Addendum to “Semistable sheaves in positive characteristic”*

By ADRIAN LANGER

In this short note we fill in the gap in [La, 3.5] and prove a few small improvements of some results of [La]. We keep the notation from [La].

All the theorems and statements in [La] remain valid and unaffected, except for Theorem 3.1, where the word “general” should be replaced with “very general”, so that $\mu_i$ and $r_i$ are well defined. The point is that if not all $D_2, \ldots, D_{n-1}$ are ample then it is not clear if $E|_D$ has the same type of the Harder–Narasimhan filtration for a general divisor $D \in |D_1|$. This difficulty vanishes if all $D_2, \ldots, D_{n-1}$ are ample since in this case semistability with respect to such a collection of divisors is an open property.

First, the author would like to mention that in the proof of Theorems 3.1, 3.2, 3.3 and 3.4 there was a tacit assumption that the base field $k$ was not countable. Since semistable sheaves are well behaved under the base field extension, the statements do not depend on the field and we could assume it.

The beginning of [La, 3.5] should be replaced with the following.

3.5.' It is sufficient to prove that $T^1(r)$ and $T^3(r-1)$ imply $T^5(r)$.

We prove this implication by induction on the dimension of $X$. If $X$ is a surface then the implication can be proved as in [La, 3.5]. So assume that the implication holds for all varieties of dimension less than $n$ for some $n \geq 3$. Take a collection $D_1, \ldots, D_{n-1}$ of very ample divisors and a strongly $(D_1, D_2, \ldots, D_{n-1})$-semistable sheaf $E$.

Assume that contrary to the implication we have $\Delta(E)D_2 \ldots D_{n-1} < 0$ and set $B_t = ((1-t)D_1 + tD_2)D_2 \ldots D_{n-1}$ for $t \in [0, 1]$.

If $E$ is strongly $B_1$-semistable then $T^1(r)$ implies that the restriction of $E$ to a general divisor in $|D_2|$ is semistable. Since $(F^k)^*E$ is also strongly semistable the restriction of $(F^k)^*E$ to a general divisor in $|D_2|$ is also semistable. Therefore the restriction of $E$ to a very general divisor $D$ in $|D_2|$ is

strongly \((D_2|_D, \ldots, D_{n-1}|_D)\)-semistable. Then by the induction assumption
we have

\[
\Delta(E)D_2D_3\ldots D_{n-1} = \Delta(E|_D)D_3\ldots D_{n-1} \geq 0,
\]
a contradiction.

If \(E\) is not strongly \(B_1\)-semistable then for sufficiently large \(k\) the sheaf
\((F^k)^*E\) is not \(B_1\)-semistable. Therefore there exists \(0 \leq t_k < 1\) such that
\((F^k)^*E\) is \(B_{t_k}\)-semistable but it is not \(B_1\)-semistable for \(t_k < t \leq 1\) (obviously,
being non-semistable is an open condition in the set of polarizations). Similarly
as in [La, 3.6] one can easily see that the Harder-Narasimhan filtration of
\(B_t\) is independent of \(t\) if the difference \((t - t_k)\) is small and positive. This
filtration provides us with a proper saturated subsheaf \(E' \subset (F^k)^*E\) such that
\(\xi_{E',(F^k)*E}B_{t_k} = 0\). Hence \(\xi_{E',E''}B_{t_k} = 0\), where \(E'' = (F^k)^*E/E'\). By the
Hodge index theorem we get

\[
\xi^2_{E',E''}D_2\ldots D_{n-1} \cdot ((1 - t_k)D_1 + t_kD_2)^2D_2\ldots D_{n-1} \leq \xi_{E',E''}B_{t_k}^2 = 0.
\]

Note that by assumption \(d(t_k) = ((1 - t_k)D_1 + t_kD_2)^2D_2\ldots D_{n-1} > 0\), so we have

\[
\xi^2_{E',E''}D_2\ldots D_{n-1} \leq 0.
\]

Set \(r' = rkE'\) and \(r'' = rkE''\) and \(\beta_r(t) = \beta_r(A; (1 - t)D_1 + tD_2, D_2, \ldots, D_{n-1})\). Since both \(E'\) and \(E''\) are \(B_{t_k}\)-semistable, \(T^3(r - 1)\) and the above
inequality imply that

\[
\frac{\Delta((F^k)^*E)D_2\ldots D_{n-1}}{r} = \frac{\Delta(E'')D_2\ldots D_{n-1}}{r'} + \frac{\Delta(E'')D_2\ldots D_{n-1}}{r''}
\]

\[
-\frac{r'r''}{r} \xi^2_{E',E''}D_2\ldots D_{n-1} \geq -\frac{1}{d(t_k)} \left( \frac{\beta_r(t_k)}{r'} + \frac{\beta_r(t_k)}{r''} \right) \geq -\frac{\beta_r(t_k)}{rd(t_k)}.
\]

This implies that

\[
\Delta(E)D_2\ldots D_{n-1} \geq -\frac{\beta_r(t_k)}{d(t_k)p^{2k}}.
\]

Since \(-\frac{\beta_r(t)}{d(t)^3}\) is a continuous function for \(t \in [0, 1]\), it can be uniformly bounded
from below. So passing with \(k\) to infinity, we get \(\Delta(E)D_2\ldots D_{n-1} \geq 0\), a
contradiction.

The statement of [La, Th. 3.12] can be simplified by the following remark.
Assume that \(\text{char } k = p\). Then

\[
\inf \left\{ \frac{\beta_r(A; D, D_2, \ldots D_{n-1})}{D^2D_2\ldots D_{n-1}} \mid D \text{ is nef and } D^2D_2\ldots D_{n-1} > 0 \right\} = \left( \frac{r(r - 1)}{p - 1} \right)^2 A^2D_2\ldots D_{n-1}.
\]
Indeed, by the Hodge index theorem we have
\[
\frac{(ADD_2 \cdots D_{n-1})^2}{D^2D_2 \cdots D_{n-1}} \geq A^2D_2 \cdots D_{n-1}
\]
and equality holds for \( D = A \).

The following theorem is an improvement of [La, Th. 4.1] (with a simplified proof).

**Theorem 4.1’.** Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \). Assume that \( n = \dim X \geq 2 \). Let \( E \) be a rank \( r \geq 2 \) torsion free sheaf on \( X \). Assume that \( H_1, \ldots, H_{n-1} \) are very ample and let \( D_l \) be a very general complete intersection in \(|H_1| \cap \cdots \cap |H_l|\). Set \( a = H_1^2H_2 \cdots H_{n-1} \). Then
\[
(L_{\max}(E|_{D_l}) - L_{\max}(E|_{D_l}))^2 \\
\leq r^l(L_{\max}(E) - L_{\min}(E))^2 + \frac{2a(r^l - 1)}{r(r - 1)} \Delta(E)H_2 \cdots, H_{n-1}
\]
for \( l = 1, \ldots, n-1 \).

**Proof.** By [La, Cor. 3.11] we have
\[
(L_{\max}(E|_D) - L_{\min}(E|_D))^2 \leq r(L_{\max}(E) - L_{\min}(E))^2 + \frac{2a}{r} \Delta(E)H_2 \cdots, H_{n-1}
\]
(see also [La, the proof of Th. 4.1]). Then one can easily get the required inequality by induction on \( l \).

Note that both the above Theorem 4.1’ and [La, Th. 4.1] can become trivial if the base field \( k \) is countable. This does not affect the proofs of [La, Th. 4.2 and Th. 4.4] since it is sufficient to prove these theorems after the base field extension. Alternatively, one can use the following analogue of [La, Cor. 3.11]:

**Corollary 3.11’.** Assume that \( D_1 \) is very ample and \( D_2, \ldots, D_{n-1} \) are ample. Let \( D \) be a general divisor in \(|D_1|\). Then
\[
\frac{r}{2}(\mu_{\max}(E|_D) - \mu_{\min}(E|_D))^2 \leq d \Delta(E)D_2 \cdots D_{n-1} + 2r^2(L_{\max} - \mu)(\mu - L_{\min}).
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**References**


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