

A geometric proof of the Berger Holonomy Theorem

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Dedicated to Ernst Heintze on the occasion of his sixtieth birthday

Abstract

We give a geometric proof of the Berger Holonomy Theorem. The proof uses Euclidean submanifold geometry of orbits and gives a link between Riemannian holonomy groups and normal holonomy groups.

1. Introduction

Holonomy groups, i.e. the orthogonal groups obtained by parallel transporting along loops, play a central role in Riemannian geometry. The holonomy group measures the deviation of a space from being flat. Moreover, the reducibility of the holonomy group representation implies, via the de Rham theorem, the local product decomposition of the space. One of the most important and beautiful results in Riemannian geometry is the following theorem:

BERGER HOLONOMY THEOREM. *Assume that the holonomy group of an irreducible Riemannian manifold M is not transitive on the sphere. Then M is locally symmetric.*

Many important results make use of the above theorem, for instance, Ballmann's proof [Ba] of the rank rigidity theorem (see also [MO, EO]). Berger's theorem follows from the classification given by Marcel Berger [Be] in 1955 of the possible holonomy groups of nonlocally symmetric irreducible spaces. After some years, James Simons gave an algebraic direct proof of this fact by defining the so-called Riemannian holonomy systems. But the proof of Simons is long and involved, except for the first general part. At some step he used case by case arguments, combined with induction on the dimension. Few math-

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ematicians went through all the details of this proof. The problem of giving a geometric proof of Berger's theorem remained. The purpose of this article is to give such a proof which depends strongly on submanifold geometry of orbits. Moreover, this proof gives a link between the holonomy groups of the normal connection of Euclidean submanifolds and the Riemannian holonomy groups. We hope that this article will serve as a motivation to study Euclidean submanifolds from a holonomic point of view.

The strategy of the proof is simple and can be summarized as follows:

1) The normal space $\nu_v(\Phi.v) \subset T_p M$ to any orbit of the holonomy group Φ of M is, via \exp_p , near 0, a totally geodesic, call it N^v , submanifold of M . This is an easy consequence of the Bianchi identity, the Ambrose-Singer theorem and a theorem of Cartan on the existence of totally geodesic submanifolds. The manifold N^v splits off the direction of v .

2) The normal holonomy group Φ^\perp of the submanifold $\Phi.v$ acts by isometries on N^v . In fact, any curve $c(t)$ in $\Phi.v$ induces, via the normal parallel transport, a perpendicular variation of totally geodesic submanifolds $N^{c(t)}$ of M . Such a variation must be by isometries. By taking loops at v we get the desired result. Moreover, the normal holonomy group must contain the connected component of the isotropy subgroup Φ_v (represented on the normal space; this is a general result for full orbits). But this isotropy group must contain the holonomy of N^v , since v is fixed by the holonomy of N^v . This implies that Φ^\perp contains the holonomy of N^v . From this it is not hard to show that N^v is locally symmetric.

3) If Φ is not transitive then any Φ -principal vector v is contained in a family of normal spaces to nontrivial orbits of Φ . Moreover, this family generates $T_p M$ (this is a general fact for irreducible group actions).

4) Now, for almost any v the geodesic γ_v is locally contained in a family of locally symmetric, totally geodesic submanifolds (whose tangent spaces generate $T_p M$). In this way, the Jacobi operator must diagonalize, with constant coefficients, in a parallel frame. Hence M is locally symmetric at p .

2. Preliminaries and basic facts

In this section we will briefly recall some basic facts from submanifold geometry that we need for the proof of the Berger holonomy theorem. The results stated without proof have a geometric proof (eventually, with simple algebraic ingredients) which can be found in [BCO]. Results which are cited without proof, for the reader's convenience, are only preceded by a number, without labeling them as lemmas, propositions or theorems.

Lie groups are always assumed to be connected.

Let G be a compact subgroup of $SO(n)$. An orbit $G.v$ is called principal if the isotropy subgroup G_v acts trivially on the normal space $\nu_v(G.v)$. Such a v is called a principal vector. The set of principal vectors is open and dense in \mathbb{R}^n . Any principal orbit must be of maximal dimension. If $G.v$ is a principal orbit then any $\xi \in \nu_v(G.v)$ extends (uniquely) to a G -invariant normal field, denoted by $\tilde{\xi}$.

If $G.v$ is not necessarily principal then the *slice representation* is the representation of the isotropy subgroup G_v on the normal space $\nu_v(G.v)$. Observe that the image of the slice representation is the set of restrictions to the normal space of elements of G_v . The connected slice representation is the restriction to $(G_v)_o$ of the slice representation.

Let X belong to the Lie algebra \mathcal{G} of G , v be a principal vector and $\xi \in \nu_v(G.v)$. Let $\gamma(t) = v + t\xi$. Then the Killing field $q \mapsto X.q$ is a (Euclidean) Jacobi field, when restricted to the geodesic γ . Such a Jacobi field will be denoted by $J_\xi(t)$. It is standard and well known that the initial conditions are $J_\xi(0) = X.v$, $J'_\xi(0) = \nabla_{X.v}^\perp \tilde{\xi} - A_\xi(X.v)$, where A denotes the shape operator of $G.v$. If v is not principal, then the initial conditions are similar, but the first term, on the right-hand side of the equality, has to be replaced by the normal covariant derivative along the curve $\text{Exp}(tX).v$ (and the normal field $\tilde{\xi}$ by the field $\text{Exp}(tX)_*\xi$ along the curve $\text{Exp}(tX).v$). Observe that Jacobi fields in \mathbb{R}^n are of the form $u + tw$.

Remark 2.1. If $\xi \in \nu_v(G.v)$, then $v \in \nu_{v+\xi}(G.(v + \xi))$. In fact,

$$\begin{aligned} \{0\} &= \langle T_v(G.v), v + \xi \rangle = \langle \mathcal{G}.v, v + \xi \rangle \\ &= -\langle v, \mathcal{G}.(v + \xi) \rangle = \langle v, T_{v+\xi}(G(v + \xi)) \rangle. \end{aligned}$$

This is nothing else than the general fact that a Killing field is perpendicular to a geodesic if and only if it is perpendicular at one point.

If some orbit $G.v$ is the round sphere $S^{n-1}(\|v\|)$, then $G.w = S^{n-1}(\|w\|)$, for any $w \neq 0$. In this case we say that the action is transitive on the sphere.

LEMMA 2.2. *Let G be a compact subgroup of $SO(n)$ which is not transitive on the sphere and let v be a principal vector. Then there exists $\xi \in \nu_v(G.v)$, not a multiple of v , such that the family of normal spaces $\nu_{\gamma(t)}(G.\gamma(t))$ spans \mathbb{R}^n , where $\gamma(t) = v + t\xi$, $t \in \mathbb{R}$ (in fact, such a ξ is generic).*

Proof. Choose $\xi \in \nu_v(G.v)$ which is not a multiple of the position vector v . Since the shape operator $A_v = -\text{Id}$ we may assume, eventually by adding to ξ a small multiple of v , that $\det(A_\xi) \neq 0$ (i.e. all eigenvalues of A_ξ are different from 0). Let $\gamma(t) = v + t\xi$ and let \mathbb{V} be the orthogonal complement of the linear span of the family $\nu_{\gamma(t)}(G.\gamma(t))$, $t \in \mathbb{R}$. We want to show that $\mathbb{V} = \{0\}$.

By construction \mathbb{V} is contained in any tangent space $T_{\gamma(t)}(G.\gamma(t))$. Let $X \in \mathcal{G}$ be such that $X.v \in \mathbb{V}$ and let $J_\xi(t)$ be its restriction to γ . If $w := J'_\xi(0)$, then $J_\xi(t) = X.v + tw$. Since $J_\xi(t)$ is tangent to the orbit $G.\gamma(t)$, we obtain that $w \perp \nu_{\gamma(t)}(G.\gamma(t))$, for $t \neq 0$. But, for small t , $\gamma(t)$ is a principal vector for the G -action and so, the normal spaces to the associated orbits converge to $\nu_v(G.v)$. Then w is also perpendicular to $\nu_v(G.v)$. Hence $w \in \mathbb{V}$. Since $J'_\xi(0) = \nabla_{X.v}^\perp \tilde{\xi} - A_\xi(X.v)$ and $X.v$ is arbitrary in \mathbb{V} , we conclude that

$$\nabla_{\mathbb{V}}^\perp \tilde{\xi} = 0, \quad A_\xi(\mathbb{V}) \subset \mathbb{V}.$$

Thus, if $\mathbb{W} = \mathbb{V}^\perp \cap T_v(G.v)$, we also have that $A_\xi(\mathbb{W}) \subset \mathbb{W}$. Let now $Y \in \mathcal{G}$ be such that $Y.v \in \mathbb{W}$. Then the Jacobi field $\bar{J}_\xi(t)$ along $\gamma(t)$, induced by Y , has initial conditions $Y.v, \nabla_{Y.v}^\perp \tilde{\xi} - A_\xi(Y.v)$ both of which lie in \mathbb{V}^\perp . Now, $\bar{J}_\xi(t) \perp \mathbb{V}$. Let $X_1, \dots, X_k \in \mathcal{G}$ be such that $X_1.v, \dots, X_k.v$ is an orthonormal basis which diagonalizes the restriction to \mathbb{V} of A_ξ . Then their associated Jacobi fields are $J_\xi^i(t) = (1 - t\lambda_i)X_i.v$, where $\lambda_i \neq 0$ is the eigenvalue of A_ξ associated to $X_i.v$ ($i = 1, \dots, k$). Now let $Z \in \mathcal{G}$ be arbitrary and write $Z = X + Y$, where X is a linear combination of X_1, \dots, X_k and $Y.v \in \mathbb{W}$. From this we obtain that the Jacobi field, induced by Z along γ , at $t = 1/\lambda_i$, is perpendicular to $X_i.v$. Since Z is arbitrary we conclude that $X_i.v \in \nu_{\gamma(1/\lambda_i)}(G.\gamma(1/\lambda_i))$ which is a contradiction, unless $\mathbb{V} = \{0\}$.

The previous lemma was inspired by methods used in a joint paper with Claudio Gorodski and Ruy Tojeiro [GOT].

The isotropy representation of a semisimple (simply connected) symmetric space is called an *s-representation*. If the orthogonal group G is orthogonally equivalent (i.e. conjugate) to (the image of) an *s-representation* then we simply say that G acts as an *s-representation*.

The following result is well known (for a proof see [BCO, p. 192]).

2.3. *If the orthogonal group G acts on \mathbb{R}^n as an s -representation then $N_o(G) = G$, where $N_o(G)$ denotes the connected component of the normalizer of G in $SO(n)$.*

If the connected group G acts on \mathbb{R}^n as an *s-representation* then there are a unique, up to order, orthogonal decomposition $\mathbb{R}^n = \mathbb{V}_0 \times \dots \times \mathbb{V}_r$ (eventually, \mathbb{V}_0 is trivial) and a decomposition $G = G_1 \times \dots \times G_r$, such that G_i acts on \mathbb{V}_i as the isotropy representation of a simple symmetric space, and G_i acts trivially on \mathbb{V}_j , $i \geq 1, j \geq 0, i \neq j$.

We now recall a result concerning the restricted normal holonomy group of a submanifold M of Euclidean space. The proof is not hard after observation that the normal curvature tensor composed with its transpose is an algebraic Riemannian curvature tensor, say \tilde{R} , on the normal bundle with, roughly

speaking, negative scalar curvature (by the Ricci identity). The normal curvature tensor at p is regarded as a linear map from $\Lambda^2(T_p M)$ to $\Lambda^2(\nu_p(M))$. Then one has to average \tilde{R} over the normal holonomy group and apply the Cartan construction of symmetric spaces, in order to obtain that the normal holonomy group corresponds to the isotropy of a symmetric space.

2.4. NORMAL HOLONOMY THEOREM ([O1]). *Let M be a submanifold of Euclidean space and let $p \in M$. Then the restricted normal holonomy group of M at p acts as an s -representation (up to its fixed set point).*

A submanifold of Euclidean space is said to be *full* if it not contained in any proper affine subspace of the ambient space.

Remark 2.5. G acts irreducibly on Euclidean space if and only if any orbit $G.v$, $v \neq 0$, is a full submanifold of Euclidean space.

2.6 ([BCO, Cor. 6.2.6]). *Let G be an orthogonal group of Euclidean space and assume that the orbit $G.v$ is full. Then the image under the connected slice representation of $(G_v)_o$ is contained in the normal holonomy group at v .*

If the (restricted) normal holonomy group of $G.v$ has no fixed set in the normal space then the above result is a consequence of 2.3, since the isotropy subgroup (represented on the normal space) must always be contained in the normalizer of the normal holonomy group (since isometries preserve geometric invariants). So, the above result follows from [O2, Cor. 4.9] after observation that curves have no connected isotropy (in the case that $G.v$ does not split off a curve then any parallel normal field must be G -invariant). A simpler proof of 2.6, in a more general context, follows from [DO2] (see [BCO, Ch. 6]).

We will need the following result of Cartan on the existence of totally geodesic submanifolds of an arbitrary Riemannian manifold. A proof can be found in Section 8.3 of [BCO].

2.7. CARTAN THEOREM ON THE EXISTENCE OF TOTALLY GEODESIC SUBMANIFOLDS. *Let \bar{M} be a Riemannian manifold, $p \in \bar{M}$ and let $\rho > 0$ be the injectivity radius at p . Let V be a subspace of $T_p \bar{M}$ and let V_ρ be the open ball of radius ρ in V . Then $\exp_p(V_\rho)$ is a totally geodesic submanifold of \bar{M} if and only if the curvature tensor of \bar{M} preserves the parallel transport of V along radial geodesics of length less than ρ , with initial condition in V .*

The following lemma is well known.

LEMMA 2.8. *Let $g_t : S \rightarrow M$, $|t| < \varepsilon$, be a smooth family of totally geodesic submanifolds of a Riemannian manifold M . If the variation field $q \mapsto \frac{\partial}{\partial t} g_t(q)$ is perpendicular to the submanifold S_t then $g_t : S_0 \rightarrow S_t$ is an isometry, where S_t is S with the metric induced by g_t .*

Proof. In fact, if γ_w is a geodesic of S_0 through q ,

$$\begin{aligned} \frac{d}{dt} \langle g_{t*q}w, g_{t*q}w \rangle &= \frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial s} \Big|_{s=0} g_t(\gamma_w(s)), \frac{\partial}{\partial s} \Big|_{s=0} g_t(\gamma_w(s)) \right\rangle \\ &= 2 \left\langle \frac{D}{\partial t} \frac{\partial}{\partial s} \Big|_{s=0} g_t(\gamma_w(s)), g_{t*q}w \right\rangle \\ &= 2 \left\langle \frac{D}{\partial s} \Big|_{s=0} \frac{\partial}{\partial t} g_t(\gamma_w(s)), g_{t*q}w \right\rangle \\ &= -2 \langle A_{\xi_t} g_{t*q}w, g_{t*q}w \rangle = 0 \end{aligned}$$

where ξ_t is the variation field at the point q of S_t and A denotes the shape operator of S_t . Then $\langle g_{t*q}w, g_{t*q}w \rangle$ does not depend on t and so $g_t : S_0 \rightarrow S_t$ is an isometry, since $g_0 : S_0 \rightarrow S_0$ is the identity map. \square

The following lemma is auxiliary and well known though it is difficult to find it in the literature.

LEMMA 2.9. *Let M be a Riemannian manifold with the property that for each $p \in M$ every (restricted) holonomy transformation of T_pM extends via the exponential map to a local isometry. Then M is locally symmetric.*

Proof. We may assume that the restricted holonomy group $H(p)$ at p acts irreducibly. Let \mathcal{L} be the Lie algebra of the Killing fields defined in a neighbourhood of $p \in M$. Since $H(p)$ leaves \mathcal{L} invariant, $\mathcal{L}.p$ is an $H(p)$ -invariant subspace of T_pM . But $\mathcal{L}.p$ cannot be trivial, since, for q close to p , $H(q)$ moves p (since $H(q)$ cannot fix any other point near q). Then $\mathcal{L}.p = T_pM$ and so M is a locally homogeneous space. Let \mathcal{N}_p be the normalizer in $\mathfrak{so}(T_pM)$ of the Lie algebra \mathcal{H}_p of $H(p)$ and consider the $\text{Ad}(H(p))$ -equivariant linear map $g : \mathcal{L} \rightarrow \mathcal{N}_p$ defined as follows:

$$g(X) = \frac{d}{dt} \Big|_{t=0} \tau_t^{-1}(\text{Exp}(tX))_{*p} = (\nabla X)_p$$

where τ_t denotes parallel translation along the curve $\text{Exp}(tX).p$. Since isometries preserve holonomy one has that $g(X) \in \mathcal{N}_p$. Moreover, if X belongs to the isotropy algebra \mathcal{L}_p then $g(X) = X$ (where \mathcal{L}_p is identified via the isotropy representation with a subalgebra of $\mathfrak{so}(T_pM)$). So, we may regard $\mathcal{L}_p \subset \mathcal{N}_p$. Observe that one has the following inclusions: $\mathcal{H}_p \subset \mathcal{L}_p \subset \mathcal{N}_p \subset \mathfrak{so}(T_pM)$. Decomposing $\mathcal{N}_p = \mathcal{L}_p \oplus (\mathcal{L}_p)^\perp$, we see that $H(p)$ must act trivially on $(\mathcal{L}_p)^\perp$ (since \mathcal{H}_p acts trivially on its complementary ideal in \mathcal{N}_p). Let $\mathfrak{m} \simeq T_pM$ be an $\text{Ad}(H(p))$ -invariant complementary subspace to \mathcal{L}_p in \mathcal{L} . Now let $\bar{g} : \mathfrak{m} \rightarrow (\mathcal{L}_p)^\perp$ be the projection to $(\mathcal{L}_p)^\perp$ of the restriction $g|_{\mathfrak{m}}$. Then \bar{g} is an $\text{Ad}(H(p))$ -equivariant map. But $H(p)$ acts irreducibly on $\mathfrak{m} \simeq T_pM$ and trivially on $(\mathcal{L}_p)^\perp$. Thus, $\bar{g} = 0$ and so $g(\mathcal{L}) \subset \mathcal{L}_p$. Therefore, for each $v \in T_pM$, there exists a unique $X \in \mathcal{L}$ with $g(X) = 0$ and $X.p = v$. This implies that $\text{Exp}(tX).p$ is a geodesic, and the parallel translation along this geodesic is given by the differential of $\text{Exp}(tX)$. From this the local symmetry follows. \square

We will need the following simple lemma but first we introduce some notation. Let M be a Riemannian manifold, $p \in M$ and let $\rho > 0$ be the injectivity radius at p . For any $v \in T_p M$ we define \mathcal{F}_v to be the family of subspaces of $T_p M$ such that, for any $W \in \mathcal{F}_v$, $v \in W$ and $\exp_p(W_\rho)$ is a totally geodesic submanifold of M which is (intrinsically) locally symmetric, where W_ρ is the Euclidean open ball of radius ρ in W .

LEMMA 2.10 (THE GLUING LEMMA). *Let M be a Riemannian manifold, let $p \in M$ and ρ be the injectivity radius at p . Assume that for any given v in some dense subset Ω of the Euclidean ball $B_\rho^E(0)$, the family \mathcal{F}_v spans $T_p M$. Then the geodesic symmetry s_p at p is an isometry of the geodesic ball $B_\rho(p)$ of M .*

Proof. If N is a locally symmetric space then its Jacobi operator $R(\cdot, \gamma'(t))\gamma'(t)$ diagonalizes with constant coefficients, in a parallel frame along the geodesic $\gamma(t)$. Then, from our assumptions, for $v \in \Omega$, it not hard to see that the Jacobi operator of M along the geodesic $\gamma_v(t)$ diagonalizes, with constant coefficients in a parallel frame. So, one can solve explicitly, in a parallel frame, the Jacobi equation to prove that $(s_p)_{*\gamma_v(1)}$ is an isometry. The lemma follows now from the density of Ω . \square

3. The normal spaces to the orbits of Riemannian holonomy groups

The general theory of holonomy groups can be found in [KN].

Let M be a Riemannian manifold, $p \in M$ and let Φ be its restricted holonomy group at p (i.e. the connected component of the holonomy group at p or, equivalently, the group obtained by parallel transporting along null-homotopic loops at p). Let \mathcal{R} be the family of algebraic curvature tensors of $T_p M$, obtained by pulling back to p , by means of parallel transport along arbitrary curves c starting at p , the curvature tensor $R_{c(1)}$. Then, by the Ambrose-Singer holonomy theorem, the Lie algebra \mathcal{G} of Φ coincides with the linear span of the skew-symmetric endomorphisms $\bar{R}(X, Y)$, $\bar{R} \in \mathcal{R}$, $X, Y \in T_p M$.

We do not assume M to be complete and so, by making M smaller, Φ can be the local holonomy group at p . We assume that Φ acts irreducibly on $T_p M$ (otherwise, by the de Rham decomposition theorem, M would be locally a product, defining the factors parallel distributions in M). It is well known that the group Φ must be compact since it is an orthogonal group acting irreducibly (see [KN, Appendice]). We now consider the orbits of Φ in $T_p M$.

The following proposition, combined with Lemma 2.2, is the key factor in the proof of the Berger Holonomy Theorem.

PROPOSITION 3.1. *Let M be a Riemannian manifold, $p \in M$ and let ρ be the injectivity radius at p . Assume that the holonomy group Φ of M at p acts irreducibly on T_pM . Let, for $v \in T_pM$, $\nu_v(\Phi.v)$ be the normal space at v of the holonomy orbit $\Phi.v$ in T_pM . Denote $N^v = \exp_p(\nu_v(\Phi.v)) \cap B_\rho^E(0)$. Then, for all $v \in T_pM$, $v \neq 0$,*

- i) N^v is a totally geodesic submanifold of M . Moreover N^v splits off, locally, the geodesic γ_v and the holonomy group Φ^v of N^v at p is contained in the image, under the slice representation, of the connected isotropy subgroup $(\Phi_v)_o$.
- ii) The normal holonomy group Φ^\perp of $\Phi.v$ at v acts by isometries on N^v in the natural way (i.e. any $g \in \Phi^\perp$ is the differential of an isometry of N^v which fixes p). Moreover, Φ^\perp contains the holonomy group Φ^v of N^v at p .
- iii) N^v is (intrinsically) locally symmetric.

Proof. (i) Let \mathcal{R} be the family of algebraic Riemannian curvature tensors at p , given by the Ambrose-Singer theorem, which generates the Lie algebra \mathcal{G} of Φ . We have that $\xi \in \nu_v(\Phi.v)$ if and only if $\langle \mathcal{G}.v, \xi \rangle = 0$ which is equivalent to $0 = \langle \bar{R}(X, Y)v, \xi \rangle = \langle \bar{R}(v, \xi)X, Y \rangle$ for all $X, Y \in T_pM$ and for all $\bar{R} \in \mathcal{R}$. Hence $\bar{R}(v, \xi) = 0$ for all $\bar{R} \in \mathcal{R}$. Let η be also in the normal space $\nu_v(\Phi.v)$ and let $\bar{R} \in \mathcal{R}$. Then, by the Bianchi identity, $\bar{R}(\xi, \eta)v = \bar{R}(v, \eta)\xi + \bar{R}(\xi, v)\eta = 0$. Thus, $\bar{R}(\xi, \eta)$ belongs to the isotropy algebra \mathcal{G}_v and $\bar{R}(\xi, \eta)\nu_v(\Phi.v) \subset \nu_v(\Phi.v)$. Then

$$\bar{R}(\nu_v(\Phi.v), \nu_v(\Phi.v))\nu_v(\Phi.v) \subset \nu_v(\Phi.v)$$

for all $\bar{R} \in \mathcal{R}$. This condition easily implies that the hypotheses of 2.7 (the Cartan theorem) are, in particular, fulfilled. Then N^v is totally geodesic.

To prove that N^v splits off the geodesic γ_v , one has only to show that v is fixed by the holonomy group Φ^v of N^v at p . In fact, let \mathcal{R}^v be the subfamily of \mathcal{R} which is obtained by pulling back to p the curvature tensor of M , but only using curves that are contained in N^v . Thus, the Lie algebra of Φ^v is given by the linear span of $\bar{R}(\xi, \eta)|_{\nu_v(\Phi.v)}$, where $\bar{R} \in \mathcal{R}^v$ and $\xi, \eta \in \nu_v(\Phi.v) = T_pN^v$. But we have shown that all these endomorphisms belong to the isotropy algebra at v . Therefore, v is fixed by the holonomy group Φ^v . Moreover, we have proved that the Lie algebra \mathcal{G}^v of Φ^v is contained in the image, under the slice representation, of the isotropy algebra \mathcal{G}_v of Φ at v . This completes the proof of (i).

(ii) Let $v \in T_pM$, $c : [0, 1] \rightarrow \Phi.v$ with $c(0) = v$ and let τ_t^\perp be the parallel transport in the normal connection along $c|_{[0, t]}$. Then $g_t : \nu_v(\Phi.v) \cap B_\rho^E(0) \rightarrow M$ defined by $g_t = \exp_p \circ \tau_t^\perp$ is a one-parameter smooth family of totally geodesic submanifolds. We wish to show that the variation field $X_t = \frac{\partial}{\partial t} g_t$

is always perpendicular to the submanifold $\exp_p(\tau_t^\perp(\nu_v(\Phi.v)) \cap B_\rho^E(0))$. By replacing v by $c(t)$, it suffices to show this at $t = 0$. Let us compute $X_0(\xi)$, $\xi \in \nu_v(\Phi.v) \cap B_\rho^E(0)$. One has that $X_0(s\xi)$ is a Jacobi field along the geodesic $\gamma_\xi(s)$ of M , with initial conditions 0 and $\frac{d}{dt}|_0 \tau_t^\perp(\xi) = -A_\xi c'(0)$ both of which are perpendicular to the tangent space $T_p N^v = \nu_v(\Phi.v)$ (A denotes the shape operator of $\Phi.v$). Then the Jacobi field $X_0(s\xi)$ must always be perpendicular to the tangent space of the totally geodesic submanifold N^v . Since ξ is arbitrary, the X_0 is always perpendicular to N^v . So, by Lemma 2.8, $g_t : N^v \rightarrow N^{c(t)}$ is an isometry. By taking arbitrary loops in $\Phi.v$ through v we obtain that the normal holonomy group acts by isometries on N^v . The last assertion follows now from (i) and 2.6.

(iii) Let $v \in T_p M$ and consider the totally geodesic submanifold N^v . Let c be a curve in N^v , starting at p . Then the parallel transport in M τ_c along c maps the (restricted) holonomy group $\Phi(p) := \Phi$ of M at p into the holonomy group $\Phi(q)$ of M at $q = c(1)$ and so it maps isometrically $\Phi.v$ into $\Phi(q).(\tau_c(v))$ (and so it maps normal spaces into normal spaces). Therefore, $\tau_c : T_p N^v \rightarrow T_q N^{\tau_c(v)}$ is well defined. Since N^v is totally geodesic, it follows that $T_q N^{\tau_c(v)} = T_q N^v$ so that $N^{\tau_c(v)}$ and N^v coincide in a neighbourhood of q . Thus, each holonomy transformation of N^v at q extends, via the exponential map, to a local isometry. Then, from Lemma 2.9, N^v is locally symmetric.

Proof of Berger's theorem. Let $p \in M$ and let \mathcal{O} be the set of principal vectors of $T_p M$. Then \mathcal{O} is open and dense. Since the holonomy group is nontransitive on the sphere we can apply Lemma 2.2 (and Remark 2.1) to obtain that there exists a line $\gamma_\xi(t) = v + t\xi$ in the normal space of $\Phi.v$ at v , which does not go through the origin, such that the normal spaces of orbits of Φ through points of γ_ξ contain v and span $T_p M$. Then, by applying Proposition 3.1 and the Gluing Lemma 2.10, we conclude that M is locally symmetric at p .

Remark 3.2. For a principal v it is easy to prove that N^v is flat and so symmetric. But in the proof of Berger's theorem we used Lemma 2.2 where in general we need to consider focal orbits of the holonomy group (in fact, this is what *a posteriori* happens in a higher rank symmetric space). So, we had to prove in Proposition 3.1 the local symmetry of N^v also for a nonprincipal v .

Remark 3.3. One can avoid the use of Lemma 2.9, in Proposition 3.1, to prove that N^v is locally symmetric. In fact, by making use of induction we have only to handle the irreducible factors with transitive isotropy. So, one can apply the well known result of Szabo on two point homogenous manifolds to conclude that this factor is locally symmetric. But we have preferred to give a more elementary and self-contained proof.

Remark 3.4. In Proposition 3.1, the restricted normal holonomy group Φ^\perp must coincide, *a posteriori*, with the holonomy of N^v . This is so, since it is true for symmetric spaces (see [HO], [BCO, p. 102]).

Final comments. The (restricted) holonomy group of a Lorentzian indecomposable manifold is always transitive, either in hyperbolic space or in a horosphere of hyperbolic space [DO1] (the proof uses the geometry of orbits in hyperbolic space). If the holonomy acts strongly irreducibly, then the holonomy group must be $\mathrm{SO}(n, 1)_o$. This last result follows from classification results of Marcel Berger (a direct proof of this can be found in [DO1]).

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