$ there exists C_ρ so that for any sufficiently large T, and $1 \le i \le 6$,*

$$(A_t \beta_i)(\Delta_T) \le C_\rho T^\rho,$$

where $t = \log T$.

Proof of claim. This argument is similar to the proofs of Proposition 5.12 and Lemma 5.13 in [EMM]. Let $q(i) = i(6-i)$. Fix $\varepsilon > 0$, and consider the linear combination

$$\beta = \sum_{i=1}^{6} \varepsilon^{q(i)} \beta_i.$$

Then β satisfies the inequality

$$A_\tau \beta \le T^{-3} + (1 + 6\varepsilon\omega^2)\beta \tag{57}$$

(cf. [EMM, eqns. (5.65)–(5.67)]). Let

$$\tilde\beta(h) = \int_K \beta(hk\Delta_T)\, dm(k).$$

Note that $\tilde\beta$ is spherically symmetric, i.e.

$$\tilde\beta(KhK) = \tilde\beta(h),\ h \in H_1. \tag{58}$$

It follows from the uniform continuity of $\log\tilde\beta$ that for a suitable neighborhood V of the identity,

$$\frac{1}{2}\tilde\beta(h) < \tilde\beta(uh) < 2\tilde\beta(h),\ h \in H_1,\ u \in V. \tag{59}$$

According to [EMM, 5.11] there exists a neighborhood U of 1 in H_1 such that $a_\tau U a_t \subset KVa_\tau a_t K$ for any $t \geq 0$ and $\tau \geq 0$. Then we get from (58) and (59) that

(60)

$$(A_\tau\tilde\beta)(a_t) = \int_K \tilde\beta(a_t k a_\tau)\, dm(k) \geq \int_{U\cap K} \tilde\beta(a_\tau k a_t)\, dm(k) > \frac{1}{2} m(U\cap K)\tilde\beta(a_\tau a_t).$$

Hence, from (57) and (60) we have

$$\tilde\beta(a_\tau a_t) \leq \frac{2}{m(U\cap K)}\left(T^{-3} + (1+6\varepsilon\omega^2)\tilde\beta(a_t)\right). \tag{61}$$

Given $\rho > 0$ we choose $\tau = \tau(\rho) > 1$ so that $\log(4/m(U\cap K))/\tau \leq \rho$, and $\varepsilon = \varepsilon(\tau) > 0$ so that $1 + 6\varepsilon\omega^2 \leq 2$ (recall $\omega = \omega(\tau)$). We assume that T is large enough so that $2T^{-3}/m(U\cap K) < 1$. Hence we get

$$\tilde\beta(a_\tau a_t) < e^{\rho\tau}\tilde\beta(a_t) + 1. \tag{62}$$

Using induction on n we get from (62) that for $n \in \mathbb{N}$,

$$\tilde\beta(a_{n\tau}) < (\tilde\beta(1) + 1)e^{n\tau\rho}, \tag{63}$$

where we have used the fact that $e^{\rho\tau} \geq 4$. Since $\{a_r \mid 0 \leq r \leq \tau\}$ belongs to V^i for some i where $V^1 = V$, $V^i = V\, V^{i-1}$, it follows that for any $t > 0$,

$$\tilde\beta(a_t) < C(\tilde\beta(1) + 1)e^{t\rho}, \tag{64}$$

where $C = C(\tau)$. Recall that $\tilde\beta(a_t) = (A_t\beta)(\Delta_T)$, and $\tilde\beta(1) = \beta(1)$. Note that $\beta(1)$ is bounded by a constant in view of (55), (52) and Lemma 10.7 (iii),(iv). Now let $t = \log T$. This shows that

$$(A_t\beta)(\Delta_T) \leq C'_\rho T^\rho,$$

from which the claim follows. ∎

Hence, in view of (55), for all i,

$$A_t\alpha_i^{(-)}(\Delta_T) \le C'_\rho T^{\rho-\delta/16} \le C'' T^{-\delta/32},$$

where we chose $\rho < \delta/32$. Now, using Proposition 10.6, we see that the number of primitive $w \in \Delta_T$ satisfying (a$'$), (b$'$), (c$'$) which are not contained in U_k is $O(T^{1-\delta/32})$. Equivalently, the number of primitive $w \in \mathbb{Z}^6$ satisfying (a),(b),(c) of Lemma 10.5 which are not contained in $g_T^{-1}U_k$ is $O(T^{1-\delta/32})$. It remains to count the number of primitive points of $g_T^{-1}U_k$ satisfying (a), (b), (c) of Lemma 10.5.

Now if $k=1$, $g_T^{-1}U_1$ has at most one primitive point. If $k=2$, then there may be at most 2 primitive points in $g_T^{-1}U_2$ satisfying (a) from Lemma 10.5 (there are two points if the restriction of $Q^{(6)}$ to $g_T^{-1}U_2$ is split, no points otherwise). If $k=3$, and Q is not EWAS then for an appropriate choice of δ, the restriction of $Q^{(6)}$ to $g_T^{-1}U_3$ must be anisotropic by Lemma 10.8. Hence there are no points satisfying (a) in $g_T^{-1}U_3$. Thus in all cases, there are at most three primitive points in $g_T^{-1}U_k$ satisfying (a)–(c). Hence, the total number of primitive vectors in $\mathbb{Z}^6$ satisfying (a)–(c) is $O(T^{1-\delta})$. □

One corollary of the argument is the following:

Proposition 10.10. *Suppose Q is any* (*possibly rational*) *quadratic form of signature* $(2,2)$. *Then the number of quasinull subspaces of norm between $T/2$ and T is $O(T\log T)$.*

In fact, in view of Lemma 10.8 part (b), we also proved the following statements:

Proposition 10.11. *There exists an absolute constant $\rho > 0$ such that the following holds*: *Suppose Q is any irrational quadratic form of signature* $(2,2)$. *Then for every sufficiently small $\delta > 0$ and for every $T > 2$ one of the following holds*:

(a) *The number of quasinull subspaces of Q of norm between $T/2$ and T is $O(T^{1-\delta})$.*

(b) *There exists a split integral quadratic form Q' and $1 \le \lambda \in \mathbb{R}$ satisfying $\|Q - \frac{1}{\lambda}Q'\| \le T^{-\rho}$ such that the number of quasinull subspaces of Q of norm between $T/2$ and T which are not isotropic subspaces of Q' is $O(T^{1-\delta})$. The coefficients of Q' are bounded by a fixed power of T^δ.*

10.3. *Proof of Theorem* 1.5.

Lemma 10.12. *Given a $\mathbb{Z}^4$-rational quasinull subspace L, if any $\mathbb{Z}$-basis for $L \cap \mathbb{Z}^4$ contains a vector of norm greater than $2T$, L is T-degenerate.*

Then for any $C < \infty$, the number of lattice points v contained in T-degenerate quasinull subspaces L with $\|v^L\| \leq CT$ such that $a < Q(v) < b$ and $\|v\| \leq T$ is $o(T)$.

Proof. It is convenient to work with the standard form B. As in Section 2, let g_Q be such that $Q(v) = B(g_Q v)$, and let $\Lambda = g_Q \mathbb{Z}^4$. For a Λ-rational subspace L, let v_L denote the shortest vector in $L \cap \Lambda$. Let Ω be the image under g_Q of the unit ball. Let f, a_t, K be as in Section 2. We know that the number of points in $L \cap \Lambda \cap T\Omega$ is bounded by

$$T^2 \sum_{v \in L \cap \Lambda} \int_K f(a_t k v)\, dk, \tag{65}$$

where $t = \log T$. Let

$$H(R) = \{k \in K \ : \ \|a_t k v_L\| < R\}.$$

Since all vectors in $L \cap \Lambda \cap T\Omega$ are integral multiples of v_L, the integral in (65) can be restricted to $H(R_f) \subset K$ where $R_f = \max\{\|v\| \ : \ f(v) \neq 0\}$. Also if $M > 0$ is any constant, the number of points in $L \cap \Lambda \cap T\Omega$ is also bounded by

$$M + T^2 \sum_{v \in L \cap \Lambda} \int_{H(R_f/M)} f(a_t k v)\, dk. \tag{66}$$

Indeed if $k \notin H(R_f/M)$, i.e $\|a_t k v_L\| > R_f/M$, then at most M integral multiples of v_L can be in the support of f. Also,

$$\sum_{v \in L \cap \Lambda} f(a_t k v) \leq \text{const} \cdot \alpha_1(a_t k \Lambda), \tag{67}$$

and for $k \in H(R_f/M)$, $\alpha_1(a_t k \Lambda) \geq M/R_f$. Substituting (67) into (66) and summing over all T-degenerate quasinull subspaces, we get

$$CMT \log T + \text{const} \cdot T^2 \int_{\mathcal{C}(M/R_f)} \alpha_1(a_t k \Lambda)\, dk,$$

where $\mathcal{C}(M/R_f) = \{k \in K \ : \ \alpha_1(a_t k \Lambda) \geq M/R_f\}$, and we have used Proposition 10.10 to bound the number of quasinull subspaces. Pick any $0 < \xi < 1$, then the above expression is bounded by

$$CMT \log T + \text{const} \cdot T^2 \left(\frac{R_f}{M}\right)^{\xi} \int_{\mathcal{C}(M/R_f)} \alpha_1(a_t k \Lambda)^{1+\xi}\, dk,$$

and the integral is bounded independently of T by Theorem 2.5. Since M is arbitrary, this implies that the total contribution of the T-degenerate quasinull subspaces is $o(T^2)$. $\square$

Proof of Theorem 1.5. In view of Lemma 10.12 we need only estimate the number of points in the non-T-degenerate quasinull subspaces. Let $M \gg 1$

be a parameter to be chosen later. Let $T \gg 1$ be arbitrary. For a $\mathbb{Z}^4$-rational subspace L, let $v^L = v_1 \wedge v_2$, where $\{v_1, v_2\}$ is any $\mathbb{Z}$-basis for $L \cap \mathbb{Z}^4$. We say that a quasinull subspace is *short* if $\|v^L\| \leq M$, and is *long* if $M \leq \|v^L\| \leq T$.

The number of short subspaces is bounded depending on M. Since we are assuming that the restriction of the quadratic form to L is not identically 0, the number of points v in $L \cap \Lambda$ such that $a < Q(v) < b$ and $\|v\| \leq T$ is $O(T)$ as $T \to \infty$. Thus, for any choice of M, the contribution of the short subspaces is $o(T^2)$.

To estimate the contribution of the long subspaces, divide them into dyadic intervals, $S/2 \leq \|v^L\| \leq S$. (the shortest interval begins at M; the longest ends at T). By Theorem 10.4, in each dyadic interval there are at most $C(\delta)S^{1-\delta}$ subspaces; since each subspace is T-nondegenerate, it contributes at most $O(T^2/S)$ points. Hence the contribution of each dyadic interval is at most $C'(\delta)T^2/S^\delta$. Letting $S = 2^k M$ and summing the resulting geometric series, we see that the total contribution of the long intervals is $C''(\delta)T^2/M^\delta$.

Since one may choose M arbitrary large, we see that the total contribution is $o(T^2)$. $\square$

Appendix A. A proof of Proposition 4.2

Notation. All the constants in this appendix are absolute. We use the notation $a \approx b$ to indicate that there exist constants $0 < c < C$ such that $c < \frac{a}{b} < C$. We write $a \gg b$ to indicate that there exists a "large" constant κ such that $a > \kappa b$. A statement of the form "there exists $c > 0$ such that if $a \gg b$ then ..." means that "there exist constants $c > 0$ and $\kappa > 1$ such that if $a > \kappa b$ then ...".

A.1. *Transition to the hyperbolic plane.* Set $G = \mathrm{SL}(2, \mathbb{R})$, $K = \mathrm{SO}(2)$, $b_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, and $B = \{b_t\}$. On $K\backslash G$ we consider the right G-invariant metric $d(\cdot,\cdot)$ normalized so that $d(Kb_t, K) = t$. This identifies $K\backslash G$ with the hyperbolic disk $\mathbb{H}^2$ of curvature -1. Let $p : G \to \mathbb{H}^2$ be the natural projection. Let $e = p(1)$ denote the origin of $\mathbb{H}^2$.

The space of the 3-dimensional representation of G can be considered to be the space S_2 of 2-by-2 symmetric matrices, with the action given by $g \cdot v = gvg^t$. This action preserves the determinant of v, which is a quadratic form of signature $(2, 1)$.

Lemma A.1. *Suppose $v \in S_2$. Then, if $\det v > 0$, there exists a point $z_v \in \mathbb{H}^2$ and a number $\lambda_v > 0$ such that*

$$\|g \cdot v\| = \lambda_v \sqrt{\cosh(2d(p(g), z_v))} \approx \lambda_v e^{d(p(g), z_v)}. \tag{68}$$

If $\det v < 0$ *there exists a geodesic* $\gamma_v \in \mathbb{H}^2$ *and a number* $\lambda_v > 0$ *such that*

$$\|g \cdot v\| = \lambda_v \sqrt{\cosh(2d(p(g), \gamma_v))} \approx \lambda_v e^{d(p(g),\gamma_v)}. \tag{69}$$

Proof. Let $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that for any $v \in S_2$ with $\det v \neq 0$ there exists $g_v \in G$ such that $g_v \cdot v = \lambda_v v_i$, where $i = 1$ if $\det v > 0$, and $i = 2$ if $\det v < 0$. Hence it is enough to prove the lemma for $v = v_1$ and for $v = v_2$.

If $v = v_1$, then the stabilizer of v is K, and $\|b_t \cdot v\| = \sqrt{(e^{2t} + e^{-2t})/2} = \sqrt{\cosh 2t}$. Since, $G = KBK$ and the norm $\|\cdot\|$ is K-invariant, (68) follows. If $v = v_2$ then the stabilizer of v_2 is the group $H = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}$. Then, as above, $\|b_t \cdot v\| = \sqrt{(e^{2t} + e^{-2t})/2} = \sqrt{\cosh 2t}$. Since $G = KBH$, for any $g \in G$ we may write $g = kb_th$ where $k \in K$ and $h \in H$; then $\|g \cdot v\| = \sqrt{\cosh 2t}$, but since $p(B)$ and $p(H)$ are orthogonal geodesics in $\mathbb{H}^2$, $t = d(p(b_t), e) = d(p(b_t), p(H)) = d(p(g), p(H))$. This proves (69). □

A.2. *Some hyperbolic geometry.* We now recall some well known lemmas from hyperbolic geometry.

LEMMA A.2. *Suppose* $\Delta \subset \mathbb{H}^2$ *is an isosceles triangle, with sides* a, a *and* b. *Let* θ *denote the angle opposite* b. *Suppose* $b \gg 1$ *and* $\theta \ll 1$. *Then,*

$$\theta \approx e^{\frac{b}{2} - a}.$$

Proof. The hyperbolic law of cosines states that

$$\cosh b = \cosh^2 a - \sinh^2 a \cos \theta.$$

This can be rewritten as

$$2 \sin^2 \frac{\theta}{2} = \frac{\cosh b - 1}{\sinh^2 a}.$$

Now the lemma easily follows. □

LEMMA A.3. *Let* Δ *be a right-angled triangle in* $\mathbb{H}^2$. *Let* t *denote the length of the side opposite the right angle. Let* θ *denote another angle of* Δ, *and let* ρ *denote the length of the side opposite* θ. *Then, if* $t \gg 1$ *and* $\theta \ll 1$,

$$\theta \approx e^{\rho - t}.$$

Proof. By the hyperbolic law of sines,

$$\frac{\sinh t}{\sin \frac{\pi}{2}} = \frac{\sinh \rho}{\sin \theta}.$$

Hence,

$$\sin \theta = \frac{\sinh \rho}{\sinh t} \approx e^{\rho - t}.$$ □

Lemma A.4. *Suppose* $d(p,e)=t$, $d(q,e)\ge t$. *Set* $p'=\overline{eq}\cap S(t,e)$ *where* $S(t,e)$ *denotes the circle of radius* t *centered at* e. *Then,*

$$d(q,p')+d(p',p)\le d(q,p)\le d(q,p')+d(p',p)+O(1).$$

Proof. This follows from the fact that the angle at p' between $\overline{p'q}$ and $\overline{p'p}$ is greater than $\pi/2$. □

Lemma A.5. *There exists an absolute constant* C *so that the following holds*: *Let* $\gamma\subset\mathbb{H}^2$ *be a geodesic. Let* q_+ *and* q_- *in* $\partial\mathbb{H}^2$ *denote the endpoints of* γ, *and let* L_+ *and* L_- *denote the geodesic rays* $\overline{eq_+}$ *and* $\overline{eq_-}$ *respectively. Let* $r=d(\gamma,e)$, *and let* $\ell_\pm$, *also geodesic rays, denote the intersection of* $L_\pm$ *with the complement of the ball* $B(r,e)$ (*see Figure* 1). *Then,*

$$\mathrm{hd}(\gamma,\ell_+\cup\ell_-)<C,$$

where hd *denotes the Hausdorff distance.*

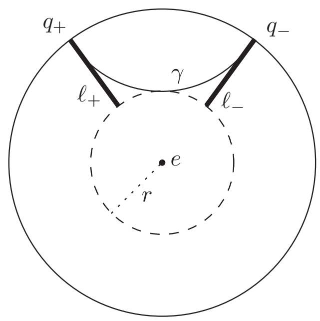

Figure 1. Lemma A.5.

Proof. Let $q=B(r,e)\cap\gamma$, $p_\pm=B(r,e)\cap\ell_\pm$. It is clear that $hd(\gamma,\ell_+\cup\ell_-)=d(q,p_+)=d(q,p_-)$. By considering the right-angled triangle (e,q,q_+) we see that $d(q,L_+)=O(1)$. Let $p'_+\in L_+$ denote the closest point to q, so that $d(p'_+,q)=O(1)$. Then, since the path $e\to p'_+\to q$ is "almost a geodesic", $d(e,p'_+)=d(e,q)-d(p'_+,q)+O(1)=r+O(1)$. Since $d(e,p_+)=r$ and both p_+ and p'_+ lie on L_+, $d(p_+,p'_+)=O(1)$. Hence $d(p_+,q)\le d(p_+,p'_+)+d(p'_+,q)=O(1)$. □

A.3. *Proof of Proposition* 4.2. For $v\in\mathbb{R}^3$ and $t>0$, let

$$m_t(v)=\min_\theta\|b_tk_\theta v\|.$$

The properties of the sets $\{k\in K:\|b_tkv\|<\delta\}$ which imply Proposition 4.2 are summarized in the following:

LEMMA A.6. *Let* $v \in \mathbb{R}^3$ *be such that* $\max_k \|b_t k v\| \gg 1$. *Then, there exist intervals* $I_t^{v,+}(\delta)$, $I_t^{v,-}(\delta)$ *such that the following properties hold*:

(a) *There exists* $c > 0$ *such that for all* $\delta > 0$ *with* $\max_k \|b_t k v\| \gg \delta \gg m_t(v)$,

$$I_t^{v,\pm}(c\delta) \subset \{\theta \;:\; \|b_t k_\theta v\| < \delta\} \subset I_t^{v,+}(c^{-1}\delta) \cup I_t^{v,-}(c^{-1}\delta). \tag{70}$$

(b) *There exists* $C < \infty$ *such that if* $\max_k \|b_i k v\| \gg \delta \gg m_t(v)$, $\tau \gg 0$ *and* $\delta \gg m_{t+\tau}(v)$,

$$I_{t+\tau}^{v,\pm}(\delta) \subset C e^{-\tau/2} I_t^{v,\pm}(\delta). \tag{71}$$

(c) *If* $\max_k \|b_t k v\| \gg \eta \geq \delta \gg m_t(v)$, *then*

$$\frac{|I_t^{v,\pm}(\delta)|}{|I_t^{v,\pm}(\eta)|} \ll \left(\frac{\delta}{\eta}\right)^{1/2}. \tag{72}$$

(d) *Either for all* δ *with* $\max_k \|b_t k v\| \gg \delta \gg m_t(v)$,

$$|I_t^{v,\pm}(\delta)| \approx e^{-t} m_t(v)^{-1/2}, \delta^{1/2}, \tag{73}$$

or for all $\tau \geq 0$,

$$m_{t+\tau}(v) \geq m_t(v). \tag{74}$$

Proof of Lemma A.6. Suppose first that $\det v > 0$. We let $I_t^{v,+}(\delta) = I_t^{v,-}(\delta) = \{k \in K \;:\; \|b_t k v\| < \delta\}$, where λ_v is as in (68). Then (a) of Lemma A.6 is trivially satisfied. In the case $\det v > 0$ we drop the $\pm$ from the notation. Let ρ be defined by the equation

$$\frac{\delta}{\lambda_v} \stackrel{\text{def}}{=} \sqrt{\cosh 2\rho} \approx e^{\rho}. \tag{75}$$

Then, in view of (68),

$$\{p(g) \in \mathbb{H}^2 \;:\; \|g \cdot v\| < \delta\} = B(z_v, \rho).$$

We may identify the K with the unit circle; then $I_t^v(\delta) = \{k \in K \colon \|b_t k \cdot v\| < \delta\}$ may be identified with the set of angles at which the set $S(t,e) \cap B(z_v, \rho)$ is visible from the origin. Thus the center of the interval $I_t^v(\delta)$ is always the same; it is the angle at which z_v is visible from e.

Let $r = d(e, z_v)$. In view of (68), $m_t(v)$ (considered as a function of t) is decreasing for $t \leq r$ and increasing for $t \geq r$. Also, in view of (68),

$$m_t(v) \approx \lambda_v e^{|t-r|}. \tag{76}$$

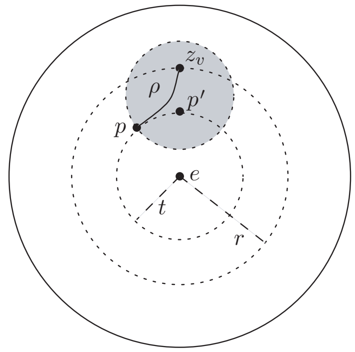

Figure 2. The case $\det v > 0$, $t \le r$.

Also, $\max_k \|b_t k v\| \approx \lambda_v e^{t+r}$. Thus, the assumption $\max_k \|b_t k v\| \gg \delta \gg m_t(v)$ implies

$$t + r \gg \rho \gg 1 + |t - r|. \tag{77}$$

Now suppose $t \le r$. Let $p \in S(t,e)$ denote a point such that $d(p, z_v) = \rho$, and let $p' = S(t,e) \cap \overline{ez_v}$ (see Fig. 2). Then, by Lemma A.4, $d(p,p') = \rho - r + t + O(1)$. In view of (77), $d(p,p') \gg 1$. Hence, by Lemma A.2 (with $b = d(p,p')$ and $a = d(e,p) = d(e,p') = t$), we get $|I_t^v(\delta)| \approx e^{\frac{\rho - r + t}{2} - t}$. Hence, using (75) we get

$$|I_t^v(\delta)| \approx \lambda_v^{-1/2} e^{-r/2} e^{-t/2} \delta^{1/2}. \tag{78}$$

In view of (76), (78) implies (73); hence (d) holds for this case.

Now suppose $t \ge r$. Since, in this range $m_t(v)$ is an increasing function of t, (d) holds. Let $q \in S(t,e)$ be a point such that $d(q, z_v) = \rho$, and let p denote the intersection of $\overline{eq}$ with $S(r,e)$ (see Fig. 3). Then, by Lemma A.4, $d(p, z_v) = \rho + t - r + O(1)$. In view of (77), $d(p, z_v) \gg 1$. Now, by Lemma A.2 (with $b = d(p, z_v)$ and $a = d(p,e) = d(z_v, e) = r$) we get $|I_t^v(\delta)| \approx e^{\frac{\rho + r - t}{2} - r}$. Hence, by (75) (78) still holds when $t \ge r$ as long as $\delta \gg m_t(v)$. Now (b) and (c) follow immediately from (78). This completes the proof of Lemma A.6 for the case $\det v > 0$.

Now suppose $\det v < 0$. Let λ_v and γ_v be as in (69), and set $r = d(\gamma_v, e)$. In view of (69), for $t \ge r$, $m_t(v) = \lambda_v$ (since $S(t,e)$ intersects γ_v). Hence if $t \ge r$ then $m_t(v)$ is not increasing and (d) is automatically satisfied.

Let ρ be as in (75). We denote by $N_\rho(\gamma_v)$ the set $\{z \in \mathbb{H}^2 \,:\, d(z, \gamma) \le \rho\}$. Then, in view of (69), the set $\{k \in K \,:\, \|b_t k v\| \le \delta\}$ may be identified with the set of angles at which $S(t,e) \cap N_\rho(\gamma_v)$ is visible from the origin e.

Let ℓ_+ and ℓ_- be as in Lemma A.5. For $\sigma \in \{+,-\}$ set $N_\rho(\ell_\sigma) = \{z \in \mathbb{H}^2 : d(z, \ell_\sigma) \le \rho\}$. We define $I_t^{v,\sigma}(\delta)$ to be the set of angles at which the set $S(t,e) \cap N_\rho(\ell_\sigma)$ is visible from the origin. Then, in view of Lemma A.5, (a) of Lemma A.6 holds.

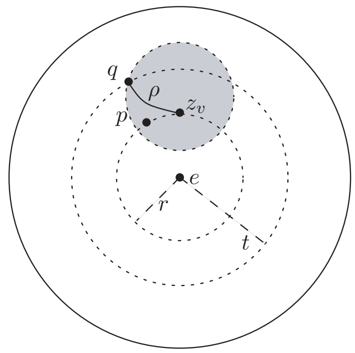

Figure 3. The case $\det v > 0$, $t \geq r$.

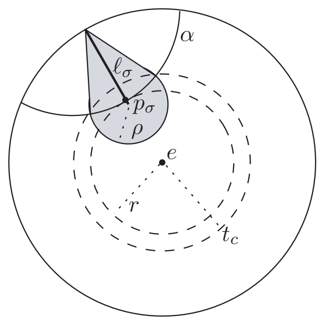

Figure 4. The neighborhood of a geodesic ray.

Let p_σ denote the endpoint of ℓ_σ; then $d(p_\sigma, e) = r$. Let α denote the geodesic orthogonal to ℓ passing through p_σ (see Fig. 4). Let $t_c > r$ be as in Figure 4; then for $t < t_c$, $|I_t^{v,\sigma}|$ is given by (78). If $t > t_c$, let p be a point on $\partial N_\rho(\ell_\sigma) \cap S(t, e)$, and let $q \neq p_\sigma$ denote the closest point on ℓ_σ to p. Then the triangle (e, q, p) is a right triangle, and by Lemma A.3, $|I_t^{v,\sigma}(\delta)| \approx e^{\rho - t} \approx \lambda_v^{-1} e^{-t}\delta$. Hence, as long as $\max_k \|b_t k v\| \gg \delta \gg m_t(v)$,

$$|I_t^{v,\sigma}(\delta)| \approx \begin{cases} \lambda_v^{-1/2} e^{-r/2} e^{-t/2} \delta^{1/2} & t < t_c \\ \lambda_v^{-1} e^{-t} \delta & t > t_c \end{cases}. \tag{79}$$

Since $|I_t^{v,\sigma}|$ is a continuous function of t, (b) and (c) of Lemma A.6 follow immediately from (79). To show (d), we note that if $t > r$ then $m_t(v) = \lambda_v$ and is thus nonincreasing. If $t < r$ then for $z \in S(t, e)$, $d(z, \ell_\sigma) = d(z, p_\sigma)$ and hence (76) and (78) hold. Thus (73) follows as in the case $\det v > 0$. This completes the proof of Lemma A.6 in the case $\det v < 0$.

The remaining case $\det v = 0$ can be considered as a limiting case of the "ball" case $\det v > 0$; we pass to the limit by sending $z_{v_i} \to \infty$ and $\lambda_{v_i} \to 0$ while keeping $\lambda_{v_i} e^{d(z_{v_i}, e)} \stackrel{\text{def}}{=} c_v$ constant. The function $\|gv\|$ becomes essentially a Buseman function, and its level sets are horocycles. It is easy to show that

(78) becomes

$$|I_t^v(\delta)| \approx c_v^{-1/2} e^{-t/2} \delta^{1/2}.$$

and (76) becomes

$$m_t(v) \approx c_v^{1/2} e^{-t}.$$

Then the assertions for the lemma follow immediately. □

Proof of Proposition 4.2. We let the rectangle $R_t^{L,\pm,\pm}(\delta_1,\delta_2)$ be the product of the intervals $I_t^{\pi_1(v^L),\pm}(\delta_1)$ and $I_t^{\pi_2(v^L),\pm}(\delta_2)$. It is clear that parts (a)–(c) of Proposition 4.2 follow immediately from the corresponding assertion of Lemma A.6. To derive (d) of Proposition 4.2 from (d) of Lemma A.6 we may argue as follows: Note that $M_t(L) = \max(m_t(\pi_1(v^L)), m_t(\pi_2(v^L)))$. Without loss of generality, we may assume that $M_t(L) = m_t(\pi_1(v^L))$. By Lemma A.6 part (d), either (73) or (74) holds. If (73) holds then (25) holds. If (74) holds, then for any $\tau > 0$, $M_{t+\tau}(L) \geq m_{t+\tau}(\pi_1(v^L)) \geq m_t(\pi_1(v^L)) = M_t(L)$, hence (26) holds. Thus, part (d) of Proposition 4.2 also holds. □

Appendix B. Proof of Lemma 8.2

In this appendix we prove Lemma 8.2. The following lemma describes the shapes of the sets $F(\eta)$:

LEMMA B.1. *There exist constants $0 < \eta_0 < \eta_1 < 1$ and $\kappa > 1$ depending only on Λ such that the following holds: Let H be either a one-dimensional or a three-dimensional Λ-rational subspace, and let $v = v^H$, where v^H is as defined in Section 2.*

Suppose $t > 0$. Suppose $\delta \stackrel{\text{def}}{=} \min_k d_{a_t k\Lambda}(a_t kH) < \eta_0$. Let $\lambda_1^2 \leq \lambda_2^2$ be the eigenvalues of $v^t v$ (where $\mathbb{R}^4$ is identified, when $\dim H = 1$ or $\wedge^3\mathbb{R}^4$, when $\dim H = 3$, with $M_2(\mathbb{R})$). Let $F^H(\eta) \subset \mathbb{R}^2$ denote the region $\{(x,y) : |\lambda_1 - \lambda_2 xy| < e^{-t}\eta,$ and $|\lambda_2 x| < \eta$ and $|\lambda_2 y| < \eta\}$. Then for $8\delta < \eta < \eta_1$,

$$F^H(\kappa^{-1}\eta) \subset \{(\tan(\alpha-\alpha_0), \tan(\beta-\beta_0)) \ : \ \|a_t(k_\alpha, k_\beta)v\| < \eta\} \subset F^H(\kappa\eta),$$

for some $\alpha_0 \in \mathbb{R}$, $\beta_0 \in \mathbb{R}$ depending only on v.

Proof of Lemma B.1. We may replace v by $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, with $|\lambda_1| < |\lambda_2|$. Then

$$kv = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \tag{80}$$

$$= \begin{pmatrix} \lambda_1\cos\alpha\cos\beta - \lambda_2\sin\alpha\sin\beta & \lambda_1\cos\alpha\sin\beta + \lambda_2\sin\alpha\cos\beta \\ -\lambda_1\sin\alpha\cos\beta - \lambda_2\cos\alpha\sin\beta & -\lambda_1\sin\alpha\sin\beta + \lambda_2\cos\alpha\cos\beta \end{pmatrix}.$$

Hence,

$$a_t k v = \begin{pmatrix} e^t(\lambda_1\cos\alpha\cos\beta - \lambda_2\sin\alpha\sin\beta) & \lambda_1\cos\alpha\sin\beta + \lambda_2\sin\alpha\cos\beta \\ -\lambda_1\sin\alpha\cos\beta - \lambda_2\cos\alpha\sin\beta & e^{-t}(-\lambda_1\sin\alpha\sin\beta + \lambda_2\cos\alpha\cos\beta) \end{pmatrix}.$$

Note that $\|a_t^{-1}\| = e^{-t}$, and $\|v\| \geq |\lambda_2|$. Hence,

$$\delta = \min_k \|a_t k v\| \geq e^{-t}|\lambda_2|. \tag{81}$$

Since $|\lambda_2|$ is comparable to the norm of v, we have $|\lambda_2| > c_0$, where c_0 depends only on Λ. If $\det v = 0$, then $\lambda_1 = 0$. If $\delta = \min_k \|a_t k v\| < \eta_0$ then $|\det v| = |\lambda_1\lambda_2| \leq \eta_0^2$. Hence, if we choose η_0 small enough, we may always assume that $|\lambda_1| \ll |\lambda_2|$.

Suppose $\|a_t k v\| < \eta < \eta_1 \ll 1$. Then from the $(1,1)$ coefficient, $|\sin\alpha\sin\beta| < 2/|\lambda_2| \ll 1$. From the $(1,2)$ coefficient, $|\sin\alpha\cos\beta\| < 2/|\lambda_2|$, and from the $(2,1)$ coefficient, $|\cos\alpha\sin\beta| < 2/|\lambda_2|$. Hence $|\sin\alpha| \ll 1$, and $|\sin\beta| \ll 1$. Thus α is near 0 or π and β is near 0 or π. After possibly applying the transformations $\alpha \to \alpha + \pi$, $\beta \to \beta + \pi$ (which have the effect of changing the signs of λ_1 and λ_2), we may assume that α and β are near 0. Thus we may assume that $|\cos\alpha| > 1/2$ and $|\cos\beta| > 1/2$. Let $C_0 = |\cos\alpha\cos\beta|^{-1}$, and let $x = \tan\alpha$, $y = \tan\beta$. The inequalities $\|a_t k v\| < \eta$ may be rewritten as:

$$|\lambda_1 - \lambda_2 xy| < C_0 e^{-t}\eta, \tag{82}$$

$$|-\lambda_1 x - \lambda_2 y| < C_0\eta, \tag{83}$$

$$|\lambda_1 y + \lambda_2 x| < C_0\eta, \tag{84}$$

$$|\lambda_2 - \lambda_1 xy| < C_0 e^t\eta, \tag{85}$$

where we note that $1 < C_0 < 4$ always.

We first observe that since $|\lambda_1| \ll |\lambda_2|$, the equations (83) and (84) are equivalent to $|x| < (\text{const})\eta/|\lambda_2|$ and $|y| < (\text{const})\eta/|\lambda_2|$. The equation (82) is equivalent to the first condition in the definition of $F^H(\eta)$. These observations imply that if κ is sufficiently large, $\{k : \|a_t k v\| < \eta\} \subset F^H(\kappa\eta)$.

To establish the other inclusion $F^H(\kappa^{-1}\eta) \subset \{k : \|a_t k v\| < \eta\}$ suppose $k = (k_\alpha, k_\beta) \in F^H(\kappa^{-1}\eta)$. As above, let $x = \tan\alpha$, $y = \tan\beta$. It is clear that if κ is sufficiently large, then (82), (83) and (84) are satisfied with $\eta/4$ in place of η. Also, since $\eta < \eta_1 \ll 1$ and $|\lambda_2| > c_0 > 0$, it follows from the second and third conditions in the definition of $F^H(\kappa^{-1}\eta)$ that $|x| < 1$ and $|y| < 1$. Then,

$$|\lambda_2 - \lambda_1 xy| \leq |\lambda_2| + |\lambda_1| \leq 2|\lambda_2| \leq 2e^t\delta \leq C_0 e^t(\eta/4),$$

where in the next-to-last estimate we used (81) and in the last estimate we used the condition $\eta > 8\delta$. Hence, $\|a_t k v\| < \eta$ as required. This shows the inclusion $F^H(\kappa^{-1}\eta) \subset \{k : \|a_t k v\| < \eta\}$ which completes the proof of the lemma. □

In view of Lemma B.1, the set $F(\eta)$ is contained in a small neighborhood of the origin. Hence we may assume that the Jacobian of the map $(\alpha, \beta) \to (\tan\alpha, \tan\beta)$ is bounded between 1/2 and 2. Lemma 8.2 would follow from the next result:

LEMMA B.2. *Let H be a Λ-rational subspace of dimension* 1 *or* 3. *For $\eta > 0$ let $F^H(\eta) \subset \mathbb{R}^2$ be as in Lemma* B.1. *Suppose R_1 and R_2 are disjoint rectangles such that for $i = 1, 2$, $|R_i \cap F(\eta)| \geq 0.6|R_i|$, and let $\lambda > 1$ be such that $3\lambda R_1$ and $3\lambda R_2$ are still disjoint and of diameter at most $\pi/8$. Then, for some $i \in \{1, 2\}$,*

$$|R_i| \leq c_3\lambda^{-1}|(\lambda R_i) \cap F^H(2\eta)|.$$

where $c_3 > 0$ depends only on Λ.

Note that $F^H(\eta) = \mathcal{F} \cap S(\eta_2)$, where $\eta_2 = \eta/|\lambda_2|$, $S(\eta_2)$ denotes the square centered at the origin of side length η_2 and $\mathcal{F}$ is a region between two hyperbolas as in Lemma B.4.

The proof of Lemma B.2 will be carried out in two steps. The following lemma allows us to reduce to the case when $R_i \subset F^H(\eta)$, $i = 1, 2$.

LEMMA B.3. *Let R be a rectangle such that $|R \cap F^H(\eta)| \geq 0.6|R|$. Then, there exists a rectangle $R' \subset R$ such that $R' \subset F^H(\eta)$ and $|R'| > \frac{1}{(100)^2}|R|$.*

Proof of Lemma B.3. After replacing R by $R \cap S(\eta_2)$ we may assume that $R \subset S(\eta_2)$. Consider a partition of R into 100×100 equal disjoint subrectangles. Each of the boundary curves of the region $\mathcal{F}$ is a monotone function. Clearly, a monotone curve can intersect at most 200 of these subrectangles. Since there are at most four boundary curves, the number of subrectangles intersected by boundary curves is at most 800. If no subrectangle is contained in $\mathcal{F}$, then $\mathcal{F} \cap R$ is covered by subrectangles intersecting boundary curves, and hence we would have $|R \cap \mathcal{F}| < 0.08|R| < 0.6|R|$, contradicting the assumption. $\square$

LEMMA B.4. *Let $P = [x_0 - a, x_0] \times [y_0 - b, y_0]$ and $Q = [x_0 - A, x_0 + A] \times [y_0 - B, y_0 + B]$ be rectangles. Suppose $x_0 > 0$, $y_0 > 0$, $0 < a < A$, and $0 < b < B$. Assume that the origin $(0, 0)$ does not belong to Q. Let $\omega_1 = (x_0 - a)(y_0 - b)$ and let $\omega_2 = x_0 y_0$. Denote $\mathcal{F}' = \{(x, y) \, : \, \omega_1 \leq xy \leq \omega_2\}$, so that $P \subset \mathcal{F}'$. Then,*

$$|Q \cap \mathcal{F}'| \geq \frac{1}{8}\min(a/A, b/B)|Q| = \frac{1}{2}\min(aB, Ab).$$

Proof. After rescaling we may assume that $\omega_2 = 1$ and $\omega_1 < 1$. Let α, β be as in Figure 5 (i.e. α is the distance from the midpoint of the top side of

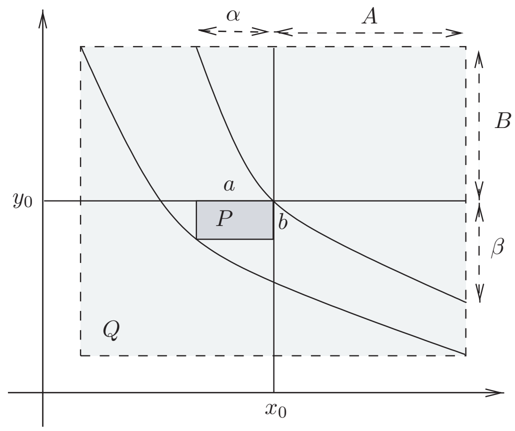

Figure 5. The proof of Lemma B.4.

Q to the intersection of the top side of Q with the hyperbola $xy = 1$; we set α equal to half the length of the top side if the hyperbola and the top side do not intersect).

Suppose $\alpha > \frac{1}{2}A$. For $x_0 - \alpha \le u \le x_0$ let $h(u)$ denote the length of the intersection of $Q \cap \mathcal{F}'$ with the line $x = u$. We note that $h(u)$ is a decreasing function on $[x_0 - \alpha, x_0]$, and since $P \subset \mathcal{F}'$, $h(x_0) \ge b$. Hence

$$|Q \cap \mathcal{F}| \ge \int_{x_0-\alpha}^{x_0} h(u)\, du \ge \alpha b \ge \tfrac{1}{2}Ab.$$

If $\beta > \frac{1}{2}B$ then a similar argument shows that $|Q \cap \mathcal{F}| \ge \frac{1}{2}bB$. Thus the lemma follows from the next statement:

Claim B.5. *Either* $\alpha \ge \frac{1}{2}A$ *or* $\beta \ge \frac{1}{2}B$.

Proof of Claim B.5. If the hyperbola $xy = 1$ does not intersect both the top and the right sides of Q, the claim is automatically satisfied. Thus, we may assume that the hyperbola intersects both the top and the right side. Hence,

$$\alpha = x_0 - \frac{1}{y_0 + B},$$
$$\beta = y_0 - \frac{1}{x_0 + A}.$$

If $B > y_0$ then $A < x_0$ (because $(0, 0) \notin Q$). Hence, $\alpha = x_0 - 1/(y_0 + B) \ge x_0 - 1/(2y_0) = x_0/2 \ge A/2$, which proves the claim in this case. Similarly, if $A > x_0$ then $B < y_0$ (because $(0, 0) \notin Q$) and then $\beta = y_0 - 1/(x_0 + A) \ge y_0 - 1/(2x_0) = y_0/2 \ge B/2$.

Finally if $A < x_0$ and $B < y_0$ we have $Ay_0 < 1$ and $Bx_0 < 1$. Then

$$\alpha\beta = \left(x_0 - \frac{1}{y_0 + B}\right)\left(y_0 - \frac{1}{x_0 + A}\right) = \frac{AB}{(1 + Ay_0)(1 + Bx_0)} \ge \frac{1}{4}AB.$$

Thus either $\alpha \geq A/2$ or $\beta \geq B/2$. This completes the proof of the claim and thus of Lemma B.4. $\square$

We now complete the proof of Lemma B.2. Let R_1 and R_2 be as in Lemma B.2. Since $3\lambda R_1$ and $3\lambda R_2$ are disjoint, we may assume that for some $j \in \{1, 2\}$, $3\lambda R_j$ does not contain the origin. Denote R_j by R, and let $R' \subset R$ be as in Lemma B.3. There is a quadrant containing at least half of R'; without loss of generality we may assume it is the first quadrant. Partition the intersection of R' with the first quadrant into 3×3 equal subrectangles, and let $P = [x_0 - a, x_0] \times [y_0 - b, y_0]$ denote the middle subrectangle. Let $Q = [x_0 - A, x_0 + A] \times [y_0 - B, y_0 + B]$ be the biggest rectangle centered around the top right corner of P and be completely contained in λR. Note that the area of Q is at least a fixed fraction of the area of λR. Let $\mathcal{F}'$ be as in Lemma B.4. We observe that $\mathcal{F}' \subset \mathcal{F}$. Thus by Lemma B.4,

$$|\lambda R \cap \mathcal{F}| \geq |\lambda R \cap \mathcal{F}'| \geq |Q \cap \mathcal{F}'| \geq \frac{1}{8} \min(a/A, b/B)|Q| \geq c\lambda^{-1}|\lambda R| = c\lambda|R|.$$

Thus, if either $\lambda R \cap \mathcal{F}' \subset S(\eta_2)$ or $\lambda R \cap \mathcal{F}' \subset S(2\eta_2)$, Lemma B.2 is proved. Suppose not; then without loss of generality we may assume that $\lambda R \cap \mathcal{F}'$ intersects both of the lines $y = \eta_2$ and $y = 2\eta_2$. Thus, λR contains the points $(\frac{1}{2}(\omega_2/\eta_2), 2\eta_2)$ and $(\omega_2/\eta_2, \eta_2)$, (where ω_2 is as in Lemma B.4). Then, $3\lambda R$ contains the origin, which is a contradiction. This completes the proof of Lemma B.2. $\square$

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