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Convergence versus integrability in Birkhoff normal form

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Abstract

We show that any analytically integrable Hamiltonian system near an equilibrium point admits a convergent Birkhoff normalization. The proof is based on a new, geometric approach to the topic.

1. Introduction

Among the fundamental problems concerning analytic (real or complex) Hamiltonian systems near an equilibrium point, one may mention the following two:

1) *Convergent Birkhoff.* In this paper, by "convergent Birkhoff" we mean a normalization, i.e., a local analytic symplectic system of coordinates in which the Hamiltonian function will Poisson commute with the semisimple part of its quadratic part.

2) *Analytic integrability.* By "analytic integrability" we mean of a complete set of local analytic, functionally independent, first integrals in involution.

These concepts have been studied by many classical and modern mathematicians, including Poincaré, Birkhoff, Siegel, Moser, Bruno, etc. In this paper, we will be concerned with the relations between the two. The starting point is that, since both the Birkhoff normal form and the first integrals are ways to simplify and solve Hamiltonian systems, these two must be very closely related. Indeed, it was known to Birkhoff [2] that, for nonresonant Hamiltonian systems, convergent Birkhoff implies analytic integrability. The inverse is also true, though much more difficult to prove [9]. What has been known to date concerning "convergent Birkhoff vs. analytic integrability" may be summarized in the following list. Denote by q ($q \geq 0$) the *degree of resonance* (see Section 2 for a definition) of an analytic Hamiltonian system at an equilibrium point. Then we have:

a) When $q = 0$ (i.e. for nonresonant systems), convergent Birkhoff is equivalent to analytic integrability. The implication is straightforward. The inverse has been a difficult problem. Under an additional nondegeneracy condition involving the momentum map, it was first proved by Rüssmann [14] in 1964 for the case with two degrees of freedom, and then by Vey [17] in 1978 for any number of degrees of freedom. Finally Ito [9] in 1989 solved the problem without any additional condition on the momentum map.

b) When $q = 1$ (i.e. for systems with a simple resonance), convergent Birkhoff is still equivalent to analytic integrability. The part "convergent Birkhoff implies analytic integrability" is again obvious. The inverse was proved some years ago by Ito [10] and Kappeler, Kodama and Némethi [11].

c) When $q \geq 2$, convergent Birkhoff *does not* imply analytic integrability. The reason is that the Birkhoff normal form in this case will give us $(n-q+1)$ first integrals in involution, where n is the number of degrees of freedom, but additional first integrals do not exist in general, not even formal ones. (A counterexample can be found in Duistermaat [6]; see also Verhulst [16] and references therein.) The question "does analytic integrability imply convergent Birkhoff?" when $q \geq 2$ has remained open until now. The powerful analytical techniques, which are based on the fast convergent method and used in [9], [10], [11], could not have been made to work in the case with nonsimple resonances.

The main purpose of this paper is to complete the above list, by giving a positive answer to the last question.

THEOREM 1.1. *Any real* (*resp.*, *complex*) *analytically integrable Hamiltonian system in a neighborhood of an equilibrium point on a symplectic manifold admits a real* (*resp.*, *complex*) *convergent Birkhoff normalization at that point.*

An important consequence of Theorem 1.1 is that we may classify degenerate singular points of analytic integrable Hamiltonian systems by their analytic Birkhoff normal forms (see, e.g., [18] and references therein).

The proof given in this paper of Theorem 1.1 works for any analytically integrable system, regardless of its degree of resonance. Our proof is based on a geometrical method involving homological cycles, period integrals, and torus actions, and it is completely different from the analytical one used in [9], [10], [11]. In a sense, our approach is close to that of Eliasson [7], who used torus actions to prove the existence of a smooth Birkhoff normal form for smooth integrable systems with a nondegenerate elliptic singularity. The role of torus actions is given by the following proposition (see Proposition 2.3 for a more precise formulation):

PROPOSITION 1.2. *The existence of a convergent Birkhoff normalization is equivalent to the existence of a local Hamiltonian torus action which preserves the system.*

We also have the following result, which implies that it is enough to prove Theorem 1.1 in the complex analytic case:

PROPOSITION 1.3. *A real analytic Hamiltonian system near an equilibrium point admits a real convergent Birkhoff normalization if and only if it admits a complex convergent Birkhoff normalization.*

Both Proposition 1.2 and Proposition 1.3 are very simple and natural. They are often used implicitly, but they have not been written explicitly anywhere in the literature, to our knowledge.

The rest of this paper is organized as follows: In Section 2 we introduce some necessary notions, and prove the above two propositions. In Section 3 we show how to find the required torus action in the case of integrable Hamiltonian systems, by searching 1-cycles on the local level sets of the momentum map, using an approximation method based on the existence of a formal Birkhoff normalization and Łojasiewicz inequalities. This section contains the proof of our main theorem, modulo a lemma about analytic extensions. This lemma, which may be useful in other problems involving the existence of first integrals of singular foliations (see [18]), is proved in Section 4, the last section.

2. Preliminaries

Let $H : U \to \mathbb{K}$, where $K = \mathbb{R}$ (resp., $K = \mathbb{C}$) be a real (resp., complex) analytic function defined on an open neighborhood U of the origin in the symplectic space $(\mathbb{K}^{2n}, \omega = \sum_{j=1}^{n} dx_j \wedge dy_j)$. When H is real, we will also consider it as a complex analytic function with real coefficients. Denote by X_H the symplectic vector field of H:

$$i_{X_H}\omega = -dH. \tag{2.1}$$

Here the sign convention is taken so that $\{H, F\} = X_H(F)$ for any function F, where

$$\{H, F\} = \sum_{j=1}^{n} \frac{dH}{dx_j}\frac{dF}{dy_j} - \frac{dH}{dy_j}\frac{dF}{dx_j} \tag{2.2}$$

denotes the standard Poisson bracket.

Assume that 0 is an equilibrium of H, i.e. $dH(0) = 0$. We may also put $H(0) = 0$. Denote by

$$H = H_2 + H_3 + H_4 + \dots \tag{2.3}$$

the Taylor expansion of H, where H_k is a homogeneous polynomial of degree k for each $k \geq 2$. The algebra of quadratic functions on $(\mathbb{K}^{2n}, \omega)$, under the standard Poisson bracket, is naturally isomorphic to the simple algebra $\mathrm{sp}(2n, \mathbb{K})$ of infinitesimal linear symplectic transformations in $\mathbb{K}^{2n}$. In particular,

$$H_2 = H_{\mathrm{ss}} + H_{\mathrm{nil}}, \tag{2.4}$$

where H_{ss} (resp., H_{nil}) denotes the semisimple (resp., nilpotent) part of H_2.

For each natural number $k \geq 3$, the Lie algebra of quadratic functions on $\mathbb{K}^{2n}$ acts linearly on the space of homogeneous polynomials of degree k on $\mathbb{K}^{2n}$ via the Poisson bracket. Under this action, H_2 corresponds to a linear operator $G \mapsto \{H_2, G\}$, whose semisimple part is $G \mapsto \{H_{\rm ss}, G\}$. In particular, H_k admits a decomposition

$$H_k = -\{H_2, L_k\} + H'_k\,, \tag{2.5}$$

where L_k is some element in the space of homogeneous polynomials of degree k, and H'_k is in the kernel of the operator $G \mapsto \{H_{\rm ss}, G\}$, i.e. $\{H_{\rm ss}, H'_k\} = 0$. Denote by ψ_k the time-one map of the flow of the Hamiltonian vector field X_{L_k}. Then $(x', y') = \psi_k(x, y)$ (where (x, y), or also (x_j, y_j), is shorthand for $(x_1, y_1, \ldots, x_n, y_n)$) is a symplectic transformation of $(\mathbb{K}^{2n}, \omega)$ whose Taylor expansion is

$$\begin{aligned} x'_j &= x_j(\psi(x,y)) = x_j - \partial L_k/\partial y_j + O(k),\\ y'_j &= y_j(\psi(x,y)) = y_j + \partial L_k/\partial x_j + O(k), \end{aligned} \tag{2.6}$$

where $O(k)$ denotes terms of order greater or equal to k. Under the new local symplectic coordinates (x'_j, y'_j), we have

$$\begin{aligned} H &= H_2(x,y) + \cdots + H_k(x,y) + O(k+1)\\ &= H_2(x'_j + \partial L_k/\partial y_j, y'_j - \partial L_k/\partial x_j) + H_3(x'_j, y'_j) + \ldots\\ &\quad \cdots + H_k(x'_j, y'_j) + O(k+1)\\ &= H_2(x'_j, y'_j) - X_{L_k}(H_2) + H_3(x'_j, y'_j) + \cdots + H_k(x'_j, y'_j) + O(k+1)\\ &= H_2(x'_j, y'_j) + H_3(x'_j, y'_j) + \cdots + H_{k-1}(x'_j, y'_j) + H'_k(x'_j, y'_j) + O(k+1). \end{aligned}$$

In other words, the local symplectic coordinate transformation $(x', y') = \psi_k(x, y)$ of $\mathbb{K}^{2n}$ changes the term H_k to the term H'_k satisfying $\{H_{\rm ss}, H'_k\} = 0$ in the Taylor expansion of H, and it leaves the terms of order smaller than k unchanged. By induction, one finds a sequence of local analytic symplectic transformations ϕ_k $(k \geq 3)$ of type

$$\phi_k(x,y) = (x,y) + \text{ terms of order } \geq k-1 \tag{2.7}$$

such that for each $m \geq 3$, the composition

$$\Phi_m = \phi_m \circ \cdots \circ \phi_3 \tag{2.8}$$

is a symplectic coordinate transformation which changes all the terms of order smaller or equal to k in the Taylor expansion of H to terms that commute with $H_{\rm ss}$.

By taking limit $m \to \infty$, we get the following classical result due to Birkhoff et al. (see, e.g., [2], [3], [15]):

Theorem 2.1 (Birkhoff et al.). *For any real* (*resp., complex*) *Hamiltonian system H near an equilibrium point with a local real* (*resp., complex*) *symplectic system of coordinates* (x, y), *there exists a formal real* (*resp.*, *complex*) *symplectic transformation* $(x', y') = \Phi(x, y)$ *such that in the coordinates* (x', y'),

$$\{H, H_{\mathrm{ss}}\} = 0, \tag{2.9}$$

where H_{ss} *denotes the semisimple part of the quadratic part of* H.

When Equation (2.9) is satisfied, one says that the Hamiltonian H is in *Birkhoff normal form*, and the symplectic transformation Φ in Theorem 2.1 is called a *Birkhoff normalization*. The Birkhoff normal form is one of the basic tools in Hamiltonian dynamics, and was already used in the 19th century by Delaunay [5] and Linstedt [12] for some problems of celestial mechanics.

When a Hamiltonian function H is in normal form, its first integrals are also normalized simultaneously to some extent. More precisely, one has the following folklore lemma, whose proof is straightforward (see, e.g., [9], [10], [11]):

LEMMA 2.2. *If* $\{H_{\mathrm{ss}}, H\} = 0$, *i.e.* H *is in Birkhoff normal form, and* $\{H, F\} = 0$, *i.e.* F *is a first integral of* H, *then* $\{H_{\mathrm{ss}}, F\} = 0$.

Recall that the simple Lie algebra $\mathrm{sp}(2n, \mathbb{C})$ has only one Cartan subalgebra up to conjugacy. In terms of quadratic functions, there is a complex linear canonical system of coordinates (x_j, y_j) of $\mathbb{C}^{2n}$ in which H_{ss} can be written as

$$H_{\mathrm{ss}} = \sum_{i=1}^{n} \gamma_j x_j y_j, \tag{2.10}$$

where γ_j are complex coefficients, called *frequencies*. (The quadratic functions $\nu_1 = x_1 y_1, \ldots, \nu_n = x_n y_n$ span a Cartan subalgebra.) The frequencies γ_j are complex numbers uniquely determined by H_{ss} up to a sign and a permutation. The reason why we choose to write $x_j y_j$ instead of $\frac{1}{2}(x_j^2 + y_j^2)$ in Equation (2.10) is that this way monomial functions will be eigenvectors of H_{ss} under the Poisson bracket:

$$\{H_{\mathrm{ss}}, \prod_{j=1}^{n} x_j^{a_j} y_j^{b_j}\} = (\sum_{j=1}^{n} (b_j - a_j)\gamma_j) \prod_{j=1}^{n} x_j^{a_j} y_j^{b_j}. \tag{2.11}$$

In particular, $\{H, H_{\mathrm{ss}}\} = 0$ if and only if every monomial term $\prod_{j=1}^{n} x_j^{a_j} y_j^{b_j}$ with a nonzero coefficient in the Taylor expansion of H satisfies the following relation, called a *resonance relation*:

$$\sum_{j=1}^{n} (b_j - a_j)\gamma_j = 0. \tag{2.12}$$

In the nonresonant case, when there are no resonance relations except the trivial ones, the Birkhoff normal condition $\{H, H_{\mathrm{ss}}\} = 0$ means that H is a function of n variables $\nu_1 = x_1y_1, \dots, \nu_n = x_ny_n$, implying complete integrability. Thus any nonresonant Hamiltonian system is formally integrable [2], [15].

More generally, denote by $\mathcal{R} \subset \mathbb{Z}^n$ the sublattice of $\mathbb{Z}^n$ consisting of elements $(c_j) \in \mathbb{Z}^n$ such that $\sum c_j\gamma_j = 0$. The dimension of $\mathcal{R}$ over $\mathbb{Z}$, denoted by q, is called the *degree of resonance* of the Hamiltonian H. Let $\mu^{(n-q+1)}, \dots, \mu^{(n)}$ be a basis of the resonance lattice $\mathcal{R}$. Let $\rho^{(1)}, \dots, \rho^{(n)}$ be a basis of $\mathbb{Z}^n$ such that $\sum_{j=1}^n \rho_j^{(k)}\mu_j^{(h)} = \delta_{kh}$ ($= 0$ if $k \neq h$ and $= 1$ if $k = h$), and set

$$F^{(k)}(x,y) = \sum_{j=1}^n \rho_j^{(k)} x_j y_j \tag{2.13}$$

for $1 \leq k \leq n$. Then we have $H_{\mathrm{ss}} = \sum_{k=1}^{n-q} \alpha_k F^{(k)}$ with no resonance relation among $\alpha_1, \dots, \alpha_{n-q}$. The equation $\{H_{\mathrm{ss}}, H\} = 0$ is now equivalent to

$$\{F_k, H\} = 0 \quad \text{for all } k = 1, \dots, n-q. \tag{2.14}$$

What is so good about the quadratic functions $F^{(k)}$ is that each $iF^{(k)}$ (where $i = \sqrt{-1}$) is a periodic Hamiltonian function; i.e., its holomorphic Hamiltonian vector field $X_{iF^{(k)}}$ is periodic with a real positive period (which is 2π or a divisor of this number). In other words, if we write $X_{iF^{(k)}} = X_k + iY_k$, where $X_k = JY_k$ is a real vector field called the real part of $X_{iF^{(k)}}$ (i.e. X_k is a vector field of $\mathbb{C}^{2n}$ considered as a real manifold; J denotes the operator of the complex structure of $\mathbb{C}^{2n}$), then the flow of X_k in $\mathbb{C}^{2n}$ is periodic. Of course, if F is a holomorphic function on a complex symplectic manifold, then the real part of the holomorphic vector field X_F is a real vector field which preserves the complex symplectic form and the complex structure.

Since the periodic Hamiltonian functions $iF^{(k)}$ commute pairwise (in this paper, when we say "periodic", we always mean with a real positive period), the real parts of their Hamiltonian vector fields generate a Hamiltonian action of the real torus $\mathbb{T}^{n-q}$ on $(\mathbb{C}^{2n}, \omega)$. (One may extend it to a complex torus $(\mathbb{C}^*)^{n-q}$-action, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, but we will only use the compact real part of this complex torus.) If H is in (analytic) Birkhoff normal form, it will Poisson-commute with $F^{(k)}$, and hence it will be preserved by this torus action.

Conversely, if there is a Hamiltonian torus action of $\mathbb{T}^{n-q}$ in $(\mathbb{C}^{2n}, \omega)$ which preserves H, then the equivariant Darboux theorem (which may be proved by an equivariant version of the Moser path method; see, e.g., [4]) implies that there is a local holomorphic canonical transformation of coordinates under which the action becomes linear (and is generated by $iF^{(1)}, \dots, iF^{(n-q)}$). Since this action preserves H, it follows that $\{H, H_{\mathrm{ss}}\} = 0$. Thus we have proved the following:

Proposition 2.3. *With the above notation, the following two conditions are equivalent*:

i) *There exists a holomorphic Birkhoff canonical transformation of coordinates* $(x', y') = \Phi(x, y)$ *for* H *in a neighborhood of* 0 *in* $\mathbb{C}^{2n}$.

ii) *There exists an analytic Hamiltonian torus action of* $\mathbb{T}^{n-q}$, *in a neighborhood of* 0 *in* $\mathbb{C}^{2n}$, *which preserves* H, *and whose linear part is generated by the Hamiltonian vector fields of the functions* $iF^{(k)} = i\sum \rho_j^{(k)} x_j y_j$, $k = 1, \ldots, n-q$.

Proof of Proposition 1.3. When H is a real analytic Hamiltonian function which admits a local complex analytic Birkhoff normalization, we will have to show that H admits a local real analytic Birkhoff normalization. Let $A : \mathbb{T}^{n-q} \times (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{2n}, 0)$ be a Hamiltonian torus action which preserves H and which has an appropriate linear part, as provided by Proposition 1.2. To prove Proposition 1.3, it suffices to linearize this action by a local real analytic symplectic transformation.

Let F be a holomorphic periodic Hamiltonian function generating a $\mathbb{T}^1$-subaction of A. Denote by F^* the function $F^*(z) = \overline{F(\bar{z})}$, where $z \mapsto \bar{z}$ is the complex conjugation in $\mathbb{C}^{2n}$. Since H is real and $\{H, F\} = 0$, we also have $\{H, F^*\} = 0$. It follows that, if H is in complex Birkhoff normal form, we will have $\{H_{\rm ss}, F^*\} = 0$, and hence F^* is preserved by the torus $\mathbb{T}^{n-q}$-action. Also, F^* is a periodic Hamiltonian function by itself (because F is), and due to the fact that H is real, the quadratic part of F^* is a real linear combination of the quadratic parts of periodic Hamiltonian functions that generate the torus $\mathbb{T}^{n-q}$-action. It follows that F^* must in fact be also the generator of an $\mathbb{T}^1$-subaction of the torus $\mathbb{T}^{n-q}$-action. (Otherwise, by combining the action of X_{F^*} with the $\mathbb{T}^{n-q}$-action, we would have a torus action of higher dimension than possible.) The involution $F \mapsto F^*$ gives rise to an involution $t \mapsto \bar{t}$ in $\mathbb{T}^{n-q}$. The torus action is reversible with respect to this involution and to the complex conjugation:

$$\overline{A(t, z)} = A(\bar{t}, \bar{z}). \tag{2.15}$$

The above equation implies that the local torus $\mathbb{T}^{n-q}$-action may be linearized locally by a real transformation of variables. Indeed, one may use the following averaging formula:

$$z' = z'(z) = \int_{\mathbb{T}^{n-q}} A_1(-t, A(t, z)) d\mu, \tag{2.16}$$

where $t \in \mathbb{T}^{n-q}$, $z \in \mathbb{C}^{2n}$, A_1 is the linear part of A (so A_1 is a linear torus action), and $d\mu$ is the standard constant measure on $\mathbb{T}^{n-q}$. The action A will be linear with respect to z': $z'(A(t, z)) = A_1(t, z'(z))$. Due to Equation (2.15), we have that $\overline{z'(z)} = z'(\overline{z})$, which means that the transformation $z \mapsto z'$ is real analytic.

After the above transformation $z \mapsto z'$, the torus action becomes linear; the symplectic structure ω is no longer constant in general, but one can use the equivariant Moser path method to make it back to a constant form (see, e.g., [4]). In order to do it, one writes $\omega - \omega_0 = d\alpha$ and considers the flow of the time-dependent vector field X_t defined by $i_{X_t}(t\omega + (1-t)\omega_0) = \alpha$, where ω_0 is the constant symplectic form which coincides with ω at point 0. One needs α to be $\mathbb{T}^{n-q}$-invariant and real. The first property can be achieved, starting from an arbitrary real analytic α such that $d\alpha = \omega - \omega_0$, by averaging with respect to the torus action. The second property then follows from Equation (2.15). Proposition 1.3 is proved. □

3. Local torus actions for integrable systems

Proof of Theorem 1.1. According to Proposition 1.3, it is enough to prove Theorem 1.1 in the complex analytic case. In this section, we will do this by finding local Hamiltonian $\mathbb{T}^1$-actions which preserve the momentum map of an analytically completely integrable system. The Hamiltonian function generating such an action will be a first integral of the system, called an *action function* (as in "action-angle coordinates"). If we find $(n-q)$ such $\mathbb{T}^1$-actions, then they will automatically commute and give rise to a Hamiltonian $\mathbb{T}^{n-q}$-action.

To find an action function, we will use the following period integral formula, known as the *Mineur-Arnold formula*:

$$P = \int_\gamma \beta\,,$$

where P denotes an action function, β denotes a primitive 1-form (i.e. $\omega = d\beta$ is the symplectic form), and γ denotes a 1-cycle (closed curve) lying on a level set of the momentum map.

To show the existence of such 1-cycles γ, we will use an approximation method, based on the existence of a formal Birkhoff normalization.

Denote by $\mathbf{G} = (G_1 = H, G_2, \ldots, G_n) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ the holomorphic momentum map germ of a given complex analytic integrable Hamiltonian system. Let $\varepsilon_0 > 0$ be a small positive number such that $\mathbf{G}$ is defined in the ball $\{z = (x_j, y_j) \in \mathbb{C}^{2n}, |z| < \varepsilon_0\}$. We will restrict our attention to what happens inside this ball. As in the previous section, we may assume that in the symplectic coordinate system $z = (x_j, y_j)$ we have

$$H = G_1 = H_{\rm ss} + H_{\rm nil} + H_3 + H_4 + \ldots \tag{3.1}$$

with

$$H_{\rm ss} = \sum_{k=1}^{n-q} \alpha_k F^{(k)},\ \ F^{(k)} = \sum_{j=1}^{n} \rho_j^{(k)} x_j y_j, \tag{3.2}$$

with no resonance relations among $\alpha_1, \dots, \alpha_{n-q}$. We will fix this coordinate system $z = (x_j, y_j)$, and all functions will be written in it.

The real and imaginary parts of the Hamiltonian vector fields of $G_1, \dots, G_n$ are in involution and define an *associated singular foliation* in the ball $\{z = (x_j, y_j) \in \mathbb{C}^{2n}, |z| < \varepsilon_0\}$. Hereafter the norm in $\mathbb{C}^n$ is given by the standard Hermitian metric with respect to the coordinate system (x_j, y_j). Similarly to the real case, the leaves of this foliation are called local *orbits of the associated Poisson action*; they are complex isotropic submanifolds, and generic leaves are Lagrangian and have complex dimension n. For each z we will denote the leaf which contains z by M_z. Recall that the momentum map is constant on the orbits of the associated Poisson action. If z is a point such that $\mathbf{G}(z)$ is a regular value for the momentum map, then M_z is a connected component of $\mathbf{G}^{-1}(\mathbf{G}(z))$.

Denote by

$$S = \{z \in \mathbb{C}^{2n}, |z| < \varepsilon_0, dG_1 \wedge dG_2 \wedge \dots \wedge dG_n(z) = 0\} \tag{3.3}$$

the singular locus of the momentum map, which is also the set of singular points of the associated singular foliation. What we need to know about S is that it is analytic and of codimension at least 1, though for generic integrable systems S is in fact of codimension 2. In particular, we have the following Łojasiewicz inequality (see [13]): there exist a positive number N and a positive constant C such that

$$|dG_1 \wedge \dots \wedge dG_n(z)| > C(d(z,S))^N \tag{3.4}$$

for any z with $|z| < \varepsilon_0$, where the norm applied to $dG_1 \wedge \dots \wedge dG_n(z)$ is some norm in the space of n-vectors, and $d(z,S)$ is the distance from z to S with respect to the Euclidean metric.

We will choose an infinite decreasing sequence of small numbers ε_m ($m = 1, 2, \dots$), as small as needed, with $\lim_{m\to\infty} \varepsilon_m = 0$, and define the following open subsets U_m of $\mathbb{C}^{2n}$:

$$U_m = \{z \in \mathbb{C}^{2n}, |z| < \varepsilon_m, d(z,S) > |z|^m\}. \tag{3.5}$$

We will also choose two infinite increasing sequence of natural numbers a_m and b_m ($m = 1, 2, \dots$), as large as needed, with $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = \infty$. It follows from Birkhoff's Theorem 2.1 and Lemma 2.2 that there is a sequence of local holomorphic symplectic coordinate transformations Φ_m, $m \in \mathbb{N}$, such that the following two conditions are satisfied:

a) The differential of Φ_m at 0 is the identity for each m, and for any two numbers m, m' with $m' > m$ we have

$$\Phi_{m'}(z) = \Phi_m(z) + O(|z|^{a_m}). \tag{3.6}$$

In particular, there is a formal limit $\Phi_\infty = \lim_{m\to\infty} \Phi_m$.

b) The momentum map is normalized up to order b_m by Φ_m. More precisely, the functions G_j can be written as

$$G_j(z) = G_{(m)j}(z) + O(|z|^{b_m}),\ j = 1, \dots, n, \tag{3.7}$$

with $G_{(m)j}$ such that

$$\{G_{(m)j}, F^{(k)}_{(m)}\} = 0 \quad \forall j = 1, \dots, n,\ k = 1, \dots, n - q. \tag{3.8}$$

Here the functions $F^{(k)}_{(m)}$ are quadratic functions

$$F^{(k)}_{(m)}(x, y) = \sum_{j=1}^{n} \rho^{(k)}_j x_{(m)j} y_{(m)j} \tag{3.9}$$

in local symplectic coordinates

$$(x_{(m)}, y_{(m)}) = \Phi_m(x, y). \tag{3.10}$$

Notice that $F^{(k)}_{(m)}$ is a quadratic function in the coordinate system $(x_{(m)}, y_{(m)})$. But from now on we will use only the original coordinate system (x, y). Then $F^{(k)}_{(m)}$ is not a quadratic function in (x, y) in general, and the quadratic part of $F^{(k)}_{(m)}$ is $F^{(k)}$. The norm in $\mathbb{C}^{2n}$, which is used in the estimates in this section, will be given by the standard Hermitian metric with respect to the original coordinate system (x, y).

Denote by $\gamma^{(k)}_m(z)$ the orbit of the real part of the periodic Hamiltonian vector field $X_{iF^{(k)}_{(m)}}$ which goes through z. Then for any $z' \in \gamma^{(k)}_m(z)$ we have $G_{(m)j}(z') = G_{(m)j}(z)$, and $|z'| \simeq |z|$; i.e. $\lim_{z\to 0} \frac{|z'|}{|z|} = 1$. (The reason is that the real part of the linear periodic Hamiltonian vector field $X_{iF^{(k)}}$ also preserves the Hermitian metric of $\mathbb{C}^{2n}$, and the linear part of $X_{iF^{(k)}_{(m)}}$ is $X_{iF^{(k)}}$.) As a consequence, we have

$$|\mathbf{G}(z') - \mathbf{G}(z)| = O(|z'|^{b_m}). \tag{3.11}$$

Note that, for each $m \in \mathbb{N}$, we can choose the numbers a_m and b_m first, then choose the radius $\varepsilon_m = \varepsilon_m(a_m, b_m)$ sufficiently small so that the equivalence $O(|z'|^{b_m}) \simeq O(|z|^{b_m})$ makes sense for $z \in U_m$.

On the other hand, we have

$$\begin{aligned} &|dG_1(z') \wedge \cdots \wedge dG_n(z')| \\ &\quad = |dG_{(m)1}(z') \wedge \cdots \wedge dG_{(m)n}(z')| + O(|z|^{b_m - 1}) \\ &\quad \simeq |dG_{(m)1}(z) \wedge \cdots \wedge dG_{(m)n}(z)| + O(|z|^{b_m - 1}) \\ &\quad = |dG_1(z) \wedge \cdots \wedge dG_n(z)| + O(|z|^{b_m - 1}). \end{aligned} \tag{3.12}$$

We can assume that $b_m - 1 > N$. Then for $|z| < \varepsilon_m$ small enough, the above inequality may be combined with Łojasiewicz inequality (3.4) to yield

$$|dG_1(z') \wedge \cdots \wedge dG_n(z')| > C_1 d(z, S)^N \tag{3.13}$$

where $C_1 = C/2$ is a positive constant (which does not depend on m).

If $z \in U_m$, and ε_m is small enough, we have $d(z, S) > |z|^m$, which may be combined with the last inequality to yield:

$$|dG_1(z') \wedge \cdots \wedge dG_n(z')| > C_1 |z|^{mN}. \tag{3.14}$$

Assuming that b_m is much larger than mN, we can use the implicit function theorem to project the curve $\gamma_m^{(k)}(z)$ on M_z as follows:

For each point $z' \in \gamma_m^{(k)}(z)$, let $D_m(z')$ be the complex n-dimensional disk centered at z', which is orthogonal to the kernel of the differential of the momentum map G at z', and which has radius equal to $|z'|^{2mN}$. Since the second derivatives of G are locally bounded by a constant near 0, it follows from the definition of $D_m(z')$ that we have, for $|z| < \varepsilon_m$ small enough:

$$|D\mathbf{G}(w) - D\mathbf{G}(z')| < |z|^{3mN/2} \ \forall w \in D_m(z') \tag{3.15}$$

where $D\mathbf{G}(w)$ denotes the differential of the momentum map at w, considered as an element of the linear space of $2n \times n$ matrices.

Inequality (3.14) together with Inequality (3.15) imply that the momentum map $\mathbf{G}$, when restricted to $D_m(z')$, is a diffeomorphism from $D(z')$ to its image, and the image of $D_m(z')$ in $\mathbb{C}^n$ under $\mathbf{G}$ contains a ball of radius $|z|^{4mN}$. (Because we have $4mN > 2mN + mN$, where $2mN$ is the order of the radius of $D_m(z')$, and mN is a majorant of the order of the norm of the differential of $\mathbf{G}$. The differential of $\mathbf{G}$ is “nearly constant” on $D_m(z')$ due to Inequality (3.15).) Thus, if $b_m > 5mN$ for example, then Inequality (3.11) implies that there is a unique point z'' on $D_m(z')$ such that $\mathbf{G}(z'') = \mathbf{G}(z)$. The map $z' \mapsto z''$ is continuous, and it maps $\gamma_m^{(k)}(z)$ to some close curve $\tilde{\gamma}_m^{(k)}(z)$, which must lie on M_z because the point z maps to itself under the projection. When b_m is large enough and ε_m is small enough, then $\tilde{\gamma}_m^{(k)}(z)$ is a smooth curve with a natural parametrization inherited from the natural parametrization of $\gamma_m^{(k)}(z)$, it has bounded derivative (we can say that its velocity vectors are uniformly bounded by 1), and it depends smoothly on $z \in U_m$.

Define the following action function $P_m^{(k)}$ on U_m:

$$P_m^{(k)}(z) = \int_{\tilde{\gamma}_m^{(k)}(z)} \beta \ , \tag{3.16}$$

where $\beta = \sum x_j dy_j$ (so that $d\beta = \sum dx_j \wedge dy_j$ is the standard symplectic form.) This function has the following properties:

i) Because the 1-form $\beta = \sum x_j dy_j$ is closed on each leaf of the Lagrangian foliation of the integrable system in U_m, $P_m^{(k)}$ is a holomorphic first integral of the foliation. (This fact is well-known in complex geometry: period integrals of holomorphic k-forms, which are closed on the leaves of a given holomorphic foliation, over p-cycles of the leaves, give rise to (local) holomorphic first integrals of the foliation.) The functions $P_m^{(1)}, \dots, P_m^{(n-q)}$ Poisson commute pairwise, because they commute with the momentum map.

ii) $P_m^{(k)}$ is uniformly bounded by 1 on U_m, because $\tilde{\gamma}_m^{(k)}(z)$ is small, together with its first derivative.

iii) Provided that the numbers a_m are chosen large enough, for any $m' > m$ we have that $P_m^{(k)}$ coincides with $P_{m'}^{(k)}$ in the intersection of U_m with $U_{m'}$. To see this important point, recall that

$$P_m^{(k)} = P_{m'}^{(k)} + O(|z|^{a_m}) \tag{3.17}$$

by construction, which implies that the curve $\gamma_{m'}^{(k)}(z)$ is $|z|^{a_m-2}$-close to the curve $\gamma_m^{(k)}(z)$ in C^1-norm. If a_m is large enough with respect to mN (say $a_m > 5mN$), then it follows that the complex n-dimensional cylinder

$$V_{m'}(z) = \{w \in \mathbb{C}^{2n} \mid d(w, \gamma_{m'}^{(k)}(z)) < |z|^{2m'N}\} \cap M_z \tag{3.18}$$

lies inside (and near the center of) the complex n-dimensional cylinder

$$V_m(z) = \{w \in \mathbb{C}^{2n} \mid d(w, \gamma_m^{(k)}(z)) < |z|^{2mN}\} \cap M_z. \tag{3.19}$$

On the other hand, one can check that $\tilde{\gamma}_m^{(k)}(z)$ is a retract of $V_m(z)$ in M_z, and the same thing is true for the index m'. It follows easily that $\tilde{\gamma}_{m'}^{(k)}(z)$ must be homotopic to $\tilde{\gamma}_m^{(k)}(z)$ in M_z, implying that $P_m^{(k)}(z)$ coincides with $P_{m'}^{(k)}(z)$.

iv) Since $P_m^{(k)}$ coincides with $P_{m'}^{(k)}$ in $U_m \bigcap U_{m'}$, we may glue these functions together to obtain a holomorphic function, denoted by $P^{(k)}$, on the union $U = \bigcup_{m=1}^{\infty} U_m$. Lemma 4.1 in the following section shows that if we have a bounded holomorphic function in $U = \cup_{m=1}^{\infty} U_m$ then it can be extended to a holomorphic function in a neighborhood of 0 in $\mathbb{C}^{2n}$. Thus our action functions $P^{(k)}$ are holomorphic in a neighborhood of 0 in $\mathbb{C}^{2n}$.

v) $P^{(k)}$ is a local periodic Hamiltonian function whose quadratic part is $iF^{(k)} = i \sum \rho_j^{(k)} x_j y_j$. To see this, note that

$$iF_m^{(k)}(z) = i \sum \rho_j^{(k)} x_{(m)j} y_{(m)j} = \int_{\gamma_m^{(k)}(z)} \beta \ , \tag{3.20}$$

for $z \in U_m$. Since the curve $\tilde{\gamma}_m^{(k)}(z)$ is $|z|^{3mN}$-close to the curve $\gamma_m^{(k)}(z)$ by construction (provided that $b_m > 4mN$),

$$P^{(k)}(z) = iF_m^{(k)}(z) + O(|z|^{3mN}) \tag{3.21}$$

for $z \in U_m$. Due to the nature of U_m (almost every complex line in $\mathbb{C}^{2n}$ which contains the origin 0 intersects with U_m in an open subset (of the line) which surrounds the point 0), it follows from the last estimate that in fact the coefficients of all the monomial terms of order $< 3mN$ of $P^{(k)}$ coincide with that of $iF_m^{(k)}$; i.e.,

$$P^{(k)}(z) = iF_m^{(k)}(z) + O(|z|^{3mN}) \tag{3.22}$$

in a neighborhood of 0 in $\mathbb{C}^{2n}$. In particular,

$$P^{(k)} = \lim_{m\to\infty} iF_m^{(k)} \ , \tag{3.23}$$

where the limit on the right-hand side of the above equation is understood as the formal limit of a Taylor series, and the left-hand side is also considered as a Taylor series. This is enough to imply that $P^{(k)}$ has $i\sum \rho_j^{(k)} x_j y_j$ as its quadratic part, and that $P^{(k)}$ is a periodic Hamiltonian of period 2π because each $iF_m^{(k)}$ is so. (If a local holomorphic Hamiltonian vector field which vanishes at 0 is formally periodic then it is periodic.)

Now we can apply Proposition 2.3 and Proposition 1.3 to finish the proof of Theorem 1.1. □

4. Holomorphic extension of action functions

The following lemma shows that the action functions $P^{(k)}$ constructed in the previous section can be extended holomorphically in a neighborhood of 0.

Lemma 4.1. *Let* $U = \bigcup_{m=1}^{\infty} U_m$, *with*

$$U_m = \{x \in \mathbb{C}^n, |x| < \varepsilon_m, d(x,S) > |x|^m\},$$

where ε_m *is an arbitrary sequence of positive numbers and* S *is a local proper complex analytic subset of* $\mathbb{C}^n$ $(\operatorname{codim}_{\mathbb{C}} S \geq 1)$. *Then any bounded holomorphic function on* U *has a holomorphic extension in a neighborhood of* 0 *in* $\mathbb{C}^n$.

Proof. Though we suspect that this lemma was known to specialists in complex analysis, we could not find it in the literature, and so we will provide a proof here. When $n = 1$ the lemma is obvious; so we will assume that $n \geq 2$. Without loss of generality, we can assume that S is a singular hypersurface. We divide the lemma into two steps:

Step 1. The case when S is contained in the union of hyperplanes $\bigcup_{j=1}^{n}\{x_j = 0\}$ where $(x_1, \ldots, x_n)$ is a local holomorphic system of coordinates. Clearly, U contains a product of nonempty annuli $\eta_j < |x_j| < \eta_j'$, hence f is defined by a Laurent series in $x_1, \cdots, x_n$ there. We will study the domain of convergence of this Laurent series, using the well-known fact that the domain of convergence of a Laurent series is logarithmically convex. More precisely, denote by π the map $(x_1, \cdots, x_n) \mapsto (\log|x_1|, \cdots, \log|x_n|)$ from $(\mathbb{C}^*)^n$ to $\mathbb{R}^n$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, and set

$$E = \{\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n \ \mid \ \pi^{-1}(\mathbf{r}) \subset U\}.$$

Denote by $\mathrm{Hull}(E)$ the convex hull of E in $\mathbb{R}^n$. Then since the function f is analytic and bounded in $\pi^{-1}(E)$, it can be extended to a bounded analytic function on $\pi^{-1}(\mathrm{Hull}(E))$. On the other hand, by definition of $U = \bigcup_{m=1}^{\infty} U_m$, there is a sequence of positive numbers K_m (tending to infinity) such that $E \supset (\bigcup_{m=1}^{\infty} E_m)$, where

$$E_m = \{(r_1, \ldots, r_n) \in \mathbb{R}^n \ \mid \ (r_j < -K_m \ \forall j) \ , \ (r_j > mr_i \ \forall j \neq i)\}.$$

It is clear that the convex hull of $\bigcup_{m=1}^{\infty} E_m$, with each E_m defined as above, contains a neighborhood of $(-\infty, \ldots, -\infty)$, i.e. a set of the type

$$\{(r_1, \ldots, r_n) \in \mathbb{R}^n \ \mid \ r_j < -K \ \forall j\}.$$

This implies that the function f can be extended to a bounded analytic function in $\mathcal{U} \cap (\mathbb{C}^*)^n$, where $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{C}^n$. Since f is bounded in $\mathcal{U} \cap (\mathbb{C}^*)^n$, it can be extended analytically on the whole $\mathcal{U}$. Step 1 is finished.

Step 2. Consider now the case with an arbitrary S. Then we can use Hironaka's desingularization theorem [8] to make it smooth. The general desingularization theorem is a very hard theorem, but in the case of a singular complex hypersurface a relatively simple constructive proof of it can be found in [1]. In fact, since the exceptional divisor will also have to be taken into account, after the desingularization process we will have a variety which may have normal crossings. More precisely, we have the following commutative diagram

$$\begin{array}{ccccc} Q & \subset & S' & \subset & M^n \\ \downarrow & & \downarrow & & \downarrow p \\ 0 & \in & S & \subset & (\mathbb{C}^n, 0) \end{array} \quad , \tag{4.1}$$

where $(\mathbb{C}^n, 0)$ denotes the germ of $\mathbb{C}^n$ at 0 presented by a ball which is small enough; M^n is a complex manifold; the projection p is surjective, and injective outside the exceptional divisor; S' denotes the union of the exceptional divisor with the smooth proper submanifold of M^n which is a desingularization of S — the only singularities in S' are normal crossings; $Q = p^{-1}(0)$ is compact. Now, M^n is obtained from $(\mathbb{C}^n, 0)$ by a finite number of blowing-ups along submanifolds.

Denote by $U' = p^{-1}(U)$ the preimage of U under the projection p. One can pull back f from U to U' to get a bounded holomorphic function on U', denoted by f'. An important observation is that the type of U persists under blowing-ups along submanifolds. (Or equivalently, the type of its complement, which may be called a *sharp-horn-neighborhood* of S because it is similar to horn-type neighborhoods of $S \setminus \{0\}$ used by singularists but it is sharp of arbitrary order, is persistent under blowing-ups.) More precisely, for each point $x \in Q$, the complement of U' in a small neighborhood of x is a "sharp-horn-neighborhood" of S' at x. Since S' only has normal crossings, the pair (U', S') satisfies the conditions of Step 1, and therefore we can extend f' holomorphically in a neighborhood of x in M^n. Since $Q = p^{-1}(0)$ is compact, we can extend f' holomorphically in a neighborhood of Q in M'. One can now project this extension of f' back to $(\mathbb{C}^n, 0)$ to get a holomorphic extension of f in a neighborhood of 0. The lemma is proved. □

Remark. The "sharp-horn" type of the complement of U in the above lemma is essential. If we replace U by U_m (for any given number m) then the lemma is false.

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