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The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation

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Abstract

We consider the critical nonlinear Schrödinger equation $iu_t = -\Delta u - |u|^{\frac{4}{N}}u$ with initial condition $u(0,x) = u_0$ in dimension $N = 1$. For $u_0 \in H^1$, local existence in the time of solutions on an interval $[0,T)$ is known, and there exist finite time blow-up solutions, that is, u_0 such that $\lim_{t\uparrow T<+\infty} |u_x(t)|_{L^2} = +\infty$. This is the smallest power in the nonlinearity for which blow-up occurs, and is critical in this sense. The question we address is to understand the blow-up dynamic. Even though there exists an explicit example of blow-up solution and a class of initial data known to lead to blow-up, no general understanding of the blow-up dynamic is known. At first, we propose in this paper a general setting to study and understand small, in a certain sense, blow-up solutions. Blow-up in finite time follows for the whole class of initial data in H^1 with strictly negative energy, and one is able to prove a control from above of the blow-up rate *below* the one of the known explicit explosive solution which has strictly positive energy. Under some positivity condition on an explicit quadratic form, the proof of these results adapts in dimension $N > 1$.

1. Introduction

1.1. *Setting of the problem.* In this paper, we consider the critical nonlinear Schrödinger equation

$$(1) \qquad \text{(NLS)} \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases}$$

with $u_0 \in H^1 = H^1(\mathbf{R}^N)$ in dimension $N \geq 1$. The problem we address is the one of formation of singularities for solutions to (1). Note that this equation is Hamiltonian and in this context few results are known.

It is a special case of the following equation

$$(2) \qquad iu_t = -\Delta u - |u|^{p-1}u$$

where $1 < p < \frac{N+2}{N-2}$ and the initial condition $u_0 \in H^1$. From a result of Ginibre and Velo [8], (2) is locally well-posed in H^1. In addition, (1) is locally well-posed in $L^2 = L^2(\mathbf{R}^N)$ from Cazenave and Weissler [5]. See also [3], [2] for the periodic case and global well posedness results. Thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$ and either $T = +\infty$, where the solution is global, or $T < +\infty$ and then $\limsup_{t\uparrow T} |\nabla u(t)|_{L^2} = +\infty$.

We first recall the main known facts about (1), (2). For $1 < p < \frac{N+2}{N-2}$, (2) admits a number of symmetries in the energy space H^1, explicitly:

- Space-time translation invariance: If $u(t,x)$ solves (2), then so does $u(t+t_0, x+x_0)$, $t_0, x_0 \in \mathbf{R}$.
- Phase invariance: If $u(t,x)$ solves (2), then so does $u(t,x)e^{i\gamma}$, $\gamma \in \mathbf{R}$.
- Scaling invariance: If $u(t,x)$ solves (2), then so does $\lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$, $\lambda > 0$.
- Galilean invariance: If $u(t,x)$ solves (2), then so does $u(t, x-\beta t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)}$, $\beta \in \mathbf{R}$.

From Ehrenfest's law or direct computation, these symmetries induce invariances in the energy space H^1, respectively:

- L^2-norm:

$$\int |u(t,x)|^2 = \int |u_0(x)|^2; \tag{3}$$

- Energy:

$$E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{p+1}\int |u(t,x)|^{p+1} = E(u_0); \tag{4}$$

- Momentum:

$$\operatorname{Im}\left(\int \nabla u \overline{u}(t,x)\right) = \operatorname{Im}\left(\int \nabla u_0 \overline{u_0}(x)\right). \tag{5}$$

The conservation of energy expresses the Hamiltonian structure of (2) in H^1.

For $p < 1 + \frac{4}{N}$, (3), (4) and the Gagliardo-Nirenberg inequality imply

$$|\nabla u(t)|^2_{L^2} \leq C(u_0)\left(|\nabla u(t)|^{2\alpha}_{L^2} + 1\right) \quad \text{for some} \quad \alpha < 1,$$

so that (2) is globally well posed in H^1:

$$\forall t \in [0,T[, \quad |\nabla u(t)|_{L^2} \leq C(u_0) \quad \text{and} \quad T = +\infty.$$

The situation is quite different for $p \geq 1 + \frac{4}{N}$. Let an initial condition $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$ and assume $E(u_0) < 0$, then $T < +\infty$ follows from the so-called virial Identity. Indeed, the quantity $y(t) = \int |x|^2|u|^2(t,x)$ is well defined for $t \in [0,T)$ and satisfies

$$y''(t) \leq C(p)E(u_0)$$

with $C(p) > 0$. The positivity of $y(t)$ yields the conclusion.

The critical power in this problem is $p = 1 + \frac{4}{N}$. From now on, we focus on it. First, note that the scaling invariance now can be written:

- Scaling invariance: If $u(t,x)$ solves (1), then so does

  $$u_\lambda(t,x) = \lambda^{\frac{N}{2}} u(\lambda x, \lambda^2 t), \quad \lambda > 0,$$

  and by direct computation

  $$|u_\lambda|_{L^2} = |u|_{L^2}.$$

Moreover, (1) admits another symmetry which is *not* in the energy space H^1, the so-called pseudoconformal transformation:

- Pseudoconformal transformation: If $u(t,x)$ solves (1), then so does

  $$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}.$$

This additional symmetry yields the conservation of the pseudoconformal energy for initial datum $u_0 \in \Sigma$ which is most frequently expressed as (see [30]):

$$\frac{d^2}{dt^2}\int |x|^2 |u(t,x)|^2 = 4\frac{d}{dt}\mathrm{Im}\left(\int x\nabla u \overline{u}\right)(t,x) = 16E(u_0). \tag{6}$$

At the critical power, special regular solutions play an important role. They are the so-called solitary waves and are of the form $u(t,x) = e^{i\omega t} W_\omega(x)$, $\omega > 0$, where W_ω solves

$$\Delta W_\omega + W_\omega |W_\omega|^{\frac{4}{N}} = \omega W_\omega. \tag{7}$$

Equation (7) is a standard nonlinear elliptic equation. In dimension $N = 1$, there exists a unique solution up to translation to (7) and infinitely many with growing L^2-norm for $N \geq 2$. Nevertheless, from [1], [7] and [11], there is a unique positive solution up to translation $Q_\omega(x)$. In addition Q_ω is radially symmetric. When $Q = Q_{\omega=1}$, then $Q_\omega(x) = \omega^{\frac{N}{4}} Q(\omega^{\frac{1}{2}} x)$ from the scaling property. Therefore, one computes

$$|Q_\omega|_{L^2} = |Q|_{L^2}.$$

Moreover, the Pohozaev identity yields $E(Q) = 0$, so that

$$E(Q_\omega) = \omega E(Q) = 0.$$

In particular, none of the three conservation laws in H^1 (3), (4), (5) of (1) sees the variation of size of the W_ω stationary solutions. These two facts are deeply related to the criticality of the problem, that is the value $p = 1 + \frac{4}{N}$. Note that in dimension $N = 1$, Q can be written explicitly

$$Q(x) = \left(\frac{3}{ch^2(2x)}\right)^{\frac{1}{4}}. \tag{8}$$

Weinstein in [29] used the variational characterization of the ground state solution Q to (7) to derive the explicit constant in the Gagliardo-Nirenberg inequality

$$\forall u \in H^1 \quad , \quad \frac{1}{2+\frac{4}{N}} \int |u|^{\frac{4}{N}+2} \leq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(\frac{\int |u|^2}{\int Q^2} \right)^{\frac{2}{N}}, \tag{9}$$

so that for $|u_0|_{L^2} < |Q|_{L^2}$, for all $t \geq 0$, $|\nabla u(t)|_{L^2} \leq C(u_0)$ and $T = +\infty$, the solution is global in H^1. In addition, blow-up in H^1 has been proved to be equivalent to "blow-up" for the L^2 theory from the following concentration result: If a solution blows up at $T < +\infty$ in H^1, then there exists $x(t)$ such that

$$\forall R > 0, \quad \liminf_{t \uparrow T} \int_{|x-x(t)| \leq R} |u(t,x)|^2 \geq |Q|^2_{L^2}.$$

See for example [18].

On the other hand, for $|u_0|_{L^2} \geq |Q|_{L^2}$, blow-up may occur. Indeed, since $E(Q) = 0$ and $\nabla E(Q) = -Q$, there exists $u_{0\varepsilon} \in \Sigma$ with $|u_{0\varepsilon}|_{L^2} = |Q|_{L^2} + \varepsilon$ and $E(u_{0\varepsilon}) < 0$, and the corresponding solution must blow-up from the virial identity (6).

The case of critical mass $|u_0|_{L^2} = |Q|_{L^2}$ has been studied in [19]. The pseudoconformal transformation applied to the stationary solution $e^{it}Q(x)$ yields an explicit solution

$$S(t,x) = \frac{1}{|t|^{\frac{N}{2}}} Q\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{4t} - \frac{i}{t}} \tag{10}$$

which blows up at $T = 0$. Note that

$$|S(t)|_{L^2} = |Q|_{L^2} \quad \text{and} \quad |\nabla S(t)|_{L^2} \sim \frac{1}{|t|}.$$

It turns out that $S(t)$ is the unique minimal mass blow-up solution in H^1 in the following sense: Let $u(-1) \in H^1$ with $|u(-1)|_{L^2} = |Q|_{L^2}$ and assume that $u(t)$ blows up at $T = 0$; then $u(t) = S(t)$ up to the symmetries of the equation.

In the case of super critical mass $\int |u_0|^2 > \int Q^2$, the situation is more complicated:

- There still exist in dimension $N = 2$ from a result by Bourgain and Wang, [4], solutions of type $S(t)$, that is, with blow-up rate $|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}$.
- Another fact suggested by numerical simulations, see Landman, Papanicolaou, Sulem, Sulem, [12], is the existence of solutions blowing up as

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\ln(|\ln|t||)}{|t|}}. \tag{11}$$

These appear to be stable with respect to perturbation of the initial data. In this frame, for $N = 1$, Perelman in [23] proves the existence of one solution which blows up according to (11) and its stability in some space $E \cap H^1$.

Results in [4] and [23] are obtained by a fixed-point-type arguments and linear estimates, our approach will be different. Note that solutions satisfying (11) are stable with respect to perturbation of the initial data from numerics, but are known to be structurally unstable. Indeed, in dimension $N = 2$, if we consider the next term in the physical approximation leading to (NLS), we get the Zakharov equation

$$\begin{cases} iu_t = -\Delta u + nu \\ \frac{1}{c_0^2} n_{tt} = \Delta n + \Delta |u|^2 \end{cases} \tag{12}$$

for some large constant c_0. Then for all $c_0 > 0$, finite time blow-up solutions to (12) satisfy

$$|\nabla u(t)|_{L^2} \geq \frac{C}{|T-t|}. \tag{13}$$

Note that this blow-up rate is the one of $S(t)$ given by (10). Using a bifurcation argument from (10), we can construct blow-up solutions to (12) with the rate of blow-up (13), and these appear to be numerically stable; see [9] and [22].

Our approach in this paper to study blow-up solutions to (1) is based on a qualitative description of the solution. We focus on the case where the nonlinear dynamic plays a role and interacts with the dispersive part of the solution. This last part will be proved to be small in L^2 for initial conditions which satisfy

$$\int Q^2 < \int |u_0|^2 < \int Q^2 + \alpha_0 \quad \text{and} \quad E(u_0) < 0 \tag{14}$$

where α_0 is small. Indeed, under assumption (14), from the conservation laws and the variational characterization of the ground state Q, the solution $u(t,x)$ remains close to Q in H^1 up to scaling and phase parameters, and also translation in the nonradial case. We then are able to define a regular decomposition of the solution of the type

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)\left(t, \frac{x - x(t)}{\lambda(t)}\right)e^{i\gamma(t)}$$

where $|\varepsilon(t)|_{H^1} \leq \delta(\alpha_0)$ with $\delta(\alpha_0) \to 0$ as $\alpha_0 \to 0$, $\lambda(t) > 0$ is *a priori* of order $\frac{1}{|\nabla u(t)|_{L^2}}$, $\gamma(t) \in \mathbf{R}$, $x(t) \in \mathbf{R}^N$. Here we use first the scaling invariance of (1), and second the fact that the Q_ω are not separated by the invariance of the equation; that is, $E(Q_\omega) = 0$ and $|Q_\omega|_{L^2} = |Q|_{L^2}$.

The problem is to understand the blow-up phenomenon under a dynamical point of view by using this decomposition, and the fact that the scaling parameter $\lambda(t)$ is such that $\frac{1}{\lambda(t)}$ is of size $|\nabla u(t)|_{L^2}$. This approach has been successfully applied in a different context for the critical generalized KdV equation

$$(15) \qquad (\mathrm{KdV}) \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbf{R} \\ u(0,x) = u_0(x), & u_0 : \mathbf{R} \to \mathbf{R} \end{cases}.$$

This equation has indeed a similar structure, except for the lack of conformal transformation which gives explicit blow-up solutions to (1). It has been proved in the papers [13], [14], [15], [16], [17] that for α_0 small enough, if $E(u_0) < 0$ and $\int |u_0|^2 < \int Q^2 + \alpha_0$, then one has:

(i) Blow-up occurence in finite or infinite time, i.e $\lambda(t) \to 0$ as $t \to T$, where $0 < T \le +\infty$.

(ii) Universality of the blow-up profile: $\int \varepsilon^2 e^{-\frac{|y|}{10}} \to 0$ as $t \to T$.

(iii) Finite time blow-up under the additional condition $\int_{x>0} x^6 |u_0|^2 < +\infty$; i.e., $T < +\infty$, and moreover $|u_x(t)|_{L^2} \le \frac{C}{T-t}$ in a certain sense.

From the proof of these results, blow-up appeared in this setting as a consequence of qualitative and dynamical properties of solutions to (15).

1.2. *Statement of the theorem.* In this paper, our goal is to derive some dynamical properties of solutions to (1) such that $\int |u_0|^2 \le \int |Q|^2 + \alpha_0$ for some small α_0, and $E(u_0) < 0$. In particular, we derive a control from *above* of the blow rate for such solutions. More precisely, we claim the following:

THEOREM 1 (Blow-up in finite time and dynamics of blow-up solutions for $N = 1$). *Let $N = 1$. There exists $\alpha^* > 0$ and a universal constant $C^* > 0$ such that the following is true. Let $u_0 \in H^1$ be such that*

$$0 < \alpha_0 = \alpha(u_0) = \int |u_0|^2 - \int Q^2 < \alpha^*$$

and

$$(16) \qquad E(u_0) < \frac{1}{2} \left(\frac{\mathrm{Im}(\int (u_0)_x \overline{u_0})}{|u_0|_{L^2}} \right)^2 .$$

Let $u(t)$ be the corresponding solution to (1), *then*:

(i) *$u(t)$ blows up in finite time, i.e. there exists $0 < T < +\infty$ such that*

$$\lim_{t \uparrow T} |u_x(t)|_{L^2} = +\infty.$$

(ii) *Moreover, for t close to T,*

$$(17) \qquad |u_x(t)|_{L^2} \le C^* \left(\frac{|\ln(T-t)|^{\frac{1}{2}}}{T-t} \right)^{\frac{1}{2}} .$$

In fact, from Galilean invariance, we view this result as a consequence of the following:

THEOREM 2. *Let $N = 1$. There exists $\alpha^* > 0$ and a universal constant $C^* > 0$ such that the following is true. Let $u_0 \in H^1$ such that*

$$0 < \alpha_0 = \alpha(u_0) = \int |u_0|^2 - \int Q^2 < \alpha^*, \tag{18}$$
$$E_0 = E(u_0) < 0,$$
$$\mathrm{Im}\left(\int (u_0)_x \overline{u_0}\right) = 0,$$

and $u(t)$ be the corresponding solution to (1), *then conclusions of Theorem* 1 *hold.*

Proof of Theorem 1 *assuming Theorem* 2. Let $N = 1$ and u_0 be as in the hypothesis of Theorem 1. We prove that up to one fixed Galilean invariance, we satisfy the hypothesis of Theorem 2. The following is well known: let $u(t,x)$ be a solution of (NLS) on some interval $[0,t_0]$ with initial condition $u_0 \in H^1$; then for all $\beta \in \mathbf{R}$, $u^\beta(t,x) = u(t, x-\beta t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)}$ is also an H^1 solution on $[0,t_0]$. Moreover,

$$\forall t \in [0,t_0], \quad \mathrm{Im}\left(\int u_x \overline{u}\right)(t) = \mathrm{Im}\left(\int u_x \overline{u}\right)(0). \tag{19}$$

We denote $u_0^\beta = u^\beta(0,x) = u_0(x)e^{i\frac{\beta}{2}x}$ and compute invariant (19)

$$\mathrm{Im}\left(\int (u_0^\beta)_x \overline{u_0^\beta}\right) = \mathrm{Im}\int \left((u_0)_x + i\frac{\beta}{2}u_0\right)\overline{u_0} = \frac{\beta}{2}\int |u_0|^2 + \mathrm{Im}\int (u_0)_x\overline{u_0}.$$

We then choose $\beta = -2\frac{\mathrm{Im}(\int (u_0)_x \overline{u_0})}{\int |u_0|^2}$ so that for this value of β

$$\mathrm{Im}\left(\int (u_0^\beta)_x \overline{u_0^\beta}\right) = 0.$$

We now compute the energy of the new initial condition u_0^β and easily evaluate from the explicit value of β and condition (16):

$$E(u_0^\beta) = \frac{1}{2}\int \left|(u_0)_x + i\frac{\beta}{2}u_0\right|^2 - \frac{1}{6}\int |u_0^\beta|^6 = E(u_0) + \frac{\beta}{4}\,\mathrm{Im}\int (u_0)_x\overline{u_0} < 0.$$

Therefore, u_0^β satisfies the hypothesis of Theorem 2. To conclude, we need only note that

$$\int |u_x(t,x)|^2 = \int |u_x^\beta(t,x)|^2 + \frac{\beta^2}{4}\int |u_0|^2 + \beta\,\mathrm{Im}\int (u_0)_x\overline{u_0}$$

so that the explosive behaviors of $u(t,x)$ and $u_\beta(t,x)$ are the same. This concludes the proof of Theorem 1 assuming Theorem 2.

Let us now consider the higher dimensional case $N \geq 2$. The proof of Theorem 1 can indeed be carried out in higher dimension assuming positivity properties of a quadratic form. See Section 4.4 for more details and comments for the case $N \geq 2$. Consider the following property:

SPECTRAL PROPERTY. *Let $N \geq 2$. Set $Q_1 = \frac{N}{2}Q + y \cdot \nabla Q$ and $Q_2 = \frac{N}{2}Q_1 + y \cdot \nabla Q_1$. Consider the two real Schrödinger operators*

$$\mathcal{L}_1 = -\Delta + \frac{2}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}y\cdot\nabla Q \quad , \quad \mathcal{L}_2 = -\Delta + \frac{2}{N}Q^{\frac{4}{N}-1}y\cdot\nabla Q, \tag{20}$$

and the quadratic form for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$:

$$H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1,\varepsilon_1) + (\mathcal{L}_2\varepsilon_2,\varepsilon_2).$$

Then there exists a universal constant $\tilde{\delta}_1 > 0$ such that for all $\varepsilon \in H^1$, if $(\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = (\varepsilon_2, \nabla Q) = 0$, then

(i) *for $N = 2$,*

$$H(\varepsilon,\varepsilon) \geq \tilde{\delta}_1\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right)$$

for some universal constant $2^- < 2$;

(ii) *for $N \geq 3$,*

$$H(\varepsilon,\varepsilon) \geq \tilde{\delta}_1 \int |\nabla\varepsilon|^2.$$

We then claim:

THEOREM 3 (Higher dimensional case). *Let $N \geq 2$ and assume the Spectral Property holds true; then there exists $\alpha^* > 0$ and a universal constant $C^* > 0$ such that the following is true. Let $u_0 \in H^1$ such that*

$$0 < \alpha_0 = \alpha(u_0) = \int |u_0|^2 - \int Q^2 < \alpha^*, \quad E_0 < \frac{1}{2}\left(\frac{|\mathrm{Im}(\int \nabla u_0 \overline{u_0})|}{|u_0|_{L^2}}\right)^2.$$

Let $u(t)$ be the corresponding solution to (1); *then $u(t)$ blows up in finite time $0 < T < +\infty$ and for t close to T:*

$$|\nabla u(t)|_{L^2} \leq C^*\left(\frac{|\ln(T-t)|^{\frac{N}{2}}}{T-t}\right)^{\frac{1}{2}}.$$

Comments on the result.

1. *Spectral conjecture*: For $N = 1$, the explicit value of the ground state Q allows us to compare the quadratic form H involved in the Spectral Property with classical known Schrödinger operators. The problem reduces then to checking the sign of some scalar products, what is done numerically. We conjecture that the Spectral Property holds true at least for low dimension.

2. *Blow-up rate*: Assume that u blows up in finite time. By scaling properties, a known *lower* bound on the blow-up rate is

$$|\nabla u(t)|_{L^2} \geq \frac{C^*}{\sqrt{T-t}}. \tag{21}$$

Indeed, consider for fixed $t \in [0,T)$

$$v^t(\tau,z) = |\nabla u(t)|_{L^2}^{-\frac{N}{2}} u\left(t + |\nabla u(t)|_{L^2}^{-2}\tau, |\nabla u(t)|_{L^2}^{-1} z\right).$$

By scaling invariance, v^t is a solution to (1). We have $|\nabla v^t|_{L^2} + |v^t|_{L^2} \leq C$, and so by the resolution of the Cauchy problem locally in time by a fixed point argument (see [10]), there exists $\tau_0 > 0$ independent of t such that v^t is defined on $[0,\tau_0]$. Therefore, $t + |\nabla u(t)|_{L^2}^{-2}\tau_0 \leq T$ which is the desired result.

The problem here is to control the blow-up rate from *above*. Our result is the first of this type for critical NLS. No upper bound on the blow-up rate was known, not even of exponential type. Note indeed that there is no Lyapounov functional involved in the proof of this result, and that it is purely a dynamical one.

We exhibit a first upper bound on the blow-up rate as

$$|\nabla u(t)|_{L^2} \leq \frac{C^*}{\sqrt{|E_0|}(T-t)} \tag{22}$$

for some universal constant $C^* > 0$. This bound is optimal for NLS in the sense that there exist blow-up solutions with this blow-up rate. Indeed, apply the pseudoconformal transformation to the stationary solutions $e^{i\omega^2 t}\omega^{\frac{N}{2}}Q(\omega x)$ to get explicit blow-up solutions

$$S_\omega(t,x) = \left(\frac{\omega^2}{|t|}\right)^{\frac{N}{2}} e^{-i\frac{\omega}{t}+i\frac{x^2}{4t}} Q\left(\frac{\omega x}{t}\right).$$

Then one easily computes

$$|S_\omega|_{L^2} = |Q|_{L^2} \quad , \quad E(S_\omega) = \frac{C}{\omega^2} \quad , \quad |\nabla S_\omega(t)|_{L^2} = \frac{\omega C}{|t|},$$

so that

$$|\nabla S_\omega(t)|_{L^2} \sim \frac{\sqrt{C}}{\sqrt{|E_0|} \quad |t|} \quad as \quad t \to 0.$$

Note nevertheless that these solutions have strictly positive energy and $\alpha_0 = 0$.

In our setting of strictly negative energy initial conditions, no solutions of this type is known, and we indeed are able to improve the upper bound by excluding any polynomial growth between the pseudoconformal blow-up (22) and the scaling estimate (21) by

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{|\ln(T-t)|^{\frac{1}{2}}}{T-t}\right)^{\frac{1}{2}}.$$

It says in particular that the blow-up rate is the one of the scaling up to a logarithmic correction. Nevertheless, we do not expect this control to be optimal in the logarithmic scale according to the expected double logarithm behavior (11). Note that the fact that the whole open set of strictly negative energy solutions shares the same dynamical behavior and in particular never sees the rate of explicit blow-up solution $S(t)$ is new and noteworthy.

We would like to point out that the improvement of blow-up rate control from estimate (22) to (17) heavily relies on algebraic cancellations deeply related to the degeneracy of the linear operator around Q which are unstable with respect to "critical" perturbations of the equation. Indeed, recall for example that all strictly negative energy solutions to the Zakharov equation (12) satisfy the lower bound (13). On the other hand, we expect the first argument to be structurally stable in a certain sense.

3. *About the exact* lnln *rate of blow-up*: We expect from the result that strictly negative energy solutions blow-up with the exact lnln law: $|\nabla u(t)|_{L^2} \sim C^* \left(\frac{\ln|\ln(T-t)|}{T-t}\right)^{\frac{1}{2}}$. There exist different formal approaches to derive this law, see [25] and references therein, all somehow based on an asymptotic expansion of the solution at very high order near blow-up time. Perelman in [23] has succeeded in dimension $N=1$ for a very specific symmetric initial data close at a very high order to these formal types of solutions in building, using a fixed point argument, an exact solution satisfying this law. Our approach is different: we consider the large set of initial data with strictly negative energy, in any dimension where formal asymptotic developments fail, and then prove *a priori* some rigidity properties of the dynamics in H^1 which yield finite time blow-up and an upper bound only on the blow-up rate. From the works on critical KdV by Martel and Merle, [14], lower bounds on the blow-up rate involve a different analysis of dispersion in L^2 which is not yet available for (1).

4. *Blow-up result*: In the situation $\int |u_0|^2 \leq \int |Q|^2 + \alpha_0$, we show that blow-up is related to local in space information, and we do not need the additional assumption $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$. Previous results were known in the symmetric case (and $N=1$) when the singularity forms at 0 (see [21]), and in the nonradial case, Nawa in [20] proved for strictly negative energy solutions the existence of a sequence of times t_n such that $\lim_{n\to+\infty} |\nabla u(t_n)|_{L^2} = +\infty$. In fact, our result decomposes into two stages:

(i) First, the solution blows up in H^1 in finite or infinite time T.

(ii) Second, a refined study of the nonlinear dynamic ensures $T < +\infty$. Note that for $E(u_0) < 0$, this last fact is unknown for critical KdV (and it is unclear whether it would be true). Note moreover that the result holds for $t < 0$ with $\overline{u}(-t, x)$ which also is a solution to (1) satisfying the hypothesis of Theorem 2.

5. *Comparison with critical* KdV: In the context of Hamiltonian systems in infinite dimension with infinite speed of propagation, the only known results of this type are for the critical generalized KdV equation, for which the proofs were delicate. The situation here is quite different. On the one hand, the existence of symmetries related to the Galilean and the pseudoconformal transformation induces more localized properties of (1) viewed in the ε variable, and we do not need to focus on exponential decay properties of the limit problem which was the key to all proofs in the study of (15).

On the other hand, from these invariances, additional degeneracies related to the underlying structure of (1) arise and tend to make the analysis of the interactions more complicated.

1.3. *Strategy of the proof.* We briefly sketch in this subsection the proof of Theorem 2. We consider equation (1) in dimension $N = 1$ for an initial datum close to Q in L^2, with strictly negative energy and zero momentum. See Section 4.4 for the higher dimensional case.

First, from the assumption of closeness to Q in L^2 and the strictly negative energy condition, variational estimates allow us to write

$$u(x,t) = \frac{e^{i\overline{\gamma}(t)}}{\overline{\lambda}^{\frac{1}{2}}(t)}(Q+\varepsilon)\left(\frac{x-\overline{x}(t)}{\lambda(t)}, t\right)$$

for some functions $\overline{\lambda}(t) > 0$, $\overline{\gamma}(t) \in \mathbf{R}$, $\overline{x}(t) \in \mathbf{R}$ such that

$$\frac{1}{\overline{\lambda}(t)} \sim |u_x(t)|_{L^2} \tag{23}$$

and ε *a priori* small in H^1.

The ε equation inherited from (1) can be written after a change of time scale $\frac{ds}{dt} = \frac{1}{\overline{\lambda}^2(t)}$:

$$i\partial_s\varepsilon + L\varepsilon = i\frac{\overline{\lambda}_s}{\overline{\lambda}}\left(\frac{Q}{2} + yQ_y\right) + \overline{\gamma}_s Q + i\frac{\overline{x}_s}{\overline{\lambda}}Q_y + R(\varepsilon)$$

with $R(\varepsilon)$ quadratic in $\varepsilon = \varepsilon_1 + i\varepsilon_2$. Using modulation theory from scaling, phase and translation invariance, we slightly modify $\overline{\lambda}(t), \overline{\gamma}(t), \overline{x}(t)$ so that ε satisfies suitable orthogonality conditions

(24)

$$\left(\varepsilon_1, \frac{Q}{2} + yQ_y\right) = (\varepsilon_1, yQ) = \left(\varepsilon_2, \frac{1}{2}\left(\frac{Q}{2} + yQ_y\right) + y\left(\frac{Q}{2} + yQ_y\right)_y\right) = 0.$$

Note that we do not use modulation theory with parameters related to the pseudoconformal transformation or to Galilean invariance, this last symmetry being used only to ensure (18).

Two noteworthy facts hold for this decomposition:

(i) Orthogonality conditions (24) are adapted to the dispersive structure of (1) for $\varepsilon \in H^1$ inherited from the virial relation (6) for $u \in \Sigma$, as they allow cancellations of some oscillatory integrals in time. Indeed, we get control of second order terms in ε of the form

$$\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|} \leq C(\varepsilon_2, \frac{Q}{2} + yQ_y)^2 \tag{25}$$

in a time-averaging sense, and for some fixed universal constant $2^- < 2$.

(ii) This decomposition is also adapted to the study of variations of size of u, or equivalently the equation governing the scaling parameter $\lambda(s)$ from (23), as we will prove

$$-\frac{\lambda_s}{\lambda} \sim (\varepsilon_2, \frac{Q}{2} + yQ_y) \tag{26}$$

in a time-averaging sense, up to quadratic terms.

Note that the same scalar product $(\varepsilon_2, \frac{Q}{2} + yQ_y)$ is involved, and in fact governs the whole dynamic, and that the ε decomposition we introduce is adapted to both (i) and (ii), while two different decompositions had to be considered in the proof of [15]. From these two facts, we exhibit the sign structure of $(\varepsilon_2, \frac{Q}{2} + yQ_y)$, which is the main key to our analysis, by showing

$$\exists s_0 \in \mathbf{R} \text{ such that } \forall s > s_0, \quad \left(\varepsilon_2, \frac{Q}{2} + yQ_y\right)(s) > 0.$$

Together with (23), (25) and (26), the almost monotonicity result of the scaling parameter $\lambda(t)$ follows:

$$\exists t_0 \in \mathbf{R} \text{ such that } \forall t' \geq t \geq t_0 \quad , \quad |u_x(t')|_{L^2} \geq \frac{1}{2}|u_x(t)|_{L^2}.$$

This property removes the difficult problem of oscillations in time of the size of the solution which had to be taken into account in the study of (15).

The proof of Theorem 2 now follows in two steps:

(i) First, we prove a finite or infinite time blow-up result; i.e., there exists $0 < T \leq +\infty$ such that

$$\lim_{t \uparrow T} |u_x(t)|_{L^2} = +\infty \quad \text{or equivalently} \quad \lim_{s \to +\infty} \lambda(s) = 0.$$

(ii) To prove blow-up in finite time and the desired upper-bound on $|u_x(t)|_{L^2}$, we study as in [15] dispersion onto intervals of slow variations of the scaling parameter. The existence of such intervals heavily relies on the first step. More precisely, we consider a sequence t_n such that

$$|u_x(t_n)|_{L^2} = 2^n \text{ or equivalently } \lambda(t_n) \sim 2^{-n}.$$

To prove an upper bound on the blow-up rate, the strategy is to exhibit two different links between the key scalar product $(\varepsilon_2, \frac{Q}{2} + yQ_y)$ and the scaling parameter λ, which formally leads according to (26) to a differential inequality for λ. We then rigorously work out this differential inequality by working on the slow variations intervals $[t_n, t_{n+1}]$.

Now, we exhibit two different ways to get pointwise control of λ by $(\varepsilon_2, \frac{1}{2}Q + yQ_y)$, which lead to two different controls on the blow-up rate:

1. A first estimate heavily relies on monotonicity results inherited from the *basic* dispersive structure of (1) in the ε variable and further dynamical arguments, and can be written for s large enough

$$|E_0|\lambda^2(s) \leq B \left(\varepsilon_2, \frac{Q}{2} + yQ_y\right)^2 (s), \tag{27}$$

for some universal constant $B > 0$. Putting together (26) and (27), we prove the integral form of the differential inequality

$$-\frac{\lambda_s}{\lambda} \geq C\sqrt{|E_0|}\lambda \quad \text{or equivalently} \quad -\lambda_t \geq C\sqrt{|E_0|}$$

from $\frac{ds}{dt} = \frac{1}{\lambda^2}$; that is, explicitly

$$t_{n+1} - t_n \leq \frac{C}{\sqrt{|E_0|}}\lambda(t_n) \leq \frac{C}{\sqrt{|E_0|}}2^{-n}.$$

This allows us to conclude the finitness of the blow-up time, and the bound

$$|u_x(t)|_{L^2} \leq \frac{C^*}{\sqrt{|E_0|}(T-t)}.$$

2. Using a degeneracy property of the linearized operator close to Q which is unstable with respect to perturbation, we exhibit a *refined* dispersive structure in the ε variable and much better control: for s large enough

$$\lambda^2(s) \leq \exp\left(-\frac{\tilde{B}}{(\varepsilon_2, \frac{Q}{2} + yQ_y)^2(s)}\right), \tag{28}$$

for some universal constant $\tilde{B}$. Putting (26) and (28) together again, we prove the integral form of the differential inequality

$$-\frac{\lambda_s}{\lambda} \geq \frac{C}{\sqrt{|\ln(\lambda(s))|}},$$

or more precisely,

$$t_{n+1} - t_n \leq C\lambda^2(t_n)|\ln(\lambda(t_n)|^{\frac{1}{2}} \leq C2^{-2n}\sqrt{n},$$

which leads to the bound

$$|u_x(t)|_{L^2} \leq C^*\left(\frac{|\ln(T-t)|^{\frac{1}{2}}}{T-t}\right)^{\frac{1}{2}}.$$

This paper is organized as follows. In Section 2, we build the regular ε decomposition adapted to dispersion with the suitable orthogonality conditions on ε. In Section 3, we exhibit the local dispersive inequality in L^2_{loc} inherited from the virial structure of (1) in Σ. The almost monotonicity of the scaling parameter then follows. In Section 4, we prove Theorem 2, and focus in Section 4.4 on the higher dimensional case. Except in Section 4.4, we shall always work with (1) in dimension $N = 1$.

2. Regular decomposition of negative energy solutions

In this section and the following, we build a general setting to study negative energy solutions to (NLS) whose L^2-norm is close enough to the one of the soliton. Here, we derive from variational estimates and conservation laws a sharp decomposition of such solutions and its basic properties.

From now on, we consider $u_0 \in H^1$ such that

$$\alpha_0 = \alpha(u_0) = \int |u_0|^2 - \int Q^2 < \alpha^*, \quad E_0 = E(u_0) < 0, \ \mathrm{Im}\left(\int (u_0)_x \overline{u_0}\right) = 0$$

for some $0 < \alpha^*$ small enough, to be chosen later.

2.1. *Decomposition of the solution and related variational structure.* Let us start with a classical lemma of proximity of the solution up to scaling, phase and translation factors to the function Q related to the variational structure of Q and the energy condition. For $u \in H^1$, we note $\alpha(u) = \int |u|^2 - \int Q^2$.

LEMMA 1. *There exists a $\alpha_1 > 0$ such that the following property is true. For all $0 < \alpha' \leq \alpha_1$, there exists $\delta(\alpha')$ with $\delta(\alpha') \to 0$ as $\alpha' \to 0$ such that for all $u \in H^1$, if*

$$0 < \alpha(u) < \alpha' \ \text{ and } \ E(u) \leq \alpha' \int |u_x|^2, \tag{29}$$

then there exist parameters $\gamma_0 \in \mathbf{R}$ and $x_0 \in \mathbf{R}$ such that

$$|Q - e^{i\gamma_0} \lambda_0^{1/2} u(\lambda_0(x + x_0))|_{H^1} < \delta(\alpha') \tag{30}$$

with $\lambda_0 = \frac{|Q_x|_{L^2}}{|u_x|_{L^2}}$.

Proof of Lemma 1. It is a classical result. See for example [14]. Let us recall the main steps. The proof is based on the variational characterization of the ground state in $H^1(\mathbf{C})$. Recall from the variational characterization of the function Q (following from the Gagliardo-Nirenberg inequality) that for $u \in H^1(\mathbf{R})$,

$$E(u) = 0, \quad \int |u|^2 = \int Q^2, \quad \int |u_x|^2 = \int {Q_x}^2, \quad u \geq 0$$

is equivalent to

$$u = Q(. + x_0) \quad \text{for some} \quad x_0 \in \mathbf{R}.$$

Now let $u \in H^1(\mathbf{C})$ be such that $E(u) = 0$ and $\int |u|^2 = \int Q^2$. Then $|u| \in H^1(\mathbf{R})$ satisfies $\int (|u|_x)^2 \leq \int |u_x|^2$, so that $E(|u|) \leq E(u) = 0$. But from Gagliardo-Nirenberg, $\int |u|^2 = \int Q^2$ implies $E(|u|) \geq 0$, so that $E(|u|) = E(u) = 0$, and $|u| = \lambda_0^{\frac{1}{2}} Q(\lambda_0(\cdot + x_0))$ for some parameters $\lambda_0 > 0$ and $x_0 \in \mathbf{R}$. Consequently, u does not vanish on $\mathbf{R}$ and one may write $u = |u|e^{i\theta}$ so that $\int |u_x|^2 = \int (|u|)_x^2 + \int |u|^2(\theta_x)^2$. From $E(|u|) = E(u)$, we conclude $\theta(x)$ is a constant. In other words, if $u \in H^1(C)$ is such that

$$E(u) = 0 \;\text{ and }\; \int |u|^2 = \int Q^2,$$

then

$$u = e^{i\gamma_0}\lambda_0^{\frac{1}{2}} Q(\lambda_0(\cdot + x_0)) \text{ for some parameters } \lambda_0 > 0, \gamma_0 \in \mathbf{R} \text{ and } x_0 \in \mathbf{R}.$$

We now prove Lemma 1 and argue by contradiction. Assume that there is a sequence $u_n \in H^1(\mathbf{C})$ such that

$$\lim_{n\to+\infty} \int |u_n|^2 = \int Q^2 \;\text{ and }\; \lim_{n\to\infty} \frac{E(u_n)}{\int |u_{nx}|^2} \leq 0.$$

Consider now $v_n = \lambda_n^{1/2} u_n(\lambda_n x)$, where $\lambda_n = \frac{|Q_x|_{L^2}}{|u_{nx}|_{L^2}}$. We have the following properties for v_n,

$$\int |v_n|^2 \to \int Q^2 \quad , \quad \int |v_{nx}|^2 = 1 \quad \text{and} \quad \lim_{n\to+\infty} E(v_n) \leq 0.$$

From Gagliardo-Nirenberg inequality $E(v_n) \geq \frac{1}{2}\left(\int |v_{nx}|^2\right)\left(1 - \left(\frac{|v_n|_{L^2}}{|Q|_{L^2}}\right)^4\right)$, we conclude $E(v_n) \to 0$. Using classical concentration compactness procedure, we are able to show that there is $x_n \in \mathbf{R}$ and $\gamma_n \in \mathbf{R}$ such that $e^{i\gamma_n} v_n(x+x_n) \to Q$ in H^1. See for example [27], [28]. This concludes the proof of Lemma 1. □

It is now natural to modulate the solution u to (1) according to the three fundamental symmetries, scaling, phase and translation, by setting

$$\varepsilon(t,y) = e^{i\gamma(t)}\lambda^{1/2}(t)u(t, \lambda(t)y + x(t)) - Q(y)$$

and to study the remainder term ε, which will be proved to be small.

Let us formally compute the equation verified by ε after the change of time scale $\frac{ds}{dt} = \frac{1}{\lambda^2(t)}$:

$$(31) \qquad i\varepsilon_s + L\varepsilon = i\frac{\lambda_s}{\lambda}\left(\frac{Q}{2} + yQ_y\right) + \gamma_s Q + i\frac{x_s}{\lambda}Q_y + R(\varepsilon).$$

$R(\varepsilon)$ is formally quadratic in ε, and L is the linear operator close to the ground state. A first strategy to understand equation (31) is to neglect the nonlinear terms $R(\varepsilon)$ which should be small according to (30), and to study the linear equation

$$i\varepsilon_s + L\varepsilon = F$$

for some fixed function F. This operator and the properties of the propagator e^{itL} have been extensively studied in [27], [28], [4].

When considering the linear equation underlying (31), the situation is as follows. The operator L, which is a matrix operator $L = (L_+, L_-)$, has a so-called generalized null space reproducing all the symmetries of (1) in H^1. This leads to the following algebraic identities:

$$\begin{aligned} L_+\left(\frac{Q}{2} + yQ_y\right) &= -2Q && \text{(scaling invariance)},\\ L_+(Q_y) &= 0 && \text{(translation invariance)},\\ L_-(Q) &= 0 && \text{(phase invariance)},\\ L_-(yQ) &= -2Q_y && \text{(Galilean invariance)}. \end{aligned}$$

An additional relation induced by the pseudoconformal transformation holds in the critical case

$$L_-(y^2Q) = -4\left(\frac{Q}{2} + yQ_y\right)$$

and leads to the existence of an additional mode in the generalized null space of L not generated by a symmetry usually denoted ρ. This solves

$$L_+\rho = -y^2Q.$$

These directions lead to the existence of growing solutions in H^1 to the linear equation. More precisely, Weinstein proved on the basis of the spectral structure of L the existence of a decomposition $H^1 = M \oplus S$, where S is finite-dimensional, with $|e^{itL}\varepsilon|_{H^1} \leq C$ for $\varepsilon \in M$ and $|e^{itL}\varepsilon|_{H^1} \sim t^3$ for $\varepsilon \in S$. The linear kind of strategies developed were then as follows: as each symmetry is at the heart of a growing direction in time for the solutions to the linear problem, one uses modulation theory, modulating on *all* the symmetries of (1), that is also Galilean invariance and pseudoconformal transformation, to *a priori* get rid of these directions. Note nevertheless that as the pseudoconformal transformation is not in the energy space and induces the additional degenerated direction ρ, the analysis is here usually very difficult. Indeed, this linear approach has been successfully applied only in [23] to build one stable blow-up solution. See [24] for other applications, and also Fibich, Papanicolaou [6] and Sulem, Sulem [25], for a more heuristic and numerical study.

Our approach is here quite different and more nonlinear. We shall use modulation theory only for the three fundamental symmetries which are scaling, phase and translation in the nonradial case. Galilean invariance is used directly on the initial data u_0 to get extra cancellation (18) which is preserved in time. Moreover, we shall make no explicit use of the pseudoconformal transformation as this symmetry is not in the energy space. In particular, we do not cover the two degenerate directions of the linearized operator induced by the pseudoconformal invariance. And when using modulation theory, the directions we should *a priori* decide to avoid are not related to the spectral structure of the linearized operator L, but to the dispersive structure in the ε variable underlying (1). This structure is not inherent to the energetic structure, that is, the study of L, but to the virial type structure related to dispersion, as was the case for the KdV equation; see the third section for more details.

2.2. *Sharp decomposition of the solution.* We now are able to have the following decomposition of the solution $u(t,x)$ for $\alpha(u_0)$ small enough. The choice of orthogonality conditions will be clear from the next section. We fix the following notation:

$$Q_1 = \frac{1}{2}Q + yQ_y \quad \text{and} \quad Q_2 = \frac{1}{2}Q_1 + y(Q_1)_y.$$

LEMMA 2 (Modulation of the solution). *There exists $\alpha_2 > 0$ such that for $\alpha_0 < \alpha_2$, there exist some continuous functions $\lambda : [0,T) \to (0,+\infty)$, $\gamma : [0,T) \to \mathbf{R}$ and $x : [0,T) \to \mathbf{R}$ such that*

$$\forall t \in [0,T) \quad , \quad \varepsilon(t,y) = e^{i\gamma(t)}\lambda^{1/2}(t)u(t,\lambda(t)y + x(t)) - Q(y) \tag{32}$$

satisfies the following properties:

(i)

$$(\varepsilon_1(t), Q_1) = 0 \quad \textit{and} \quad (\varepsilon_1(t), yQ) = 0 \tag{33}$$

and

$$(\varepsilon_2(t), Q_2) = 0, \tag{34}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$ in terms of real and imaginary parts.

(ii)

(35)
$$|1 - \lambda(t)\frac{|u_x(t)|_{L^2}}{|Q_x|_{L^2}}| + |\varepsilon(t)|_{H^1} \leq \delta(\alpha_0) \quad , \quad \textit{where} \quad \delta(\alpha_0) \to 0 \quad \textit{as} \quad \alpha_0 \to 0.$$

Proof of Lemma 2. The proof is similar to that of Lemma 1 in [14]. Let us briefly recall it. By conservation of the energy, we have for all $t \in [0,T)$, $E(u(t)) = E_0 < 0$ and condition (29) is fulfilled. Therefore, by Lemma 1, for all $t \in [0,T)$, there exists $\gamma_0(t) \in \mathbf{R}$ and $x_0(t) \in \mathbf{R}$ such that, with $\lambda_0(t) = \frac{|Q_x|_{L^2}}{|u_x(t)|_{L^2}}$,

$$\left|Q - e^{i\gamma_0(t)}\lambda_0(t)^{1/2}u\left(\lambda_0(t)(x + x_0(t))\right)\right|_{H^1} < \delta(\alpha_0).$$

Now we sharpen the decomposition as in Lemma 2 in [14]; i.e., we choose $\lambda(t) > 0$, $\gamma(t) \in \mathbf{R}$ and $x(t) \in \mathbf{R}$ close to $\lambda_0(t)$, $\gamma_0(t)$ and $x_0(t)$ such that $\varepsilon(t,y) = e^{i\gamma(t)}\lambda^{1/2}(t)u(t,\lambda(t)y+x(t))-Q(y)$ is small in H^1 and satisfies suitable orthogonality conditions

$$(\varepsilon_1(t), Q_1) = (\varepsilon_1(t), yQ) = 0 \quad \text{and} \quad (\varepsilon_2(t), Q_2) = 0. \tag{36}$$

The existence of such a decomposition is a consequence of the implicit function theorem (see [14] for more details). For $\alpha > 0$, let

$$U_\alpha = \{u \in H^1(\mathbf{C}); \quad |u - Q|_{H^1} \le \alpha\},$$

and for $u \in H^1(\mathbf{C})$, $\lambda_1 > 0$, $\gamma_1 \in \mathbf{R}$, $x_1 \in \mathbf{R}$, define

$$\varepsilon_{\lambda_1,\gamma_1,x_1}(y) = e^{i\gamma_1}\lambda_1^{1/2}u(\lambda_1 y + x_1) - Q. \tag{37}$$

We claim that there exist $\overline{\alpha} > 0$ and a unique C^1 map : $U_{\overline{\alpha}} \to (1-\overline{\lambda}, 1+\overline{\lambda}) \times (-\overline{\gamma}, \overline{\gamma}) \times (-\overline{x}, \overline{x})$ such that if $u \in U_{\overline{\alpha}}$, there is a unique $(\lambda_1, \gamma_1, x_1)$ such that $\varepsilon_{\lambda_1,\gamma_1,x_1}$, defined as in (37), is such that

$$(\varepsilon_{\lambda_1,\gamma_1,x_1})_1 \perp Q_1, \quad (\varepsilon_{\lambda_1,\gamma_1,x_1})_1 \perp yQ \quad \text{and} \quad (\varepsilon_{\lambda_1,\gamma_1,x_1})_2 \perp Q_2 \tag{38}$$

where $\varepsilon_{\lambda_1,\gamma_1,x_1} = (\varepsilon_{\lambda_1,\gamma_1,x_1})_1 + i(\varepsilon_{\lambda_1,\gamma_1,x_1})_2$. Moreover, there exist a constant $C_1 > 0$ such that if $u \in U_{\overline{\alpha}}$, then

$$|\varepsilon_{\lambda_1,\gamma_1,x_1}|_{H^1} + |\lambda_1 - 1| + |\gamma_1| + |x_1| \le C_1\alpha.$$

Indeed, we define the following functionals of $(\lambda_1, \gamma_1, x_1)$:

$$\rho^1(u) = \int (\varepsilon_{\lambda_1,\gamma_1,x_1})_1 Q_1, \rho^2(u) = \int (\varepsilon_{\lambda_1,\gamma_1,x_1})_1 yQ, \rho^3(u) = \int (\varepsilon_{\lambda_1,\gamma_1,x_1})_2 Q_2.$$

We compute at $(\lambda_1, \gamma_1, x_1) = (1, 0, 0)$:

$$\frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1}}{\partial x_1} = u_x, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1}}{\partial \lambda_1} = \frac{u}{2} + yu_x, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1}}{\partial \gamma_1} = iu,$$

and obtain at the point $(\lambda_1, \gamma_1, x_1, u) = (1, 0, 0, Q)$,

$$\begin{array}{lll} \dfrac{\partial \rho^1}{\partial \lambda_1} = \displaystyle\int Q_1^2, & \dfrac{\partial \rho^1}{\partial \gamma_1} = 0, & \dfrac{\partial \rho^1}{\partial x_1} = 0, \\ \dfrac{\partial \rho^2}{\partial \lambda_1} = 0, & \dfrac{\partial \rho^2}{\partial \gamma_1} = 0, & \dfrac{\partial \rho^2}{\partial x_1} = -\dfrac{1}{2}\displaystyle\int Q^2, \\ \dfrac{\partial \rho^3}{\partial \lambda_1} = 0, & \dfrac{\partial \rho^3}{\partial \gamma_1} = -\displaystyle\int Q_1^2, & \dfrac{\partial \rho^3}{\partial x_1} = 0. \end{array}$$

The Jacobian of the above functional is $\frac{1}{2}|Q_1|_{L^2}^4|Q|_{L^2}^2$, so that by the implicit function theorem, there exist $\overline{\alpha} > 0$, a neighborhood $V_{1,0,0}$ of $(1,0,0)$ in $\mathbf{R}^3$ and a unique C^1 map $(\lambda_1, \gamma_1, x_1) : \{u \in H^1; \ |u - Q|_{H^1} < \overline{\alpha}\} \to V_{1,0,0}$, such that (38) holds. Now consider $\alpha_2 > 0$ such that $\delta(\alpha_2) < \overline{\alpha}$. For all time, there are parameters $x_0(t) \in \mathbf{R}$, $\gamma_0(t) \in \mathbf{R}$, $\lambda_0(t) > 0$ such that

$$\left| Q - e^{i\gamma_0(t)}\lambda_0(t)^{1/2}u\left(\lambda_0(t)(x + x_0(t))\right)\right|_{H^1(\mathbf{C})} < \overline{\alpha}.$$

Now existence and local uniqueness follow from the previous result applied to the function $e^{i\gamma_0(t)}\lambda_0(t)^{1/2}u(\lambda_0(t)(x+x_0(t)))$. Smallness estimates follow from direct calculations. Note also that for fixed t, $\gamma_0(t)$ and $x_0(t)$ are continuous functions of u from (33) and (34), so that the continuity of u with respect to t yields the continuity in time of $\gamma_0(t)$ and $x_0(t)$. This concludes the proof of Lemma 2.

2.3. *Smallness estimate on ε.* In this section, we prove a smallness result on the remainder term ε of the above regular decomposition. The argument relies only on the conservation of the two first invariants in H^1, namely the L^2-norm and energy. The third invariant, momentum, will be used in the next subsection. We claim:

LEMMA 3 (Smallness property on ε). *There exists $\alpha_3 > 0$ and a universal constant $C > 0$ such that for $\alpha_0 < \alpha_3$,*

$$\forall t, \quad |\varepsilon(t)|_{H^1} \leq C\sqrt{\alpha_0}. \tag{39}$$

Remark 1. Note that we have already proved a smallness estimate on ε (35): $|\varepsilon|_{H^1} \leq \delta(\alpha_0)$. This estimate was a consequence of the variational characterization of the ground state Q. In this sense, (39) is a refinement of (35) and is obtained by exhibiting coercive properties of L, that is of the linearized structure of the energy close to Q. Nevertheless, we could carry out the whole proof of Theorem 2 with (35) only.

Proof of Lemma 3. Let us recall that L is a matrix operator, $L = (L_+, L_-)$:

$$L_+ = -\Delta + 1 - 5Q^4 \quad , \quad L_- = -\Delta + 1 - Q^4. \tag{40}$$

Now the conservation of the L^2-norm can be written

$$\int \varepsilon_1^2 + \varepsilon_2^2 + 2\int \varepsilon_1 Q = \alpha_0 \tag{41}$$

and the conservation of energy yields for $E_0 < 0$,

(42)
$$\int |\varepsilon_{1y}|^2 - 5\int Q^4\varepsilon_1^2 - 2\int \varepsilon_1 Q + \int |\varepsilon_{2y}|^2 - \int Q^4\varepsilon_2^2 = -2\lambda^2|E_0| + \frac{1}{3}\int F(\varepsilon)$$

with

$$F(\varepsilon) = |\varepsilon + Q|^6 - Q^6 - 6Q^5\varepsilon_1 - 15Q^4\varepsilon_1^2 - 3Q^4\varepsilon_2^2. \tag{43}$$

We use the notation $(L\varepsilon, \varepsilon) = (L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2)$. Combining (41) and (42), we get

$$(L\varepsilon, \varepsilon) \leq \alpha_0 + F(\varepsilon) \leq \alpha_0 + C|\varepsilon|_{H^1}|\varepsilon|_{L^2}^2. \tag{44}$$

Let us now recall the following spectral properties of L. The following lemma combines results from [27] and [14].

Lemma 4 (Spectral structure of L). (i) *Algebraic relations*:

$$L_+(Q^3) = -8Q^3, L_+(Q_1) = -2Q, L_+(Q_y) = 0$$

and

$$L_-(Q) = 0, L_-(xQ) = -2Q_y.$$

(ii) *Coercivity of* L:

(45)
$$\forall \varepsilon_1 \in H^1, \quad \text{if} \ \ (\varepsilon_1, Q^3) = 0 \ \ \text{and} \ \ (\varepsilon_1, Q_y) = 0 \ \ \text{then} \ \ (L_+\varepsilon_1, \varepsilon_1) \geq (\varepsilon_1, \varepsilon_1),$$

$$\forall \varepsilon_2 \in H^1, \quad \text{if} \ \ (\varepsilon_2, Q) = 0 \ \ \text{then} \ \ (L_-\varepsilon_2, \varepsilon_2) \geq (\varepsilon_2, \varepsilon_2). \tag{46}$$

Note that orthogonality conditions (33) and (34) are not sufficient *a priori* to ensure the coerciveness of L. Nevertheless, we argue as follows.

Let an auxiliary function

$$\tilde{\varepsilon} = \varepsilon - aQ_1 - bQ_y - icQ.$$

On the real part, we have $(\tilde{\varepsilon_1}, Q^3) = (\tilde{\varepsilon_1}, Q_y) = 0$ with $a = 4\frac{(\varepsilon_1, Q^3)}{\int Q^4}$ (note that $\int Q_1 Q^3 = \int Q^4$) and $b = \frac{(\varepsilon_1, Q_y)}{\int Q_y^2}$. Now using the orthogonality conditions on ε_1 (33), we also have $a = -\frac{(\tilde{\varepsilon_1}, Q_1)}{\int Q_1^2}$ and $b = 2\frac{(\tilde{\varepsilon_1}, xQ)}{\int Q^2}$ (note that $\int yQQ_y = -\frac{1}{2}\int Q^2$). On the imaginary part, $(\tilde{\varepsilon_2}, Q) = 0$ with $c = \frac{(\varepsilon_2, Q)}{\int Q^2}$. Moreover, $(Q, Q_2) = -\int Q_1^2$ so that by the orthogonality condition on ε_2, $c = \frac{(\tilde{\varepsilon_2}, Q_2)}{\int Q_1^2}$. Therefore, we have for some constant $K > 0$

$$\frac{1}{K}(\varepsilon, \varepsilon) \leq (\tilde{\varepsilon}, \tilde{\varepsilon}) \leq K(\varepsilon, \varepsilon).$$

Moreover, two noteworthy facts are

$$(\tilde{\varepsilon_1}, Q) = (\varepsilon_1, Q) \quad , \quad (L_+\tilde{\varepsilon_1}, \tilde{\varepsilon_1}) = (L_+\varepsilon_1, \varepsilon_1) + 4a(\varepsilon_1, Q)$$

and

$$(L_-\tilde{\varepsilon_2}, \tilde{\varepsilon_2}) = (L_-\varepsilon_2, \varepsilon_2).$$

Thus, from (44), (45) and (46),

$$\frac{1}{K}(\varepsilon,\varepsilon)\leq(\tilde{\varepsilon},\tilde{\varepsilon})\leq(L\tilde{\varepsilon},\tilde{\varepsilon})\leq\alpha_0+4|a||(\varepsilon_1,Q)|+C|\varepsilon|^2_{L^2}|\varepsilon|_{H^1}.$$

Now $|a|\leq C|\varepsilon|_{H^1}$ from its expression, and from the conservation of the L^2 mass (41), $2|(\varepsilon_1,Q)|\leq\alpha_0+|\varepsilon|^2_{L^2}$, so that

$$\frac{1}{K}(\varepsilon,\varepsilon)\leq 2\alpha_0+C|\varepsilon|_{H^1}|\varepsilon|^2_{L^2}.$$

Now recall *a priori* estimate (35): $|\varepsilon|_{H^1}\leq\delta(\alpha_0)$; then for $\alpha_0<\alpha_3$ small enough

$$\frac{1}{K}(\varepsilon,\varepsilon)\leq 2\alpha_0+\frac{1}{2K}(\varepsilon,\varepsilon)\text{ so that }(\varepsilon,\varepsilon)\leq 4K\alpha_0.$$

We conclude from (44)

$$|\varepsilon|^2_{H^1}\leq(L\varepsilon,\varepsilon)+5\int Q^4\varepsilon_1^2+\int Q^4\varepsilon_2^2\leq C\alpha_0+C\alpha_0|\varepsilon|_{H^1}$$

so that

$$|\varepsilon|_{H^1}\leq C\sqrt{\alpha_0}.$$

This concludes the proof of Lemma 3. □

2.4. *Properties of the decomposition.* We now are in position to prove additional properties of the regular decomposition in ε and estimates on the modulated parameters $\lambda(t)$, $\gamma(t)$ and $x(t)$. These estimates rely on the equation verified by ε, which is inherited from (1), and on smallness estimate (39). Moreover, using Galilean invariance (18), we will prove an additional degeneracy which will be the heart of the proof when showing the effect of nonradial symmetries in the energy space, that is, translation and Galilean invariances.

We first introduce a new time scale

$$s=\int_0^t\frac{dt'}{\lambda^2(t')},\quad\text{or equivalently}\quad\frac{ds}{dt}=\frac{1}{\lambda^2}.$$

Now ε, λ, γ and x are functions of s. Let $(T_1,T_2)\in(0,+\infty]^2$ be respectively the negative and positive blow-up times of $u(t)$. Let us check that when $t\in(-T_1,T_2)$, $\{s(t)\}=(-\infty,+\infty)$. On the one hand, the strictly negative energy condition together with Gagliardo-Nirenberg inequality imply that λ is bounded from above and if u is defined for $t>0$ then the conclusion follows. If u blows up in finite time T_2, the scaling estimate (21) implies $\lambda(t)\geq C(T_2-t)^{\frac{1}{2}}$ and again $s(t)>0$ is defined. We argue in the same way for $t<0$. From now on, we let $T\in(0,+\infty]$ the positive blow-up time.

We first fix once and for all for the rest of this paper in dimension $N=1$ a constant $2^-=\frac{9}{5}$. As will be clear from further analysis, we shall not need the exact value of 2^-, only the fact that

$$2^-<2.$$

We now claim:

LEMMA 5 (Properties of the decomposition). *There exists $\alpha_4 > 0$ such that for $\alpha_0 < \alpha_4$, $\{\lambda(s), \gamma(s), x(s)\}$ are $\mathcal{C}^1$ functions of s on $\mathbf{R}$, with the following properties*:

(i) *Equations of $\varepsilon(s)$*: *$\varepsilon(s)$ satisfies for $s \in \mathbf{R}$, $y \in \mathbf{R}$ the following system of coupled partial differential equations*:

(47)
$$\partial_s\varepsilon_1 - L_-\varepsilon_2 = \frac{\lambda_s}{\lambda}Q_1 + \frac{x_s}{\lambda}Q_y + \frac{\lambda_s}{\lambda}\left(\frac{\varepsilon_1}{2} + y(\varepsilon_1)_y\right) + \frac{x_s}{\lambda}(\varepsilon_1)_y + \tilde{\gamma}_s\varepsilon_2 - R_2(\varepsilon)$$

$$\partial_s\varepsilon_2 + L_+\varepsilon_1 = -\tilde{\gamma}_s Q - \tilde{\gamma}_s\varepsilon_1 + \frac{\lambda_s}{\lambda}\left(\frac{\varepsilon_2}{2} + y(\varepsilon_2)_y\right) + \frac{x_s}{\lambda}(\varepsilon_2)_y + R_1(\varepsilon) \tag{48}$$

where $\tilde{\gamma}(s) = -s - \gamma(s)$ and the functionals R_1 and R_2 are given by

$$\begin{aligned} R_1(\varepsilon) &= (\varepsilon_1 + Q)|\varepsilon + Q|^4 - Q^5 - 5Q^4\varepsilon_1 \\ &= 10Q^3\varepsilon_1^2 + 2\varepsilon_2^2 Q^3 + 10Q^2\varepsilon_1^3 + 5Q\varepsilon_1^4 + \varepsilon_1^5 \\ &\quad + \varepsilon_2^4(\varepsilon_1 + Q) + 2\varepsilon_2^2(\varepsilon_1^3 + 3Q^2\varepsilon_1 + 3Q\varepsilon_1^2), \end{aligned} \tag{49}$$

$$\begin{aligned} R_2(\varepsilon) &= \varepsilon_2|\varepsilon + Q|^4 - \varepsilon_2 Q^4 \\ &= \varepsilon_2(4Q^3\varepsilon_1 + 6Q^2\varepsilon_1^2 + 4Q\varepsilon_1^3 + \varepsilon_1^4 + \varepsilon_2^4 + 2\varepsilon_2^2(\varepsilon_1 + Q)^2). \end{aligned} \tag{50}$$

(ii) *Invariance induced estimates*: *for all $s \in \mathbf{R}$,*

$$\left|\lambda^2(s)E_0 + (\varepsilon_1, Q)\right| \leq C\left(\int|\varepsilon_y|^2 + \int|\varepsilon|^2 e^{-2^-|y|}\right), \tag{51}$$

$$|(\varepsilon_2, Q_y)|(s) \leq C\sqrt{\alpha_0}\left(\int|\varepsilon_y|^2\right)^{\frac{1}{2}}. \tag{52}$$

(iii) *A priori estimates on the modulation parameters*:

$$|\frac{\lambda_s}{\lambda}| + |\tilde{\gamma}_s| \leq C\left(\int|\varepsilon_y|^2 + \int|\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}, \tag{53}$$

$$|\frac{x_s}{\lambda}| \leq C\sqrt{\alpha_0}\left(\int|\varepsilon_y|^2 + \int|\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}. \tag{54}$$

Remark 2. Let us draw attention to the two last estimates above. Comparing (53) and (54), one sees that the order size of the parameter $\frac{x_s}{\lambda}$ induced by translation invariance is of smaller order by a factor $\sqrt{\alpha_0}$ than one of the parameters $\frac{\lambda_s}{\lambda}$ and $\tilde{\gamma}_s$ induced by scaling and phase invariance, radial symmetries. This fact will be both related to our choice of orthogonality condition

$(\varepsilon_1, yQ) = 0$ and to our use of Galilean invariance, relation (18). Such a decoupling of the effect of radial versus nonradial symmetries is known for other types of equations like the nonlinear heat equation, but is exhibited for the first time in the setting of (1).

Before stating the proof, we need to draw attention to estimates which we will use in the paper without explicitly mentioning them. We let $R(\varepsilon) = R_1(\varepsilon) + iR_2(\varepsilon)$ given by (49), (50), $F(\varepsilon)$ given by (43) and $\tilde{R}_1(\varepsilon) = R_1(\varepsilon) - 10Q^3\varepsilon_1^2 - 2Q^3\varepsilon_2^2$ the formally cubic part of $R_1(\varepsilon)$. We claim:

LEMMA 6 (Control of nonlinear interactions). *Let $P(y)$ be a polynomial with an integer $0 \le k \le 3$, then*:

(i) *Control of linear terms*:

$$\left|\left(\varepsilon_{1,2}, P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \le C_{P,k}\left(\int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}.$$

(ii) *Control of second order terms*:

$$\left|\left(R(\varepsilon), P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \le C\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right).$$

(iii) *Control of higher order terms*:

$$\int |F(\varepsilon)| + \left|\left(\tilde{R}_1(\varepsilon), P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \le C\sqrt{\alpha_0}\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right).$$

Proof of Lemma 6. (i) follows from Cauchy-Schwarz and the uniform estimate $|P(y)\frac{d^k}{dy^k}Q(y)| \le C_{P,k}e^{-1^-|y|}$ for any number $1^- < 1$.

(ii) follows from

$$|R(\varepsilon)| \le C(|\varepsilon|^2 Q^3 + |\varepsilon|^5),$$

so that

$$\begin{aligned}\left|\left(R(\varepsilon), P(y)\frac{d^k}{dy^k}Q(y)\right)\right| &\le C\left(\int |\varepsilon|^2 e^{-2^-|y|}\right) + C\int |\varepsilon|^5 e^{1^-|y|}\\ &\le C\left(\int |\varepsilon|^2 e^{-2^-|y|}\right) + C\left(\int |\varepsilon|^8\right)^{\frac{1}{2}}\left(\int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}\\ &\le C\left(\int |\varepsilon|^2 e^{-2^-|y|} + |\varepsilon|^3_{L^\infty}|\varepsilon|_{L^2}\left(\int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}\right)\end{aligned}$$

which implies the desired result from $|\varepsilon|_{L^\infty} \le C|\varepsilon_y|^{\frac{1}{2}}_{L^2}|\varepsilon|^{\frac{1}{2}}_{L^2}$.

(iii) follows from
$$|F(\varepsilon)| \leq C(|\varepsilon|^3 Q^3 + |\varepsilon|^6)$$
and the Gagliardo-Nirenberg inequality. $|\left(\tilde{R}_1(\varepsilon), P(y)\frac{d^k}{dy^k}Q(y)\right)|$ is controlled similarly, and Lemma 6 is proved. □

Proof of Lemma 5. (i) We compute the equation of ε by simply injecting (32) into (1) and write the result as a coupled system of partial differential equations on the real and imaginary part of ε as stated. Note that if $Q(x)$ is the ground state, then $Q(x)e^{it}$ is a solution to (1). This is why we set $\tilde{\gamma}(s) = -s - \gamma(s)$.

(ii) This is an easy consequence of smallness estimate (39) and of the conservation of energy and the momentum. Let us first recall the conservation of the energy (42):
$$\int |\varepsilon_{1y}|^2 - 5\int Q^4\varepsilon_1^2 - 2\int \varepsilon_1 Q + \int |\varepsilon_{2y}|^2 - \int Q^4\varepsilon_2^2 = -2\lambda^2|E_0| + \frac{1}{3}\int F(\varepsilon)$$
with
$$\int |F(\varepsilon)| \leq C\sqrt{\alpha_0}\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right).$$
This yields (51).

We rewrite (18) in the ε variable:

(55)
$$0 = \mathrm{Im}(\int u_x\overline{u}) = \frac{1}{\lambda}Im\left(\int (\varepsilon+Q)_y\overline{(\varepsilon+Q)}\right) = \frac{1}{\lambda}\left\{\mathrm{Im}(\int \varepsilon_y\overline{\varepsilon}) - 2(\varepsilon_2, Q_y)\right\}$$
so that with (39), (52) follows.

(iii) We prove (iii) thanks to the orthogonality conditions verified by ε and the conservation law (18) for the nonradial term induced by Galilean invariance.

Indeed, we take the inner product of (47) with the well-localized function Q_1 and integrate by parts. From the first relation of (33), we get
$$\frac{\lambda_s}{\lambda}(|Q_1|_{L^2}^2 - (\varepsilon_1, Q_2)) = -(\varepsilon_2, L_-(Q_1)) + \frac{x_s}{\lambda}(\varepsilon_1, (Q_1)_y) - \tilde{\gamma}_s(\varepsilon_2, Q_1) + (R_2(\varepsilon), Q_1).$$
We now take the inner product of (48) with Q_2 and use (34) to get

$$\tilde{\gamma}_s(|Q_1|_{L^2}^2 - (\varepsilon_1, Q_2)) = (\varepsilon_1, L_+(Q_2)) - \frac{\lambda_s}{\lambda}(\varepsilon_2, \frac{1}{2}Q_2 + y(Q_2)_y) + \frac{x_s}{\lambda}(\varepsilon_2, (Q_2)_y) - (R_1(\varepsilon), Q_2). \tag{56}$$

Note that in the above expression, one term formally involves a fourth order derivative of Q, that is, in the term $(\varepsilon_1, L_+(Q_2))$. We shall estimate for this term

$$|(\Delta\varepsilon_1, Q_2)| = |(\varepsilon_1)_y, (Q_2)_y)| \le C\left(\int |\varepsilon_y|^2\right)^{\frac{1}{2}}.$$

Last, using $L_-(yQ) = -2Q_y$ and the second relation of (33), take the inner product of (47) with yQ,

$$\text{(57)}\quad \frac{x_s}{\lambda}\left(\frac{1}{2}|Q|_{L^2}^2 + (\varepsilon_1, (yQ)_y)\right) = -2(\varepsilon_2, Q_y) - \frac{\lambda_s}{\lambda}\left(\varepsilon_1, \frac{1}{2}yQ + y(yQ)_y\right) + \tilde{\gamma}_s(\varepsilon_2, yQ) - (R_2(\varepsilon), yQ).$$

Summing the three equalities above, we get

$$|\frac{\lambda_s}{\lambda}| + |\tilde{\gamma}_s| + |\frac{x_s}{\lambda}| \le C\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}$$

and (53) is proved. We now inject (52) and (53) into (57) to get (54) and Lemma 5 is proved.

3. L^2_{loc} dispersion and almost monotonicity properties

Our aim in this section is to exhibit the dispersive structure underlying (NLS) in the vicinity of the ground state Q. So far indeed, variational estimates and the conservation of both energy and the L^2-norm have allowed us to build a regular decomposition of solutions close to the ground state up to some invariances of the equation and to estimate the smallness of the remainder term ε in H^1 and the size of the modulation parameters $\lambda(s)$, $\gamma(s)$, $x(s)$. We now shall make heavy use of the symmetries of the equation and of its dispersive properties.

In the two first subsections, we rewrite the virial relation (6) in terms of ε and use all the symmetries of (NLS) in the energy space H^1 to deduce from the obtained relation a dispersive structure in the ε variable. This strategy is similar to the one used for the study of the KdV equation. Then in the last subsection, using this inequality and the equation governing the scaling parameter, we eventually prove a result of almost monotonicity of the scaling parameter for negative energy solutions, which is the heart of the proof of the main theorem.

3.1. *Dispersion in variable u and virial identity.* At this point, we have fully used the ε-version of the three fundamental conservation laws, that is L^2-norm, energy and momentum. In this section, we derive the ε-version in H^1 of the virial relation on u in Σ

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 = 16E(u_0),$$

or equivalently the dispersive effect of equation (1) in the u variable. The virial relation we obtain makes heavy use of the structure underlying the (NLS) equation and is the main key to our analysis, in particular, to obtain monotonicity results around Q as was the case for the KdV equation. Note that the monotonicity result we obtain is of a different nature from the one exhibited in the study of KdV.

Before stating the result, let us first make a formal computation to exhibit the natural quantities to investigate. Indeed, if $\int |x|^2|u_0(x)|^2 < +\infty$, then the virial relation can be written

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 = 4\frac{d}{dt}\,\mathrm{Im}\left(\int xu_x\overline{u}\right) = 16E(u_0)$$

or equivalently

$$\frac{d}{dt}\int |x|^2|u(t,x)|^2 = 4\,\mathrm{Im}\left(\int xu_x\overline{u}\right) = -16|E(u_0)|t + c_0 \quad .$$

Therefore, it is natural to look for a virial-type relation in ε by formally computing the time derivative in s of the quantity $\Psi(\varepsilon)(s) = \mathrm{Im}(\int y\varepsilon_y\overline{\varepsilon})(s)$. This approach is indeed successful provided this quantity is *a priori* defined, which it is not in the hypothesis of our theorem. A fundamental way to avoid this difficulty is to observe that the quantity $\Psi(u)(t) = \mathrm{Im}(\int xu_x\overline{u})(t)$ is scaling and phase invariant. In addition, it is also translation invariant thanks to (18)

$$\mathrm{Im}\left(\int u_x\overline{u}\right) = 0.$$

In other words,

$$\Psi(u)(t) = \mathrm{Im}\left(\int y(\varepsilon+Q)_y(\overline{\varepsilon+Q})\right)(s),$$

or by expanding the last term we see that

$$\Psi(\varepsilon)(s) - 2(\varepsilon_2, Q_1)(s) = -4|E(u_0)|t + \frac{c_0}{4}.$$

Taking the derivative of the above relation in time s and using $\frac{dt}{ds} = \lambda^2(s)$, we get

$$\left(\Psi(\varepsilon)\right)_s(s) = 2(\varepsilon_2, Q_1)_s(s) - 4\lambda^2(s)|E(u_0)|.$$

In other words, the expected virial type relation in ε on the *nonlocal* term $\Psi(\varepsilon)$ may be replaced by a similar relation on the *well localized* term (ε_2, Q_1). This simple but fundamental fact explains why we shall never need more for the proof of the theorem than $u_0 \in H^1$.

According to the above formal heuristic, we are led to compute $(\varepsilon_2, Q_1)_s$. The result is the following:

LEMMA 7 (Local virial identity). *Under the assumptions of Theorem* 2,

(58)

$$(\varepsilon_2, Q_1)_s = H(\varepsilon,\varepsilon) + 2\lambda^2|E_0| - \tilde{\gamma}_s(\varepsilon_1, Q_1) - \frac{\lambda_s}{\lambda}(\varepsilon_2, Q_2) - \frac{x_s}{\lambda}(\varepsilon_2, (Q_1)_y) + G(\varepsilon)$$

with

$$G(\varepsilon) = -\frac{1}{3}\int F(\varepsilon) + (\tilde{R}_1(\varepsilon), Q_1) \tag{59}$$

where $\tilde{R}_1(\varepsilon) = R_1(\varepsilon) - 10Q^3\varepsilon_1^2 - 2Q^3\varepsilon_2^2$ *and* $R_1(\varepsilon)$ *is given by* (49), *and where the quadratic form* $H(\varepsilon,\varepsilon)$ *is decoupled in the variables* $\varepsilon_1, \varepsilon_2$. *Explicitly,* $H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1, \varepsilon_1) + (\mathcal{L}_2\varepsilon_2, \varepsilon_2)$, *where* $(\mathcal{L}_i)_{i=1,2}$ *are linear real Schrödinger operators given by*

$$\mathcal{L}_1 = -\Delta + 10yQ^3Q_y \quad \text{and} \quad \mathcal{L}_2 = -\Delta + 2yQ^3Q_y. \tag{60}$$

Proof of Lemma 7. We take the inner product of (48) with Q_1 and use $L_+(Q_1) = -2Q$ and the critical relation $(Q, Q_1) = 0$. We get, after integration by parts,

(61)

$$(\varepsilon_2, Q_1)_s = 2(\varepsilon_1, Q) - \frac{\lambda_s}{\lambda}(\varepsilon_2, Q_2) - \tilde{\gamma}_s(\varepsilon_1, Q_1) - \frac{x_s}{\lambda}(\varepsilon_2, (Q_1)_y) + (R_1(\varepsilon), Q_1).$$

We now recall the conservation of energy (42) to expand the term $2(\varepsilon_1, Q)$ in (61),

$$2(\varepsilon_1, Q) = \int |\varepsilon_y|^2 - 5\int Q^4\varepsilon_1^2 - \int Q^4\varepsilon_2^2 + 2\lambda^2|E_0| - \frac{1}{3}\int F(\varepsilon)$$

and $F(\varepsilon)$ given by (43). We get

$$\begin{aligned}(\varepsilon_2, Q_1)_s = &\int |\varepsilon_{1y}|^2 - 5\int Q^4\varepsilon_1^2 + \int |\varepsilon_{2y}|^2 - \int Q^4\varepsilon_2^2 + (R_1(\varepsilon), Q_1)\\ &+ 2\lambda^2|E_0| - \tilde{\gamma}_s(\varepsilon_1, Q_1) - \frac{\lambda_s}{\lambda}(\varepsilon_2, Q_2) - \frac{x_s}{\lambda}(\varepsilon_2, (Q_1)_y) - \frac{1}{3}\int F(\varepsilon).\end{aligned}$$

We now focus on the second order terms in ε on the right-hand side of the above relation. To do so, we use the explicit form of $R_1(\varepsilon)$ given by (49): $R_1(\varepsilon) = 10Q^3\varepsilon_1^2 + 2Q^3\varepsilon_2^2 + \tilde{R}_1(\varepsilon)$, $\tilde{R}_1$ cubic in ε. Note that $F(\varepsilon)$ given by (43) is also cubic in ε, and $G(\varepsilon) = -\frac{1}{3}\int F(\varepsilon) + (\tilde{R}_1(\varepsilon), Q_1)$. An elementary computation yields (58) and concludes the proof of Lemma 7.

3.2. *Symmetries and modulation theory.* In this subsection, we explain how to use the whole system of symmetries to extract a dispersive type information from (58):

$$\begin{aligned}(\varepsilon_2, Q_1)_s = &H(\varepsilon,\varepsilon) + 2\lambda^2|E_0| - \tilde{\gamma}_s(\varepsilon_1, Q_1) - \frac{\lambda_s}{\lambda}(\varepsilon_2, Q_2)\\ &- \frac{x_s}{\lambda}(\varepsilon_2, (Q_1)_y) + (G(\varepsilon), Q_1).\end{aligned}$$

Different kinds of terms appear in this expression:

(i) The Schrödinger operator H: Note that the quadratic form H is decoupled in the variables $\varepsilon_1, \varepsilon_2$. On each coordinate, a classical elliptic Schrödinger operator with an exponentially decreasing potential underlies the quadratic form. There is then a classical theorem that such a quadratic form has only a finite number of negative directions.

(ii) The Energy term: Note that the term $\lambda^2|E_0|$ appears with the + sign in (58). This heavily relies on our assumption $E_0 < 0$.

(iii) Scalar product terms: three *a priori* second order in ε scalar product terms appear in (58), and each of them is related to our choice of modulation parameters on the initial solution u, namely scaling, phase and translation.

(iv) The last term $G(\varepsilon)$ is formally cubic in ε, and then of smaller order size and controlled according to Lemma 6.

We now precisely detail how to use the symmetries and conservation laws in the energy space H^1 to exhibit from (58) the dispersive structure in the ε variable. This approach is completely different from the linear kind of approach previously studied and was based on the linearized structure of the energy. On the contrary, we develop a more nonlinear approach by focusing on the dispersive relations inherited from the virial structure. This will make clear the choice of orthogonality conditions (33) and (34), which indeed allows us to cancel in equality (58) some oscillatory integrals in time. We now claim:

Proposition 1 (Dispersive structure in the ε variable). *There exist a universal constant $\delta_1 > 0$ and $\alpha_5 > 0$ such that for $\alpha_0 < \alpha_5$, for all s:*

$$(\varepsilon_2, Q_1)_s \geq \frac{\delta_1}{2}\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right) + 2\lambda^2|E_0| - \frac{2}{\delta_1}\left((\varepsilon_1, Q)^2 + (\varepsilon_2, Q_1)^2\right). \tag{62}$$

Proof of Proposition 1. A) Modulation theory for phase and scaling. From the symmetry of (NLS) with respect to scaling and phase, we have been able through modulation theory to build a regular decomposition of the initial solution u and the corresponding ε. Working out the implicit function theorem, we have seen that one may assume that two scalar products are zero for all time, provided the corresponding matrix has an inverse. The choice of orthogonality conditions (33) and (34) has been made to cancel the two first second order scalar products in (58). This somehow treats the case of radial symmetries in the energy space.

B) Modulation theory for translation invariance. We now focus on nonradial symmetries. On the one hand, Galilean invariance has been used directly on the initial solution u to ensure (18). This led to crucial estimate (52)

$$|(\varepsilon_2, Q_y)|(s) \leq C\sqrt{\alpha_0}(\int |\varepsilon_y|^2)^{\frac{1}{2}}.$$

On the other hand, we applied modulation theory to the translation parameter. The choice of orthogonality condition

$$(\varepsilon_1, yQ) = 0$$

has been made to ensure a relation of the type $\frac{x_s}{\lambda} \sim -(\varepsilon_2, Q_y)$, and this together with (52) yields (54). Therefore, we are in position to estimate the term $\frac{x_s}{\lambda}(\varepsilon_2, (Q_1)_y)$ in (58) as

$$\left|\frac{x_s}{\lambda}(\varepsilon_2, (Q_1)_y)\right| \leq C\sqrt{\alpha_0}\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right). \tag{63}$$

C) Control of the negative directions of the quadratic form H. The spectral structure of the quadratic form H is proved in dimension $N = 1$ only and conjectured in higher dimension. Note that this study is precisely the *only part* of the proof where we use the low dimension hypothesis. See Section 4.4. It turns out that the Schrödinger linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ given by (60) have the following spectral structure:

(i) $\mathcal{L}_1$ has two strictly negative eigenvalues. In one dimension, this corresponds to one negative direction for even functions, and one for odd. It turns out that for even functions, the choice $(\varepsilon_1, Q_1) = 0$ does not suffice to ensure the positivity of H_1 and a negative direction along Q has to be taken into account. On the contrary, for odd functions, a miracle happens, which is that the choice $(\varepsilon_1, yQ) = 0$ suffices to ensure the positivity of H_1.

(ii) $\mathcal{L}_2$ has one strictly negative eigenvalue. Once again, the choice $(\varepsilon_2, Q_2) = 0$ does not suffice to ensure its positivity, and a negative direction along Q_1 has to be taken into account.

Nevertheless, a key to our analysis is that the negative directions of H which we cannot control *a priori* from modulation theory appear to correspond to two key scalar products, (ε_1, Q) and (ε_2, Q_1), related to the Hamiltonian structure of (1) and its dynamical properties.

More precisely, we prove in Appendix A the following:

Proposition 2 (Spectral structure of the linear virial operator). *Let $2^- = \frac{9}{5}$. There exists a universal constant $\tilde{\delta}_1 > 0$ such that for all $\varepsilon \in H^1$, if*

$$(\varepsilon_1, Q) = (\varepsilon_1, yQ) = 0 \quad \textit{and} \quad (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = 0,$$

then

$$H(\varepsilon, \varepsilon) \geq \tilde{\delta}_1 \left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right).$$

Now let $\varepsilon \in H^1$ with $(\varepsilon_1, Q_1) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_2) = 0$, and set

$$\varepsilon = \tilde{\varepsilon} + aQ + ibQ_1.$$

Note that $(\tilde{\varepsilon}_1, Q_1) = (\tilde{\varepsilon}_1, yQ) = (\tilde{\varepsilon}_2, Q_2) = 0$, and $(\tilde{\varepsilon}_1, Q) = 0$ with $a = \frac{(\varepsilon_1, Q)}{\int Q^2}$ and $(\tilde{\varepsilon}_2, Q_1) = 0$ with $b = \frac{(\varepsilon_2, Q_1)}{\int Q_1^2}$. We heavily used both critical relations $(Q, Q_1) = (Q_1, Q_2) = 0$. Therefore, $\tilde{\varepsilon}$ satisfies the hypothesis of Proposition 2 and one easily evaluates:

(64)

$$\begin{aligned} H(\varepsilon, \varepsilon) &= H(\tilde{\varepsilon}, \tilde{\varepsilon}) + 2a(\tilde{\varepsilon}_1, \mathcal{L}_1 Q) + 2b(\tilde{\varepsilon}_2, \mathcal{L}_2 Q_1) + a^2 H_1(Q, Q) + b^2 H_2(Q_1, Q_1) \\ &\geq \frac{\tilde{\delta}_1}{2}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) - C(a^2 + b^2) \\ &\geq \delta_1\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right) - \frac{1}{\delta_1}\left((\varepsilon_1, Q)^2 + (\varepsilon_2, Q_1)^2\right) \end{aligned}$$

for some fixed universal constant $\delta_1 > 0$ small enough.

D) Conclusion. Using orthogonality conditions (33) and (34), estimate (63), estimate (64) and estimating directly $G(\varepsilon)$ from (59) and Lemma 6, we get

$$\begin{aligned} (\varepsilon_2, Q_1)_s &\geq H(\varepsilon, \varepsilon) + 2\lambda^2 |E_0| - C\sqrt{\alpha_0}\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right) \\ &\geq \delta_1\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right) + 2\lambda^2 |E_0| \\ &\quad - C\sqrt{\alpha_0}\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right) - \frac{1}{\delta_1}\left((\varepsilon_1, Q)^2 + (\varepsilon_2, Q_1)^2\right) \end{aligned}$$

and (62) is proved for $\alpha_0 < \alpha_5$ small enough. This concludes the proof of Proposition 1.

3.3. *Transformation of the dispersive relation.* We are now in position to prove the dispersive result for solutions to (NLS) in the ε variable in order to prove Theorem 2. One can see from (62) that two quantities play an important role, i.e. (ε_1, Q) and (ε_2, Q_1). It turns out that the first one may be removed using dynamical properties of equation (1), and we are left with only one leading order term to understand, namely (ε_2, Q_1). We claim

PROPOSITION 3 (Local virial estimate in ε). *There exists a universal constant $\delta_0 > 0$ and $\alpha_6 > 0$ such that for $\alpha_0 < \alpha_6$,*

(i) *for all $s \in \mathbf{R}$,*

$$\left\{\left(1 + \frac{1}{4\delta_0}(\varepsilon_1, Q)\right)(\varepsilon_2, Q_1)\right\}_s \geq \delta_0\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right) + 2\lambda^2 |E_0| - \frac{1}{\delta_0}(\varepsilon_2, Q_1)^2. \tag{65}$$

(ii) *for all* $s_2 \geq s_1$,

(66)
$$\left[\left(1+\frac{1}{4\delta_0}(\varepsilon_1,Q)(s)\right)(\varepsilon_2,Q_1)(s)\right]_{s_1}^{s_2} \geq \delta_0\int_{s_1}^{s_2}\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right) +2\int_{s_1}^{s_2}\lambda^2|E_0| - \frac{1}{\delta_0}\int_{s_1}^{s_2}(\varepsilon_2,Q_1)^2.$$

Proof of Proposition 3. (i) Recall (62),

$$(\varepsilon_2,Q_1)_s \geq \frac{1}{2}\delta_1\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right) +2\lambda^2|E_0| - \frac{2}{\delta_1}\left((\varepsilon_1,Q)^2+(\varepsilon_2,Q_1)^2\right).$$

We now note that the term $(\varepsilon_1,Q)^2$ is the derivative in time of a well localized scalar product up to small quadratic terms.

Indeed, take the inner product of (47) with Q. From $L_-(Q)=0$ and $(Q,Q_1)=0$, we get

$$(\varepsilon_1,Q)_s = -\frac{\lambda_s}{\lambda}(\varepsilon_1,Q_1) - \frac{x_s}{\lambda}(\varepsilon_1,Q_y)+\tilde\gamma_s(\varepsilon_2,Q)-(R_2(\varepsilon),Q).$$

We then recall (61),

$$(\varepsilon_2,Q_1)_s = 2(\varepsilon_1,Q)-\frac{\lambda_s}{\lambda}(\varepsilon_2,Q_2)-\tilde\gamma_s(\varepsilon_1,Q_1)-\frac{x_s}{\lambda}(\varepsilon_2,(Q_1)_y)+(R_1(\varepsilon),Q_1)$$

and estimate from (39), (53) and (63)

$$|(\varepsilon_1,Q)_s|+|(\varepsilon_2,Q_1)_s-2(\varepsilon_1,Q)| \leq C\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right).$$

It follows,

$$|\left\{(\varepsilon_1,Q)(\varepsilon_2,Q_1)\right\}_s - 2(\varepsilon_1,Q)^2| \leq C\sqrt{\alpha_0}\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right).$$

Injecting this relation into (62) yields

$$\begin{aligned}(\varepsilon_2,Q_1)_s &+ \frac{4}{\delta_1}\left\{(\varepsilon_1,Q)(\varepsilon_2,Q_1)\right\}_s\\ &\geq \frac{1}{2}\delta_1\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right)+2\lambda^2|E_0|\\ &\quad -\frac{1}{\delta_1}(\varepsilon_2,Q_1)^2 - C\sqrt{\alpha_0}\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right)\end{aligned}$$

and (65) is proved for $\alpha_0<\alpha_6$ small enough and $\delta_0=\frac{\delta_1}{4}$. This concludes the proof of (i).

(ii) We simply integrate (65) on the time interval $[s_1, s_2]$. This concludes the proof of Proposition 3.

We draw attention to the strength of estimate (66).

- On the one hand, we get dispersive control of $\int |\varepsilon_y|^2$, and by Gagliardo-Nirenberg on $\int |\varepsilon|^6$.
- On the other hand, we do not control the global L^2 norm of ε, which anyway cannot tend to 0 from conservation laws, but only a local L^2-norm of the form $\int |\varepsilon|^2 e^{-2^-|y|}$ which allows us to control all the second order and higher terms which correspond to scalar products with well-localized functions; see Lemma 6.

Remark 3. Let us summarize the strategy we have used to derive the virial dispersive estimate (65). We start with the exact dispersive relation (6) in the variable u thus a nonlinear conservation law. We then inject geometrical decomposition (32) into this conservation law and note that it is in some sense invariant through this transformation. Let us focus on the fact that this property is destroyed when approximating the ε equation by the purely linear one. This relation links a linear term and a quadratic term. We then use linear types of estimates on the quadratic terms to derive an estimate for the first order term.

3.4. *Almost monotonicity of the scaling parameter.* In this section, we prove a result of almost monotonicity of the scaling parameter for negative energy solutions to (NLS). The proof heavily relies on the local virial estimates of Proposition 3 proved in the previous subsection. We first exhibit from (65) and energy condition $E_0 < 0$ the sign structure of

$$(\varepsilon_2, Q_1).$$

In a certain sense, this inner product has thus parabolic behavior and satisfies the typical maximum principle property.

From dispersive inequality (66), (ε_2, Q_1) also governs the size of ε in L^2-loc in a time-averaging sense. On the other hand, we will see that the scaling parameter λ is governed by an equation of the form

$$\frac{\lambda_s}{\lambda} \sim -(\varepsilon_2, Q_1)$$

in a time-averaging sense in L^2-loc again. On the basis of these two facts, we prove a surprising result of almost monotonicity of the scaling parameter.

Proposition 4 (Almost monotonicity of the scaling parameter). *There exists $\alpha_7 > 0$ such that for $\alpha_0 < \alpha_7$, there exists a unique $s_0 \in \mathbf{R}$ such that:*

(i)

$$\forall s < s_0 \quad , \quad (\varepsilon_2, Q_1)(s) < 0, \qquad (67)$$

$$(\varepsilon_2, Q_1)(s_0) = 0,$$

$$\forall s > s_0 \quad , \quad (\varepsilon_2, Q_1)(s) > 0.$$

(ii) *Moreover, for all* $s_2 \geq s_1 \geq s_0$,

$$3\int_{s_1}^{s_2}(\varepsilon_2, Q_1) - C(\delta_0)\sqrt{\alpha_0} \leq -|yQ|_{L^2}^2 \ln\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \leq 5\int_{s_1}^{s_2}(\varepsilon_2, Q_1) + C(\delta_0)\sqrt{\alpha_0} \qquad (68)$$

and

$$\lambda(s_2) < 2\lambda(s_1). \qquad (69)$$

Proof of Proposition 4.

Step 1. *Integral form of the equation for the scaling parameter.* First, we assume $\alpha_0 < \alpha_7$ small enough so that

$$\frac{1}{2} \leq 1 + \frac{1}{4\delta_0}(\varepsilon_1, Q) \leq \frac{3}{2}. \qquad (70)$$

We then claim: for all $s_2 \geq s_1$,

$$\left|4\int_{s_1}^{s_2}(\varepsilon_2, Q_1) + |yQ|_{L^2}^2 \ln\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right)\right| \leq C(\delta_0)\sqrt{\alpha_0} + \int_{s_1}^{s_2}|(\varepsilon_2, Q_1)|. \qquad (71)$$

This relation follows from the equation governing the scaling parameter λ and the dispersive inequality (66). This equation is found by taking the inner product of (47) with the well-localized function y^2Q. Recall $L_-(y^2Q) = -4Q_1$. We get

$$4(\varepsilon_2, Q_1) + |yQ|_{L^2}^2\frac{\lambda_s}{\lambda} + (\varepsilon_1, y^2Q)_s = -\frac{\lambda_s}{\lambda}\left(\varepsilon_1, \frac{1}{2}y^2Q + y(y^2Q)_y\right) + \tilde{\gamma}_s(\varepsilon_2, y^2Q) - \frac{x_s}{\lambda}(\varepsilon_1, (y^2Q)_y) - (R_2(\varepsilon), y^2Q).$$

Using again estimates (39), (53) and (54), we easily conclude that for some universal constant C

$$\left|4(\varepsilon_2, Q_1) + |yQ|_{L^2}^2\frac{\lambda_s}{\lambda} + (\varepsilon_1, y^2Q)_s\right| \leq C\left(\int|\varepsilon_y|^2 + \int|\varepsilon|^2 e^{-2^-|y|}\right).$$

We integrate the above inequality between s_1 and $s_2 \geq s_1$:

$$\left|4\int_{s_1}^{s_2}(\varepsilon_2, Q_1) + |yQ|_{L^2}^2 \ln\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right)\right| \leq C\sqrt{\alpha_0} + C\int_{s_1}^{s_2}\left(\int|\varepsilon_y|^2 + \int|\varepsilon|^2 e^{-2^-|y|}\right). \qquad (72)$$

We now use (66)

$$\delta_0\int_{s_1}^{s_2}\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right)\leq\frac{1}{\delta_0}\int_{s_1}^{s_2}(\varepsilon_2,Q_1)^2+\frac{3}{2}|(\varepsilon_2,Q_1)|(s_2)$$
$$+\frac{3}{2}|(\varepsilon_2,Q_1)|(s_1)-2\int_{s_1}^{s_2}\lambda^2|E_0|$$
$$\leq\frac{1}{\delta_0}\int_{s_1}^{s_2}(\varepsilon_2,Q_1)^2+C\sqrt{\alpha_0}$$

and estimate, for $\alpha_0<\alpha_7$ small enough, from (72)

$$\left|4\int_{s_1}^{s_2}(\varepsilon_2,Q_1)+|yQ|_{L^2}^2\ln\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right)\right|\leq C(\delta_0)\sqrt{\alpha_0}+C(\delta_0)\int_{s_1}^{s_2}(\varepsilon_2,Q_1)^2$$
$$\leq C(\delta_0)\sqrt{\alpha_0}+\int_{s_1}^{s_2}|(\varepsilon_2,Q_1)|,$$

and (71) is proved.

Step 2. *Proof of* (i). We now claim as a consequence of (65) the following property: assume that for some $s_2\in R$, $(\varepsilon_2,Q_1)(s_2)=0$, then $(\varepsilon_2,Q_1)_s(s_2)>0$. We argue by contradiction assuming that for some $s_2\in\mathbf{R}$, $(\varepsilon_2,Q_1)(s_2)=0$ and $(\varepsilon_2,Q_1)_s(s_2)\leq 0$. Then

$$\left\{\left(1+\frac{1}{4\delta_0}(\varepsilon_1,Q)\right)(\varepsilon_2,Q_1)\right\}_s(s_2)\leq 0.$$

Injecting this into (65) yields $(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|})(s_2)\leq 0$, that is $\varepsilon(s_2)=0$. A contradiction follows from (51), the strictly negative energy condition and the fact that $\lambda(s)>0,\forall s$.

Consequently, the $\mathcal{C}^1$ function of time (ε_2,Q_1) may vanish at most once in $\mathbf{R}$ at some point s_0, and then is strictly negative at the left of this point, and positive at its right. We want to prove that such a time s_0 must indeed exist.

Assume for example for the sake of contradiction that

$$\forall s\in\mathbf{R},(\varepsilon_2,Q_1)(s)<0. \tag{73}$$

We look for a contradiction to (73) by looking at asymptotic properties of the solution as $s\to+\infty$. Inject the sign condition (73) into (71): for all $s\geq 0$,

$$-|yQ|_{L^2}^2\ln\left(\frac{\lambda(s)}{\lambda(0)}\right)\leq 3\int_0^s(\varepsilon_2,Q_1)+C(\delta_0)\sqrt{\alpha_0}.$$

Suppose now $\int_0^{+\infty}(\varepsilon_2,Q_1)=-\infty$; then the above relation implies $\lim_{s\to+\infty}\lambda(s)=+\infty$, so that with (35), we get $\lim_{t\to T}|u_x(t)|_{L^2}=0$. This contradicts by Gagliardo-Nirenberg the energy constraint $E_0<0$ on u_0. We thus have proved

$$\left|\int_0^{+\infty}(\varepsilon_2,Q_1)\right|<+\infty. \tag{74}$$

By (71) again,

$$\forall s \geq 0, \quad 0 < \lambda_1 \leq \lambda(s) \leq \lambda_2. \tag{75}$$

Consider now the $\mathcal{C}^1$ function of time $(\varepsilon_2, Q_1)(s)$. Then from (61), for some constant C, $|(\varepsilon_2, Q_1)_s| < C$ uniformly in s. Recall $(\varepsilon_2, Q_1)(s) < 0$ from (73). These two facts together with (74) yield

$$(\varepsilon_2, Q_1)(s) \to 0 \text{ as } s \to +\infty. \tag{76}$$

Consider now the pointwise virial relation (65) to compare (ε_2, Q_1) and the local norm of ε:

$$\left\{ \left(1 + \frac{1}{4\delta_0}(\varepsilon_1, Q)\right)(\varepsilon_2, Q_1) \right\}_s \geq \delta_0 \left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|} \right) - \frac{1}{\delta_0}(\varepsilon_2, Q_1)^2.$$

The left-hand side of this relation is the time derivative of a uniformly bounded function in time s, so that from (76), for some sequence $\tilde{s}_n \to +\infty$,

$$\lim_{n\to+\infty} \left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|} \right)(\tilde{s}_n) = 0. \tag{77}$$

This contradicts the energy constraint $E_0 < 0$. Indeed, from (51),

$$\begin{aligned} \lambda^2(s)|E_0| &\leq C \left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|} \right) + 2|(\varepsilon_1, Q)| \\ &\leq C \left(\int |\varepsilon_y|^2 + (\int |\varepsilon|^2 e^{-2^-|y|})^{\frac{1}{2}} \right), \end{aligned}$$

so that $\lambda(\tilde{s}_n) \to 0$, which contradicts (75). By looking at asymptotic properties of u as $s \to -\infty$, we prove in the same way that for all $s \in \mathbf{R}, (\varepsilon_2, Q_1)(s) > 0$ leads to a contradiction. This concludes the proof of (i).

Step 3. *Proof of* (ii). Once (ε_2, Q_1) is known to be strictly positive for $s > s_0$, estimate (68) follows from (71).

It remains to prove (69). For the sake of contradiction, assume, for some times $s_0 \leq s_1 < s_2$, that $\lambda(s_2) > 2\lambda(s_1)$. Then from (68), we estimate

$$|yQ|^2_{L^2} \ln\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) - C(\delta_0)\sqrt{\alpha_0} \leq -3\int_{s_1}^{s_2} (\varepsilon_2, Q_1) < 0$$

so that for $\alpha_0 < \alpha_7$ small enough, we get $\frac{1}{2}\|yQ\|_2^2 \ln(2) < 0$, a contradiction. This ends the proof of Proposition 4. □

Now note that using the invariance of the ε equation by translation in time, we may always assume that s_0, as defined as in Proposition 4, is such that

$$s_0 = 0.$$

4. Finite time blow-up and control of the blow-up rate

This section is devoted to the proof of Theorem 2. We consider $u_0 \in H^1$ such that

$$0 < \alpha_0 = \int |u|^2 - \int Q^2 \quad , \quad E(u_0) < 0 \quad , \quad \operatorname{Im}\left(\int (u_0)_x \overline{u_0}\right) = 0,$$

assuming α_0 small enough so that the results of the two previous sections apply. The proof is in two steps:

(i) We first prove in Section 4.1, as a consequence of both the almost monotonicity of the scaling parameter and the energetic constraint $E_0 < 0$, a result of finite or infinite time blow-up, or equivalently

$$\lim_{s \to +\infty} \lambda(s) = 0.$$

(ii) We then prove two different ways of exhibiting a differential inequality for the scaling parameter: the first one in Section 4.2 based on a refined version of the almost monotonicity of the scaling parameter which will imply through dynamical properties blow-up in finite time and a first upper-bound on the rate of growth

$$|u_x(t)|_{L^2} \leq \frac{C^*}{\sqrt{|E_0|}(T-t)},$$

the second one in Section 4.3 based on a refined version of virial inequality (65) which leads to the announced bound

$$|u_x(t)|_{L^2} \leq C^* \left(\frac{|\ln(T-t)|^{\frac{1}{2}}}{T-t}\right)^{\frac{1}{2}}.$$

We then focus in Section 4.4 on the N^{th} dimensional case.

4.1. *Finite or infinite time blow-up.* We claim $\lim_{t \uparrow T} |u_x(t)|_{L^2} = +\infty$ for some $0 < T \leq +\infty$, or equivalently

$$\lim_{s \to +\infty} \lambda(s) = 0. \tag{78}$$

We argue by contradiction assuming that for some sequence $s_n \to +\infty$

$$\forall n > 0, \quad \lambda(s_n) \geq \lambda_0 > 0.$$

We apply the almost monotonicity of the scaling parameter: let $s > 0$ and n be such that $s_n > s$; then (69) reads: $\lambda(s) > \frac{1}{2}\lambda(s_n)$, so that

$$\forall s > 0 \quad , \quad \lambda(s) > \frac{1}{2}\lambda_0 > 0. \tag{79}$$

From (68), we conclude

$$0 < \int_0^{+\infty} (\varepsilon_2, Q_1) < +\infty. \tag{80}$$

The proof now is similar to the one of Step 3 in the proof of Proposition 4. Let us recall the argument. Consider first the $\mathcal{C}^1$ function of time $(\varepsilon_2, Q_1)(s)$; then from (61) and (67), for some constant C and all $s > 0$,

$$|(\varepsilon_2, Q_1)_s| < C \quad \text{and} \quad (\varepsilon_2, Q_1)(s) > 0.$$

These two facts together with (80) yield

$$(\varepsilon_2, Q_1)(s) \to 0 \quad \text{as} \quad s \to +\infty. \tag{81}$$

Consider then the pointwise virial relation (65):

$$\left\{\left(1 + \frac{1}{4\delta_0}(\varepsilon_1, Q)\right)(\varepsilon_2, Q_1)\right\}_s \geq \delta_0 \left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right) - \frac{1}{\delta_0}(\varepsilon_2, Q_1)^2.$$

The left-hand side of this relation is the time derivative of a uniformly bounded function in time s, so that from (81), for some sequence $\tilde{s}_n \to +\infty$,

$$\lim_{n\to+\infty} \left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right)(\tilde{s}_n) = 0.$$

This contradicts the energy constraint $E_0 < 0$. Indeed, from (51)

$$\lambda^2(s)|E_0| \leq C \left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}},$$

so that $\lambda(\tilde{s}_n) \to 0$, which contradicts (79). This concludes the proof of finite or infinite time blow-up.

4.2. *Finite time blow-up and first upper bound on the blow-up rate.* In this subsection, we prove a weaker but more structurally stable version of Theorem 2:

Proposition 5. *Let $N = 1$. There exist $\alpha^* > 0$ and a universal constant $C^* > 0$ such that the following is true. Let $u_0 \in H^1$ with*

$$0 < \alpha_0 = \alpha(u_0) = \int |u_0|^2 - \int Q^2 < \alpha^*,$$
$$E_0 = E(u_0) < 0, \quad \operatorname{Im}\left(\int (u_0)_x \overline{u_0}(x)\right) = 0.$$

Let $u(t)$ be the corresponding solution to (1)*; then $u(t)$ blows up in finite time $0 < T < +\infty$ and for t close to T:*

$$|u_x(t)|_{L^2} \leq \frac{C^*}{\sqrt{|E_0|}(T-t)}.$$

Three facts are at the heart of the proof of this result:

(i) First, recall the virial relation (65):

$$\left\{\left(1+\frac{4}{\delta_0}(\varepsilon_1,Q)\right)(\varepsilon_2,Q_1)\right\}_s \geq \delta_0\left(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|}\right)+2\lambda^2(s)|E_0|-\frac{1}{\delta_0}(\varepsilon_2,Q_1)^2.$$

This pointwise estimate gives control of the oscillatory function of time $(\int|\varepsilon_y|^2+\int|\varepsilon|^2e^{-2^-|y|})(s)$ by the quantity $(\varepsilon_2,Q_1)^2(s)$ in a time-averaging sense. Now the problem is to relate the two key parameters $\lambda(s)$ and (ε_2,Q_1).

(ii) Second, the equation governing the scaling parameter has been proved to give

$$\frac{\lambda_s}{\lambda}\sim-(\varepsilon_2,Q_1)$$

up to oscillatory integrals controlled by $\int|\varepsilon|^2e^{-2^-|y|}$.

(iii) Third, on the basis of these two facts, we are able to prove a refined version of the almost monotonicity property of the scaling parameter. This result allows us to prove a new link between the two quantities (ε_2,Q_1) and $\lambda(s)$. More precisely, we claim the following pointwise *uniform* estimate

PROPOSITION 6 (Uniform control of the scaling parameter by (ε_2,Q_1)). *There exists a universal constant B and $\alpha_8>0$ such that for $\alpha_0<\alpha_8$, there exists $\tilde{s_0}\geq 0$ such that*

$$\forall s\geq\tilde{s}_0,\quad |E_0|\lambda^2(s)\leq B(\varepsilon_2,Q_1)^2(s). \tag{82}$$

A key fact in our analysis is that the ε decomposition, i.e. the choice of orthogonality conditions

$$(\varepsilon_1,Q_1)=(\varepsilon_1,yQ)=0\quad\text{and}\quad(\varepsilon_2,Q_2)=0$$

adapted to (i), study of dispersion, and to (ii), evolution of the scaling parameter, turn out to be the same. This is a noteworthy fact for the study of the dynamic of (1). Recall for example that in the study of (15), two different decompositions had to be taken into account.

Let us now finish the proof of Proposition 5 which is a fairly easy consequence of the three above facts.

Proof of Proposition 5 *assuming Proposition* 6. As for the proof of [15], we first use the finite or infinite time blow-up result (78) and consider a sequence of times t_n be such that

$$\lambda(t_n)=2^{-n} \tag{83}$$

and $s_n = s(t_n)$ the corresponding sequence. Let $\tilde{t}_0$ be such that $s(\tilde{t}_0) = \tilde{s}_0$ given by Proposition 6. Note that we may assume $n \geq n_0$ such that $t_n \geq \tilde{t}_0$. Note that $0 < t_n < t_{n+1}$ from (69), and so $0 < s_n < s_{n+1}$. Moreover, $t_n \to T$, and from (69),

$$\forall s \in [s_n, s_{n+1}], \quad 2^{-n-1} \leq \lambda(s) \leq 2^{-(n-1)}.$$

We now claim that blow-up in finite time follows from a control from above of the size of the intervals $[t_n, t_{n+1}]$.

First write (82) using (67) for n large enough

$$\forall s_n \leq s \leq s_{n+1}, \quad 0 < \lambda(s) \leq \frac{\sqrt{B}}{\sqrt{|E_0|}}(\varepsilon_2, Q_1)(s)$$

and integrate this relation between s_n and s_{n+1}

$$\int_{s_n}^{s_{n+1}} \lambda(s)ds \leq \frac{\sqrt{B}}{\sqrt{|E_0|}} \int_{s_n}^{s_{n+1}} (\varepsilon_2, Q_1)(s)ds.$$

Moreover, from informations of type (i) and (ii), we have derived (68) which implies for $\alpha_0 < \alpha^*$ small enough

$$3\int_{s_n}^{s_{n+1}} (\varepsilon_2, Q_1) \leq C(\delta_0)\sqrt{\alpha_0} + |yQ|^2_{L^2}\ln(2) \leq 3|yQ|^2_{L^2}\ln(2).$$

Therefore

$$\int_{s_n}^{s_{n+1}} \lambda(s)ds \leq \frac{\sqrt{B}}{\sqrt{|E_0|}}|yQ|^2_{L^2}\ln(2).$$

Now we change variables in the integral at the left of the above inequality according to $\frac{ds}{dt} = \frac{1}{\lambda^2(s)}$ and estimate with the use of (69) and (83)

$$\begin{aligned} \frac{\sqrt{B}}{\sqrt{|E_0|}}|yQ|^2_{L^2}\ln(2) &\geq \int_{s_n}^{s_{n+1}} \frac{1}{\lambda(s)}\lambda^2(s)ds \\ &\geq 2^{n-1}\int_{s_n}^{s_{n+1}} \lambda^2(s)ds = 2^{n-1}(t_{n+1} - t_n) \end{aligned}$$

so that for $n \geq n_0$

$$t_{n+1} - t_n \leq \frac{C}{\sqrt{|E_0|}}2^{-(n+1)}.$$

Summing this inequality in n yields $T = \lim_{n\to+\infty} t_n < +\infty$ and blow-up in finite time is proved. Moreover, the summation also gives the estimate for n large

$$T - t_n \leq \frac{C}{\sqrt{|E_0|}}2^{-(n+1)} \leq \frac{C}{\sqrt{|E_0|}}\lambda(t_{n+1}).$$

Now let $T > t > t_{n_0}$, then $t_n \leq t < t_{n+1}$ for some n and the above inequality together with (69) ensures

$$T - t \leq T - t_n \leq \frac{C}{\sqrt{|E_0|}}\lambda(t_{n+1}) \leq 2\frac{C}{\sqrt{|E_0|}}\lambda(t), \tag{84}$$

which together with estimate (35), which concludes the proof of Theorem 2. □

Let us now prove Proposition 6.

Proof of Proposition 6. We use here an idea which was first introduced for the study of the KdV equation in [14], and which is that the study of dispersion has to be made on time intervals of slow variations of the scaling parameter. Again, the existence of such intervals heavily relies on the finite or infinite time blow-up result. On such intervals, we are able to prove a monotonicity kind of result on the key quantity (ε_2, Q_1). Together with the dispersive effect of (1) in the ε variable, this last result will allow us to exhibit a lower bound on the size of the slow variations intervals constructed. Note that such a lower bound is unknown for the KdV equation, and will yield the result in our setting.

Let δ_0 be as in (65) and $C(\delta_0)$ be the fixed constant of estimate (66). We first fix a constant $k_0 > 1$ such that

$$0 < \ln(k_0) < \frac{\delta_0}{10|yQ|_{L^2}^2}, \tag{85}$$

and assume that α_8 in Proposition 6 is small enough so that

$$\frac{2}{|yQ|_{L^2}^2} C(\delta_0)\sqrt{\alpha_8} \leq \ln(k_0). \tag{86}$$

(81) holds with $B = 2\left(\frac{60}{\|yQ\|_2^2 \ln(k_0)} + \frac{16}{\delta_0}\right)$.

Step 1. *Construction and properties of slow variations time intervals.* Let us recall the finite or infinite time blow-up result:

$$\lambda(s) \to 0 \quad \text{as} \quad s \to +\infty.$$

This result easily implies the following : there exists $\tilde{s_0} \geq 0$ such that for all $s_2 \geq \tilde{s}_0$, there exists $s_1(s_2) \in (0, s_2)$ such that

$$\lambda(s_1) = k_0\lambda(s_2) \quad \text{and} \quad \forall s \in [s_1, s_2], \quad \lambda(s) \leq k_0\lambda(s_2).$$

We note $I(s_2) = [s_1, s_2]$ such an interval. A first key to our analysis is the following lemma:

Lemma 8 (Control of the parameters on $I(s_2)$). *Let* $s_2 \geq \tilde{s}_0$ *and* $s \in I(s_2)$, *then*

$$\text{(i)} \qquad \frac{\lambda(s_2)}{2} \leq \lambda(s) \leq k_0\lambda(s_2), \tag{87}$$

$$\text{(ii)} \qquad (\varepsilon_2, Q_1)(s) \leq 4(\varepsilon_2, Q_1)(s_2). \tag{88}$$

Proof. Let $s_2 \geq \tilde{s}_0$ and $I(s_2) = [s_1, s_2]$. (i) follows from the definition of $I(s_2)$ and the almost monotonicity of the scaling parameter (69)

$$\forall s \leq s_2 \quad , \quad \lambda(s_2) \leq 2\lambda(s).$$

(ii) Let $K = K(s_2) = \sup_{s\in[s_1,s_2]} \frac{(\varepsilon_2,Q_1)(s)}{(\varepsilon_2,Q_1)(s_2)} > 0$. We fix $s \in I(s_2)$ and apply the L^2_{loc} dispersive inequality (66) with (70) on the time interval $[s, s_2]$:

$$\frac{1}{2}(\varepsilon_2, Q_1)(s) \leq \frac{3}{2}(\varepsilon_2, Q_1)(s_2) + \frac{1}{\delta_0}\int_{s_1}^{s_2}(\varepsilon_2, Q_1)^2.$$

From the definition of K and (67), we get: for all $s \in I(s_2)$,

$$(\varepsilon_2, Q_1)(s) \leq (\varepsilon_2, Q_1)(s_2)\left(3 + \frac{2K}{\delta_0}\int_{s_1}^{s_2}(\varepsilon_2, Q_1)\right).$$

Taking the sup in $s \in I(s_2)$ in the above inequality, we conclude

$$K \leq 3 + \frac{2K}{\delta_0}\int_{s_1}^{s_2}(\varepsilon_2, Q_1).$$

We now apply (68), (86) and (87) on the interval $[s_1, s_2]$ to get

$$\begin{aligned}3\int_{s_1}^{s_2}(\varepsilon_2, Q_1) &\leq C(\delta_0)\sqrt{\alpha_0} - |yQ|^2_{L^2}\ln\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right)\\ &\leq \frac{1}{2}|yQ|^2_{L^2}\ln(k_0) + |yQ|^2_{L^2}\ln(k_0) = \frac{3}{2}|yQ|^2_{L^2}\ln(k_0)\end{aligned}$$

so that $K(s_2) = 3 + \frac{K}{\delta_0}|yQ|^2_{L^2}\ln(k_0) \leq 3 + \frac{1}{10}K$, and $K \leq 4$. This concludes the proof of Lemma 8.

Step 2. *Conclusion.* The conclusion follows from the differential inequality satisfied by (ε_2, Q_1) on $I(s_2)$ and the sign condition $(\varepsilon_2, Q_1) > 0$. Let $s_2 \geq \tilde{s}_0$, $I(s_2) = [s_1, s_2]$ with a constant B such that

$$\lambda^2(s_2)|E_0| \geq B(\varepsilon_2, Q_1)^2(s_2). \tag{89}$$

First express (65)

$$\left\{\left(1 + \frac{1}{4\delta_0}(\varepsilon_1, Q)\right)(\varepsilon_2, Q_1)\right\}_s(s) \geq 2\lambda^2(s)|E_0| - \frac{1}{\delta_0}(\varepsilon_2, Q_1)(s)^2.$$

From Lemma 8 and (89), we estimate for $s \in I(s_2)$:

$$\begin{aligned}2\lambda^2(s)|E_0| - \frac{1}{\delta_0}(\varepsilon_2, Q_1)^2(s) &\geq \frac{2|E_0|}{4}\lambda^2(s_2) - \frac{16}{\delta_0}(\varepsilon_2, Q_1)^2(s_2)\\ &\geq \left(\frac{B}{2} - \frac{16}{\delta_0}\right)(\varepsilon_2, Q_1)^2(s_2).\end{aligned}$$

If $\frac{B}{2} - \frac{16}{\delta_0} \leq 0$ then the proof is finished. If not, integrating (65) with the help of the above inequality on the time interval $[s_1, s_2]$, we get

$$\frac{3}{2}(\varepsilon_2, Q_1)(s_2) - \left(\frac{B}{2} - \frac{16}{\delta_0}\right)(s_2 - s_1)(\varepsilon_2, Q_1)^2(s_2) \geq \frac{1}{2}(\varepsilon_2, Q_1)(s_1) > 0$$

so that

$$\frac{3}{2} > \left(\frac{B}{2} - \frac{16}{\delta_0}\right)(s_2 - s_1)(\varepsilon_2, Q_1)(s_2). \tag{90}$$

To conclude, we therefore need a lower-bound for the size of $I(s_2)$. This lower bound is a consequence of the uniform backward control of (ε_2, Q_1) on $I(s_2)$, Lemma 8, and of the equation governing the scaling parameter.

First, we recall estimate (68) together with (87)

$$\begin{aligned}|yQ|_{L^2}^2 \ln(k_0) &= -|yQ|_{L^2}^2 \ln\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right)\\ &\leq 5\int_{s_1}^{s_2}(\varepsilon_2, Q_1) + C(\delta_0)\sqrt{\alpha_0} \leq 5\int_{s_1}^{s_2}(\varepsilon_2, Q_1) + \frac{1}{2}|yQ|_{L^2}^2\ln(k_0),\end{aligned}$$

the last estimate following from (86). This last inequality together with (88) yields

$$|yQ|_{L^2}^2 \ln(k_0) \leq 10\int_{s_1}^{s_2}(\varepsilon_2, Q_1) \leq 40(s_2 - s_1)(\varepsilon_2, Q_1)(s_2).$$

It suffices now to inject this last estimate into (90) to get

$$\frac{3}{2} \geq \frac{|yQ|_{L^2}^2}{40}\ln(k_0)\left(\frac{B}{2} - \frac{16}{\delta_0}\right) \quad \text{and} \quad B \leq 2\left(\frac{60}{|yQ|_{L^2}^2\ln(k_0)} + \frac{16}{\delta_0}\right)$$

which concludes the proof of Proposition 6.

4.3. *Refined upper bound on the blow-up rate.*. In this section, we finish the proof of Theorem 2 by proving the announced upper bound on the blow-up rate

$$|u_x(t)|_{L^2} \leq C\left(\frac{|\ln(T-t)|^{\frac{1}{2}}}{T-t}\right)^{\frac{1}{2}}.$$

We assume that blow-up in finite or infinite time is already proved (see Section 4.1); i.e., $\lambda(s) \to 0$ as $s \to +\infty$. This further estimate is derived on the basis of a refinement of dispersive inequality (65) and is related to the very specific algebraic structure of the virial linearized operator $\mathcal{L}$ around Q of Lemma 7. Indeed, let us make the following formal computation. In the limit $\alpha_0 \to 0$, ε satisfies the linear limit equation

$$\begin{cases} \partial_s\varepsilon_1 - L_-\varepsilon_2 = l(s)Q_1 + X(s)Q_y \\ \partial_s\varepsilon_2 + L_+\varepsilon_1 = g(s)Q \end{cases}$$

for some parameters $l(s), X(s), g(s)$. In the spirit of the linear Liouville Theorem, Theorem 3 of [13], one can prove from (65) that the space of uniformly bounded solutions in time in $H^1 \cap \Sigma$ of this linear equation which satisfy orthogonality conditions (33) and (34) is in fact one dimensional and generated by the stationary solution

$$\varepsilon = iW \quad \text{with} \quad W = y^2 Q + \mu Q$$

where μ is such that

$$(W, Q_2) = 0 \quad \text{i.e.} \quad \mu = \frac{2}{|yQ|_{L^2}^2}(y^2 Q, Q_2). \tag{91}$$

The existence of such a solution corresponds to an additional degeneracy of the linear operator close to the ground state L, and is very specific to the Schrödinger equation.

The idea to refine dispersive estimate (65) is therefore to express it in terms of a new variable

$$\tilde{\varepsilon} = \varepsilon + ib(s)W$$

for some function $b(s)$ to be chosen, that is, to introduce the first term in the asymptotic formal expansion of ε as $s \to +\infty$.

We note

$$W_1 = \frac{1}{2}W + yW_y$$

and claim the following refined dispersive inequality:

Proposition 7 (Refined local virial estimate in ε). *Let* $\tilde{\varepsilon} = \varepsilon + i\frac{(\varepsilon_2, Q_1)}{|yQ|_{L^2}^2}W$. *There exist universal constants* $\tilde{\delta}_0$, $C > 0$ *and* $\alpha_9 > 0$ *such that for* $\alpha_0 < \alpha_9$, *there exists* $\tilde{s}_1$ *such that*: *for all* $s \geq \tilde{s}_1$,

$$\left\{\left(1 + \frac{1}{|yQ|_{L^2}^2}(\varepsilon_1, W_1)\right)(\varepsilon_2, Q_1)\right\}_s + C(\varepsilon_2, Q_1)^4 \geq \tilde{\delta}_0\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) + \lambda^2 |E_0|. \tag{92}$$

Remark 4. Compare (65) and (92). The first one says that in a time-averaging sense and with the suitable norm, ε^2 is of order $(\varepsilon_2, Q_1)^2$, whereas the second one says that $\tilde{\varepsilon}^2$ is of order $(\varepsilon_2, Q_1)^4$, so that $\varepsilon = -i\frac{(\varepsilon_2, Q_1)}{|yQ|_{L^2}^2}W + \tilde{\varepsilon}$ with $\tilde{\varepsilon}$ of smaller order is a formal asymptotic development of ε as $s \to +\infty$.

Let us assume Proposition 7. We now are in position to considerably refine estimate (82) of Proposition 6 by showing:

Proposition 8 (Refined uniform control of the scaling parameter by (ε_2, Q_1)). *There exists a universal constant* $\tilde{B}$ *and* $\alpha_{10} > 0$ *such that for*

$\alpha_0 < \alpha_{10}$, *there exists* $\tilde{s}_4$ *such that for all* $s \geq \tilde{s}_4$,

(93)
$$\lambda^2(s) \leq \exp\left(-\frac{\tilde{B}}{(\varepsilon_2, Q_1)^2(s)}\right) \quad \textit{or equivalently} \quad (\varepsilon_2, Q_1)(s) \geq \frac{\tilde{B}}{|\ln(\lambda(s))|^{\frac{1}{2}}}.$$

Proof of Proposition 8. The proof is simply derived from (92) and the almost monotonicity of the scaling parameter (68). First recall from Proposition 4 that $(\varepsilon_2, Q_1)(s) > 0$ for $s > 0$. Therefore, for $\alpha_0 < \alpha_{10}$ small enough, the function $f(s) = \left(1 + \frac{1}{|yQ|^2_{L^2}}(\varepsilon_1, W_1)\right)(\varepsilon_2, Q_1)$ satisfies

$$\frac{1}{2}(\varepsilon_2, Q_1) \leq f(s) \leq 2(\varepsilon_2, Q_1) \tag{94}$$

and so does not vanish for $s > 0$, and estimate (92) may be viewed as a differential inequality
$$f_s + Cf^4 \geq 0.$$

We integrate this inequality from the nonvanishing property of f and get for $s \geq \tilde{s}_1$ of Proposition 7:
$$\frac{1}{f^3(s)} \leq C(s - \tilde{s}_1) + \frac{1}{f^3(\tilde{s}_1)} \leq 2Cs$$

for $s \geq \tilde{s}_2$. From (94), we get for some universal constant

$$\forall s \geq \tilde{s}_2, \quad (\varepsilon_2, Q_1)(s) \geq \frac{C}{s^{\frac{1}{3}}}. \tag{95}$$

We now recall (68) on the time interval $[\tilde{s}_2, s]$,
$$3\int_{\tilde{s}_2}^{s}(\varepsilon_2, Q_1) \leq -|yQ|^2_{L^2}\ln\left(\frac{\lambda(s)}{\lambda(\tilde{s}_2)}\right) + C(\delta_0)\sqrt{\alpha_0} \leq -\frac{1}{2}|yQ|^2_{L^2}\ln\left(\frac{\lambda(s)}{\lambda(\tilde{s}_2)}\right)$$

for $s \geq \tilde{s}_3$ large enough, from the fact that $\lambda(s) \to 0$ as $s \to +\infty$. We now inject (95) into the above inequality and get for $s \geq \tilde{s}_3$,
$$C(s^{\frac{2}{3}} - \tilde{s}_2^{\frac{2}{3}}) \leq -\ln\left(\frac{\lambda(s)}{\lambda(\tilde{s}_2)}\right) \quad \text{i.e.} \quad \frac{C}{2}s^{\frac{2}{3}} \leq -\ln(\lambda(s)) = |\ln(\lambda(s))|$$

for some universal constant $C > 0$ and $s \geq \tilde{s}_4$. Injecting (95) into the above inequality, we conclude for $s \geq \tilde{s}_4$,
$$|\ln(\lambda(s))| \geq Cs^{\frac{2}{3}} \geq \frac{C}{(\varepsilon_2, Q_1)^2(s)} \quad \text{i.e.} \quad (\varepsilon_2, Q_1)(s) \geq \frac{C}{|\ln(\lambda(s))|^{\frac{1}{2}}}$$

and Proposition 8 is proved. □

We now easily conclude the proof of Theorem 2 as in Section 4.2.

Proof of Theorem 2. The proof is very similar to the one of finite time blow-up and we briefly sketch the argument.

Again let t_n be a sequence of times such that $\lambda(t_n) = 2^{-n}$, $s_n = s(t_n)$ be the corresponding sequence, and $\tilde{t}_4$ such that $s(\tilde{t}_4) = \tilde{s}_4$ of Proposition 8. We may assume $n \geq n_0$ so that $t_n \geq \tilde{t}_4$. Note that $t_n \to T$, the blow-up time. Recall also from (68) that for all $s \in [s_n, s_{n+1}]$, $2^{-(n+1)} \leq \lambda(s) \leq 2^{-(n-1)}$. We then get, from (68), the definition of the sequence t_n, the relation $\frac{ds}{dt} = \frac{1}{\lambda^2}$ and estimate (93), the following: for all $n \geq n_0$,

$$C \geq \int_{s_n}^{s_{n+1}} (\varepsilon_2, Q_1) ds \geq \int_{s_n}^{s_{n+1}} \frac{C ds}{|\ln(\lambda(s))|^{\frac{1}{2}}} \geq \int_{t_n}^{t_{n+1}} \frac{C dt}{\lambda^2(t)|\ln(\lambda(t))|^{\frac{1}{2}}}$$

so that

$$\forall n \geq n_0, \quad C\lambda^2(t_n)|\ln(\lambda(t_n))|^{\frac{1}{2}} \geq t_{n+1} - t_n.$$

From $\lambda(t_n) = 2^{-n}$, and by summing the above inequality in n, we get

$$\begin{aligned} C(T - t_n) &\leq \sum_{k\geq n} 2^{-2k}\sqrt{k} = \sum_{n\leq k\leq 2n} 2^{-2k}\sqrt{k} + \sum_{k\geq 2n} 2^{-2k}\sqrt{k} \\ &\leq C2^{-2n}\sqrt{n} + 2^{-4n}\sqrt{n}\sum_{k\geq 0} 2^{-2k}\sqrt{2+\frac{k}{n}} \\ &\leq C2^{-2n}\sqrt{n} + C2^{-4n}\sqrt{n} \leq C2^{-2n}\sqrt{n} \leq C\lambda^2(t_n)|\ln(\lambda(t_n))|^{\frac{1}{2}}. \end{aligned}$$

Now since $t \geq \tilde{t}_4$, for some $n \geq n_0$, $t \in [t_n, t_{n+1}]$, and from $\frac{1}{4}\lambda(t_n) = \frac{1}{2}\lambda(t_{n+1}) \leq \lambda(t) \leq 2\lambda(t_n)$, we conclude

$$\lambda^2(t)|\ln(\lambda(t))|^{\frac{1}{2}} \geq C\lambda^2(t_n)|\ln(\lambda(t_n))|^{\frac{1}{2}} \geq C(T-t_n) \geq C(T-t).$$

Now note that the function $f(x) = x^2|\ln(x)|^{\frac{1}{2}}$ is nondecreasing in a neighborhood of $x = 0$, and moreover

$$f\left(\frac{C\sqrt{T-t}}{|\ln(T-t)|^{\frac{1}{4}}}\right) = C(T-t)\left(1 - C\frac{\ln(|\ln(T-t)|)}{|\ln(T-t)|^{\frac{1}{2}}}\right) \leq C(T-t)$$

for t close enough to T, so that we get for some universal constant C^*:

$$f(\lambda(t)) \geq f\left(\frac{C^*\sqrt{T-t}}{|\ln(T-t)|^{\frac{1}{4}}}\right) \quad \text{i.e.} \quad \lambda(t) \geq C^*\frac{\sqrt{T-t}}{|\ln(T-t)|^{\frac{1}{4}}}$$

and Theorem 2 is proved. □

It now remains to prove Proposition 7.

Proof of Proposition 7. We proceed in several steps.

Step 1. *Structure of the virial linearized operator* $\mathcal{L}_2$. We use here some noteworthy cancellation of oscillatory integrals in (58). Let L_- as in (40) and

$\mathcal{L}_2$ be the linear operator introduced in Lemma 7. For any function f, we denote $f_1 = \frac{1}{2}f + yf_y$ and $f_2 = (f_1)_1$. Note again $(f, g_1) = -(f_1, g)$. From direct verification,

$$\mathcal{L}_2(f) = \frac{1}{2}\left\{L_-(f_1) - (L_-(f))_1\right\}$$

and

$$H_2(\varepsilon_2, \varepsilon_2) = (\mathcal{L}_2\varepsilon_2, \varepsilon_2) = \left(L_-\varepsilon_2, \frac{1}{2}\varepsilon_2 + y\varepsilon_{2y}\right).$$

Let

$$W = y^2Q + \mu Q$$

with μ such that $(W, Q_2) = 0$. We claim

$$\mathcal{L}_2(W) = \frac{1}{2}L_-(W_1) + 2Q_2 \quad \text{and} \quad H_2(W, W) = 0.$$

Indeed, from $L_-(Q) = 0$ and $L_-(y^2Q) = -4Q_1$, we compute $L_-W = -4Q_1$ so that $\mathcal{L}_2(W) = \frac{1}{2}\left\{L_-(W_1) - (L_-(W))_1)\right\} = \frac{1}{2}L_-(W_1) + 2Q_2$. Now

$$H_2(W, W) = (L_-W, W_1) = (-4Q_1, W_1) = 4(W, Q_2) = 0$$

from (91).

We now consider

$$\tilde{\varepsilon}_2 = \varepsilon_2 + bW \quad \text{with} \quad b = \frac{(\varepsilon_2, Q_1)}{|yQ|^2_{L^2}}$$

so that

$$(\tilde{\varepsilon}_2, Q_2) = 0 \quad \text{and} \quad (\tilde{\varepsilon}_2, Q_1) = 0. \tag{96}$$

Indeed, the first relation holds from (34) and (91), the second one directly follows from the definition of $\tilde{\varepsilon}$ and $(y^2Q, Q_1) = -|yQ|^2_{L^2}$. We now compute

$$\begin{aligned} H_2(\tilde{\varepsilon}_2, \tilde{\varepsilon}_2) &= H_2\left(\varepsilon_2 + \frac{(\varepsilon_2, Q_1)}{|yQ|^2_{L^2}}W, \varepsilon_2 + \frac{(\varepsilon_2, Q_1)}{|yQ|^2_{L^2}}W\right) \\ &= H_2(\varepsilon_2, \varepsilon_2) + 2\frac{(\varepsilon_2, Q_1)}{|yQ|^2_{L^2}}(\varepsilon_2, \mathcal{L}_2W) \\ &= H_2(\varepsilon_2, \varepsilon_2) + \frac{2}{|yQ|^2_{L^2}}(\varepsilon_2, Q_1)\left(\varepsilon_2, \frac{1}{2}L_-W_1 + 2Q_2\right) \\ &= H_2(\varepsilon_2, \varepsilon_2) + \frac{1}{|yQ|^2_{L^2}}(\varepsilon_2, Q_1)(\varepsilon_2, L_-W_1) \end{aligned}$$

where in the last step we used the orthogonality condition $(\varepsilon_2, Q_2) = 0$ from (34).

Step 2. *The first estimate on cubic terms.* We now rewrite virial equality (58) with (33) and (34) as

$$\begin{aligned} (\varepsilon_2, Q_1)_s = {} & 2\lambda^2|E_0| + H_1(\varepsilon_1, \varepsilon_1) + H_2(\tilde{\varepsilon}_2, \tilde{\varepsilon}_2) \\ & - \frac{x_s}{\lambda}(\varepsilon_2, Q_{1y}) - \frac{1}{|yQ|^2_{L^2}}(\varepsilon_2, Q_1)(\varepsilon_2, L_-W_1) + G(\varepsilon) \end{aligned}$$

with $G(\varepsilon)$ as given in (59). From (96) and Proposition 2, we then estimate

$$(\varepsilon_2, Q_1)_s \geq 2\lambda^2|E_0| + \frac{\delta_1}{2}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) - \frac{1}{|yQ|_{L^2}^2}(\varepsilon_2, Q_1)(\varepsilon_2, L_-W_1)\frac{x_s}{\lambda}(\varepsilon_2, Q_{1y}) - \frac{2}{\delta_1}(\varepsilon_1, Q)^2 + G(\varepsilon).$$

We directly estimate the three formally cubic terms in the above expression:

- We can write $G(\varepsilon) = G(\tilde{\varepsilon} - ibW)$ and easily estimate from (59)

$$|G(\varepsilon)| \leq \frac{16}{\delta_1}b^4 + \frac{\delta_1}{16}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right).$$

- Recall (51)

$$\left|\lambda^2(s)|E_0| + 2(\varepsilon_1, Q)\right| \leq C\left(\int |\varepsilon_y|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right),$$

  then from $\lambda(s) \to 0$ as $s \to +\infty$, we get

$$(\varepsilon_1, Q)^2 \leq \frac{16}{\delta_1}b^4 + \frac{\delta_1}{16}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|} + \lambda^2|E_0|\right) \tag{97}$$

  for some $\tilde{s}_1 > 0$ and $s \geq \tilde{s}_1$.

- We treat the nonradial term $\frac{x_s}{\lambda}(\varepsilon_2, Q_{1y})$. To do so, we first recall (55)

  (98)
$$(\varepsilon_2, Q_y) = \frac{1}{2}\mathrm{Im}\left(\int \varepsilon_y\overline{\varepsilon}\right) = \frac{1}{2}\left\{\mathrm{Im}\left(\int \tilde{\varepsilon}_y\overline{\tilde{\varepsilon}}\right) + \frac{2}{|yQ|_{L^2}^2}(\varepsilon_2, Q_1)(\varepsilon_1, W_y)\right\}.$$

  We now recall (57):

$$\frac{|Q|_{L^2}^2}{2}\frac{x_s}{\lambda} = -2(\varepsilon_2, Q_y) - \frac{x_s}{\lambda}(\varepsilon_1, (yQ)_y) - \frac{\lambda_s}{\lambda}\left(\varepsilon_1, \frac{1}{2}yQ + y(yQ)_y\right) + \tilde{\gamma}_s(\varepsilon_2, yQ) - (R_2(\varepsilon), yQ),$$

  and then compute from $(W, Q_{1y}) = 0$

$$\frac{|Q|_{L^2}^2}{2}\frac{x_s}{\lambda}(\varepsilon_2, Q_{1y}) = -\mathrm{Im}\left(\int \tilde{\varepsilon}_y\overline{\tilde{\varepsilon}}\right)(\tilde{\varepsilon}_2, Q_{1y}) + G^{(1)}(\varepsilon)$$

  with

$$G^{(1)}(\varepsilon) = -\frac{2}{|yQ|_{L^2}^2}(\varepsilon_2, Q_1)(\varepsilon_1, W_y)(\varepsilon_2, Q_{1y}) + (\varepsilon_2, Q_{1y})\left\{-\frac{x_s}{\lambda}(\varepsilon_1, (yQ)_y) - \frac{\lambda_s}{\lambda}\left(\varepsilon_1, \frac{1}{12}yQ + y(yQ)_y\right)\right\} + (\varepsilon_2, Q_{1y})\left\{\tilde{\gamma}_s(\varepsilon_2, yQ) - (R_2(\varepsilon), yQ)\right\}$$

cubic in ε. Now express $G^{(1)}(\varepsilon) = G^{(1)}(\tilde{\varepsilon} - bW)$ and note from (56) that $\left(\frac{|yQ|_2^2}{2}\tilde{\gamma}_s - (\varepsilon_1, L_+Q_2)\right)$ is quadratic in ε, so that each cubic term in $G^{(1)}(\varepsilon)$ of the form of three scalar products contains at least one term (ε_1, V) for some well localized function V. Therefore the estimate

$$|G^{(1)}(\varepsilon)| \leq \frac{16}{\delta_1}b^4 + \frac{\delta_1}{16}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right)$$

easily follows. Moreover,

$$\begin{aligned}\left|\mathrm{Im}\left(\int \tilde{\varepsilon}_y \overline{\tilde{\varepsilon}}\right)(\tilde{\varepsilon}_2, Q_{1y})\right| &\leq C|\tilde{\varepsilon}|_{L^2}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right)\\ &\leq \frac{\delta_1}{16}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right)\end{aligned}$$

for $\alpha_0 < \alpha_9$ small enough.

Putting together the three estimates above, we have so far proved

$$\begin{aligned}(\varepsilon_2, Q_1)_s + Cb^4 \geq{}& \frac{3}{2}\lambda^2|E_0| + \frac{5\delta_1}{16}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right)\\ &-\frac{1}{|yQ|_{L^2}^2}(\varepsilon_2, Q_1)(\varepsilon_2, L_-W_1).\end{aligned} \tag{99}$$

Step 3. *Transformation of the dispersive inequality.* We now inject dynamical information to handle the term $(\varepsilon_2, Q_1)(\varepsilon_2, L_-W_1)$ in (99). To do so, we take the inner product of (47) with W_1. Note that $(W_1, Q_1) = -(W, Q_2) = 0$, so that

$$\begin{aligned}(\varepsilon_2, L_-W_1) = {}&\partial_s(\varepsilon_1, W_1) + \frac{\lambda_s}{\lambda}\left(\varepsilon_1, \frac{1}{2}W_1 + yW_{1y}\right)\\ &+\frac{x_s}{\lambda}(\varepsilon_1, W_{1y}) - \tilde{\gamma}_s(\varepsilon_2, W_1) + (R_2(\varepsilon), W_1).\end{aligned}$$

Injecting this into (99) and integrating by parts in time, we get

$$\begin{aligned}&\left\{\left(1 + \frac{1}{|yQ|_{L^2}^2}(\varepsilon_1, W_1)\right)(\varepsilon_2, Q_1)\right\}_s + Cb^4 \geq \frac{3}{2}\lambda^2|E_0|\\ &\quad+ \frac{5\delta_1}{16}\left(\int |\tilde{\varepsilon}_y|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) + \frac{1}{|yQ|_{L^2}^2}(\varepsilon_1, W_1)\partial_s(\varepsilon_2, Q_1) + G^{(2)}(\varepsilon)\end{aligned}$$

where $G^{(2)}(\varepsilon)$ is formally cubic in ε and explicitly is

$$\begin{aligned}G^{(2)}(\varepsilon) = -\frac{1}{|yQ|_{L^2}^2}(\varepsilon_2, Q_1)\Big\{&-\frac{\lambda_s}{\lambda}(\varepsilon_1, W_2) + \frac{x_s}{\lambda}(\varepsilon_{1y}, W_1)\\ &- \tilde{\gamma}_s(\varepsilon_2, W_1) + (R_2(\varepsilon), W_1)\Big\}.\end{aligned}$$

We now inject (61) into the above inequality to get

$$\left\{\left(1+\frac{1}{|yQ|_{L^2}^2}(\varepsilon_1,W_1)\right)(\varepsilon_2,Q_1)\right\}_s \geq \frac{3}{2}\lambda^2|E_0| + \frac{5\delta_1}{16}\left(\int|\tilde{\varepsilon}_y|^2+\int|\tilde{\varepsilon}|^2e^{-2^-|y|}\right) + \frac{2}{|yQ|_{L^2}^2}(\varepsilon_1,W_1)(\varepsilon_1,Q)+G^{(3)}(\varepsilon)$$

with

$$G^{(3)}(\varepsilon)=G^{(2)}(\varepsilon)+\frac{1}{|yQ|_{L^2}^2}(\varepsilon_1,W_1)\left(-\frac{x_s}{\lambda}(\varepsilon_2,Q_{1y})+(R_1(\varepsilon),Q_1)\right)$$

still cubic in ε. We now estimate:

- Similarly as for $G^{(1)}(\varepsilon)$,

$$|G^{(3)}(\varepsilon)| \leq \frac{16}{\delta_1}b^4+\frac{\delta_1}{16}\left(\int|\tilde{\varepsilon}_y|^2+\int|\tilde{\varepsilon}|^2e^{-2^-|y|}\right).$$

- Using (97), we estimate

$$|(\varepsilon_1,W_1)(\varepsilon_1,Q)| \leq \frac{16}{\delta_1}b^4+\frac{\delta_1}{16}\left(\int|\tilde{\varepsilon}_y|^2+\int|\tilde{\varepsilon}|^2e^{-2^-|y|}+2\lambda^2|E_0|\right)$$

  for $s \geq \tilde{s}_1$.

Putting together these two estimates yields (92) and Proposition 7 is proved.

4.4. *The higher dimensional case.* In this section, we explain how to adapt the proof of Theorem 2 in higher dimension $N \geq 2$ to get Theorem 3. It turns out that provided some slight modifications explicitly detailed here, the whole proof adapts except the positivity property of the linear virial operator H, Proposition 2, which we can prove only in dimension $N=1$.

Let us now briefly explain what modifications have to be taken into account, and how to handle them. We consider in this section a solution to (1) in dimension $N \geq 2$,

$$\begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x)\in[0,T)\times\mathbf{R}^N \\ u(0,x)=u_0(x), & u_0:\mathbf{R}^N\to\mathbf{C}, \end{cases}$$

for an initial condition u_0 which satisfies

$$0<\alpha_0=\int|u_0|^2-\int Q^2<\alpha^*,\quad E_0=E(u_0)<0,\ \mathrm{Im}\left(\int\nabla u_0\overline{u_0}\right)=0,$$

for some α^* small enough.

A) *Sharp decomposition of the solution.* In dimension N, (1) admits $2N+2$ symmetries in the energy space H^1, that is 2 for scaling and phase, N for translation and N for Galilean invariance which have been directly used to ensure $\mathrm{Im}(\int \nabla u_0 \overline{u_0}) = 0$. We therefore use modulation theory to build a regular decomposition

$$\varepsilon(t,y) = e^{i\gamma(t)}\lambda^{\frac{N}{2}}(t)u(t,\lambda(t)y + x(t)) - Q(y)$$

where $x(t)$ is an N-dimensional vector $x(t) = (x_i(t))_{1\leq i\leq N}$. From the variational characterization of the ground state Q, the energy condition and the implicit function theorem, we build ε such that:

(100)

(i) $$\left|1 - \lambda(t)\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}\right| + |\varepsilon(t)|_{H^1} \leq \delta(\alpha_0), \quad \text{where} \quad \delta(\alpha_0) \to 0 \quad \text{as} \quad \alpha_0 \to 0;$$

(ii) the following orthogonality conditions hold:

$$(\varepsilon_1, Q_1) = (\varepsilon_2, Q_2) = 0 \quad \text{and} \quad \forall 1 \leq i \leq N, (\varepsilon_1, y_i Q) = 0,$$

where $Q_1 = \frac{N}{2}Q + y\cdot\nabla Q$ and $Q_2 = \frac{N}{2}Q_1 + y\cdot\nabla Q_1$.

B) *Algebraic relations for the linearized operator L.* From Weinstein [28], the linearized operator L close to the ground state is $L = (L_+, L_-)$ with

$$L_+ = -\Delta + 1 - \left(\frac{4}{N} + 1\right)Q^{\frac{4}{N}} \quad \text{and} \quad L_- = -\Delta + 1 - Q^{\frac{4}{N}},$$

and the following algebraic relations hold:

$$L_+(Q_1) = -2Q \quad , \quad L_+(\nabla Q) = 0,$$
$$L_-(Q) = 0 \quad , \quad L_-(yQ) = -2\nabla Q \quad , \quad L_-(|y|^2 Q) = -4Q_1.$$

From [28] and Lemma 2 in [16] with Q^3 replaced by the first vector of L_+, one could also prove a coercive result on L similar to Lemma 4. Nevertheless, from Remark 1, a smallness estimate on ε (100) suffices for our analysis.

C) *Control of nonlinear interactions.* The ε equation inherited from (1) can now be written:

(101)
$$\partial_s\varepsilon_1 - L_-\varepsilon_2 = \frac{\lambda_s}{\lambda}Q_1 + \frac{x_s}{\lambda}\cdot\nabla Q + \frac{\lambda_s}{\lambda}\left(\frac{N}{2}\varepsilon_1 + y\cdot\nabla\varepsilon_1\right) + \frac{x_s}{\lambda}\cdot\nabla\varepsilon_1 + \tilde{\gamma}_s\varepsilon_2 - R_2(\varepsilon),$$

(102)
$$\partial_s\varepsilon_2 + L+\varepsilon_1 = -\tilde{\gamma}_s Q - \tilde{\gamma}_s\varepsilon_1 + \frac{\lambda_s}{\lambda}\left(\frac{N}{2}\varepsilon_2 + y\cdot\nabla\varepsilon_2\right) + \frac{x_s}{\lambda}\cdot\nabla\varepsilon_2 + R_1(\varepsilon),$$

with $R_1(\varepsilon) = (\varepsilon_1 + Q)|\varepsilon + Q|^{\frac{4}{N}} - Q^{\frac{4}{N}+1} - \left(\frac{4}{N}+1\right)Q^{\frac{4}{N}}\varepsilon_1$ and $R_2(\varepsilon) = \varepsilon_2(|\varepsilon + Q|^{\frac{4}{N}} - Q^{\frac{4}{N}})$.

All along the proof of Theorem 2, we need to estimate nonlinear interaction terms with respect to some local L^2-norm and $|\nabla\varepsilon|_{L^2}$. First note that elliptic estimates easily imply:

$$\forall 0 \le k \le 3, \quad \left|\frac{d^k}{dr^k}Q(r)\right| \le Ce^{-1^- r}$$

for any number $1^- < 1$. Three different order sizes appear from its very proof, all of them involving derivatives of Q of order k, $0 \le k \le 3$:

- First order terms are scalar products, terms of the form

$$\left(\varepsilon_{1,2}, P(y)\frac{d^k}{dy^k}Q(y)\right)$$

for some integer k and polymonial P. We need an estimate

$$\left|\left(\varepsilon_{1,2}, P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \le C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}.$$

- Second order terms are either products of first order terms, and are estimated so, or of the form $\left(R(\varepsilon), P(y)\frac{d^k}{dy^k}Q(y)\right)$ with

$$R(\varepsilon) = R_1(\varepsilon) + iR_2(\varepsilon).$$

- Third order terms are either products of second by first order terms, and then are easily estimated, or of two other forms, one term being inherited from the conservation of the energy

$$\begin{aligned} F(\varepsilon) = {} & |\varepsilon + Q|^{\frac{4}{N}+2} - Q^{\frac{4}{N}+2} - \left(\frac{4}{N}+2\right)Q^{\frac{4}{N}+1}\varepsilon_1 \\ & - \left(1+\frac{2}{N}\right)\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}}\varepsilon_1^2 - \left(1+\frac{2}{N}\right)Q^{\frac{4}{N}}\varepsilon_2^2, \end{aligned}$$

and the other one corresponding to the introduction of the virial linear operator in the computation of $(\varepsilon_2, Q_1)_s$ as in Lemma 7 and so to the formally cubic order term of the real part of $R(\varepsilon)$, i.e.,

$$\begin{aligned} \tilde{R}_1(\varepsilon) = {} & (\varepsilon_1 + Q)|\varepsilon_1 + Q|^{\frac{4}{N}} - Q^{\frac{4}{N}+1} - \left(\frac{4}{N}+1\right)Q^{\frac{4}{N}}\varepsilon_1 \\ & - \frac{2}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}\varepsilon_1^2 - \frac{2}{N}Q^{\frac{4}{N}-1}\varepsilon_2^2. \end{aligned}$$

We need an estimate

$$|F(\varepsilon)| + \left|\left(\tilde{R}_1(\varepsilon), P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \le \delta(\alpha_0)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right)$$

with $\delta(\alpha_0) \to 0$ as $\alpha_0 \to 0$.

We recall the Gagliardo-Nirenberg estimate in dimension N

$$\int |\varepsilon|^{\frac{4}{N}+2} \leq C\left(\int |\nabla\varepsilon|^2\right)\left(\int |\varepsilon|^2\right)^{\frac{2}{N}}.$$

Let us also recall an estimate $|\varepsilon|_{H^1} \leq \delta(\alpha_0)$ from Step A. In what follows, k denotes any positive integer, and $P(y)$ a polynomial in $y = (y_i)_{1\leq i\leq N}$.

Moreover, note that the ground state Q is no longer explicit in dimension $N \geq 2$, but the uniform asymptotic estimate $|Q(y)| \leq Ce^{-1^-|y|}$ for any number $1^- < 1$ is easily derived from the equation of Q, so that $|P(y)\frac{d^k}{dy^k}Q(y)| \leq C_{P,k}e^{-1^-|y|}$. We now consider three different cases depending on N:

$N = 2$: Let 2^- be any strictly positive number $2^- < 2$.

- First order terms: from Cauchy-Schwarz,

$$\left|\left(\varepsilon_{1,2}, P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \leq C_{P,k}\left(\int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}}.$$

- Second order terms:

  $\forall z \in \mathbf{C}, \quad |(1+z_1)|1+z|^2 - 1 - 3z_1 + iz_2(|1+z|^2-1)| \leq C(|z|^3+|z|^2),$

  so that

$$\forall \varepsilon \in H^1 \quad |R(\varepsilon)| \leq C(|\varepsilon|^3 + Q|\varepsilon|^2).$$

  Now using Gagliardo-Nirenberg, we estimate

$$\begin{aligned}\int |R(\varepsilon)e^{-1^-|y|}| &\leq C\left(\int |\varepsilon|^3 e^{-1^-|y|} + \int |\varepsilon|^2 Q e^{-1^-|y|}\right)\\ &\leq C\left(\int |\varepsilon|^4\right)^{\frac{1}{2}}\left(\int\int |\varepsilon|^2 e^{-2^-|y|}\right)^{\frac{1}{2}} + C\int |\varepsilon|^2 e^{-2^-|y|}\\ &\leq C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right).\end{aligned}$$

- Third order terms: for all $z \in \mathbf{C}$, $\quad ||1+z|^4 - 1 - 4z_1 - 10z_1^2 - 2z_2^2| \leq C(|z|^4+|z|^3)$ so that $|F(\varepsilon)| \leq C(|\varepsilon|^4 + Q|\varepsilon|^3)$ and

$$\begin{aligned}\int |F(\varepsilon)| &\leq C\left(\int |\varepsilon|^4 + \int |\varepsilon|^3 Q\right)\\ &\leq C\left(\int |\nabla\varepsilon|^2\right)\left(\int |\varepsilon|^2\right) + C\left(\int |\varepsilon|^4\right)^{\frac{1}{2}}\left(\int |\varepsilon|^2 Q^2\right)^{\frac{1}{2}}\\ &\leq C|\varepsilon|_{H^1}\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right)\end{aligned}$$

where we implicitly used an *a priori* smallness estimate on ε (100). Moreover, for all $z \in \mathbf{C}$, $|(1+z_1)|1+z|^2 - 1 - 3z_1 - 3z_1^2 - z_2^2| \leq C|z|^3$ so that $|\tilde{R}_1(\varepsilon)| \leq CQ|\varepsilon|^3$, and

$$\begin{aligned}\int |\tilde{R}_1(\varepsilon)e^{1^-|y|}| &\leq C\left(\int |\varepsilon|^3 e^{-1^-|y|}\right)\\ &\leq C\left(\int |\varepsilon|^4\right)^{\frac{1}{2}}\left(\int |\varepsilon|^2 Q^2\right)^{\frac{1}{2}}\\ &\leq C|\varepsilon|_{H^1}\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-2^-|y|}\right).\end{aligned}$$

$N = 3$: We recall Sobolev injection $|\varepsilon|_{L^6} \leq C|\nabla\varepsilon|_{L^2}$.

- First order terms are estimated according to

$$\left|\left(\varepsilon_{1,2}, P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \leq C_{P,k}|\varepsilon|_{L^6} \leq C_{P,k}|\nabla\varepsilon|_{L^2}.$$

- Second order terms:

  $\forall z \in \mathbf{C}$, $|(1+z_1)|1+z|^{\frac{4}{3}} - 1 - \dfrac{7}{3}z_1 + iz_2(|1+z|^{\frac{4}{3}} - 1) \leq C(|z|^{\frac{4}{3}+1} + |z|^2)$,

  so that

$$\forall \varepsilon \in H^1 \quad |R(\varepsilon)| \leq C(|\varepsilon|^{\frac{4}{3}+1} + Q^{\frac{1}{3}}|\varepsilon|^2),$$

  and

$$\begin{aligned}\int |R(\varepsilon)e^{-1^-|y|}| &\leq C\left(\int |\varepsilon|^{\frac{7}{3}} e^{-1^-|y|} + \int |\varepsilon|^2 e^{-(\frac{4}{3})^-|y|}\right)\\ &\leq C(|\varepsilon|_{L^6})^{\frac{7}{3}} + C(|\varepsilon|_{L^6})^2 \leq C\left(\int |\nabla\varepsilon|^2\right).\end{aligned}$$

- Third order terms: for all $z \in \mathbf{C}$, $||1+z|^{\frac{4}{3}+2} - 1 - \frac{10}{3}z_1 - \frac{35}{9}z_1^2 - \frac{5}{3}z_2^2| \leq C(|z|^{\frac{4}{3}+2} + |z|^3)$ so that

$$\begin{aligned}\int |F(\varepsilon)| &\leq C\left(\int |\varepsilon|^{\frac{4}{3}+2}\right) + \int |\varepsilon|^3 Q^{\frac{1}{3}}\\ &\leq C\left(\int |\varepsilon|^2\right)^{\frac{2}{3}}\left(\int |\nabla\varepsilon|^2\right) + |\varepsilon|^3_{L^6} \leq C|\varepsilon|_{H^1}\left(\int |\nabla\varepsilon|^2\right),\end{aligned}$$

  where we implicitly used (100).

  Moreover, for all $z \in \mathbf{C}$, $|(1+z_1)|1+z|^{\frac{4}{3}} - 1 - \frac{7}{3}z_1 - \frac{14}{9}z_1^2 - \frac{2}{3}z_2^2| \leq C|z|^3$, so that

$$\int |\tilde{R}_1(\varepsilon)e^{-1^-|y|}| \leq C\int |\varepsilon|^3 e^{-\frac{2}{3}^-|y|} \leq C|\varepsilon|^3_{L^6} \leq C\left(\int |\nabla\varepsilon|^2\right)^{\frac{3}{2}}.$$

$N \geq 4$: Let $2^* = \frac{2N}{N-2}$ the critical Sobolev exponent; then $|\varepsilon|_{L^{2^*}} \leq C|\nabla\varepsilon|_{L^2}$.

- First order terms are estimated as for $N = 3$,

$$\left|\left(\varepsilon_{1,2}, P(y)\frac{d^k}{dy^k}Q(y)\right)\right| \leq C_{P,k}|\varepsilon|_{L^{2^*}} \leq C_{P,k}|\nabla\varepsilon|_{L^2}.$$

- Second order terms from $\frac{4}{N} \leq 1$:

$$\forall z \in \mathbf{C},\ \left|(1+z_1)|1+z|^{\frac{4}{N}} - 1 - \left(\frac{4}{N}+1\right)z_1 + iz_2(|1+z|^{\frac{4}{N}} - 1)\right| \leq C|z|^2$$

so that

$$\forall \varepsilon \in H^1 \quad |R(\varepsilon)| \leq C|\varepsilon|^2 Q^{\frac{4}{N}-1},$$

and

$$\int |R(\varepsilon)e^{-1^-|y|}| \leq C\int |R(\varepsilon)e^{-1^-\frac{4}{N}|y|}| \leq C|\varepsilon|^{\frac{2}{2^*}}_{L^{2^*}} \leq C\left(\int |\nabla\varepsilon|^2\right).$$

- Third order terms: for all $z \in \mathbf{C}$,

$$\left||1+z|^{\frac{4}{N}+2} - 1 - \left(\frac{4}{N}+2\right)z_1 - \left(\frac{2}{N}+1\right)\left(\frac{4}{N}+1\right)z_1^2 - \left(\frac{2}{N}+1\right)z_2^2\right| \leq C|z|^{\frac{4}{N}+2}$$

still from $\frac{4}{N} \leq 1$, and then $|F(\varepsilon)| \leq C|\varepsilon|^{\frac{4}{N}+2}$ so that

$$\int |F(\varepsilon)| \leq C\int |\varepsilon|^{\frac{4}{N}+2} \leq |\varepsilon|^{\frac{4}{N}}_{H^1}\left(\int |\nabla\varepsilon|^2\right).$$

Moreover, for all $z \in \mathbf{C}$,

$$|(1+z_1)|1+z|^{\frac{4}{N}} - 1 - \left(\frac{4}{N}+1\right)z_1 - \frac{2}{N}\left(\frac{4}{N}+1\right)z_1^2 - \frac{2}{N}z_2^2| \leq C|z|^{2+\frac{2}{N}},$$

and so we estimate

$$\int |\tilde{R}_1(\varepsilon)e^{-1^-|y|}| \leq C\left(\int |\varepsilon|^{2+\frac{2}{N}}\frac{Q^{\frac{4}{N}+1}}{Q^{2+\frac{2}{N}}}e^{-1^-|y|}\right) \leq C\left(\int |\varepsilon|^{2+\frac{2}{N}}e^{-\frac{2^-}{N}|y|}\right)$$
$$\leq C|\varepsilon|^{2+\frac{2}{N}}_{L^{2^*}} \leq C\left(\int |\nabla\varepsilon|^2\right)^{1+\frac{1}{N}}$$

from $2 + \frac{2}{N} < 2^* = 2 + \frac{4}{N-2}$.

D) *The local virial estimate.* Arguing as in Section 3 and using equations (101), (102), we then exhibit the following local virial inequality

$$(\varepsilon_2, Q_1)_s \geq H(\varepsilon,\varepsilon) + 2\lambda^2|E_0| - \delta(\alpha_0)\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2 e^{-2^-|y|}\right), \tag{103}$$

with $H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1, \varepsilon_1) + (\mathcal{L}_2\varepsilon_2, \varepsilon_2)$ and

$$\mathcal{L}_1 = -\Delta + \frac{2}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}y\cdot\nabla Q \quad , \quad \mathcal{L}_2 = -\Delta + \frac{2}{N}Q^{\frac{4}{N}-1}y\cdot\nabla Q. \tag{104}$$

We now conjecture that the same spectral properties of H as in Proposition 2 hold true, at least for low dimension; that is, we assume the Spectral Property announced in Section 1.2 holds true:

Spectral property. Let $N \geq 2$. There exists a universal constant $\tilde{\delta}_1 > 0$ such that for all $\varepsilon \in H^1$, if $(\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = (\varepsilon_2, \nabla Q) = 0$, then

(i) for $N = 2$,

$$H(\varepsilon,\varepsilon) \geq \tilde{\delta}_1\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2 e^{-2^-|y|}\right)$$

for some universal constant $2^- < 2$;

(ii) for $N \geq 3$,

$$H(\varepsilon,\varepsilon) \geq \tilde{\delta}_1\int|\nabla\varepsilon|^2.$$

Let us say a word about the structure of $H(\varepsilon,\varepsilon)$. For $\varepsilon \in H^1$ and $\lambda > 0$, set $\varepsilon_\lambda = \lambda^{\frac{N}{2}}\varepsilon(\lambda y)$. From direct computation, we have

$$\begin{aligned} H(\varepsilon,\varepsilon) &= \left(L_+\varepsilon_1, \frac{N}{2}\varepsilon_1 + y\cdot\nabla\varepsilon_1\right) + \left(L_-\varepsilon_2, \frac{N}{2}\varepsilon_2 + y\cdot\nabla\varepsilon_2\right) \\ &= \frac{1}{2}\frac{d}{d\lambda}(L\varepsilon_\lambda, \varepsilon_\lambda)_{|\lambda=1}. \end{aligned}$$

Now recall that none of the three conservation laws in H^1 sees the variation of size of the ground states $Q_{\lambda,\gamma,x}(y) = \lambda^{\frac{N}{2}}e^{i\gamma}Q(\lambda(y-x))$ where $\lambda > 0$, $\gamma \in \mathbf{R}$ and $x \in \mathbf{R}^N$. Note, from [28], that the functional $J_N(u) = \frac{(\int|\nabla u|^2)(\int|u|^2)^{\frac{2}{N}}}{\int|u|^{2+\frac{4}{N}}}$ attains its infimum in H^1 at the points $Q_{\lambda,\gamma,x}$, and the Hessian of this functional is

$$\frac{d^2}{d\eta^2}J(Q+\eta\varepsilon)_{|\eta=0} = (L\varepsilon,\varepsilon) + S(\varepsilon)$$

where $S(\varepsilon)$ is a sum of terms of the form $(\varepsilon_{1,2}, V_1)(\varepsilon_{1,2}, V_2)$ for some well localized functions V_1, V_2.

From this point of view, to exhibit a positivity property on H is equivalent to comparing the Hessian matrices of J_N at the points $Q_{\lambda,\gamma,x}$, and thus to separate these functions. Unfortunately, the analysis of the operator H is more complicated in dimension $N \geq 2$ because the function Q is no longer explicit.

E) *Refined blow-up rate.* We now claim the following proposition which implies Theorem 3 from Galilean invariance:

PROPOSITION 9. *Let $N \geq 2$ and assume the Spectral Property holds true; then there exists $\alpha^* > 0$ and a universal constant C^* such that the following is true. Let $u_0 \in H^1$ such that*

$$\alpha_0 = \alpha(u_0) = \int |u_0|^2 - \int Q^2 < \alpha^*, \quad E_0 = E(u_0) < 0, \quad \mathrm{Im}\left(\int \nabla u_0 \overline{u_0}(x)\right) = 0.$$

Let $u(t)$ be the corresponding solution to (1); *then $u(t)$ blows up in finite time $0 < T < +\infty$ and for t close to T:*

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{|\ln(T-t)|^{\frac{N}{2}}}{T-t}\right)^{\frac{1}{2}}.$$

As for the one dimensional case, the heart of the proof of Proposition 9 is the local virial inequality. From (103) and the Spectral Property, we indeed first get (65)

$$\left\{\left(1 + \frac{1}{4\delta_0}(\varepsilon_1, Q)\right)(\varepsilon_2, Q_1)\right\}_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-2^{-}|y|}\right) + 2\lambda^2 |E_0| - \frac{1}{\delta_0}(\varepsilon_2, Q_1)^2.$$

Now the whole proof of the refinement of the blow-up rate adapts in dimension $N \geq 2$. For a given function f, set $f_1 = \frac{N}{2} f + y \cdot \nabla f$; then one easily checks that $\mathcal{L}_2$ given by (20) satisfies

$$\mathcal{L}_2(f) = \frac{1}{2}\{L_-(f_1) - (L_-(f))_1\}$$

and

$$H_2(\varepsilon_2, \varepsilon_2) = (\mathcal{L}_2 \varepsilon_2, \varepsilon_2) = \left(L_- \varepsilon_2, \frac{N}{2}\varepsilon_2 + y \cdot \nabla \varepsilon_2\right).$$

Set then $W = |y|^2 Q + \mu Q$ with μ so that $(W, Q_2) = 0$; then $H_2(W, W) = 0$, $(W, Q_1) = -|yQ|^2_{L^2}$ and the whole algebra of the proof follows. The only point to check is which refinement of the blow-up rate is attained. The answer to this question depends on the control we are able to prove on formally cubic terms in dimension N. This is the point we now investigate.

More precisely, letting $\tilde{\varepsilon} = \varepsilon + i\frac{(\varepsilon_2, Q_1)}{|yQ|^2_{L^2}} W$ and $b = \frac{(\varepsilon_2, Q_1)}{|yQ|^2_{L^2}}$, we express

$$\left\{\left(1 + \frac{1}{|yQ|^2_{L^2}}(\varepsilon_1, W_1)\right)(\varepsilon_2, Q_1)\right\}_s \geq H(\tilde{\varepsilon}, \tilde{\varepsilon}) + G^{(4)}(\varepsilon)$$

with $G^{(4)}(\varepsilon)$ formally cubic in ε, and we claim

$$|G^{(4)}(\varepsilon)| = |G^{(4)}(\tilde{\varepsilon} - bW)| \leq \delta(\alpha_0)\left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) + Cb^{2+\frac{2}{N}}.$$

From the fact that $|G^{(4)}(\varepsilon)|$ is formally cubic in ε, it is composed with three kind of terms in the terminology of Step C: products of three scalar products, products of a scalar product with a second order term, third order term $F(\varepsilon)$ and $\tilde{R}_1(\varepsilon)$. The first two kinds are directly estimated we focus on the last kind, and argue differently depending on the dimension, and implicitly recall the corresponding estimates of Step C:

$N = 2$:

$$\begin{aligned}\int |F(\tilde{\varepsilon} - bW)| &\leq C\left(\int |\tilde{\varepsilon} - bW|^4 + \int |\tilde{\varepsilon} - bW|^3 Q\right)\\ &\leq C\left(\int |\tilde{\varepsilon}|^4 + \int |\tilde{\varepsilon}|^3 Q\right) + Cb^3\\ &\leq C|\tilde{\varepsilon}|_{H^1}\left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) + Cb^3,\end{aligned}$$

and similarly,

$$\begin{aligned}\int |\tilde{R}_1(\tilde{\varepsilon} - bW)e^{1^-|y|}| &\leq C\left(\int |\tilde{\varepsilon} - bW|^3 e^{-1^-|y|}\right) \leq C\left(\int |\tilde{\varepsilon}|^3 e^{-1^-|y|}\right) + Cb^3\\ &\leq C|\tilde{\varepsilon}|_{H^1}\left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) + Cb^3.\end{aligned}$$

$N = 3$:

$$\begin{aligned}\int |F(\tilde{\varepsilon} - bW)| &\leq C\left(\int |\tilde{\varepsilon} - bW|^{\frac{4}{3}+2} + \int |\tilde{\varepsilon} - bW|^3 Q^{\frac{1}{3}}\right)\\ &\leq C\left(\int |\tilde{\varepsilon}|^{\frac{4}{3}+2} + \int |\tilde{\varepsilon}|^3 Q^{\frac{1}{3}}\right) + Cb^3\\ &\leq C|\tilde{\varepsilon}|_{H^1}\left(\int |\nabla\tilde{\varepsilon}|^2\right) + Cb^3\end{aligned}$$

and similarly,

$$\int |\tilde{R}_1(\tilde{\varepsilon} - bW)e^{1^-|y|}| \leq C\left(\int |\nabla\tilde{\varepsilon} - bW|^2\right)^{\frac{3}{2}} \leq C\left(\int |\nabla\tilde{\varepsilon}|^2\right)^{\frac{3}{2}} + Cb^3.$$

$N \geq 4$:

$$\begin{aligned}\int |F(\tilde{\varepsilon} - bW)| &\leq C\int |\tilde{\varepsilon} - bW|^{\frac{4}{N}+2} \leq C\int |\tilde{\varepsilon}|^{\frac{4}{N}+2} + Cb^{2+\frac{4}{N}}\\ &\leq |\tilde{\varepsilon}|_{H^1}^{\frac{4}{N}}\left(\int |\nabla\tilde{\varepsilon}|^2\right) + Cb^{2+\frac{4}{N}}\end{aligned}$$

and similarly,

$$\int |\tilde{R}_1(\tilde{\varepsilon} - bW)e^{1^-|y|}| \leq C \int |\tilde{\varepsilon} - bW|^{2+\frac{2}{N}} e^{-\frac{2^-}{N}|y|} \leq C \int |\tilde{\varepsilon}|^{2+\frac{2}{N}} e^{-\frac{2^-}{N}|y|} + Cb^{2+\frac{2}{N}} \leq C \left(\int |\nabla\tilde{\varepsilon}|^2\right)^{1+\frac{1}{N}} + Cb^{2+\frac{2}{N}}. \tag{105}$$

Therefore, the N^{th} dimensional version of Proposition 7 can be written:

PROPOSITION 10. *Let* $\tilde{\varepsilon} = \varepsilon + i\frac{(\varepsilon_2, Q_1)}{|yQ|^2_{L^2}}W$. *There exist universal constants* $\tilde{\delta}_0$, $C > 0$ *and* $\alpha_{11} > 0$ *such that for* $\alpha_0 < \alpha_{11}$, *there exists* $\tilde{s}_6$ *such that for all* $s \geq \tilde{s}_6$,

$$\left\{\left(1 + \frac{1}{|yQ|^2_{L^2}}(\varepsilon_1, W_1)\right)(\varepsilon_2, Q_1)\right\}_s + C(\varepsilon_2, Q_1)^{2+\frac{2}{N}} \geq \tilde{\delta}_0 \left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-2^-|y|}\right) + \lambda^2 |E_0|.$$

Integrating the obtained differential inequality easily leads to the announced control

$$\lambda(t) \geq C \left(\frac{T-t}{|\ln(T-t)|^{\frac{N}{2}}}\right)^{\frac{1}{2}}$$

and the proofs of Proposition 9 and Theorem 3 are complete.

Appendix A: Proof of Proposition 2

We prove Proposition 2 with $2^- = \frac{9}{5}$ and $\tilde{\delta_1} = \frac{1}{10}$. The proof is similar to the one of Appendix C in [13].

We note $H(\varepsilon, \varepsilon) = H_1(\varepsilon_1, \varepsilon_1) + H_2(\varepsilon_2, \varepsilon_2)$ with

$$H_1(\varepsilon_1, \varepsilon_1) = \int \varepsilon_{1y}^2 + 10 \int yQ^3 Q_y \varepsilon_1^2 \quad \text{and} \quad H_2(\varepsilon_2, \varepsilon_2) = \int \varepsilon_{2y}^2 + 2 \int yQ^3 Q_y \varepsilon_2^2.$$

We recall that from direct computation

$$H_1(\varepsilon_1, \varepsilon_1) = \left(L_+ \varepsilon_1, \frac{\varepsilon_1}{2} + y\varepsilon_{1y}\right) \tag{106}$$

and

$$H_2(\varepsilon_2, \varepsilon_2) = \left(L_- \varepsilon_2, \frac{\varepsilon_2}{2} + y\varepsilon_{2y}\right).$$

Next we set

$$\overline{H}_1(\varepsilon_1, \varepsilon_1) = (\overline{\mathcal{L}}_1 \varepsilon_1, \varepsilon_1) = \int \varepsilon_{1y}^2 + \frac{10}{9}\left(\int 10yQ^3 Q_y \varepsilon_1^2 - \frac{1}{10}\int \frac{\varepsilon_1^2}{ch^2(\frac{9}{10}y)}\right)$$

and

$$\overline{H}_2(\varepsilon_2,\varepsilon_2) = (\overline{\mathcal{L}}_2\varepsilon_2,\varepsilon_2) = \int \varepsilon_{2y}^2 + \frac{10}{9}\left(\int 2yQ^3Q_y\varepsilon_2^2 - \frac{1}{10}\int \frac{\varepsilon_2^2}{ch^2(\frac{9}{10}y)}\right)$$

so that

(107)

$$H(\varepsilon,\varepsilon) - \frac{1}{10}\left(\int |\varepsilon_y|^2 + \int \frac{1}{ch^2\left(\frac{9}{10}y\right)}|\varepsilon|^2\right) = \frac{9}{10}\left(\overline{H}_1(\varepsilon_1,\varepsilon_1) + \overline{H}_2(\varepsilon_2,\varepsilon_2)\right).$$

We prove that under orthogonality conditions (33) and (34), $\overline{H}_1$ and $\overline{H}_2$ are positive, which concludes the proof of Proposition 2. □

We give a definition of the index of a bilinear form. Let B a bilinear form on a vector space V. Let us define the index of B on V as:

$$\mathrm{ind}_V(B) = \max\{k \in \mathbf{N}/ \text{ there exists a subspace } P \text{ of codimension } k \text{ such that } B_{|P} \text{ is positive}\}.$$

Let H_e^1 (respectively H_0^1) denote the subspace of even (respectively odd) H^1 functions. Assume that H_e^1 is B-orthogonal to H_0^1. We say that B defined on H^1 has index $i+j$ if $\mathrm{ind}_{H_e^1} = i$ and $\mathrm{ind}_{H_o^1} = j$.
The proof proceeds in several steps:

Step 1. $\overline{H}_1$ *has index* $1+1$, $\overline{H}_2$ *has index* $1+0$. This is achieved by comparing $\overline{H}_1$ and $\overline{H}_2$ by a simpler quadratic form of classical type. Here we use the dimension $N=1$ hypothesis.

LEMMA 9 (Lower bound on H). (i) *For all* $\varepsilon_1 \in H^1$,

$$\overline{H}_1(\varepsilon_1,\varepsilon_1) \geq (\tilde{L}_1\varepsilon_1,\varepsilon_1) + 2\int \frac{1}{ch^2(\frac{9}{10}y)}\varepsilon_1^2 \tag{108}$$

where $\tilde{L}_1 = -\Delta - \frac{243}{25}\frac{1}{ch^2(\frac{9}{10}y)}$.

(ii) *For all* $\varepsilon_2 \in H^1$,

$$H_2(\varepsilon_2,\varepsilon_2) \geq (\tilde{L}_2\varepsilon_2,\varepsilon_2) + \frac{1}{30}\int \frac{1}{ch^2(\frac{9}{10}y)}\varepsilon_2^2 \tag{109}$$

where $\tilde{L}_2 = -\Delta - \frac{81}{50}\frac{1}{ch^2(\frac{9}{10}y)}$.

Proof of Lemma 9. (i) and (ii) are both a consequence of the following inequality: for all $y \in \mathbf{R}$,

$$10yQ^3Q_y \geq -\frac{13}{2}\frac{1}{ch^2(\frac{9}{10}y)}. \tag{110}$$

From the explicit value of Q (8), this is implied by the following inequality

$$\forall y \geq 0, \quad \frac{60}{13} y \frac{ch^2(y)}{ch^2(2y)} th(2y) \leq 1,$$

which can be checked similarly as in [13].

We then collect

$$\begin{aligned}\overline{H}_1(\varepsilon_1, \varepsilon_1) &\geq \int \varepsilon_{1y}^2 - \frac{10}{9}\left(\frac{13}{2} + \frac{1}{10}\right) \int \frac{1}{ch^2\left(\frac{9}{10}y\right)} \varepsilon_1^2 \\ &\geq 2 \int \frac{1}{ch^2(\frac{9}{10}y)} \varepsilon_1^2 + (\tilde{L}_1 \varepsilon_1, \varepsilon_1),\end{aligned}$$

$$\begin{aligned}\overline{H}_2(\varepsilon_2, \varepsilon_2) &\geq \int \varepsilon_{2y}^2 - \frac{10}{9}\left(\frac{13}{10} + \frac{1}{10}\right) \int \frac{1}{ch^2(\frac{9}{10}y)} \varepsilon_2^2 \\ &\geq \frac{1}{30} \int \frac{1}{ch^2\left(\frac{9}{10}y\right)} \varepsilon_2^2 + (\tilde{L}_2 \varepsilon_2, \varepsilon_2).\end{aligned}$$

This ends the proof of Lemma 9. □

From the fact that operators $\tilde{L}_1, \tilde{L}_2$ introduced in Lemma 9 have a known explicit spectral structure, we claim

LEMMA 10. *$\overline{H}_1$ has index $1+1$, $\overline{H}_2$ has index $1+0$.*

Remark 5. Consequently, the operator $\overline{\mathcal{L}}_1$ has exactly two strictly negative eigenvalues λ_1, λ_2 associated to the respectively even and odd eigenfunctions ψ_1, ψ_2 and continuous spectrum on $[0, +\infty)$. Moreover, $\overline{H}_1$ is positive on $[\text{span}(\psi_1, \psi_2)]^\perp$. Similarly, the operator $\overline{\mathcal{L}}_2$ has exactly one strictly negative eigenvalue λ_3 associated to the even eigenfunction ψ_3 and continuous spectrum on $[0, +\infty)$. Moreover, $\overline{H}_2$ is positive on $[\text{span}(\psi_3)]^\perp$.

Proof of Lemma 10. From [26], the operator $L_n = -\Delta - \frac{n(n+1)}{4ch^2(\frac{y}{2})}$ has exactly $\left[\frac{n}{2}\right]+1$ strictly negative eigenvalues, and continuous spectrum on $[0, +\infty)$. Now we note that when $\varepsilon(y) = \tilde{\varepsilon}(\frac{9}{5}y)$,

$$\tilde{L}_1(\varepsilon_1)(y) = \frac{81}{25}(L_3 \tilde{\varepsilon}_1)\left(\frac{9}{5}y\right) \quad \text{and} \quad \tilde{L}_2(\varepsilon_2)(y) = \frac{81}{25}(L_1 \tilde{\varepsilon}_2)\left(\frac{9}{5}y\right).$$

Therefore, $\overline{H}_1$ has index at most 1+1, and $\overline{H}_2$ has index at most 1+0. Moreover,

$$\overline{H}_1(Q_y, Q_y) < 0 \quad , \quad \overline{H}_1(Q, Q) < 0 \quad , \quad \overline{H}_2(Q, Q) < 0,$$

which allows us to conclude the proof of Lemma 10. Indeed, $H_1(Q_y, Q_y) = 0$ follows from (106) and $L_+(Q_y) = 0$. Now compute

$$H_1(Q, Q) = \left(L_+ Q, \frac{Q}{2} + yQ_y\right) = \left(-4Q^5, \frac{Q}{2} + yQ_y\right) = -\frac{4}{3} \int Q^6 < 0.$$

$H_2(Q,Q) = 0$ follows from (106) and $L_- Q = 0$. From (107), $\overline{H} < H$ and Lemma 10 is proved. □

Step 2. *The positivity property on H^1.* We show that if $\varepsilon_1 \in H^1$ is such that $(\varepsilon_1, Q) = (\varepsilon_1, yQ) = 0$, then $\overline{H}_1(\varepsilon_1, \varepsilon_1) \geq 0$, and that if $\varepsilon_2 \in H^1$ is such that $(\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = 0$, then $\overline{H}_2(\varepsilon_2, \varepsilon_2) \geq 0$.

LEMMA 11 (Numerical estimates). (i) *There exists a unique regular even function $\phi_1 \in L^\infty$ such that $\overline{\mathcal{L}}_1 \phi_1 = Q$. Moreover,*

$$\int (\phi_{1y})^2 < +\infty \tag{111}$$

and

$$-(\phi_1, Q)\left(1 - \overline{H}_1(Q,Q)\frac{(\phi_1, Q)}{(Q,Q)^2}\right) > 0. \tag{112}$$

(ii) *There exists a unique regular odd function $\phi_2 \in L^\infty$ such that $\overline{\mathcal{L}}_1 \phi_2 = yQ$. Moreover,*

$$\int (\phi_{2y})^2 < +\infty \quad \text{and} \quad -(\phi_2, yQ)\left(1 - \overline{H}_1(Q_y, Q_y)\frac{(\phi_2, yQ)}{(Q_y, yQ)^2}\right) > 0. \tag{113}$$

(iii) *Let $Q_3 = Q_1 + \frac{1}{2} Q_2$. There exists a unique regular even function $\phi_3 \in L^\infty$ such that $\overline{\mathcal{L}}_2 \phi_3 = Q_3$. Moreover,*

$$\int (\phi_{3y})^2 < +\infty \quad \text{and} \quad -(\phi_3, Q_3)\left(1 - \overline{H}_2(Q,Q)\frac{(\phi_3, Q_3)}{(Q,Q_3)^2}\right) > 0. \tag{114}$$

Remark 6. Note that (112), (113) and (114) are checked numerically.

Proof of Lemma 11. Note that existence and uniqueness of ϕ_1, ϕ_2, ϕ_3 are not given by the Lax-Milgram theorem and these functions are not in H^1. We prove the existence and uniqueness of ϕ_1. The proof is similar for ϕ_2 and ϕ_3.

A) *Uniqueness.* This follows from

LEMMA 12 (Coercivity of $\overline{\mathcal{L}}_1$). *Let $u \in L^\infty$ be a regular even function such that $\overline{\mathcal{L}}_1 u = 0$. Then $u = 0$.*

Proof. The proof is based on estimate (108). Let u be as in Lemma 12; we want to prove $u = 0$, and argue by contradiction assuming $u \neq 0$. First, note that

$$\int (u_y)^2 < \infty. \tag{115}$$

Indeed, u satisfies $u'' + V(y)u = 0$ for some well localized positive potential $V(y)$. Integrating this equation using $u \in L^\infty$ yields $|u'(y)| \leq C$. Then multiplying the equation by u and integrating by parts ensure $\int (u_y)^2 < \infty$.

Now assume u is not zero. Let $\psi_1 \in H^1_e$ be the eigenvector associated to the strictly negative eigenvalue λ_1 of $\overline{\mathcal{L}}_1$ in H^1_e. Then $(\overline{\mathcal{L}}_1 u, \psi_1) = \lambda_1 (u, \psi_1) = 0$ from assumption $\overline{\mathcal{L}}_1 u = 0$, and therefore

$$(u, \psi_1) = 0. \tag{116}$$

Let now χ be a regular even cutoff function $\chi(y) = 1$ for $|y| \leq 1$ and $\chi(y) = 0$ for $|y| \geq 2$, and for $A > 0$, $\chi_A(y) = \chi(\frac{y}{A})$. We set $u_A = \chi_A u \in H^1_e$. Consider then $V = \text{span}(\psi_1, u)$ and $V_A = \text{span}(\psi_1, u_A) \subset H^1_e$. Consider now the quadratic form $\tilde{H}_1(u, u) = (\tilde{L}_1 u, u)$ and the two by two symmetric matrices $M = \text{mat}_V(\tilde{H}_1)$ and $M_A = \text{mat}_{V_A}(\tilde{H}_1)$. Then from (116) and (108), M is diagonal, definite, negative. Moreover, from (115), it is a trivial task to verify $M_A \to M$ as $A \to +\infty$, so that M_A and M have the same signature for A large enough. From Lemma 10, $\text{ind}_{H^1_e} \tilde{H}_1 = 1$, so that for A large enough, $\dim V_A = 1$ and $u_A = \lambda_A \psi_1$. Now, from $u_A \to u$ in L^∞_{loc} and (116), we conclude $\lambda_A \to 0$, so that $u = 0$, and a contradiction follows. This concludes the proof of Lemma 12, and the uniqueness part of the proof of Lemma 11.

B) *Existence.* We now prove the existence of ϕ_1 as in Lemma 11. Note that $\overline{\mathcal{L}}_1 = -\Delta - V(y)$ for some regular well localized even potential $V(y)$. We want to prove the existence of a regular even solution $u \in L^\infty$ to $\overline{\mathcal{L}}_1 u = f$ for some regular even and well localized function f with exponential decay. This is a classical result. We recall its proof using a fixed point argument.

First let ρ be a regular solution to $\overline{\mathcal{L}}_1 \rho = 0$. We claim

$$\forall y \in \mathbf{R} \quad , \quad |\rho(y)| \leq C|y|. \tag{117}$$

Indeed, note from the decay properties of V that it is a trivial task to build ρ_1 and ρ_2 solutions to the integral equation

$$\rho_1(y) = y + \int_y^{+\infty} \int_s^{+\infty} V(\tau)\rho_1(\tau) d\tau ds$$

and

$$\rho_2(y) = 1 + \int_y^{+\infty} \int_s^{+\infty} V(\tau)\rho_2(\tau) d\tau ds.$$

Now, ρ_1, ρ_2 are solutions to the homogeneous linear equation $(\overline{\mathcal{L}}_1 \rho_i)_{i=1,2} = 0$ locally on $(A, +\infty)$ for some A large enough, and can be extended to $\mathbf{R}$ from linear theory. Moreover, from their behavior at $+\infty$, they are linearly independent. Therefore, any solution ρ to $\overline{\mathcal{L}} \rho = 0$ belongs to $\text{span}(\rho_1, \rho_2)$, and consequently $|\rho(y)| \leq C|y|$ as $y \to +\infty$. We argue similarly for $y \to -\infty$, and (117) is proved.

Again using a fixed point argument, we consider $\tilde{\rho}_1, \tilde{\rho}_2$ solutions to the integral equations

$$\tilde{\rho}_1(y) = 1 + \int_{-\infty}^{y} \int_{-\infty}^{s} V(\tau)\tilde{\rho}_1(\tau) d\tau ds$$

and

$$\tilde{\rho}_2(y) = 1 + \int_{y}^{+\infty} \int_{s}^{+\infty} V(\tau)\tilde{\rho}_2(\tau) d\tau ds.$$

Note again that $(\overline{\mathcal{L}}_1 \rho_i)_{i=1,2} = 0$. Then $\tilde{\rho}_1, \tilde{\rho}_2$ are linearly independent of Lemma 12. Therefore, their Wronskian $D = \tilde{\rho}_1 \tilde{\rho}_{2y} - \tilde{\rho}_2 \tilde{\rho}_{1y}$ is a nonzero constant. The method of variation of the constant gives an explicit regular solution u to $\overline{\mathcal{L}}_1 u = f$ with

$$u(y) = -\left\{ \tilde{\rho}_1(y) \int_{y}^{+\infty} \frac{f(\tau)\tilde{\rho}_2(\tau)}{D} d\tau + \tilde{\rho}_2(y) \int_{-\infty}^{y} \frac{f(\tau)\tilde{\rho}_1(\tau)}{D} d\tau \right\}.$$

Now we may change u to $\frac{1}{2}(u(y) + u(-y))$ to get an even solution. Note that $u \in L^\infty$ follows from the asymptotic behavior of f and $\tilde{\rho}_1$, $\tilde{\rho}_2$ at respectively $-\infty$ and $+\infty$, together with (117).

It remains to prove (111), which follows from direct verification. Also, u satisfies $-u'' - V(y)u = f(y)$ and $u \in L^\infty$. By integration of the equation, we get $|u_y| \leq C$ for all $y \in \mathbf{R}$. Then multiplying the equation by u and integrating by parts yields the result. This ends the proof of existence and uniqueness of ϕ_1, ϕ_2, ϕ_3 of Lemma 11.

C) *Numerical estimates.* It remains to prove estimates (112), (113) and (114). These are checked numerically. We compute

$$-(\phi_1, Q)\left(1 - \overline{H}_1(Q,Q)\frac{(\phi_1, Q)}{(Q,Q)^2}\right) \sim 0.2,$$

$$-(\phi_2, yQ)\left(1 - \overline{H}_1(Q_y,Q_y)\frac{(\phi_2, yQ)}{(Q_y,yQ)^2}\right) \sim 0.8,$$

$$-(\phi_3, Q_3)\left(1 - \overline{H}_2(Q,Q)\frac{(\phi_3, Q_3)}{(Q,Q_3)^2}\right) \sim 0.2,$$

and Lemma 11 is proved. □

Remark 7. These calcululations were made with the software MAPLE.

LEMMA 13 (The positivity property of $\overline{H}$ in H^1). (i) *If* $\varepsilon_1 \in H^1$ *satisfies* $(\varepsilon_1, Q) = (\varepsilon_1, yQ) = 0$, *then* $\overline{H}_1(\varepsilon_1, \varepsilon_1) \geq 0$.

(ii) *If* $\varepsilon_2 \in H^1$ *satisfies* $(\varepsilon_2, Q_3) = 0$, *then* $\overline{H}_2(\varepsilon_2, \varepsilon_2) \geq 0$

Remark 8. Note that orthogonality condition $(\varepsilon_2, Q_1) = 0$ does not suffice to ensure the positivity of $\overline{H}_2$. Indeed, $\overline{H}_2(Q,Q) < H_2(Q,Q) = 0$.

Proof of Lemma 13. We prove (i) for $\varepsilon_1 \in H^1_e$. The proof is similar for the two other directions, with the help of Lemma 10.

The proof is also similar to the one of Lemma 27 in [13] with a regularizing argument on the function ϕ_1. We indeed consider a regular even cutoff function $\chi_A(y) = \chi(\frac{y}{A})$, $\chi(y) = 1$ for $0 \le y \le 1$, $\chi(y) = 0$ for $y \ge 2$. We set $(\phi_1)_A = \chi_A \phi_1$. Let $\|f\| = (\int |f_y|^2 + \int |V||f|^2)^{\frac{1}{2}}$ where $\overline{\mathcal{L}}_1 = -\Delta + V$. One easily estimates

$$|\overline{H}_1(f,g)| \le \|f\|\|g\| \text{ and } \|(\phi_1)_A - \phi_1\| \to 0 \text{ as } A \to +\infty. \tag{118}$$

First, we consider the plane $(P_1)_A$ spanned by Q and $(\phi_1)_A$ in H^1_e, and show that $\overline{H}_1$ restricted to $(P_1)_A$ is not degenerate for A large enough.

Next, we define $(P_1)_A^\perp$ the orthogonal of $(P_1)_A$ in H^1_e for the quadratic form $\overline{H}_1$. By an index argument, we show that $\overline{H}_1$ is nonnegative on $(P_1)^\perp_A$.

Finally, we show that for $\varepsilon_1 \in H^1_e$ nonzero and $(\varepsilon_1, Q) = 0$, one has $\overline{H}_1(\varepsilon_1, \varepsilon_1) \ge 0$.

(α) Let $(P_1)_A = \text{span}(Q, (\phi_1)_A)$; then

$$\begin{vmatrix} \overline{H}_1(Q,Q) & \overline{H}_1(Q,(\phi_1)_A) \\ \overline{H}_1(Q,(\phi_1)_A) & \overline{H}_1((\phi_1)_A,(\phi_1)_A) \end{vmatrix} = -(Q,Q)^2\left(1 - \overline{H}_1(Q,Q)\frac{(\phi_1,Q)}{(Q,Q)^2}\right) + o(1) \neq 0$$

for A large enough by (112) and (118). We conclude that $\overline{H}_1$ restricted to $(P_1)_A$ is not degenerate. It follows that $H^1_e = (P_1)_A \oplus (P_1)^\perp_A$.

(β) Since the index of $\overline{H}_1$ in H^1_e is 1 and $\overline{H}_1(Q,Q) < 0$, we conclude that $\overline{H}_1 \ge 0$ on $(P_1)^\perp_A$.

(γ) There exists $A_0 > 0$ such that for all $A \ge A_0$, $\forall \varepsilon_1 \in (P_1)_A$ nonzero with $(\varepsilon_1, Q) = 0$, then $\overline{H}_1(\varepsilon_1,\varepsilon_1) > 0$. Indeed, when $\varepsilon_1 = \alpha Q + \beta(\phi_1)_A$, then from $(\varepsilon_1, Q) = 0$, we have $\beta \neq 0$ and $\frac{\alpha}{\beta} = -\frac{(Q,(\phi_1)_A)}{(Q,Q)}$. Then

$$\begin{aligned}\frac{\overline{H}_1(\varepsilon_1,\varepsilon_1)}{\beta^2} &= \left(\frac{\alpha}{\beta}\right)^2 \overline{H}_1(Q,Q) + 2\left(\frac{\alpha}{\beta}\right)((\phi_1)_A, Q) + \overline{H}_1((\phi_1)_A,(\phi_1)_A) \\ &= -(Q,\phi_1)\left(1 - \overline{H}_1(Q,Q)\frac{(Q,\phi_1)}{(Q,Q)^2}\right) + o(1) \text{ as } A \to +\infty.\end{aligned}$$

From (112), we conclude $\overline{H}_1(\varepsilon_1,\varepsilon_1) > 0$ for A large enough. Moreover, arguing similarly, one has: when a sequence $A_n \to +\infty$ and $\varepsilon_{A_n} \in (P_1)_{A_n}$ such that $(\varepsilon_{A_n}, Q) \to 0$ as $n \to +\infty$, then $\liminf H(\varepsilon_{A_n}, \varepsilon_{A_n}) \ge 0$.

(δ) Now let $\varepsilon_1 \in H^1_e$ be nonzero such that $(\varepsilon_1, Q) = 0$ and $A \ge A_0$; then $\varepsilon_1 = \varepsilon^{(1)}_A + \varepsilon^{(2)}_A$, where $\varepsilon^{(1)}_A \in (P_1)_A$ and $\varepsilon^{(2)}_A \in (P_1)^\perp_A$. By definition, we have $\overline{H}_1(\varepsilon_1,\varepsilon_1)=\overline{H}_1(\varepsilon_A^{(1)},\varepsilon_A^{(1)})+\overline{H}_1(\varepsilon_A^{(2)},\varepsilon_A^{(2)})\geq\overline{H}_1(\varepsilon_A^{(1)},\varepsilon_A^{(1)})$ from (β). The conclusion will then follow from (γ) and

$$(\varepsilon_A^{(1)},Q)\to 0 \quad\text{as}\quad A\to+\infty. \tag{119}$$

We now prove (119). Indeed, from $(\varepsilon_1,Q)=0$, we compute $(\varepsilon_A^{(1)},Q)=-(\varepsilon_A^{(2)},Q)$. Now by definition, $0=\overline{H}_1(\varepsilon_A^{(2)},(\phi_1)_A)=(\varepsilon_A^{(2)},\overline{\mathcal{L}}_1(\phi_1)_A)$, so that $|(\varepsilon_A^{(2)},Q)|=|(\varepsilon_A^{(2)},\overline{\mathcal{L}}_1\phi_1)|=|(\varepsilon_A^{(2)},\overline{\mathcal{L}}_1(\phi_1-(\phi_1)_A))|\leq\|\varepsilon_A^{(2)}\|\|\phi_1-(\phi_1)_A\|$ from (118). The conclusion follows from $\|\varepsilon_A^{(2)}\|\leq C|\varepsilon_1|_{H^1}$. Indeed, writing $\varepsilon_1=\alpha_A Q+\beta_A(\phi_1)_A+\varepsilon_A^{(2)}$ and using the nondegeneracy of $\overline{H}_1$ on $(P_1)_A$, step (α), one easily evaluates $|\alpha_A|+|\beta_A|\leq C|\varepsilon_1|_{H^1}$, and this concludes the proof of Lemma 13, and of Proposition 2.

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