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Random k-surfaces

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Abstract

Invariant measures for the geodesic flow on the unit tangent bundle of a negatively curved Riemannian manifold are a basic and well-studied subject. This paper continues an investigation into a 2-dimensional analog of this flow for a 3-manifold N. Namely, the article discusses 2-dimensional surfaces immersed into N whose product of principal curvature equals a constant k between 0 and 1, surfaces which are called *k-surfaces*. The "2-dimensional" analog of the unit tangent bundle with the geodesic flow is a "space of pointed k-surfaces", which can be considered as the space of germs of complete k-surfaces passing through points of N. Analogous to the 1-dimensional lamination given by the geodesic flow, this space has a 2-dimensional lamination. An earlier work [1] was concerned with some topological properties of chaotic type of this lamination, while this present paper concentrates on ergodic properties of this object. The main result is the construction of infinitely many mutually singular transversal measures, ergodic and of full support. The novel feature compared with the geodesic flow is that most of the leaves have exponential growth.

1. Introduction

We associated in [1] a compact space laminated by 2-dimensional leaves, to every compact 3-manifold N with curvature less than -1. Considered as a "dynamical system", its properties generalise those of the geodesic flow.

In this introduction, I will just sketch the construction of this space, and will be more precise in Section 2. Let $k \in]0,1[$. A *k-surface* is an immersed surface in N, such that the product of the principal curvatures is k. If N has constant curvature K, a k-surface has curvature $K+k$. Analytically, k-surfaces are described by elliptic equations.

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When dealing with ordinary differential solutions, one is led to introduce the *phase space* consisting of pairs (γ, x) where γ is a trajectory solution of the O.D.E., and x is a point on γ. We recover the dynamical picture by moving x along γ.

We can mimic this construction in our situation in which a P.D.E. replaces the O.D.E. More precisely, we can consider the space of pairs (Σ, x) where Σ is a k-surface, and x a point of Σ.

We proved in [4] that this construction actually makes sense. More precisely, we proved the space just described can be compactified by a space, called the *space of k-surfaces.* Furthermore, the boundary is finite dimensional and related in a simple way to the geodesic flow. This space, which we denote by $\mathcal{N}$, is laminated by 2-dimensional leaves, in particular by those obtained by moving x along a k-surface Σ. A lamination means that the space has a local product structure.

The purpose of this article is to study transversal measures, ergodic and of full support on this space of k-surfaces. At the present stage, let us just notice that since many leaves are hyperbolic (cf. Theorem 2.4.1), one cannot produce transversal measures by Plante's argument. Our strategy will be to "code" by a combinatorial model on which it will be easier to build transversal measures.

This article is organised as follows.

2. *The space of all k-surfaces.* We describe more precisely the *space of k-surfaces* we are going to work with, and state some of its properties proved in [1].

3. *Transversal measures.* We present our main result, Theorem 3.2.1, discuss other constructions and questions, and sketch the main construction.

4. *A combinatorial model.* In this section, we explain a combinatorial construction. Starting from *configuration data*, we consider "configuration spaces". These are spaces of mappings from $\mathbb{QP}^1$ to a space W. We produce invariant and ergodic measures under the action of $\mathrm{PSL}(2, \mathbb{Z})$ by left composition.

5. *Configuration data and the boundary at infinity of a hyperbolic 3-manifold.* We exhibit a combinatorial model associated to hyperbolic manifolds. In this context, the previous W is going to be $\mathbb{CP}^1$.

6. *Convex surfaces and configuration data.* We prove here that the combinatorial model constructed in the previous section actually codes for the space of k-surfaces.

7. *Conclusion.* We summarise our constructions and prove our main result, Theorem 3.2.1.

I would like to thank W. Goldman for references about $\mathbb{CP}^1$-structures, and R. Kenyon for discussions.

2. The space of all k-surfaces

The aim of this section is to present in a little more detail the space "of k-surfaces" that we are going to work with.

Let N be a compact 3-manifold with curvature less than -1. Let $k \in]0,1[$ be a real number. All definitions and results are expounded in [1].

2.1. *k-surfaces, tubes.* If S is an immersed surface in N, it carries several natural metrics. By definition, the *u-metric* is the metric induced from the immersion in the unit tangent bundle given by the Gauss map. We shall say a surface is *u-complete* if the u-metric is complete.

A *k-surface* is an immersed u-complete connected surface such that the determinant of the shape operator (i.e. the product of the principal curvatures) is constant and equal to k.

We described in [1] various ways to construct k-surfaces. In Section 6.3, we summarise results of [1] which allow us to obtain k-surfaces as solutions of an *asymptotic Plateau problem*.

Since k-surfaces are solutions of an elliptic problem, the germ of a k-surface determines the k-surface. It follows that a k-surface is determined by its image, up to coverings. More precisely, for every k-surface S immersed by f in N, there exists a unique k-surface Σ, the *representative of S*, immersed by ϕ, such that for every k-surface $\bar{S}$ immersed by $\bar{f}$ satisfying $f(S) = \bar{f}(\bar{S})$, there exists a covering $\pi : \bar{S} \to \Sigma$ such that $\bar{f} = \phi \circ \pi$.

By a slight abuse of language, the expression "k-surface" will generally mean "representative of a k-surface".

The *tube* of a geodesic is the set of normal vectors to this geodesic. It is a 2-dimensional submanifold of the unit tangent bundle.

2.2. *The space of k-surfaces.* The *space of k-surfaces* is the space of pairs (Σ, x) where $x \in \Sigma$ and Σ is either the representative of a k-surface or a tube. We denote it by $\mathcal{N}$. Alternatively, we can think of it as the space of germs of *u-complete* k-surfaces, or by analytic continuation as the space of ∞-jet of complete k-surfaces. If we denote by $J^k(2, UN)$ the finite dimensional manifold of k-jets of surfaces in UN, $\mathcal{N}$, can be seen as a subset of the projective limit $J^\infty(2, UN)$; this point of view is interesting, but one should stress it seems hard to detect from the germ (or the jet) if a k-surface is complete or not.

The space $\mathcal{N}$ inherits a topology coming from the topology of pointed immersed 2-manifolds in the unit tangent bundle (cf. Section 2.3 of [1]); alternatively, this topology coincides with the topology induced by the embedding in $J^\infty(2, UN)$.

We describe now the structure of a lamination of $\mathcal{N}$. First notice that each k-surface (or tube) S_0 determines a *leaf* $\mathcal{L}_{S_0}$ defined by

$$\mathcal{L}_{S_0} = \{(S_0, x)/x \in S_0\}.$$

We proved in [4] that $\mathcal{N}$ is compact. Furthermore, the partition of $\mathcal{N}$ into leaves is a lamination, i.e. admits a local product structure. Notice that $\mathcal{N}$ has two parts:

(1) a dense set which turns out to be infinite dimensional, and which truly consists of k-surfaces,

(2) a "boundary" consisting of the union of tubes; this "boundary" is closed, finite dimensional, and is an S^1 fibre bundle over the geodesic flow.

Therefore, in some sense, $\mathcal{N}$ is an extension of the geodesic flow. To enforce this analogy, one should also notice that the 1-dimensional analogue, namely the space of curves of curvature k in a hyperbolic surface, is precisely the geodesic flow.

2.3. *Examples of k-surfaces.* In order to give a little more flesh to our discussion, we give some examples of k-surfaces.

Equidistant surfaces to totally geodesic planes in $\mathbb{H}^3$. If we suppose N is of constant curvature, or equivalently that the universal cover of N is $\mathbb{H}^3$, a surface equidistant to a geodesic plane is a k-surface. It follows the subset of $\mathcal{N}$ corresponding to such k-surfaces (with an orientation) in N is identified with the unit tangent bundle of the hyperbolic space $UN = S^1\backslash\mathrm{PSL}(2,\mathbb{C})/\pi_1(N)$. The lamination structure comes from the right action of $\mathrm{PSL}(2,\mathbb{R})$ on $S^1\backslash\mathrm{PSL}(2,\mathbb{C})$.

Solutions to the asymptotic Plateau problem. Let M be a simply connected negatively curved 3-manifold $\partial_\infty M$. An oriented surface Σ possesses a *Minkowski-Gauss map*, N_Σ, with values in the boundary at infinity, namely the map which associates to a point, the point at infinity of the exterior normal geodesic. Since a k-surface is locally convex, this map is a local homeomorphism. We define an *asymptotic Plateau problem* to be a couple (S, ι) such that ι is a local homeomorphism from S to $\partial_\infty M$. A *solution* is a k-surface Σ homeomorphic by ϕ to S, such that $\phi \circ \iota = N_\Sigma$. For instance, an equisdistant surface, as discussed in the previous paragraph, is the solution of the asymptotic Plateau problem given by the injection of a "circular" disc in $\partial_\infty\mathbb{H}^3 = \mathbb{CP}^1$. We proved in [1] that there exists at most one solution to a given asymptotic problem. Furthermore many asymptotic problems admit solutions, and in Section 6.3 we explain some of the results obtained in [1]. The general heuristic idea to keep in mind is that, most of the time, an asymptotic Plateau problem has a solution, at least as often as a Riemann surface is hyperbolic instead of being parabolic. We give three examples from [1]. In all these examples M is assumed to be a negatively curved 3-manifold with bounded geometry, for instance with a compact quotient.

THEOREM C. *If (S, ι) is an asymptotic Plateau problem such that $\partial_\infty M \setminus i(S)$ contains at least three points then (S, ι) admits a solution.*

THEOREM D. *Let Γ be a group acting on S, such that S/Γ is a compact surface of genus greater than 2. Let ρ be a representation of Γ in the isometry group of M. If ι satisfies*

$$\forall \gamma \in \Gamma, \qquad \iota \circ \gamma = \rho(\gamma) \circ \iota,$$

then (S, ι) admits a solution.

THEOREM E. *Let (U, ι) be an asymptotic Plateau problem. Let S be a relatively compact open subset of U, then (S, ι) admits a solution.*

2.4. *Dynamics of the space of k-surfaces.* The main Theorem of [1] which we quote now shows that $\mathcal{N}$, with is lamination considered as a dynamical system, enjoys the chaotic properties of the geodesic flow:

THEOREM 2.4.1. *Let $k \in]0, 1[$. Let N be a compact 3-manifold. Let h be a Riemannian metric on N with curvature less than -1. Let $\mathcal{N}_h$ be the space of k-surfaces of N. Then*

(i) *a generic leaf of $\mathcal{N}_h$ is dense,*

(ii) *for every positive number g, the union of compact leaves of $\mathcal{N}_h$ of genus greater than g is dense,*

(iii) *if $\bar{h}$ is close to h, then there exists a homeomorphism from $\mathcal{N}_h$ to $\mathcal{N}_{\bar{h}}$ sending leaves to leaves.*

This last property will be called the *stability* property.

To conclude this presentation, we show yet another point of view on this space, which will make it belong to a family of more familiar spaces. Assume N has constant curvature, and, for just a moment, let's vary k between 0 and ∞, the range for which the associated P.D.E. is elliptic.

For $k > 1$, k-surfaces are geodesic spheres. Therefore the space of k-surfaces is just the unit tangent bundle, foliated by unit spheres.

For $k = 1$, k-surfaces are either horospheres, or equidistant surfaces to a geodesic. The space of 1-surfaces is hence described the following way: first we take the S^1-bundle over the unit tangent bundle, where the fibre over u is the set of unit vectors orthogonal to u. This space is foliated by 2-dimensional leaves which are inverse images of geodesics. Then, we take the product of this space by $[0, \infty[$. The number $r \in [0, \infty[$ represents the distance to the geodesic. We now complete the space by adding horospheres, when r goes to infinity.

Our construction allows us to continue deforming k below 1. However passing through this barrier, the space of k-surfaces undergoes dramatic change; in particular, it becomes infinite dimensional and "chaotic" as we just said.

3. Transversal measures

Let N be a compact 3-manifold with curvature less than minus 1. Let $k \in]0, 1[$ be a real number. Let $\mathcal{N}$ be the space of k-surfaces of N.

3.1. *First examples.* Let us first show some simple examples of natural transversal measures on $\mathcal{N}$. The first three are ergodic. They all come from the existence of natural finite dimensional subspaces in $\mathcal{N}$.

- *Dirac measures* supported on closed leaves. By Theorem 2.4.1(ii), there are plenty of them.

- *Ergodic measures for the geodesic flow.* Indeed, ergodic and invariant measures for the geodesic flow give rise to transversal measures on the space of tubes, hence on the space of k-surfaces.

- *Haar measures for totally geodesic planes.* Assume N has constant curvature. Then, the space of oriented totally geodesic planes carries a transverse invariant measure. Indeed, the Haar measure for $\mathrm{SL}(2, \mathbb{C})/\pi_1(N)$ is invariant under the $\mathrm{SL}(2, \mathbb{R})$ action. But every oriented totally geodesic plane gives rise to a k-surface, namely the one equidistant to the geodesic plane. This way, we can construct an ergodic transversal measure on $\mathcal{N}$, when N has constant curvature. Its support is finite dimensional.

- *Measures on spaces of ramified coverings.* We sketch briefly here a construction yielding transversal, but nonergodic, measures on $\mathcal{N}$. Let $\partial_\infty M$ be the boundary at infinity of the universal cover M of N. Let Σ be an oriented surface of genus g. Let π be topological ramified covering of Σ into $\partial_\infty M$. Let S_π be the set of singular points of π and s_π its cardinal. Let S be a set of extra marked points of cardinal s. Assume $2g + s_\pi + s \geq 3$, so that the surface with $s_\pi + s$ deleted points is hyperbolic. One can show following the ideas of the proof of Theorem 7.3.3 of [1] that such a ramified covering can be represented by a k-surface. More precisely, there exists a unique solution to the asymptotic Plateau problem (as described in Paragraph 6.3) represented by $(\pi, \Sigma \setminus (S_\pi \cup S))$. To be honest, this last result is not stated as such in [1]. However, one can prove it using the ideas contained in the article. Let now $[\pi]$ be the space of ramified coverings equivalent up to homeomorphisms of the target to π, modulo homeomorphisms of Σ. More precisely, let $\mathcal{H}$ be the group of homeomorphism of $\partial_\infty M$, let $\mathcal{F}$ be the group of homeomorphism of Σ preserving the set $S \cup S_\pi$. Notice that both $\mathcal{H}$ and $\mathcal{H}$ act on $C^0(\Sigma, \partial_\infty M)$. Then

$$[\pi] = \mathcal{H}.\pi.\mathcal{F}/\mathcal{F}.$$

The group $\pi_1(N)$ acts properly on $[\pi]$, and explicit invariant measures can be obtained using equivariant families of measures (cf. Section 5.1.1) and configuration spaces of finite points. Since $[\pi]/\pi_1(N)$ is a space of leaves of $\mathcal{N}$, this yields transversal measures on this latter space.

None of these examples has full support, and they all have finite dimensional support. So far, apart from these and the construction I will present in this article, I do not know of other examples of transversal measures which are easy to construct.

3.2. *Main Theorem.* We now state our main theorem.

THEOREM 3.2.1. *Let N be a compact 3-manifold with curvature less than minus 1. Assume the metric on N can be deformed, through negatively curved metrics, to a constant curvature 1. Then the space of k-surfaces admits infinitely many mutually singular, ergodic transversal finite measures of full support.*

3.3. *First remarks.*

3.3.1. *Restriction to the constant curvature case.* The restriction upon the metric is a severe one. Actually, thanks to the stability property (iii) of Theorem 2.4.1, in order to prove our main result, it suffices to show the existence of transversal ergodic finite measures of full support in the case of constant curvature manifolds.

3.3.2. *Choices made in the construction.* The measure we construct on $\mathcal{N}$ depends on several choices, and various choices lead to mutually singular measures.

We describe now one of the crucial choice needed in the construction. Let M be the universal cover of N. Let $\partial_\infty M$ be its boundary at infinity. Let $\mathcal{P}(\partial_\infty \mathcal{M})$ be the space of probability Radon measures on $\partial_\infty M$. Let

$$O_3 = \{(x, y, z) \in \partial_\infty M^3 | x \neq y \neq z \neq x\}.$$

The construction requires a map ν, invariant under the natural action of $\pi_1(N)$,

$$O_3 \xrightarrow{\nu} \mathcal{P}(\partial_\infty M).$$

Here, $\nu(x,y,z)$ is assumed to be of full support, and to fall in the same measure class, independently of (x,y,z). Such maps are easily obtained through *equivariant families of measures* (also described in F. Ledrappier's article [5] as *Gibbs current*, *crossratios etc.*) and a barycentric construction as shown in Paragraph 5.1.

3.4. *Strategy of proof.* As we said in the introduction, the construction is obtained through a coding of the space of k-surfaces. We give now a heuristic, nonrigorous, outline of the proof, which is completed in the last section.

From the stability property, we can assume N has constant curvature. Our first step (§6) is to associate to (almost) every k-surface a locally convex pleated surface, analogous to a "convex core boundary". It turns out that this way we can describe a dense subset of k-surfaces, by locally convex pleated surfaces, and in particular by their *pleating loci* at infinity. Such pleating loci are described as special maps from $\mathbb{QP}^1$ to $\mathbb{CP}^1$. This is the aim of Sections 5 and 6. Identifying $\mathbb{QP}^1$ with the space of connected components of $\mathbb{H}^2$ minus a trivalent tree, we build invariant measures on this space of maps as projective limits of measures on finite configuration spaces of points on $\mathbb{CP}^1$. This is done in Section 4.

3.5. *Comments and questions.*

3.5.1. *General negatively curved* 3*-manifolds.* As we have seen before, the construction only works in the case of constant curvature manifolds, extending to other cases through the stability. Of course, it would be more pleasant to obtain transversal measures without any restriction on the metric. Some parts of the construction do not require any hypothesis on the metric, and we tried to keep, sometimes at the price of slightly longer proof, the proof as general as possible.

3.5.2. *Equidistribution of closed leaves.* Keeping in mind the analogy with the geodesic flow and the construction of the Bowen-Margulis measure, we have a completely different attempt to exhibit transversal measures, without any initial assumption on the metric. Define the *H-area* of a k-surface to be the integral of its mean curvature. It is not difficult to show that for any real number A, the number $N(A)$ of k-surfaces in $\mathcal{N}$ of H-area less than A is bounded. Starting from this fact, one would like to know if closed leaves are equidistributed in some sense, i.e. that some average μ_n of measures supported on closed leaves of area less than n weakly converges as n goes to infinity. We can be more specific and ask about closed leaves of a given genus, or closed leaves whose π_1 surjects onto a given group. This is a whole range of questions on which I am afraid to say I have no hint of answer. However, the constructions in this article should be related to equidistribution of ramified coverings of the boundary at infinity by spheres.

4. A combinatorial model

In general, $\mathcal{P}(X)$ will denote the space of probability Radon measures on the topological space X, $\delta_x \in \mathcal{P}(X)$ will be the Dirac measure concentrated at $x \in X$, and $\mathbb{I}_S$ will be the characteristic function of the set S.

In this section, we shall describe *restricted infinite configuration spaces* (4.0.3), which are, roughly speaking, spaces of infinite sets of points on a topological space W, associated to *configuration data* (4.1). Our main result is Theorem 4.2.1 which defines invariant ergodic measures of full support on these spaces, starting from measures defined on configuration data as in 4.1.2. One may think of these restricted infinite configuration spaces as analogues of subshifts of finite type, where the analogue of the Bernoulli shift is the space of maps of $\mathbb{QP}^1$ (instead of $\mathbb{Z}$) into a space W with the induced action of $\mathrm{PSL}(2,\mathbb{Z})$. We call this latter space the infinite configuration space as described in the first paragraph, as well as related notions. The role of the *configuration data* is that of local transition rules.

4.0.1. *The trivalent tree.* We consider the infinite trivalent tree T, with a fixed cyclic ordering on the set of edges stemming from any vertex. Alternatively we can think of this ordering as defining a proper embedding of the tree in the real plane $\mathbb{R}^2$, such that the cyclic ordering agrees with the orientation. Another useful picture to keep in mind is to consider the periodic tiling of the hyperbolic plane $\mathbb{H}^2$ by ideal triangles, and our tree is the dual to this picture (Figure 1). The group F of symmetries of that picture, which we abusively call the *ideal triangle group*, acts transitively on the set of vertices. It is isomorphic to $F = \mathbb{Z}_2 * \mathbb{Z}_3 = \mathrm{PSL}(2,\mathbb{Z})$.

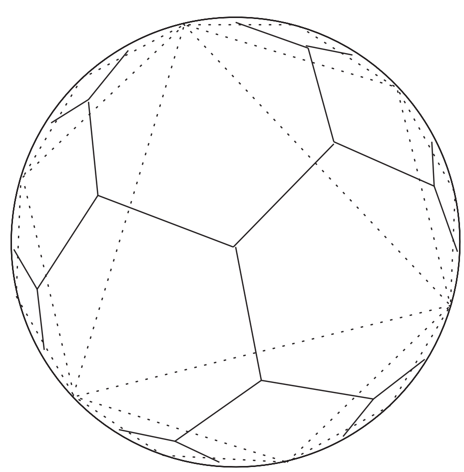

Figure 1: The infinite trivalent tree dual to the ideal triangulation

We now consider the set B of connected components of $\mathbb{H}^2 \setminus T$. In our tiling picture this set B is in one-to-one correspondence with the set of vertices of triangles, and it follows that the ideal triangle group F acts also transitively on B. Actually B can be identified with $\mathbb{QP}^1$ and this identification agrees with the action of $\mathrm{PSL}(2,\mathbb{Z})$.

4.0.2. *Quadribones, tribones.* Every edge of T defines a set of four points in B, namely the connected components of $\mathbb{H}^2 \setminus T$ that touch this edge; we shall call these particular sets *quadribones*. We consider this set as an oriented set, i.e. up to signature 1 permutations, as labelled in Figure 2. Also, every vertex of the tree defines special subsets of three points in B, that we shall call *tribones*. Obviously every quadribone contains two tribones corresponding to the extremities of the corresponding edge, and again these quadribones are oriented sets. When our quadribone is given by (a, b, c, d) the two corresponding tribones are (a, b, c) and (d, c, b).

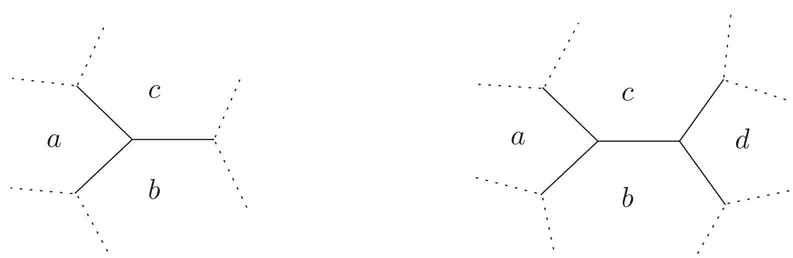

Figure 2: tribone (a, b, c) and quadribone (a, b, c, d)

4.0.3. *Infinite configuration spaces.* We define the *infinite configuration space* of W to be the space, denoted $\mathcal{B}_\infty$, of maps from B to W.

Notice that every tribone t (resp. quadribone q) of B defines a natural map from $\mathcal{B}_\infty$ to W^3 (resp. W^4) given respectively by $f \mapsto f(t)$ and $f \mapsto f(q)$; we call these maps *associated maps to the tribone* t (resp. *to the quadribone* q).

4.1. *Local rules.* For the construction of our combinatorial model, we need the following definitions.

4.1.1. *Configuration data.* We shall say that (W, Γ, O_3, O_4) defines (3,4)-*configuration data* if:

(a) W is a metrisable topological space;

(b) Γ is a discrete group acting continuously on W.

We deduce from that a (diagonal) action of Γ on W^n which commutes with the action of the n^{th}-symmetric group σ_n. Let σ_n^+ be the subgroup of σ_n of signature $+1$. Let $\lambda_3 = \sigma_3^+$, and $\lambda_4 \subset \sigma_4^+$ be the subgroup generated by $(a, b, c, d) \mapsto (d, c, b, a)$. Let $\Delta_n = \{(x_1, \ldots, x_n) | \exists i \neq j,\ x_i = x_j\}$. Assume furthermore that:

(c) O_n are open $\lambda_n \times \Gamma$-invariant subsets of $W^n \backslash \Delta_n$, on which Γ acts properly.

(d) $p(O_4) = O_3$, where p is the projection from W^4 to W^3 defined by

$$(a, b, c, d) \mapsto (a, b, c).$$

We shall also say configuration data are *Markov* if they satisfy the following extra hypothesis:

(e) There exists some constant $p \in \mathbb{N}$, such that if (a, b, c) and (d, e, f) both belong to O_3, then there exists a sequence $(q_1, \ldots, q_j)$ of elements of O_4, where $j \leq p$ and $q_n = (q_n^1, q_n^2, q_n^3, q_n^4)$, satisfying:
   - $(q_1^1, q_1^2, q_1^3) = (a, b, c)$;
   - $(q_j^2, q_j^3, q_j^4) = (d, e, f)$;
   - $(q_n^2, q_n^4, q_n^3) = (q_{n+1}^1, q_{n+1}^2, q_{n+1}^3)$ or $(q_n^3, q_n^2, q_n^4) = (q_{n+1}^1, q_{n+1}^2, q_{n+1}^3)$.

In 4.3.4, this property will have a geometric consequence.

4.1.2. *Measured configuration data.* Our next goal is to associate measures to this situation. We shall say $(W, \Gamma, O_3, O_4, \mu^3, \mu^4)$ is a (3,4)-*measured configuration data* if:

(f) μ^n are $\lambda_n \times \Gamma$-invariant measures, such that $p_* \mu^4 = \mu^3$.

(g) The pushforward measures on O_n/Γ are probability measures.

We shall say that the measured configuration data are *regular* if they satisfy the following extra condition:

(h) The measure μ_4 is in the measure class of $\mathbb{I}_{O_4} m \otimes m \otimes m \otimes m$ where m is of full support in W. It follows that μ_3 is in the measure class of $\mathbb{I}_{O_3} m \otimes m \otimes m$.

We also say two regular measured configuration data $(W, \Gamma, O_3, O_4, \mu^3, \mu^4)$ and $(W, \Gamma, O_3, O_4, \bar{\mu}^3, \bar{\mu}^4)$, defined on the same configuration data, are *mutually singular* if μ^3 and $\bar{\mu}^3$ are mutually singular.

4.1.3. *Remarks.* (i) From disintegration of measures, it follows from the hypotheses (f) and (g) that for μ^3-almost every triple of points (a, b, c)

in W, we have a probability measure $\nu_{(a,b,c)}$ on W such that for every positive measurable function f on W^4:

$$\int_{W^4} f(a,b,c,d)d\mu^4(a,b,c,d) = \int_{W^3}\left(\int_W f(a,b,c,d)d\nu_{(a,b,c)}(d)\right)d\mu^3(a,b,c)$$

which we can rewrite as

$$d\mu^4 = \int_{W^3}\left(\delta_{(a,b,c)} \otimes d\nu_{(a,b,c)}\right)d\mu^3(a,b,c).$$

(ii) Conversely, there is a way to build regular measured configuration data starting from configuration data (W, Γ, O_3, O_4), *if we assume that O_4 is invariant under σ_4^+.*

Assume we have:

- a Γ-invariant measure $\bar{\mu}^3$ on W^3 in the measure class of $\mathbb{I}_{O_3} m \otimes m \otimes m$ where m has full support, such that the pushforward on O_3/Γ is a probability measure;
- a Γ-equivariant map $\bar{\nu}$:

  $$\begin{cases} O_3 & \to \mathcal{P}_m(W) \\ (a,b,c) & \mapsto \bar{\nu}_{(a,b,c)} \end{cases}$$

  where $\mathcal{P}_m(W)$ is the set of finite Radon measures on W in the measure class of m.

Then, we can build μ^3 and μ^4 which will fulfil the requirements of the definition. Let us describe the procedure:

Firstly, we define a probability measure $\bar{\mu}^4$ on O_4 to be proportional to

$$\mathbb{I}_{O_4}\int_{W^3}\left(\delta_{(a,b,c)} \otimes \bar{\nu}_{(a,b,c)}\right)d\bar{\mu}^3(a,b,c).$$

Secondly, we average $\bar{\mu}^4$ using the group σ_4^+ and obtain a finite measure μ^4 on O_4/Γ, and we ultimately take $\mu^3 = p_*\mu^4$.

It is routine now that μ^3 and μ^4 defined this way satisfy our needs.

Furthermore, if $\bar{\mu}^3$ has full support in O_3 as well as $\nu_{(a,b,c)}$ for $\bar{\mu}^3$-almost every (a,b,c) in W_3, then μ^3 and μ^4 have full support.

4.1.4. *Example.* In the sequel, we only wish to study one example that we describe briefly now and more precisely in Section 5. Our specific interest lies in the following situation.

- Γ is a cocompact discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$;
- $W = \mathbb{CP}^1$ with the canonical action of Γ; it is a well-known fact that Γ acts properly on

  $$U_n = \{(x_1, \dots, x_n) \in (\mathbb{CP}^1)^n |\ x_i \neq x_j \text{ if } i \neq j\}.$$

  Actually Γ acts properly on U_3.

- $O_3 = U_3$,

- O_4 is the set of points whose crossratios have a nonzero imaginary part; it will satisfy hypothesis (e) for $N = 1000$ (cf. 5.2).

This is Markov configuration data and furthermore in this specific situation O_4 is invariant under σ_4^+. We will explain in subsection 5.1 how to attach measures to this situation, and discuss the case of general negatively curved 3-manifolds.

4.1.5. *Final remark.* Even though we only wish to study this specific class of examples, it is a little more comfortable to work in a more general setting, since very little of the geometry is used at this stage.

4.2. *Restricted infinite configuration spaces and the main result.* Let now (W, Γ, O_i) be (3,4)-configuration data (cf. 4.1).

We define the *restricted infinite configuration space* of W to be the subset $\bar{\mathcal{B}}_\infty$ of $\mathcal{B}_\infty$, consisting of those maps such that the image of every tribone lies in O_3, and the image of every quadribone is in O_4.

$$\bar{\mathcal{B}}_\infty = \{f \in \mathcal{B}_\infty \,|\, \text{for all tribone } t, \text{quadribone } q, f(t) \in O_3, f(q) \in O_4\}.$$

Let also $\mathcal{B}^0_\infty$ be the open set of the infinite configuration space such that the image of at least one tribone lies in O_3. Let us call this subset the *nondegenerate configuration space*, and notice that Γ acts properly on this open subset of $\mathcal{B}_\infty$.

Now we can state the theorem we wish to prove:

THEOREM 4.2.1. *Let (W, Γ, O_i, μ^i) be $(3,4)$-measured configuration data. Then there exists a Γ-invariant measure μ on the infinite configuration space of W, which is invariant by the action of the ideal triangle group, such that:*

(i) *The restricted infinite configuration space $\bar{\mathcal{B}}_\infty$ is of full measure and μ has full support on it provided the data are regular;*

(ii) *The pushforward of μ on $\mathcal{B}^0_\infty/\Gamma$ is finite, where $\mathcal{B}^0_\infty$ is the nondegenerate infinite configuration space;*

(iii) *Given any tribone or quadribone, the pushforward of μ by the associated maps on W^3 and W^4 is our original μ^3, μ^4;*

(iv) *Two regular, mutually singular, measured configuration data give rise to mutually singular measures;*

(v) *If the configuration data are Markov and regular, then the pushforward of μ on $\mathcal{B}^0_\infty/\Gamma$ is ergodic with respect to the action of the infinite triangle group.*

Essentially, this measure is built by a Markov-type procedure.

4.3. *Construction of the measure.* Let (W, Γ, O_i, μ^i) be (3,4)-configuration data. We shall use the notation and definitions of the preceding sections.

Also in our constructions, for every $(x, y, z) \in O_3$, we shall denote by $\nu_{(x,y,z)}$ the probability measure coming from the disintegration of μ^4 over μ^3 as defined in 4.1.3.

4.3.1. *Connected sets*, *P-bones*, *P-disconnected sets.* For our constructions, we require terminology for some subsets of B which roughly corresponds to certain subtrees of T.

A subset A of B will be called *connected* if it is a union of quadribones such that the union $e(A)$ of the associated edges is connected; if v is a vertex, it will be called *v-connected* if furthermore $e(A)$ contains v. In other words a connected subset of B is the union of the connected components of $\mathbb{H}^2 \setminus T$ touching the edges of a connected subtree of T.

A subset A of B will be called a *P-bone* if it is connected and the union of fewer than P quadribones; two subsets A and C will be called *P-disconnected* if there is no P-bone which intersects both A and C.

4.3.2. *Relative configuration spaces.* If A is a subset of B, we shall denote:

- $\mathcal{W}(A)$ the set of maps from A to W; in particular, $\mathcal{W}(B) = \mathcal{B}_\infty$.

- $\bar{\mathcal{W}}(A)$ the set of maps such that the image of every tribone of A lies in O_3, and the image of every quadribone is in O_4; if A is finite, $\bar{\mathcal{W}}(A)$ is an open set on which Γ acts properly. Again, $\bar{\mathcal{W}}(B) = \bar{\mathcal{B}}_\infty$.

4.3.3. *Finite construction.* We can now prove:

Proposition 4.3.1. *Let A be a finite v_0-connected subset of B. Then, there exists a Radon measure μ^{A,v_0} on $\bar{\mathcal{W}}(A)$ enjoying the following properties*:

(i) *The pushforward of μ^{A,v_0} on $\bar{\mathcal{W}}(A)/\Gamma$ is finite*; *it is of full support if the data are regular*;

(ii) *Let t_0 be the tribone corresponding to the vertex v_0*; *also let t_0 be the associated map from A to W^3*; *then $t_* \mu^{A,v_0} = \mu^3$.*

(iii) *Let q be a v_0-connected quadribone*; *assume $q \subset A$*; *let q be the associated map from A to W^4*; *then $q_* \mu^{A,v_0} = \mu^4$.*

(iv) *Assume there exist a tribone $t \subset A$, some element $a \in B \setminus A$, such that $q = t \cup \{a\}$ is a quadribone*; *let now $C = A \cup \{a\}$ and identify $\mathcal{W}(C)$ with*

$\mathcal{W}(A) \times W$; *then*

$$\mu^{C,v_0} = \int_{\mathcal{W}(A)} \big(\delta_f \otimes \nu_{f(t)}\big) d\mu^{A,v_0}(f).$$

(v) *Let* $A \subset C$; *let* p *be the natural restriction from* $\bar{\mathcal{W}}(C)$ *to* $\bar{\mathcal{W}}(A)$. *Then* $p_*\mu^{C,v_0} = \mu^{A,v_0}$.

(vi) *If* (μ^3, μ^4) *and* $(\bar{\mu}^3, \bar{\mu}^4)$ *are regular and mutually singular, then the corresponding measures* μ^{A,v_0} *and* $\bar{\mu}^{A,v_0}$ *are mutually singular.*

One should notice that the listed properties define μ^{A,v_0} uniquely. We shall also say in the sequel that if C and A are as in (iv), that C is obtained from A by *gluing a quadribone along a tribone*, as in Figure 3.

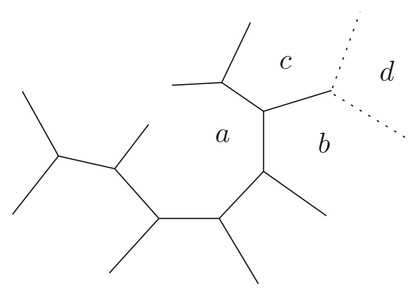

Figure 3: Gluing a quadribone (a, b, c, d) along a tribone (a, b, c)

We have a useful consequence of the previous proposition:

Corollary 4.3.2. *Let* A *be a finite set and let* v *and* w, *such that* A *is both* v-*connected and* w-*connected*; *then* $\mu^{A,v} = \mu^{A,w}$.

Now of course, we may write $\mu^A = \mu^{A,v}$.

Our last proposition exhibits some kind of "Markovian" property of our measure.

Proposition 4.3.3. *Assume the configuration data are Markov and regular. There exists an integer* P, *such that if* A_0 *and* A_1 *are two* P-*disconnected subsets of a finite set* $C \subset B$, *then* $(p^0, p^1)_*\mu^C$ *and* $p^0_*\mu^C \otimes p^1_*\mu^C$ *are in the same measure class. Here,* $p^i : \mathcal{W}(C) \to \mathcal{W}(A_i)$ *are the natural restriction maps.*

We will now prove the results stated in this section.

4.3.4. *Proof of Proposition* 4.3.1. We introduce first some notation with respect to a vertex v. By definition $B_n(v)$ will denote the union of all v-connected n-bones; also, for any subset A of B, we put $A_n(v) = B_n(v) \cap A$.

For the moment, we will work with a fixed v_0 and will omit the dependence in v_0 in the notation for the sake of simplicity; in particular $A_n = A_n(v_0)$. We will construct this measure by an induction procedure.

Our first task is to build for every $n \in \mathbb{N}$, a map:

$$\nu^{A,n} : \begin{cases} \bar{\mathcal{W}}(A_n) & \rightarrow & \mathcal{P}\big(\mathcal{W}(A_{n+1} \setminus A_n)\big) \\ f & \mapsto & \nu_f^{A,n}. \end{cases}$$

Let us do it. If $a \in A_{n+1} \setminus A_n$, it belongs to a unique quadribone $q_a \subset A_{n+1}$. Let $t_a = q_a \setminus \{a\}$; notice that t_a is a subset of A_n. Let $A_{n+1} \setminus A_n = \{a_1, \dots, a_q\}$. In particular, $\mathcal{W}(A_{n+1} \setminus A_n)$ is identified with W^q. Let $T_n^A = \cup_{i=1}^{i=q} t_{a_i}$. We have a natural restriction map

$$i^{A,n} : \bar{\mathcal{W}}(A_n) \longrightarrow \bar{\mathcal{W}}(T_n^A),$$

and define

$$\bar{\nu}^{A,n} : \begin{cases} \bar{\mathcal{W}}(T_n^A) & \rightarrow & \mathcal{P}(W^q) = \mathcal{P}\big(\mathcal{W}(A_{n+1} \setminus A_n)\big) \\ f & \mapsto & \bigotimes_i \nu_{f(t_i)}. \end{cases}$$

Finally, we set: $\nu^{A,n} = \bar{\nu}^{A,n} \circ i^{A,n}$.

Next, notice the following fact. Let $f \in \bar{\mathcal{W}}(A_n)$ and $\bar{\mathcal{W}}_f(A_{n+1})$ be the fibre, over f, of the restriction map. We use the identification

$$\mathcal{W}(A_{n+1}) = \mathcal{W}(A_{n+1} \setminus A_n) \times \mathcal{W}(A_n).$$

Then, $\bar{\mathcal{W}}_f(A_{n+1})$ has full measure for $\nu_f^{A,n} \otimes \delta_f$.

We can now define our measure on $\bar{\mathcal{W}}(A_{n+1})$ by an induction procedure:

- $\bar{\mathcal{W}}(A_0)$ is identified with O_3 using t_0; we define $\mu^{A_0} = (t_0^{-1})_* \mu^3$;
- Assuming by induction that μ^{A_n} is defined on $\mathcal{W}(A_n)$ such that $\bar{\mathcal{W}}(A_n)$ has full measure, we set

$$\mu^{A,n+1} = \int_{\bar{\mathcal{W}}(A_n)} \big(\nu_f^{A,n} \otimes \delta_f\big) d\mu^{A,n}(f).$$

From the previous observation, we deduce that $\bar{\mathcal{W}}(A_{n+1})$ has full measure. Furthermore, if the μ^i have full support, then $\mu^{A,n+1}$ has full support.

Finally, there exists $p \in \mathbb{N}$ such that $A = A_p$, and

$$\mu^{A,v_0} = \mu^{A,p}.$$

Properties (i), (ii), (iii), and (vi) are immediately checked. Let us prove property (iv).

Notice first that a lies in exactly one quadribone q of C. Let d be the unique tribone of C that contains a. Then, there exists p_0 such that

$$C_p = A_p \text{ for } p < p_0,$$
$$C_p = A_p \cup \{a\} \text{ for } p \geq p_0.$$

By construction, using the obvious identifications, we have

$$(*)\qquad \begin{aligned} \mu^{C,p} &= \mu^{A,p}, \quad \text{for } p < p_0, \\ \mu^{C,p} &= \textstyle\int_{\mathcal{W}(A_p)} (\delta_f \otimes \nu_{f(q\setminus a)}) d\mu^{A,p}(f), \quad \text{for } p = p_0. \end{aligned}$$

To conclude the proof of (iv), it remains to prove $(*)$ for $p > p_0$. By induction, this follows from the fact that, for $p > p_0$, $T_p^A = T_p^C$. We check this step by step. By definition,

$$\mu^{C,p+1} = \int_{\mathcal{W}(C_p)} \big(\nu_{f(T_p)} \otimes \delta_f\big) d\mu^{C,p}(f).$$

But, by induction

$$\mu^{C,p} = \int_{\mathcal{W}(A_p)} (\delta_g \otimes \nu_{g(q\setminus a)}) d\mu^{A,p}(g).$$

Combining the two last equalities, and using $T_p^A = T_p^C$, we get

$$\begin{aligned} \mu^{C,p+1} &= \int_{\mathcal{W}(A_p)} (\delta_g \otimes \nu_{g(q\setminus a)} \otimes \nu_{g(T_p^A)}) d\mu^{A,p}(g) \\ &= \int_{\mathcal{W}(A_{p+1})} (\delta_g \otimes \nu_{g(q\setminus a)}) d\mu^{A,p+1}(g). \end{aligned}$$

This is what we wanted to prove.

Property (v) is an immediate consequence of (iv). Indeed, if C contains A, it is obtained inductively from A by "gluing quadribones along tribones" as in (v). □

4.3.5. *Proof of Corollary* 4.3.2. Obviously, it suffices to prove this whenever v and w are the extremities of a common edge e. Let q be the associated quadribone. Since we can build A from q by successively "gluing quadribones along tribones", using property (v) of Proposition 4.3.1, it suffices to show that

$$\mu^{q,v} = \mu^{q,w}.$$

Thanks to Proposition 4.3.1 (iii), this follows from the invariance of μ^4 under the permutation $(a, b, c, d) \mapsto (d, c, b, a)$. □

4.3.6. *A consequence of hypothesis* (e) *of* 4.1. Using the previous notation, we have:

PROPOSITION 4.3.4. *Assume the configuration data are Markov. Then, there exists an integer P, such that if A_0 and A_1 are connected and P-disconnected, and if C is a connected set that contains both, then*

$$(p^0, p^1)(\bar{\mathcal{W}}(C)) = \bar{\mathcal{W}}(A_0) \times \bar{\mathcal{W}}(A_1).$$

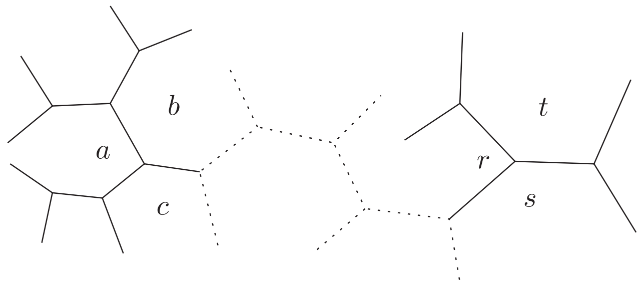

Figure 4: $t_0 = (a, b, c)$, $t_1 = (r, s, t)$

Proof. Let A_0 and A_1 be two P-disconnected subsets. Then there exists an N-bone K, where $N > P$, such that K intersects each A_i exactly along one tribone t_i as in Figure 4. Let $D = A_0 \cup K \cup A_1$.

Let f_0 (resp. f_1) be an element of $\bar{\mathcal{W}}(A_0)$ (resp. $\bar{\mathcal{W}}(A_1)$). Hypothesis (e) of 4.1 implies there exists some element g of $\bar{\mathcal{W}}(K)$ such that g coincides with f_0 (resp. f_1) on t_0 (resp. t_1) provided N is greater than p. Gluing together g and the f_i, we obtain an element h of $\bar{\mathcal{W}}(D)$, whose restriction to A_i is f_i. In other words, the restriction from $\bar{\mathcal{W}}(D)$ to $\bar{\mathcal{W}}(A_0) \times \bar{\mathcal{W}}(A_1)$ is surjective. To conclude, its suffices to notice that since D is connected, the restriction from $\bar{\mathcal{W}}(C)$ to $\bar{\mathcal{W}}(D)$ is surjective. □

4.3.7. *Proof of Proposition* 4.3.3. The first point to notice is that if μ^3 and μ^4 are in the measure class of $\mathbb{I}_{O_3} m \otimes m \otimes m$ and $\mathbb{I}_{O_4} m \otimes m \otimes m \otimes m$ respectively, then for m-almost every tribone t, ν_t is also in the measure class of $\mathbb{I}_{O_t} m$ where O_t is such that $\{t\} \times O_t = p^{-1}(t) \cap O_4$. It follows that if A is connected then μ^A is actually in the measure class of

$$\mathbb{I}_{\bar{\mathcal{W}}(A)} m^{\otimes \# A}.$$

We prove this last assertion by induction: let C_n, $1 \leq n \leq p$ be an increasing sequence of sets such that C_1 is quadribone, $C_p = A$ and C_{n+1} is obtained from C_n by gluing a quadribone along a tribone t_n as in Proposition 4.3.1(iv). Then

$$\bar{\mathcal{W}}(C_{n+1}) = \bigcup_{f \in \bar{\mathcal{W}}(C_n)} \{f\} \times O_{f(t_n)},$$

where we identified $\mathcal{W}(C_{n+1})$ and $\mathcal{W}(C_n) \times W$. An inductive use of 4.3.1(iv) implies our statement.

Assume now the configuration data are Markov. Then according to Proposition 4.3.4,

$$(p^0, p^1)(\bar{\mathcal{W}}(C)) = \bar{\mathcal{W}}(A_0) \times \bar{\mathcal{W}}(A_1).$$

Hence Proposition 4.3.3 is proved. □

4.3.8. *Infinite construction, and proof of properties* (i), ... ,(iv) *of Theorem* 4.2.1. We first define a measure μ on $\mathcal{B}_\infty$. We consider as before the set $B_n = B_n(v_0)$, and put $\mu_n = \mu^{B_n}$.

The set $\mathcal{B}_\infty$ equipped with the product topology is the projective limit of the sequence $(\mathcal{W}(B_n)$. We define μ as the projective limit of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. If μ^3, μ^4 have full support on O_3 and O_4 respectively, then the measure μ_n has full support on $\bar{\mathcal{W}}(B_n)$. It follows that μ has full support on $\bar{\mathcal{B}}_\infty$.

The only nonimmediate property of μ is the invariance under the ideal triangle group $\mathrm{PSL}(2,\mathbb{Z})$.

Notice first that if g belongs to the stabiliser of the vertex v_0, then $g_*\mu_n = \mu_n$: this follows from the invariance of μ^3 under cyclic permutations, and from property (iv) of Proposition 4.3.1.

Then, because of the symmetries of T and the uniqueness of our construction, we have that if $g \in F$, $g_*\mu^{A,v} = \mu^{g(A),g(v)}$, and therefore $g_*\mu^A = \mu^{g(A)}$, because of Corollary 4.3.2.

It follows that $g_*\mu$ is the projective limit measure of the projective limit of $\{\mathcal{W}(g(B_n)\}_{n\in\mathbb{N}}$, which is also $\mathcal{B}_\infty$.

To conclude, we just have to remark that, thanks to (v) of Proposition 4.3.1, for whatever sequence of finite v-connected set $\{D_n\}_{n\in\mathbb{N}}$ in B, such that $D_{n+1} \subset D_n$ and $\cup_n D_n = B$, the projective limit measure associated with the sequence of $\{\mu^{D_n}\}_{n\in\mathbb{N}}$ coincides with μ. □

4.4. *Ergodicity.* We shall now prove property (vi) of Theorem 4.2.1. We first introduce some definitions.

4.4.1. *Hyperbolic elements, pseudo-Markov measure.* Let $F = \mathrm{PSL}(2,\mathbb{Z})$ be the ideal triangle group, which we consider embedded in the isometry group of the Poincaré disk. We shall say $\gamma \in F$ is *hyperbolic*, if γ is a hyperbolic isometry. Notice that since F is Zariski dense, it contains many hyperbolic elements.

We also say a measure on $\mathcal{B}^0_\infty/\Gamma$ is *pseudo-Markov* if it satisfies the following property: There exists an integer P, such that for any P-disconnected and connected subsets A and C in B, if p_A and p_B are the associated projections, then $p^A_*\mu \otimes p^B_*\mu$ and $(p^A, p^B)_*\mu$ are in the same measure class. By Proposition 4.3.3, the measure we constructed in the last section enjoys that property.

4.4.2. *Main result.* To conclude it suffices to prove:

Proposition 4.4.1. *Let μ be an F-invariant finite measure on $\mathcal{B}^0_\infty/\Gamma$, which is the pushforward of a pseudo-Markov measure. Then μ is ergodic for the action of any hyperbolic element of F, hence ergodic for F itself.*

The proof is closely related to the proof of the ergodicity of subshifts of finite type, and is an avatar of Hopf's argument. We introduce *stable* and *unstable* leaves in 4.4.4, using *vanishing sequences* of sets defined in 4.4.3. We finally conclude using the Birkhoff ergodic theorem.

4.4.3. *Hyperbolic elements of F.* When X is a topological space and $\gamma \in C^0(X, X)$, we shall say for short that a sequence of nonempty subsets $\{V_n\}_{n\in\mathbb{N}}$ is a *vanishing sequence* for γ if:

(i) $V_{n+1} \subset V_n$;

(ii) $\bigcap_{n\in\mathbb{N}} V_n = \emptyset$;

(ii) For all compact subsets K of X, and $n \in \mathbb{N}$, there exists $p \in \mathbb{N}$, such that $\gamma^p(K) \subset V_n$.

LEMMA 4.4.2. *Let γ be a hyperbolic element in F. Then, there exist two families of connected subsets of B, $\{U_n^+\}_{n\in\mathbb{N}}$ and $\{U_n^-\}_{n\in\mathbb{N}}$, which are respectively vanishing sequences for γ and for γ^{-1}, such that $U_0^+ \cap U_0^- = \emptyset$.*

Proof. This is a consequence of elementary hyperbolic geometry. Indeed, if we consider F as a subgroup of the hyperbolic plane, the fixed points of γ on the boundary at infinity are not vertices of the tiling by ideal triangles, and the lemma follows. □

4.4.4. *Contractions.* Let now γ, $\{U_n^\pm\}_{n\in\mathbb{N}}$ be as in Lemma 4.4.2. We first introduce equivalence relations among elements of $\mathcal{B}^0_\infty$. We say $f \sim_n^+ g$, if $f|_{U_n^+} = g|_{U_n^+}$. If $f \in \mathcal{B}^0_\infty$, let $\mathcal{F}_n^+(f)$ be the equivalence class of f. Finally define $f \sim^+ g$, if there exists n such that $f \sim_n^+ g$ and $\mathcal{F}^+(f)$ is the equivalence class of f. Observe that

$$\mathcal{F}^+(f) = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n^+(f),$$

and define $\sim^-$, and $\mathcal{F}^-(f)$ in a symmetric way. These equivalence classes are going to play the role of the stable and unstable leaves of hyperbolic systems.

We shall prove:

PROPOSITION 4.4.3. *There exists a Γ-invariant metric on $\mathcal{B}^0_\infty$ inducing the natural topology, such that for all $f \in \mathcal{B}^0_\infty$ and $g \in \mathcal{F}^+(f)$,*

$$\lim_{p\to+\infty} d(\gamma^p(f), \gamma^p(g)) = 0,$$

and similarly if $g \in \mathcal{F}^-(f)$ then

$$\lim_{p\to+\infty} d(\gamma^{-p}(f), \gamma^{-p}(g)) = 0.$$

Proof. We first define a metric on $\mathcal{B}^0_\infty$ depending on the choice of a vertex v_0 of the tree T. Let $B_n \subset B$ be defined as in 4.3.4. Let $\mathcal{T}_n$ be the set of tribones of B_n and let t be a tribone; then

- let $\mathcal{B}^t_\infty$ be the set of maps from B to W, such that the image of t lies in O^3;

- if $t \in \mathcal{T}_n$, let $\mathcal{B}^t_n$ be the set of maps from B_n to W, such that the image of t lies in O^3; notice that Γ acts properly on $\mathcal{B}^t_n$.

Next,

- let δ^t_n be a Γ-invariant distance of diameter less than 1 on $\mathcal{B}^t_n$ which induces the product topology;

- let d^t_n be the semi-distance on $\mathcal{B}^t_\infty$, induced from δ^t_n by the canonical projection; notice that the product topology of $\mathcal{B}^t_\infty$ is induced by the family of semi-distances $\{d^t_n\}_{n\in\mathbb{N},t\in\mathcal{T}_n}$.

By definition,

$$\mathcal{B}^0_\infty = \bigcup_{\text{tribones } t} \mathcal{B}^t_\infty.$$

If $t \in \mathcal{T}_n$, we extend d^t_n to $\mathcal{B}^0$ in the following way:

$$\begin{cases} d^t_n(x,y) \ = \ 0, & \text{if } x,y \in \mathcal{B}^0_\infty \setminus \mathcal{B}^t_\infty, \\ d^t_n(x,y) \ = \ 1, & \text{if } y \notin \mathcal{B}^t_\infty, x \in \mathcal{B}^t_\infty. \end{cases}$$

Ultimately, we define a Γ-invariant metric d on $\mathcal{B}^0_\infty$, by the formula

$$d(x,y) = \sum_{n\in\ltimes} \frac{1}{2^n} \sum_{t\in\mathcal{T}_n} \frac{1}{\#\mathcal{T}_n} d^t_n(x,y).$$

By construction of this distance, if f and g coincide on B_n then $d(f,g) \leq (\frac{1}{2})^{n-1}$. In particular, since

$$\forall q,n \in \mathbb{N}\ , \exists p \in \mathbb{N} \ \text{ such that } \ \gamma_{p_n}(B_n) \subset U_q,$$

it follows that for every n, if $f \sim^+_q g$, then there exists $p \in \mathbb{N}$, such that

$$d(\gamma^p(f), \gamma^p(g)) \leq (\frac{1}{2})^{n-1}.$$

This ends the proof of the proposition. $\square$

4.4.5. *Preliminary steps for proof of ergodicity.* Define for every bounded function ϕ on $\mathcal{B}^0_\infty$

$$\phi^+ = \limsup_{n\to+\infty}(\phi \circ \gamma^n),$$

and

$$\phi^- = \limsup_{n\to+\infty}(\phi \circ \gamma^{-n}).$$

We first prove:

Proposition 4.4.4. *Let ϕ be a continuous Γ-invariant function on $\mathcal{B}^0_\infty$, such that the quotient function on $\mathcal{B}^0_\infty/\Gamma$ is compactly supported. Then $f \sim^+ g$ implies $\phi^+(f) = \phi^+(g)$ and, $f \sim^- g$ implies $\phi^-(f) = \phi^-(g)$.*

Proof. Notice first that ϕ is bounded and uniformly continuous. Hence, the proposition follows at once from Proposition 4.4.3. ☐

A second preliminary step is:

Proposition 4.4.5. *Let μ be a locally finite pseudo-Markov measure of full support on $\mathcal{B}^0_\infty$. Let E be a set of μ-full measure. Then for μ-almost every f, there exists a set F_f of μ-full measure such that*

$$\forall g \in F_f, \quad \exists h \in E, \quad \text{such that } f \sim^+ h \sim^- g.$$

Proof. We should first notice that the set of equivalence classes of $\sim^+_n$ is precisely $\mathcal{W}(U^+_n)$, the space of maps from U^-_n to W. A similar statement holds for $\sim^-_n$. Fix some integer n, for which U^+_n and U^-_n are P-disconnected. Let p^+ be the natural continuous projection

$$\mathcal{B}^0_\infty \mapsto \mathcal{W}(U^+_n).$$

Define p^- a similar way. At last, let $p = (p^+, p^-)$.

If E has full measure, then $p(E)$ has full measure for $p_*\mu$. Hence by the pseudo-Markov property, it has full measure for $p^+_*\mu \otimes p^-_*\mu$.

From Fubini's theorem, we deduce there is a set of full measure A in $\mathcal{W}(U^+_n)$, such that for every $a \in A$, the set

$$V_a = \{c \in \mathcal{W}(U^-_n),\ (a, c) \in p(E)\}$$

has full measure for $p^-_*\mu$.

In particular, for every $f \in (p^+)^{-1}(A)$, the set $F_f = (p^-)^{-1}(V_{p^+(f)})$ has full measure.

Now, by construction if $f \in (p^+)^{-1}(A)$ and $g \in F_f$, then $p^-(g) = p^-(h)$, where $h \in E$ and $p^+(h) = p^+(f)$. This is exactly what we wanted to prove. ☐

4.4.6. *End of the proof of ergodicity.* In this subsction, we will prove Proposition 4.4.1. Let γ be some hyperbolic element in F. Let μ be the F-invariant measure on $\mathcal{B}^0_\infty/\Gamma$, constructed in 4.2.1. From the ergodic decomposition theorem,

$$\mu = \int_Z \nu_z d\lambda(z),$$

where for λ-almost every z in Z, ν_z is an ergodic measure for γ.

To conclude, it suffices to show that for any continuous and compactly supported function ψ on $\mathcal{B}^0_\infty/\Gamma$, and for every z and u in Z, we have $\int \psi dv_z = \int \psi dv_u$.

Let now $\phi = \psi \circ \pi$, where π is the natural projection from $\mathcal{B}^0_\infty$ to $\mathcal{B}^0_\infty/\Gamma$. Define, as for Proposition 4.4.4, the measurable functions ϕ^+ and ϕ^-. From the Birkhoff ergodic theorem, we deduce that for ν_z-almost every x, if $\pi(y) = x$,

$$(*) \qquad \phi^+(y) = \phi^-(y) = \int_{\mathcal{B}^0_\infty/\Gamma} \psi d\nu_z.$$

In particular, there exists a set of μ-full measure E on which $\phi^+ = \phi^-$.

Now, we apply Proposition 4.4.5, and deduce that for μ-almost every x, there exists a set of full measure F_x with the following property: if $b \in F_x$ then there exists $a \in E$ such that $x \sim^+ a \sim^- b$.

From Proposition 4.4.4, we deduce that $\phi^+(a) = \phi^+(x)$ and $\phi^-(b) = \phi^-(a)$. From the definition of E, we get that ϕ^- is constant and equal to ϕ^- on F_x, hence μ-almost everywhere.

Using $(*)$, we ultimately get that for almost every $z,\ u \in Z$,

$$\int_{\mathcal{B}^0_\infty/\Gamma} \psi d\nu_z = \int_{\mathcal{B}^0_\infty/\Gamma} \psi d\nu_u,$$

which is what we wanted to prove. $\square$

5. Configuration data and the boundary at infinity of a hyperbolic 3-manifold

We describe here our main, and actually unique useful example: the Markov configuration data associated to a hyperbolic 3-manifold.

Let in general $\partial_\infty M$ be the boundary at infinity of a negatively curved 3-manifold M. Let Γ be a discrete, torsion-free and cocompact group of isometries of M.

Unless otherwise specified, we shall assume M is the hyperbolic 3-space $\mathbb{H}^3$. Then, $\partial_\infty M = \partial_\infty \mathbb{H}^3$ is canonically identified with $\mathbb{CP}^1$. In this identification, the action of the group of isometries of M on $\partial_\infty M$ coincides with the action of $\mathrm{PSL}(2, \mathbb{C})$ on $\mathbb{CP}^1$.

As explained in 4.1.4, the (3,4)-configuration data we shall study are the following:

- $W = \partial_\infty M = \mathbb{CP}^1$.
- O_3 is the subset of $\partial_\infty M^3$ consisting of triples of different points:

$$O_3 = \{(x, y, z) \in \partial_\infty M \ / \ x \neq z \neq y \neq x\}.$$

- O_4 is the set of points whose cross ratios have a nonzero imaginary part;

Now, we have,

Proposition 5.0.1. *The quadruple* $(\mathbb{CP}^1, \Gamma, O_3, O_4)$ *is a Markov* (3,4)-*configuration data.*

It is obvious. The only point that requires a check is hypothesis (e). In the last paragraph 5.2, we will devise a fancy (and far too long) proof of this fact. Of course, a straightforward check would give that these configuration data satisfy (e) for $N = 10$, instead of $N = 1000$, provided by our proof. However, I hope the scheme of this proof might be useful in more general situations.

In the next subsection we explain how to turn this example into regular measured configuration data in many ways, using the *equivariant family of measures* (cf. 5.1.1).

5.1. *Measured configuration data.* In view of 4.1.3(ii) we need to produce $\bar{\mu}^3$ in the Lebesgue class of $m \otimes m \otimes m$ for some measure class m of full support, such that the pushforward of $\bar{\mu}^3$ on O_3/Γ is finite. Then we have to build a Γ-equivariant map $\bar{\nu}$:

$$\left\{ \begin{array}{cl} O_3 & \to \mathcal{P}_m(W) \\ (a,b,c) & \mapsto \bar{\nu}_{(a,b,c)} \end{array} \right.$$

where $\mathcal{P}_m(W)$ is the set of finite Radon measures on W in the measure class of m.

We shall do this using the notion of the equivariant family of measures described by F. Ledrappier in [5], which is a generalisation of work on conformal densities due to D. Sullivan [6].

5.1.1. *An equivariant family of measures.* An *equivariant family of measures on the boundary* is a map μ which associates to every $x \in M$ a finite measure μ_x on $\partial_\infty M$ such that:

(i) For all γ in Γ, $\mu_{\gamma x} = \gamma_* \mu_x$.

(ii) For all $x, y \in M$, μ_x and μ_y are in the same Lebesgue class.

In particular we can write $d\mu_x(a) = e^{-\gamma_a(x,y)} d\mu_y(a)$. Actually, the original definition requires some regularity of the function $(a, x, y) \mapsto \gamma_a(x, y)$, which we shall not need in the sequel.

A typical example arises when one associates to a point x the pushforward by the exponential map of the Liouville measure on the unit sphere at x.

When $c_\eta(x, y) = \delta B_\eta(x, y)$, where $B_\eta(x, y)$ is the the *Busemann function* defined by

$$B_\eta(x, y) = \lim_{z \to \eta} (d(x, z) - d(y, z)),$$

the corresponding equivariant family of measures is called a *conformal density of ratio* δ. Among these is the Patterson-Sullivan measure.

In [5] which also contains many references to related results, F. Ledrappier discusses various ways of building equivariant families of measures, and in particular relates them to other notions like *cross ratios*, *Gibbs currents*, *transverse invariant measures to the horospherical foliations etc.* As a conclusion, there exist numerous examples of equivariant families of measures, all mutually singular.

5.1.2. *End of the construction.* Let us go back to our construction now. First let β

$$(a,b,c) \mapsto \beta_{a,b,c}$$

be a Γ-equivariant map from O_3 to M. For instance, we can take the barycentre of the sum of the three Dirac measures concentrated at a, b, and c. Now define, if $x \in M$,

$$d\bar{\mu}^3(a,b,c) = e^{c_a(x,\beta_{a,b,c})+c_b(x,\beta_{a,b,c})+c_c(x,\beta_{a,b,c})} d\mu_x \otimes d\mu_x \otimes d\mu_x(a,b,c).$$

It follows from the definition of equivariant families of measure that this definition is independent on x. If Γ is a group of isometries then $\bar{\mu}^3$ is Γ-invariant. Furthermore, if Γ is cocompact then the corresponding measure is finite on O_3/Γ.

For $\bar{\nu}$, we can now just take the map $(a,b,c) \mapsto \mu_{\beta_{a,b,c}}$.

5.1.3. *Negatively curved* 3*-manifolds.* We have not used previously the hyperbolic structure. Let us take for a general negatively curved M and cocompact group of isometries Γ

- $W = \partial_\infty M$,
- $O_3 = \partial_\infty M^3 \setminus \Delta_3$,
- O_4 is any λ_4-invariant subset such that $p(O_4) = O_3$.

Then the previous construction works provided that O_4 is invariant under all σ_4^+. For instance, we could take $O_4 = U_4$, but the corresponding construction seems to be of no use for our problem. For the moment, I have not been able to construct configuration data adapted to the problem, for general negatively curved 3-manifolds.

5.2. *Complex cross ratio.* Let $[a;b;c;d]$ be the complex cross ratio of four points of $\mathbb{CP}^1$, such that $[0;1;\infty;z] = z$. Let $\Im(\alpha)$ be the imaginary part of the complex number α.

We will single out the geometric properties of the cross ratio which are actually used in Proposition 5.0.1.

5.2.1. *Disks.* We associate to every triple (a,b,c), a *disk* $D(a,b,c)$, defined by

$$D(a,b,c) = \{z \in \mathbb{CP}^1 \ \mid\ \Im([a;b;c;z]) < 0\}.$$

If we consider D as a map from O_3 to the set of subsets of $\partial_\infty M$, it enjoys the following properties:

(i) The map D is Γ-equivariant;

(ii) $D(a,b,c) \cup D(a,c,b)$ is dense;

(iii) For every (a,b,c) in O_3, a belongs to the closure of $D(a,b,c)$.

(iv) Let $O_4 = \{(a,b,c,d) \mid (a,b,c) \in O_3\ ,\quad d \in D(a,b,c)\}$, then O_4 is an open set invariant under the oriented permutation $(a,b,c,d) \mapsto (d,c,b,d)$.

In our particular case, all these properties are easily checked from the invariance of the cross ratio and its behaviour under permutations.

We will prove:

Proposition 5.2.1. *Let D satisfy properties* (i) *to* (iv) *of* 5.2.1. *Then the quadruple* $(\partial_\infty M, \Gamma, O_3, O_4)$ *represents Markov* (3,4)*-configuration data.*

In our very precise situation, we represents could devise a quick proof of that fact. However, we will give a somewhat longer proof: the idea is to stress the importance of properties (i) to (v) of 5.2.1, and forget a while the complex structure on $\partial_\infty M$.

Proof of the proposition. it only remains to prove (e) of Definition 4.1 which characterises Markov configuration data. Let us recall it:

(e) There exists some constant $p \in \mathbb{N}$, such that if (a,b,c) and (d,e,f) both belong to O_3, then there exists a sequence $(q_1, \dots, q_j)$ of elements of O_4, where $j \leq p$, such that if $q_n = (q_n^1, q_n^2, q_n^3, q_n^4)$ then $(q_1^1, q_1^2, q_1^3) = (a,b,c)$, $(q_j^3, q_j^2, q_j^4) = (d,e,f)$ and at last $(q_n^2, q_n^4, q_n^3) = (q_{n+1}^1, q_{n+1}^2, q_{n+1}^3)$ or $(q_n^3, q_n^2, q_n^4) = (q_{n+1}^1, q_{n+1}^2, q_{n+1}^3)$.

In order to proceed, we shall write that $(a,b,c) \overset{p}{\rightsquigarrow} (d,e,f)$ if (a,b,c) and (d,e,f) satisfy condition (e). With this notation at hand, one immediately checks:

- $(a,b,c) \overset{p}{\rightsquigarrow} (u,v,w)$ implies $(w,u,v) \overset{p}{\rightsquigarrow} (b,c,a)$;
- *composition rule*: if $t_1 \overset{p}{\rightsquigarrow} (a,b,c)$, and $(a,b,c) \overset{q}{\rightsquigarrow} t_3$ or $(b,c,a) \overset{q}{\rightsquigarrow} t_3$ then $t_1 \overset{p+q}{\rightsquigarrow} t_3$;
- $(a,b,c,d) \in O_4$ exactly means that $(a,b,c) \overset{1}{\rightsquigarrow} (c,b,d)$.

We are going to proceed through various steps.

Step 1. *For any* (a,b,c) *there exists* (a_1,b_1,c_1) *arbitrarily close to* (a,b,c) *such that* $(a,b,c) \overset{3}{\rightsquigarrow} (a_1,c_1,b_1)$.

We shall prove this using property (iii) of the Definition 5.2.1. First, using (iii) of 5.2.1, we choose b_1 arbitrarily close to b such that $(b,c,a,b_1) \in O_4$.

Next, using (iii) again, we choose c_1 arbitrarily close to c such that $(c,b,a,c_1) \in O_4$, and still, because O_4 is open, $(b,c_1,a,b_1) \in O_4$.

At last, using (iii) again, we choose a_1 arbitrarily close to a such that $(a,b,c,a_1) \in O_4$ and still (b,c_1,a_1,b_1) and (c,b,a_1,c_1) in O_4.

It follows that we have

$$
\begin{aligned}
(a,b,c) &\rightsquigarrow (c,b,a_1)\\
(c,b,a_1) &\rightsquigarrow (a_1,b,c_1)\\
(b,c_1,a_1) &\rightsquigarrow (a_1,c_1,b_1).
\end{aligned}
$$

The composition rule implies now that $(a,b,c) \overset{3}{\rightsquigarrow} (a_1,c_1,b_1)$.

Step 2. *For any* (a,b,c) *there exists* (a_1,b_1,c_1) *arbitrarily close to* (a,b,c) *such that* $(a,b,c) \overset{3}{\rightsquigarrow} (b_1,a_1,c_1)$.

The proof is symmetric: first we notice, using (iii), that we can find c_1 arbitrarily close to c such that

$$(c,a,b) \rightsquigarrow (b,a,c_1).$$

We choose b_1, arbitrarily close to b, such that

$$
\begin{aligned}
(b,a,c) &\rightsquigarrow (c,a,b_1)\\
(c,a,b_1) &\rightsquigarrow (b_1,a,c_1).
\end{aligned}
$$

Lastly, we choose a_1, arbitrarily close to a, such that

$$
\begin{aligned}
(a,b,c) &\rightsquigarrow (c,b,a_1)\\
(b,a_1,c) &\rightsquigarrow (c,a_1,b_1)\\
(c,a_1,b_1) &\rightsquigarrow (b_1,a_1,c_1).
\end{aligned}
$$

The composition rule implies the desired statement.

Step 3. *For any* (a_1,b_1,c_1) *close enough to* (a,b,c), $(a,b,c) \overset{36}{\rightsquigarrow} (a_1,b_1,c_1)$.

It suffices to prove that $(a,b,c) \overset{36}{\rightsquigarrow} (a,b,c)$. From the first step, we have that given (a,b,c) there exists (a_4,b_4,c_4) arbitrarily close to (a,b,c) such that

$$(a,b,c) \overset{3}{\rightsquigarrow} (a_4,c_4,b_4).$$

Next applying the first step one more time, we can choose (a_3,b_3,c_3) arbitrarily close to (a_4,b_4,c_4) such that

$$(a_4,c_4,b_4) \overset{3}{\rightsquigarrow} (a_3,b_3,c_3),$$

and this leads to

$$(a, b, c) \overset{6}{\rightsquigarrow} (a_3, b_3, c_3).$$

Actually we can choose (a_3, b_3, c_3) close enough to (a_4, b_4, c_4) such that still

$$(a, b, c) \overset{3}{\rightsquigarrow} (a_3, c_3, b_3).$$

At last, we choose thanks to Step 2, (a_2, b_2, c_2) arbitrarily close to (a_3, b_3, c_3) such that

$$(a_3, c_3, b_3) \overset{3}{\rightsquigarrow} (c_2, a_2, b_2),$$

and this implies

$$(a, b, c) \overset{6}{\rightsquigarrow} (c_2, a_2, b_2).$$

As before, we can choose (a_2, b_2, c_2) close enough so that we still have

$$(a, b, c) \overset{6}{\rightsquigarrow} (a_2, b_2, c_2).$$

From this last relation, we obtain

$$(c_2, a_2, b_2) \overset{6}{\rightsquigarrow} (b, c, a).$$

Now

$$(a, b, c) \overset{12}{\rightsquigarrow} (b, c, a).$$

Hence

$$(a, b, c) \overset{36}{\rightsquigarrow} (a, b, c).$$

Step 4. *For any a, b, c and any permutation σ,*

$$(a, b, c) \overset{100}{\rightsquigarrow} (\sigma(a), \sigma(b), \sigma(c)).$$

This follows easily from the previous steps.

Step 5. *For any a, b, c there exists an open dense set of d such that*

$$(a, b, c) \overset{300}{\rightsquigarrow} (b, c, d).$$

Indeed, from hypothesis (ii) of 5.2.1, there exists an open dense case we are done — or $(a, c, b) \overset{1}{\rightsquigarrow} (b, c, d)$ and we obtain our assertion using Step 4 twice.

Final step. For any (a, b, c, d, e, f), we have $(a, b, c) \overset{1000}{\rightsquigarrow} (d, e, f)$.

Using Step 5 three times, we have an open dense set of (u, v, w) such that $(a, b, c) \overset{900}{\rightsquigarrow} (u, v, w)$, hence our conclusion, thanks to Step 3.

The proof is complete although, obviously, 1000 is not the optimal constant. Also this proof is far too complicated in our case, but one of my hopes is to build a map D satisfying (i), (ii), (iii) (iv) and (v) of 5.2.1 for a general negatively curved 3-manifold. $\square$

6. Convex surfaces and configuration data

Again let $N = M/\Gamma$ be a compact hyperbolic 3-manifold. In the last section we built configuration data associated to that situation, and we can extend that to measured configuration data in many ways (cf. 4.1.4).

We consider now the restricted configuration space $\bar{\mathcal{B}}_\infty$, associated to that situation. According to the construction of Theorem 4.2.1, this space comes equipped with a measure μ, invariant under Γ and ergodic under the action of the ideal triangle group F.

We turn $\bar{\mathcal{B}}_\infty$ into an ergodic Riemannian lamination by a suspension procedure; namely we consider

$$\mathcal{F} = (\bar{\mathcal{B}}_\infty \times \mathbb{H}^2)/F,$$

where F acts as an isometry group on $\mathbb{H}^2$ and diagonally on $\bar{\mathcal{B}}_\infty \times \mathbb{H}^2$.

The ergodic and $\Gamma \times F$-invariant measure μ gives rise to a transversal Γ-invariant and ergodic measure on $\mathcal{F}$ that we shall also call μ.

Our aim is now to prove:

PROPOSITION 6.0.1. *There exists a continuous leaf-preserving map Φ with dense image from $\mathcal{F}/\Gamma$ to $\mathcal{N}$, the space of k-surfaces in N.*

This proposition, loosely speaking, explains our combinatorial construction codes for convex surfaces. As a corollary, we obtain our main theorem

THEOREM 6.0.2. *Let $N = M/\Gamma$ be a compact negatively curved 3-manifold whose metric can be deformed through negatively curved metrics to a hyperbolic one. Then there exist infinitely many mutually singular ergodic transversal measures of full support on $\mathcal{N}$, the space of k-surfaces of N.*

6.1. *Bent and pleated surfaces.* Recall that a $\mathbb{CP}^1$-*surface* is a surface locally modelled on $\mathbb{CP}^1$.

We shall recall facts about (locally convex) pleated surfaces, mostly without demonstrations, especially when dealing with the relation between measured geodesic laminations and $\mathbb{CP}^1$-structures which has been described by W. Thurston. A useful reference is [7], where H. Tanigawa gives a description and some results about this relation.

The main fact about this construction is the following: to every hyperbolic surface S (maybe incomplete) and every measured geodesic lamination μ we can associate

- a pleated locally convex surface in the hyperbolic space,
- a 3-manifold $B(S, \mu)$, the *end* of (S, μ),
- a $\mathbb{CP}^1$-surface Σ which will be be the boundary at infinity of the end.

The map which associates to (S, μ) the $\mathbb{CP}^1$-surface Σ is called the *Thurston map*, and we shall denote it by Θ. Notice that since S is not assumed to be complete, this map has no reason to be injective.

6.1.1. *An example.* For the sake of completeness, we briefly recall Thurston's construction in a special case, which will be the one actually needed.

Let S be a open subset of $\mathbb{H}^2$ (maybe incomplete) which is the union of totally geodesic ideal polygons. To every edge e of this tiling, we associate a positive number θ_e less than π. The datum μ consisting of the edges of the tiling and of the assigned positive numbers is a specific example of a geodesic lamination.

We may think of every polygon T as totally geodesically embedded in $\mathbb{H}^3$. Let n_T be the exterior normal field along T. Let p_T be the map from $T\times\ ]0, \infty[$ defined by

$$p_T \ : \ (x, s) \mapsto \exp((sn_T(x)).$$

Let P_T be the *prism* over T, i.e. the image of p_T.

Let e be an edge of the tiling of S, the intersection of two polygons T_0^e and T_1^e and considered as a geodesic in $\mathbb{H}^3$. The θ_e-*wedge* over e is the closed set delimited by the two half-planes whose boundary is e and which forms an angle θ_e.

Finally, the *end* $B(S, \mu)$ of (S, μ) is the union of all prisms and edges. Notice that there is a canonical isometric local homeomorphism from M_Σ to $\mathbb{H}^3$. In this special case, S is isometrically immersed in $\mathbb{H}^3$ as a pleated surface.

6.1.2. *Facts.* The following propositions, whose proofs follow from results explained in [7], summarised the property of Thurston's construction needed in the sequel. All these properties rely on the next observation:

OBSERVATION 6.1.1. *Let S be a hyperbolic surface* (*maybe incomplete*) *and μ a geodesic lamination. Let D be an embedded $\mathbb{CP}^1$-disk in $\Theta(S, \mu)$. Then there is an embedding of the* (*hyperbolic*) *half space P in $B(S, \mu)$ such that, the boundary at infinity of this embedded P is precisely D.*

From this observation, we deduce easily the following results.

Proposition 6.1.2. *Let S be a (maybe incomplete) hyperbolic surface and μ_1 a measured geodesic lamination on S. Let μ_2 be a measured geodesic lamination on $\mathbb{H}^2$ supported on finitely many geodesics. Assume $\Theta(\mathbb{H}^2, \mu_2)$ injects by f (as a $\mathbb{CP}^1$-surface) in $\Theta(S, \mu_1)$. Then $B(\mathbb{H}^2, \mu_2)$ injects in $B(S, \mu_1)$ in such a way that the associated injection between the boundaries at infinity is f.*

Proposition 6.1.3. *Let M be a $\mathbb{CP}^1$-surface. Then there exists an exhaustion of M by relatively compact $\mathbb{CP}^1$-surfaces M_i such that $M_i = \Theta(\mathbb{H}^2, \mu_i)$ where μ_i is supported on finitely many geodesics.*

Let S be a locally convex immersed surface in $\mathbb{H}^3$. Let n be its exterior normal field. We define the *end* to be the 3-manifold B_S diffeomorphic to $S\times]0, \infty[$ equipped with the hyperbolic metric induced by the immersion $(s, t) \mapsto \exp(tn(s))$. In particular, the boundary at infinity S_∞ of B_S is a $\mathbb{CP}^1$-surface.

Proposition 6.1.4. *Let S be a locally convex immersed surface in $\mathbb{H}^3$ such that B_∞ injects as a $\mathbb{CP}^1$-surface in $\Theta(\mathbb{H}^2, \mu)$. Then B_S injects in $B(\mathbb{H}^2, \mu)$.*

6.2. *Tilings and related definitions.* We shall denote by $T(a, b, c)$ the ideal triangle in $\mathbb{H}^2$ whose vertices are a, b and c in $\partial_\infty \mathbb{H}^2 = \mathbb{RP}^1$.

Recall that we consider $\mathbb{H}^2$ periodically tiled by ideal triangles. Let us denote $\mathcal{T}^0$ the collection of ideal triangles of this triangulation. The set $B = \mathbb{QP}^1$ is the set of vertices at infinity of this triangulation (cf. 4.0.1).

Notice now that every monotone map g from $B = \mathbb{QP}^1$ to $\partial_\infty \mathbb{H}^2 = \mathbb{RP}^1$ defines a tiling by ideal triangles of an open set U_g of $\mathbb{H}^2$. This triangulation is given by the collection $\mathcal{T}^g$ of triangles defined by

$$\mathcal{T}^g = \{T(g(a), g(b), g(c))/T(a, b, c) \in \mathcal{T}^0\}.$$

With this notation we have:

$$U_g = \bigcup_{T \in \mathcal{T}^g} T.$$

6.2.1. *Pleated surfaces and tilings.* We will prove the following elementary proposition:

Proposition 6.2.1. *For every $f \in \bar{\mathcal{B}}_\infty$, there exists a unique monotone map $g(f)$ from B to $\partial_\infty \mathbb{H}^2$, a unique map ψ_f from $U_{g(f)}$ to $\mathbb{H}^3$, such that its restriction to every tile $T(a, b, c)$ is totally geodesic, and the ideal triangle $\psi_f(T(a, b, c))$ has $f(a)$, $f(b)$ and $f(c)$ as vertices at infinity. Furthermore, $S_f^0 = \psi_f$ is locally convex and ψ_f depends continuously on f.*

Using this proposition, we introduce the following notation: we shall denote by μ_f the geodesic lamination on $U_{g(f)}$ whose support is the set of edges of $\mathcal{T}^g$, each edge being labelled with the angle of the two corresponding triangles in $\mathbb{H}^3$. Then we shall write B_f for $B(U_{g(f)}, \mu_f)$ and S_f^∞ for $\Theta(U_{g(f)}, \mu_f)$.

Proof. The construction of ψ_f is described in the statement of the proposition. The only point to check is the local convexity of S_f^0. This follows at once from the next observation: three points (a, b, c) at infinity in $\mathbb{H}^3$ determine an oriented totally geodesic plane P in $\mathbb{H}^3$, and the points in $\mathbb{CP}^1$ "below" P are precisely those points d such that $\Im(a, b, c, d) < 0$. □

6.2.2. *The tiling map.* Later on, we shall need a technical device, called a *tiling map*, associated to every element of $\bar{\mathcal{B}}_\infty$.

Let $C^0(\mathbb{H}^2, \mathbb{CP}^1)$ be the space of continuous maps from $\mathbb{H}^2$ to $\mathbb{CP}^1$ with the topology of uniform convergence on every compact set. We use the notation of Section 6.1.

The following proposition is obvious.

Proposition 6.2.2. *There exists a continuous map ξ*

$$\begin{cases} \bar{\mathcal{B}}_\infty & \rightarrow & C^0(\mathbb{H}^2, \mathbb{CP}^1) \\ f & \mapsto & \xi_f. \end{cases}$$

which satisfies the following properties:

(i) *There exists a homeomorphism h_f from $\mathbb{H}^2$ to S_f^∞, such that $\xi_f = i_f \circ h_f$.*

(ii) *For every $T(a_1, a_2, a_3)$ in $\mathcal{T}^0$, the map $\xi_f|_{T(a_1,a_2,a_3)}$ extends continuously to $\{a_1, a_2, a_3\}$ in such a way that $\xi_f(a_i) = f(a_i)$.*

(iii) *For every element γ in F, $\xi_{f\circ\gamma} = \xi_f \circ \gamma$.*

By definition, ξ_f is a *tiling map* associated to f.

6.3. *k-surfaces and asymptotic Plateau problems.* We recall definitions and results from [1] that we specialise in the case of $\mathbb{H}^3$.

Let S be a locally convex surface immersed in $\mathbb{H}^3$. Let ν_S be the exterior normal vector field to S. The *Gauss-Minkowski* (Figure 5) map from S to $\partial_\infty\mathbb{H}^3$ is the local homeomorphism n_S:

$$\begin{cases} S & \rightarrow & \partial_\infty\mathbb{H}^3 \\ x & \mapsto & n_S(x) = \exp(\infty\nu_S(x)). \end{cases}$$

An *asymptotic Plateau problem* is a pair (i, U) where U is a surface, and i is a local homeomorphism from U to $\partial_\infty\mathbb{H}^3$. A k-solution to an asymptotic Plateau problem (i, U), is a k-surface S immersed in $\mathbb{H}^3$, such that there exists a homeomorphism g from U to S such that $i = n_s \circ g$.

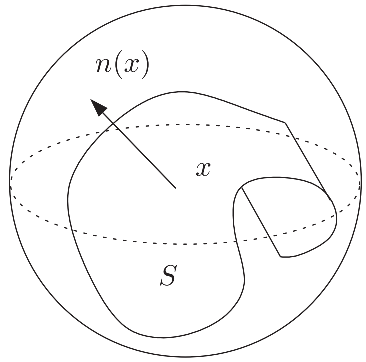

Figure 5: The Gauss-Minkowski map

We proved (Theorem A of [1]) that there exists at most one solution of a given asymptotic Plateau problem. We also proved (Theorem E of [1]) that if (i, U) is an asymptotic Plateau problem, and if O is a relatively compact open set of U, then (i, O) admits a solution.

We need the following proposition which uses the notation of Section 6.1.

PROPOSITION 6.3.1. *Let f be an element of $\bar{\mathcal{B}}_\infty$; then the asymptotic Plateau problem (i_f, S_f^∞) admits a k-solution.*

Proof. Using Proposition 6.1.3, we have an exhaustion of S_f^∞, by $\mathbb{CP}^1$-surfaces M_i, such that $M_i = \Theta(\mathbb{H}^2, \mu_i)$ where μ_i is supported on finitely many geodesics. Let $B_i = B(\mathbb{H}^2, \mu_i)$. According to 6.1.2, we have

$$B_i \subset B_{i+1} \subset B_f.$$

Since M_i is relatively compact in S_f^∞, there exists, according to Theorem E of [1], a k-solution Σ_i to the asymptotic Plateau problem defined by M_i. According to Proposition 6.1.4,

$$B_{\Sigma_i} \subset B_i \subset B_f.$$

Let

$$W = \bigcup_{i \in \mathbb{N}} B_{\Sigma_i}.$$

We wish now to prove that ∂W the boundary of W is a k-surface solution of the asymptotic Plateau problem defined by S_f^∞.

First we should notice that since $B_{\Sigma_i} \subset B_f$, there exists a constant A just depending on f, such that every ball $\Sigma_i(x, A)$ of centre x and radius A in Σ_i, when considered immersed in $\mathbb{H}^3$, is a subset of the boundary of a convex set. Hence, according to Lemma 5.4(iii) of [2], there exists a constant C such that if H is the mean curvature of Σ_i and $d\sigma$ the area element:

$$\int_{\Sigma_i(x,A)} H d\sigma \leq C.$$

Let now x_i be a point in Σ_i. Assume the sequence $\{x_i\}_{i\in\mathbb{N}}$ converges in the metric completion of B_f to a point x_0. We conclude from Theorem D of [2] that $\{(\Sigma_i, x_i)\}_{i\in\mathbb{N}}$ converges smoothly to a pointed k-surface (Σ_∞, x_0). It follows that x_0 is in the interior of B_f. Indeed, let ∂B_f be the boundary of the metric completion of B_f. Notice that every point of ∂B_f is included in an open geodesic segment drawn on ∂B_f. If x_0 belongs to ∂B_f, it would follow that the corresponding open segment is actually drawn on Σ_∞ and this is impossible.

This argument finally shows that ∂W is a k-surface, and that every half infinite geodesic joining a point of ∂B_f to a point of S_f^∞ intersects ∂W. The conclusion follows. □

6.4. *Construction of the map* Φ. In this section, we summarise the previous sections and build a continuous map Φ from $\mathcal{F}$ to $\mathcal{N}$.

Let $f \in \bar{\mathcal{B}}_\infty$. Let ξ_f be the tiling map of f (cf. 6.2.2). Let Σ_f be the k-solution of the asymptotic Plateau problem (i_f, S_f^∞) (cf. 6.3.1). Let n_f be the Gauss-Minkowski map of Σ_f. We define Φ by

$$\Phi([f,x]) = (\Sigma_f, n_f^{-1}(\xi_f(x))).$$

Continuity follows from the uniqueness of the solution of an asymptotic Plateau problem.

6.5. *Density of the image of* Φ. The only point left to be proved in Proposition 6.0.1 is the density of the image of Φ.

We start with an observation. Let S be a compact surface, $\tilde{S}$ its universal cover. Let μ_1 be a measured lamination on S supported on finitely many geodesics. Assume the weight of every geodesic is strictly less than π. Then, from the construction explained above we deduce that $\Theta(\tilde{S}, \mu_1)$ lies in the image of Φ.

According to 2.4.1, the union of compact leaves of $\mathcal{N}$ is dense. It therefore suffices to prove that every compact k-surface belongs to the closure of the image of Φ.

Let S be such a compact k-surface in N. The underlying surface admits a $\mathbb{CP}^1$-structure induced by the Minkowski-Gauss map. According to Thurston's parametrisation theorem [7], such a surface is of the form $\Theta(S, \mu_0)$ for a certain measured lamination μ_0. We proved, using different words (Corollary 1 of [3]) that the map which associates to every $\mathbb{CP}^1$-structure on a compact surface, the k-surface solution of the corresponding asymptotic Plateau problem, is continuous. To complete our proof, we just have to note that the set of measured geodesic laminations with finite support and such that the weight of every geodesic is strictly less than π is dense in the space of all measured geodesic laminations.

7. Conclusion

It remains to combine the main propositions of the previous sections to obtain the proof of our main result.

From the stability property, it suffices to build a transverse invariant measure of full support on $\mathcal{N}$, whenever N has constant curvature. Let $\Gamma = \pi_1(N)$, and $F = \mathrm{PSL}(2,\mathbb{Z})$.

We consider the restricted configuration space $\bar{\mathcal{B}}_\infty$, subset of the space of maps from $\mathbb{QP}^1$ to $\mathbb{CP}^1$, as defined in subsection 4.0.3, and associated to the Markov configuration data (as defined in 4.1) coming from the complex cross ratio on $\partial_\infty M\mathbb{H}^3$, according to Section 5 and Proposition 5.2.1.

We can now turn these configuration data into measured ones as shown in 5.1.

Thanks now to the main result of Section 4, Theorem 4.2.1, we obtain a finite F-invariant ergodic measure of full support on $\bar{\mathcal{B}}_\infty/\Gamma$. Here, the action of F is by right composition.

Furthermore, choices of mutually singular equivariant families of measures lead to mutually singular transversal measures.

Next, we suspend the action of F on $\bar{\mathcal{B}}_\infty$. Namely, we consider the Riemannian lamination.

$$\mathcal{F} = (\bar{\mathcal{B}}_\infty \times \mathbb{H}^2)/F,$$

where F acts as an isometry group on $\mathbb{H}^2$ and diagonally on $\bar{\mathcal{B}}_\infty \times \mathbb{H}^2$.

The finite ergodic and F-invariant measure on $\bar{\mathcal{B}}_\infty/\Gamma$ gives rise to a transversal Γ-invariant and ergodic measure on $\mathcal{F}$ called μ.

Finally, Proposition 6.0.1 defines a map Φ from $\mathcal{F}/\Gamma$ to $\mathcal{N}$, which is leaf-preserving, continuous with a dense image. Therefore, we can pushforward μ using Φ to obtain a transversal ergodic finite measure of full support.

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