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Addendum to "Semistable sheaves in positive characteristic"*

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In this short note we fill in the gap in [La, 3.5] and prove a few small improvements of some results of [La]. We keep the notation from [La].

All the theorems and statements in [La] remain valid and unaffected, except for Theorem 3.1, where the word "general" should be replaced with "very general", so that μ_i and r_i are well defined. The point is that if not all $D_2, \ldots, D_{n-1}$ are ample then it is not clear if $E|_D$ has the same type of the Harder–Narasimhan filtration for a general divisor $D \in |D_1|$. This difficulty vanishes if all $D_2, \ldots, D_{n-1}$ are ample since in this case semistability with respect to such a collection of divisors is an open property.

First, the author would like to mention that in the proof of Theorems 3.1, 3.2, 3.3 and 3.4 there was a tacit assumption that the base field k was not countable. Since semistable sheaves are well behaved under the base field extension, the statements do not depend on the field and we could assume it.

The beginning of [La, 3.5] should be replaced with the following.

3.5.′ It is sufficient to prove that $T^1(r)$ and $T^3(r-1)$ imply $T^5(r)$.

We prove this implication by induction on the dimension of X. If X is a surface then the implication can be proved as in [La, 3.5]. So assume that the implication holds for all varieties of dimension less than n for some $n \geq 3$. Take a collection $D_1, \ldots, D_{n-1}$ of very ample divisors and a strongly $(D_1, D_2, \ldots, D_{n-1})$-semistable sheaf E.

Assume that contrary to the implication we have $\Delta(E)D_2 \ldots D_{n-1} < 0$ and set $B_t = ((1-t)D_1 + tD_2)D_2 \ldots D_{n-1}$ for $t \in [0,1]$.

If E is strongly B_1-semistable then $T^1(r)$ implies that the restriction of E to a general divisor in $|D_2|$ is semistable. Since $(F^k)^*E$ is also strongly semistable the restriction of $(F^k)^*E$ to a general divisor in $|D_2|$ is also semistable. Therefore the restriction of E to a very general divisor D in $|D_2|$ is

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strongly $(D_2|_D, \dots, D_{n-1}|_D)$-semistable. Then by the induction assumption we have

$$\Delta(E)D_2D_3\dots D_{n-1} = \Delta(E|_D)D_3\dots D_{n-1} \geq 0,$$

a contradiction.

If E is not strongly B_1-semistable then for sufficiently large k the sheaf $(F^k)^*E$ is not B_1-semistable. Therefore there exists $0 \leq t_k < 1$ such that $(F^k)^*E$ is B_{t_k}-semistable but it is not B_t-semistable for $t_k < t \leq 1$ (obviously, being non-semistable is an open condition in the set of polarizations). Similarly as in [La, 3.6] one can easily see that the Harder-Narasimhan filtration of B_t is independent of t if the difference $(t - t_k)$ is small and positive. This filtration provides us with a proper saturated subsheaf $E' \subset (F^k)^*E$ such that $\xi_{E',(F^k)^*E}B_{t_k} = 0$. Hence $\xi_{E',E''}B_{t_k} = 0$, where $E'' = (F^k)^*E/E'$. By the Hodge index theorem we get

$$\xi^2_{E',E''}D_2\dots D_{n-1} \cdot ((1-t_k)D_1 + t_kD_2)^2D_2\dots D_{n-1} \leq (\xi_{E',E''}B_{t_k})^2 = 0.$$

Note that by assumption $d(t_k) = ((1-t_k)D_1 + t_kD_2)^2D_2\dots D_{n-1} > 0$, so we have

$$\xi^2_{E',E''}D_2\dots D_{n-1} \leq 0.$$

Set $r' = \operatorname{rk} E'$ and $r'' = \operatorname{rk} E''$ and $\beta_r(t) = \beta_r(A; (1-t)D_1 + tD_2, D_2\dots, D_{n-1})$. Since both E' and E'' are B_{t_k}-semistable, $T^3(r-1)$ and the above inequality imply that

$$\begin{aligned} \frac{\Delta((F^k)^*E)D_2\dots D_{n-1}}{r} &= \frac{\Delta(E')D_2\dots D_{n-1}}{r'} + \frac{\Delta(E'')D_2\dots D_{n-1}}{r''} \\ -\frac{r'r''}{r}\xi^2_{E',E''}D_2\dots D_{n-1} &\geq -\frac{1}{d(t_k)}\left(\frac{\beta_{r'}(t_k)}{r'} + \frac{\beta_{r''}(t_k)}{r''}\right) \geq -\frac{\beta_r(t_k)}{rd(t_k)}. \end{aligned}$$

This implies that

$$\Delta(E)D_2\dots D_{n-1} \geq -\frac{\beta_r(t_k)}{d(t_k)p^{2k}}.$$

Since $\frac{-\beta_r(t)}{d(t)}$ is a continuous function for $t \in [0,1]$, it can be uniformly bounded from below. So passing with k to infinity, we get $\Delta(E)D_2\dots D_{n-1} \geq 0$, a contradiction. □

The statement of [La, Th. 3.12] can be simplified by the following remark. Assume that $\operatorname{char} k = p$. Then

$$\inf\left\{\frac{\beta_r(A; D, D_2, \dots D_{n-1})}{D^2D_2\dots D_{n-1}}\ D \text{ is nef and } D^2D_2\dots D_{n-1} > 0\right\} = \left(\frac{r(r-1)}{p-1}\right)^2 A^2D_2\dots D_{n-1}.$$

Indeed, by the Hodge index theorem we have

$$\frac{(ADD_2 \dots D_{n-1})^2}{D^2 D_2 \dots D_{n-1}} \geq A^2 D_2 \dots D_{n-1}$$

and equality holds for $D = A$.

The following theorem is an improvement of [La, Th. 4.1] (with a simplified proof).

THEOREM 4.1′. *Let X be a smooth projective variety defined over an algebraically closed field k. Assume that $n = \dim X \geq 2$. Let E be a rank $r \geq 2$ torsion free sheaf on X. Assume that $H_1, \dots, H_{n-1}$ are very ample and let D_l be a very general complete intersection in $|H_1| \cap \dots \cap |H_l|$. Set $a = H_1^2 H_2 \dots H_{n-1}$. Then*

$$(L_{\max}(E|_{D_l}) - L_{\max}(E|_{D_l}))^2 \leq r^l (L_{\max}(E) - L_{\min}(E))^2 + \frac{2a(r^l-1)}{r(r-1)} \Delta(E) H_2 \dots, H_{n-1}$$

for $l = 1, \dots, n-1$.

Proof. By [La, Cor. 3.11] we have

$$(L_{\max}(E|_D) - L_{\min}(E|_D))^2 \leq r(L_{\max}(E) - L_{\min}(E))^2 + \frac{2a}{r} \cdot \Delta(E) H_2 \dots, H_{n-1}$$

(see also [La, the proof of Th. 4.1]). Then one can easily get the required inequality by induction on l. □

Note that both the above Theorem 4.1′ and [La, Th. 4.1] can become trivial if the base field k is countable. This does not affect the proofs of [La, Th. 4.2 and Th. 4.4] since it is sufficient to prove these theorems after the base field extension. Alternatively, one can use the following analogue of [La, Cor. 3.11]:

COROLLARY 3.11′. *Assume that D_1 is very ample and $D_2, \dots, D_{n-1}$ are ample. Let D be a general divisor in $|D_1|$. Then*

$$\frac{r}{2}(\mu_{\max}(E|_D) - \mu_{\min}(E|_D))^2 \leq d\Delta(E) D_2 \dots D_{n-1} + 2r^2 (L_{\max} - \mu)(\mu - L_{\min}).$$

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REFERENCES

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